Stabilization of DLA in a wedge

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Abstract

We consider Diffusion Limited Aggregation (DLA) in a two-dimensional wedge. We prove that if the angle of the wedge is smaller than $\pi/4$, there is some $a > 2$ such that almost surely, for all R large enough, after time R^a all new particles attached to the DLA will be at distance larger than R from the origin. Furthermore, we provide estimates on the size of R under which this holds. This means that DLA stabilizes in growing balls, thus allowing a definition of the infinite DLA in a wedge via a finite time process.

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1 Introduction

Diffusion Limited Aggregation (DLA) was introduced in 1983 by E. Witten and L. M. Sander [WS83] in order to study the geometry and dynamics of physical aggregation systems governed by diffusive laws. On the Euclidean lattice $\mathbb{Z}^2$, DLA is a random process $(A_n)_{n \geq 0}$ of growing subsets of $\mathbb{Z}^2$, which are defined recursively. Typically, one fixes $A_0 := \{(0,0)\}$, and given A_n, defines $A_{n+1} := A_n \cup \{a_{n+1}\}$, where a_{n+1} is a point sampled according to the harmonic measure of ∂A_n from infinity. More precisely, a_{n+1} is the first hitting place of ∂A_n (the outer boundary of A_n) by a simple random walk started from distance R, in the limit $R \to \infty$ (See Section 3 for the precise definition).

In this paper we study DLA in a two-dimensional wedge

$$W_{\theta_1, \theta_2} = \left\{ (x,y) \in \mathbb{Z}^2 : \arctan(y/x) \in [\theta_1, \theta_2],\ x \geq 0 \right\},$$

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where $-\pi/2 \leq \theta_1 < \theta_2 \leq \pi/2$. Here we used the convention that $(0,0)$ belongs to all wedges and that $\arctan(y/0)$ equals $\pi/2$ for $y > 0$ and $-\pi/2$ for $y < 0$. In Appendix A we prove the existence of the harmonic measure in W_{θ_1,θ_2} which is needed for the definition of the DLA in the wedge.

For $R > 0$, let $B_R = \{(x,y) \in \mathbb{Z}^2 \ : \ x^2 + y^2 < R^2\}$ be the discrete Euclidean ball of radius R around the origin, and define $W_{\theta_1,\theta_2}^R = W_{\theta_1,\theta_2} \cap B_R$. Throughout the paper, we consider W_{θ_1,θ_2} and W_{θ_1,θ_2}^R for $R > 0$ as graphs, with vertices W_{θ_1,θ_2} and W_{θ_1,θ_2}^R respectively and edges induced from the graph $\mathbb{Z}^2$. We denote by $\mathbf{P}_{\theta_1,\theta_2}^x$ the law of a simple random walk $(S_n)_{n \geq 0}$ in the graph W_{θ_1,θ_2}, starting from x and for $D \subset W_{\theta_1,\theta_2}$, denote by $\tau_D^+ = \inf\{n \geq 1 \ : \ S_n \in D\}$, the first return time of the random walk into the set D. Finally, we set $\mathbb{P} = \mathbb{P}_{\theta_1,\theta_2}$ to be the law of the DLA $(A_n)_{n \geq 0}$ in W_{θ_1,θ_2} (see Section 3 for the formal definition). For future use, for $n \geq 1$ we denote by a_n the particle added to the aggregate at time n, namely the unique vertex in W_{θ_1,θ_2} such that $A_n = A_{n-1} \cup \{a_n\}$.

Our main result is the stabilization of the DLA in sufficiently sharp wedges.

Theorem 1.1. *Assume $-\pi/2 \leq \theta_1 < \theta_2 \leq \pi/2$ satisfy $\theta_2 - \theta_1 < \pi/4$ and fix $a > \frac{2\pi+4(\theta_2-\theta_1)}{\pi-4(\theta_2-\theta_1)}$. Then $\mathbb{P}_{\theta_1,\theta_2}$-almost surely, for every $R > 0$ sufficiently large, the random sets $(A_n \cap B_R)_{n \geq R^a}$ are all the same. In other words, for all R sufficiently large, none of the particles $(a_n)_{n \geq R^a}$ added to the system after time R^a will attach to the aggregate inside W_{θ_1,θ_2}^R. Furthermore, for a fixed $a > 2$, $\varepsilon > 0$, there exists $C > 0$ such that for all $R > 0$*

$$\mathbb{P}_{\theta_1,\theta_2}(\exists n \geq R^a \text{ such that } a_n \in B_R) \leq C R^{a+1-\left(\frac{a}{2}-1\right)\left(\frac{\pi}{2(\theta_2-\theta_1)}-\varepsilon\right)} \log R. \tag{1.1}$$

The main tool in proving Theorem 1.1 is a discrete Beurling estimate for random walk in a wedge, which enables us to control the harmonic measure of finite, connected subsets of W_{θ_1,θ_2}. Unlike in the work of H. Kesten [Kes87b], the proof of the discrete Beurling estimate here does not rely on Green function calculations. Furthermore, our discrete version takes the form of penetration probability bounds rather than Kesten's maximal harmonic measure bound, which is non trivial to prove for reflected random walk.

For $A \subset W_{\theta_1,\theta_2}$ denote by $\partial A = \{y \in W_{\theta_1,\theta_2} \setminus A \ : \ \exists x \in A \text{ such that } \|x - y\|_1 = 1\}$ the outer boundary of A.

Theorem 1.2. *Fix $-\pi/2 \leq \theta_1 < \theta_2 \leq \pi/2$. For every $\varepsilon > 0$, there exists $M \in \mathbb{N}$ and $C \in (0,\infty)$ such that for every $r, L \in \mathbb{N}$ satisfying $r \geq M$ and $L/r \geq M$, every $R > 0$ sufficiently large (depending on ε and L), every connected subset $A \subset W_{\theta_1,\theta_2}$, such that $0 \in A$, $A \cap \partial W_{\theta_1,\theta_2}^L \neq \emptyset$ and every $x \in \partial W_{\theta_1,\theta_2}^R$*

$$\mathbf{P}_{\theta_1,\theta_2}^x \left(\tau_{\partial W_{\theta_1,\theta_2}^r}^+ \leq \tau_{\partial A}^+ \right) \leq C \left(\frac{r}{L} \right)^{\frac{\pi}{2(\theta_2-\theta_1)}-\varepsilon} r \log L. \tag{1.2}$$

2 Discussion and open problems

Since the introduction of DLA in 1983, rigorous understanding of the model was limited. The main exception being H. Kesten's upper bound on the growth rate [Kes87b], see also [BY17]. Lately, similar results were obtained for DLA in the upper half plane with Dirichlet boundary conditions [PZ19, PZ17]. The main technical difference between this paper and previous works is that the former does not use any Green function calculations. The main reason is the lack of control over the discrete Green function in the wedge that would allow hitting probability calculations (See [GP17] for the best known control in the case of general Neumann boundary conditions).

There are many interesting open questions regarding DLA. First natural questions are about the growth rate, the fractal dimension and the relation between the two (see

[HMP86]). For these important questions our paper does not add to the discussion. Another natural question is about the number of arms in DLA growing in a wedge (or in $\mathbb{Z}^2$). The physics literature does not provide clear conjectures or even definitions for the number of arms in a wedge. In [KOO$^+$98], D. A. Kessler, Z. Olami, J. Oz, I. Procaccia, E. Somfai and L. M. Sander claim evidence for a critical angle ν between 120 and 140 degrees which guarantees coexistence of two arms in a wedge of angle ν.
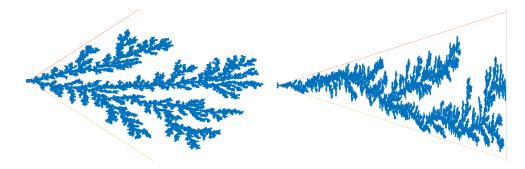
One immediate contribution of our result is to provide a method to sampling the DLA in W_{θ_1,θ_2}^R, for every finite $R > 0$, via a finite time random process. By Theorem 1.1 there is some $a > 0$ such that for any R large enough almost surely the sets $(A_n \cap B_R)_{n \geq R^a}$ are all the same. As a result, for all $R > 0$ sufficiently large, we can define the DLA in W_{θ_1,θ_2}^R to be $A_{R^a} \cap B_R$, which is a finite time random process. For a fixed R, an estimate for the probability that no particle will ever penetrate the ball of radius R after time R^a is presented in (4.1).
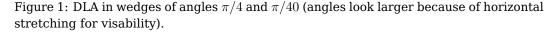
Returning to discuss the number of arms, since the sets $(A_{R^a} \cap B_R)$ are monotonic increasing in R, we can define

$$A_\infty := \bigcup_{n=0}^\infty A_n = \lim_{R \to \infty} (A_{R^a} \cap B_R).$$

Let ⅂ be an infinite graph. The number of ends of ⅂ is defined to be the supremum on the number of infinite, connected components of ⅂ $\setminus K$, where we run over all finite $K \subset$ ⅂. Hence, one can define the number of arms of the DLA as the number of ends of the graph ⅂ $= A_\infty$. Due to the fact that A_∞ can be written as the limit of the sets $A_{R^a} \cap B_R$, we can erase a finite set K in finite times and only look on the dynamics after such times.

Conjecture 2.1. *There exists $\theta_0 \in (0, 2\pi)$ such that if $0 < \theta_2 - \theta_1 < \theta_0$, A_∞ has one arm.*

Remark 2.2. Computer simulations seem to suggest that θ_0 is smaller than $\pi/4$. See Figure 1.

Figure 1: DLA in wedges of angles $\pi/4$ and $\pi/40$ (angles look larger because of horizontal stretching for visability).

From our results one can deduce that if in a wedge of angle smaller than $\pi/4$, there are two connected components and one is behind the other by a polynomial order, then asymptotically the smaller component will cease to grow. However, without a lower bound on the growth rate this is not enough to prove the conjecture. That being said, we would like to suggest an open problem for the DLA in a wedge. Define the growth rate of $(A_n)_{n \geq 0}$, denote $\text{gr}((A_n)_{n \geq 0})$ by

$$\text{gr}((A_n)_{n \geq 0}) = \sup \left\{ \beta \geq 1/2 : \limsup_{n \to \infty} \frac{\text{diam}(A_n)}{n^\beta} > 0 \right\},$$

where diam stands for the diameter of the set in the Euclidean distance.

Conjecture 2.3. *For every $-\frac{\pi}{2} < \theta_1 < \theta_2 < \frac{\pi}{2}$, the growth rate is $\mathbb{P}_{\theta_1,\theta_2}$-almost surely a constant $\mathrm{gr}_{\theta_1,\theta_2} = \mathrm{gr}(\theta_2 - \theta_1)$. Furthermore $\limsup_{\theta_2-\theta_1\to 0} \mathrm{gr}(\theta_2 - \theta_1) < 1$.*

3 Formal definition of the DLA process in a wedge

This section is devoted to the formal definition of the DLA process in a wedge. In particular, we state a result regarding the existence of the harmonic measure from infinity in it, whose proof we postpone to Appendix A.

For $x = (x_1, x_2) \in \mathbb{Z}^2$ we denote by $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$ its Euclidean distance from the origin. Fix $-\pi/2 \le \theta_1 < \theta_2 \le \pi/2$. Recall that for $x \in W_{\theta_1,\theta_2}$ we denote by $\mathbf{P}^x_{\theta_1,\theta_2}$ the law of a simple random walk $(S_n)_{n\ge 0}$ in W_{θ_1,θ_2} staring from x, and for $A \subset W_{\theta_1,\theta_2}$ and $y \in \mathbb{Z}^2$, define $\mathcal{H}_A(x, y) = \mathbf{P}^x_{\theta_1,\theta_2}(S_{\tau_A} = y)$.

Theorem 3.1. *For every $A \subset W_{\theta_1,\theta_2}$ and $y \in \mathbb{Z}^2$, the following limit, called the harmonic measure of A from infinity at y exists*

$$\mathcal{H}^\infty_A(y) := \lim_{\|x\|_2 \to \infty} \mathcal{H}_A(x, y).$$

Unlike the analouge problem in the whole plane $\mathbb{Z}^2$, c.f. [LL10, Proposition 6.6.1], the existence of the limiting harmonic measure in a wedge is delicate. The main issue here is that, as of right now, there seems to be no discrete Green function approximations on the wedge (or part of it) that is precise enough to match our needs in applying that same approach. A detailed proof for the existence of the limit can be found in Appendix A.

Remark 3.2. Through private communications with Asaf Nachmias we learned that there is another approach to Theorem 3.1 using uniform spanning trees. By [BLPS01, Theorem 14.2] the limit in Theorem 3.1 exists if the uniform spanning tree in the wedge is one ended. This can be proved using electric network calculations similar to the ones in this paper. Another approach more similar to the one presented in this paper was suggested to us by Gregory F. Lawler using similar couplings to the ones presented in [HLLS18].

