Electronic Journal of Probability

Electron. J. Probab. **25** (2020), article no. 79, 1–20.
ISSN: 1083-6489 https://doi.org/10.1214/20-EJP482

Convergence to scale-invariant Poisson processes and applications in Dickman approximation*

Chinmoy Bhattacharjee[†] Ilya Molchanov[†]

Abstract

We study weak convergence of a sequence of point processes to a scale-invariant simple point process. For a deterministic sequence $(z_n)_{n\in\mathbb{N}}$ of positive real numbers increasing to infinity as $n \to \infty$ and a sequence $(X_k)_{k\in\mathbb{N}}$ of independent non-negative integer-valued random variables, we consider the sequence of point processes

$$\nu_n = \sum_{k=1}^{\infty} X_k \delta_{z_k/z_n}, \quad n \in \mathbb{N},$$

and prove that, under some general conditions, it converges vaguely in distribution to a scale-invariant Poisson process η_c on $(0,\infty)$ with the intensity measure having the density ct^{-1}, $t \in (0,\infty)$. An important motivating example from probabilistic number theory relies on choosing $X_k \sim \mathrm{Geom}(1-1/p_k)$ and $z_k = \log p_k$, $k \in \mathbb{N}$, where $(p_k)_{k\in\mathbb{N}}$ is an enumeration of the primes in increasing order. We derive a general result on convergence of the integrals $\int_0^1 t\nu_n(dt)$ to the integral $\int_0^1 t\eta_c(dt)$, the latter having a generalized Dickman distribution, thus providing a new way of proving Dickman convergence results.

We extend our results to the multivariate setting and provide sufficient conditions for vague convergence in distribution for a broad class of sequences of point processes obtained by mapping the points from $(0,\infty)$ to $\mathbb{R}^d$ via multiplication by i.i.d. random vectors. In addition, we introduce a new class of multivariate Dickman distributions which naturally extends the univariate setting.

Keywords: Poisson processes; Vague convergence; scale invariance; random measures; Dickman distributions.
AMS MSC 2010: 60G55; 11K99; 60F05; 60G57.
Submitted to EJP on November 27, 2019, final version accepted on June 8, 2020.
Supersedes arXiv:1911.06229.

*Supported by the Swiss National Science Foundation Grant No. 200021_175584.

[†]Institut für Mathematische Statistik und Versicherungslehre, University of Bern, Switzerland.
E-mail: chinmoy.bhattacharjee@stat.unibe.ch,ilya.molchanov@stat.unibe.ch

1 Introduction

Consider a locally compact separable metric space S with Borel σ-algebra $\mathfrak{S}$. Let $\mathcal{M}(S)$ denote the space of all locally finite non-negative measures on S. This space is endowed with the *vague* topology generated by assuming continuity of the integration maps $\mu \mapsto \mu f = \int_S f(x)\mu(dx)$ for all f from the family $\widehat{C}_S$ of bounded non-negative continuous functions on S with relatively compact support. A *random measure* ξ is a random element in $\mathcal{M}(S)$, equivalently, $\xi A = \xi \mathbb{1}_A$ is a random variable for each relatively compact Borel set A. The associated notion of convergence in distribution of random measures is called *vague convergence* in distribution, denoted hereafter by $\xrightarrow{d}$, see [11, 12]. When considering *point processes*, we restrict ourselves to the subclass $\mathcal{N}(S) \subset \mathcal{M}(S)$ of counting measures (that is, taking values in $\mathbb{N}_0$, the set of non-negative integers). A random measure ξ is said to have a finite intensity if $\mathbf{E}(\xi A) < \infty$ for every relatively compact Borel set A.

In this paper, we are particularly interested in vague convergence in distribution to scale-invariant Poisson processes. A random measure ξ on S is *scale-invariant* if its distribution is invariant with respect to a group of scaling transformations of S. Even though convergence to stationary Poisson processes has been extensively studied in the literature, studies regarding convergence to scale-invariant processes seem to be rare. Distributional properties of scale-invariant Poisson processes on the half-line $(0, \infty)$ are surveyed in [2]. While a simple transformation relates a scale-invariant Poisson process on $(0, \infty)$ to a stationary Poisson processes on the line, such a transformation is not readily available in general Euclidean spaces.

Throughout the sequel, we take $S = \mathbb{R}^d \setminus \{0\}$, $d \in \mathbb{N}$, that is, the Euclidean space with the origin removed. On the half-line, for $c > 0$, we denote by η_c the scale-invariant Poisson process on $(0, \infty)$ with intensity measure $ct^{-1}dt$, and we will simply write η for η_1.

Scale-invariant processes naturally arise as limits of point processes when a scaling is applied to the support points of the point processes. For measures, this amounts to scaling of their arguments, namely, the scaling of $\nu \in \mathcal{M}(S)$ by $t > 0$ is defined as

$$T_t\nu(A) = \nu(t^{-1}A), \quad A \in \mathfrak{S}. \tag{1.1}$$

We call this operation *intrinsic scaling*. In Section 2, we show that random measures when intrinsically scaled, naturally yield scale-invariant measures as limits. As an application, we generalize a result in [10] proving that the intrinsically scaled process of jump sizes in a pure-jump subordinator converges vaguely in distribution to a scale-invariant Poisson process, and as a consequence, the sum of small jumps in the process converges to a Dickman distribution.

In this paper, our basic objects of interest are point processes on $(0, \infty)$ of the following type. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence of positive *deterministic* numbers with $z_n \uparrow \infty$ as $n \to \infty$. For a sequence $(X_k)_{k\in\mathbb{N}}$ of independent random variables in $\mathbb{N}_0$, define the point process

$$\nu = \sum_{k=1}^{\infty} X_k \delta_{z_k},$$

where δ_x denotes the Dirac measure at x. Rescaling the support points of ν by $(z_n)_{n\in\mathbb{N}}$ yields the sequence of point processes

$$\nu_n A = T_{z_n}\nu(A) = \nu(z_n^{-1}A), \quad A \in \mathfrak{S}, n \in \mathbb{N}. \tag{1.2}$$

In Section 3, we study the convergence of such processes; these results are extended to point processes in multidimensional Euclidean spaces in Section 4.

Our interest in the scale-invariant Poisson process η_c also stems from its connection to the *Dickman distributions*. It is well known that the sum of points of η_c lying in the interval $(0,1)$ is distributed as a generalized Dickman random variable denoted hereafter by D_c for $c>0$, with $D=D_1$ being a standard Dickman random variable. The generalized Dickman distribution with parameter $c>0$ can be defined as the unique non-negative fixed point of the distributional transformation $W \mapsto W^*$ given by

$$W^* =_d Q^{1/c}(W+1),$$

where $=_d$ denotes equality in distribution and Q is a uniformly distributed random variable on $[0,1]$ independent of W. It was introduced in the work of Dickman [13] in the context of smooth numbers and since then has appeared, sometimes curiously, in various areas including probabilistic number theory [9, 23], minimal directed spanning trees [8, 21], quickselect sorting algorithm [15, 16] and log-combinatorial structures [4, 6].

Given the various application, not surprisingly, there have been many works studying weak convergence to Dickman distributions [16, 21, 23] and, more recently, Stein's method has been used to provide non-asymptotic bounds for Dickman approximations [1, 9, 15]. In [22], Pinsky provided some general conditions under which certain randomly weighted Bernoulli sums converge to a generalized Dickman random variable. But, to the best of our knowledge, there has been no other attempt to characterize the domain of attraction of the Dickman distributions. Elaborating on [3], one aim of this work is to identify a broad class of random variables which asymptotically behave like a Dickman random variable. To do this, we make use of the fact that

$$D_c =_d \int_0^1 t\eta_c(dt) = \sum_{t\in\eta_c\cap(0,1)} t.$$

Hence, if a sequence of point processes converges vaguely in distribution to η_c, then, under certain natural additional conditions, sums of their points in the interval $(0,1)$ converge in distribution to the Dickman random variable D_c. Thus, our approach via scale-invariant Poisson processes yields a new tool to prove Dickman convergences and provides useful insights into why such convergences occur. We note here that a similar approach concerning limit theorems for point processes in relation to the behaviour of sums of their points has previously been discussed in [5]. Also, the simpler case of Poisson processes converging to η_c on $(0,\infty)$ was considered in [10]. Scale-invariant Poisson processes also arise in limit theorems for records, see e.g. [7] and references therein.

In Section 5, we characterize scale-invariant Poisson processes in general dimension d, and show that any such process can be obtained by independently multiplying each point of a scale-invariant Poisson process on $(0,\infty)$ with independent and identically distributed unit vectors in $\mathbb{R}^d$. Such a characterization naturally leads to a multivariate generalization of the Dickman distribution. Analogous to the univariate case, these multivariate Dickman distributions are fixed points of a distributional transform

$$W^* =_d Q^{1/c}(W+U),$$

where Q is a uniform random variable on $[0,1]$ and U a unit random vector in $\mathbb{R}^d$, independent of everything else.

Some results concerning weak convergence of general point processes (not necessarily scale-invariant) are collected in the Appendix.

2 Intrinsic scaling of random measures

Let $\widehat{\mathfrak{S}} \subseteq \mathfrak{S}$ denote the family of relatively compact Borel sets in $S=\mathbb{R}^d\setminus\{0\}$ for some $d\in\mathbb{N}$. A subclass $\mathcal{U}\subset\widehat{\mathfrak{S}}$ is called *dissecting* if every open set can be expressed as a countable union of sets from $\mathcal{U}$ and every set in $\widehat{\mathfrak{S}}$ can be covered by finitely many sets in $\mathcal{U}$. Recall that a subclass $\mathcal{I} \subset \widehat{\mathfrak{S}}$ is a *ring* if it is closed under proper differences and under finite unions and intersections. In the special case of $(0,\infty)$, we will often take the dissecting ring $\mathcal{U}$ to be the family of finite unions of semi-open intervals $(a,b]$ with $0 < a < b < \infty$.

Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of point processes in S. It is well known that the vague convergence in distribution $\xi_n \xrightarrow{d} \xi$ for a simple ξ follows from the one-dimensional weak convergences $\xi_n A \xrightarrow{d} \xi A$ for all A from the dissecting ring

$$\mathcal{U} \subset \widehat{\mathfrak{S}}_{\mathbf{E}\xi} = \{B \in \widehat{\mathfrak{S}} : \mathbf{E}\,\xi(\partial B) = 0\},$$

where ∂B denotes the boundary of B, see e.g. [19, Chapter 4]. A measure $\mu \in \mathcal{M}(S)$ is said to be *scale-invariant* if $T_c\mu = \mu$ for all $c > 0$, where T_c is defined at (1.1). The next result shows that the limit of the sequence of random measures obtained by intrinsic scalings of a given random measure ν is necessarily scale-invariant under some mild conditions on the normalizing constants. For deterministic measures, similar results are known, see e.g. [20, Theorem 3.1]. We write $\mathbb{S}^{d-1}$ for the d-dimensional unit sphere and B_r for the closed ball of radius $r > 0$ around the origin.

Lemma 2.1. *Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers increasing to infinity with $\lim_{n\to\infty} s_{n-1}/s_n = 1$, and let $\mu, \nu \in \mathcal{M}(S)$ be random measures with finite intensities such that $T_{s_n}\nu \xrightarrow{d} \mu$ as $n \to \infty$. Then $T_t\nu \xrightarrow{d} \mu$ as $t \to \infty$, and the limiting measure μ is scale-invariant.*

Proof. Since μ has finite intensity, the family of sets

$$\mathcal{U} = \left\{A \times [a,b] : \mathbf{E}\,\mu[\partial A \times (0,\infty)] = \mathbf{E}\,\mu[\partial(B_a) \cup \partial(B_b)] = 0, A \subseteq \mathbb{S}^{d-1}, 0 < a < b < \infty\right\}$$

forms a dissecting semi-ring. Hence, the first claim will follow (see [17, Theorem 1.1]) by establishing that

$$(T_t\nu(A_i \times [a_i,b_i]))_{i\in[k]} \xrightarrow{d} (\mu(A_i \times [a_i,b_i]))_{i\in[k]} \quad \text{as } n \to \infty \tag{2.1}$$

for all $k \in \mathbb{N}$ and $A_i \times [a_i,b_i] \in \mathcal{U}$, $i = 1,\dots,k$.

To simplify the argument, assume that $k = 1$; for general $k \in \mathbb{N}$, one can argue similarly. For $t > 0$, let $n(t)$ be the integer such that $s_{n(t)} < t \le s_{n(t)+1}$. Fix a Borel set $A \subseteq \mathbb{S}^{d-1}$ and $0 < a < b < \infty$ with $A \times [a,b] \in \mathcal{U}$ and $\varepsilon \in (0, b-a)$. Since $\lim_{n\to\infty} s_{n-1}/s_n = 1$ and $n(t) \to \infty$ as $t \to \infty$,

$$\frac{a}{s_{n(t)+1}} > \frac{a-\varepsilon}{s_{n(t)}} \quad \text{and} \quad \frac{b}{s_{n(t)+1}} > \frac{b-\varepsilon}{s_{n(t)}}$$

for all sufficiently large t. Hence, for t large enough, we have

$$T_t\nu(A \times [a,b]) \le \nu(A \times [a/s_{n(t)+1}, b/s_{n(t)}]) \le T_{s_{n(t)}}\nu(A \times [a-\varepsilon, b]).$$

A similar argument yields a lower bound, so that

$$T_{s_{n(t)}}\nu(A \times [a, b-\varepsilon]) \le T_t\nu(A \times [a,b]) \le T_{s_{n(t)}}\nu(A \times [a-\varepsilon, b])$$

for all sufficiently large t. Since $n(t) \to \infty$ as $t \to \infty$ and $T_{s_n}\nu \xrightarrow{d} \mu$ as $n \to \infty$, we obtain that

$$\limsup_{t\to\infty} \mathbf{P}\{T_t\nu(A \times [a,b]) \le x\} \le \mathbf{P}\{\mu(A \times [a, b-\varepsilon]) \le x\}$$

and

$$\liminf_{t\to\infty} \mathbf{P}\{T_t\nu(A\times[a,b])\le x\} \ge \mathbf{P}\{\mu(A\times[a-\varepsilon,b])\le x\}$$

for $x\ge 0$. Since $\mathbf{E}\,\mu[\partial(B_a)\cup\partial(B_b)]=0$,

$$\lim_{\varepsilon\to 0}\mathbf{P}\{\mu(A\times[a,b-\varepsilon])\le x\}=\lim_{\varepsilon\to 0}\mathbf{P}\{\mu(A\times[a-\varepsilon,b])\le x\}=\mathbf{P}\{\mu(A\times[a,b])\le x\},$$

which, together with the two inequalities above yield (2.1), proving the first claim.

Finally, let $v:S\to\mathbb{R}$ be a bounded continuous function with relatively compact support. For $c>0$, since $T_t\nu\xrightarrow{d}\mu$ as $t\to\infty$,

$$\lim_{t\to\infty}T_cT_t\nu(v)=\lim_{t\to\infty}\int_S v(x)T_{ct}\nu(dx)=\lim_{t\to\infty}\int_S v(cx)T_t\nu(dx)=\int_S v(cx)\mu(dx)=T_c\mu(v),$$

which implies that

$$T_cT_t\nu\xrightarrow{d}T_c\mu\quad\text{as}\ \ t\to\infty.$$

On the other hand, $T_cT_t\nu=T_{ct}\nu$ converges vaguely in distribution to μ as $t\to\infty$ by our assumption. Hence we obtain $T_c\mu=\mu$, proving the scale invariance of μ. □

The following theorem proves Dickman convergence for the sums of small jump sizes in a pure-jump subordinator; we note here that the Dickman limit result is not new and has been proved in [10]. We prove a stronger result that the scaled point process of jump sizes converges to a scale-invariant Poisson process on $(0,\infty)$.

Let $Y=(Y(t))_{t\ge 0}$ be a pure-jump subordinator with infinite Lévy measure σ and for $\varepsilon>0$, let Y_ε be the process obtained by removing the jumps of size larger than ε in the Lévy-Ito decomposition of Y. For $t>0$, let Π_t denote the point process of jump sizes occurring in the time interval $[0,t]$. The scaled process $T_{1/\varepsilon}\Pi_t$ consists of the points of Π_t scaled by ε. Recall, D_c denotes a Dickman distributed random variable with parameter $c>0$.

Theorem 2.2. *If $\varepsilon^{-1}\int_0^\varepsilon x\sigma(dx)\to c>0$ as $\varepsilon\to 0$, then for any $t>0$,*

$$T_{1/\varepsilon}\Pi_t\xrightarrow{d}\eta_{ct}\quad\text{as}\ \ \varepsilon\to 0.$$

Moreover,

$$\varepsilon^{-1}Y_\varepsilon(t)\xrightarrow{d}D_{ct}\quad\text{as}\ \ \varepsilon\to 0.$$

Proof. Arguing as in the proof of [10, Theorem 2.1], letting ψ and ψ_ε, $\varepsilon>0$ be the measures given by $\psi(dx)=\mathbb{1}_{(0,1]}(x)c\,dx$ and $\psi_\varepsilon(dx)=x\cdot T_{1/\varepsilon}\sigma(dx)=x\sigma(\varepsilon dx)$ respectively, for any $p\in(0,1)$, we have

$$\psi_\varepsilon((0,p])=\int_0^p x\sigma(\varepsilon dx)=\frac{1}{\varepsilon}\int_0^{p\varepsilon}z\sigma(dz)\to cp=\psi((0,p])\quad\text{as}\ \ \varepsilon\to 0.$$

By Lemma A.2,

$$T_{1/\varepsilon}\sigma((p,1])=\int_p^1 x^{-1}\psi_\varepsilon(dx)\to\int_p^1 x^{-1}\psi(dx)=c\log(1/p)\quad\text{as}\ \ \varepsilon\to 0,$$

which yields that the Poisson process on $(0,\infty)$ with intensity measure $T_{1/\varepsilon}\sigma$ converges vaguely in distribution to η_c as $\varepsilon\to 0$. Since Y is a Lévy process with Lévy measure σ, the jump process Π_t is distributed as a Poisson process on $(0,\infty)$ with intensity measure $t\sigma$; this proves the first claim.

Finally, note that $\varepsilon^{-1}Y_\varepsilon(t) = \int_0^1 x\ (T_{1/\varepsilon}\Pi_t)(dx)$. To prove the last claim, by Lemma A.3, it suffices to check that

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\mathbf{E}\int_0^\delta x\ (T_{1/\varepsilon}\Pi_t)(dx) = 0. \tag{2.2}$$

Since Π_t is a Poisson process with intensity measure $t\sigma$, we have that $T_{1/\varepsilon}\Pi_t$ is distributed as a Poisson process on $(0,\infty)$ with intensity measure $tT_{1/\varepsilon}\sigma$. Thus, using the Mecke equation in the first equality and that $\varepsilon^{-1}\int_0^\varepsilon x\sigma(dx) \to c$ as $\varepsilon \to 0$ in the third, we obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0}\mathbf{E}\int_0^\delta x\ (T_{1/\varepsilon}\Pi_t)(dx) &= \limsup_{\varepsilon\to 0} t\int_0^\delta xT_{1/\varepsilon}\sigma(dx)\\ &= \limsup_{\varepsilon\to 0} t\varepsilon^{-1}\int_0^{\varepsilon\delta} x\sigma(dx) = ct\delta\end{aligned}$$

which implies (2.2), concluding the proof. □

3 Convergence to scale-invariant Poisson processes

Now we move our attention to proving convergence to scale-invariant Poisson processes for sequences of general (not necessarily Poisson) point processes. The necessary and sufficient conditions for vague convergence in distribution of point processes to a simple point process given by Theorem A.1, when applied to ν_n given by (1.2) with η_c being the limit, translate to the following simpler condition. For convenience, denote

$$q_k^0 = \mathbf{P}\{X_k = 0\} \quad \text{and} \quad q_k^1 = \mathbf{P}\{X_k = 1\},\ k \ge 1.$$

Condition 3.1. *There exists $c > 0$ such that for all $0 < a < b < \infty$,*

(i) $\prod_{k:az_n<z_k\le bz_n} q_k^0 \to (a/b)^c$ *as* $n\to\infty$.

(ii) $\liminf_{n\to\infty}\sum_{k:az_n<z_k\le bz_n} q_k^1/q_k^0 \ge c\log(b/a)$.

