Functional inequalities for weighted Gamma distribution on the space of finite measures[*]

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Abstract

Let $\mathbb{M}$ be the space of finite measures on a locally compact Polish space, and let $\mathcal{G}$ be the Gamma distribution on $\mathbb{M}$ with intensity measure $\nu \in \mathbb{M}$. Let ∇^{ext} be the extrinsic derivative with tangent bundle $T\mathbb{M} = \cup_{\eta \in \mathbb{M}} L^2(\eta)$, and let $\mathcal{A} : T\mathbb{M} \to T\mathbb{M}$ be measurable such that $\mathcal{A}_\eta$ is a positive definite linear operator on $L^2(\eta)$ for every $\eta \in \mathbb{M}$. Moreover, for a measurable function V on $\mathbb{M}$, let $\mathrm{d}\mathcal{G}^V = \mathrm{e}^V \mathrm{d}\mathcal{G}$. We investigate the Poincaré, weak Poincaré and super Poincaré inequalities for the Dirichlet form

$$\mathcal{E}_{\mathcal{A},V}(F, G) := \int_{\mathbb{M}} \langle \mathcal{A}_\eta \nabla^{ext} F(\eta), \nabla^{ext} G(\eta) \rangle_{L^2(\eta)} \, \mathrm{d}\mathcal{G}^V(\eta),$$

which characterize various properties of the associated Markov semigroup. The main results are extended to the space of finite signed measures.

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1 Introduction

Let $\mathbb{M}$ be the class of finite measures on a locally compact Polish space E, which is again a Polish space under the weak topology. Recall that a sequence of finite measures $\eta_n \to \eta$ weakly if $\eta_n(f) \to \eta(f)$ for $f \in C_b(E)$, where and in what follows, for a measure η we denote

$$\eta(f) := \int f \mathrm{d}\eta, \quad f \in L^1(\eta). \tag{1.1}$$

Since M is locally compact, the Borel σ-algebra $\mathcal{B}(\mathbb{M})$ induced by the weak topology coincides with that induced by the vague topology. Let $\nu \in \mathbb{M}$ with $\nu(E) > 0$. The Gamma distribution $\mathcal{G}$ with intensity measure ν is the unique probability measure on $\mathbb{M}$ such that for any finitely many disjoint measurable subsets $\{A_1, \cdots, A_n\}$ of E, $\{\eta(A_i)\}_{1 \le i \le n}$ are

independent Gamma random variables with shape parameters $\{\nu(A_i)\}_{1\leq i\leq n}$ and scale parameter 1; that is,

$$\int_{\mathbb{M}} f(\eta(A_1),\cdots,\eta(A_n))\mathcal{G}(\mathrm{d}\eta) = \int_{[0,\infty)^n} f(x_1,\cdots,x_n)\prod_{i=1}^n \gamma_{\nu(A_i)}(\mathrm{d}x_i), \ \ f\in\mathcal{B}_b(E), \quad (1.2)$$

where $\mathcal{B}_b(E)$ is the class of bounded measurable functions on E, for a constant $r>0$

$$\gamma_r(\mathrm{d}s) := 1_{[0,\infty)}(s)\frac{s^{r-1}\mathrm{e}^{-s}}{\Gamma(r)}\,\mathrm{d}s, \ \ \Gamma(r) := \int_0^\infty x^{r-1}\mathrm{e}^{-x}\mathrm{d}x, \quad\quad (1.3)$$

and $\gamma_0 := \delta_0$ is the Dirac measure at point 0. It is well known that $\mathcal{G}$ is concentrated on the class of finite discrete measures

$$\mathbb{M}_{dis} := \Big\{ \sum_{i=1}^\infty s_i\delta_{x_i} : \ s_i\geq 0, x_i\in E, \sum_{i=1}^\infty s_i < \infty\Big\}.$$

Consider the weighted Gamma distribution $\mathcal{G}^V(\mathrm{d}\eta) := \mathrm{e}^{V(\eta)}\mathcal{G}(\mathrm{d}\eta)$, where V is a measurable function on $\mathbb{M}$. We will investigate functional inequalities for the Dirichlet form induced by $\mathcal{G}^V(\mathrm{d}\eta)$ and a positive definite linear map $\mathcal{A}$ on the tangent space of the extrinsic derivative. See [7] and references therein for Dirichlet forms induced by both extrinsic and intrinsic derivatives, where the intensity measure ν is the Lebesgue measure on $\mathbb{R}^d$ such that the Gamma distribution $\mathcal{G}$ is concentrated on the space of infinite Radon measures on $\mathbb{R}^d$. In this paper, we only consider finite intensity measure ν.

Definition 1.1 ([11])**.** A measurable real function F on $\mathbb{M}$ is called extrinsically differentiable at $\eta\in\mathbb{M}$, if

$$\nabla^{ext}F(\eta)(x) := \frac{\mathrm{d}}{\mathrm{d}s}F(\eta + s\delta_x)\Big|_{s=0} \ \text{ exists for all } x\in E,$$

such that

$$\|\nabla^{ext}F(\eta)\| := \|\nabla^{ext}F(\eta)(\cdot)\|_{L^2(\eta)} < \infty.$$

If F is extrinsically differentiable at all $\eta\in\mathbb{M}$, we denote $F\in\mathcal{D}(\nabla^{ext})$ and call it extrinsically differentiable on $\mathbb{M}$.

Regarding $L^2(\eta)$ as the extrinsic tangent space at $\eta\in\mathbb{M}$, we define the directional derivatives by

$$\nabla^{ext}_\phi F(\eta) := \langle\nabla^{ext}F(\eta),\phi\rangle_{L^2(\eta)} = \eta\big(\phi\nabla^{ext}F(\eta)\big), \ \ \phi\in L^2(\eta).$$

When ϕ is bounded, this coincides with the directional derivative under multiplicative actions:

$$\nabla^{ext}_\phi F(\eta) = \frac{\mathrm{d}}{\mathrm{d}s}F(\mathrm{e}^{s\phi}\eta)\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}F((1+s\phi)\eta)\Big|_{s=0}, \ \ \phi\in\mathcal{B}_b(E),$$

where $h\eta$ for $h\in\mathcal{B}_b(E)$ is a finite signed measure given by

$$(h\eta)(A) := \eta(1_A h) = \int_A h\,\mathrm{d}\eta, \ \ A\in\mathcal{B}(E).$$

To introduce the Dirichlet form induced by the extrinsic derivative and the weighted Gamma distribution $\mathcal{G}^V$, we consider the class $\mathcal{F}C_0^\infty$, which consists of cylindrical functions functions of type

$$F(\eta) := f(\eta(A_1),\cdots,\eta(A_n)), \ \ n\geq 1, f\in C_0^\infty(\mathbb{R}^n), \{A_i\}_{1\leq i\leq n}\in\mathcal{I}(E),$$

where $\mathcal{I}(E)$ is the set of all measurable partitions of E. Obviously, such a function F is extrinsically differentiable with

$$\nabla^{ext}F(\eta) = \sum_{i=1}^{n}(\partial_i f)(\eta(A_1), \cdots, \eta(A_n)) \cdot 1_{A_i}. \tag{1.4}$$

We consider the square field

$$\Gamma_{\mathcal{A}}(F, G) := \langle \mathcal{A}_\eta \nabla^{ext}F(\eta), \nabla^{ext}G(\eta) \rangle_{L^2(\eta)} = \int_E \left[\mathcal{A}_\eta \nabla^{ext}F(\eta)\right] \cdot \left[\nabla^{ext}G(\eta)\right] d\eta,$$

and the pre-Dirichlet form

$$\mathcal{E}_{\mathcal{A},V}(F, G) := \int_{\mathbb{M}} \Gamma_{\mathcal{A}}(F, G) \, d\mathcal{G}^V, \quad F, G \in \mathcal{F}C_0^\infty,$$

where $\mathcal{A}$ and V satisfy the following assumption.

(H) For any $\eta \in \mathbb{M}$, let $\mathcal{A}_\eta$ be a bounded linear operator on $L^2(\eta)$ such that

$$\langle \mathcal{A}_\eta h, h \rangle_{L^2(\eta)} \geq 0, \quad h \in L^2(\eta), \tag{1.5}$$

for any $A \in \mathcal{B}(E)$ and $x \in E$, $\mathcal{A}_\eta 1_A(x)$ is measurable in $(\eta, x) \in \mathbb{M} \times E$ and is extrinsically differentiable in η with

$$\sup_{\eta(E) \leq r} \left\{\|\mathcal{A}_\eta\|_{L^2(\eta)}^2 + \|\nabla^{ext}[\mathcal{A}_\eta 1_A]\|_{L^2(\eta)}\right\} < \infty, \quad r \in (0, \infty), \tag{1.6}$$

where $\|\cdot\|_{L^2(\eta)}$ is the norm (or the operator norm for linear operators) in $L^2(\eta)$. Moreover, $V \in \mathcal{D}(\nabla^{ext})$ such that

$$\sup_{\eta(E) \leq r} \left\{|V(\eta)| + \|\nabla^{ext}V(\eta)\|_{L^2(\eta)}\right\} < \infty, \quad r \in (0, \infty). \tag{1.7}$$

Condition (1.5) is essential for the nonnegativity of $\mathcal{E}_{\mathcal{A},V}$, where conditions (1.6) and (1.7) ensure the boundedness of $\mathcal{A}, V$ and their extrinsic derivatives on the level sets $\{\eta(E) \leq r\}$ for $r > 0$. These conditions are standard for establishing functional inequalities by using perturbation argument, see [14, 24] for the study of the finite-dimensional models.

We write $\mathcal{A} = \mathbf{1}$ if $\mathcal{A}_\eta$ is the identity map on $L^2(\eta)$ for every $\eta \in \mathbb{M}$. According to Theorem 3.1 below, the assumption **(H)** implies that $(\mathcal{E}_{\mathcal{A},V}, \mathcal{F}C_0^\infty)$ is closable in $L^2(\mathcal{G}^V)$ and the closure $(\mathcal{E}_{\mathcal{A},V}, \mathcal{D}(\mathcal{E}_{\mathcal{A},V}))$ is a symmetric Dirichlet form. If moreover

$$\int_{\mathbb{M}} \left(1 + \frac{\|\mathcal{A}_\eta\|_{L^2(\eta)}}{1 + \eta(E)}\right) \mathcal{G}^V(d\eta) < \infty, \tag{1.8}$$

then $1 \in \mathcal{D}(\mathcal{E}_{\mathcal{A},V})$ with $\mathcal{E}_{\mathcal{A},V}(1, 1) = 0$. Let $(\mathcal{L}_{\mathcal{A},V}, \mathcal{D}(\mathcal{L}_{\mathcal{A},V}))$ be the associated generator. We aim to investigate functional inequalities for the Dirichlet form $\mathcal{E}_{\mathcal{A},V}$ and the spectral gap of the generator $\mathcal{L}_{\mathcal{A},V}$.

We first consider the Poincaré inequality

$$\mathcal{G}^V(F^2) \leq \frac{1}{\lambda}\mathcal{E}_{\mathcal{A},V}(F, F) + \mathcal{G}^V(F)^2, \quad F \in \mathcal{D}(\mathcal{E}_{\mathcal{A},V}), \tag{1.9}$$

where $\lambda > 0$ is a constant. The spectral gap of $\mathcal{L}_{\mathcal{A},V}$, denoted by $\mathrm{gap}(\mathcal{L}_{\mathcal{A},V})$, is the largest constant $\lambda > 0$ such that (1.9) holds. If (1.9) is invalid, i.e. there is no any constant $\lambda > 0$ satisfying the inequality, we write $\mathrm{gap}(\mathcal{L}_{\mathcal{A},V}) = 0$ and say that $\mathcal{L}_{\mathcal{A},V}$ does not have

spectral gap. It is well known that (1.9) is equivalent to the exponential convergence of the associated Markov semigroup $P_t^{\mathcal{A},V}$:

$$\|P_t^{\mathcal{A},V}F - \mathcal{G}^V(F)\|_{L^2(\mathcal{G}^V)} \le \mathrm{e}^{-\lambda t}\|F\|_{L^2(\mathcal{G}^V)}, \quad t \ge 0, F \in L^2(\mathcal{G}^V).$$

When $\mathrm{gap}(\mathcal{L}_{\mathcal{A},V}) = 0$, the following weak Poincaré inequality was introduced in [13]:

$$\mathcal{G}^V(F^2) \le \alpha(r)\mathcal{E}_{\mathcal{A},V}(F,F) + r\|F\|_\infty^2, \quad F \in \mathcal{D}(\mathcal{E}_{\mathcal{A},V}), \ \mathcal{G}^V(F) = 0, r > 0, \quad (1.10)$$

where $\alpha : (0,\infty) \to (0,\infty)$ corresponds to a non-exponential convergence rate of $P_t^{\mathcal{A},V}$ as $t \to \infty$, see [13, Theorems 2.1 and 2.3]. In particular, (1.10) implies

$$\|P_t^{\mathcal{A},V} - \mathcal{G}^V\|_{L^\infty(\mathcal{G}^V) \to L^2(\mathcal{G}^V)} \le \inf\left\{r > 0 : \alpha(r)\log r^{-1} \le 2t\right\} \downarrow 0 \text{ as } t \uparrow \infty.$$

We also consider the super Poincaré inequality

$$\mathcal{G}^V(F^2) \le r\mathcal{E}_{\mathcal{A},V}(F,F) + \beta(r)\mathcal{G}^V(|F|)^2, \quad r > 0, F \in \mathcal{D}(\mathcal{E}_{\mathcal{A},V}), \quad (1.11)$$

where $\beta : (0,\infty) \to (0,\infty)$ is a decreasing function. The existence of super Poincaré inequality is equivalent to the uniform integrability of $P_t^{\mathcal{A},V}$ for $t > 0$, and, when $P_t^{\mathcal{A},V}$ has an asymptotic density with respect to $\mathcal{G}^V$, it is also equivalent to the compactness of $P_t^{\mathcal{A},V}$ in $L^2(\mathcal{G}^V)$, see [24, Theorem 3.2.1] for details. According to [24, Definition 3.1.2], $P_t^{\mathcal{A},V}$ is said to have an asymptotic density, if $\|P_t^{\mathcal{A},V} - P_n\|_{L^2(\mathcal{G}^V)} \to 0$ for a sequence of bounded linear operators $\{P_n\}_{n \ge 1}$ having densities with respect to $\mathcal{G}^V$. We say that $\mathcal{E}_{\mathcal{A},V}$ does not satisfy the super Poincaré inequality, if there is no $\beta : (0,\infty) \to (0,\infty)$ satisfying (1.11). In particular, (1.11) holds with $\beta(r) = \mathrm{e}^{cr^{-1}}$ for some constant $c > 0$ if and only if the log-Sobolev inequality

$$\mathcal{G}^V(F^2 \log F^2) \le C\mathcal{E}_{\mathcal{A},V}(F,F), \quad F \in \mathcal{D}(\mathcal{E}_{\mathcal{A},V}), \mathcal{G}^V(F^2) = 1 \quad (1.12)$$

holds for some constant $C > 0$. It is well known (see [2, 6]) that (1.12) is equivalent to the hypercontractivity of $P_t^{\mathcal{A},V}$:

$$\|P_t^{\mathcal{A},V}\|_{L^2(\mathcal{G}^V) \to L^4(\mathcal{G}^V)} = 1 \text{ for large } t > 0,$$

as well as the exponential convergence in entropy:

$$\mathcal{G}^V((P_t^{\mathcal{A},V}F)\log P_t^{\mathcal{A},V}F) \le \mathrm{e}^{-2t/C}\mathcal{G}^V(F \log F), \quad t \ge 0, F \ge 0, \mathcal{G}^V(F) = 1.$$

See [21, 22, 23] or [24] for more results on the super Poincaré inequalities, for instance, estimates on the semigroup $P_t^{\mathcal{A},V}$ and higher order eigenvalues of the generator $\mathcal{L}_{\mathcal{A},V}$ using the function β in (1.11).

The remainder of the paper is organised as follows. In section 2, we state the main results of the paper, and illustrate these results by a typical example with specific interactions. In Section 3, we establish the integration by parts formula which implies the closability of $(\mathcal{E}_{\mathcal{A},V}, \mathcal{F}C_0^\infty)$. Then the main results are proved in Section 4, and extended in Section 5 to the space $\mathbb{M}_s$ of finite signed measures.

2 Main results and an example

We first consider $\mathcal{E}_{\mathbf{1},0}$ in $L^2(\mathcal{G})$ whose restriction on $\mathbb{M}_1 := \{\mu \in \mathbb{M} : \mu(E) = 1\}$ gives rise to the Dirichlet form of the Fleming–Viot process. Corresponding to results of [16, 17] for the Fleming–Viot process, we have the following result. See also [12, 26] for functional inequalities of different type measure-valued processes.

Theorem 2.1. Let $V = 0$ and $\mathcal{A} = \mathbf{1}$.

(1) $\text{gap}(\mathcal{L}_{\mathbf{1},0}) = 1$, i.e. $\lambda = 1$ is the largest constant such that (1.9) holds for $V = 0$ and $\mathcal{A} = \mathbf{1}$.

(2) If $\text{supp}\,\nu$ contains infinitely many points, then $\mathcal{E}_{\mathbf{1},0}$ does not satisfy the super Poincaré inequality.