Combining Theorem 1.2 and Theorem 3.1 we obtain

Corollary 3.3. *Fix $-\pi/2 \le \theta_1 < \theta_2 \le \pi/2$. For every $\varepsilon > 0$, there exists $M \in \mathbb{N}$ and $C \in (0, \infty)$ such that for all $r, L \in \mathbb{N}$ satisfying $r \ge M$ and $L/r \ge M$ and every connected subset $A \subset W_{\theta_1,\theta_2}$, such that $W^r_{\theta_1,\theta_2} \subset A$ and $A \cap \partial W^L_{\theta_1,\theta_2} \ne \emptyset$.*

$$\mathcal{H}^\infty_{\partial A}(\partial W^r_{\theta_1,\theta_2}) \le C \left(\frac{r}{L}\right)^{\frac{\pi}{2(\theta_2-\theta_1)} - \varepsilon} r \log L. \tag{3.1}$$

Using the existence of the limit $\mathcal{H}^\infty_{\partial A}(y)$, we can now formally define the DLA in the wedge, denoted $(A_n)_{n\ge 0}$, to be a sequence of random subsets of W_{θ_1,θ_2} such that

$$A_0 = \{(0, 0)\} \qquad \text{and} \qquad A_{n+1} = A_n \cup \{a_{n+1}\}, \tag{3.2}$$

where, given A_n, the vertex $a_{n+1} \in W_{\theta_1,\theta_2}$ is sampled according to $\mathcal{H}^\infty_{\partial A_n}(\cdot)$. Note that $(A_n)_{n\ge 0}$ is an increasing family of subsets in W_{θ_1,θ_2}, and we denote its limit by $A_\infty = \bigcup_{n=0}^\infty A_n$.

4 Proof of Theorem 1.1

Throughout the remainder of the paper we fix $-\pi/2 \le \theta_1 < \theta_2 \le \pi/2$. In this section we assume Corollary 3.3 and turn to prove Theorem 1.1. Let $M > 0$ and $R \in \mathbb{N}$ such that $R > M$. Denote by $(L_n)_{n\ge 0}$ a sequence such that $L_0 \ge R$ and $L_{n+1}/L_n \ge M$ for every $n \ge 1$, and by σ_i the sequence of successive first exit times of $(A_n)_{n\ge 0}$ from B_{L_i}, namely,

$$\sigma_i \equiv \sigma(L_i) := \inf\{n \ge 0 : A_n \not\subset B_{L_i}\}.$$

Lemma 4.1. *Let $\varepsilon > 0$. For all M sufficiently large (depending only on ε) there exists a constant $C = C(M, \varepsilon) \in (0, \infty)$ such that for all $R > 0$ sufficiently large*

$$\mathbb{E}[|(A_\infty \setminus A_{\sigma_1}) \cap B_R|] \leq C \sum_{i=1}^\infty L_{i+1}^2 \left(\frac{R}{L_i}\right)^{\frac{\pi}{2(\theta_2 - \theta_1)} - \varepsilon} R \log L_i.$$

Proof. We rewrite $\mathbb{E}[|(A_\infty \setminus A_{\sigma_1}) \cap B_R|]$ as

$$\mathbb{E}[|(A_\infty \setminus A_{\sigma_1}) \cap B_R|] = \sum_{i=1}^\infty \mathbb{E}\left[|(A_{\sigma_{i+1}} \setminus A_{\sigma_i}) \cap B_R|\right],$$

and turn to bound each of the terms on the right hand side.

Fix $i \geq 1$. Note that $|W_{\theta_1, \theta_2}^{L_{i+1}}| \leq |B_{L_{i+1}}| \leq \pi L_{i+1}^2$ and therefore $\sigma_{i+1} \leq \pi L_{i+1}^2$. For every $\sigma_i \leq j \leq \sigma_{i+1}$, denote $G_j = A_j \cup W_{\theta_1, \theta_2}^R$ and note that the random set G_j is a connected set containing W_{θ_1, θ_2}^R such that $G_j \cap \partial W_{\theta_1, \theta_2}^{L_j} \neq \emptyset$. Therefore, by Corollary 3.3, for every $\sigma_i \leq j < \sigma_{i+1}$,

$$\mathcal{H}_{\partial A_j}^\infty(W_{\theta_1, \theta_2}^R) \leq \mathcal{H}_{\partial G_j}^\infty(\partial W_{\theta_1, \theta_2}^R) \leq C \left(\frac{R}{L_i}\right)^{\frac{\pi}{2(\theta_2 - \theta_1)} - \varepsilon} R \log L_i,$$

where for the first inequality we used the fact that a particle starting from a sufficiently far point must hit $\partial W_{\theta_1, \theta_2}^R$ before hitting W_{θ_1, θ_2}^R.

Consequently, the random variable $|(A_{\sigma_{i+1}} \setminus A_{\sigma_i}) \cap B_R|$ is stochastically dominated by a sum of πL_{i+1}^2 independent Bernoulli random variables with success probability $C(R/L_i)^{\frac{\pi}{2(\theta_2 - \theta_1)} - \varepsilon} R \log L_i$, whose expectation is $C L_{i+1}^2 (R/L_i)^{\frac{\pi}{2(\theta_2 - \theta_1)} - \varepsilon} R \log L_i$. □

Fix some $b > 1$ and choose the sequence $L_i := M^i R^b$. By Lemma 4.1, for every $\varepsilon > 0$ and $M, R > 0$ sufficiently large

$$\begin{aligned} & \mathbb{E}[|(A_\infty \setminus A_{\sigma_1}) \cap B_R|] \\ & \leq C \sum_{i=1}^\infty L_{i+1}^2 \left(\frac{R}{L_i}\right)^{\frac{\pi}{2(\theta_2 - \theta_1)} - \varepsilon} R \log(L_i) \\ & = C R^{2b + 1 - (b-1)\left(\frac{\pi}{2(\theta_2 - \theta_1)} - \varepsilon\right)} \sum_{i=1}^\infty M^2 (i \log M + b \log R) M^{\left(2 + \varepsilon - \frac{\pi}{2(\theta_2 - \theta_1)}\right)i}. \end{aligned} \tag{4.1}$$

Since $\varepsilon > 0$ can be chosen to be arbitrary small, if $\theta_2 - \theta_1 < \pi/4$, then the sum on the right hand side is finite and equals

$$\mathbb{E}[|(A_\infty \setminus A_{\sigma_1}) \cap B_R|] \leq C R^{2b + 1 - (b-1)\left(\frac{\pi}{2(\theta_2 - \theta_1)} - \varepsilon\right)} (\log M + b \log R) M^{\left(4 + \varepsilon - \frac{\pi}{2(\theta_2 - \theta_1)}\right)}. \tag{4.2}$$

Furthermore, for every fixed $b > 1$, the expectation goes to zero as $R \to \infty$ as soon as

$$\theta_2 - \theta_1 < \frac{\pi(b-1)}{4b+2}. \tag{4.3}$$

As $b \to \infty$, this assumption coincides with the previous one, namely $\theta_2 - \theta_1 < \pi/4$. In fact, for θ_1, θ_2 such that $\theta_2 - \theta_1 < \pi/4$, the power of R on the right hand side can be made arbitrarily small by increasing b, and in particular, if

$$\theta_2 - \theta_1 < \frac{\pi(b-1)}{4(b+1)} \tag{4.4}$$

the power is strictly smaller than -1. Assuming (4.4) it follows that from the Markov inequality that

$$\sum_{R=1}^{\infty} \mathbb{P}_{\theta_1,\theta_2}(|A_\infty \setminus A_{\sigma(MR^b)} \cap B_R| \geq 1) < \infty\,,$$

and therefore, using the Borel Cantelli lemma, that $\mathbb{P}_{\theta_1,\theta_2}$-almost surely, for all $R > 0$ sufficiently large, there are no particles hitting B_R after time $\sigma_1 = \sigma(MR^b)$, namely, after the aggregate reaches distance MR^b. Taking $a > 2b$ and noting that for sufficiently large R we have $\sigma_1 \leq |B_{L_1}| \leq \pi L_1^2 = \pi M^2 R^{2b} < R^a$ the result follows.

Finally, estimate (1.1) follows from Markov's inequality and (4.2). $\qquad\square$

5 Discrete Beurling estimate in a wedge

The goal of this section is to prove Theorem 1.2. We start by describing the proof strategy.

1. Applying the strong Markov property, and the fact that for $r < L < R$ a random walk starting from radius R must hit radius L before r, we conclude that it suffices to consider random walks starting from radius L.

2. Using time reversibility of the random walk, we rewrite the hitting probability as the ratio between the escape probability from radius r to radius L while avoiding A, and the probability starting from radius L to hit A before returning to the starting point.

3. We bound the probability to hit A before returning to the starting point from below using the theory of electrical networks.

4. Finally, we bound the escape probability from radius r to radius L from above with the help of the invariance principle and geometric observations on the set A for which the escape probability is maximal.

5.1 Proof of Theorem 1.2 – Reversibility and key lemmas

Recall that for a set $D \subset W_{\theta,\theta_2}$ we denote by $\tau_D^+ = \inf\{n \geq 1 \ : \ S_n \in D\}$ the first return time to D and let $\tau_D = \inf\{n \geq 0 \ : \ S_n \in D\}$ be the first hitting time of D. Also, define $\overline{D} = D \cup \partial D$ to be the closure of D.

We start our proof by rewriting the hitting probabilities from a far away point appearing in (1.2) as escaping probabilities. First, note that for all $y \in W_{\theta_1,\theta_2}^r$ and $x \in \partial W_{\theta_1,\theta_2}^R$ and $A \subset W_{\theta_1,\theta_2}$ as in Theorem 1.2, the hitting probability $\mathbf{P}_{\theta_1,\theta_2}^x(S_{\tau_{\overline{A}}} = y)$, can only increase if we replace A with $A \cap W_{\theta_1,\theta_2}^L$. Thus, without loss of generality, we can assume that $A \subset W_{\theta_1,\theta_2}^L$ and hence $\overline{A} \subset \overline{W}_{\theta_1,\theta_2}^L$.

For any $R > L + 1$ and $x \in \partial W_{\theta_1,\theta_2}^R$, observe that a random walk starting at x must hit $\partial W_{\theta_1,\theta_2}^L$ before hitting $\overline{A}$. Thus, by the strong Markov property, for any $y \in W_{\theta_1,\theta_2}^r$

$$\begin{aligned}
\mathbf{P}_{\theta_1,\theta_2}^x(\tau_{\overline{A}} = \tau_y) &= \sum_{u \in \partial W_{\theta_1,\theta_2}^L} \mathbf{P}_{\theta_1,\theta_2}^x\big(S_{\tau_{\partial W_{\theta_1,\theta_2}^L}} = u\big)\mathbf{P}_{\theta_1,\theta_2}^u(\tau_{\overline{A}} = \tau_y) \\
&\leq \sup_{u \in \partial W_{\theta_1,\theta_2}^L \setminus \overline{A}} \mathbf{P}_{\theta_1,\theta_2}^u(\tau_{\overline{A}} = \tau_y).
\end{aligned} \tag{5.1}$$

For any $u \in \partial W_{\theta_1,\theta_2}^L \setminus \overline{A}$, we now rewrite the hitting probability from u to y as the escaping probabiity from y to u. This is done using the reversibility property of simple random walk on graphs. The method of replacing the hitting probability with the escaping probability was used in H. Kesten [Kes87a] work on the DLA (see also [PZ19] for the case of a domain with a boundary).

First, note that for each $u \in \partial W_{\theta_1,\theta_2}^L \setminus \overline{A}$ and $y \in \partial W_{\theta_1,\theta_2}^r$,

$$
\begin{aligned}
\mathbf{P}_{\theta_1,\theta_2}^u(\tau_{\overline{A}} = \tau_y) &= \sum_{n=1}^{\infty} \mathbf{P}_{\theta_1,\theta_2}^u(\tau_{\overline{A}} = \tau_y = n) \\
&= \sum_{n=1}^{\infty} \mathbf{P}_{\theta_1,\theta_2}^u(S_1 \notin \overline{A}, \cdots, S_{n-1} \notin \overline{A}, S_n = y).
\end{aligned}
\tag{5.2}
$$

Here we used the fact that A is a connected set and that $u \notin \overline{A}$. As a result, the whole path must stay outside of the set $\overline{A}$ until it first hit y.