Theorem 3.2. *A sequence of point processes $(\nu_n)_{n\in\mathbb{N}}$ given by (1.2) converges vaguely in distribution to η_c for some $c > 0$ as $n\to\infty$ if and only if $(q_k^0, q_k^1)_{k\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ satisfy Condition 3.1.*

Proof. Condition 3.1(i) for the dissecting ring composed of finite unions of semi-open intervals is equivalent to condition (i) in Theorem A.1. Condition (ii) in Theorem A.1 is equivalent to

$$\liminf_{n\to\infty}\left[\left(1+\sum_{k:az_n<z_k\le bz_n} q_k^1/q_k^0\right)\prod_{l:az_n<z_l\le bz_n} q_l^0\right] \ge \left(\frac{a}{b}\right)^c\left(1+c\log\frac{b}{a}\right),$$

which, given Condition 3.1(i), simplifies to Condition 3.1(ii), proving the result. □

The next result concerns vague convergence to scale-invariant Poisson processes for a large class of point processes ν_n of the form (1.2) and, as a consequence, establishes weak convergence of sums of the points in $(0,1)$ of ν_n to a generalized Dickman distributed random variable D_c. Note that such a convergence does not readily follow from the vague convergence since η_c has infinitely many points in any neighbourhood of zero.

Theorem 3.3. *For a monotone sequence of positive numbers $(z_k)_{k\ge 0}$ increasing to infinity with $\lim_{k\to\infty} z_k/z_{k-1} = 1$, let $(X_k)_{k\in\mathbb{N}}$ be independent random variables in $\mathbb{N}_0$ with*

$$q_k^0 = (z_{k-1}/z_k)^c \quad \textit{and} \quad q_k^1 = q_k^0(1-q_k^0)$$

for some $c > 0$*. Then the sequence* $(\nu_n)_{n\in\mathbb{N}}$ *defined at* (1.2) *converges vaguely in distribution to* η_c *as* $n \to \infty$*. If, in addition,* $\mathbf{E}X_k = \mathcal{O}(q_k^1)$*, then*

$$\frac{1}{z_n}\sum_{k=1}^{n} z_k X_k \xrightarrow{d} D_c \quad \text{as } n \to \infty. \tag{3.1}$$

Proof. Fix $0 < a < b < \infty$. Let $M = \inf\{k : az_n < z_k \le bz_n\}$ and $N = \sup\{k : az_n < z_k \le bz_n\}$. Letting $\delta_n = az_n - z_{M-1}$ and $\delta'_n = bz_n - z_N$, one has

$$\frac{z_{M-1}}{z_N} = \frac{a - \delta_n/z_n}{b - \delta'_n/z_n}.$$

Since $\lim_{k\to\infty} z_k/z_{k+1} = 1$ and $M \to \infty$ as $n \to \infty$,

$$\limsup_{n\to\infty} \frac{\delta_n}{z_n} \le \lim_{n\to\infty} \frac{z_M - z_{M-1}}{z_M} \cdot \frac{z_M}{z_n} = 0,$$

and a similar argument shows that $\limsup_{n\to\infty} \delta'_n/z_n = 0$. Thus,

$$\prod_{k:az_n<z_k\le bz_n} q_k^0 = \prod_{k:az_n<z_k\le bz_n} \left(\frac{z_{k-1}}{z_k}\right)^c = \left(\frac{z_{M-1}}{z_N}\right)^c \to \left(\frac{a}{b}\right)^c \quad \text{as } n \to \infty.$$

Also,

$$\begin{aligned}\liminf_{n\to\infty} \sum_{k:az_n<z_k\le bz_n} \frac{q_k^1}{q_k^0} &\ge \liminf_{n\to\infty}\left(\frac{z_{n-1}}{z_n}\right)^c \liminf_{n\to\infty} \sum_{k:az_n<z_k\le bz_n} \frac{z_k^c - z_{k-1}^c}{z_{k-1}^c} \\ &\ge \liminf_{n\to\infty} \int_{z_{M-1}^c}^{z_N^c} \frac{1}{t}dt = c \liminf_{n\to\infty} \log\frac{z_N}{z_{M-1}} = c\log\frac{b}{a}.\end{aligned}$$

Hence, Condition 3.1 is satisfied and the first claim follows by Theorem 3.2.

If $\mathbf{E}X_k = \mathcal{O}(q_k^1)$, then there exists $C > 0$ such that $\mathbf{E}X_k \le Cq_k^1$ for all $k \in \mathbb{N}$. Denoting by $\lceil\cdot\rceil$ the ceiling function and using the simple inequality that $1 - (1-x)^c \le 2^{\lceil c\rceil}x$ for $x \in [0,1]$ in the penultimate step, we have

$$\begin{aligned}\mathbf{E}\int_0^\varepsilon t\nu_n(dt) &= \frac{1}{z_n}\sum_{k:z_k\le z_n\varepsilon} z_k\mathbf{E}X_k \le \frac{C}{z_n}\sum_{k:z_k\le z_n\varepsilon} z_k q_k^1 \\ &\le \frac{C}{z_n}\sum_{k:z_k\le z_n\varepsilon} z_k\left(1 - \left(1 - \frac{z_k - z_{k-1}}{z_k}\right)^c\right) \\ &\le \frac{C}{z_n}\sum_{k:z_k\le z_n\varepsilon} 2^{\lceil c\rceil}(z_k - z_{k-1}) \le C2^{\lceil c\rceil}\varepsilon.\end{aligned}$$

Therefore,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} \mathbf{E}\int_0^\varepsilon t\nu_n(dt) = 0. \tag{3.2}$$

Thus, invoking Lemma A.3 we obtain

$$\int_0^1 t\nu_n(dt) = \frac{1}{z_n}\sum_{k=1}^n z_k X_k \xrightarrow{d} D_c \quad \text{as } n \to \infty.$$

□

Remark 3.4. Recall that X is a geometric random variable with parameter $p \in (0,1)$ if $\mathbf{P}\{X = m\} = (1-p)^m p$ for $m \ge 0$; we then write $X \sim \mathrm{Geom}(p)$. For $(z_k)_{k\in\mathbb{N}}$ as in Theorem 3.3, clearly $X_k \sim \mathrm{Geom}(q_k^0)$ satisfies the conditions therein. We can also take the random variables $X_k \sim \mathrm{Ber}(q_k^0)$ with q_k^0 as in Theorem 3.3, i.e. X_k is a $\{0,1\}$-valued random variable with $\mathbf{P}\{X_k = 0\} = q_k^0$. In this case, a similar proof shows that

$$\nu_n = \sum_{k=1}^{\infty} X_k \delta_{z_k/z_n} \xrightarrow{d} \eta_c \quad \text{as } n \to \infty.$$

Since $\mathbf{E}X_k = q_k^1$, arguing like in Theorem 3.3, one can establish (3.1) in this case as well.

Remark 3.5. Even though under Condition 3.1 the sequence ν_n converges vaguely in distribution to a simple process, it is not necessarily true that the X_k's are $\{0,1\}$-valued almost surely for all sufficiently large n. Consider the sequence ν_n as in Theorem 3.3 with $c = 1$ and z_k defined sequentially by letting $z_0 = z_1 = 1$ and $z_n/z_{n-1} = \sqrt{n}/(\sqrt{n} - 1)$ for $n \geq 2$. Since

$$z_n = \frac{\sqrt{n}}{\sqrt{n}-1} z_{n-1} \geq \frac{n}{n-1} z_{n-1} \geq \cdots \geq n z_1 = n,$$

Theorem 3.3 yields that $\nu_n \xrightarrow{d} \eta$ as $n \to \infty$. Furthermore,

$$\sum_{k=1}^{\infty} \mathbf{P}\{X_k \geq 2\} = \sum_{k=1}^{\infty} (1 - q_k^0 - q_k^1) = \sum_{k=2}^{\infty} (1 - z_{k-1}/z_k)^2 \geq \sum_{k=2}^{\infty} k^{-1},$$

which diverges. By the Borel-Cantelli lemma, X_k is strictly greater than 1 for infinitely many k. However, after rescaling, the number of points with multiplicities more than 1 in any bounded interval $[a,b] \subset (0,\infty)$ converges to zero.

The processes in Theorem 3.3 do not necessarily satisfy (A.4), since only q_k^0 and q_k^1 are specified there and one can allocate the rest of the probability on a large number to make $\mathbf{E}X_k$ sufficiently large so that (A.4) does not hold. Hence, an additional condition like $\mathbf{E}X_k = \mathcal{O}(q_k^1)$ is essential. Note that, for $X_k \sim \mathrm{Geom}(q_k^0)$, we have $q_k^1 = q_k^0(1-q_k^0)$ and

$$\mathbf{E}X_k = (1 - q_k^0)/q_k^0 = (1/q_k^0)^2 q_k^1 = \mathcal{O}(q_k^1),$$

since $q_k^0 \to 1$ as $k \to \infty$.

Next, we describe a sequence of point processes arising in probabilistic number theory which satisfies Condition 3.1, and hence, converges to the scale-invariant Poisson process η by Theorem 3.2 and the sums of points in $(0,1)$ converge to the standard Dickman distribution. For an enumeration $(p_k)_{k\in\mathbb{N}}$ of the prime numbers in increasing order, let Ω_n denote the set of positive integers having all its prime factors less than or equal to the n^{th} prime p_n. Let M_n be a random variable distributed according to the probability mass function Θ_n with $\Theta_n(m)$ being proportional to the inverse of m for $m \in \Omega_n$. Then one can show that (see e.g. [23])

$$\frac{\log M_n}{\log p_n} =_d \frac{1}{\log p_n} \sum_{k=1}^{n} X_k \log p_k, \tag{3.3}$$

where $X_1, \ldots, X_n$ are independent with $X_k \sim \mathrm{Geom}(1 - 1/p_k)$ for $1 \leq k \leq n$. The distributional convergence of the right-hand side of (3.3) to the standard Dickman distribution was proved in [23] with optimal convergence rates provided in [9] using Stein's method. We prove that this convergence is a consequence of the underlying sequence of point processes converging to η.