(3) There exists a constant $c_0 > 0$ such that when $\text{supp}\,\nu$ is a finite set, the log-Sobolev inequality

$$\mathcal{G}(F^2 \log F^2) \le \frac{c_0}{1 \wedge \delta} \mathcal{E}_{\mathbf{1},0}(F, F), \quad F \in \mathcal{D}(\mathcal{E}_{\mathbf{1},0}), \mathcal{G}(F^2) = 1 \tag{2.1}$$

holds, where $\delta := \min\{\nu(\{x\}) : \ x \in \text{supp}\,\nu\}$.

To extend this result to $\mathcal{E}_{\mathcal{A},V}$, we will adopt a split argument by making perturbations to $\mathcal{E}_{\mathbf{1},0}$ on bounded sets and estimating the principal eigenvalue of $\mathcal{L}_{\mathcal{A},V}$ outside. To this end, we take

$$\rho(\eta) = 2\sqrt{\eta(E)}, \quad \eta \in \mathbb{M}$$

and let $\mathbf{B}_N = \{\eta \in \mathbb{M} : \rho(\eta) \le N\}$ for $N > 0$. Since (1.4) implies

$$\nabla^{ext}\rho(\eta) = \frac{1}{\sqrt{\eta(E)}}, \quad \eta \in \mathbb{M} \setminus \{0\}, \tag{2.2}$$

we have

$$\Gamma_{\mathbf{1}}(\rho, \rho) := \eta\big(|\nabla^{ext}\rho(\eta)|^2\big) = \frac{\eta(E)}{\eta(E)} = 1. \tag{2.3}$$

According to (3.1) below, we set

$$\mathcal{L}_{\mathcal{A},V}\rho(\eta) = \frac{2}{\rho(\eta)}\big[(\nu - \eta)(\mathcal{A}_\eta 1) + \eta\big(\nabla^{ext}[\mathcal{A}_\eta 1(\cdot)](\cdot)\big) + \nabla^{ext}_{\mathcal{A}_\eta 1}V(\eta)\big] - \frac{4}{\rho(\eta)^2}\eta(\mathcal{A}_\eta 1), \tag{2.4}$$

where

$$\eta\big(\nabla^{ext}[\mathcal{A}_\eta 1(\cdot)](\cdot)\big) := \int_E \nabla^{ext}[\mathcal{A}_\eta 1(x)](x)\,\eta(\mathrm{d}x).$$

Let

$$\begin{aligned}
\xi(r) &= \sup_{\rho(\eta)=r} \mathcal{L}_{\mathcal{A},V}\rho(\eta), \quad \underline{a}(r) = \inf_{\rho(\eta)=r} \inf_{\|\phi\|_{L^2(\eta)}=1} \langle \mathcal{A}_\eta \phi, \phi \rangle_{L^2(\eta)}, \\
\bar{a}(r) &= \sup_{\rho(\eta)=r} \sup_{\|\phi\|_{L^2(\eta)}=1} \langle \mathcal{A}_\eta \phi, \phi \rangle_{L^2(\eta)}, \quad r > 0.
\end{aligned} \tag{2.5}$$

Under **(H)**, $|V(\eta)| + \|\mathcal{A}_\eta\|_{L^2(\eta)}$ is bounded on $\mathbf{B}_r := \{\rho \le r\}$ for $r \in (0, \infty)$. So, these functions are bounded on $[k, K]$ for any constants $K > k > 0$. Moreover, define

$$\sigma_k := \sup_{t \ge k} \int_t^\infty \mathrm{e}^{\int_k^r \frac{\xi(s)}{\underline{a}(s)}\mathrm{d}s}\mathrm{d}r \int_k^t \frac{1}{\underline{a}(r)}\mathrm{e}^{-\int_k^r \frac{\xi(s)}{\underline{a}(s)}\mathrm{d}s}\mathrm{d}r, \quad k > 0. \tag{2.6}$$

Obviously, σ_k is non-increasing in k and might be infinite. We will see in Theorem 2.2(1) that under certain conditions $\sigma_k < \infty$ implies the validity of Poincaré inequality.

We have the following extension of Theorem 2.1 to $\mathcal{E}_{\mathcal{A},V}$. When $\text{supp}\,\nu$ is finite the model reduces to finite-dimensional diffusions, for which one may derive super Poincaré inequalities by making perturbations to (2.1). As the present study mainly focusses on the infinite-dimensional model, we exclude this case in the following result.

Theorem 2.2. Assume **(H)** and (1.8). Suppose that $\underline{a}(r)^{-1}$ is locally bounded in $r \in [0, \infty)$ and

$$\psi(s) := \int_0^s [\bar{a}(r)]^{-1/2}\,\mathrm{d}r \uparrow \infty \text{ as } s \uparrow \infty. \tag{2.7}$$

Then the following assertions hold.

(1) If $\lim_{k\to\infty} \sigma_k < \infty$ (equivalently, $\sigma_k < \infty$ for all $k > 0$), then

$$\mathrm{gap}(\mathcal{L}_{\mathcal{A},V}) \geq \sup\left\{\frac{1}{2\Phi\big(\psi^{-1}(\psi(k) + 32\sigma_k + 1)\big) + 32\sigma_k} : k > 0\right\} > 0,$$

where

$$\Phi(N) := \left(1 \vee \frac{N^2}{4\nu(E)}\right)\exp\left[\sup_{\rho \leq N} V - \inf_{\rho \leq N} V\right]\sup_{r \leq N} \underline{a}(r)^{-1}, \quad N > 0.$$

(2) If $\mathrm{supp}\,\nu$ contains infinitely many points, then $\mathcal{E}_{\mathcal{A},V}$ does not satisfy the super Poincaré inequality.

(3) The weak Poincaré inequality (1.10) holds for

$$\alpha(r) := \inf\left\{2\Phi(N) : \mathcal{G}^V(\rho > N) \leq \frac{r}{1+r}\right\}, \quad r > 0.$$

The following result shows that the condition in Theorem 2.2(1) is sharp when $\mathcal{A}_\eta$ and $V(\eta)$ depend only on $\rho(\eta)$.

Corollary 2.3. Assume **(H)** and (1.8). Let $V(\eta) = v(\rho(\eta))$ and $\mathcal{A}_\eta = a(\rho(\eta))\mathbf{1}$ for large $\rho(\eta)$ and some $a, v \in C^1([0, \infty))$ with $\underline{a}(r) > 0$ for $r \geq 0$. Then

$$\xi(r) := \sup_{\rho(\eta)=r} \mathcal{L}_{\mathcal{A},V}\rho(\eta) = a(r)\left(\frac{1}{r} + v'(r) - \frac{r}{2}\right) + \frac{r}{2}a(r), \quad \text{for large } r > 0,$$

and $\mathrm{gap}(\mathcal{L}_{\mathcal{A},V}) > 0$ if and only if $\lim_{k\to\infty}\sigma_k < \infty$.

As in the proof of [14, Corollary 1.3] using [14, Theorem 1.1], it is easy to see that Theorem 2.2(2) implies the following result.

Corollary 2.4. Assume **(H)** and (1.8). If $\inf_{r \geq 0} \underline{a}(r) > 0$ and $\limsup_{r\to\infty} \frac{\xi(r)}{a(r)} < 0$, then $\mathrm{gap}(\mathcal{L}_{\mathcal{A},V}) > 0$.

The above two corollaries are concerned with the validity of Poincaré inequality. On the other hand, according to Theorem 2.2(3), the weak Poincaré inequality always holds under **(H)**, (1.8) and (2.7). We will see in the proof that the rate function α is derived by comparing $\mathcal{E}_{\mathcal{A},V}$ with $\mathcal{E}_{1,0}$ on bounded sets $\mathbf{B}_N$, $N > 0$. However, when these two Dirichlet forms are far away, this α is less sharp. As a principle, to derive a sharper weak Poincaré inequality, one should compare $\mathcal{E}_{\mathcal{A},V}$ with a closer Dirichlet form which satisfies the Poincaré inequality. In this spirit, we present below an alternative result on the weak Poincaré inequality. To state the result, we introduce the class $\mathcal{H}$ as follows.

Class $\mathcal{H}$ We denote $h \in \mathcal{H}$, if $0 \leq h \in C^1([0, \infty))$ with $h'(r) > 0$ for $r > 0$, such that

$$\xi_h(r) := \xi(r) - \frac{2}{r}h(r)\inf_{\rho(\eta)=r}\eta(\mathcal{A}_\eta \mathbf{1}), \quad r > 0 \tag{2.8}$$

satisfies

$$\sigma_{1,h} := \sup_{t \geq 1}\int_t^\infty e^{\int_1^r \frac{\xi_h(s)}{\underline{a}(s)}\mathrm{d}s}\mathrm{d}r\int_1^t \frac{1}{\underline{a}(r)}e^{-\int_1^r \frac{\xi_h(s)}{\underline{a}(s)}\mathrm{d}s}\mathrm{d}r < \infty. \tag{2.9}$$

It is easy to see that $\mathcal{H} \neq \emptyset$ under the conditions of Theorem 2.2 and $\inf \underline{a} > 0$. For any $h \in \mathcal{H}$, let $V_h = V - h(\rho) + c(h)$, where $c(h) \in \mathbb{R}$ is such that $\mathcal{G}^{V_h}$ is a probability measure on $\mathbb{M}$. By Theorem 2.2(1) with $k = 1$, for any $h \in \mathcal{H}$, the Poincaré inequality

$$\mathcal{G}^{V_h}(F^2) \leq C(h)\mathcal{E}_{\mathcal{A},V_h} + \mathcal{G}^{V_h}(F)^2, \quad F \in \mathcal{D}(\mathcal{E}_{\mathcal{A},V_h}) \tag{2.10}$$

holds for

$$C(h) := 2\Phi_{1,h}\big(\psi^{-1}(\psi(1) + 32\sigma_{1,h} + 1)\big) + 32\sigma_{1,h}, \quad h \in \mathcal{H}. \tag{2.11}$$

Theorem 2.5. Assume **(H)**, (1.8) and (2.7). If $\mathcal{H} \neq \emptyset$, then (1.10) holds for

$$\alpha(r) := \inf \left\{ C(h) \mathrm{e}^{h(N)} : \ h \in \mathcal{H}, \ N > 0 \text{ with } \mathcal{G}^V(\rho > N) \leq \frac{r}{1+r} \right\}, \ \ r > 0,$$

where $C(h)$ is given by (2.9) and (2.11).

To conclude this section, we present below a simple example to illustrate the main results. For simplicity, we only consider $\mathcal{A}_\eta = \mathbf{1}$. But by a simple comparison argument, the assertions apply also to $\mathcal{A}_\eta$ with $\langle \mathcal{A}_\eta \phi, \phi \rangle_{L^2(\eta)} \geq c \|\phi\|^2_{L^2(\eta)}$ for some constant $c > 0$ and all $\eta \in \mathbb{M}$, $\phi \in L^2(\eta)$.

Example 2.6. Consider the following potential V_0 with interactions given by $\psi_i \in \mathcal{B}_b(E \times E), i = 1, 2, 3$:

$$V_0(\eta) = \frac{2(\eta \times \eta)(\psi_1)}{3\eta(E)^{3/2}} + \frac{(\eta \times \eta)(\psi_2)}{\eta(E)} + (\eta \times \eta)(\psi_3) - p \log(1 + \eta(E)),$$

where $p \in \mathbb{R}$ is a constant. Let $\theta_i = \sup \psi_i, 1 \leq i \leq 3$. Assume that one of the following conditions hold:

(1) $\min \left\{ \theta_3, \theta_2 - 1, \theta_1 \cdot 1_{\{\theta_2 = 1\}} \right\} < 0$;

(2) $\theta_1 = \theta_2 - 1 = \theta_3 = 0$ and $p > \nu(E)$.

Then $Z := \mathcal{G}(\mathrm{e}^{V_0}) \leq \frac{1}{\Gamma(\nu(E))} \int_0^\infty (1+s)^{-p} s^{\nu(E)-1} \mathrm{e}^{\theta_1 s^{1/2} - (1-\theta_2)s + \theta_3 s^2} \, \mathrm{d}s < \infty$, so that $\mathcal{G}^V$ for $V := V_0 - \log Z$ is a probability measure on $\mathbb{M}$, and the following assertions hold:

(a) Condition (1) implies $\mathrm{gap}(\mathcal{L}_{\mathbf{1},V}) > 0$;

(b) Under condition (2), let

$$\theta = \max \left\{ 12 \times 1_{\{\|\psi_3\|_\infty > 0\}}, \ 8 \times 1_{\{\|\psi_2 - 1\|_\infty > 0\}}, \ 6 \times 1_{\{\|\psi_1\|_\infty > 0\}}, \ 5 \right\}.$$

Then there exists a constant $c > 0$ such that the weak Poincaré inequality (1.10) holds for

$$\alpha(r) = c r^{-\frac{\theta}{2(p-\nu(E))}}, \ \ r > 0.$$

Proof. Obviously, the assumptions in Theorem 2.2 hold for V and $\mathcal{A}_\eta = \mathbf{1}$. By definition it is easy to see that

$$\nabla^{ext} V(\eta)(x) = \frac{\eta(E)\eta(\psi_1(x,\cdot) + \psi_1(\cdot,x)) - (\eta \times \eta)(\psi_1)}{\eta(E)^{5/2}} + \eta(\psi_3(x,\cdot) + \psi_3(\cdot,x))$$
$$+ \frac{\eta(E)\eta(\psi_2(x,\cdot) + \psi_2(\cdot,x)) - (\eta \times \eta)(\psi_2)}{\eta(E)^2} + \frac{p}{1 + \eta(E)}.$$

Then

$$\nabla_1^{ext} V(\eta) := \eta\big(\nabla^{ext} V(\eta)\big) \leq \theta_1 \sqrt{\eta(E)} + \theta_2 \eta(E) + \theta_3 \eta(E)^2 + \frac{p\eta(E)}{1 + \eta(E)}$$
$$= \frac{\theta_1 \rho(\eta)}{2} + \frac{\theta_2 \rho(\eta)^2}{4} + \frac{\theta_3 \rho(\eta)^4}{8} + \frac{p\rho(\eta)^2}{4 + \rho(\eta)^2}.$$

(a) If (1) holds, then $\theta_3 < 0$, or $\theta_2 < 1$, or $\theta_3 = \theta_2 - 1 = 0$ and $\theta_1 < 0$. In any case, we have

$$\limsup_{\rho(\eta) \to \infty} \mathcal{L}_{\mathbf{1},V} \rho(\eta) = \limsup_{\rho(\eta) \to \infty} \frac{2}{\rho(\eta)} \Big(\nu(E) - \frac{\rho(\eta)^2}{4} + \nabla_1^{ext} V(\eta)\Big) < 0,$$

so that Corollary 2.4 implies $\mathrm{gap}(\mathcal{L}_{\mathbf{1},V}) > 0$.

(b) Under condition (2), we prove the weak Poincaré inequality for the desired $\alpha(r)$. Since one may always take $\alpha(r) \leq 1$ in (1.10) due to $\mathcal{G}^V(F^2) \leq \|F\|_\infty^2$, it suffices to prove for small $r > 0$, say $r \in (0,1]$.

It is easy to see that

$$\mathcal{G}^V(\rho > N) \leq c_0 N^{\nu(E)-p}, \quad N > 0 \tag{2.12}$$

holds for some constant $c_0 > 0$. For $\varepsilon \in (0,1]$, we take $h_\varepsilon(s) = \varepsilon\sqrt{s}$. Since $a = 1$, it is easy to check that

$$\sigma_{1,h_\varepsilon} \leq c_1 \varepsilon^{-2}$$

for some constant $c_1 > 0$ independent of $\varepsilon \in (0,1]$. Moreover, there is a constant c_2 independent of $\varepsilon \in (0,1]$ such that

$$\sup_{\rho \leq N} V_{h_\varepsilon} - \inf_{\rho \leq N} V_{h_\varepsilon} \leq c_2 \big[\|\psi_3\|_\infty N^4 + \|\psi_2 - 1\|_\infty N^2 + \|\psi_1\|_\infty N + \varepsilon N + \log(1+N)\big].$$

Combining this with (2.11), we may find constants $c_3, c_4 > 0$ independent of $\varepsilon \in (0,1]$ such that

$$C(h_\varepsilon) \leq c_3\big(\|\psi_3\|_\infty \varepsilon^{-12} + \|\psi_2 - 1\|_\infty \varepsilon^{-8} + \|\psi_1\|_\infty \varepsilon^{-6} + \varepsilon^{-5}\big) \leq c_4 \varepsilon^{-\theta}.$$

Taking this into account and applying Theorem 2.5 for

$$N = N_r := \left(\frac{2c_0}{r}\right)^{\frac{1}{p-\nu(E)}},$$

such that (2.12) implies $\mathcal{G}^V(\rho > N) \leq \frac{r}{2}$ as required for $r \in (0,1]$, we conclude that the weak Poincaré inequality holds for

$$\alpha(r) := \inf_{\varepsilon \in (0,1]} C(h_\varepsilon) \mathrm{e}^{h_\varepsilon(N_r)} \leq \inf_{\varepsilon \in (0,1]} c_4 \varepsilon^{-\theta} \exp\big[\varepsilon(2c_0 r^{-1})^{\frac{1}{2(p-\nu(E))}}\big], \quad r \in (0,1].$$

Therefore, by taking $\varepsilon = 1 \wedge r^{\frac{1}{2(p-\nu(E))}}$, we prove (1.10) for the desired $\alpha(r)$. $\qquad\square$

3 The Dirichlet form

For any $F \in \mathcal{F}C_0^\infty$, let

$$\begin{aligned}
\mathcal{L}_{\mathcal{A},V}F(\eta) := &\int_E \mathcal{A}_\eta[\nabla^{ext}F(\eta)](x)(\nu - \eta)(\mathrm{d}x) \\
&+ \int_E \nabla^{ext}\big[\mathcal{A}_\eta(\nabla^{ext}F(\eta))(x)\big](x)\,\eta(\mathrm{d}x) + \big\langle\nabla^{ext}V(\eta), \mathcal{A}_\eta[\nabla^{ext}F(\eta)]\big\rangle_{L^2(\eta)}.
\end{aligned} \tag{3.1}$$

It is easy to see from (1.4) that when $F(\eta) = f(\eta(A_1), \cdots, \eta(A_n))$ for some $n \geq 1$, $f \in C_0^\infty(\mathbb{R}^n)$ and a measurable partition $\{A_i\}_{1 \leq i \leq n}$ of E, we have

$$\begin{aligned}
\mathcal{L}_{\mathcal{A},V}F(\eta) = \bigg(&\sum_{i=1}^n \Big[(\nu - \eta)(\mathcal{A}_\eta 1_{A_i}) + \eta\big(\nabla^{ext}[\mathcal{A}_\eta 1_{A_i}(\cdot)](\cdot)\big) + \nabla^{ext}_{\mathcal{A}_\eta 1_{A_i}} V(\eta)\Big]\partial_i f \\
&+ \sum_{i,j=1}^n \eta(1_{A_i}\mathcal{A}_\eta 1_{A_j})(\partial_i\partial_j f)\bigg)(\eta(A_1), \cdots, \eta(A_n)).
\end{aligned}$$

Theorem 3.1. Assume **(H)**. Then

$$\mathcal{E}_{\mathcal{A},V}(F,G) = -\int_{\mathbb{M}}(G\mathcal{L}_{\mathcal{A},V}F)\mathrm{d}\mathcal{G}^V, \quad F, G \in \mathcal{F}C_0^\infty. \tag{3.2}$$

Consequently, $(\mathcal{E}_{\mathcal{A},V}, \mathcal{F}C_0^\infty)$ is closable in $L^2(\mathbb{M}, \mathcal{G}^V)$ whose closure $(\mathcal{E}_{\mathcal{A},V}, \mathcal{D}(\mathcal{E}_{\mathcal{A},V}))$ is a symmetric Dirichlet form with generator $(\mathcal{L}_{\mathcal{A},V}, \mathcal{D}(L_{\mathcal{A},V}))$ being the Friedrichs extension of $(\mathcal{L}_{\mathcal{A},V}, \mathcal{F}C_0^\infty)$. If moreover (1.8) holds, then $1 \in \mathcal{D}(\mathcal{E}_{\mathcal{A},V})$ and $\mathcal{E}_{\mathcal{A},V}(1,1) = 0$.