Let $\Gamma_{u,y}^{A,n}$ be the collection of all paths $\gamma = (x_0, x_1, x_2, \cdots, x_n)$ of length n in W_{θ_1,θ_2} such that $x_0 = u$, $x_n = y$ and $x_1, \cdots, x_{n-1} \notin \overline{A}$. For $\gamma = (x_0, \ldots, x_n) \in \Gamma_{u,y}^{A,n}$ denote by $\widehat{\gamma} = (x_n, \ldots, x_0) \in \Gamma_{y,u}^{A,n}$ the path in the reverse direction. Then, the reversibility of simple random walk implies

$$
\mathbf{P}_{\theta_1,\theta_2}^u((S_0, \ldots, S_n) = \gamma) = \frac{\deg(y)}{\deg(u)} \mathbf{P}_{\theta_1,\theta_2}^y((S_0, \ldots, S_n) = \widehat{\gamma}) \qquad \forall \gamma \in \Gamma_{u,y}^{A,n},
$$

where for $z \in W_{\theta_1,\theta_2}$ we denote by $\deg(z)$ its degree in the graph W_{θ_1,θ_2}. Since $\deg(z) \in \{1, 2, 3, 4\}$ for every $z \in W_{\theta_1,\theta_2}$, we conclude that

$$
\mathbf{P}_{\theta_1,\theta_2}^u((S_0, \ldots, S_n) = \gamma) \leq 4 \mathbf{P}_{\theta_1,\theta_2}^y((S_0, \ldots, S_n) = \widehat{\gamma}) \qquad \forall \gamma \in \Gamma_{u,y}^{A,n}.
\tag{5.3}
$$

Combining (5.2) and (5.3),

$$
\begin{aligned}
\mathbf{P}_{\theta_1,\theta_2}^u(\tau_{\overline{A}} = \tau_y) &= \sum_{n=1}^{\infty} \mathbf{P}_{\theta_1,\theta_2}^u(S_1 \notin \overline{A}, \cdots, S_{n-1} \notin \overline{A}, S_n = y) \\
&\leq 4 \sum_{n=1}^{\infty} \mathbf{P}_{\theta_1,\theta_2}^y(S_1 \notin \overline{A}, \cdots, S_{n-1} \notin \overline{A}, S_n = u) \\
&= 4 \mathbf{E}_{\theta_1,\theta_2}^y \big[|\{n \in [0, \tau_{\overline{A}}^+) : S_n = u\}| \big].
\end{aligned}
\tag{5.4}
$$

By the strong Markov property for the stopping time $\tau_{\overline{A}}^+ \wedge \tau_u^+$

$$
\begin{aligned}
&\mathbf{E}_{\theta_1,\theta_2}^y \big[|\{n \in [0, \tau_{\overline{A}}^+) : S_n = u\}| \big] \\
&= \mathbf{P}_{\theta_1,\theta_2}^y(\tau_u^+ < \tau_{\overline{A}}^+) \cdot \mathbf{E}_{\theta_1,\theta_2}^u \big[|\{n \in [0, \tau_{\overline{A}}^+) : S_n = u\}| \big] = \frac{\mathbf{P}_{\theta_1,\theta_2}^y(\tau_u^+ < \tau_{\overline{A}}^+)}{\mathbf{P}_{\theta_1,\theta_2}^u(\tau_{\overline{A}}^+ \leq \tau_u^+)},
\end{aligned}
\tag{5.5}
$$

where in the last step we used the fact that the number of visits to u before hitting A when starting from u is a geometric random variables with parameter $\mathbf{P}_{\theta_1,\theta_2}^u(\tau_{\overline{A}}^+ < \tau_u^+)$.

Combining (5.1), (5.4) and (5.5) together with the fact that $|\partial W_{\theta_1,\theta_2}^r| \leq Cr$ for some universal constant $C \in (0, \infty)$ gives

$$
\begin{aligned}
\mathbf{P}_{\theta_1,\theta_2}^x \big(\tau_{\partial W_{\theta_1,\theta_2}^r} = \tau_{\overline{A}} \big) &= \sum_{y \in \partial W_{\theta_1,\theta_2}^r} \mathbf{P}_{\theta_1,\theta_2}^x(\tau_y = \tau_{\overline{A}}) \\
&\leq \sum_{y \in \partial W_{\theta_1,\theta_2}^r} \sup_{u \in \partial W_{\theta_1,\theta_2}^L \setminus \overline{A}} \mathbf{P}_{\theta_1,\theta_2}^u(\tau_{\overline{A}} = \tau_y) \\
&\leq Cr \sup_{y \in \partial W_{\theta_1,\theta_2}^r} \sup_{u \in \partial W_{\theta_1,\theta_2}^L \setminus \overline{A}} \mathbf{P}_{\theta_1,\theta_2}^u(\tau_{\overline{A}} = \tau_y) \\
&\leq Cr \sup_{y \in \partial W_{\theta_1,\theta_2}^r} \sup_{u \in \partial W_{\theta_1,\theta_2}^L \setminus \overline{A}} \frac{\mathbf{P}_{\theta_1,\theta_2}^y(\tau_u^+ < \tau_{\overline{A}}^+)}{\mathbf{P}_{\theta_1,\theta_2}^u(\tau_{\overline{A}}^+ \leq \tau_u^+)}.
\end{aligned}
$$

Consequently, in order to prove Theorem 1.2, it suffices to show that there is a constant $C < \infty$ such that for every $\varepsilon > 0$, $y \in \partial W^r_{\theta_1,\theta_2}$ and every $u \in \partial W^L_{\theta_1,\theta_2} \setminus \overline{A}$,

$$\frac{\mathbf{P}^y_{\theta_1,\theta_2}(\tau^+_u < \tau^+_{\overline{A}})}{\mathbf{P}^u_{\theta_1,\theta_2}(\tau^+_{\overline{A}} \leq \tau^+_u)} \leq C \left(\frac{r}{L}\right)^{\frac{\pi}{2(\theta_2-\theta_1)}-\varepsilon} \log L \,, \tag{5.6}$$

for sufficiently large L. The last inequality is an immediate corollary of the following two lemmas. $\qquad\square$

Lemma 5.1. *There exists a constant $c \in (0,\infty)$ independent of r and L such that*

$$\mathbf{P}^u_{\theta_1,\theta_2}(\tau^+_{\overline{A}} \leq \tau^+_u) \geq \frac{c}{\log L}$$

uniformly for all $A \subset W^L_{\theta_1,\theta_2}$ as in Theorem 1.2 and all $u \in \partial W^L_{\theta_1,\theta_2} \setminus \overline{A}$.

Lemma 5.2. *For every $\varepsilon > 0$, there exists a constant $C \in (0,\infty)$ such that for every r, L sufficiently large, every $y \in \partial W^r_{\theta_1,\theta_2}$, every $A \subset W^L_{\theta_1,\theta_2}$ satisfying $W^r_{\theta_1,\theta_2} \subset A$, $A \cap \partial W^L_{\theta_1,\theta_2} \neq 0$ and every $u \in \partial W^L_{\theta_1,\theta_2} \setminus \overline{A}$*

$$\mathbf{P}^y_{\theta_1,\theta_2}(\tau^+_u < \tau^+_{\overline{A}}) \leq \mathbf{P}^y_{\theta_1,\theta_2}(\tau^+_{\partial W^L_{\theta_1,\theta_2}} < \tau^+_{\overline{A}}) \leq C \left(\frac{r}{L}\right)^{\frac{\pi}{2(\theta_2-\theta_1)}-\varepsilon} \,.$$

The proof of Lemma 5.1 is presented in Subsection 5.2 and the proof of Lemma 5.2 can be found in Subsection 5.3.

5.2 Proof of Lemma 5.1

We will use the theory of electrical networks in order to estimate the effective resistance from u to 0 in the graph W_{θ_1,θ_2}. Since $0 \in W^r_{\theta_1,\theta_2} \subset A$ it follows that $\tau^+_{\overline{A}} \leq \tau_0$, and therefore

$$\mathbf{P}^u_{\theta_1,\theta_2}(\tau^+_{\overline{A}} < \tau^+_u) \geq \mathbf{P}^u_{\theta_1,\theta_2}(\tau_0 < \tau^+_u).$$

Recall that the effective resistance from a vertex $v \in W_{\theta_1,\theta_2}$ to $Z \subset W_{\theta_1,\theta_2}$ is given by

$$\mathcal{R}(v \to Z) = \frac{1}{\deg(v)\mathbf{P}^v_{\theta_1,\theta_2}(\tau^+_v < \tau_Z)}.$$

Repeating the argument in [LP17][Proposition 2.15] for the graph W_{θ_1,θ_2} instead of $\mathbb{Z}^2$ shows that for every $-\pi/2 \leq \theta_1 < \theta_2 \leq \pi/2$, there exists a constant $C \in (0,\infty)$ so that

$$\mathcal{R}(0 \to x) \in [C^{-1}\log \|x\|_2, C \log \|x\|_2], \qquad \forall x \in W_{\theta_1,\theta_2}\,.$$

Noting that $\deg(x) \in \{1,2,3,4\}$ for every $x \in W_{\theta_1,\theta_2}$, the result follows. $\qquad\square$

5.3 Proof of Lemma 5.2 – Escaping probability to distance L

Having completed the proof of Lemma 5.1, we turn to the proof of Lemma 5.2. The proof of the latter contains several sub-claims: (i) Finding the set A that maximizes the escape probability, (ii) explicit calculation for the continuous counterpart of the upper bound obtained in step (i) and (iii) using the invariance principle to compare the discrete and the continuous probabilities. We now turn to implement this strategy.

5.3.1 Proof of Lemma 5.2 part (i) – The worst choice for A

We start by showing that the worst choice for A, namely the set which maximizes the probability $\mathbf{P}^y_{\theta_1,\theta_2}(\tau^+_{\partial W^L_{\theta_1,\theta_2}} < \tau^+_{\overline{A}})$ among all sets $A \subset W^L_{\theta_1,\theta_2}$ as in Theorem 1.2 is given by one of the lines along the wedge boundary.
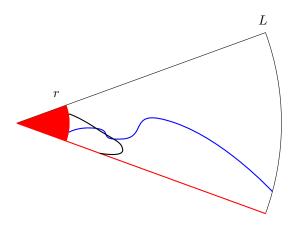
Figure 2: Illustration of proof argument for Lemma 5.3

To this end we define the **discrete upper and lower boundaries** of W_{θ_1,θ_2} by

$$\Gamma^u_{\theta_1,\theta_2} = \left\{ x \in W_{\theta_1,\theta_2} \ : \ \exists y \in \mathbb{Z}^2 \text{ satisfying } \|y - x\|_\infty = 1 \text{ and } \arctan(y_2/y_1) > \theta_2 \right\}$$

and

$$\Gamma^l_{\theta_1,\theta_2} = \left\{ x \in W_{\theta_1,\theta_2} \ : \ \exists y \in \mathbb{Z}^2 \text{ satisfying } \|y - x\|_\infty = 1 \text{ and } \arctan(y_2/y_1) < \theta_1 \right\},$$

where we use the same convention regarding the function $\arctan$ as in the introduction. Also, for $0 < r < L$ and $\alpha \in \{u, l\}$, we denote

$$\Gamma^{\alpha,r,L} = \Gamma^{\alpha,r,L}_{\theta_1,\theta_2} := \overline{W^r_{\theta_1,\theta_2}} \cup (\Gamma^\alpha_{\theta_1,\theta_2} \cap W^L_{\theta_1,\theta_2}).$$

Lemma 5.3. *For all sufficiently large r, all sufficiently large $L > r$, every set A as in Lemma 5.2 and every $y \in \partial W^r_{\theta_1,\theta_2}$*

$$\begin{aligned}
&\mathbf{P}^y_{\theta_1,\theta_2}(\tau^+_{\partial W^L_{\theta_1,\theta_2}} < \tau^+_A) \\
&\leq \max\left\{ \mathbf{P}^y_{\theta_1,\theta_2}\big(\tau^+_{\partial W^L_{\theta_1,\theta_2}} < \tau^+_{\Gamma^{u,r,L}}\big), \mathbf{P}^y_{\theta_1,\theta_2}\big(\tau^+_{\partial W^L_{\theta_1,\theta_2}} < \tau^+_{\Gamma^{l,r,L}}\big) \right\}.
\end{aligned} \tag{5.7}$$

Proof. Recall that $W^r_{\theta_1,\theta_2} \subset A$, $\partial W^L_{\theta_1,\theta_2} \cap A \neq \emptyset$ and that A is a connected subset within the wedge W_{θ_1,θ_2}. Therefore there exists a connected path $\gamma = (x^0, x^1, \ldots, x^k)$ within $\overline{A}$ from $W^r_{\theta_1,\theta_2}$ to $\partial W^L_{\theta_1,\theta_2}$. Without loss of generality we can assume that $x^k \in \partial W^L_{\theta_1,\theta_2}$, $x^0 \in \partial W^r_{\theta_1,\theta_2}$ and that $x^1, x^2, \cdots, x^{k-1} \in W_{\theta_1,\theta_2} \cap (\overline{W}^r_{\theta_1,\theta_2} \cup \partial W^L_{\theta_1,\theta_2})^c$.

