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Are random permutations spherically uniform?

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Abstract

For large q, does the (discrete) uniform distribution on the set of $q!$ permutations of the vector $\bar{\mathbf{x}}^q = (1, 2, \ldots, q)'$ closely approximate the (continuous) uniform distribution on the $(q-2)$-sphere that contains them? These permutations comprise the vertices of the regular permutohedron, a $(q-1)$-dimensional convex polyhedron. The answer is emphatically no: these permutations are confined to a negligible portion of the sphere, and the regular permutohedron occupies a negligible portion of the ball. However, $(1, 2, \ldots, q)$ is not the most favorable configuration for spherical uniformity of permutations. A more favorable configuration $\hat{\mathbf{x}}^q$ is found, namely that which minimizes the normalized surface area of the largest empty spherical cap among its $q!$ permutations. Unlike that for $\bar{\mathbf{x}}^q$, the normalized surface area of the largest empty spherical cap among the permutations of $\hat{\mathbf{x}}^q$ approaches 0 as $q \to \infty$. Nonetheless the permutations of $\hat{\mathbf{x}}^q$ do not approach spherical uniformity either. The existence of an asymptotically spherically uniform permutation sequence remains an open question.

Keywords: permutations; uniform distribution; spherical cap discrepancy; largest empty cap; regular configuration; regular permutohedron; L-minimal configuration; L-minimal permutohedron; normal configuration; normal permutohedron; majorization; spherical code; permutation code.
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This paper is dedicated to the memory of Ingram Olkin, my teacher, mentor, and friend, who introduced so many of us to the joy of majorization.

1 Introduction

Column vectors denoted by Roman letters appear in bold type, their components in plain type; thus $\mathbf{x} = (x_1, \ldots, x_q)' \in \mathbb{R}^q$. For any nonzero $\mathbf{x} \in \mathbb{R}^q$ ($q \geq 2$) let $\Pi(\mathbf{x})$ denote the set of all $q!$ permutations of $\mathbf{x}$, that is

$$\Pi(\mathbf{x}) = \{P\mathbf{x} \mid P \in \mathcal{P}^q\}, \tag{1.1}$$

where $\mathcal{P}^q$ is the set of all $q \times q$ permutation matrices. In this paper the following question is examined:

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Question 1. *For large q, do there exist nonzero vectors $\mathbf{x} \in \mathbb{R}^q$ such that the (discrete) uniform distribution on $\Pi(\mathbf{x})$ closely approximates the (continuous) uniform distribution on the $(q-2)$-sphere in which $\Pi(\mathbf{x})$ is contained? More precisely, do there exist sequences*[1] $\{\mathbf{x}^q \in \mathbb{R}^q\}$ *such that $\Pi(\mathbf{x}^q)$ approaches spherical uniformity as $q \to \infty$?*

This question arose in a statistical context: how representative of a general design matrix is the matrix whose columns consist of permutations of a single vector $\mathbf{x} = (x_1, \dots, x_q)'$? This statistical viewpoint is mentioned briefly in Remark 4.3.

In coding theory, a finite set N of points on a d-sphere is called a *spherical code*, cf. [13, 14]. A question of substantial interest has been the construction of sequences of spherical codes that are asymptotically spherically uniform, that is, closely approximate the uniform distribution on the sphere, as $|N| \to \infty$ with d held fixed. Departure from spherical uniformity is usually measured by the *normalized spherical cap discrepancy (NSCD)* or by the weaker *largest empty cap discrepancy (LECD)*. These discrepancies are defined in Definition 3.2; in general LECD $\leq$ NSCD, cf. (3.20).

A good deal of previous work has been aimed at determining the rate at which the optimal (smallest possible) discrepancy for spherical codes of size n approaches 0 as $n \to \infty$ with d fixed; cf. [5, 17, 19]. A recent survey of the literature appears in [6].