To prove this result, we introduce the divergence operator corresponding to ∇^{ext}. To this end, we formulate the Gamma distribution $\mathcal{G}$ by using the Poisson measure $\pi_{\hat{\nu}}$ with intensity $\hat{\nu}(\mathrm{d}x, \mathrm{d}s) := s^{-1}\mathrm{e}^{-s}\nu(dx)\mathrm{d}s$ on $\hat{E} := E \times (0, \infty)$. Recall that $\pi_{\hat{\nu}}$ is the unique probability measure on the configuration space

$$\mathbf{\Gamma}(\hat{E}) := \left\{ \gamma = \sum_{i=1}^{\infty} \delta_{(x_i, s_i)} : \ \gamma(K) < \infty \text{ for compact } K \subset \hat{E}, (x_i, s_i) \in \hat{E} \right\}$$

such that for any disjoint relatively compact subsets $\{\hat{A}_i\}_{1 \le i \le n}$ of $\hat{E}$, $\{\gamma \mapsto \gamma(\hat{A}_i)\}_{1 \le i \le n}$ are independent random Poisson random variables with parameters $\{\hat{\nu}(\hat{A}_i)\}_{1 \le i \le n}$. Since $S(\gamma) := \sum_{i=1}^{\infty} s_i$ for $\gamma = \sum_{i=1}^{\infty} s_i \delta_{x_i} \in \mathbf{\Gamma}(\hat{E})$ satisfies

$$\int_{\mathbf{\Gamma}(\hat{E})} S(\gamma) \pi_{\hat{\nu}}(\mathrm{d}\gamma) = \int_{\hat{E}} s\hat{\nu}(\mathrm{d}x, \mathrm{d}s) = \nu(E) < \infty,$$

the measure $\pi_{\hat{\nu}}$ is concentrated on the S-finite configuration space

$$\mathbf{\Gamma}_f(\hat{E}) := \left\{ \gamma = \sum_{i=1}^{\infty} \delta_{(x_i, s_i)} \in \mathbf{\Gamma}(\hat{E}) : S(\gamma) := \sum_{i=1}^{\infty} s_i < \infty \right\}.$$

Lemma 3.2. The map $\Phi : \mathbf{\Gamma}_f(\hat{E}) \ni \gamma = \sum_{i=1}^{\infty} s_i \delta_{x_i} \mapsto \sum_{i=1}^{\infty} s_i \delta_{x_i} \in \mathbb{M}$ is measurable with

$$\mathcal{G} = \pi_{\hat{\nu}} \circ \Phi^{-1}. \tag{3.3}$$

Moreover,

$$\begin{aligned}
&\int_{\mathbb{M}} \mathcal{G}(\mathrm{d}\eta) \int_E F(\eta, x) \eta(\mathrm{d}x) \\
&= \int_{\mathbb{M}} \mathcal{G}(\mathrm{d}\eta) \int_{\hat{E}} \mathrm{e}^{-s} F(\eta + s\delta_x, x) \nu(\mathrm{d}x)\mathrm{d}s, \quad F \in L^1\big(\mathbb{M} \times E, \mathcal{G}(\mathrm{d}\eta)\eta(\mathrm{d}x)\big).
\end{aligned} \tag{3.4}$$

Proof. Formula (3.3) was proved in [8, Theorem 6.2] for $E = \mathbb{R}^d$ and $\nu(\mathrm{d}x) = \theta\mathrm{d}x$ (which is an infinite measure) with $\theta > 0$, by identifying the Laplace transforms of $\mathcal{G}$ and $\pi_{\hat{\nu}} \circ \Phi^{-1}$. Below we explain that the same argument works to the present setting.

Firstly, the Laplace transform of $\mathcal{G}$ is

$$\int_{\mathbb{M}} \mathrm{e}^{-\eta(h)} \mathcal{G}(\mathrm{d}\eta) = \mathrm{e}^{-\nu(\log(1+h))}, \quad h \in \mathcal{B}^+(E), \tag{3.5}$$

where $\mathcal{B}^+(E)$ is the class of nonnegative measurable functions on E. This was given by [18, (7)] when ν is atomless. In general, we decompose ν into $\nu = \nu_0 + \sum_{i=1}^{\infty} c_i \delta_{x_i}$, where ν_0 is an atomless finite measure on E, $x_i \in E$ with $x_i \neq x_j$ for $i \neq j$, and $c_i \ge 0$ with $\sum_{i=1}^{\infty} c_i < \infty$. Let $E_0 = E \setminus \{x_i : i \ge 1\}$. By the definition of Gamma distribution,

$$\eta(h \cdot 1_{E_0}), \ \ \eta(h \cdot 1_{\{x_i\}}), \quad i \ge 1$$

are independent under $\mathcal{G}$, the distribution of $\eta(h \cdot 1_{E_0})$ under $\mathcal{G}$ coincides with that under $\mathcal{G}_0$ (the Gamma distribution with intensity measure ν_0), and the distribution of $\eta(\{x_i\})$ under $\mathcal{G}$ coincides with the one-dimensional Gamma distribution γ_{c_i} with shape parameter c_i. So, applying (3.5) for ν_0 replacing ν due to [18, (7)], and using the Laplace transform for Gamma distributions on $\mathbb{R}_+$, we derive

$$\int_{\mathbb{M}} \mathrm{e}^{-\eta(h)} \mathcal{G}(\mathrm{d}\eta) = \left(\int_{\mathbb{M}} \mathrm{e}^{-\eta(h \cdot 1_{E_0})} \mathcal{G}(\mathrm{d}\eta) \right) \cdot \prod_{i=1}^{\infty} \int_{\mathbb{M}} \mathrm{e}^{-h(x_i)\eta(\{x_i\})} \mathcal{G}(\mathrm{d}\eta)$$

$$= \mathrm{e}^{-\nu_0(\log(1+h))} \cdot \prod_{i=1}^{\infty} \mathrm{e}^{-c_i \log(1+h(x_i))} = \mathrm{e}^{-\nu(\log(1+h))}.$$

Therefore, (3.5) holds.

On the other hand, the Laplace transform for $\pi_{\hat{\nu}}$ (see for instance [1]) is

$$\int_{\boldsymbol{\Gamma}_{pf}(\hat{E})} \mathrm{e}^{-\gamma(\hat{h})} \pi_{\hat{\nu}}(\mathrm{d}\gamma) = \exp\big[-\hat{\nu}(1-\mathrm{e}^{-\hat{h}})\big], \quad \hat{h} \in \mathcal{B}^+(\hat{E}).$$

By letting $\hat{h}(x,s) = sh(x)$ for $(x,s) \in \hat{E}$, we arrive at

$$\int_{\mathbb{M}} \mathrm{e}^{-\eta(h)} (\pi_{\hat{\nu}} \circ \Phi^{-1})(\mathrm{d}\eta) = \int_{\boldsymbol{\Gamma}_{pf}(\hat{E})} \mathrm{e}^{-\gamma(\hat{h})} \pi_{\hat{\nu}})(\mathrm{d}\gamma)$$
$$= \exp\big[-\hat{\nu}\big(1-\mathrm{e}^{-\hat{h}}\big)\big] = \mathrm{e}^{-\nu(\log(1+h))}, \quad h \in \mathcal{B}^+(E).$$

Combining this with (3.5) we prove (3.3).

Finally, (3.4) follows from (3.3) and the Mecke formula [10, Satz 3.1] for Poisson measures. $\qquad\square$

To establish the integration by parts formula for $\nabla_\phi^{ext} F$, we introduce the divergence operator div^{ext} as follows.

Let $\phi : \mathbb{M} \times E \to \mathbb{R}$ be measurable. If for any $x \in E$, $\phi(\cdot, x) \in \mathcal{D}(\nabla^{ext})$ such that

$$(\mathcal{G} \times \nu)(|\phi|) + \int_{\mathbb{M}} \eta\big(|\phi(\eta, \cdot)| + |\nabla^{ext}\phi(\eta, \cdot)(\cdot)|\big) \mathcal{G}(\mathrm{d}\eta) < \infty,$$

where $\eta(\cdot)$ stands for the integral with respect to η as in (1.1), then we write $\phi \in \mathcal{D}(\mathrm{div}^{ext})$ and denote

$$\mathrm{div}^{ext}(\phi)(\eta) = (\eta - \nu)\big(\phi(\eta, \cdot)\big) - \eta\big(\nabla^{ext}\phi(\eta, \cdot)(\cdot)\big). \tag{3.6}$$

When $\phi(\eta, x) = \phi(x)$ does not depend on η, the following integration by parts formula follows from [9, Theorem 14]. We include below a complete proof for the η-dependent ϕ.

Lemma 3.3. Let $\phi \in \mathcal{D}(\mathrm{div}^{ext})$. Then

$$\int_{\mathbb{M}} (\nabla_\phi^{ext} F) \, \mathrm{d}\mathcal{G} = \int_{\mathbb{M}} [F \mathrm{div}^{ext}(\phi)] \, \mathrm{d}\mathcal{G}, \quad F \in \mathcal{F}C_0^\infty. \tag{3.7}$$

Proof. By (3.4) and the Dominated Convergence Theorem, we obtain

$$\int_{\mathbb{M}} (\nabla_\phi^{ext} F) \, \mathrm{d}\mathcal{G} = \int_{\mathbb{M} \times E} \left(\lim_{\varepsilon \downarrow 0} \frac{F(\eta + \varepsilon\delta_x) - F(\eta)}{\varepsilon} \right) \phi(\eta, x) \, \eta(\mathrm{d}x) \mathcal{G}(\mathrm{d}\eta)$$
$$= \int_{\mathbb{M}} \mathcal{G}(\mathrm{d}\eta) \lim_{\varepsilon \downarrow 0} \int_{\hat{E}} \frac{1}{\varepsilon} \mathrm{e}^{-s} \big[F(\eta + (s+\varepsilon)\delta_x) - F(\eta + s\delta_x) \big] \phi(\eta + s\delta_x, x) \, \nu(\mathrm{d}x)\mathrm{d}s$$
$$= \int_{\mathbb{M}} \mathcal{G}(\mathrm{d}\eta) \int_{\hat{E}} \mathrm{e}^{-s} \big[\partial_s F(\eta + s\delta_x, x) \big] \phi(\eta + s\delta_x, x) \, \nu(\mathrm{d}x)\mathrm{d}s$$
$$= \int_{\mathbb{M}} \mathcal{G}(\mathrm{d}\eta) \int_{\hat{E}} \Big(\partial_s \big[\mathrm{e}^{-s} F(\eta + s\delta_x) \phi(\eta + s\delta_x, x) \big] - F(\eta + s\delta_x) \partial_s \big[\mathrm{e}^{-s} \phi(\eta + s\delta_x, x) \big] \Big) \, \nu(\mathrm{d}x)\mathrm{d}s.$$

Noting that $F \in \mathcal{F}C_0^\infty$ implies $F(\eta + s\delta_x) = 0$ for large s, we have

$$\int_0^\infty \partial_s \big[\mathrm{e}^{-s} F(\eta + s\delta_x) \phi(\eta + s\delta_x, x) \big] \, \mathrm{d}s = -F(\eta)\phi(\eta, x).$$

Hence, by using (3.4) again,

$$
\int_{\mathbb{M}} (\nabla^{ext}_\phi F) \, \mathrm{d}\mathcal{G} + \int_{\mathbb{M}} F(\eta)\nu(\phi(\eta, \cdot)) \, \mathcal{G}(\mathrm{d}\eta)
$$

$$
= -\int_{\mathbb{M}} \mathcal{G}(\mathrm{d}\eta) \int_{\hat{E}} F(\eta + s\delta_x)\mathrm{e}^{-s} \big[\partial_s \phi(\eta + s\delta_x, x) - \phi(\eta + s\delta_x, x) \big] \nu(\mathrm{d}x)\mathrm{d}s
$$

$$
= \int_{\mathbb{M}} \mathcal{G}(\mathrm{d}\eta) \int_{\hat{E}} \big[\phi(\eta + s\delta_x, x) - \nabla^{ext}\phi(\cdot, x)(\eta + s\delta_x)(x) \big] \mathrm{e}^{-s} F(\eta + s\delta_x)\nu(\mathrm{d}x)\mathrm{d}s
$$

$$
= \int_{\mathbb{M}} F(\eta)\mathcal{G}(\mathrm{d}\eta) \int_E [\phi(\eta, x) - \nabla^{ext}\phi(\eta, x)(x)]\eta(\mathrm{d}x).
$$

Therefore, (3.7) holds. $\qquad\square$

Proof of Theorem 3.1. We first prove (3.2), which implies the closability of $(\mathcal{E}_{\mathcal{A},V}, \mathcal{F}C_0^\infty)$ and that the closure is a symmetric Dirichlet form in $L^2(\mathcal{G}^V)$, see [4]. By the definition of $\mathcal{E}_{\mathcal{A},V}$ and Lemma 3.3, for any $F, G \in \mathcal{F}C_0^\infty$ we have

$$
\mathcal{E}_{\mathcal{A},V}(F, G) = \int_{\mathbb{M}} \Gamma_{\mathcal{A}}(F, G)\mathrm{d}\mathcal{G}^V = \int_{\mathbb{M}} \big(\nabla^{ext}_{\mathrm{e}^{V(\eta)}\mathcal{A}_\eta \nabla^{ext} F(\eta)} G \big)(\eta) \, \mathcal{G}(\mathrm{d}\eta)
$$

$$
= \int_{\mathbb{M}} G(\eta)\mathrm{div}^{ext}\big(\mathrm{e}^{V(\eta)}\mathcal{A}_\eta[\nabla^{ext} F(\eta)](\cdot) \big) \, \mathcal{G}(\mathrm{d}\eta).
$$

Therefore, by (3.6), (3.2) holds for

$$
\mathcal{L}_{\mathcal{A},V} F(\eta) := -\mathrm{e}^{-V(\eta)}\mathrm{div}^{ext}\big(\mathrm{e}^{V(\eta)}\mathcal{A}_\eta[\nabla^{ext} F(\eta)](\cdot) \big)
$$

$$
= \int_E \Big([\nabla^{ext} V(\eta)(x)]\mathcal{A}_\eta[\nabla^{ext} F(\eta)](x) + \nabla^{ext}\big(\mathcal{A}_\eta[\nabla^{ext} F(\eta)](x))(x) \Big) \eta(\mathrm{d}x)
$$

$$
+ \int_E \mathcal{A}_\eta[\nabla^{ext} F(\eta)](x) \, (\nu - \eta)(\mathrm{d}x).
$$

Next, assume that (1.8) holds. It remains to find a sequence $\{F_n\}_{n \geq 1} \subset \mathcal{D}(\mathcal{E}_{\mathcal{A},V})$ such that

$$
\lim_{n \to \infty} \big[\mathcal{G}^V(|F_n - 1|^2) + \mathcal{E}_{\mathcal{A},V}(F_n, F_n) \big] = 0.
$$

To this end, we consider $\rho_n := \sqrt{n^{-1} + \rho^2}$, $n \geq 1$. By (2.3), we have $\rho_n \in \mathcal{D}(\nabla^{ext})$ with

$$
\Gamma_{\mathbf{1}}(\rho_n, \rho_n) = \frac{\rho^2}{\rho_n^2} \leq 1.
$$

Let $h \in C_0^\infty([0, \infty))$ such that $h(r) = 1$ for $r \leq 1$ and $h(r) = 0$ for $r \geq 2$. We have

$$
F_n := h\big(n^{-1} \log[1 + \rho_n] \big) \subset \mathcal{F}C_0^\infty, \quad n \geq 1.
$$

It is easy to see that $\mathcal{G}^V(|F_n - 1|^2) \to 0$ as $n \to \infty$ and due to (1.8),

$$
\limsup_{n \to \infty} \mathcal{E}_{\mathcal{A},V}(F_n, F_n) \leq \limsup_{n \to \infty} \int_{\mathbb{M}} \frac{\|\mathcal{A}_\eta\|_{L^2(\eta)}\|h'\|_\infty^2}{n^2(1 + \rho)^2} \, \mathcal{G}^V(\mathrm{d}\eta) = 0. \qquad\square
$$

4 Proofs of the main results

In this section, we prove Theorems 2.1, 2.2, 2.5 and Corollary 2.3.