Let $A' = \overline{W^r_{\theta_1,\theta_2}} \cup \gamma$. Since $A' \subset \overline{A}$ we know that $\tau^+_A \leq \tau^+_{A'}$ and therefore

$$\mathbf{P}^y_{\theta_1,\theta_2}(\tau^+_{\partial W^L_{\theta_1,\theta_2}} < \tau^+_A) \leq \mathbf{P}^y_{\theta_1,\theta_2}(\tau^+_{\partial W^L_{\theta_1,\theta_2}} < \tau^+_{A'}).$$

Next, we turn to compare the hitting probability in A' to the hitting probability in $\Gamma^{u,r,L}$ and $\Gamma^{l,r,L}$. We separate the proof into three cases according to the position of y with respect to the path γ:

(1) The point y satisfies $\arctan(y_2/y_1) > \arctan(x^0_2/x^0_1)$.

(2) The point y satisfies $\arctan(y_2/y_1) < \arctan(x^0_2/x^0_1)$.

(3) The point y satisfies $y = x^0$.

We start with case (1), since W_{θ_1,θ_2} is a connected, planar graph and γ is a connected path in it from $\partial W_{\theta_1,\theta_2}^r$ to $\partial W_{\theta_1,\theta_2}^L$, every path from y to $\Gamma^{l,r,L}$ must hit A'. In particular, paths starting in y that hit $\Gamma^{l,r,L}$ before hitting $\partial W_{\theta_1,\theta_2}^L$ must hit A' before hitting $\partial W_{\theta_1,\theta_2}^L$. This implies that $\mathbf{P}_{\theta_1,\theta_2}^y(\tau_{\Gamma^{l,r,L}}^+ \leq \tau_{\partial W_{\theta_1,\theta_2}^L}^+) \leq \mathbf{P}_{\theta_1,\theta_2}^y(\tau_{A'}^+ \leq \tau_{\partial W_{\theta_1,\theta_2}^L}^+)$ and therefore the required inequality. Similarly, in case (2), every path from y to $\Gamma^{u,r,L}$ must hit A' and therefore paths hitting $\Gamma^{u,r,L}$ before $\partial W_{\theta_1,\theta_2}^L$ must also hit A' before $\partial W_{\theta_1,\theta_2}^L$. Hence $\mathbf{P}_{\theta_1,\theta_2}^y(\tau_{\partial W_{\theta_1,\theta_2}^L}^+ \leq \tau_{A'}^+) \leq \mathbf{P}_{\theta_1,\theta_2}^y(\tau_{\partial W_{\theta_1,\theta_2}^L}^+ \leq \tau_{\Gamma^{u,r,L}}^+)$.

Finally, we turn to deal with case (3). Using the Markov property, after one step of the random walk we have $\mathbf{P}_{\theta_1,\theta_2}^y(\tau_{\partial W_{\theta_1,\theta_2}^L}^+ \leq \tau_{A'}^+) = \frac{1}{\deg(y)}\sum_{z\in W_{\theta_1,\theta_2}\ s.t.\|z-y\|_1=1}\mathbf{P}_{\theta_1,\theta_2}^z(\tau_{\partial W_{\theta_1,\theta_2}^L} \leq \tau_{A'})$. Since for $z\in A'$ we have $\mathbf{P}_{\theta_1,\theta_2}^z(\tau_{\partial W_{\theta_1,\theta_2}^L} \leq \tau_{A'}) = 0$ and for $z\notin A'$ we can repeat the argument in (1) and (2) we conclude that

$$\mathbf{P}_{\theta_1,\theta_2}^y(\tau_{\partial W_{\theta_1,\theta_2}^L}^+ \leq \tau_{A'}^+)$$
$$\leq \frac{1}{\deg(y)}\sum_{\substack{z\in W_{\theta_1,\theta_2}\\\|z-y\|_1=1}} \max\left\{\mathbf{P}_{\theta_1,\theta_2}^z\big(\tau_{\partial W_{\theta_1,\theta_2}^L}^+ < \tau_{\Gamma^{u,r,L}}^+\big), \mathbf{P}_{\theta_1,\theta_2}^z\big(\tau_{\partial W_{\theta_1,\theta_2}^L}^+ < \tau_{\Gamma^{l,r,L}}^+\big)\right\}$$
$$= \max\left\{\mathbf{P}_{\theta_1,\theta_2}^y\big(\tau_{\partial W_{\theta_1,\theta_2}^L}^+ < \tau_{\Gamma^{u,r,L}}^+\big), \mathbf{P}_{\theta_1,\theta_2}^y\big(\tau_{\partial W_{\theta_1,\theta_2}^L}^+ < \tau_{\Gamma^{l,r,L}}^+\big)\right\}$$
$$\leq \max\left\{\mathbf{P}_{\theta_1,\theta_2}^y\big(\tau_{\partial W_{\theta_1,\theta_2}^L}^+ < \tau_{\Gamma^{u,0,L}}^+\big), \mathbf{P}_{\theta_1,\theta_2}^y\big(\tau_{\partial W_{\theta_1,\theta_2}^L}^+ < \tau_{\Gamma^{l,0,L}}^+\big)\right\},$$

as required. $\qquad\square$

Due to the last lemma it is enough to obtain bounds on

$$\mathbf{P}_{\theta_1,\theta_2}^y(\tau_{\partial W_{\theta_1,\theta_2}^L}^+ \leq \tau_{\Gamma^{l,0,L}}^+) \qquad \text{and} \qquad \mathbf{P}_{\theta_1,\theta_2}^y(\tau_{\partial W_{\theta_1,\theta_2}^L}^+ \leq \tau_{\Gamma^{u,0,L}}^+)$$

for all $y \in \partial W_{\theta_1,\theta_2}^r$ and all sufficiently large r and L. We focus here on the estimation for $\Gamma^{l,r,L}$, the bound for $\Gamma^{u,r,L}$ is obtained in the same manner.

Let K be a constant to be chosen later on, for $i \geq 0$, define $M_i = rK^i$ and let $N = N(r,L,K)$ be the largest integer such that $M_N \leq L$. Then

$$\mathbf{P}_{\theta_1,\theta_2}^y\left(\tau_{\partial W_{\theta_1,\theta_2}^L}^+ \leq \tau_{\Gamma^{l,0,L}}^+\right) \leq \mathbf{P}_{\theta_1,\theta_2}^y\left(\tau_{\partial W_{\theta_1,\theta_2}^{M_N}}^+ \leq \tau_{\Gamma^{l,0,L}}^+\right),$$

and by strong Markov property

$$\mathbf{P}_{\theta_1,\theta_2}^y\left(\tau_{\partial W_{\theta_1,\theta_2}^{M_N}}^+ \leq \tau_{\Gamma^{l,0,L}}^+\right) \leq \prod_{i=0}^{N-1} \sup_{z\in\partial W_{\theta_1,\theta_2}^{M_i}} \mathbf{P}_{\theta_1,\theta_2}^z\left(\tau_{\partial W_{\theta_1,\theta_2}^{M_{i+1}}} \leq \tau_{\Gamma^{l,M_i,M_{i+1}}}\right). \tag{5.8}$$

In order to estimate each of the probabilities on the right hand side we first turn to evaluate their continuous counterpart.

5.3.2 Proof of Lemma 5.2 part (ii) – Calculating the continuous analogue of the probability

Define the continuous wedge between the angles $-\pi/2 \leq \theta_1 < \theta_2 \leq \pi/2$ by

$$\mathcal{W}_{\theta_1,\theta_2} = \left\{(x_1,x_2)\in\mathbb{R}^2 \ : \ \arctan(x_2/x_1)\in[\theta_1,\theta_2]\right\},$$

with the same convention as in the discrete case regarding $\arctan$. Furthermore, define the intersection of $\mathcal{W}_{\theta_1,\theta_2}$ with the ball of radius K around the origin by

$$\mathcal{W}_{\theta_1,\theta_2}^K = \mathcal{W}_{\theta_1,\theta_2} \cap \left\{x\in\mathbb{R}^2 \ : \ \|x\|_2 \leq K\right\}$$

and the lower, upper and front boundaries of $\mathcal{W}^K_{\theta_1,\theta_2}$ respectively, by

$$\partial^l \mathcal{W}^K_{\theta_1,\theta_2} = \{x \in \mathcal{W}^K_{\theta_1,\theta_2} \ : \ \arctan(x_2/x_1) = \theta_1\},$$
$$\partial^u \mathcal{W}^K_{\theta_1,\theta_2} = \{x \in \mathcal{W}^K_{\theta_1,\theta_2} \ : \ \arctan(x_2/x_1) = \theta_2\},$$

and

$$\partial^f \mathcal{W}^K_{\theta_1,\theta_2} = \{x \in \mathcal{W}^K_{\theta_1,\theta_2} \ : \ \|x\|_2 = K\} \,.$$

Let $(\mathcal{B}(t))_{t\geq 0} = (\mathcal{B}^K_{\theta_1,\theta_2}(t))_{t\geq}$ denote a reflected Brownian motion in $\mathcal{W}^K_{\theta_1,\theta_2}$ and denote by $\mathcal{T}_K$ the hitting time of $\partial^l \mathcal{W}^K_{\theta_1,\theta_2} \cup \partial^f \mathcal{W}^K_{\theta_1,\theta_2}$ and by $\mathcal{P}^x_{\theta_1,\theta_2} = \mathcal{P}^{x,K}_{\theta_1,\theta_2}$ the law of $(\mathcal{B}^K_{\theta_1,\theta_2}(t))_{t\geq 0}$ with starting point x as well. In this subsection, our goal is to estimate the continuous analogue of the discrete probability

$$\mathbf{P}^z_{\theta_1,\theta_2}\big(\tau_{\partial W^{KL}_{\theta_1,\theta_2}} \leq \tau_{\Gamma^{l,L,KL}}\big), \qquad \forall z \in \partial W^L_{\theta_1,\theta_2},$$

given by

$$\mathcal{P}^x_{\theta_1,\theta_2}(|\mathcal{B}(\mathcal{T}_K)| = K)\,,$$

for all x of the form $(\cos\theta, \sin\theta)$ with $\theta \in [\theta_1,\theta_2]$.

Due to the rotation invariance of (reflected) Brownian motion we can replace the angles θ_1, θ_2 by the angles $-(\theta_2 - \theta_1), 0$. Furthermore, using the reflection principle (see Figure 3 for an illustration), denoting by $\widehat{\mathcal{T}}_K$ the hitting time of $\partial \mathcal{W}^K_{\theta_1,\theta_2} = \partial \mathcal{W}^l_{\theta_1,\theta_2} \cup \partial \mathcal{W}^u_{\theta_1,\theta_2} \cup \partial \mathcal{W}^f_{\theta_1,\theta_2}$ and by $\varphi = \theta_2 - \theta_1$, the difference of the angles, we conclude that

$$\mathcal{P}^{(\cos\theta,\sin\theta)}_{\theta_1,\theta_2}(|\mathcal{B}(\mathcal{T}_K)| = K) = \mathcal{P}^{(\cos(-\theta+\theta_1),\sin(-\theta+\theta_1))}_{-\varphi,\varphi}(|\mathcal{B}(\widehat{\mathcal{T}}_K)| = K)\,.$$
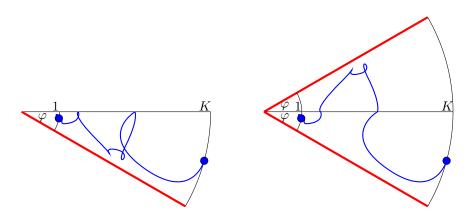

Figure 3: Reflecting the Dirichlet boundary conditions

The probability on the right hand side was calculated in [MP10] for the case $\theta = \theta_1$

Lemma 5.4 ([MP10] Theorem 7.24). *Let $\varphi \in (0,\pi]$ and $K > 1$. Then*

$$\mathcal{P}^{(1,0)}_{-\varphi,\varphi}(|\mathcal{B}(\widehat{\mathcal{T}}_K)| = K) = \frac{2}{\pi} \arctan\left(\frac{2K^{\pi/2\varphi}}{K^{\pi/\varphi} - 1}\right).$$

In addition, using conformal maps (or alternatively the Beurling estimation) one can verify that

$$\mathcal{P}^x_{-\varphi,\varphi}(|\mathcal{B}(\widehat{\mathcal{T}}_K)| = K) \leq \mathcal{P}^{(1,0)}_{-\varphi,\varphi}(|\mathcal{B}(\widehat{\mathcal{T}}_K)| = K), \tag{5.9}$$

for all x of the form $(\cos\psi, \sin\psi)$ with $\psi \in [-\varphi, \varphi]$.