The set $\Pi(\mathbf{x})$, consisting of all permutations of $\mathbf{x}$, is a spherical code of special type, which we shall call a *permutation code*. The second part of Question 1 can be re-stated as follows: which if any sequences $\{N^q\} \equiv \{\Pi(\mathbf{x}^q)\}$ of permutation codes are asymptotically spherically uniform, that is, have small NSCD and/or small LECD as $q \to \infty$? Here, however, if the components of $\mathbf{x}^q$ are distinct then $|N^q| = q!$ and $d = q-2$, so both $|N^q| \to \infty$ and $d \to \infty$. Thus general results of the form $\mathrm{discrepancy}(N) \sim c(d)g(|N|)$ as $|N| \to \infty$, where the constant $c(d)$ is not known explicitly, are not helpful for permutation codes. Instead, the permutation structure of $\Pi(\mathbf{x})$ can be exploited.

In this paper three configuration sequences $\{\mathbf{x}^q\}$ are studied in detail: the *regular configuration* $\bar{\mathbf{x}}^q = (1, 2, \dots, q)'$ in Section 4, the *L-minimal configuration* $\hat{\mathbf{x}}^q$ (which minimizes the LECD of $\Pi(\mathbf{x}^q)$) in Section 5, and the *normal configuration* $\breve{\mathbf{x}}^q = (1, 2, \dots, q)'$ (constructed from standard normal quantiles) in Section 6.

Proposition 4.2 shows that the regular configurations deviate greatly from spherical uniformity, measured by LECD, as $q \to \infty$. Proposition 5.3 shows that the LECD of the L-minimal configuration does approach 0, but the univariate marginal distributions of this configuration do not converge to normality (Proposition 5.5), which is a necessary condition for asymptotic spherical uniformity (Proposition 3.10). The univariate marginal distributions of the normal configuration do approach normality (by construction), but, like the regular configuration, the LECD of the normal configuration does not approach 0 (Proposition 6.1).

These results are summarized and compared in Section 7 (see Table 1). Some comments relating these results to the volumes of permutohedra appear in Section 8. The existence or non-existence of an asymptotically spherically uniform sequence of permutation codes, as measured by NSCD, is left as an open question.

2 Translation and rotation of coordinates

Because $\Pi(\mathbf{x})$ is invariant under permutations of $\mathbf{x}$, we may always assume that the components of $\mathbf{x}$ and $\mathbf{x}^q$ are ordered, i.e., $\mathbf{x}, \mathbf{x}^q \in \mathbb{R}^q_{\leq}$, where

$$\mathbb{R}^q_{\leq} := \{\mathbf{x} \in \mathbb{R}^q \mid x_1 \leq \cdots \leq x_q\}. \tag{2.1}$$

[1] Superscripts denote indices, not exponents, unless the contrary is evident.

Clearly $\|P\mathbf{x}\| = \|\mathbf{x}\|$ for all $P \in \mathcal{P}^q$, so

$$\Pi(\mathbf{x}) \subset \mathcal{S}^{q-1}_{\|\mathbf{x}\|} \cap \mathcal{M}^{q-1}_{\mathbf{x}}, \tag{2.2}$$

where $\mathcal{S}^{q-1}_{\rho}$ denotes the 0-centered $(q-1)$-sphere of radius ρ in $\mathbb{R}^q$ and

$$\mathcal{M}^{q-1}_{\mathbf{x}} := \{\mathbf{v} \in \mathbb{R}^q \mid \mathbf{v}'\mathbf{e}^q = \mathbf{x}'\mathbf{e}^q\} \tag{2.3}$$

is the $(q-1)$-dimensional hyperplane containing $\mathbf{x}$ that is orthogonal to $\mathbf{e}^q := (1, \dots, 1)'$. Because $\mathcal{M}^{q-1}_{\mathbf{x}}$ does not contain the origin but we wish to work with 0-centered spheres, we shall translate $\mathcal{M}^{q-1}_{\mathbf{x}}$ to

$$\tilde{\mathcal{M}}^{q-1} \equiv \{\mathbf{v} \in \mathbb{R}^q \mid \mathbf{v}'\mathbf{e}^q = 0\}, \tag{2.4}$$

the $(q-1)$-dimensional linear subspace parallel to $\mathcal{M}^{q-1}_{\mathbf{x}}$ and orthogonal to $\mathbf{e}^q$.