4.1 Proof of Theorem 2.1 and a local Poincaré inequality

Proof of Theorem 2.1. The invalidity of the super Poincaré inequality will be included in the proof of Theorem 2.2(3) for a more general case. So, we only prove (1) and (3).

(a) We first prove $\mathrm{gap}(\mathcal{L}_{\mathbf{1},0}) = 1$, i.e. $\lambda = 1$ is the optimal constant for the Poincaré inequality

$$\mathcal{G}(F^2) \leq \frac{1}{\lambda} \mathcal{E}_{\mathbf{1},0}(F, F) + \mathcal{G}(F)^2, \quad F \in \mathcal{F}C_0^\infty \tag{4.1}$$

to hold. Let $F(\eta) = f(\eta(A_1), \cdots, \eta(A_n))$ for some $f \in C_0^\infty(\mathbb{R}^n)$ and disjoint $A_1, \cdots, A_n$. This Poincaré inequality reduces to

$$\mu^n(f^2) - \mu^n(f)^2 \leq \mu^n\Big(\sum_{i=1}^n x_i |\partial_i f(x_1, \cdots, x_n)|^2 \Big),$$

where according to (1.2),

$$\mu^n(\mathrm{d}x) := \prod_{i=1}^n \mu_i(\mathrm{d}x_i), \quad \mu_i(\mathrm{d}s) = \gamma_{\nu(A_i)}(\mathrm{d}s) := 1_{[0,\infty)}(s) \frac{s^{\nu(A_i)-1} \mathrm{e}^{-s}}{\Gamma(\nu(A_i))} \mathrm{d}s, \ 1 \leq i \leq n. \tag{4.2}$$

By the additive property of the Poincaré inequality, it suffices to prove that for every $1 \leq i \leq n$, $\lambda = 1$ is the largest constant satisfying

$$\mu_i(f^2) - \mu_i(f)^2 \leq \frac{1}{\lambda} \int_0^\infty r f'(r)^2 \mu_i(\mathrm{d}r), \quad f \in C_0^\infty([0,\infty)).$$

This follows from the fact that the generator of the Dirichlet form

$$\mathcal{E}_i(f, g) := \int_0^\infty r f'(r) g'(r) \mu_i(\mathrm{d}r), \quad f, g \in W^{1,2}([0,\infty), \mu_i)$$

is

$$\mathcal{L}_i f(r) := r f''(r) + (\nu(A_i) - r) f'(r), \quad r \in [0, \infty),$$

which has spectral gap 1 with the first eigenfunction $u_i(r) = r - \nu(A_i)$.

(b) Let $\mathrm{supp}\, \nu = \{x_1, \cdots, x_n\}$, we have $\delta = \min\{\nu(\{x_i\}) : 1 \leq i \leq n\} > 0$. It suffices to find a universal constant $c_0 > 0$ such that (2.1) holds for

$$F(\eta) := f(\eta(\{x_1\}), \cdots, \eta(\{x_n\})), \quad f \in C_0^\infty(\mathbb{R}^n).$$

Letting μ^n and μ_i be as in (4.2) for $A_i = \{x_i\}$, (2.1) for this F becomes

$$\mu^n(f^2 \log f^2) \leq \frac{c_0}{1 \wedge \delta} \sum_{i=1}^n \int_{[0,\infty)^n} s_i (\partial_i f)^2 (s_1, \cdots, s_n) \mu^n(\mathrm{d}s_1, \cdots, \mathrm{d}s_n) + \mu^n(f^2) \log \mu^n(f^2).$$

By the additive property of the log-Sobolev inequality, this follows from the following Lemma 4.1. □

Lemma 4.1. For any $a, b > 0$, let $\mu_a(\mathrm{d}s) := 1_{[0,\infty)}(s) \frac{s^{a-1} \mathrm{e}^{-s}}{\Gamma(a)} \mathrm{d}s$ and $\mu_{a,b}(\mathrm{d}s) := 1_{[0,b]}(s) \frac{\mu_a(\mathrm{d}s)}{\mu_a([0,b])}$. Then there exists a constant $c_0 > 0$ such that for any $a, b > 0$,

$$\mu_{a,b}(f^2 \log f^2) \leq \frac{c_0}{a \wedge 1} \int_0^b s f'(s)^2 \mu_{a,b}(\mathrm{d}s), \quad f \in C^1([0,b]), \mu_{a,b}(f^2) = 1. \tag{4.3}$$

Proof. (a) Let $a \geq 2$. We will use the Bakry–Émery criterion on Riemannian manifolds with convex boundary which in particular includes $[0, b]$ for $b > 0$. More precisely, let

$\mathcal{L}_a f(s) = s f''(s) + (a - s)f'(s)$ and $\Gamma_1(f, g)(s) = s f'(s)g'(s)$. By [25, Theorem 1.1(4)] with $\sigma = 0$ and $t \to \infty$, if

$$\Gamma_2(f, f) := \frac{1}{2}\mathcal{L}_a \Gamma_1(f, f) - \Gamma_1(\mathcal{L}_a f, f) \geq K\Gamma_1(f, f)$$

holds for some constant $K > 0$ and all $f \in C^2([0, b])$, then

$$\mu_{a,b}(f^2 \log f^2) \leq \frac{2}{K}\int_0^b s f'(s)^2 \mu_{a,b}(\mathrm{d}s), \quad f \in C^1([0, b]), \mu_{a,b}(f^2) = 1.$$

So, the desired inequality (4.3) with $c_0 = 4$ follows since

$$\begin{aligned}
\Gamma_2(f, f)(s) &= s^2 f''(s)^2 + \frac{a + s}{2}f'(s)^2 + 2s f'(s)f''(s) \\
&\geq \frac{a + s - 2}{2s}\Gamma_1(f, f)(s), \quad s \geq 0,
\end{aligned} \tag{4.4}$$

so that $\Gamma_2(f, f) \geq \frac{1}{2}\Gamma_1(f, f)$ when $a \geq 2$.

(b) Let $a \in (0, \frac{1}{2}]$. By (4.4) we have $\Gamma_2(f, f)(s) \geq \frac{a \wedge 2}{4}\Gamma_1(f, f)(s)$ for $s \geq 2$. So, by the Bakry–Émery criterion,

$$\mu_{a,b_1}(1_{[2,b_1]}f^2 \log f^2) \leq \frac{8}{a \wedge 2}\mu_{a,b_1}(1_{[2,b_1]}\Gamma_1(f, f)) + \mu_{a,b_1}(1_{[2,b_1]}f^2)\log \mu_{a,b_1}(1_{[2,b_1]}f^2) \tag{4.5}$$

holds for any $b_1 > 2$ and all $f \in C^1([0, b_1])$.

On the other hand, for any $b_2 > 0$ and $f \in C^1([0, b_2])$ with $\mu_{a,b_2}(f) = 0$, there exists $r_0 \in [0, b_2]$ such that $f(r_0) = 0$. So, for any $r \in [0, b_2]$ we have

$$\begin{aligned}
|f(r)| = \left|\int_{r_0}^r f'(s)\mathrm{d}s\right| &\leq \left(\int_0^{b_2} s f'(s)^2 \mu_{a,b_2}(\mathrm{d}s)\right)^{\frac{1}{2}}\left(\int_0^{b_2} s^{-a}\mathrm{e}^s\Gamma(a)\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq \left(\frac{\Gamma(a)b_2^{1-a}\mathrm{e}^{b_2}}{1 - a}\mu_{a,b_2}(\Gamma_1(f, f))\right)^{\frac{1}{2}}, \quad r \in [0, b_2].
\end{aligned}$$

Therefore, for $\mu_{a,b_2}(f^2) = 1$ with $\mu_{a,b_2}(f) = 0$ we have

$$\begin{aligned}
\mu_{a,b_2}(f^2 \log f^2) &\leq \mu_{a,b_2}(f^2)\log\left[\frac{\Gamma(a)b_2^{1-a}\mathrm{e}^{b_2}}{1 - a}\mu_{a,b_2}(\Gamma_1(f, f))\right] \\
&\leq \frac{\Gamma(a)b_2^{1-a}\mathrm{e}^{b_2}}{1 - a}\mu_{a,b_2}(\Gamma_1(f, f)) - 1.
\end{aligned}$$

This implies

$$\begin{aligned}
&\mu_{a,b_2}(f^2 \log f^2) - \mu_{a,b_2}(f^2)\log \mu_{a,b_2}(f^2) \\
&\leq \frac{\Gamma(a)b_2^{1-a}\mathrm{e}^{b_2}}{1 - a}\mu_{a,b_2}(\Gamma_1(f, f)) - \mu_{a,b_2}(f^2), \quad f \in C^1([0, b_2]), \ \mu_{a,b_2}(f) = 0.
\end{aligned} \tag{4.6}$$

In general, for a non-zero function $f \in C^1([0, b_2])$, let $\tilde{f} = f - \mu_{a,b_2}(f)$. We have (see [2])

$$\begin{aligned}
&\mu_{a,b_2}(f^2 \log f^2) - \mu_{a,b_2}(f^2)\log \mu_{a,b_2}(f^2) \\
&\leq \mu_{a,b_2}(\tilde{f}^2 \log \tilde{f}^2) - \mu_{a,b_2}(\tilde{f}^2)\log \mu_{a,b_2}(\tilde{f}^2) + 2\mu_{a,b_2}(\tilde{f}^2).
\end{aligned} \tag{4.7}$$

Combining this with (4.6) and using the Poincaré inequality (4.15) below, we arrive at

$$\begin{aligned}
&\mu_{a,b_2}(f^2 \log f^2) - \mu_{a,b_2}(f^2)\log \mu_{a,b_2}(f^2) \\
&\leq \left(\frac{\Gamma(a)b_2^{1-a}\mathrm{e}^{b_2}}{1 - a} + 1\right)\mu_{a,b_2}(\Gamma_1(f, f)), \quad b_2 > 0, f \in C_b^1([0, b_2]).
\end{aligned} \tag{4.8}$$

In conclusion, when $b \leq 4$, the desired inequality (4.3) for $a \in (0, \frac{1}{2}]$ follows from (4.8). Finally, for $b \geq 4$ we deduce from (4.5) and (4.8) that for any $f \in C^1([0, b])$ with $\mu_{a,b}(f^2) = 1$,

$$\mu_{a,b}(f^2 \log f^2) = \frac{\int_0^2 s^{a-1} \mathrm{e}^{-s} \mathrm{d}s}{\int_0^b s^{a-1} \mathrm{e}^{-s} \mathrm{d}s} \mu_{a,2}(f^2 \log f^2) + \mu_{a,b}(1_{[2,b]} f^2 \log f^2)$$

$$\leq \Big(\frac{\Gamma(a) 2^{1-a} \mathrm{e}^2}{1-a} + 1 \Big) \mu_{a,b}(1_{[0,2]} \Gamma_1(f, f)) + \mu_{a,b}(1_{[0,2]} f^2) \log \frac{\Gamma(a)}{\int_0^2 s^{a-1} \mathrm{e}^{-s} \mathrm{d}s}$$

$$+ \frac{8}{a \wedge 2} \mu_{a,b}(1_{[2,b]} \Gamma_1(f, f)) + \mu_{a,b}(1_{[2,b]} f^2) \log \frac{\Gamma(a)}{\int_2^b s^{a-1} \mathrm{e}^{-s} \mathrm{d}s}$$

$$\leq \frac{c_1}{a} \int_0^b s f'(s)^2 \, \mu_{a,b}(\mathrm{d}s) + \frac{c_1}{a} \mu_{a,b}(f^2),$$

where $c_1 > 0$ is a universal constant independent of $a \in (0, \frac{1}{2}]$ and $b \geq 4$. Combining this with (4.7) and the Poincaré inequality (4.15) below, we prove the inequality (4.3) for some universal constant $c_0 > 0$ and all $a \in (0, \frac{1}{2}]$ and $b \geq 4$.

(c) Let $a \in (\frac{1}{2}, 2)$. In this case, we have $a' := \frac{a}{4} \in (0, \frac{1}{2}]$, so that by (b) there exists a constant $c_0 > 0$ such that

$$\mu_{a',b}(f^2 \log f^2) \leq \frac{c_0}{a} \int_0^b s f'(s)^2 \mu_{a',b}(\mathrm{d}s), \quad a \in \Big(\frac{1}{2}, 2 \Big), f \in C^1([0, b]), \mu_{a',b}(f^2) = 1. \quad (4.9)$$

Let $\bar{\mu}_{a',\infty}(\mathrm{d}s_1, \mathrm{d}s_2, \mathrm{d}s_3, \mathrm{d}s_4) = \prod_{i=1}^4 \mu_{a',\infty}(\mathrm{d}s_i)$, where $\mu_{a',\infty} := \lim_{b \to \infty} \mu_{a',b}$ is the Gamma distribution with parameter a'. By the property of Gamma distributions we have

$$\int_{[0,\infty)^n} f(s_1 + s_2 + s_3 + s_4) \bar{\mu}_{a',\infty}(\mathrm{d}s_1, \mathrm{d}s_2, \mathrm{d}s_3, \mathrm{d}s_4) = \int_{[0,\infty)} f(s) \mu_{a,\infty}(\mathrm{d}s), \quad f \in \mathcal{B}_b([0, \infty)).$$

Using (4.9) with $b \to \infty$ and the additivity property of the log-Sobolev inequality, we obtain

$$\bar{\mu}_{a',\infty}(F^2 \log F^2) - \bar{\mu}_{a',\infty}(F^2) \log \bar{\mu}_{a',\infty}(F^2)$$

$$\leq \frac{c_0}{a} \int_0^b \sum_{i=1}^4 s_i \partial_i F(s_1, \cdots, s_4)^2 \bar{\mu}_{a',b}(\mathrm{d}s_1, \cdots, \mathrm{d}s_4), \quad F \in C_b^1([0, \infty)^4).$$

By an approximation argument we may apply this inequality to

$$F(s_1, \cdots, s_4) := f(b \wedge (s_1 + \cdots + s_4))$$

for $f \in C^1([0, b])$, so that (4.3) is derived. $\qquad \square$

To prove Theorem 2.2, we consider the local Poincaré inequality for $\mathcal{E}_{1,0}$ on the set $\mathbf{B}_N$, by decomposing η into the radial part $\eta(E)$ and the simplicial part $\bar{\eta} := \frac{\eta}{\eta(E)}$. It is well known that under $\mathcal{G}$ these two parts are independent with

$$\mathcal{G}(\eta(E) < r, \bar{\eta} \in \mathbf{A}) = \mathbf{Dir}(\mathbf{A}) \gamma_{\nu(E)}([0, r)), \quad r > 0, \mathbf{A} \in \mathcal{B}(\mathbb{M}_1), \quad (4.10)$$

where $\gamma_{\nu(E)}(\mathrm{d}s) := 1_{[0,\infty)}(s) \frac{s^{\nu(E)-1} \mathrm{e}^{-s}}{\Gamma(\nu(E))} \mathrm{d}s$, and $\mathbf{Dir}$ is the Dirichlet measure with intensity measure ν, see for instance [17] for details. According to [16] (see also [17, Proposition 3.3]), we have the Poincaré inequality

$$\mathbf{Dir}(F^2) \leq \mathbf{Dir}(\Gamma^D(F, F)) + \mathbf{Dir}(F)^2, \quad F \in \mathcal{F}C_0^\infty, \quad (4.11)$$

where for $F(\eta) = f(\eta(A_1), \cdots, \eta(A_n))$ and $\eta \in \mathbb{M}_1$,

$$\Gamma^D(F, F)(\eta) := \sum_{i,j=1}^{n} \left[\delta_{ij}\eta(A_i) - \eta(A_i)\eta(A_j)\right] \cdot \left[(\partial_i f)(\partial_j f)\right](\eta(A_1), \cdots, \eta(A_n)). \quad (4.12)$$

Lemma 4.2. For any $N > 0$,

$$\mathcal{G}(1_{\mathbf{B}_N} F^2) \leq \left(\frac{N^2}{4\nu(E)} \vee 1\right)\mathcal{G}(1_{\mathbf{B}_N}\Gamma_{\mathbf{1}}(F, F)), \quad F \in \mathcal{F}C_0^\infty, \mathcal{G}(1_{\mathbf{B}_N}F) = 0. \quad (4.13)$$