5.3.3 Proof of Lemma 5.2 part (iii) – from continuous to discrete and completion of the proof

In this subsection, we use Lemma 5.4 in order to find an upper bound on the discrete probability

$$\mathbf{P}^z_{\theta_1,\theta_2}\Big(\tau_{\partial W^{KL}_{\theta_1,\theta_2}} \leq \tau_{\Gamma^{l,L,KL}}\Big), \qquad \forall z \in \partial W^L_{\theta_1,\theta_2}$$

Lemma 5.5. *Let $K > 0$. For every $\varepsilon > 0$, there exists a constant $L_0 \in (0,\infty)$ such that for all $L \geq L_0$, and all $z \in \partial W^L_{\theta_1,\theta_2}$*

$$\mathbf{P}^z_{\theta_1,\theta_2}\Big(\tau_{\partial W^{KL}_{\theta_1,\theta_2}} \leq \tau_{\Gamma^{l,L,KL}_{\theta_1,\theta_2}}\Big) \leq \mathcal{P}^{(1,0)}_{-\varphi,\varphi}(|\mathcal{B}(\widehat{\mathcal{T}}_K)| = K) + \varepsilon, \tag{5.10}$$

where φ, $\mathcal{B}$ and $\widehat{\mathcal{T}}_K$ are defined as in the previous subsection.

The main ingredient in the proof of lemma 5.5 is the invariance principle for simple random walk in a wedge.

Lemma 5.6 (Lemma 2.1 in [BQ06], Theorem 6.3 of [SV71])**.** *Let D be a bounded connected open set with an analytic (smooth) boundary in $\mathbb{R}^d$, $d \geq 2$ and let $D_\varepsilon = \varepsilon\mathbb{Z}^d \cap D$. Suppose that $x_\varepsilon \in D_\varepsilon$ and $x_\varepsilon \to x_0 \in \overline{D}$ as $\varepsilon \to 0$. Let $\{W^\varepsilon_t,\ t \geq 0\}$ be continuous time simple random walk on D_ε with $W^\varepsilon_0 = x_\varepsilon$. Then $(W^\varepsilon_t)_{t\geq 0}$ converge weakly to reflected Brownian motion on $\overline{D}$ starting from x_0 as $\varepsilon \to 0$ (in the weak topology on the space of continuous functions).*

Remark 5.7. In [BQ06], it was assumed throughout the paper that D is a bounded, connected, open set with analytic boundary. The analyticity assumption is made for technical reason, needed in proving their main result. However, it is noted in the paper that the lemma above is derived from Theorem 6.3 of [SV71], which holds also for non analytic domains.

Proof of Lemma 5.5. Suppose the lemma does not hold. Then, there exists $\varepsilon_0 > 0$ and an increasing sequence $(L_n)_{n\geq 1}$ going to infinity together with a sequence of points $(z_n)_{n\geq 1}$ such that $z_n \in \partial W^{L_n}_{\theta_1,\theta_2}$ for all $n \geq 1$ such that

$$\mathbf{P}^{z_n}_{\theta_1,\theta_2}\Big(\tau_{\partial W^{KL_n}_{\theta_1,\theta_2}} \leq \tau_{\Gamma^{l,L_n,KL_n}_{\theta_1,\theta_2}}\Big) \geq \mathcal{P}^{(1,0)}_{-\varphi,\varphi}(|\mathcal{B}(\widehat{\mathcal{T}}_K)| = K) + \varepsilon_0. \tag{5.11}$$

Noting that $|z_n/L_n| \to 1$ as $n \to \infty$, it follows that there exists a subsequence k_n such that

$$\lim_{n\to\infty} z_{k_n}/L_{k_n} = z_0 \in \{x \in \mathcal{W}_{\theta_1,\theta_2} \ :\ \|x\|_2 = 1\}.$$

Consequently, by Lemma 5.6 and (5.9) we have

$$\lim_{n\to\infty} \mathbf{P}^{z_{k_n}}_{\theta_1,\theta_2}\Big(\tau_{\partial W^{KL_{k_n}}_{\theta_1,\theta_2}} \leq \tau_{\Gamma^{l,L_{k_n},KL_{k_n}}_{\theta_1,\theta_2}}\Big) = \mathcal{P}^{z_0}_{\theta_1,\theta_2}\big(|\mathcal{B}^{\theta_1,\theta_2}_k(\widehat{\mathcal{T}}_K)| = K\big)$$
$$\leq \mathcal{P}^{(1,0)}_{-\varphi,\varphi}(|\mathcal{B}(\widehat{\mathcal{T}}_K)| = K) \tag{5.12}$$

contradicting (5.11). $\qquad\square$

At this point we have all the ingredients needed in order to complete the proof of Theorem 1.2. Recalling Lemma 5.4, we observe that there exists a universal constant $C \in (0,\infty)$ such that

$$\mathcal{P}^{(1,0)}_{-\varphi,\varphi}(|\mathcal{B}(\widehat{\mathcal{T}}_K)| = K) = \frac{2}{\pi}\arctan\left(\frac{2K^{\pi/2\varphi}}{K^{\pi/\varphi} - 1}\right) \leq \frac{C}{K^{\pi/2\varphi}}, \qquad \forall K \geq 2. \tag{5.13}$$

Furthermore, since $\lim_{K \to \infty} \log_K(C+1) \to 0$, there exists $K_0 \in (1, \infty)$ such that

$$0 < \log_{K_0}(C+1) < \frac{\varepsilon}{4}. \tag{5.14}$$

Taking $\varepsilon_{K_0} = K_0^{-\pi/2\varphi}$ in Lemma 5.5, we conclude that exists $R_0 \in (0, \infty)$ such that for all $r \geq R_0$ and all $z \in \partial W_{\theta_1,\theta_2}^r$

$$\mathbf{P}_{\theta_1,\theta_2}^z \left(\tau_{\partial W_{\theta_1,\theta_2}^{rK_0}} \leq \tau_{\Gamma^{l,r,rK_0}} \right) \leq \mathcal{P}_{-\varphi,\varphi}^{(1,0)}(|\mathcal{B}(\widehat{\mathcal{T}}_{K_0})| = K_0) + \varepsilon_{K_0} \leq \frac{C+1}{K_0^{\pi/2\varphi}}. \tag{5.15}$$

Next, recall that N_{K_0} was defined to be the largest integer n such that $rK_0^n \leq L$, which implies that $N_{K_0} = \lfloor \log_{K_0}(L/r) \rfloor$. Consequently, for all $r \geq R_0$, and L sufficiently large so that

$$\left(\frac{L}{r} \right)^{\varepsilon/2} \geq K_0^{\pi/2\varphi},$$

we have

$$\begin{aligned}
\mathbf{P}_{\theta_1,\theta_2}^y \left(\tau_{\partial W_{\theta_1,\theta_2}^{M_{N(K_0)}}} \leq \tau_{B_{\theta_1,\theta_2}^{l,0,L}} \right) &\leq \prod_{i=0}^{N-1} \sup_{z \in \partial W_{\theta_1,\theta_2}^{M_i}} \mathbf{P}_{\theta_1,\theta_2}^z \left(\tau_{\partial W_{\theta_1,\theta_2}^{M_{i+1}}} \leq \tau_{\Gamma^{l,M_i,M_{i+1}}} \right) \\
&\leq \left(\frac{C+1}{K_0^{\pi/2\varphi}} \right)^{\log_{K_0}(L/r)-1} \leq K_0^{\pi/2\varphi} \left(\frac{r}{L} \right)^{\pi/2\varphi - \log_{K_0}(C+1)} \\
&\leq K_0^{\pi/2\delta} \left(\frac{r}{L} \right)^{\pi/2\varphi - \frac{1}{4}\varepsilon} \leq \left(\frac{r}{L} \right)^{\pi/2\varphi - \varepsilon}.
\end{aligned} \tag{5.16}$$

Thus, the proof of Lemma 5.2 is complete. $\qquad\square$

A Existence of infinite harmonic measure

In this section, we prove Theorem 3.1. The convergence is proved by showing that for any $y \in W_{\theta_1,\theta_2}$ and any sequence $(x_n)_{n \geq 1}$ in W_{θ_1,θ_2} such that $\lim_{n \to \infty} \|x_n\|_2 = \infty$, the sequence $(\mathcal{H}_A(x_n, y))_{n \geq 1}$ is Cauchy.

Let $(x_n)_{n \geq 1}$ be a sequence as above. Since A is finite one can find $r > 0$ such that $A \subset W_{\theta_1,\theta_2}^r$ and thus $\mathcal{H}_A(x, y) = 0$ for all $y \notin W_{\theta_1,\theta_2}^r$ and $x \in W_{\theta_1,\theta_2}$. Hence, we can restrict attention to $y \in W_{\theta_1,\theta_2}^r$. Since $\lim_{n \to \infty} \|x_n\|_2 = \infty$, we can assume without loss of generality that $\|x_n\|_2 > r$ for all $n \geq 1$

Note that for any $m, n \in \mathbb{N}$ such that $\|x_n\|_2 < \|x_m\|_2$, a random walk starting in x_m must hit $\partial W_{\theta_1,\theta_2}^{\|x_n\|_2}$ before hitting A. Thus it is enough to prove that for every $y \in W_{\theta_1,\theta_2}^r$

$$\lim_{R \to \infty} \max_{x_1,x_2 \in \partial W_{\theta_1,\theta_2}^R} |\mathcal{H}_A(x_1, y) - \mathcal{H}_A(x_2, y)| = 0. \tag{A.1}$$

As mentioned in Section 3, there is no discrete Green function approximation on the wedge which is accurate enough to allow us to follow the proof outline of H. Kesten in $\mathbb{Z}^2$. Instead, we will prove the result using the following strategy

1. Show that the number of steps needed for a random walk, starting from $\partial W_{\theta_1,\theta_2}^R$, to reach A is asymptotically bigger than R^2.
2. Show that the mixing time for a random walk started from $x \in \partial W_{\theta_1,\theta_2}^R$ is much smaller than the hitting time of A and therefore, using a coupling argument, that two random walks starting from x_1 and x_2 in $\partial W_{\theta_1,\theta_2}^R$ respectively, will coincide with high probability before hitting A.

Note that carrying out the strategy above requires careful choices of parameters in the proof. This is the content of the following subsections.

A.1 Coupling, two key propositions and the proof of Theorem 3.1

For $R > 0$ and $x_1, x_2 \in \partial W^R_{\theta_1,\theta_2}$, denote by $\mathbb{P}^{x_1,x_2}_R$ a coupling of two continuous time Markov processes $(B^R_1(t), B^R_2(t))_{t\geq 0}$ each with state space $W_{\theta_1,\theta_2}/R := \{x/R : x \in W_{\theta_1,\theta_2}\}$ defined as follows:

1. $B^R_1(t)$ is a continuous time, simple random walk on $W_{\theta_1,\theta_2}/R$, starting at x_1/R, with fixed jump rate $2R^2$.
2. $B^R_2(t)$ is a continuous time, simple random walk on $W_{\theta_1,\theta_2}/R$, starting at x_2/R, with fixed jump rate $2R^2$.
3. $(B^R_1(t))_{t\geq 0}$ and $(B^R_2(t))_{t\geq 0}$ are coupled according to the maximum coupling, see [LL10, Appendix A.4.2].

For $i \in \{1,2\}$, we define $(S^R_i(n))_{n\geq 0}$ to be the embedded, discrete time, simple random walk in $(B^R_i(t))_{t\geq 0}$, and for $s \geq 0$, denote by $N^R_i(s)$, the number of jumps made by the Markov process $(B^R_i(t))_{t\geq 0}$ up to time s. It follows from the definitions above that $B^R_i(s) = S^R_i(N^R_i(s))$ for $i \in \{1,2\}$, $R > 0$ and $s \geq 0$. Denoting by $\tau^i_A = \inf\{t \geq 0 : B^R_i(t) \in A/R\}$, the hitting time of $(B^R_i(n))_{n\geq 0}$ in A/R, it follows from the definition of $\mathcal{H}_A(\cdot,\cdot)$ and the relation between B^R_i and S^R_i that for every $y \in W^r_{\theta_1,\theta_2}$

$$\mathcal{H}_A(x_i, y) = \mathbf{P}^{x_i}\big(S_i(\tau_A) = y\big) = \mathbb{P}^{x_1,x_2}_R\big(B^R_i(\tau_A) = y\big), \qquad \forall i \in \{1,2\}.$$

Define the stopping time

$$\mathrm{T} = \inf\{t \geq 0 : B^R_1(t) = B^R_2(t)\},$$

and note that from the definition of the coupling and the stopping time $B_1(s) = B_2(s)$ for all $s \geq \mathrm{T}$.