Theorem 3.6. *Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of point processes defined at (1.2) with $z_k = \log p_k$ and $X_k \sim \mathrm{Geom}(1 - 1/p_k)$ for $k \in \mathbb{N}$. Then $\nu_n \xrightarrow{d} \eta$ as $n \to \infty$ and*

$$\frac{1}{\log p_n} \sum_{k=1}^{n} X_k \log p_k \xrightarrow{d} D \quad \text{as } n \to \infty.$$

Proof. For the first part, by Theorem 3.2, we only need to check Condition 3.1. Since $q_k^0 = (1 - 1/p_k)$, for $0 < a < b < \infty$, by Merten's formula (see e.g. [25, Prop. 1.51]),

$$\prod_{k: az_n < z_k \le bz_n} q_k^0 = \prod_{k: p_n^a < p_k \le p_n^b} \left(1 - \frac{1}{p_k}\right) \to \frac{a}{b} \quad \text{as} \ n \to \infty.$$

Hence, Condition 3.1(i) is satisfied. For Condition 3.1(ii), since $q_k^1 = p_k^{-1}(1 - p_k^{-1})$, Merten's formula yields that

$$\sum_{k: az_n < z_k \le bz_n} q_k^1 / q_k^0 = \sum_{k: p_n^a < p_k \le p_n^b} \frac{1}{p_k} \to \log \frac{b}{a} \quad \text{as} \ n \to \infty.$$

Theorem 3.2 now yields the first part of the result.

For the second part, by Lemma A.3, it suffices to check (A.4). Since

$$\sum_{p_k \le n} p_k^{-1} \log p_k = \log n + \mathcal{O}(1)$$

(see [25, Prop. 1.51]), it follows that for $\varepsilon > 0$,

$$\mathbf{E} \int_0^\varepsilon t \nu_n(dt) = \frac{1}{\log p_n} \mathbf{E} \sum_{k=1}^\infty X_k \log p_k \mathbb{1}_{\{1 < p_k \le p_n^\varepsilon\}} \le \frac{2}{\log p_n} [\log p_n^\varepsilon + \mathcal{O}(1)],$$

which converges to ε as $n \to \infty$. Thus, $(\nu_n)_{n \ge 1}$ satisfies (A.4), proving the result. □

Remark 3.7. Let $X_k \sim \mathrm{Ber}(1/(1 + p_k))$, where p_k is the k^{th} prime number and consider $(\nu_n)_{n \in \mathbb{N}}$ defined in Theorem 3.6. One can argue as in the proof of Theorem 3.6 to show that $\nu_n \xrightarrow{d} \eta$ as $n \to \infty$ and

$$\frac{1}{\log p_k} \sum_{k=1}^n X_k \log p_k \xrightarrow{d} D \quad \text{as} \ n \to \infty.$$

As mentioned above, if the X_k's are distributed as geometric random variables given in Theorem 3.6, the induced distribution on $M_n = \prod_{k=1}^n p_k^{X_k}$ is the reciprocal distribution on the set Ω_n of positive integers with all prime factors less than or equal to p_n. If $X_k \sim \mathrm{Ber}(1/(1 + p_k))$, the induced distribution on M_n turns out to be the reciprocal distribution on the set of square-free positive integers with all its prime factors less than or equal to p_n.

Next, we provide a few more examples that arise as special cases of the class of point processes considered in Theorem 3.2 and in Remark 3.4.

Example 3.8. Let $X_k \sim \mathrm{Ber}(1/k)$, $k \ge 1$, be independent and $\nu_n = \sum_{k=1}^\infty X_k \delta_{k/n}$. In this case, one can easily check that Condition 3.1 and (A.4) are satisfied. Hence, $\nu_n \xrightarrow{d} \eta$ and $n^{-1} \sum_{k=1}^n k X_k \xrightarrow{d} D$ as $n \to \infty$. This is a well-known example arising in the context of counting sums of 'records' in a random permutation. For a uniformly random permutation σ of $\{1, \dots, n\}$, let S_n be the sum of records, which are positions k such that $\sigma(k) > \max_{i \in [k-1]} \sigma(i)$. One can check that S_n is indeed distributed as $\sum_{k=1}^n k X_k$.

Example 3.9. Let ν_n be as in (1.2) with $z_k = \log k$ and independent $X_k \sim \mathrm{Geom}(1 - 1/(k \log k))$, $k \in \mathbb{N}$. In this case, it is straightforward to check that the conclusions of Theorem 3.3 hold. Heuristically, this is equivalent to Theorem 3.6, since, by the prime number theorem, one has that the k^{th} prime number p_k is asymptotically of the order $k \log k$.

Example 3.10. Theorem 3.2 and Lemma A.3 apply if X_k's are independent Poisson random variables with mean $1/p_k$ and ν_n is given by (1.2) with $z_k = \log p_k$, $k \in \mathbb{N}$.

4 Convergence of uplifted point processes

In this section, we consider convergence of certain general point processes to scale-invariant Poisson processes in dimension d. These point processes are obtained by first taking a point process on $(0,\infty)$ and transforming (uplifting) its points to $\mathbb{R}^d$ by multiplying them with random vectors taking values in $S = \mathbb{R}^d \setminus \{0\}$. We start with a point process $\xi = \sum_{k=1}^{\infty} X_k \delta_{Z_k}$ with finite intensity on the positive half-line. Let V be a random vector in S with i.i.d. copies $(V_k)_{k\in\mathbb{N}}$ which are independent of ξ. Define the uplifted process ξ^V as

$$\xi^V = \sum_{k=1}^{\infty} X_k \delta_{V_k Z_k}. \tag{4.1}$$

We need to impose some conditions on ξ and V to ensure that ξ^V is locally finite on S. To this end, throughout this section, we assume for any uplifted process ξ^V that ξ and V satisfy

$$\mathbf{E} \sum_{k=1}^{\infty} X_k \mathbb{1}_{\{Z_k \|V_k\| \in [a,b]\}} < \infty \quad \text{for all } \ 0 < a < b < \infty, \tag{4.2}$$

where $\|\cdot\|$ denotes the Euclidean norm. Since ξ has a finite intensity, this condition is always satisfied if V is bounded away from 0 and ∞. In Lemma 5.1, we show that any scale-invariant Poisson process in S has the same distribution as the uplifted process η_c^U for some $c > 0$ and a unit random vector U in $\mathbb{R}^d$. Thus, our uplifting scheme is a natural choice to recover all scale-invariant point processes in S.

It is well known that, if $\xi_n \xrightarrow{d} \xi$ as $n \to \infty$, then (see e.g. [19, Theorem 4.11])

$$\mathbf{E}\, e^{-\xi_n f} \to \mathbf{E}\, e^{-\xi f} \quad \text{as } \ n \to \infty \tag{4.3}$$

for any $f \in \widehat{C}_S$. In order to handle uplifting transformations by a possibly unbounded random vector V, we need to consider test functions f with unbounded support. The following result extends (4.3) to more general functions.

Lemma 4.1. *Let $(\xi_n)_{n\in\mathbb{N}}$ and ξ be point processes on a locally compact separable metric space Ω with ξ having a finite intensity, such that $\xi_n \xrightarrow{d} \xi$ as $n \to \infty$. Let h be a non-negative continuous function on Ω such that for any $\varepsilon > 0$, there exists a relatively compact set K_ε with*

$$\limsup_{n\to\infty} \mathbf{E} \int_{K_\varepsilon^c} h(x)\xi_n(dx) \le \varepsilon. \tag{4.4}$$

Then

$$\mathbf{E}\, e^{-\xi_n h} \to \mathbf{E}\, e^{-\xi h} \quad \textit{as } \ n \to \infty.$$

For a proof, see the Appendix. For $f \in \widehat{C}_S$, define the function $h_f : \mathbb{N}_0 \times (0,\infty) \to \mathbb{R}$ as

$$h_f(x,y) = -\log \mathbf{E}\, e^{-xf(Vy)}. \tag{4.5}$$

Note that by Jensen's inequality, one has

$$h_f(x,y) \le x\mathbf{E}\, f(Vy). \tag{4.6}$$

Define the map $M : \mathcal{N}((0,\infty)) \to \mathcal{N}(\mathbb{N}_0 \times (0,\infty))$ at $\xi = \sum_{k=1}^{\infty} a_k \delta_{z_k}$ as

$$M(\xi) = \sum_{k=1}^{\infty} \delta_{(a_k, z_k)}. \tag{4.7}$$

This map turns a counting measure with possibly multiple points into a simple counting measure in the product space $\mathbb{N}_0 \times (0,\infty)$.

Theorem 4.2. *Assume that a sequence of point processes* $\xi_n = \sum_{k=1}^{\infty} X_k \delta_{Z_k^n}$*,* $n \in \mathbb{N}$ *converges vaguely in distribution to a simple point process* ξ *with finite intensity in* $\mathcal{N}((0,\infty))$ *as* $n \to \infty$*. Moreover, let* V *be a random vector in* S *with i.i.d. copies* $(V_k)_{k\in\mathbb{N}}$ *such that for every* $f \in \widehat{C}_S$ *and* $\varepsilon > 0$*, there exists a compact set* $K_{f,\varepsilon} \subseteq \mathbb{N}_0 \times (0,\infty)$ *such that*

$$\limsup_{n\to\infty} \mathbf{E} \sum_{(X_k, Z_k^n) \in K_{f,\varepsilon}^c} X_k f(V_k Z_k^n) \le \varepsilon. \tag{4.8}$$

Then $\xi_n^V \xrightarrow{d} \xi^V$ *as* $n \to \infty$*.*

Proof. Fix $f \in \widehat{C}_S$. Then

$$\begin{aligned} \mathbf{E}\, e^{-\xi_n^V f} &= \mathbf{E}\left[\prod_{k=1}^{\infty} \mathbf{E}\left[\exp\{-X_k f(V_k Z_k^n)\} \,\Big|\, \xi_n\right]\right] \\ &= \mathbf{E}\, \exp\Big\{ -\sum_{k=1}^{\infty} h_f(X_k, Z_k^n)\Big\} = \mathbf{E}\, e^{-\widetilde{\xi}_n h_f}, \end{aligned}$$

where $\widetilde{\xi}_n = M(\xi_n)$ and h_f is given by (4.5). Since $\xi_n \xrightarrow{d} \xi$ as $n \to \infty$ with ξ being simple, Lemma A.4 and the continuous mapping theorem yield that $\widetilde{\xi}_n \xrightarrow{d} \widetilde{\xi} = M(\xi)$. Clearly, h_f is continuous as f is such. Also note that by (4.6) and (4.8), we have that h_f satisfies (4.4) with respect to the processes $(\widetilde{\xi}_n)_{n\in\mathbb{N}}$. By Lemma 4.1,

$$\mathbf{E}\, e^{-\xi_n^V f} = \mathbf{E}\, e^{-\widetilde{\xi}_n h_f} \to \mathbf{E}\, e^{-\widetilde{\xi} h_f} \quad \text{as } n \to \infty.$$