Proof. Since $\mathbf{B}_N = \{\eta(E) \leq N^2/4\}$, (4.10) implies

$$\int_{\mathbb{M}} \left[1_{\mathbf{B}_N} F\right](\eta)\mathcal{G}(\mathrm{d}\eta) = \int_{\mathbb{M}_1 \times [0, N^2/4]} F(s\bar{\eta})\mathbf{Dir}(\mathrm{d}\bar{\eta})\gamma_{\nu(E)}(\mathrm{d}s), \quad F \in L^1(1_{\mathbf{B}_N}\mathcal{G}). \quad (4.14)$$

We observe that (2.1) implies

$$\gamma_{\nu(E)}(1_{[0,r]}f^2) \leq \int_0^r sf'(s)^2\gamma_{\nu(E)}(\mathrm{d}s), \quad r > 0, f \in C^1([0, r]), \gamma_{\nu(E)}(1_{[0,r]}f) = 0. \quad (4.15)$$

Indeed, applying the Poincaré inequality

$$\mathcal{G}(F^2) \leq \mathcal{E}_{\mathbf{1},0}(F, F) + \mathcal{G}(F)^2$$

to $F(\eta) := f(\eta(E) \wedge r)$, and noting that for $\tilde{f}(s) := f(s \wedge r)$ we have

$$\mathcal{G}(F^i) = \gamma_{\nu(E)}(\tilde{f}^i) = \gamma_{\nu(E)}(1_{[0,r]}f^i) + \gamma_{\nu(E)}((r, \infty))f(r), \quad i = 1, 2,$$

$$\mathcal{E}_{\mathbf{1},0}(F, F) = \int_0^\infty s\tilde{f}'(s)^2\mathrm{d}s = \int_0^N sf'(s)^2\mathrm{d}s,$$

it follows that

$$\gamma_{\nu(E)}(1_{[0,r]}f^2) = \gamma(\tilde{f}^2) - \gamma_{\nu(E)}((r, \infty))f(r)^2$$
$$\leq \int_0^N sf'(s)^2\mathrm{d}s + \gamma_{\nu(E)}((r, \infty))^2 f(r)^2 - \gamma_{\nu(E)}((r, \infty))f(r)^2 \leq \int_0^N sf'(s)^2\mathrm{d}s.$$

By the additivity property of the Poincaré inequality, (4.11), (4.14) and (4.15), we obtain that for any $F \in \mathcal{F}C_0^\infty$ with $\mathcal{G}(1_{\mathbf{B}_N}F) = 0$,

$$\mathcal{G}(1_{\mathbf{B}_N}F^2) \leq \int_{\mathbb{M}_1 \times [0, N^2/4]} \left[\frac{1}{\nu(E)}\Gamma^D(F(s\cdot), F(s\cdot))(\bar{\eta}) + s\left|\frac{\partial}{\partial s}F(s\bar{\eta})\right|^2\right]\mathbf{Dir}(\mathrm{d}\bar{\eta})\gamma_{\nu(E)}(\mathrm{d}s)$$

$$= \int_{\mathbf{B}_N} \left[\frac{1}{\nu(E)}\Gamma^D(F(\eta(E)\cdot), F(\eta(E)\cdot))(\bar{\eta}) + \eta(E)\left|\frac{\partial}{\partial\eta(E)}F(\eta(E)\bar{\eta})\right|^2\right]\mathcal{G}(\mathrm{d}\eta).$$

So, it remains to prove

$$I(\eta) := \frac{1}{\nu(E)}\Gamma^D(F(\eta(E)\cdot), F(\eta(E)\cdot))(\bar{\eta}) + \eta(E)\left|\frac{\partial}{\partial\eta(E)}F(\eta(E)\bar{\eta})\right|^2$$
$$\leq \left(\frac{N^2}{4\nu(E)} \vee 1\right)\Gamma_{\mathbf{1}}(F, F)(\eta), \quad \eta(E) \leq \frac{N^2}{4}. \quad (4.16)$$

For $F \in \mathcal{F}C_0^\infty$ with $F(\eta) = f(\eta(A_1), \cdots, f(A_n)) = f(\eta(E)\bar{\eta}(A_1), \cdots, \eta(E)\bar{\eta}(A_n))$, by (4.12) we have

$$\Gamma^D\big(F(\eta(E)\cdot), F(\eta(E)\cdot)\big)(\bar{\eta})$$
$$= \sum_{i,j=1}^{n} \left[\delta_{ij}\bar{\eta}(A_i) - \bar{\eta}(A_i)\bar{\eta}(A_j)\right]\eta(E)^2[(\partial_i f)(\partial_j f)](\eta(A_1), \cdots, \eta(A_n))$$
$$= \sum_{i,j=1}^{n} \left[\delta_{ij}\eta(A_i)\eta(E) - \eta(A_i)\eta(A_j))[(\partial_i f)(\partial_j f)](\eta(A_1), \cdots, \eta(A_n)).$$

Moreover,

$$
\eta(E)\Big|\frac{\partial}{\partial\eta(E)}F(\eta(E)\bar{\eta})\Big|^2 = \eta(E)\Big|\sum_{i=1}^{n}\bar{\eta}(A_i)(\partial_i f)(\eta(A_1),\cdots,\eta(A_n))\Big|^2
$$
$$
= \frac{1}{\eta(E)}\sum_{i,j=1}^{n}\eta(A_i)\eta(A_j)[(\partial_i f)(\partial_j f)](\eta(A_1),\cdots,\eta(A_n)).
$$

So, when $\eta(E) \leq \frac{N^2}{4}$ (i.e. $\rho(E) \leq N$),

$$
I(\eta) \leq \frac{\eta(E)\vee\nu(E)}{\nu(E)}\Big(\frac{1}{\eta(E)}\Gamma^D(F(\eta(E)\cdot),F(\eta(E)\cdot))(\bar{\eta}) + \eta(E)\Big|\frac{\partial}{\partial\eta(E)}F(\eta(E)\bar{\eta})\Big|^2\Big)
$$
$$
= \Big(1\vee\frac{N^2}{4\nu(E)}\Big)\Gamma_\mathbf{1}(F,F)(\eta).
$$

This implies (4.16), and hence finishes the proof. $\qquad\square$

4.2 Proofs of Theorem 2.2 and Corollary 2.3

Proof of Theorem 2.2. We will make a standard split argument by using the local Poincaré inequality (4.13) and the principal eigenvalue of $\mathcal{L}_{\mathcal{A},V}$ outside $\mathbf{B}_N$. To estimate the principal eigenvalue, we recall Hardy's criterion for the first mixed eigenvalue. Consider the following differential operator on $[0,\infty)$:

$$
\mathcal{L}f(r) = \underline{a}(r)f''(r) + \gamma(r)f'(r), \quad r \geq 0.
$$

For any $k > 0$ and $n \geq 1$, let $\lambda_{k,n}$ be the first mixed eigenvalue of $\mathcal{L}$ on $[k,k+n]$ with Dirichlet boundary condition at k and Neumann boundary condition at $k+n$. Define

$$
\sigma_{k,n} = \sup_{t\in(k,n+k)}\int_t^{n+k}\mathrm{e}^{\int_k^r\frac{\gamma(s)}{\underline{a}(s)}\mathrm{d}s}\mathrm{d}r\int_k^t\frac{1}{\underline{a}(r)}\mathrm{e}^{-\int_k^r\frac{\gamma(s)}{\underline{a}(s)}\mathrm{d}s}\mathrm{d}r.
$$

By Hardy's criterion, see for instance [24, Theorem 1.4.2], we have

$$
\frac{1}{\sigma_{k,n}} \geq \lambda_{k,k+n} \geq \frac{1}{4\sigma_{k,n}}, \quad n \geq 1, k > 0. \tag{4.17}
$$

Below we prove assertions (1)–(3) respectively.

(1) By (4.13) and a standard perturbation argument, we have

$$
\mathcal{G}^V(1_{\{\rho\leq N\}}F^2) \leq \mathcal{G}^V(1_{\{\rho\leq N\}}F)^2 + \Phi(N)\mathcal{G}^V(1_{\{\rho\leq N\}}\Gamma_\mathcal{A}(F,F)), \quad F \in \mathcal{F}C_0^2. \tag{4.18}
$$

If $\sigma_k < \infty$ for some $k > 0$, it suffices to prove the Poincaré inequality

$$
\mathcal{G}^V(F^2) \leq C\mathcal{G}^V(\Gamma_\mathcal{A}(F,F)), \quad F \in \mathcal{F}C_0^2, \mathcal{G}^V(F) = 0 \tag{4.19}
$$

for

$$
C = 2\Phi\big(\psi^{-1}\big(\psi(k) + 8\lambda_k^{-1} + 1\big) + \lambda_k\big) + 8\lambda_k^{-1},
$$

where according to (4.17),

$$
\lambda_k := \lim_{n\to\infty}\lambda_{k,n} \geq \frac{1}{4\sigma_k}. \tag{4.20}
$$

Let $F \in \mathcal{F}C_0^2$ such that $\mathrm{supp}\, F \subset \mathbf{B}_{N_1}$ for some constant $N_1 > k$. For any $N \geq k$, let

$$
F_N = F[(\psi(\rho) - \psi(N))^+ \wedge 1].
$$

Then $F_N = 0$ for $\rho \le N$ and $F_N = F$ for $\psi(\rho) \ge \psi(N) + 1$. For $n > N_1$, let $u_n \ge 0$ be the first mixed eigenfunction of $\mathcal{L}$ on $[k, k+n]$ with Dirichlet boundary condition at k and Neumann boundary condition at $k+n$, such that

$$u_n(k) = u_n'(k+n) = 0, \quad u_n'(r) > 0 \text{ for } r \in (k, k+n), \quad \mathcal{L}u_n = -\lambda_{k,n} u_n \le 0.$$

Combining this with the definition of $\mathcal{L}$ we obtain

$$\mathcal{L}_{\mathcal{A},V}(u_n \circ \rho) \ge (\mathcal{L}u_n) \circ \rho, \quad \rho \in [k, k+n].$$

So,

$$\begin{aligned}
\lambda_{k,n} \mathcal{G}^V(F_N^2) &= -\int_{\{k < \rho < n+k\}} \frac{F_N^2}{u_n^2 \circ \rho}(-\mathcal{L}u_n) \circ \rho \, \mathrm{d}\mathcal{G}^V \\
&\le -\int_{\{k < \rho < n+k\}} \frac{F_N^2}{u_n \circ \rho}\big[-\mathcal{L}_{\mathcal{A},V}(u_n \circ \rho)\big] \mathrm{d}\mathcal{G}^V.
\end{aligned}$$
(4.21)

To apply the integration by parts formula, we approximate u_n as follows. Since $u_n(k) = u_n'(k+n) = 0$, we may construct a sequence $\{u_{n,m}\}_{m \ge 1} \subset C^\infty([0, \infty))$ such that

$$\begin{aligned}
&u_{n,m}(r) = u_n(r) \text{ for } r \in [k + m^{-1}, k + n - m^{-1}], \\
&u_{n,m}(r) = 0 \text{ for } r \le k, \quad u_{n,m}'(r) = 0 \text{ for } r \ge k + n, \\
&\sup_{m \ge 1} \sup_{r \ge k} \big(|u_{n,m}'(r)| + |u_{n,m}''(r)|\big) < \infty.
\end{aligned}$$

Since $F_N = 0$ for $\rho \le N$, (4.21) implies that for any $k < N$,

$$\begin{aligned}
\lambda_{k,n} \mathcal{G}^V(F_N^2) &= -\lim_{m \to \infty} \int_{\{k < \rho < n+k\}} \frac{F_N^2}{u_{n,m}^2 \circ \rho}(-\mathcal{L}u_{n,m}) \circ \rho \, \mathrm{d}\mathcal{G}^V \\
&= \lim_{m \to \infty} \int_{\mathbb{M}} \left\langle \mathcal{A}_\eta \nabla^{ext} \frac{F_N^2}{u_{n,m} \circ \rho}(\eta), \nabla^{ext}(u_{n,m} \circ \rho)(\eta) \right\rangle_{L^2(\eta)} \mathrm{d}\mathcal{G}^V.
\end{aligned}$$
(4.22)

On the other hand, since $\mathcal{A}_\eta$ is positive definite due to **(H)**, for any $u \in C^2([0, \infty))$ with $u(r) > 0$ for $r \ge N$, we have

$$\begin{aligned}
&\left\langle \mathcal{A}_\eta \nabla^{ext} \frac{F_N^2}{u \circ \rho}(\eta), \nabla^{ext}(u \circ \rho)(\eta) \right\rangle_{L^2(\eta)} \\
&= \left\langle \mathcal{A}_\eta \nabla^{ext} F_N(\eta), \nabla^{ext} F_N(\eta) \right\rangle_{L^2(\eta)} \\
&\quad - \left\langle \mathcal{A}_\eta \Big[\nabla^{ext} F_N - \frac{F_N}{u \circ \rho} \nabla^{ext}(u \circ \rho) \Big](\eta), \nabla^{ext} F_N(\eta) - \frac{F_N}{u \circ \rho} \nabla^{ext}(u \circ \rho)(\eta) \right\rangle_{L^2(\eta)} \\
&\le \left\langle \mathcal{A}_\eta \nabla^{ext} F_N(\eta), \nabla^{ext} F_N(\eta) \right\rangle_{L^2(\eta)}.
\end{aligned}$$

Combining this with (4.22) and the definition of F_N, we obtain

$$\begin{aligned}
\lambda_{k,n} \mathcal{G}^V(F_N^2) &\le \int_{\mathbb{M}} \langle \mathcal{A}_\eta \nabla^{ext} F_N, \nabla^{ext} F_N \rangle_{L^2(\eta)} \mathrm{d}\mathcal{G}^V \\
&\le 2\mathcal{E}_{\mathcal{A},V}(F, F) + 2\int_{\{\psi(N) < \psi(\rho) < \psi(N)+1\}} F^2 \Gamma_{\mathcal{A}}(\psi(\rho), \psi(\rho)) \mathrm{d}\mathcal{G}^V.
\end{aligned}$$

Multiplying by $\lambda_{k,n}^{-1}$ and letting $n \to \infty$ leads to

$$\int_{\mathbb{M}} F_N^2 \mathrm{d}\mathcal{G}^V \le \frac{2}{\lambda_k} \mathcal{E}_{\mathcal{A},V}(F, F) + \frac{2}{\lambda_k} \int_{\{\psi(N) < \psi(\rho) < \psi(N)+1\}} F^2 \Gamma_{\mathcal{A}}(\psi(\rho), \psi(\rho)) \mathrm{d}\mathcal{G}^V. \quad (4.23)$$

By the definition of ψ and $\bar{a}$, and noting that $\Gamma_1(\rho, \rho) = 1$, we have

$$\Gamma_{\mathcal{A}}(\psi(\rho), \psi(\rho))(\eta) = \frac{\langle \mathcal{A}_\eta \nabla^{ext} \rho(\eta), \nabla^{ext} \rho(\eta) \rangle_{L^2(\eta)}}{\bar{a}(\rho(\eta))} \le \Gamma_1(\rho, \rho)(\eta) = 1. \qquad (4.24)$$

So, (4.23) implies

$$\int_{\mathbb{M}} F_N^2 \mathrm{d}\mathcal{G}^V \le \frac{2}{\lambda_k} \mathcal{E}_{\mathcal{A},V}(F, F) + \frac{2}{\lambda_k} \int_{\{\psi(N) < \psi(\rho) < \psi(N)+1\}} F^2 \mathrm{d}\mathcal{G}^V. \qquad (4.25)$$

Letting $\lfloor s \rfloor = \sup\{k \in \mathbb{Z} : k \le s\}$ be the integer part of a real number s, we have

$$\int_{\mathbb{M}} F^2 \mathrm{d}\mathcal{G}^V \ge \sum_{i=1}^{1+\lfloor 8\lambda_k^{-1} \rfloor} \int_{\{\psi(k)+i-1 < \psi(\rho) < \psi(k)+i\]} F^2 \mathrm{d}\mathcal{G}^V.$$

Then there exists $N \in \left[k, \psi^{-1}\left(\psi(k) + 8\lambda_k^{-1} \right) \right]$ such that

$$\int_{\{\psi(N) < \psi(\rho) < \psi(N)+1\}} F^2 \mathrm{d}\mathcal{G}^V \le \frac{\lambda_k}{8} \int_{\mathbb{M}} F^2 \mathrm{d}\mathcal{G}^V,$$

so that (4.25) yields

$$\int_{\mathbb{M}} F_N^2 \mathrm{d}\mathcal{G}^V \le \frac{2}{\lambda_k} \mathcal{E}_{\mathcal{A},V}(F, F) + \frac{1}{4} \int_{\mathbb{M}} F^2 \mathrm{d}\mathcal{G}^V. \qquad (4.26)$$

Combining this with (4.18) and noting that $\mathcal{G}^V(F) = 0$, we may find $N \in [k, \psi^{-1}(\psi(k) + 8\lambda_k^{-1})]$ such that

$$\begin{aligned}
\int_{\mathbb{M}} F^2 \mathrm{d}\mathcal{G}^V &\le \int_{\psi(\rho) \le \psi(N)+1} F^2 \mathrm{d}\mathcal{G}^V + \int_{\mathbb{M}} F_N^2 \mathrm{d}\mathcal{G}^V \\
&\le \Phi(\psi^{-1}(\psi(N)+1)) \mathcal{E}_{\mathcal{A},V}(F, F) + \mathcal{G}^V(1_{\{\psi(\rho) \ge \psi(N)+1\}} F)^2 + \int_{\mathbb{M}} F_N^2 \mathrm{d}\mathcal{G}^V \\
&\le \Phi(\psi^{-1}(\psi(N)+1)) \mathcal{E}_{\mathcal{A},V}(F, F) + 2 \int_{\mathbb{M}} F_N^2 \mathrm{d}\mathcal{G}^V \\
&\le \left(\Phi(\psi^{-1}(\psi(N)+1)) + \frac{4}{\lambda_k} \right) \mathcal{E}_{\mathcal{A},V}(F, F) + \frac{1}{2} \int_{\mathbb{M}} F^2 \mathrm{d}\mathcal{G}^V.
\end{aligned}$$

Since $\Phi(N)$ is increasing in $N \in \left[k, \psi^{-1}\left(\psi(k) + 8\lambda_k^{-1} \right) \right]$, this implies (4.19) with

$$C = 2\Phi(\psi^{-1}(\psi(N)+1)) + \frac{8}{\lambda_k} \le 2\Phi\left(\psi^{-1}\left(\psi(k) + 8\lambda_k^{-1} + 1 \right) + \lambda_k \right) + 8\lambda_k^{-1}.$$

Then the proof is finished by (4.20).