For this purpose consider the $q \times q$ Helmert orthogonal matrix

$$\begin{aligned}
\Gamma^q &\equiv (\gamma^q_1, \gamma^q_2, \dots, \gamma^q_q) \\
&\equiv \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & \cdots & 1 \\ 1 & 0 & -2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & \cdots & -(q-1) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{q}} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{1\cdot 2}} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\sqrt{2\cdot 3}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{(q-1)q}} \end{pmatrix} \\
&\equiv (\gamma^q_1, \Gamma^q_2),
\end{aligned}$$

where $\gamma^q_1 \equiv \frac{1}{\sqrt{q}}\mathbf{e}^q$ is the $(q \times 1)$-dimensional unit vector along the direction of $\mathbf{e}^q$. By the orthogonality of Γ^q,

$$(\Gamma^q_2)'\gamma^q_1 = 0, \tag{2.5}$$
$$(\Gamma^q_2)'\Gamma^q_2 = I_{q-1}, \tag{2.6}$$
$$\Gamma^q_2(\Gamma^q_2)' = I_q - \gamma^q_1(\gamma^q_1)' =: \Omega_q, \tag{2.7}$$

where I_q denotes the $q \times q$ identity matrix. Here Ω_q is the projection matrix of rank $q-1$ that projects $\mathbb{R}^q$ onto $\tilde{\mathcal{M}}^{q-1}$, so that $\Omega_q \mathcal{M}^{q-1}_{\mathbf{x}} = \tilde{\mathcal{M}}^{q-1}$.

Let $\mathbf{y}$ be the projection of $\mathbf{x}$ onto $\tilde{\mathcal{M}}^{q-1}$:

$$\mathbf{y} = \Omega_q \mathbf{x} = \mathbf{x} - \bar{x}\mathbf{e}^q, \tag{2.8}$$

where

$$\bar{x} \equiv \tfrac{1}{q}\mathbf{x}'\mathbf{e}^q = \tfrac{1}{q}\textstyle\sum_{i=1}^q x_i$$

is the average of the q components of $\mathbf{x}$. Then $\bar{y} = 0$ since $\mathbf{y}'\mathbf{e}^q = 0$, so

$$\|\mathbf{y}\|^2 = \|\mathbf{x} - \bar{x}\mathbf{e}^q\|^2 = \textstyle\sum_{i=1}^q (x_i - \bar{x})^2 = \sum_{i=1}^q (y_i - \bar{y})^2, \tag{2.9}$$

which is proportional to their sample variance. Note that $y_1 \le \cdots \le y_q$, so

$$\mathbf{y} \in \tilde{\mathcal{M}}^{q-1}_{\le} := \tilde{\mathcal{M}}^{q-1} \cap \mathbb{R}^q_{\le}. \tag{2.10}$$

Because $\Omega_q P = P\Omega_q$ for all $P \in \mathcal{P}^q$,

$$\Pi(\mathbf{y}) = \Pi(\Omega_q \mathbf{x}) = \Omega_q(\Pi(\mathbf{x})), \tag{2.11}$$

so $\Pi(\mathbf{y})$ is a rigid translation of $\Pi(\mathbf{x})$ and satisfies

$$\Pi(\mathbf{y}) \subset \mathcal{S}^{q-1}_{\|\mathbf{y}\|} \cap \tilde{\mathcal{M}}^{q-1} =: \tilde{\mathcal{S}}^{q-2}_{\|\mathbf{y}\|}, \tag{2.12}$$

the 0-centered $(q-2)$-sphere of radius $\|\mathbf{y}\|$ in $\tilde{\mathcal{M}}^{q-1}$. If the (discrete) uniform distribution on $\Pi(\mathbf{y})$ is denoted by $\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}}$ and the (continuous) uniform distribution on $\tilde{\mathcal{S}}^{q-2}_{\|\mathbf{y}\|}$ denoted by $\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}\|}$, then Question 1 can be restated equivalently as follows:

Question 2. *For large q, do there exist nonzero vectors $\mathbf{y} \in \tilde{\mathcal{M}}^{q-1}_{\leq}$ such that $\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}}$ closely approximates $\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}\|}$? More precisely, do there exist sequences $\{\mathbf{y}^q \in \tilde{\mathcal{M}}^{q-1}_{\leq}\}$ such that the discrepancy between $\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}^q}$ and $\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}^q\|}$ approaches zero as $q \to \infty$?*

3 Measures of spherical discrepancy

If we abuse notation by letting $\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}}$ and $\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}\|}$ also denote random vectors having these distributions, then the possible existence of the vectors $\mathbf{y}$ and sequences $\{\mathbf{y}^q\}$ in Question 2 is supported by the fact that the first and second moments of $\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}}$ and $\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}\|}$ coincide:

$$\mathrm{E}(\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}}) = E(\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}\|}) = 0, \tag{3.1}$$

$$\mathrm{Cov}(\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}}) = \mathrm{Cov}(\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}\|}) = \tfrac{\|\mathbf{y}\|^2}{q(q-1)}(qI_q - \mathbf{e}^q(\mathbf{e}^q)'). \tag{3.2}$$

(In fact all odd moments agree since these are 0 by symmetry.) Three measures of the discrepancy between $\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}}$ and $\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}\|}$ will be considered.

For nonzero $\mathbf{w} \in \tilde{\mathcal{M}}^{q-1}$, $-1 \leq t < 1$, and $\rho > 0$ define

$$C(\mathbf{w}; t) := \big\{\mathbf{v} \in \tilde{\mathcal{S}}^{q-2}_{\|\mathbf{w}\|} \;\big|\; \mathbf{v}'\mathbf{w} > \|\mathbf{w}\|^2 t\big\}, \tag{3.3}$$

$$\tilde{\mathcal{C}}^{q-2}_{\rho} := \big\{C(\mathbf{w}; t) \;\big|\; \mathbf{w} \in \tilde{\mathcal{S}}^{q-2}_{\rho},\ -1 \leq t < 1\big\}. \tag{3.4}$$

Thus $C(\mathbf{w}; t)$ is the open spherical cap in $\tilde{\mathcal{S}}^{q-2}_{\|\mathbf{w}\|}$ of angular half-width $\cos^{-1}(t)$ centered at $\mathbf{w}$, while $\tilde{\mathcal{C}}^{q-2}_{\rho}$ is the set of all such spherical caps in $\tilde{\mathcal{S}}^{q-2}_{\rho}$.

If $\mathbf{U}$ is uniformly distributed over the unit $(q-2)$-sphere in $\mathbb{R}^{q-1}$ then for any $(q \times 1)$-dimensional unit vector $\mathbf{u}$,

$$(\mathbf{u}'\mathbf{U})^2 \stackrel{d}{=} \mathrm{Beta}\big(\tfrac{1}{2}, \tfrac{q-2}{2}\big). \tag{3.5}$$

Thus, if $0 \leq t < 1$ then the normalized $(q-2)$-dimensional surface area of the spherical cap $C(\mathbf{w}; t) \subset \tilde{\mathcal{S}}^{q-2}_{\|\mathbf{w}\|}$ is given by

$$\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{w}\|}(C(\mathbf{w}; t)) = \tfrac{1}{2}\Pr\big[\mathrm{Beta}\big(\tfrac{1}{2}, \tfrac{q-2}{2}\big) > t^2\big] \tag{3.6}$$

$$= \tfrac{\Gamma(\frac{q-1}{2})}{2\Gamma(\frac{1}{2})\Gamma(\frac{q-2}{2})} \int_{t^2}^1 w^{-\frac{1}{2}}(1-w)^{\frac{q}{2}-2}dw \tag{3.7}$$

$$= \tfrac{\Gamma(\frac{q-1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{q-2}{2})} \int_t^1 (1-v^2)^{\frac{q}{2}-2}dv \tag{3.8}$$

$$=: \beta^{q-2}(t) \tag{3.9}$$

a strictly decreasing smooth function of t.