Finally, noticing that

$$\mathbf{E}\, e^{-\xi^V f} = \mathbf{E}\big[\mathbf{E}(e^{-\xi^V f} | \xi)\big] = \mathbf{E}\, e^{-\widetilde{\xi} h_f},$$

we obtain

$$\mathbf{E}\, e^{-\xi_n^V f} \to \mathbf{E}\, e^{-\xi^V f} \quad \text{as } n \to \infty$$

for all $f \in \widehat{C}_S$, which proves that $\xi_n^V \xrightarrow{d} \xi^V$ as $n \to \infty$. □

The condition (4.8) in Theorem 4.2 that $(V_n)_{n\in\mathbb{N}}$ and $(\xi_n)_{n\in\mathbb{N}}$ are required to satisfy can be hard to check in general. In some special cases, one can find some easily verifiable conditions on $(\xi_n)_{n\in\mathbb{N}}$ and V so that (4.8) is satisfied. Throughout, $\|\cdot\|_\infty$ denotes the supremum norm on $\widehat{C}_S$.

Lemma 4.3. *Let* $(\xi_n)_{n\in\mathbb{N}}$ *be a sequence of simple point processes in* $(0,\infty)$*. Let* V *be such that for some* $\alpha > 0$*,*

$$\limsup_{t\to\infty} t\mathbf{P}\{\|V\| \ge t\} < \infty \quad \textit{and} \quad \limsup_{t\to\infty} t^{\alpha}\mathbf{P}\{\|V\| \le 1/t\} < \infty. \tag{4.9}$$

Moreover, assume that

$$\lim_{r\to 0} \limsup_{n\to\infty} \mathbf{E} \int_0^r t\xi_n(dt) = 0 \quad \textit{and} \quad \lim_{r\to\infty} \limsup_{n\to\infty} \mathbf{E} \int_r^{\infty} t^{-\alpha}\xi_n(dt) = 0. \tag{4.10}$$

Then the processes $(\xi_n)_{n\in\mathbb{N}}$ *and i.i.d. copies* $(V_n)_{n\in\mathbb{N}}$ *of* V *satisfy* (4.8)*.*

Proof. Since ξ_n is simple, for $f \in \widehat{C}_S$, it suffices to check that $h(y) = \mathbf{E} f(Vy)$ satisfies

$$\begin{cases} \lim_{r\to 0} \limsup_{n\to\infty} \mathbf{E} \int_0^r h(y)\xi_n(dy) = 0, \\ \lim_{r\to\infty} \limsup_{n\to\infty} \mathbf{E} \int_r^{\infty} h(y)\xi_n(dy) = 0. \end{cases} \tag{4.11}$$

Since f is compactly supported, there exist $0 < a < b < \infty$ such that $f(z) = 0$ for $\|z\| < a$ or $\|z\| > b$. Thus, using (4.9) in the last step, we have

$$\begin{aligned}\limsup_{y \searrow 0} \frac{h(y)}{y} &= \limsup_{y \searrow 0} \frac{\mathbf{E}\ f(Vy)}{y} = \limsup_{y \searrow 0} \frac{\mathbf{E}\left[f(Vy)\mathbb{1}_{\{\|V\| \geq a/y\}}\right]}{y} \\ &\leq \|f\|_\infty \limsup_{y \searrow 0} y^{-1}\mathbf{P}\{\|V\| \geq a/y\} < \infty.\end{aligned}$$

Arguing similarly and using (4.9),

$$\begin{aligned}\limsup_{y \to \infty} y^\alpha h(y) &= \limsup_{y \to \infty} y^\alpha \mathbf{E}\ f(Vy) \\ &= \limsup_{y \to \infty} y^\alpha \mathbf{E}\left[f(Vy)\mathbb{1}_{\{\|Vy\| \leq b\}}\right] \\ &\leq \|f\|_\infty \limsup_{y \to \infty} y^\alpha \mathbf{P}\{\|V\| \leq b/y\} < \infty.\end{aligned}$$

Thus, $\limsup_{y \searrow 0} h(y)/y < \infty$ and $h(y) = \mathcal{O}(y^{-\alpha})$ as $y \to \infty$. Together with (4.10), this implies that h satisfies (4.11). □

Corollary 4.4. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of simple point processes converging vaguely in distribution to η_c as $n \to \infty$. Assume that a random vector V in S and $(\xi_n)_{n\in\mathbb{N}}$ satisfy* (4.9) *and* (4.10)*, respectively, for some $\alpha > 0$. Then $\xi_n^V \xrightarrow{d} \eta_c^V$ as $n \to \infty$.*

Remark 4.5. Fix $\alpha > 0$. For a sequence of point processes $(\nu_n)_{n\in\mathbb{N}}$ as in Theorem 3.3 with $\mathbf{E}X_k = \mathcal{O}(q_k^1) \leq Cq_k^1$ for some $C > 0$, by (3.2) in the proof of Theorem 3.3, the first condition in (4.10) is satisfied. Letting $N = \inf\{k : z_k > z_n r\}$ for $r > 0$ yields that

$$\begin{aligned}\mathbf{E}\int_r^\infty t^{-\alpha}\nu_n(dt) = z_n^\alpha \sum_{k: z_k > z_n r} z_k^{-\alpha}\mathbf{E}X_k &\leq C z_n^\alpha (\sup_k q_k^0) \sum_{k=N}^\infty \frac{z_k^c - z_{k-1}^c}{z_k^{c+\alpha}} \\ &\leq C z_n^\alpha \left[\frac{z_N^c - z_{N-1}^c}{z_N^{c+\alpha}} + \int_{(z_n r)^c}^\infty \frac{1}{x^{(c+\alpha)/c}} dx\right].\end{aligned}$$

Since the right-hand side converges to $C(c/\alpha)r^{-\alpha}$ as $n \to \infty$,

$$\lim_{r \to \infty} \limsup_{n \to \infty} \mathbf{E}\int_r^\infty t^{-\alpha}\nu_n(dt) = 0.$$

Hence, these point processes satisfy (4.10).

Example 4.6. Consider the sequence of point processes $(\nu_n)_{n\in\mathbb{N}}$ given by (1.2) with $z_k = \log p_k$ and $X_k \sim \mathrm{Geom}(1 - 1/p_k)$. Since $p_k > k\log k$ (see e.g. [25]) and $\log p_k < 2\log k$ for $k \geq 6$, (see e.g. [14, Lem. 1]), we have that $N_n = \inf\{k : p_k > p_n^r\} > n^{r/2}$ for n large enough. Hence,

$$\begin{aligned}\sum_{k: p_k > p_n^r} \frac{1}{p_k(\log p_k)^\alpha} &\leq \sum_{k=N_n}^\infty \frac{1}{k\log k(\log k)^\alpha} \\ &\leq \int_{N_n - 1}^\infty \frac{1}{x(\log x)^{1+\alpha}} dx = \frac{(\log(N_n - 1))^{-\alpha}}{\alpha} \leq \frac{2^\alpha(\log n)^{-\alpha}}{\alpha r^\alpha}.\end{aligned}$$

Therefore,

$$\begin{aligned}\limsup_{n \to \infty} \mathbf{E}\int_r^\infty t^{-\alpha}\nu_n(dt) &= \limsup_{n \to \infty} (\log p_n)^\alpha \sum_{k: p_k > p_n^r} \frac{2}{p_k(\log p_k)^\alpha} \\ &\leq \limsup_{n \to \infty} (2\log n)^\alpha \frac{2^{1+\alpha}(\log n)^{-\alpha}}{\alpha r^\alpha} = \frac{2^{1+2\alpha}}{\alpha r^\alpha},\end{aligned}$$

which yields

$$\lim_{r\to\infty}\limsup_{n\to\infty}\mathbf{E}\int_r^\infty t^{-\alpha}\nu_n(dt)=0.$$

The other condition in (4.10) is easy to check using Merten's formula. Hence, as $\nu_n \xrightarrow{d} \eta$ as $n\to\infty$ by Theorem 3.6, for V satisfying (4.9), Corollary 4.4 yields that $\nu_n^V \xrightarrow{d} \eta^V$ as $n\to\infty$.

We now return to our basic example of point processes given by (1.2). For a point process on $(0,\infty)$ with support points in a deterministic set, we can generalize the notion of uplifting. For $(\nu_n)_{n\in\mathbb{N}}$ given by (1.2), consider its uplifting by independent vectors $\mathbf{V}=(V_k)_{k\in\mathbb{N}}$ in S which are possibly non-identically distributed, allowing for possible dependence within the pairs (V_k,X_k) for any $k\in\mathbb{N}$. Assume that the conditional distribution of V_k given X_k is a function $V(X_k)$ that does not depend on k, i.e.,

$$V(x)=_d (V_k|X_k=x),\quad k\in\mathbb{N}. \tag{4.12}$$

For instance, this is the case if the random vectors $(V_k)_{k\in\mathbb{N}}$ are i.i.d. and independent of the random variables $(X_k)_{k\in\mathbb{N}}$. We also assume that the random vectors $(V_k)_{k\in\mathbb{N}}$ are uniformly bounded away from 0 and ∞ and define the *uplifted* process $\nu_n^{\mathbf{V}}$ as

$$\nu_n^{\mathbf{V}}=\sum_{k=1}^\infty X_k\delta_{V_k z_k/z_n}.$$

Finally, we assume that the random variables $(X_k)_{k\in\mathbb{N}}$ are $\{0,1\}$-valued with high probability, i.e.,

$$\prod_{k=1}^\infty (q_k^0+q_k^1)>0. \tag{4.13}$$

Theorem 4.7. *For $(\nu_n)_{n\in\mathbb{N}}$ given by (1.2), assume that the X_k's satisfy (4.13) and $\nu_n \xrightarrow{d} \eta_c$ for some $c>0$. Let $\mathbf{V}=(V_k)_{k\in\mathbb{N}}$ be a sequence of random vectors in S satisfying (4.12) with $\varepsilon\le\|V(x)\|\le r$ almost surely for all $x\in\mathbb{N}$ for some $0<\varepsilon<r<\infty$. Then $\nu_n^{\mathbf{V}} \xrightarrow{d} \eta_c^{V(1)}$ as $n\to\infty$, where $\eta_c^{V(1)}$ is defined as in (4.1).*