(2) Assume that $\mathrm{supp}\,\nu$ is an infinite set. To disprove the super Poincaré inequality, it suffices to construct a sequence $\{F_n\} \subset \mathcal{D}(\mathcal{E}_{\mathcal{A},V})$ such that $\mathcal{G}^V(F_n^2) > 0$ and

$$C := \sup_{n \ge 1} \frac{\mathcal{E}_{\mathcal{A},V}(F_n, F_n)}{\mathcal{G}^V(F_n^2)} < \infty, \quad \lim_{n \to \infty} \frac{\mathcal{G}^V(|F_n|)^2}{\mathcal{G}^V(F_n^2)} = 0. \qquad (4.27)$$

Indeed, if (1.11) holds for some $\beta : (0, \infty) \to (0, \infty)$, then

$$1 \le r \frac{\mathcal{E}_{\mathcal{A},V}(F, F_n)}{\mathcal{G}^V(F_n^2)} + \beta(r) \frac{\mathcal{G}^V(|F_n|)^2}{\mathcal{G}^V(F_n^2)}, \quad n \ge 1, r > 0.$$

Combining this with (4.27) and letting $n \to \infty$, we obtain $1 \le rC$ for all $r > 0$ which is impossible.

We now show that (4.27) holds for $F_n(\eta) := (1 - \eta(E))^+ \frac{\eta(A_n)}{\eta(E)}$, where $\{A_n\}_{n \geq 1}$ are measurable subsets of E such that $\frac{1}{2}\nu(E) > p_n := \nu(A_n) \downarrow 0$ as $n \uparrow \infty$, which exist since $\operatorname{supp}\nu$ is an infinite set.

Obviously, $\{F_n\}_{n \geq 1} \subset \mathcal{D}(\mathcal{E}_{\mathcal{A},V})$. Since $\|\mathcal{A}_\eta\|_{L^2(\eta)} + e^{V(\eta)} + e^{-V(\eta)}$ is bounded on the set $\{\eta(E) \leq 1\}$, we may find constants $K_i, C_i > 0, i = 1, 2, 3$ such that (4.10) implies for all $n \geq 1$ that

$$\mathcal{G}^V(F_n^2) \geq K_1 \mathcal{G}(F_n^2) = K_1 \int_0^1 \frac{(1-s)^2 s^{\nu(E)-1} e^{-s}}{\Gamma(\nu(E))} \, \mathrm{d}s \int_0^1 \frac{t^{p_n+1}(1-t)^{\nu(E)-p_n-1}}{\Gamma(p_n)\Gamma(\nu(E)-p_n)} \, \mathrm{d}t \geq C_1 p_n,$$

$$\mathcal{G}^V(|F_n|)^2 \leq K_2 \mathcal{G}(|F_n|)^2 = K_2 \left(\int_0^1 \frac{(1-s)s^{\nu(E)-1} e^{-s}}{\Gamma(\nu(E))} \, \mathrm{d}s \int_0^1 \frac{t^{p_n}(1-t)^{\nu(E)-p_n-1}}{\Gamma(p_n)\Gamma(\nu(E)-p_n)} \, \mathrm{d}t \right)^2$$

$$\leq C_2 p_n^2,$$

$$\mathcal{E}_{\mathcal{A},V}(F_n, F_n) \leq K_3 \mathcal{G}(\|\nabla^{ext} F_n\|_{L^2(\eta)}^2) = K_3 \int_{\{\eta(E) \leq 1\}} \mathcal{G}(\mathrm{d}\eta) \int_E |(1 - \eta(E)) - \eta(A_n)|^2 \mathrm{d}\eta$$

$$\leq 2K_3 \int_{\{\eta(E) \leq 1\}} \left[(1-\eta(E))^2 \eta(A_n) + \eta(A_n)^2 \eta(E) \right] \mathcal{G}(\mathrm{d}\eta)$$

$$\leq 4K_3 \int_{\{\eta(E) \leq 1\}} \frac{\eta(A_n)}{\eta(E)} \mathcal{G}(\mathrm{d}\eta) \leq C_3 p_n.$$

Since $p_n \downarrow 0$ as $n \uparrow \infty$, we prove (4.27).

(3) The local Poincaré inequality (4.13) implies that for any $F \in \mathcal{F}C_0^\infty$ with $\mathcal{G}^V(F) = 0$,

$$\mathcal{G}^V(F^2) = \mathcal{G}^V(\mathbf{1}_{\mathbf{B}_N} F^2) + \mathcal{G}^V(F^2 \cdot \mathbf{1}_{\mathbf{B}_N^c})$$

$$\leq 2\Phi(N)\mathcal{E}_{\mathcal{A},V}(F, F) + \frac{1}{\mathcal{G}^V(\mathbf{B}_N)}\mathcal{G}^V(\mathbf{1}_{\mathbf{B}_N^c} F)^2 + \mathcal{G}^V(\rho > N)\|F\|_\infty^2$$

$$\leq 2\Phi(N)\mathcal{E}_{\mathcal{A},V}(F, F) + \left(\frac{\mathcal{G}^V(\rho > N)^2}{1 - \mathcal{G}^V(\rho > N)} + \mathcal{G}^V(\rho > N) \right)\|F\|_\infty^2, \quad N > 0.$$

So, for any $r > 0$, taking $N > 0$ such that $\frac{\mathcal{G}^V(\rho > N)^2}{1 - \mathcal{G}^V(\rho > N)} + \mathcal{G}^V(\rho > N) \leq r$, i.e. $\mathcal{G}^V(\rho > N) \leq \frac{r}{1+r}$, we prove (1.10). $\qquad\square$

Proof of Corollary 2.3. Let $r_0 \in (0, \infty)$ such that $\mathcal{A}_\eta = a(\rho(\eta))\mathbf{1}$ and $V(\eta) = v(\rho(\eta))$ for large $\rho(\eta) \geq r_0$. By (2.2) we have

$$(\nu - \eta)(\mathcal{A}_\eta 1) = a(\rho(\eta))\big(\nu(E) - \eta(E)\big) = a(\rho(\eta))\Big(\nu(E) - \frac{\rho(\eta)^2}{4}\Big),$$

$$\eta\big(\nabla^{ext}[\mathcal{A}_\eta 1(\cdot)](\cdot)\big) = a'(\rho(\eta))\eta\Big(\frac{1}{\sqrt{\eta(E)}}\Big) = \frac{\rho(\eta)}{2}a'(\rho(\eta)),$$

$$\nabla_{\mathcal{A}_\eta 1}^{ext} V(\eta) := \eta\big([\mathcal{A}_\eta 1]\nabla^{ext} V(\eta)\big) = (av')(\rho(\eta))\frac{2\eta(E)}{\rho(\eta)} = \frac{\rho(\eta)}{2}(av')(\rho(\eta)),$$

$$\frac{4}{\rho(\eta)^3}\eta(\mathcal{A}_\eta 1) = \frac{4a(\rho(\eta))\eta(E)}{\rho(\eta)^3} = \frac{a(\rho(\eta))}{\rho(\eta)}, \quad \rho(\eta) \geq r_0.$$

Then (2.4) implies

$$\mathcal{L}_{\mathcal{A}_\eta, V}\rho(\eta) = \xi(\rho(\eta)) \tag{4.28}$$

for the given function ξ. So, when $\sigma_k < \infty$ for some $k > 0$, Theorem 2.2(1) implies $\operatorname{gap}(\mathcal{L}_{\mathcal{A},V}) > 0$.

On the other hand, let $\sigma_k = \infty$ for all $k > 0$. We have

$$\lambda_k := \lim_{n \to \infty} \lambda_{k,n} = 0, \quad k > 0, \tag{4.29}$$

where $\lambda_{k,n}$ is given in the proof of Theorem 2.2. Let $u_{k,n}$ be the corresponding first mixed eigenfunction of $\mathcal{L}$ on $[k, k+n]$ with $u_{k,n}(r) > 0$ in $(k, k+n)$, and let

$$\Theta_v(\mathrm{d}s) = \frac{\mathrm{e}^{v(s)-s} s^{\nu(E)-1}}{\Gamma(\nu(E))} \, \mathrm{d}s,$$

such that $\mathcal{L}$ is symmetric in $L^2([k, k+n], \Theta_v)$ under the mixed boundary conditions. Then

$$\int_k^{k+n} u_{k,n}(r)^2 \Theta_v(\mathrm{d}r) = \frac{1}{\lambda_{k,n}} \int_k^{k+n} r\underline{a}(r) |u'_{k,n}(r)|^2 \Theta_v(\mathrm{d}r).$$

Letting $F_{k,n}(\eta) = u_{k,n}((\eta(E) \vee k) \wedge (k+n))$, for large enough $k > 0$ such that $\mathcal{A}_\eta = a(\rho(\eta))\mathbf{1}$ and $V(\eta) = v(\rho(\eta))$ for $\eta(E) \geq k$, the above formula implies

$$\mathcal{G}^V(F_{k,n}^2) - \mathcal{G}^V(F_{k,n})^2 \geq \mathcal{G}^V(F_{k,n}^2 \cdot 1_{\{k \leq \rho \leq k+n\}}) = \frac{1}{\lambda_{k,n}} \mathcal{E}_{\mathcal{A},V}(F_{n,k}, F_{n,k}), \quad n \geq 1.$$

Obviously, due to (4.29) this implies $\mathrm{gap}(\mathcal{L}_{\mathcal{A},V}) = 0$. $\qquad\square$

4.3 Proof of Theorem 2.5

Let $h \in \mathcal{H}$, i.e. $h \in C^1([0,\infty))$ with $h(r), h'(r) > 0$ for $r > 0$ such that (2.8) and (2.9) hold. By (2.10) and noting that $V_h = V - h(\rho) + c(h)$ where $c(h)$ is a constant such that $\mathcal{G}^{V_h}$ is a probability measure, for any $F \in \mathcal{F}C_0^\infty$ we have

$$\mathcal{G}^V(F^2 \cdot 1_{\mathbf{B}_N}) - \frac{\mathcal{G}^V(F \cdot 1_{\mathbf{B}_N})^2}{\mathcal{G}^V(\mathbf{B}_N)} = \inf_{c \in \mathbb{R}, |c| \leq \|F\|_\infty} \mathcal{G}^V(|F - c|^2 \cdot 1_{\mathbf{B}_N})$$

$$\leq \mathrm{e}^{h(N)-c(h)} \inf_{c \in \mathbb{R}, |c| \leq \|F\|_\infty} \mathcal{G}^{V_h}(|F - c|^2 \cdot 1_{\mathbf{B}_N}) \leq \mathrm{e}^{h(N)-c(h)} \inf_{c \in \mathbb{R}, |c| \leq \|F\|_\infty} \mathcal{G}^{V_h}(|F - c|^2)$$

$$= \mathrm{e}^{h(N)-c(h)} \left[\mathcal{G}^{V_h}(F^2) - \mathcal{G}^{V_h}(F)^2 \right] \leq C(h) \mathrm{e}^{h(N)-c(h)} \mathcal{G}^{V_h}(\Gamma_{\mathcal{A}}(F, F))$$

$$\leq C(h) \mathrm{e}^{h(N)} \mathcal{G}^V(\Gamma_{\mathcal{A}}(F, F)) = C(h) \mathrm{e}^{h(N)} \mathcal{E}_{\mathcal{A},V}(F, F).$$

This implies

$$\mathcal{G}^V(F^2 \cdot 1_{\mathbf{B}_N}) \leq C(h) \mathrm{e}^{h(N)} \mathcal{E}_{\mathcal{A},V}(F, F) + \frac{\mathcal{G}^V(F \cdot 1_{\mathbf{B}_N})^2}{\mathcal{G}^V(\mathbf{B}_N)}, \quad F \in \mathcal{D}(\mathcal{E}_{\mathcal{A},V}).$$

Then for any $F \in \mathcal{D}(\mathcal{E}_{\mathcal{A},V})$ with $\mathcal{G}^V(F) = 0$, we have $\mathcal{G}^V(F \cdot 1_{\mathbf{B}_N})^2 = \mathcal{G}^V(F \cdot 1_{\{\rho > N\}})^2$ and

$$\mathcal{G}^V(F^2) \leq \mathcal{G}^V(F^2 \cdot 1_{\mathbf{B}_N}) + \mathcal{G}^V(F^2 \cdot 1_{\{\rho > N\}})$$

$$\leq C(h) \mathrm{e}^{h(N)} \mathcal{E}_{\mathcal{A},V}(F, F) + \frac{\mathcal{G}^V(F \cdot 1_{\mathbf{B}_N})^2}{\mathcal{G}^V(\mathbf{B}_N)} + \mathcal{G}^V(F^2 \cdot 1_{\{\rho > N\}})$$

$$\leq C(h) \mathrm{e}^{h(N)} \mathcal{E}_{\mathcal{A},V}(F, F) + \left(\frac{\mathcal{G}^V(\rho > N)^2}{\mathcal{G}^V(\mathbf{B}_N)} + \mathcal{G}^V(\rho > N) \right) \|F\|_\infty^2$$

$$\leq C(h) \mathrm{e}^{h(N)} \mathcal{E}_{\mathcal{A},V}(F, F) + \frac{\mathcal{G}^V(\rho > N)}{\mathcal{G}^V(\mathbf{B}_N)} \|F\|_\infty^2.$$

So, for any $r > 0$ and $N > 0$ such that $\frac{\mathcal{G}^V(\rho > N)}{\mathcal{G}^V(\mathbf{B}_N)} \leq r$, equivalently $\mathcal{G}^V(\rho > N) \leq \frac{r}{1+r}$, we have

$$\mathcal{G}^V(F^2) \leq C(h) \mathrm{e}^{h(N)} \mathcal{E}_{\mathcal{A},V}(F, F) + r\|F\|_\infty^2, \quad F \in \mathcal{D}(\mathcal{E}_{\mathcal{A},V}), \mathcal{G}^V(F) = 0, h \in \mathcal{H}.$$

Therefore, the weak Poincaré inequality (1.10) holds for

$$\alpha(r) := \inf \left\{ C(h) \mathrm{e}^{h(N)} : h \in \mathcal{H}, \mathcal{G}^V(\rho > N) \leq \frac{r}{1+r} \right\}, \quad r > 0.$$

5 Extensions to the space of finite signed measures

Consider the space of finite signed measures

$$\mathbb{M}_{\mathbf{s}} := \{\eta - \eta' : \ \eta, \eta' \in \mathbb{M}\}$$

equipped with the topology induced by the map

$$\eta \mapsto (\eta^+, \eta^-) \in \mathbb{M} \times \mathbb{M},$$

where η^+ and η^- are the positive and negative parts of η in the Hahn decomposition respectively, and $\mathbb{M} \times \mathbb{M}$ is equipped with the weak topology. So, under this topology $\mathbb{M}_{\mathbf{s}}$ is a Polish space. Note that this topology maybe different from the weak topology, i.e. $\eta_n \to \eta$ if $\eta_n(f) := \int_E f \mathrm{d}\eta_n \to \eta(f)$ holds for any $f \in C_b(E)$, since the latter on $\mathbb{M}_{\mathbf{s}}$ might be not metrizable, see [19].

To extend the Dirichlet form $(\mathcal{E}_{\mathcal{A},V}, \mathcal{D}(\mathcal{E}_{\mathcal{A},V}))$ from $L^2(\mathcal{G}^V)$ to $L^2(\mathcal{G}_{\mathbf{s}}^V)$ for a probability measure $\mathcal{G}_{\mathbf{s}}^V$ with a potential V on $\mathbb{M}_{\mathbf{s}}$, we introduce below the measure $\mathcal{G}_{\mathbf{s}}^V$, the extrinsic derivative and the operator $\mathcal{A}$ respectively.

In [18], an analogue to the Lebesgue measure was introduced on $\mathbb{M}_{\mathbf{s}}$ by using the convolution of two weighted Gamma distributions. In the same spirit, we extend the measure $\mathcal{G}$ to $\mathcal{G}_{\mathbf{s}}$ on $\mathbb{M}_{\mathbf{s}}$ as follows:

$$\int_{\mathbb{M}_{\mathbf{s}}} f(\eta) \mathcal{G}_{\mathbf{s}}(\mathrm{d}\eta) = \int_{\mathbb{M} \times \mathbb{M}} f(\eta^+ - \eta^-) \mathcal{G}(\mathrm{d}\eta^+) \mathcal{G}(\mathrm{d}\eta^-), \ \ f \in \mathcal{B}_b(\mathbb{M}_{\mathbf{s}}). \tag{5.1}$$

Let $\tau(\eta) = \{x \in E : \eta(\{x\}) \neq 0\}$. To ensure that $\tau(\eta^+)$ and $\tau(\eta^-)$ are disjoint such that $\eta = \eta^+ - \eta^-$ is the Hahn decomposition of η, we will assume that ν is atomless. In this case, $\tau(\eta^+) \cap \tau(\eta^-) = \emptyset$ for $\mathcal{G} \times \mathcal{G}$-a.e. (η^+, η^-).