For $t_0 > 0$ and $R > 0$, define the event

$$I_{t_0,R} = \{\tau^1_A > t_0, \ \tau^2_A > t_0, \ \mathrm{T} \leq t_0\}.$$

Then, by the Markov property

$$\{B^R_1(\tau^1_A) = y\} \cap I_{t_0,R} = \{B^R_2(\tau^2_A) = y\} \cap I_{t_0,R},$$

and therefore

$$
\begin{aligned}
&|\mathcal{H}_A(x_1, y) - \mathcal{H}_A(x_2, y)| \\
&= \left| \mathbb{P}^{x_1,x_2}_R\big(\{B^R_1(\tau^1_A) = y\} \cap I^c_{t_0,R}\big) - \mathbb{P}^{x_1,x_2}_R\big(\{B^R_2(\tau^2_A) = y\} \cap I^c_{t_0,R}\big) \right| \\
&\leq \mathbb{P}^{x_1,x_2}_R(I^c_{t_0,R}) \leq \mathbb{P}^{x_1,x_2}_R(\tau^1_A \leq t_0) + \mathbb{P}^{x_1,x_2}_R(\tau^2_A \leq t_0) + \mathbb{P}^{x_1,x_2}_R(\mathrm{T} > t_0)
\end{aligned}
\tag{A.2}
$$

Also, note that for $i \in \{1,2\}$

$$
\begin{aligned}
\mathbb{P}^{x_1,x_2}_R(\tau^i_A \leq t_0) &\leq \mathbb{P}^{x_1,x_2}_R(N_i(t_0) \geq 4t_0R^2) + \mathbb{P}^{x_1,x_2}_R(N_i(\tau^i_A) \leq 4t_0R^2) \\
&\leq e^{-2(\log(4)-1)t_0R^2} + \mathbf{P}^{x_i}_{\theta_1,\theta_2}(\tau_A \leq 4t_0R^2),
\end{aligned}
\tag{A.3}
$$

where in the last inequality we used large deviation estimate for the random variable $N_1(t_0) \sim \mathrm{Pois}(2t_0R^2)$, the fact that the paths of (B^R_i) and (S^R_i) are the same and that (S^R_i) as the same law as a simple random walk in the wedge started in x_i, i.e., the its law is given by $\mathbf{P}^{x_i}_{\theta_1,\theta_2}$.

Combining all of the above, we conclude that for every $t_0 > 0$, $R > 0$, $x_1, x_2 \in W^R_{\theta_1,\theta_2}$ and $y \in W^r_{\theta_1,\theta_2}$

$$
\begin{aligned}
|\mathcal{H}_A(x_1, y) - \mathcal{H}_A(x_2, y)| \leq{}& 2 \max_{x \in W^R_{\theta_1,\theta_2}} \mathbf{P}^x_{\theta_1,\theta_2}(\tau_A \leq 4t_0R^2) + \mathbb{P}^{x_1,x_2}_R(\mathrm{T} > t_0) \\
&+ 2e^{-2(\log(4)-1)t_0R^2}.
\end{aligned}
$$

Consequently, the proof of Theorem 3.1 is an immediate consequence of the following two propositions:

Proposition A.1. For every finite set $A \in W_{\theta_1,\theta_2}$ and every $t \in (0,\infty)$

$$\lim_{R \to \infty} \max_{x \in \partial W_{\theta_1,\theta_2}^R} \mathbf{P}_{\theta_1,\theta_2}^x(\tau_A \leq tR^2) = 0\,.$$

Proposition A.2. It holds that

$$\lim_{t_0 \to \infty} \lim_{R \to \infty} \max_{x_1,x_2 \in \partial W_{\theta_1,\theta_2}^R} \mathbb{P}_R^{x_1,x_2}(\mathrm{T} > t_0) = 0\,.$$

Proof of Theorem 3.1. Let $A \subset W_{\theta_1,\theta_2}$ be a finite set. The discussion above combined with Proposition A.1 and Proposition A.2 implies that (A.1) holds. As a result, the limit $\lim_{\|x\|_2 \to \infty} \mathcal{H}_A(x,y)$ exists for every $y \in W_{\theta_1,\theta_2}$ and thus the existence of the Harmonic measure follows. $\qquad\square$

A.2 Lower bound on the hitting time – Proof of Proposition A.1

We start with some results for the continuous analogue of reflected Brownian motion in the continuous wedge $\mathcal{W}_{\theta_1,\theta_2}$, whose law when starting in $u \in \mathcal{W}_{\theta_1,\theta_2}$ we denote by $\mathcal{P}_{\theta_1,\theta_2}^u$. For $L > 0$ denote by $\mathcal{S}_L$ the hitting time of the reflected Brownian motion in $\partial \mathcal{W}_{\theta_1,\theta_2}^L$.

Lemma A.1. *For every $\varepsilon > 0$, $C > 1$ and $u \in \mathcal{W}_{\theta_1,\theta_2}$ such that $\|u\|_2 = 1$*

$$\mathcal{P}_{\theta_1,\theta_2}^u(\mathcal{S}_{C^{-1}} < \mathcal{S}_{C^\varepsilon}) = \frac{\varepsilon}{1+\varepsilon}\,. \tag{A.4}$$

Proof. The proof follows from the fact that $\log|x|$ is the Green function in $\mathbb{R}^2$ and therefore, if $(\mathcal{B}(t))_{t \geq 0}$ is a standard two-dimensional Brownian motion, then $(\log|\mathcal{B}(t)|)_{t \geq 0}$ is a martingale.

Indeed, note that the reflected Brownian motion in a smooth region is conformally invariant up to a time change, c.f. Theorem 9.3 of [LSW03]. By the conformal mapping theorem we can map the wedge into $\mathbb{C} \setminus [0,\infty)$, which transforms the Brownian motion in $\mathcal{W}_{\theta_1,\theta_2}$ to a Brownian motion in $\mathbb{C}$ reflected on the line $\{(x,0) \ : \ x \geq 0\}$. Note that the original event $\{\mathcal{S}_{C^{-1}} < \mathcal{S}_{C^\varepsilon}\}$ is mapped under this transformation to the event $\{\widehat{\mathcal{S}}_{C^{-2\pi/(\theta_2-\theta_1)}} < \widehat{\mathcal{S}}_{C^{2\pi\varepsilon/(\theta_2-\theta_1)}}\}$, where $\widehat{\mathcal{S}}_L$ is the first time that a Brownian motion reflected on the link $\{(x,0) \ : \ x \geq 0\}$ is at distance L from the origin. Next, observe that by the reflection principle, the reflection on the line $\{(x,0) \ : \ x \geq 0\}$ does not change the probability of the event $\{\widehat{\mathcal{S}}_{C^{-2\pi/(\theta_2-\theta_1)}} < \widehat{\mathcal{S}}_{C^{2\pi\varepsilon/(\theta_2-\theta_1)}}\}$.

Due to the fact that $(\log(|\mathcal{B}(t)|))_{t \geq 0}$ is a martingale, where $(\mathcal{B}(t))_{t \geq 0}$ is a standard two-dimensional Brownian motion. It follows from the optional stopping theorem for the stopping time $\widehat{\mathcal{S}} = \min\{\widehat{\mathcal{S}}_{C^{-2\pi/(\theta_2-\theta_1)}}, \widehat{\mathcal{S}}_{C^{2\pi\varepsilon/(\theta_2-\theta_1)}}\}$ that

$$-\frac{2\pi}{\theta_2-\theta_1}\mathcal{P}_{\theta_1,\theta_2}^u(\mathcal{S}_{C^{-1}} < \mathcal{S}_{C^\varepsilon})\log(C) + \frac{2\pi\varepsilon}{\theta_2-\theta_1}(1 - \widehat{\mathcal{P}}_{\theta_1,\theta_2}^u(\mathcal{S}_{C^{-1}} < \mathcal{S}_{C^\varepsilon}))\log C = 0$$

which proves the result. $\qquad\square$

Lemma A.2. *For every $\varepsilon > 0$, there is a constant $C_\varepsilon \in (0,\infty)$ such that for all $u \in \mathcal{W}_{\theta_1,\theta_2}$ satisfying $\|u\|_2 = 1$ and all $t_0 > 0$*

$$\mathcal{P}_{\theta_1,\theta_2}^u(\mathcal{S}_{t_0^{1/2+\varepsilon}} \leq t_0) \leq \frac{C_\varepsilon}{t_0}.$$

Proof. It suffices to prove the result for all sufficiently large t. Let $\varphi := (\theta_2 - \theta_1)/\pi$ and define $f(z) = z^\varphi$, be the conformal map from the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ onto the wedge $\mathcal{W}_{\theta_1, \theta_2}$. Recall that by Ito's formula, if $(\mathcal{B}^{\mathbb{H}}(t))_{t \geq 0}$ is a reflected Brownian motion in $\mathbb{H}$, then $(f(\mathcal{B}^{\mathbb{H}}(t)))_{t \geq 0}$ is a time changed reflected Brownian motion in the wedge $\mathcal{W}_{\theta_1, \theta_2}$, and that the time change of the process is given by

$$\zeta(t) = \int_0^t |f'(\mathcal{B}^{\mathbb{H}}(s))|^2 ds,$$

We denote the law of $\mathcal{B}^{\mathbb{H}}(s)$ starting from u by $\mathcal{P}^{u,\mathbb{H}}$.

For $t_0 > 0$, define
$$\mathcal{S}^{\mathbb{H}}_{t_0} = \inf\{t \geq 0 : \|\mathcal{B}^{\mathbb{H}}(t)\|_2 = t_0\}.$$

Then
$$\mathcal{P}^u_{\theta_1, \theta_2}(\mathcal{S}_{t_0^{1/2+\varepsilon}} \leq t_0)$$
$$\leq \mathcal{P}^{u^{1/\varphi},\mathbb{H}}(\mathcal{S}^{\mathbb{H}}_{t_0^{(1/2+\varepsilon)/\varphi}} \leq t_0^{(1+\varepsilon/2)/\varphi}) + \mathcal{P}^{u^{1/\varphi},\mathbb{H}}(\zeta(t_0^{(1+\varepsilon/2)/\varphi}) \leq t_0). \tag{A.5}$$

Starting from the first term on the right hand side of (A.5), note that $\|u^{1/\varphi}\|_2 = 1$ for every $u = (u_1, u_2)$ such that $\|u\|_2 = 1$. Also, observe that a reflected Brownian motion in the upper half plane can be constructed by replacing the y-coordinate of a standard 2-dimensional Brownian motion by its absolute value. Since, $\|\mathcal{B}^{\mathbb{H}}(t)\|_2 \geq t_0$ implies that at least one of the coordinates of $\mathcal{B}^{\mathbb{H}}(t)$ is bigger than $t_0/2$ it follows that

$$\mathcal{P}^{u^{1/\varphi},\mathbb{H}}(\mathcal{S}^{\mathbb{H}}_{t_0^{(1/2+\varepsilon)/\varphi}} \leq t_0^{(1+\varepsilon/2)/\varphi})$$
$$\leq 2 \sup_{|a| \leq 1} \mathcal{P}^a\left(\max_{t \leq t_0^{(1+\varepsilon/2)/\varphi}} |\mathcal{B}^1(t)| \geq t_0^{(1/2+\varepsilon)/\varphi}/2\right), \tag{A.6}$$

where $\mathcal{B}(t)$ is a one-dimensional Brownian motion started at a, whose law is denoted by $\mathcal{P}^a$. By reflection principle for one dimensional Brownian motion

$$\mathcal{P}^a\left(\max_{t \leq t_0^{(1+\varepsilon/2)/\varphi}} |\mathcal{B}(t)| \geq t_0^{(1/2+\varepsilon)/\varphi}/2\right)$$
$$\leq 4\mathcal{P}^a(\mathcal{B}(t_0^{(1+\varepsilon/2)/\varphi}) \geq t_0^{(1/2+\varepsilon)/\varphi}/2) \leq 4\exp(-t_0^{\varepsilon/2}) < \frac{1}{2t_0}$$

for all $|a| \leq 1$ and all $t_0 > 0$ sufficiently large. Thus it remains to control the second term on the right hand side of (A.5), and show that for every $\varepsilon > 0$ there exists $C_\varepsilon \in (0, \infty)$ such that
$$\mathcal{P}^{u^{1/\varphi},\mathbb{H}}\left(\zeta(t_0^{(1+\varepsilon/2)/\varphi}) \leq t_0\right) \leq \frac{C_\varepsilon}{t_0}. \tag{A.7}$$