The following two bounds for $\beta^{q-2}(t)$, $0 \le t < 1$, will be used. From (3.7),

$$\begin{aligned}\beta^{q-2}(t) &< \tfrac{\Gamma(\frac{q-1}{2})}{2\Gamma(\frac{1}{2})\Gamma(\frac{q-2}{2})} \cdot \tfrac{1}{t} \int_0^{1-t^2} u^{\frac{q}{2}-2} du \\ &= \tfrac{\Gamma(\frac{q-1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{q-2}{2})} \cdot \tfrac{(1-t^2)^{\frac{q}{2}-1}}{t(q-2)} \\ &\le \sqrt{\tfrac{q-2}{2\pi}} \cdot \tfrac{(1-t^2)^{\frac{q}{2}-1}}{t(q-2)} \qquad (3.10) \\ &= \tfrac{(1-t^2)^{\frac{q}{2}-1}}{t\sqrt{2\pi(q-2)}}. \qquad (3.11)\end{aligned}$$

The inequality used to obtain (3.10) appears in [20]. Second, from (3.8) and Wendell's inequality,

$$\begin{aligned}\tfrac{1}{2} - \beta^{q-2}(t) &= \tfrac{1}{2} \Pr\left[0 \le \mathrm{Beta}\left(\tfrac{1}{2}, \tfrac{q-2}{2}\right) \le t^2\right] \qquad (3.12) \\ &= \tfrac{\Gamma(\frac{q-1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{q-2}{2})} \int_0^t (1-v^2)^{\frac{q}{2}-2} dv \\ &\le \tfrac{\Gamma(\frac{q-1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{q-2}{2})} \int_0^t e^{-\frac{v^2(q-4)}{2}} dv \\ &= \tfrac{\Gamma(\frac{q-1}{2})\sqrt{2}}{\Gamma(\frac{q-2}{2})\sqrt{q-4}} \cdot \tfrac{1}{\sqrt{2\pi}} \int_0^{t\sqrt{q-4}} e^{-\frac{z^2}{2}} dz \\ &\le \sqrt{\tfrac{q-2}{q-4}} \cdot \left[\Phi\left(t\sqrt{q-4}\right) - \tfrac{1}{2}\right], \qquad (3.13)\end{aligned}$$

where Φ denotes the standard normal cumulative distribution function.

Lemma 3.1. *Let $\{t^q\}$ be a sequence in $[0,1)$ and let $0 \le \lambda \le \infty$. Then*

$$\lim_{q\to\infty} \beta^{q-2}(t^q) = 1 - \Phi(\lambda) \iff \lim_{q\to\infty} t^q\sqrt{q} = \lambda. \qquad (3.14)$$

Proof. Let X_1 and X_{q-2} denote independent chi-square variates with 1 and $q-2$ degrees of freedom. From (3.6) and (3.9),

$$\begin{aligned}\beta^{q-2}(t^q) &= \tfrac{1}{2} \Pr\left[\mathrm{Beta}\left(\tfrac{1}{2}, \tfrac{q-2}{2}\right) > (t^q)^2\right] \\ &= \tfrac{1}{2} \Pr\left[\tfrac{X_1}{X_1+X_{q-2}} > (t^q)^2\right] \\ &= \tfrac{1}{2} \Pr\left[\tfrac{X_1}{X_{q-2}/(q-2)} > \tfrac{(t^q)^2(q-2)}{1-(t^q)^2}\right].\end{aligned}$$