Proof. Let $\widetilde{X}_k=\mathbb{1}_{\{X_k>0\}}$. Let $(m(n))_{n\in\mathbb{N}}$ be such that $m(n)\to\infty$ and $z_{m(n)}=o(z_n)$ as $n\to\infty$. Denote

$$E_n=\{X_k=\widetilde{X}_k \text{ for all } k\ge m(n)\},\quad n\in\mathbb{N}.$$

By Kolmogorov's zero-one law and (4.13),

$$\lim_{n\to\infty}\mathbf{P}(E_n)=1.$$

Fix $f\in\widehat{C}_S$. Then, recalling that $V(1)=_d (V_k|X_k=1)$, we have

$$\begin{aligned}\mathbf{E}\left[e^{-\nu_n^{\mathbf{V}}f}|E_n\right]&=\mathbf{E}\left[\exp\Big\{-\sum_{k=1}^\infty X_k f(V_k z_k/z_n)\Big\}\Big|E_n\right]\\ &=\mathbf{E}\left[\exp\Big\{-\sum_{k=1}^{m(n)-1} X_k f(V_k z_k/z_n)\Big\}\right]\mathbf{E}\left[\prod_{k=m(n),X_k=1}^\infty \mathbf{E}\, e^{-f(V(1)z_k/z_n)}\Big|E_n\right].\end{aligned} \tag{4.14}$$

Since $z_{m(n)}=o(z_n)$, the process $\sum_{k=1}^{m(n)-1}X_k\delta_{z_k/z_n}$ converges vaguely in distribution to the zero process in $\mathcal{M}((0,\infty))$ as $n\to\infty$. Combined with our assumption that $\varepsilon\le\|V(x)\|\le r$ almost surely for all $x\in\mathbb{N}$, this implies that the first factor on the right-hand side of (4.14) converges to 1 as $n \to \infty$. For the second factor in (4.14), we have

$$\mathbf{E}\left[\prod_{k=m(n),X_k=1}^{\infty} \mathbf{E}\, e^{-f(V(1)z_k/z_n)}\Big|E_n\right] = \mathbf{E}\,\exp\Big\{-\sum_{k=m(n)}^{\infty} Y_k \widetilde{h}(z_k/z_n)\Big\},$$

where $Y_k \sim \mathrm{Ber}(q_k^1/(q_k^0+q_k^1))$, $k \geq m(n)$, has the same distribution as X_k conditional on E_n, and

$$\widetilde{h}(t) = -\log \mathbf{E}\, e^{-f(V(1)t)}.$$

Consider the point process $\widetilde{\nu}_n = \sum_{k=1}^{\infty} Y_k \delta_{z_k/z_n}$. Using (4.13) for the first equality, we have that for any $0 < a < b < \infty$,

$$\lim_{n\to\infty} \prod_{k: az_n < z_k \leq bz_n} \frac{q_k^0}{q_k^0 + q_k^1} = \lim_{n\to\infty} \prod_{k: az_n < z_k \leq bz_n} q_k^0 = \left(\frac{a}{b}\right)^c,$$

where in the last equality we have used our assumption that $\nu_n \xrightarrow{d} \eta_c$ and Theorem 3.2. Hence, $(\widetilde{\nu}_n)_{n\in\mathbb{N}}$ satisfies Condition 3.1(i). That $(\widetilde{\nu}_n)_{n\in\mathbb{N}}$ satisfies Condition 3.1(ii) follows trivially by noticing that $(\nu_n)_{n\in\mathbb{N}}$ satisfies Condition 3.1(ii). Thus, $\widetilde{\nu}_n$ converges vaguely in distribution to η_c as $n \to \infty$ by Theorem 3.2. Again, we can ignore the first $m(n)-1$ terms of the sum $\widetilde{\nu}_n$ as it converges to a zero process, whence

$$\sum_{k=m(n)}^{\infty} Y_k \delta_{z_k/z_n} \xrightarrow{d} \eta_c \quad \text{as } n \to \infty.$$

By our assumption that $V(1)$ is bounded away from 0 and ∞ and that f is compactly supported, it follows that the function $\widetilde{h}$ has a relatively compact support in $(0,\infty)$. Clearly, $\widetilde{h}$ is continuous and bounded. Hence by (4.3),

$$\mathbf{E}\,\exp\Big\{-\sum_{k=m(n)}^{\infty} Y_k \widetilde{h}(z_k/z_n)\Big\} \to \mathbf{E}\, e^{-\eta_c \widetilde{h}} \quad \text{as } n \to \infty.$$

By (4.14),

$$\mathbf{E}\left[e^{-\nu_n^{\mathbf{V}} f}|E_n\right] \to \mathbf{E}\, e^{-\eta_c \widetilde{h}} \quad \text{as } n \to \infty.$$

Finally, noticing that

$$\mathbf{E}\, e^{-\eta_c \widetilde{h}} = \mathbf{E}\Big[\mathbf{E}(e^{-\eta_c^{V(1)} f}|\eta_c)\Big] = \mathbf{E}\, e^{-\eta_c^{V(1)} f},$$

and that $\mathbf{P}(E_n) \to 1$ as $n \to \infty$, we have

$$\mathbf{E}\, e^{-\nu_n^{\mathbf{V}} f} = \mathbf{E}\left[e^{-\nu_n^{\mathbf{V}} f}|E_n\right]\mathbf{P}(E_n) + \mathbf{E}\left[e^{-\nu_n^{\mathbf{V}} f}|E_n^c\right]\mathbf{P}(E_n^c) \to \mathbf{E}\, e^{-\eta_c^{V(1)} f}$$

as $n \to \infty$ for any $f \in \widehat{C}_S$, which yields that $\nu_n^{\mathbf{V}} \xrightarrow{d} \eta_c^{V(1)}$ as $n \to \infty$. □

Example 4.8. For $z_k = \log p_k$ and $X_k \sim \mathrm{Geom}(1-1/p_k)$ or $\mathrm{Ber}(1/(p_k+1))$, one can easily see that the conditions of Theorem 4.7 are satisfied by $(\nu_n)_{n\in\mathbb{N}}$, and hence, for $\mathbf{V}$ as in Theorem 4.7, the conclusion of the result holds.

Note, if V_k is independent of X_k for all $k \in \mathbb{N}$, then they are necessarily i.i.d. by (4.12). Now we consider an example when $(X_k)_{k\in\mathbb{N}}$ and $\mathbf{V}$ are dependent.

Example 4.9. Let $d \geq 2$ and $m \in \mathbb{N}$ be positive integers. Let $X_k \sim \mathrm{Geom}(1 - 1/p_k)$ be independent and $V_k = (mX_k)^{-1}(X_k^1, \dots, X_k^d)\mathbb{1}_{\{X_k>0\}}$ for $k \in \mathbb{N}$, where $(X_k^1, \dots, X_k^d)$ is multinomially distributed with the number of experiments mX_k and the probabilities of outcomes $q_1, \dots, q_d$ with $\sum_{i=1}^d q_i = 1$. Let

$$\nu_n = \sum_{k=1}^{\infty} X_k \delta_{\log p_k / \log p_n} \quad \text{and} \quad \nu_n^{\mathbf{V}} = \sum_{k=1}^{\infty} X_k \delta_{V_k \log p_k / \log p_n},$$

where $(p_k)_{k\in\mathbb{N}}$ is an enumeration of the primes. Clearly, the random variables $(X_k)_{k\in\mathbb{N}}$ satisfy (4.13). For each k, the random vector V_k and hence $V(x)$ is almost surely bounded away from 0 and ∞ when $X_k = x > 0$. Since by Theorem 3.6 we have that ν_n converges vaguely in distribution to η as $n \to \infty$, Theorem 4.7 yields that $\nu_n^{\mathbf{V}} \xrightarrow{d} \eta^{V(1)}$ as $n \to \infty$, where $mV(1)$ is distributed as a multinomial random variable with m experiments and probabilities of outcomes $q_1, \dots, q_d$.

5 Scale-invariant Poisson processes in higher dimensions and multivariate Dickman distributions

In this section, we study and classify scale-invariant Poisson processes in higher dimensions and extend the generalized Dickman distributions in one dimension to its multivariate counterpart. For a simple point process ξ in $(0,\infty)$ and a random vector V taking values in $S = \mathbb{R}^d \setminus \{0\}$ bounded away from 0 and ∞ with i.i.d. copies $(V_k)_{k\in\mathbb{N}}$, recall that the uplifted point process ξ^V is given by

$$\xi^V =_d \sum_{k=1}^{\infty} \delta_{V_k Z_k},$$

where $(Z_k)_{k\in\mathbb{N}}$ is an enumeration of the points in ξ.