Recalling that $f(z) = z^\varphi$, we have $|f'(\mathcal{B}^{\mathbb{H}}(s))|^2 = \varphi^2|\mathcal{B}^{\mathbb{H}}(s)|^{2(\varphi-1)}$. Define $\delta_\varepsilon = \varphi\varepsilon/(3+\varepsilon)$. For any $n \in \mathbb{N}$, using similar argument as in (A.6), we have

$$\mathcal{P}^{u^{1/\varphi},\mathbb{H}}\left(\max_{t \leq n} \|\mathcal{B}^{\mathbb{H}}(t)\|_2 \geq n^{(1+\delta_\varepsilon)/2}\right)$$
$$\leq 2\max_{|a| \leq 1} \mathcal{P}^a\left(\max_{t \leq n} |\mathcal{B}(t)| \geq n^{(1+\delta_\varepsilon)/2}/2\right) \leq 8\exp(-n^{\delta_\varepsilon/4}) \tag{A.8}$$

for all sufficiently large n. Consequently

$$\sum_{n=\lfloor t_0^{(1+\varepsilon/2)/\varphi}/2 \rfloor}^{\infty} \mathcal{P}^{u^{1/\varphi},\mathbb{H}}\left(\max_{t \leq n} \|\mathcal{B}^{\mathbb{H}}(t)\|_2 \geq n^{(1+\delta_\varepsilon)/2}\right) \leq \frac{C_\varepsilon}{t_0}. \tag{A.9}$$

Thus it suffices to show that $\zeta(t_0^{1+\varepsilon/2}) > t_0$ on the event

$$A = \bigcap_{n=\lfloor t_0^{(1+\varepsilon/2)/\varphi}/2 \rfloor}^{\infty} \left\{ \max_{t \leq n} \|\mathcal{B}^{\mathbb{H}}(t)\|_2 < n^{(1+\delta_\varepsilon)/2} \right\}.$$

This indeed holds since for all $n \geq \lfloor t_0^{(1+\varepsilon/2)/\varphi}/2 \rfloor - 1$

$$\int_{n-1}^{n} \varphi^2 |CB^{\mathbb{H}}(s)|^{2(\varphi-1)} ds > \varphi^2 n^{(1+\delta_\varepsilon)(\varphi-1)} \geq \varphi^2 n^{-1+\varphi-\delta_\varepsilon} = \varphi^2 n^{-1+\varphi/(1+\varepsilon/3)}$$

and therefore

$$\zeta(t_0^{(1+\varepsilon/2)/\varphi}) \geq \varphi^2 \int_{\frac{t_0^{(1+\varepsilon/2)/\varphi}}{2}}^{t_0^{(1+\varepsilon/2)/\varphi}} |\mathcal{B}^{\mathbb{G}}(s)|^{2(\varphi-1)} ds \quad \geq \varphi^2 \sum_{n=\lfloor \frac{t_0^{(1+\varepsilon/2)/\varphi}}{2} \rfloor}^{\lfloor t_0^{(1+\varepsilon/2)/\varphi} \rfloor - 1} n^{-1+\varphi/(1+\varepsilon/3)} \tag{A.10}$$

$$\geq c t_0^{(1+\varepsilon/2)/(1+\varepsilon/3)} > t_0. \quad \square$$

With Lemma A.2 at hand, by choosing $C = t_0^{1+\frac{1}{2\varepsilon}}$ one obtains:

Lemma A.3. *For every $\varepsilon > 0$, every $t_0 \in (0,\infty)$ and any $u \in \mathcal{W}_{\theta_1,\theta_2}$ satisfying $\|u\|_2 = 1$,*

$$\lim_{C \to \infty} \mathcal{P}_{\theta_1,\theta_2}^{u}(\mathcal{S}_{C^\varepsilon} \leq t_0) = 0. \tag{A.11}$$

Next, using the invariance principle and an argument similar to the one in the proof of Lemma 5.5, we prove analogue results to the ones in Lemmas A.1–A.3, for simple random walk in W_{θ_1,θ_2}.

For $\varepsilon > 0$, $R \geq 1$ and $C > 0$, let

$$(S_n^{R,C,\varepsilon})_{n \geq 0} = \left(S_{n \wedge \tau_{\partial W_{\theta_1,\theta_2}^{2C^\varepsilon R}} \wedge \tau_{\partial W_{\theta_1,\theta_2}^{R/2C}}} \right)_{n \geq 0}$$

be a simple random walks in W_{θ_1,θ_2}, starting from $x_R \in \partial W_{\theta_1,\theta_2}^{R}$, stopped at the first hitting time of $\partial W_{\theta_1,\theta_2}^{R/2C}$ or $\partial W_{\theta_1,\theta_2}^{2C^\varepsilon R}$. Due to the invariance principle, for every sequence of points $(x_R)_{R \geq 1}$ for which the limit $x_\infty := \lim_{R \to \infty} x_R/R$ exists, the linear interpolation of $(S_n^{R,C,\varepsilon})_{n \geq 0}$ converges weakly to reflected Brownian motion in the wedge $(\mathcal{B}_{\theta_1,\theta_2}(t))_{t \geq 0}$, starting from $x_\infty \in \mathcal{W}_{\theta_1,\theta_2}$ (satisfying $\|x_\infty\|_2 = 1$) until the stopping time

$$\inf\{t \geq 0 \;:\; |\mathcal{B}_{\theta_1,\theta_2}(t)| \in \{2C^{-1}, 2C^\varepsilon\}\}.$$

Lemma A.4. *For every $\varepsilon > 0$, $t_0 \in (0,\infty)$ and $C > 1$, there exists $C_\varepsilon \in (0,\infty)$ such that the following holds*

$$\lim_{R \to \infty} \max_{y \in \partial W_{\theta_1,\theta_2}^{R}} \mathbf{P}_{\theta_1,\theta_2}^{y}\left(\tau_{\partial W_{\theta_1,\theta_2}^{R/C}} < \tau_{\partial W_{\theta_1,\theta_2}^{RC^\varepsilon}} \right) \leq \varepsilon, \tag{A.12}$$

$$\lim_{R \to \infty} \max_{y \in \partial W_{\theta_1,\theta_2}^{R}} \mathbf{P}_{\theta_1,\theta_2}^{y}\left(\tau_{\partial W^{T^{1/2+\varepsilon}R}} \leq t_0 R^2 \right) \leq \frac{C_\varepsilon}{t_0}, \tag{A.13}$$

and

$$\lim_{C \to \infty} \lim_{R \to \infty} \max_{y \in \partial W_{\theta_1,\theta_2}^{R}} \mathbf{P}_{\theta_1,\theta_2}^{y}\left(\tau_{\partial W_{\theta_1,\theta_2}^{C^\varepsilon R}} \leq t_0 R^2 \right) = 0. \tag{A.14}$$

Proof. As alluded above, we use a similar argument to the one in the proof of Lemma 5.5. Suppose (A.12) does not hold. Then there is a sequence $(R_n)_{n \geq 1}$ going to infinity and a sequence $(x_n)_{n \geq}$ such that for every $n \geq 1$, $x_n \in \partial W_{\theta_1,\theta_2}^{R_n}$ and

$$\mathbf{P}_{\theta_1,\theta_2}^{x_n}\left(\tau_{\partial W_{\theta_1,\theta_2}^{R/C}} < \tau_{\partial W_{\theta_1,\theta_2}^{RC^\varepsilon}} \right) \geq \varepsilon$$

Since x_n/R_n is a bounded sequence in $\mathbb{R}^2$, there exists a subsequence $(n_k)_{k \geq 1}$ such that $\lim_{k \to \infty} x_{n_k}/R_{n_k} = x'_\infty \in \mathcal{W}_{\theta_1,\theta_2}$, satisfying $\|x'_\infty\|_2 = 1$. Thus by the invariance principle and Lemma Lemma A.1,

$$\varepsilon \leq \lim_{k \to \infty} \mathbf{P}^{x_{n_k}}_{\theta_1,\theta_2}\big(\tau_{\partial W^{R_{n_k}/C}_{\theta_1,\theta_2}} < \tau_{\partial W^{R_{n_k}C^\varepsilon}_{\theta_1,\theta_2}}\big) = \mathcal{P}^{x'_\infty}_{\theta_1,\theta_2}(\mathcal{S}_{C^{-1}} < \mathcal{S}_{C^\varepsilon}) = \frac{\varepsilon}{1+\varepsilon}$$

which contradicts the assumption. Repeating the argument with Lemma A.2 and Lemma A.3 replacing Lemma A.1, yields (A.13) and (A.14) respectively. $\qquad\square$

Proof of Proposition A.1. Since A is a fixed finite set, for every $C > 1$ and R sufficiently large (depending only on C and A) we have $A \subset W^{R/C}_{\theta_1,\theta_2}$. Hence, for every $x \in \partial W^R_{\theta_1,\theta_2}$ and R sufficiently large

$$\tau_{C^\varepsilon R} \wedge \tau_{R/C} \leq \tau_{R/C} \leq \tau_A, \qquad \mathbf{P}^x_{\theta_1,\theta_2} - a.s$$

and therefore, for every $t_0 > 0$, $C > 1$, $\varepsilon > 0$ and $x \in \partial W^R_{\theta_1,\theta_2}$

$$\mathbf{P}^x_{\theta_1,\theta_2}(\tau_A \leq t_0 R^2) \leq \mathbf{P}^x_{\theta_1,\theta_2}\big(\tau_{\partial W^{C^\varepsilon R}_{\theta_1,\theta_2}} \wedge \tau_{\partial W^{R/C}_{\theta_1,\theta_2}} \leq t_0 R^2\big)$$
$$\leq \mathbf{P}^x_{\theta_1,\theta_2}\big(\tau_{\partial W^{R/C}_{\theta_1,\theta_2}} < \tau_{\partial W^{C^\varepsilon R}_{\theta_1,\theta_2}}\big) + \mathbf{P}^x_{\theta_1,\theta_2}\big(\tau_{\partial W^{C^\varepsilon R}_{\theta_1,\theta_2}} \leq t_0 R^2\big).$$

Taking the maximum over $x \in \partial W^R_{\theta_1,\theta_2}$, then the limit $R \to \infty$, then the limit $C \to \infty$ and finally the limit $\varepsilon \to 0$, the result follows from Lemma A.4. $\qquad\square$

A.3 Proof of Proposition A.2

Since $\mathbb{P}^{x_1,x_2}_R$ couples the two random walks via a maximum coupling for Markov chains, it follows that

$$\mathbb{P}^{x_1,x_2}_R(\mathrm{T} > t_0) \leq d_{\mathrm{TV}}(B_1^R(t_0), B_2^R(t_0))$$
$$:= \sum_{y \in W_{\theta_1,\theta_2}/R} \big|\mathbb{P}^{x_1,x_2}_R(B_1^R(t_0) = y) - \mathbb{P}^{x_1,x_2}_R(B_2^R(t_0) = y)\big|$$

For $M_0 \in (0, \infty)$, one can split the sum over y into two parts, those satisfying $\|y\|_2 \geq M_0$ and the ones satisfying $\|y\|_2 < M_0$. We turn to estimate each of the sums separately.
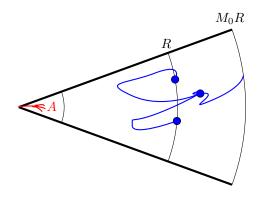

Figure 4: Max coupling occurs before hitting $\partial W^{M_0 R}_{\theta_1,\theta_2}$.

For the first sum, note that a similar argument the one in (A.3) yields

$$\sum_{\substack{y \in W_{\theta_1,\theta_2}/R \\ \|y\|_2 \geq M_0}} \left| \mathbb{P}_R^{x_1,x_2}(B_1^R(t_0) = y) - \mathbb{P}_R^{x_1,x_2}(B_2^R(T_0) = y) \right|$$

$$\leq \mathbb{P}_R^{x_1,x_2}(\|B_1^R(t_0)\|_2 \geq M_0) + \mathbb{P}_R^{x_1,x_2}(\|B_2^R(t_0)\|_2 \geq M_0)$$

$$\leq 2e^{-2(\log(4)-1)T_0 R^2} + 2 \max_{x \in \partial W_{\theta_1,\theta_2}^R} \mathbf{P}_{\theta_1,\theta_2}^x(\tau_{M_0 R} \leq 4t_0 R^2).$$

Fixing some $\varepsilon > 0$ and defining $M_0 = t_0^{1/2+\varepsilon}$, it follows from Lemma A.4 (see (A.13)) that the sum is bounded by

$$2e^{-2(\log(4)-1)t_0 R^2} + \frac{3C_\varepsilon}{t_0} \leq \frac{4C_\varepsilon}{t_0},$$

provided R is sufficiently large.