Next, we define the extrinsic derivative operator $(\nabla^{ext}, \mathcal{D}(\nabla^{ext}))$ as in Definition 1.1 for $\mathbb{M}_{\mathbf{s}}$ replacing $\mathbb{M}$:

$$\nabla^{ext} F(\eta)(x) = \lim_{0 \neq s \to 0} \frac{F(\eta + s\delta_x) - F(\eta)}{s}, \ \ \eta \in \mathbb{M}_{\mathbf{s}}. \tag{5.2}$$

Let $\mathcal{F}_{\mathbf{s}} C_0^\infty$ be the class of cylindrical functions of type

$$F(\eta) := f(\eta^+(A_1), \cdots, \eta^+(A_n), \eta^-(A_1), \cdots, \eta^-(A_n)), \ \ n \geq 1, f \in C_0^\infty(\mathbb{R}^{2n}), \tag{5.3}$$

where $\{A_i\}_{1 \leq i \leq n}$ is a measurable partition of E, and $\eta = \eta^+ - \eta^-$ is the Hahn decomposition. Let

$$A_\eta := \{x \in E : \ \eta(\{x\}) < 0\}, \ \ \eta \in \mathbb{M}_{\mathbf{s}}. \tag{5.4}$$

It is easy to see that such a function F is extrinsically differentiable with

$$\nabla^{ext} F(\eta) = \sum_{i=1}^{2n} (1 - 2 \cdot 1_{\{i > n\}})(\partial_i f)(\eta^+(A_1), \cdots, \eta^+(A_n), \eta^-(A_1), \cdots, \eta^-(A_n)) 1_{A_\eta^{i,n}}, \tag{5.5}$$

where

$$A_\eta^{i,n} := \begin{cases} A_i \cap A_\eta^c, & \text{if } i \leq n, \\ A_i \cap A_\eta, & \text{if } i > n. \end{cases}$$

Since for any $\eta \in \mathbb{M}_{\mathbf{s}}$, $A_{\eta + \varepsilon \delta_x} = A_\eta$ holds for small $\varepsilon > 0$ and all $x \in E$, $\nabla^{ext} F(\eta)(x)$ is again extrinsically differentiable in η with

$$\nabla^{ext}[\nabla^{ext} F(\eta)(x)](y) = \sum_{i,j=1}^{2n} \Big[(1 - 2 \cdot 1_{\{i > n\}})(1 - 2 \cdot 1_{\{j > n\}}) \tag{5.6}$$
$$\times (\partial_i \partial_j f)(\eta^+(A_1), \cdots, \eta^+(A_n), \eta^-(A_1), \cdots, \eta^-(A_n)) 1_{A_\eta^{i,n}}(x) 1_{A_\eta^{j,n}}(y) \Big].$$

Finally, for any $\eta \in \mathbb{M}_\mathbf{s}$, let $\mathcal{A}_\eta$ be a positive definite bounded linear operator on $L^2(|\eta|)$, where $|\eta| := \eta^+ + \eta^-$ is the total variation of η. Consider the pre-Dirichlet form

$$\mathcal{E}^\mathbf{s}_{\mathcal{A},V}(F,G) := \int_{\mathbb{M}_\mathbf{s}} \langle \mathcal{A}_\eta \nabla^{ext} F(\eta), \nabla^{ext} G(\eta) \rangle_{L^2(|\eta|)} \, \mathrm{d}\mathcal{G}^V_\mathbf{s}, \quad F, G \in \mathcal{F}C^\infty_0. \tag{5.7}$$

To ensure the closability of this bilinear form, we assume

(H') ν is atomless, $V \in \mathcal{D}(\nabla^{ext})$ such that $\mathcal{G}^V_\mathbf{s}$ is a probability measure. Moreover, for any $A \in \mathcal{B}(E)$ and $x \in E$, $\mathcal{A}_\eta 1_{A \cap A^c_\eta}(x)$ and $\mathcal{A}_\eta 1_{A \cap A_\eta}(x)$ are extrinsically differentiable in η with

$$\int_{\mathbb{M}_\mathbf{s}} \Big[|\eta| \big(|\nabla^{ext}[\mathcal{A}_\eta 1_{A \cap A^c_\eta}]| + |\nabla^{ext}[\mathcal{A}_\eta 1_{A \cap A_\eta}]| \big)$$
$$+ |\eta| \big((|\mathcal{A}_\eta 1_{A \cap A^c_\eta}| + |\mathcal{A}_\eta 1_{A \cap A_\eta}|) |\nabla^{ext} V(\eta)| \big) \Big] \mathcal{G}^V_\mathbf{s}(\mathrm{d}\eta) < \infty.$$

Obviously, this assumption is satisfied if $\mathcal{A}_\eta = F(\eta)\mathbf{1}$ for some positive bounded extrinsically differentiable function F such that $\mathcal{G}^V_\mathbf{s}$ is a probability measure with

$$\int_{\mathbb{M}_\mathbf{s}} |\eta| (|\nabla^{ext} F(\eta)| + |\nabla^{ext} V(\eta)|) \mathcal{G}^V_\mathbf{s}(\mathrm{d}\eta) < \infty.$$

5.1 Integration by parts formula

Theorem 5.1. Assume **(H')**. Then

$$\mathcal{E}^\mathbf{s}_{\mathcal{A},V}(F,G) = -\int_{\mathbb{M}_\mathbf{s}} (G \mathcal{L}^\mathbf{s}_{\mathcal{A},V} F) \, \mathrm{d}\mathcal{G}^V_\mathbf{s}, \quad F, G \in \mathcal{F}_\mathbf{s} C^\infty_0 \tag{5.8}$$

holds for

$$\mathcal{L}^\mathbf{s}_{\mathcal{A},V} F(\eta) := \int_E \Big(\big[\nabla^{ext} V(\eta)(x) \big] \mathcal{A}_\eta [\nabla^{ext} F(\eta)](x) + \nabla^{ext} \big(\mathcal{A}_\eta [\nabla^{ext} F(\eta)](x))(x) \big) |\eta|(\mathrm{d}x)$$
$$- \int_E \mathcal{A}_\eta [\nabla^{ext} F(\eta)](x) \eta(\mathrm{d}x).$$

Consequently, $(\mathcal{E}^\mathbf{s}_{\mathcal{A},V}, \mathcal{F}_\mathbf{s} C^\infty_0)$ is closable in $L^2(\mathcal{G}^V_\mathbf{s})$ and its closure $(\mathcal{E}^\mathbf{s}_{\mathcal{A},V}, \mathcal{D}(\mathcal{E}^\mathbf{s}_{\mathcal{A},V}))$ is a symmetric Dirichlet form with $1 \in \mathcal{D}(\mathcal{E}^\mathbf{s}_{\mathcal{A},V})$ and $\mathcal{E}^\mathbf{s}_{\mathcal{A},V}(1,1) = 0$.

To prove this result, we introduce the divergence operator associated with ∇^{ext}.

Definition 5.1. A measurable function ϕ on $\mathbb{M}_\mathbf{s} \times E$ is said in the domain $\mathcal{D}(\mathrm{div}^{ext}_\mathbf{s})$, if for any $x \in E$ we have $\phi(\cdot, x) \in \mathcal{D}(\nabla^{ext})$ and

$$\int_{\mathbb{M}_\mathbf{s}} \left(\int_E \big(|\nabla^{ext} \phi(\eta, x)(x)| + |\phi(\eta, x)| \big) |\eta|(\mathrm{d}x) \right) \mathcal{G}_\mathbf{s}(\mathrm{d}\eta) < \infty. \tag{5.9}$$

In this case, the divergence operator is given by

$$\mathrm{div}^{ext}_\mathbf{s}(\phi)(\eta) := \int_E \phi(\eta, x) \eta(\mathrm{d}x) - \int_E \nabla^{ext} \phi(\eta, x)(x) |\eta|(\mathrm{d}x), \quad \eta \in \mathbb{M}_\mathbf{s}. \tag{5.10}$$

We have the following integration by parts formula for the directional derivative

$$\nabla^{ext}_\phi F(\eta) := \int_E \big[\phi(\eta, x) \nabla^{ext} F(\eta)(x) \big] |\eta|(\mathrm{d}x), \quad \phi \in \mathcal{D}(\mathrm{div}^{ext}_\mathbf{s}), F \in \mathcal{D}(\nabla^{ext}).$$

Lemma 5.2. Let $\phi \in \mathcal{D}(\mathrm{div}^{ext}_\mathbf{s})$. Then

$$\int_{\mathbb{M}_\mathbf{s}} (\nabla^{ext}_\phi F) \, \mathrm{d}\mathcal{G}_\mathbf{s} = \int_{\mathbb{M}_\mathbf{s}} [F \mathrm{div}^{ext}_\mathbf{s}(\phi)] \, \mathrm{d}\mathcal{G}_\mathbf{s}, \quad F \in \mathcal{F}_\mathbf{s} C^\infty_0.$$

Proof. By a simple approximation argument, we may and do assume that ϕ is bounded so that $(\mathcal{G}_{\mathbf{s}} \times \nu)(|\phi|) < \infty$. For $F \in \mathcal{F}_{\mathbf{s}} C_0^\infty$, (5.2) implies

$$\nabla^{ext} F(\eta)(x) = \nabla^{ext} F(\cdot - \eta^-)(\eta^+)(x) = -\nabla^{ext} F(\eta^+ - \cdot)(\eta^-)(x), \quad F \in \mathcal{D}(\nabla^{ext}). \quad (5.11)$$

Next, for any $\eta' \in \mathbb{M}$, let

$$\phi_{+,\eta'}(\eta, x) := \phi(\eta' - \eta, x), \quad \phi_{-,\eta'}(\eta, x) := \phi(\eta - \eta', x), \quad (\eta, x) \in \mathbb{M} \times E.$$

By (3.6) and (5.11) we obtain

$$\mathrm{div}^{ext}(\phi_{-,\eta^-})(\eta^+) - \mathrm{div}^{ext}(\phi_{+,\eta^+})(\eta^-)$$

$$= \int_E \left[\phi(\eta^+ - \eta^-, x) - \nabla^{ext}\phi(\cdot - \eta^-, x)(\eta^+)(x)\right]\eta^+(\mathrm{d}x) - \nu(\phi(\eta, \cdot))$$

$$\quad - \int_E \left[\phi(\eta^+ - \eta^-, x) + \nabla^{ext}\phi(\eta^+ - \cdot, x)(\eta^-)(x)\right]\eta^-(\mathrm{d}x) - \nu(\phi(\eta, \cdot))$$

$$= \int_E \phi(\eta^+ - \eta^-, x)(\eta^+ - \eta^-)(\mathrm{d}x) - \int_E \left[\nabla^{ext}(\cdot, x)(\eta^+ - \eta^-)(x)\right](\eta^+ + \eta^-)(\mathrm{d}x)$$

$$= \mathrm{div}_{\mathbf{s}}^{ext}(\phi)(\eta), \quad \eta = \eta^+ - \eta^- \text{ with } \tau(\eta^+) \cap \tau(\eta^-) = \emptyset.$$

Combining this with Lemma 3.3, (5.1) and (5.11), we obtain

$$\int_{\mathbb{M}_{\mathbf{s}}} (\nabla_\phi^{ext} F) \, \mathrm{d}\mathcal{G}_{\mathbf{s}}$$

$$= \int_{\mathbb{M} \times \mathbb{M}} \mathcal{G}(\mathrm{d}\eta^+)\mathcal{G}(\mathrm{d}\eta^-) \int_E \left[\phi(\eta^+ - \eta^-, x)\nabla^{ext} F(\eta^+ - \eta^-)(x)\right](\eta^+ + \eta^-)(\mathrm{d}x)$$

$$= \int_{\mathbb{M}} \mathcal{G}(\eta^-) \int_{\mathbb{M} \times E} \left[\phi(\eta^+ - \eta^-, x)\nabla^{ext} F(\cdot - \eta^-)(\eta^+)(x)\right]\eta^+(\mathrm{d}x)$$

$$\quad - \int_{\mathbb{M}} \mathcal{G}(\eta^+) \int_{\mathbb{M} \times E} \left[\phi(\eta^+ - \eta^-, x)\nabla^{ext} F(\eta^+ - \cdot)(\eta^-)(x)\right]\eta^-(\mathrm{d}x)$$

$$= \int_{\mathbb{M} \times \mathbb{M}} F(\eta^+ - \eta^-)\left[\mathrm{div}^{ext}(\phi_{-,\eta^-})(\eta^+) - \mathrm{div}^{ext}(\phi_{+,\eta^+})(\eta^-)\right]\mathcal{G}(\mathrm{d}\eta^+)\mathcal{G}(\eta^-)$$

$$= \int_{\mathbb{M}_{\mathbf{s}}} F(\eta)\mathrm{div}_{\mathbf{s}}^{ext}(\phi)(\eta) \, \mathcal{G}_{\mathbf{s}}(\mathrm{d}\eta). \qquad \qquad \square$$

Proof of Theorem 5.1. Let $F \in \mathcal{F}_{\mathbf{s}} C_0^\infty$ be given in (5.3), and let

$$\phi(\eta, x) := \mathrm{e}^{V(\eta)}\mathcal{A}_\eta[\nabla^{ext} F(\eta)](x)$$

$$= \mathrm{e}^{V(\eta)} \sum_{i=1}^{2n}(1 - 2 \cdot 1_{\{i>n\}})(\partial_i f)(\eta(A_1), \cdots, \eta(A_n))\mathcal{A}_\eta 1_{A_i^\eta}(x), \quad (\eta, x) \in \mathbb{M}_{\mathbf{s}} \times E.$$

Then **(H')** and (5.5) imply $\phi \in \mathcal{D}(\mathrm{div}_{\mathbf{s}}^{ext})$. By the definition of $\mathcal{E}_{\mathcal{A},V}^{\mathbf{s}}$ and Lemma 5.2, for any $G \in \mathcal{F}_{\mathbf{s}} C_0^\infty$ we have

$$\mathcal{E}_{\mathcal{A},V}^{\mathbf{s}}(F, G) = \int_{\mathbb{M}_{\mathbf{s}}} \left\langle \mathcal{A}_\eta \nabla^{ext} F(\eta), \nabla^{ext} G(\eta) \right\rangle_{L^2(|\eta|)} \mathcal{G}_{\mathbf{s}}^V(\mathrm{d}\eta)$$

$$= \int_{\mathbb{M}_{\mathbf{s}}} \left\langle \phi(\eta, \cdot), \nabla^{ext} G(\eta) \right\rangle_{L^2(|\eta|)} \mathcal{G}_{\mathbf{s}}(\mathrm{d}\eta) = \int_{\mathbb{M}_{\mathbf{s}}} G(\eta)\mathrm{div}_{\mathbf{s}}^{ext}(\phi) \, \mathcal{G}_{\mathbf{s}}(\mathrm{d}\eta).$$

This together with (5.10) implies (5.8) for

$$\mathcal{L}_{\mathcal{A},V}^{\mathbf{s}} F(\eta) := -\mathrm{e}^{-V(\eta)}\mathrm{div}_{\mathbf{s}}^{ext}(\phi) = -\mathrm{e}^{-V(\eta)}\mathrm{div}_{\mathbf{s}}^{ext}\left(\mathrm{e}^{V(\eta)}\mathcal{A}_\eta[\nabla^{ext} F(\eta)](\cdot)\right)$$

$$= \int_E \left(\left[\nabla^{ext} V(\eta)(x)\right]\mathcal{A}_\eta[\nabla^{ext} F(\eta)](x) + \nabla^{ext}\left(\mathcal{A}_\eta[\nabla^{ext} F(\eta)](x)\right)(x)\right)|\eta|(\mathrm{d}x)$$

$$\quad - \int_E \mathcal{A}_\eta[\nabla^{ext} F(\eta)](x)\eta(\mathrm{d}x).$$

Next, to prove that $1 \in \mathcal{D}(\mathcal{E}_{\mathcal{A},V}^{\mathbf{s}})$ with $\mathcal{E}_{\mathcal{A},V}^{\mathbf{s}}(1,1) = 0$, we take $\{f_n\}_{n \geq 1} \subset C_0^{\infty}(\mathbb{R})$ such that $f_n(s) = 1$ for $|s| \leq n$, $0 \leq f_n \leq 1$ and $\|f_n'\|_{\infty} \leq 1$. Let $F_n(\eta) := f_n(\eta(E)), n \geq 1$. Then $F_n \in \mathcal{F}C_0^{\infty}$. By **(H')** we have $\mathcal{G}_{\mathbf{s}}^V(|F_n - 1|^2) \to 0$ as $n \to \infty$, and

$$\limsup_{n \to \infty} \mathcal{E}_{\mathcal{A},V}^{\mathbf{s}}(F_n, F_n) = \limsup_{n \to \infty} \int_{\{|\eta(E)| > n\}} \|\mathcal{A}_\eta 1\|_{L^1(|\eta|)} \, \mathcal{G}_{\mathbf{s}}^V(\mathrm{d}\eta) = 0.$$

Therefore, $1 \in \mathcal{D}(\mathcal{E}_{\mathcal{A},V}^2)$ and $\mathcal{E}_{\mathcal{A},V}^{\mathbf{s}}(1,1) = 0$. $\qquad\square$

5.2 Functional inequalities for $\mathcal{E}_{\mathbf{1},0}^{\mathbf{s}}$

For any $N > 0$, let $\tilde{\mathbf{B}}_N^{\mathbf{s}} = \{\eta \in \mathbb{M}_{\mathbf{s}} : \eta^+(E) \vee \eta^-(E) \leq N\}$.