Lemma 5.1. *Any scale-invariant Poisson process in S has the same distribution as η_c^U for some $c > 0$ and unit random vector U in $\mathbb{R}^d$. Moreover, for any random vector V in S with η_c and $(V_k)_{k\in\mathbb{N}}$ satisfying (4.2), the uplifted point process η_c^V has the same distribution as η_c^U with $U = V/\|V\|$.*

Proof. Let ν be a scale-invariant Poisson process in S. Hence $\nu(tB) =_d \nu(B)$ for every Borel set $B \in \mathfrak{S}$ and $t > 0$. Represent each point $x \in S$ as a pair $(u,r) \in \mathbb{S}^{d-1} \times (0,\infty)$, where $u = x/\|x\|$ and $r = \|x\|$. For a measurable subset $A \subseteq \mathbb{S}^{d-1}$ and $0 < a < b < \infty$, by scale invariance one has

$$\mathbf{E}\, \nu(A \times [a,b]) = \mathbf{E}\, \nu(A \times [a/b, 1]). \tag{5.1}$$

For $p \in (0,1)$ and $A \subseteq \mathbb{S}^{d-1}$, define $\gamma_\nu(p,A) = \mathbf{E}\, \nu(A \times [p,1])$ and $\gamma_\nu(1,A) = 0$. For every fixed $A \subseteq \mathbb{S}^{d-1}$, notice that γ_ν satisfies

$$\gamma_\nu(p,A) + \gamma_\nu(q,A) = \gamma_\nu(pq,A), \quad p,q \in (0,1).$$

By monotonicity, $\gamma_\nu(p,A) = -\gamma_\nu(A)\log p$ for $p \in (0,1]$, where γ_ν is a locally finite measure on $\mathbb{S}^{d-1}$ not depending on p. By (5.1),

$$\mathbf{E}\, \nu(A \times [a,b]) = \gamma_\nu(A)\log(b/a).$$

For a random vector U in the unit sphere $\mathbb{S}^{d-1}$ with distribution μ, the uplifted process η_c^U is also a Poisson process. Its intensity measure is given by

$$\mathbf{E}\,\eta_c^U(A\times[a,b]) = \int_{u\in A}\int_a^b ct^{-1}dt\mu(du) = c\mu(A)\log(b/a) \tag{5.2}$$

for all Borel $A\subseteq\mathbb{S}^{d-1}$ and $0<a<b<\infty$. It is immediately seen that η_c^U is scale-invariant. By comparing the two equations above., we obtain that ν has the same intensity measure as η_c^U with $c=\gamma_\nu(\mathbb{S}^{d-1})$ and U is distributed according to $\mu=\gamma_\nu/c$. Thus $\nu =_d \eta_c^U$ proving the first claim.

Next, for a random vector V distributed on S according to a probability measure ψ with η_c and $(V_k)_{k\in\mathbb{N}}$ satisfying (4.2), let $U=V/\|V\|$. Clearly, η_c^V is also a Poisson process. For all $A\subseteq\mathbb{S}^{d-1}$ and $0<a<b<\infty$, using the substitution $z=\|v\|t$ in the second step, the intensity of η_c^V can be expressed as

$$\begin{aligned}\mathbf{E}\,\eta_c^V(A\times[a,b]) &= \int_{v/\|v\|\in A,\|v\|t\in[a,b]} ct^{-1}dt\psi(dv)\\ &= \int_{v/\|v\|\in A,z\in[a,b]} cz^{-1}dz\psi(dv) = c\log(b/a)\mathbf{P}\{U\in A\} = \mathbf{E}\,\eta_c^U(A\times[a,b]),\end{aligned}$$

where in the last step we have used (5.2). Hence, $\eta_c^V =_d \eta_c^U$. □

Recall that the generalized Dickman random variable D_c with parameter $c>0$ has the same distribution as the sum of points of η_c in the interval $(0,1)$. One can naturally generalize this definition to dimensions $d\ge 2$ by considering a scale-invariant Poisson process in S, which by Lemma 5.1 is of the form η_c^U for some $c>0$ and unit random vector U in $\mathbb{R}^d$, and summing its points lying inside the unit ball B_1. The following definition makes this precise.

Definition 5.2. *For a unit random vector U in $\mathbb{R}^d$ and $c>0$, the* multivariate Dickman *random variable D_c^U with parameters (c,U) is defined by*

$$D_c^U = \int_{B_1} x\eta_c^U(dx) = \sum_{x\in\eta_c^U\cap B_1} x. \tag{5.3}$$

Note that the points of η_c in the interval $(0,1)$ are distributed as the collection $\{Q_1^{1/c},(Q_1Q_2)^{1/c},\dots\}$, where $(Q_k)_{k\in\mathbb{N}}$ are independent copies of a random variable Q which is uniformly distributed on $[0,1]$. Thus, letting $(U_i)_{i\in\mathbb{N}}$ be i.i.d. copies of U, we can write

$$D_c^U =_d \sum_{x\in\eta_c^U} x\mathbb{1}_{\{\|x\|<1\}} =_d \sum_{k=1}^\infty U_k\prod_{i=1}^k Q_i^{1/c} =_d Q^{1/c}(D_c^U+U')$$

for Q uniformly distributed on $[0,1]$ and $U' =_d U$ independent of D_c^U. Thus, the random variable D_c^U is the unique fixed point of the distributional transformation $W\mapsto W^*$ given by

$$W^* =_d Q^{1/c}(W+U)$$

with Q, U and W mutually independent.

By Lemma 5.1, the sum of points from any scale-invariant Poisson process lying inside the unit ball is distributed as D_c^U for some $c>0$ and unit random vector U. In particular, for a general random vector V in S, by Lemma 5.1, it is straightforward to see that the sum of points of η_c^V inside the unit ball is distributed as D_c^U with $U=V/\|V\|$.

Also note that

$$D_c^U =_d \sum_{k=1}^{\infty} e^{-Z_k} U_k,$$

where $(U_k)_{k\geq 1}$ are i.i.d. copies of U and $(Z_k)_{k\in\mathbb{N}}$ is an enumeration of the points of a homogeneous Poisson process on the interval $(0,1)$ with intensity c. In particular, D_c^U is self-decomposable, see [24].

We finish this section with an example of weak convergence to a multivariate Dickman distribution as defined in (5.3). Consider the setting of Example 4.9 with $d = 2$ and $m = 1$. Let p_k, X_k and V_k be as in Example 4.9. For $p \in (0,1)$, let $X_k^1 \sim \mathrm{Bin}(X_k, p)\mathbb{1}_{\{X_k>0\}}$ and $X_k^2 = X_k - X_k^1$. Define

$$W_n = \sum_{k=1}^{n} X_k V_k \log p_k / \log p_n = \frac{1}{\log p_n} \sum_{k=1}^{n} (X_k^1, X_k^2) \log p_k, \tag{5.4}$$

where $V_k = X_k^{-1}(X_k^1, X_k^2)\mathbb{1}_{\{X_k>0\}}$. Let D_1^U denote a Dickman random variable defined at (5.3), where $U = (X, 1-X)$ with $X \sim \mathrm{Ber}(p)$.

Theorem 5.3. *Let W_n be given by* (5.4). *Then $W_n \xrightarrow{d} D_1^U$ as $n \to \infty$.*

Proof. Define

$$\overline{\nu}_n = \sum_{k=\log n}^{\infty} Y_k \delta_{\log p_k / \log p_n},$$

where the random variables $(Y_k)_{k\in\mathbb{N}}$ are independent with $Y_k \sim \mathrm{Ber}((1+p_k)^{-1})$ for $k \in \mathbb{N}$. Notice that $\sum_{k=1}^{\log n - 1} Y_k \delta_{\log p_k / \log p_n}$ converges vaguely in distribution to the zero process on $(0,\infty)$ as $n \to \infty$. By Remark 3.7, the process $\sum_{k=1}^{\infty} Y_k \delta_{\log p_k / \log p_n}$ converges vaguely in distribution to η as $n \to \infty$, hence, so does $(\overline{\nu}_n)_{n\in\mathbb{N}}$. By Theorem 4.7, we obtain that $\overline{\nu}_n^U \xrightarrow{d} \eta^U$ as $n \to \infty$.

Let $E_n = \{X_k = \widetilde{X}_k \text{ for all } k \geq \log n\}$, where $\widetilde{X}_k = \mathbb{1}_{\{X_k>0\}}$. Notice that for each k, the random variable Y_k has the same law as X_k conditional on the event $X_k = \widetilde{X}_k$. Hence, for each n, conditional on E_n, the point process $(X_k V_k \log p_k / \log p_n)_{\log n \leq k \leq n}$ has the same law as $\overline{\nu}_n^U$ restricted to the unit ball B_1. Therefore, the conditional law of

$$Z_n = \sum_{k=\log n}^{n} X_k V_k \log p_k / \log p_n$$

given E_n is the same as that of $\int_{B_1} x d\overline{\nu}_n^U$. Using [25, Prop. 1.51], notice that

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbf{E} \int_{B_\varepsilon} x \overline{\nu}_n^U(dx) \leq \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{\log p_n} \sum_{k: p_k \leq p_n^\varepsilon} \frac{\log p_k}{1+p_k} = 0.$$

Since $\overline{\nu}_n^U \xrightarrow{d} \eta^U$ as $n \to \infty$, using [18, Theorem 4.28] and Lemma A.2, it is not hard to see that

$$(Z_n | E_n) =_d \int_{B_1} x \overline{\nu}_n^U(dx) \xrightarrow{d} \int_{B_1} x \eta^U(dx) =_d D_1^U \quad \text{as} \quad n \to \infty.$$

Since $\mathbf{P}(E_n) \to 1$ as $n \to \infty$, this yields that $Z_n \xrightarrow{d} D_1^U$ as $n \to \infty$.

Finally, taking expectation and using [25, Prop. 1.51], it is straightforward to see that

$$\sum_{k=1}^{\log n - 1} X_k V_k \log p_k / \log p_n \to 0 \quad \text{as} \quad n \to \infty$$

in L^1, hence, in probability as $n \to \infty$. An application of Slutsky's theorem yields the result. ☐

A Results on vague convergence in distribution

Let S be a locally compact separable metric space. The following result provides a necessary and sufficient condition for the vague convergence in distribution of a sequence of point processes to a simple point process. Recall that a *semi-ring* $\mathcal{I}$ is a family of sets closed under finite intersections such that any proper difference of sets in $\mathcal{I}$ is a finite, disjoint union of $\mathcal{I}$-sets.

Theorem A.1 (see [19, Theorem 4.15]). *Let $(\xi_n)_{n\ge 1}$ be point processes on S, and fix a dissecting ring $\mathcal{U} \subset \widehat{\mathfrak{S}}_{\mathbf{E}\xi}$ and a semi-ring $\mathcal{I} \subset \mathcal{U}$. Then $\xi_n \xrightarrow{d} \xi$ in $\mathcal{N}(S)$ as $n \to \infty$ for a simple point process ξ if and only if*

(i) $\lim_{n\to\infty} \mathbf{P}\{\xi_n A = 0\} = \mathbf{P}\{\xi A = 0\}$ *for all $A \in \mathcal{U}$, and*

(ii) $\limsup_{n\to\infty} \mathbf{P}\{\xi_n B > 1\} \le \mathbf{P}\{\xi B > 1\}$ *for all $B \in \mathcal{I}$.*

Recall that $\xi_n \xrightarrow{d} \xi$ is equivalent to (see e.g. [19, Theorem 4.11])

$$\int f(x)\xi_n(dx) \xrightarrow{d} \int f(x)\xi(dx) \quad \text{as } n \to \infty \tag{A.1}$$

for all $f \in \widehat{C}_S$. By a standard argument, approximating an indicator function with a continuous function, it is straightforward to derive the following result.