Next, we turn to estimate the second term, namely

$$\sum_{\substack{y \in W_{\theta_1,\theta_2}/R \\ \|y\|_2 < M_0}} \left| \mathbb{P}_R^{x_1,x_2}(B_1^R(t_0) = y) - \mathbb{P}_R^{x_1,x_2}(B_2^R(t_0) = y) \right| \tag{A.15}$$

The strategy for bounding the last sum is to use known bounds on the mixing time and total variation distance for random walks on finite graphs, obtained by intersecting scaled version of $\mathbb{Z}^2$ with some bounded and sufficiently regular domains in $\mathbb{R}^2$. Note however, that the continuous time, simple random walk in (A.15) is defined on the cone $W_{\theta_1,\theta_2}/R$, which is not bounded. Thus our first step is to show that the last sum can be well approximated by a corresponding sum for a continuous time, simple random walk in $W_{\theta_1,\theta_2}^{M_0 R}/R$, with M_0 chosen (as before) to be $t_0^{1/2+\varepsilon}$.

To this end, for $M_0 = t_0^{1/2+\varepsilon} > 1$, $R > 0$ and $x_1, x_2 \in \partial W_{\theta_1,\theta_2}^R$, denote by $\mathbb{P}_R^{x_1,x_2,M_0}$ a coupling of two continuous time, simple random walks on $W_{\theta_1,\theta_2}^{M_0 R}/R$, denoted $(B_1^{R,M_0}(t))_{t\geq 0}$ and $(B_2^{R,M_0}(t))_{t\geq 0}$, defined as follows:

1. $B_1^{R,M_0}(t)$ is a continuous time, simple random walk on $W_{\theta_1,\theta_2}^{M_0 R}/R$, starting at x_1/R, with fixed jump rate of $2R^2$.

2. $B_2^{R,M_0}(t)$ is a continuous time, simple random walk on $W_{\theta_1,\theta_2}^{M_0 R}/R$, starting at x_2/R, with fixed jump rate of $2R^2$.

3. $(B_1^{R,M_0}(t))_{t\geq 0}$ and $(B_2^{R,M_0}(t))_{t\geq 0}$ are coupled according to the maximum coupling, see [LL10, Appendix A.4.2].

Furthermore, for $i \in \{1,2\}$ and $s \geq 0$, we denote by $N_i^{R,M_0}(s)$, the number of jumps made by the Markov process $(B_i^{R,M_0}(t))_{t\geq 0}$ up to time s and for $D \subset W_{\theta_1,\theta_2}^{M_0 R}$, define

$$\tau_D^{i,M_0} = \inf\{t \geq 0 \; : \; B_i^{R,M_0}(t) \in D/R\}.$$

Lemma A.5. *Fix $\varepsilon > 0$ and $t_0 > 1$, and let $M_0 = t_0^{1/2+\varepsilon}$. Then for every $R > 0$ sufficiently large and every $x_1, x_2 \in \partial W_{\theta_1,\theta_2}^R$*

$$\left| \sum_{y \in W_{\theta_1,\theta_2}^{M_0 R}/R} \left| \mathbb{P}_R^{x_1,x_2}(B_1^R(t_0) = y) - \mathbb{P}_R^{x_1,x_2}(B_2^R(t_0) = y) \right| - d_{\mathrm{TV}}(B_1^{R,M_0}(t_0), B_2^{R,M_0}(t_0)) \right| \leq \frac{16C_\varepsilon}{t_0}$$

Proof. Fix $x_1, x_2 \in \partial W_{\theta_1,\theta_2}^R$ and note that for $i \in \{1,2\}$

$$\mathbb{P}_R^{x_1,x_2}\big(\tau_{\partial W_{\theta_1,\theta_2}^{M_0 R/2}}^i \le t_0\big) \le \mathbb{P}_R^{x_1,x_2}\big(N_i(t_0) \ge 4t_0 R^2\big) + \mathbb{P}_R^{x_1,x_2}\big(N_i(\tau_{\partial W_{\theta_1,\theta_2}^{M_0 R/2}}^i) \le 4t_0 R^2\big)$$

$$\le e^{-2(\log(4)-1)t_0 R^2} + \mathbf{P}_{\theta_1,\theta_2}^{x_i}\big(\tau_{\partial W_{\theta_1,\theta_2}^{M_0 R/2}} \le 4t_0 R^2\big) \le \frac{4C_\varepsilon}{t_0} \,,$$

where in the last step we used Lemma A.4(A.13). A similar argument shows that for $i \in \{1,2\}$

$$\mathbb{P}_R^{x_1,x_2,M_0}\Big(\tau_{\partial W_{\theta_1,\theta_2}^{M_0 R/2}}^i \le t_0\Big) \le \frac{4C_\varepsilon}{t_0} \,.$$

Next, notice that the laws of $(B_i^R(t \wedge \tau_{\partial W_{\theta_1,\theta_2}^{M_0 R/2}}^i))_{t \ge 0}$ and $(B_i^{R,M_0}(t \wedge \tau_{\partial W_{\theta_1,\theta_2}^{M_0 R/2}}^{i,M_0}))_{t \ge 0}$ are equal and therefore

$$\mathbb{P}_R^{x_1,x_2}\big(B_i^R(t_0) = y, \ \tau_{\partial W_{\theta_1,\theta_2}^{M_0 R/2}}^i > t_0\big) = \mathbb{P}_R^{x_1,x_2,M_0}\big(B_i^{R,M_0}(t_0) = y, \ \tau_{\partial W_{\theta_1,\theta_2}^{M_0 R/2}}^{i,M_0} > t_0\big).$$

Combining all of the above, together with the fact that

$$d_{\mathrm{TV}}(B_1^{R,M_0}(t_0), B_2^{R,M_0}(t_0))$$
$$= \sum_{y \in W_{\theta_1,\theta_2}^{M_0 R}/R} \big|\mathbb{P}_R^{x_1,x_2,M_0}(B_1^{R,M_0}(t_0) = y) - \mathbb{P}_R^{x_1,x_2,M_0}(B_2^{R,M_0}(t_0) = y)\big|$$

yields the result. $\qquad\square$

Proof of Proposition A.2. Combining the estimation for the sum over $y \in (W_{\theta_1,\theta_2} \setminus W_{\theta_1,\theta_2}^{M_0 R})/R$ together with Lemma A.5 implies that for every $x_1, x_2 \in \partial W_{\theta_1,\theta_2}^R$

$$|\mathbb{P}_R^{x_1,x_2}(\mathrm{T} > t_0) - d_{\mathrm{TV}}(B_1^{R,M_0}(t_0), B_2^{R,M_0}(t_0))| \le \frac{20C_\varepsilon}{t_0} \,,$$

where $(B_1^{R,M_0}(t_0)), B_2^{R,M_0}(t_0)))$ is distributed according to the coupling $\mathbb{P}_R^{x_1,x_2,M_0}$. Therefore, it suffices to show that

$$\lim_{t_0 \to \infty} \lim_{R \to \infty} \sup_{x_1,x_2 \in \partial W_{\theta_1,\theta_2}^R} d_{\mathrm{TV}}(B_1^{R,M_0}(t_0), B_2^{R,M_0}(t_0)) = 0 \,.$$

Recall that $(B_1^{R,M_0}(t))_{t \ge 0}$ and $(B_2^{R,M_0}(t))_{t \ge 0}$ are continuous time, simple random walks on $W_{\theta_1,\theta_2}^{M_0 R}/R$ with law $\mathbb{P}_R^{x_1,x_2,M_0}$, and in particular that they start in x_1/R and x_2/R respectively.

We finish the proof using one last rescaling. For $i \in \{1,2\}$ and $R > 0$, define

$$\widetilde{B}_i^{R,M_0}(t) = \frac{1}{M_0} B_i^{R,M_0}(M_0^2 t), \qquad \forall t \ge 0, \forall i \in \{1,2\}.$$

Note that $(\widetilde{B}_i^{R,M_0}(t))_{t \ge 0}$, for $i \in \{1,2\}$, is a continuous time, simple random walks in $W_{\theta_1,\theta_2}^{M_0 R}/(M_0 R)$ with constant jump rate $2(M_0 R)^2$ started in $x_i/M_0 R$. In addition, observe that $W_{\theta_1,\theta_2}^{M_0 R}/(M_0 R)$ is also the intersection of the rescaled lattice $(M_0 R)^{-1}\mathbb{Z}^2$ and the continuous wedge $\mathcal{D} = \mathcal{W}_{\theta_1,\theta_2} \cap \{\|y\|_2 < 1\}$. Thus for any $y \in W_{\theta_1,\theta_2}/R$ such that $\|y\|_2 < M_0$,

$$\mathbb{P}_R^{x_1,x_2,M_0}\big(B_i^{R,M_0}(t_0) = y\big) = \mathbb{P}_R^{x_1,x_2,M_0}\left(\widetilde{B}_i^{R,M_0}\left(\frac{t_0}{M_0^2}\right) = \frac{y}{M_0}\right). \tag{A.16}$$

Let $\widetilde{t}_0 = t_0/M_0^2 = t_0^{-2\varepsilon}$. Then

$$
\begin{aligned}
&d_{\mathrm{TV}}\big(B_1^{R,M_0}(t_0), B_2^{R,M_0}(t_0)\big) \\
&= \sum_{\substack{y \in W_{\theta_1,\theta_2}/R \\ \|y\|_2 < M_0}} \Big|\mathbb{P}_R^{x_1,x_2,M_0}\big(B_1^{R,M_0}(t_0) = y\big) - \mathbb{P}_R^{x_1,x_2,M_0}\big(B_2^{R,M_0}(t_0) = y\big)\Big| \\
&= \sum_{z \in W_{\theta_1,\theta_2}^{M_0 R}/M_0 R} \Big|\mathbb{P}_R^{x_1,x_2,M_0}\big(\widetilde{B}_1^{R,M_0}(\widetilde{t}_0) = z\big) - \mathbb{P}_R^{x_1,x_2,M_0}\big(\widetilde{B}_2^{R,M_0}(\widetilde{t}_0) = z\big)\Big|.
\end{aligned}
\tag{A.17}
$$

We are now ready to use the aforementioned known bound on the mixing time for continuous-time random walks on bounded domains in $\mathbb{R}^2$. Note that $\mathcal{D}$ is a bounded, Lipschitz domain in $\mathbb{R}^2$, and therefore, by (2.8) and Theorem 2.11 in [CF17], for $\varepsilon > 0$ sufficiently small, there exists a constant, $C' \in (0, \infty)$ which is independent of R and z, such that for all sufficiently large R and $z \in W_{\theta_1,\theta_2}^{M_0 R}/M_0 R$

$$
\begin{aligned}
(M_0 R)^2 &\big|\mathbb{P}_R^{x_1,x_2,M_0}\big(\widetilde{B}_1^{R,M_0}(\widetilde{t}_0) = z\big) - \mathbb{P}_R^{x_1,x_2,M_0}\big(\widetilde{B}_2^{R,M_0}(\widetilde{t}_0) = z\big)\big| \\
&\leq C' \frac{|\widetilde{B}_1^{R,M_0}(0) - \widetilde{B}_2^{R,M_0}(0)|^{8\varepsilon}}{\widetilde{t}_0^{1+4\varepsilon}}.
\end{aligned}
\tag{A.18}
$$

Recalling that $\widetilde{B}_i^{R,M_0}(0) = x_i/(M_0 R)$ for $i \in \{1,2\}$ and that $x_1, x_2 \in \partial W_{\theta_1,\theta_2}^R$, we conclude that $|\widetilde{B}_1^{R,M_0}(0) - \widetilde{B}_2^{R,M_0}(0)| \leq C M_0^{-1} = C t_0^{-1/2-\varepsilon}$, and therefore

$$
(M_0 R)^2 \big|\mathbb{P}_R^{x_1,x_2,M_0}\big(\widetilde{B}_1^{R,M_0}(\widetilde{t}_0) = z\big) - \mathbb{P}_R^{x_1,x_2,M_0}\big(\widetilde{B}_2^{R,M_0}(\widetilde{t}_0) = z\big)\big| \leq \frac{C'}{t_0^{2\varepsilon}}
$$

Noting that $|W_{\theta_1,\theta_2}^{M_0 R}/M_0 R| = O((M_0 R)^2)$, and that the bound is uniform in $x_1, x_2 \in \partial W_{\theta_1,\theta_2}^R$, the proof of Proposition A.2 (and thus of Theorem 3.1 as well) is complete. $\qquad\square$

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