Theorem 5.3. Let $\mathcal{A} = \mathbf{1}$ and $V = 0$.

(1) $\mathrm{gap}(\mathcal{L}_{\mathcal{A},V}^{\mathbf{s}}) = 1$, i.e. the following Poincaré inequality

$$\mathcal{G}_{\mathbf{s}}(F^2) \leq \mathcal{E}_{\mathbf{1},0}^{\mathbf{s}}(F, F) + \mathcal{G}_{\mathbf{s}}(F)^2, \quad F \in \mathcal{D}(\mathcal{E}_{\mathbf{1},0}^{\mathbf{s}}) \tag{5.12}$$

holds, and the constant 1 in front of $\mathcal{E}_{\mathbf{1},0}^{\mathbf{s}}(F, F)$ is optimal.

(2) If $\mathrm{supp}\,\nu$ is infinite, then $\mathcal{E}_{\mathbf{1},0}^{\mathbf{s}}$ does not satisfy the super Poincaré inequality. On the other hand, there exists a constant $c_0 > 0$ such that when $\mathrm{supp}\,\nu$ is a finite set, the log-Sobolev inequality

$$\mathcal{G}_{\mathbf{s}}(F^2 \log F^2) \leq \frac{c_0}{1 \wedge \delta} \mathcal{E}_{\mathbf{1},0}^{\mathbf{s}}(F, F), \quad F \in \mathcal{D}(\mathcal{E}_{\mathbf{1},0}), \mathcal{G}(F^2) = 1 \tag{5.13}$$

holds, where $\delta := \min\{\nu(\{x\}) : x \in \mathrm{supp}\,\nu\}$.

(3) For any $N > 0$ and $F \in \mathcal{F}C_0^{\infty}$ with $\mathcal{G}_{\mathbf{s}}(1_{\tilde{\mathbf{B}}_N^{\mathbf{s}}} F) = 0$,

$$\mathcal{G}_{\mathbf{s}}(1_{\tilde{\mathbf{B}}_N^{\mathbf{s}}} F^2) \leq \left(2 \vee \frac{N^2}{2\nu(E)}\right) \mathcal{G}_{\mathbf{s}}\left(1_{\tilde{\mathbf{B}}_N^{\mathbf{s}}} \|\nabla^{ext} F\|_{L^2(|\eta|)}^2\right). \tag{5.14}$$

Proof. By taking $F(\eta)$ depending only on η^+, it is easy to see that a Poincaré inequality for $\mathcal{E}_{\mathbf{1},0}^{\mathbf{s}}$ implies the same inequality for $\mathcal{E}_{\mathbf{1},0}$. So, the optimality of (5.12), and the invalidity of the super Poincaré inequality when $\mathrm{supp}\,\nu$ is infinite, follow from Theorem 2.1. It remains to prove the inequalities (5.12), (5.13) and (5.14). According to the additivity property of the Poincaré and log-Sobolev inequalities, these inequalities follow from the corresponding ones of $\mathcal{E}_{\mathbf{1},0}$. For simplicity, below we only prove the first inequality.

Let $\mathcal{F} \in \mathcal{F}_{\mathbf{s}}C_0^{\infty}$. By Theorem 2.1, (5.1), (5.7) for $\mathcal{A} = \mathbf{1}$ and $V = 0$, and using (5.11), we obtain

$$\mathcal{G}_{\mathbf{s}}(F^2) = \int_{\mathbb{M}} \mathcal{G}(\mathrm{d}\eta^-) \int_{\mathbb{M}} F(\eta^+ - \eta^-)^2 \mathcal{G}(\mathrm{d}\eta^+)$$

$$\leq \int_{\mathbb{M} \times \mathbb{M}} \|\nabla^{ext} F(\cdot - \eta^-)(\eta^+)\|_{L^2(\eta^+)}^2 \mathcal{G}(\mathrm{d}\eta^+)\mathcal{G}(\mathrm{d}\eta^-) + \int_{\mathbb{M}} \left(\int_{\mathbb{M}} F(\eta^+ - \eta^-)\mathcal{G}(\mathrm{d}\eta^+)\right)^2 \mathcal{G}(\mathrm{d}\eta^-)$$

$$\leq \int_{\mathbb{M}_{\mathbf{s}}} \|\nabla^{ext} F(\eta)\|_{L^2(\eta^+)}^2 \mathcal{G}_{\mathbf{s}}(\mathrm{d}\eta) + \left(\int_{\mathbb{M} \times \mathbb{M}} F(\eta^+ - \eta^-)\mathcal{G}(\mathrm{d}\eta^+)\mathcal{G}(\mathrm{d}\eta^-)\right)^2$$

$$+ \int_{\mathbb{M}} \left\|\nabla^{ext}\left[\int_{\mathbb{M}} F(\eta^+ - \cdot)\mathcal{G}(\mathrm{d}\eta^+)\right](\eta^-)\right\|_{L^2(\eta^-)}^2 \mathcal{G}(\mathrm{d}\eta^-).$$

By the Jensen inequality, we have

$$\left\|\nabla^{ext}\left[\int_{\mathbb{M}} F(\eta^+ - \cdot)\mathcal{G}(\mathrm{d}\eta^+)\right](\eta^-)\right\|_{L^2(\eta^-)}^2 \leq \int_{\mathbb{M}} \|\nabla^{ext} F(\eta^+ - \cdot)(\eta^-)\|_{L^2(\eta^-)}^2 \mathcal{G}(\mathrm{d}\eta^+).$$

Therefore,

$$\mathcal{G}_{\mathbf{s}}(F^2) \leq \mathcal{G}_{\mathbf{s}}(F)^2 + \int_{\mathbb{M}_{\mathbf{s}}} \|\nabla^{ext} F(\eta)\|_{L^2(\eta^+ + \eta^-)}^2 \mathcal{G}_{\mathbf{s}}(\mathrm{d}\eta) = \mathcal{G}_{\mathbf{s}}(F)^2 + \mathcal{E}_{\mathbf{1},0}^{\mathbf{s}}(F, F). \qquad\square$$

5.3 Functional inequalities for $\mathcal{E}_{\mathcal{A},V}^{\mathbf{s}}$

According to the proof of Theorem 2.2 and the local Poincaré inequality (5.14), it seems that we should take

$$\tilde{\rho}_{\mathbf{s}}(\eta) := 2\sqrt{\eta^+(E) \vee \eta^-(E)}, \ \ \eta \in \mathbb{M}_{\mathbf{s}}$$

to replace the function ρ on M. But by (5.5) we have

$$\nabla^{ext}\tilde{\rho}_{\mathbf{s}}(\eta)(x) = \frac{2}{\tilde{\rho}_{\mathbf{s}}(\eta)}\Big(1_{\{\eta(E)\geq 0\}}1_{A_\eta^c}(x) - 1_{\{\eta(E)<0\}}1_{A_\eta}(x)\Big),$$

which is however not extrinsically differentiable in η, so that $\mathcal{L}_{\mathcal{A},V}\tilde{\rho}_{\mathbf{s}}$ is not well defined as required. To avoid this problem, below we will use both $\tilde{\rho}_{\mathbf{s}}$ and

$$\rho_{\mathbf{s}}(\eta) := 2\sqrt{|\eta|(E)}, \ \ \eta \in \mathbb{M}_{\mathbf{s}}, \tag{5.15}$$

which satisfies $\|\nabla^{ext}\rho_{\mathbf{s}}(\eta)\|_{L^2(|\eta|)} = 1$ according to the following lemma.

Lemma 5.4. Let ρ be defined in (5.15) and let $\mathbf{s}(\eta,\cdot) := 1 - 2 \cdot 1_{A_\eta}$ for A_η in (5.4). Then

$$\nabla^{ext}\rho_{\mathbf{s}}(\eta) = \frac{2\mathbf{s}(\eta,\cdot)}{\rho_{\mathbf{s}}(\eta)}, \ \ \nabla^{ext}\mathbf{s}(\eta,x)(y) = 0, \ \ \eta \in \mathbb{M}_{\mathbf{s}}, x, y \in E.$$

Consequently, if $\mathcal{A}_\eta\mathbf{s}(\eta,\cdot)$ is extrinsically differentiable in $\eta \in \mathbb{M}_{\mathbf{s}}$ with

$$\sup_{|\eta|(E)\leq r} |\eta|\big(|\mathcal{A}_\eta\mathbf{s}(\eta,\cdot)| + |\nabla^{ext}[\mathcal{A}_\eta\mathbf{s}(\eta,\cdot)](\cdot)|\big) < \infty, \ \ r \in (0,\infty), \tag{5.16}$$

then

$$\mathcal{L}_{\mathcal{A},V}^{\mathbf{s}}\rho_{\mathbf{s}}(\eta) = \frac{2}{\rho_{\mathbf{s}}(\eta)}\Big[|\eta|\Big([\nabla^{ext}V(\eta)]\mathcal{A}_\eta\mathbf{s}(\eta,\cdot)) + \nabla^{ext}[\mathcal{A}_\eta\mathbf{s}(\eta,\cdot)](\cdot)\Big) - \eta\big(\mathcal{A}_\eta\mathbf{s}(\eta,\cdot)\big)\Big]. \tag{5.17}$$

This lemma can be proved by simple calculations using (5.2) and the definition of $\mathcal{L}_{\mathcal{A},V}$ in Theorem 5.1, so we omit the details.

By Lemma 5.4, we have

$$\Gamma_{\mathbf{1}}^{\mathbf{s}}(\rho_{\mathbf{s}},\rho_{\mathbf{s}}) := |\eta|(|\nabla^{ext}\rho_{\mathbf{s}}|^2) = 1, \ \ \nabla^{ext}[\nabla^{ext}\rho_{\mathbf{s}}(\eta)(x)](x) = -\frac{4}{\rho_{\mathbf{s}}(\eta)^3}.$$

These coincide with the corresponding properties of ρ on $\mathbb{M}$.

Similarly to (2.5) and (2.6), let

$$\xi_{\mathbf{s}}(r) = \inf_{\rho_{\mathbf{s}}(\eta)=r} \mathcal{L}_{\mathcal{A},V}^{\mathbf{s}}\rho_{\mathbf{s}}(\eta), \ \ \underline{a}_{\mathbf{s}}(r) = \inf_{\rho_{\mathbf{s}}(\eta)=r} \inf_{\|\phi\|_{L^2(|\eta|)}=1} \langle \mathcal{A}_\eta\phi, \phi\rangle_{L^2(|\eta|)},$$

$$\bar{a}_{\mathbf{s}}(r) = \sup_{\rho_{\mathbf{s}}(\eta)=r} \sup_{\|\phi\|_{L^2(|\eta|)}=1} \langle \mathcal{A}_\eta\phi, \phi\rangle_{L^2(|\eta|)}, \ \ r > 0,$$

$$\sigma_{k,\mathbf{s}} = \sup_{t\geq k} \int_t^\infty e^{\int_k^r \frac{\xi_{\mathbf{s}}(s)}{\bar{a}_{\mathbf{s}}(s)}ds}dr \int_k^t \frac{1}{\underline{a}_{\mathbf{s}}(r)}e^{-\int_k^r \frac{\xi_{\mathbf{s}}(s)}{\bar{a}_{\mathbf{s}}(s)}ds}dr, \ \ k > 0. \tag{5.18}$$

Assume that

$$\psi(t) := \int_0^t [\bar{a}_{\mathbf{s}}(r)]^{-\frac{1}{2}}dr \uparrow \infty \text{ as } t \uparrow \infty. \tag{5.19}$$

As in the proof of Theorem 2.2, we may use $\sigma_{k,\mathbf{s}}$ to estimate $\mathcal{G}_{\mathbf{s}}^V(F_N^2)$ for

$$F_N := \big[(\psi(\rho_{\mathbf{s}}) - \psi(N))^+ \wedge 1\big] \cdot F, \ \ N > 0, F \in \mathcal{F}_{\mathbf{s}}C_0^\infty.$$

More precisely, as in (4.20) and (4.26) we conclude that for any $k > 0$ there exists $N \in [k, \psi^{-1}(\psi(k) + 32\sigma_{k,\mathbf{s}})]$ such that

$$
\begin{aligned}
\int_{\mathbb{M}_{\mathbf{s}}} F_N^2 \mathrm{d}\mathcal{G}_{\mathbf{s}}^V &\leq \frac{2}{\lambda_k} \mathcal{E}_{\mathcal{A},V}^{\mathbf{s}}(F,F) + \frac{1}{4} \int_{\mathbb{M}} F^2 \mathrm{d}\mathcal{G}_{\mathbf{s}}^V \\
&\leq 8\sigma_{k,\mathbf{s}} \mathcal{E}_{\mathcal{A},V}^{\mathbf{s}}(F,F) + \frac{1}{4} \int_{\mathbb{M}} F^2 \mathrm{d}\mathcal{G}_{\mathbf{s}}^V.
\end{aligned}
\tag{5.20}
$$

On the other hand, we estimate $\mathcal{G}_{\mathbf{s}}^V(F^2 \cdot 1_{\{\rho_{\mathbf{s}} \leq N\}})$ by using the local Poincaré inequality (5.14). Since the bounded set in (5.14) is $\tilde{\mathbf{B}}_N^{\mathbf{s}} := \{\tilde{\rho}_{\mathbf{s}} \leq N\}$ rather than $\mathbf{B}_N^{\mathbf{s}} := \{\rho_{\mathbf{s}} \leq N\}$, we change the definition of $\Phi(N)$ into

$$
\Phi_{\mathbf{s}}(N) := \left(2 \vee \frac{N^2}{2\nu(E)}\right) \exp\left[\sup_{\tilde{\rho}_{\mathbf{s}} \leq N} V - \inf_{\tilde{\rho}_{\mathbf{s}} \leq N} V\right] \sup_{\tilde{\rho}_{\mathbf{s}}(\eta) \leq N} \sup_{\|\phi\|_{L^2(|\eta|)}=1} \frac{1}{\langle \mathcal{A}_\eta \phi, \phi \rangle_{L^2(|\eta|)}}, \quad N > 0.
$$

Noting that $1_{\{\rho_{\mathbf{s}} \leq N\}} \leq 1_{\{\tilde{\rho}_{\mathbf{s}} \leq N\}}$, we may apply Theorem 5.3 to bound $\mathcal{G}_{\mathbf{s}}^V(F^2 \cdot 1_{\{\rho_{\mathbf{s}} \leq N\}})$. For instance, corresponding to (4.18) we have

$$
\mathcal{G}_{\mathbf{s}}^V(F^2 \cdot 1_{\{\rho_{\mathbf{s}} \leq N\}}) \leq \mathcal{G}_{\mathbf{s}}^V(F^2 \cdot 1_{\{\tilde{\rho}_{\mathbf{s}} \leq N\}}) \leq \mathcal{G}_{\mathbf{s}}^V(1_{\{\tilde{\rho}_{\mathbf{s}} \leq N\}} F)^2 + \Phi_{\mathbf{s}}(N) \mathcal{E}_{\mathcal{A},V}^{\mathbf{s}}(F,F).
$$

Combining this with (5.20) we may extend assertions of Theorem 2.2 to the present setting as follows, where when $\mathrm{supp}\,\nu$ is infinite the super Poincaré can be disproved as in the proof of Theorem 2.2(2) by taking $F_n(\eta) = (1 - \eta^+(E))^+ \frac{\eta^+(A_n)}{\eta^+(E)}$ for $0 < \nu(A_n) \downarrow 0$. Moreover, one may also extend Corollaries 2.3–2.4 and Theorem 2.5. We omit the details to save space.

Theorem 5.5. In addition to **(H')**, assume that $\mathcal{A}_\eta \mathbf{s}(\eta, \cdot)$ is extrinsically differentiable in η such that (5.16) holds. Moreover, assume that $\underline{a}_{\mathbf{s}}$ and $\bar{a}_{\mathbf{s}}$ in (5.18) are such that $\underline{a}_{\mathbf{s}}^{-1}(r)$ is locally bounded in $r \geq 0$ and (5.19) holds.

(1) If $\lim_{k \to \infty} \sigma_{k,\mathbf{s}} < \infty$, then

$$
\mathrm{gap}(\mathcal{L}_{\mathcal{A},V}^{\mathbf{s}}) \geq \sup\left\{\frac{1}{2\Phi_{\mathbf{s}}\big(\psi^{-1}(\psi(k) + 32\sigma_{k,\mathbf{s}} + 1)\big) + 32\sigma_{k,\mathbf{s}}} : k > 0\right\} > 0.
$$

(2) If $\mathrm{supp}\,\nu$ contains infinitely many points, then $\mathcal{E}_{\mathcal{A},V}^{\mathbf{s}}$ does not satisfy the super Poincaré inequality.

(3) The weak Poincaré inequality (1.10) holds for $(\mathcal{E}_{\mathcal{A},V}^{\mathbf{s}}, \mathcal{G}_{\mathbf{s}}^V)$ replacing $(\mathcal{E}_{\mathcal{A},V}, \mathcal{G}^V)$ and

$$
\alpha(r) := \inf\left\{2\Phi_{\mathbf{s}}(N) : \mathcal{G}_{\mathbf{s}}^V(\rho > N) \leq \frac{r}{1+r}\right\}, \quad r > 0.
$$

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