Lemma A.2. *Let $(\xi_n)_{n\ge 1}, \xi$ be random measures in S such that $\xi_n \xrightarrow{d} \xi$ as $n \to \infty$. For a relatively compact measurable set K, let $f : S \to \mathbb{R}$ be a non-negative function which is continuous when restricted to K and $f(x) = 0$ for $x \notin K$. If $\mathbf{E}\xi(\partial K) = 0$, then* (A.1) *holds.*

Next we prove Lemma 4.1.

Proof of Lemma 4.1. Fix $\varepsilon > 0$ and K_ε satisfying (4.4). Since ξ has a finite intensity, without loss of generality, we can assume that $\mathbf{E}\,\xi(\partial K_\varepsilon) = 0$. By Lemma A.2,

$$\int_{K_\varepsilon} h(x)\xi_n(dx) \xrightarrow{d} \int_{K_\varepsilon} h(x)\xi(dx) \quad \text{as } n \to \infty.$$

Hence,

$$\mathbf{E}\,\exp\left\{-\int_{K_\varepsilon} h(x)\xi_n(dx)\right\} \to \mathbf{E}\,\exp\left\{-\int_{K_\varepsilon} h(x)\xi(dx)\right\} \tag{A.2}$$

as $n \to \infty$. Since $e^{\mathbf{E}X} \le \mathbf{E}\,e^X$,

$$\log\,\mathbf{E}\,\exp\left\{-\int_{K_\varepsilon^c} h(x)\xi_n(dx)\right\} \ge -\mathbf{E}\int_{K_\varepsilon^c} h(x)\xi_n(dx).$$

Thus, by (4.4), we have that

$$\lim_{\varepsilon\downarrow 0}\liminf_{n\to\infty}\mathbf{E}\,\exp\left\{-\int_{K_\varepsilon^c} h(x)\xi_n(dx)\right\} = e^0 = 1. \tag{A.3}$$

The same holds for the upper limit trivially. Combining (A.2) and (A.3) yields the desired result. □

The following result which is a direct consequence of [18, Theorem 4.28] and Lemma A.2 provides conditions under which the vague convergence of a general sequence of point processes $(\nu_n)_{n\in\mathbb{N}}$ to η_c implies the convergence of the sum of points in $(0, 1)$.

Lemma A.3. *Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of point processes in $(0,\infty)$ with $\nu_n \xrightarrow{d} \eta_c$ for $c>0$. If*

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\mathbf{E}\int_0^\varepsilon t\nu_n(dt)=0, \tag{A.4}$$

then

$$\int_0^1 t\nu_n(dt) \xrightarrow{d} \int_0^1 t\eta_c(dt) =_d D_c.$$

Lemma A.4 (Continuity of M restricted to simple counting measures)**.** *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of counting measures (deterministic) in $\mathcal{N}((0,\infty))$ such that ξ_n converges vaguely to ξ as $n\to\infty$ for a simple counting measure ξ. If M is given by* (4.7)*, then $M(\xi_n)$ converges vaguely to $M(\xi)$ as $n\to\infty$.*

Proof. Denote $\widetilde{\xi}_n = M(\xi_n)$ and $\widetilde{\xi} = M(\xi)$. Note that it suffices to show that, for all $0<a<b<\infty$ and $k\in\mathbb{N}_0$,

$$\widetilde{\xi}_n([k,\infty)\times[a,b]) \to \widetilde{\xi}([k,\infty)\times[a,b]) \quad \text{as} \;\; n\to\infty. \tag{A.5}$$

Since ξ is simple,

$$\widetilde{\xi}([k,\infty)\times[a,b]) = \begin{cases}\xi([a,b]) & \text{for } k=0,1,\\ 0 & \text{for } k>1.\end{cases}$$

Note that $\widetilde{\xi}_n([k,\infty)\times[a,b]) = \xi_n([a,b])$ for $k=0,1$. Hence, (A.5) holds for $k=0,1$ by our assumption that $\xi_n\to\xi$ as $n\to\infty$. Fix $k>1$. Let $\xi([a,b])=m$ for some $m\ge 0$. If $m=0$, by our assumption we have $\xi_n([a,b])\to\xi([a,b])=0$ as $n\to\infty$, which yields $\widetilde{\xi}_n([k,\infty)\times[a,b])\to 0$ as $n\to\infty$, showing (A.5). Next, assume that $m\ge 1$. Since ξ is a locally finite counting measure, there are disjoint intervals $(I_i)_{1\le i\le m}$ such that $\xi(I_i)=1$ for $1\le i\le m$ and $\cup_{i=1}^m I_i=[a,b]$. By our assumption, $\xi_n(I_i)\to\xi(I_i)=1$ as $n\to\infty$ for $1\le i\le m$. Since $k>1$, we have $\widetilde{\xi}_n([k,\infty)\times I_i)\to 0$ as $n\to\infty$. Taking union over the m sets $[k,\infty)\times I_i$, $1\le i\le m$ proves (A.5), concluding the proof. □

References

[1] Arras, B., Mijoule, G., Poly, G. and Swan, Y.: A new approach to the Stein–Tikhomirov method: with applications to the second Wiener chaos and Dickman convergence. *arXiv preprint* arXiv:1605.06819, (2016).

[2] Arratia, R.: On the central role of scale invariant Poisson processes on $(0,\infty)$. In *Microsurveys in Discrete Probability (Princeton, NJ, 1997)*, Amer. Math. Soc., Providence, RI, (1998), 21–41. MR-1630407

[3] Arratia, R.: Personal communications, (2017).

[4] Arratia, R., Barbour, A. D. and Tavaré, S.: *Logarithmic Combinatorial Structures: a Probabilistic Approach*. European Mathematical Society (EMS), Zürich, 2003. MR-2032426

[5] Arratia, R., Garibaldi, S. and Kilian, J.: Asymptotic distribution for the birthday problem with multiple coincidences, via an embedding of the collision process. *Random Structures Algorithms*, **48**, (2016), 480–502. MR-3481270

[6] Barbour, A. D. and Nietlispach, B.: Approximation by the Dickman distribution and quasi-logarithmic combinatorial structures. *Electron. J. Probab.*, **16**, (2011), 880–902. MR-2793242

[7] Basrak, B., Planinić, H. and Soulier, P.: An invariance principle for sums and record times of regularly varying stationary sequences. *Probab. Theory Related Fields*, **172**, (2018), 869–914. MR-3877549

[8] Bhatt, A. G. and Roy, R.: On a random directed spanning tree. *Adv. in Appl. Probab.*, **36**, (2004), 19–42. MR-2035772

[9] Bhattacharjee, C. and Goldstein, L.: Dickman approximation in simulation, summations and perpetuities. *Bernoulli*, **25**, (2019), 2758–2792. MR-4003564

[10] Covo, S.: On approximations of small jumps of subordinators with particular emphasis on a Dickman-type limit. *J. Appl. Probab.*, **46**, (2009), 732–755. MR-2562319

[11] Daley, D. J. and Vere-Jones, D.: *An Introduction to the Theory of Point Processes. Vol. I: Elementary Theory and Methods*. Springer, New York, second edition, 2003. MR-1950431

[12] Daley, D. J. and Vere-Jones, D.: *An Introduction to the Theory of Point Processes. Vol. II: General Theory and Structure*. Springer, New York, second edition, 2008. MR-2371524

[13] Dickman, K.: On the frequency of numbers containing prime factors of a certain relative magnitude. *Arkiv for matematik, astronomi och fysik*, **22**, (1930), 1–14.

[14] Dusart, P.: The kth prime is greater than $k(\ln k + \ln\ln k - 1)$ for $k \geq 2$. *Math. Comp.*, **68(225)**, (1999), 411–415. MR-1620223

[15] Goldstein, L.: Non-asymptotic distributional bounds for the Dickman approximation of the running time of the Quickselect algorithm. *Electron. J. Probab.*, **23**, (2018), 13 pp. MR-3862615

[16] Hwang, H. K. and Tsai, T. H.: Quickselect and the Dickman function. *Combin. Probab. Comput.*, **11**, (2002), 353–371. MR-1918722

[17] Kallenberg, O.: Characterization and convergence of random measures and point processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **(27)**, (1973), 9–21. MR-0431374

[18] Kallenberg, O.: *Foundations of Modern Probability*. Springer, New York, second edition, 2002. MR-1876169

[19] Kallenberg, O.: *Random Measures, Theory and Applications*. Springer, Cham, 2017. MR-3642325

[20] Lindskog, F., Resnick, S. I. and Roy, J.: Regularly varying measures on metric spaces: hidden regular variation and hidden jumps. *Probab. Surv.*, **11**, (2014), 270–314. MR-3271332

[21] Penrose, M. D. and Wade, A. R.: Random minimal directed spanning trees and Dickman-type distributions. *Adv. in Appl. Probab.*, **36**, (2004), 691–714. MR-2079909

[22] Pinsky, R. G.: On the strange domain of attraction to generalized Dickman distribution for sums of independent random variables. *Electron. J. Probab.*, **23**, (2018), 17 pp. MR-3751078

[23] Pinsky, R. G.: A natural probabilistic model on the integers and its relation to Dickman-type distributions and Buchstab's function. In *Probability and Analysis in Interacting Physical Systems — in Honor of S.R.S. Varadhan*, Springer, (2019), 267–294. MR-3968515

[24] Sato, K. i.: Class L of multivariate distributions and its subclasses. *J. Multivariate Anal.*, **10**, (1980), 207–232. MR-0575925

[25] Tao, T. and Vu, V.: *Additive Combinatorics*. Cambridge University Press, Cambridge, 2006. MR-2289012

Acknowledgments. We thank Matthias Schulte for a number of useful comments on the manuscript that greatly improved the paper. We would also like to thank Richard Arratia for pointing out the possible connection between convergence of sums and the convergence of underlying point processes.