Thus, because $\frac{X_{q-2}}{q-2} \xrightarrow{p} 1$ by the Law of Large Numbers and $X_1 \stackrel{d}{=} [N(0,1)]^2$,

$$\lim_{q\to\infty} \beta^{q-2}(t^q) = 1 - \Phi(\lambda) \iff \lim_{q\to\infty} \tfrac{t^q\sqrt{q-2}}{\sqrt{1-(t^q)^2}} = \lambda. \qquad (3.15)$$

It is straightforward to show that

$$\lim_{q\to\infty} \tfrac{t^q\sqrt{q-2}}{\sqrt{1-(t^q)^2}} = \lambda \iff \lim_{q\to\infty} t^q\sqrt{q} = \lambda \qquad (3.16)$$

(consider the cases $0 \le \lambda < \infty$ and $\lambda = \infty$ separately), hence (3.14) holds. □

For nonzero $\mathbf{y} \in \tilde{\mathcal{M}}^{q-1}_{\le}$ and any nonempty finite subset $N \subset \tilde{\mathcal{S}}^{q-2}_{\|\mathbf{y}\|}$, let $\tilde{\mathbf{U}}^{q-2}_N$ denote the (discrete) uniform distribution on N; thus $\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}} = \tilde{\mathbf{U}}^{q-2}_{\Pi(\mathbf{y})}$.

Definition 3.2. *The* normalized spherical cap discrepancy (NSCD) *of N in $\tilde{\mathcal{S}}^{q-2}_{\|\mathbf{y}\|}$ is defined as*[2]

$$\begin{aligned}D^{q-2}(N) &:= \sup\left\{\left|\tilde{\mathbf{U}}^{q-2}_N(C) - \tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}\|}(C)\right| \;\middle|\; C \in \tilde{\mathcal{C}}^{q-2}_{\|\mathbf{y}\|}\right\} \qquad (3.17) \\ &= \sup\left\{\left|\tfrac{|N\cap C|}{|N|} - \tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}\|}(C)\right| \;\middle|\; C \in \tilde{\mathcal{C}}^{q-2}_{\|\mathbf{y}\|}\right\}, \qquad (3.18)\end{aligned}$$

[2] See [13] Def. 2.11.5, [14] Section 1, [1] Section 1.2. Unlike [1] we divide $|N \cap C|$ by $|N|$ to be able to compare NSCD's of differing dimensions. Thus $D^{q-2}(N)$ is a distance measure between the probability distributions $\tilde{\mathbf{U}}^{q-2}_N$ and $\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}\|}$.

where $|N \cap C|$ *and* $|N|$ *are the cardinalities of* $N \cap C$ *and* N*. The* largest empty cap discrepancy (LECD) *of* N *in* $\tilde{\mathcal{S}}^{q-2}_{\|\mathbf{y}\|}$ *is defined as*[3]

$$\begin{aligned} L^{q-2}(N) &:= \sup\Big\{\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}\|}(C) \;\Big|\; C \in \tilde{\mathcal{C}}^{q-2}_{\|\mathbf{y}\|},\, N \cap C = \emptyset\Big\} \\ &= \sup\Big\{\beta^{q-2}(t) \;\Big|\; \exists \mathbf{w} \in \tilde{\mathcal{S}}^{q-2}_{\|\mathbf{y}\|},\, C(\mathbf{w};\, t) \in \tilde{\mathcal{C}}^{q-2}_{\|\mathbf{y}\|},\, N \cap C(\mathbf{w};\, t) = \emptyset\Big\}. \quad \square \qquad (3.19) \end{aligned}$$

Obviously

$$0 \le L^{q-2}(N) \le D^{q-2}(N) \le 1. \qquad (3.20)$$

Note that the suprema in (3.17)–(3.19) must be maxima, i.e., must be attained. This follows by applying the Blashke Selection Theorem to

$$\big\{\overline{\mathrm{co}}(C) \;\big|\; C \in \tilde{\mathcal{C}}^{q-2}_{\|\mathbf{y}\|}\big\},$$

a collection of closed convex subsets of the closed ball $\tilde{B}$ bounded by $\tilde{\mathcal{S}}^{q-2}_{\|\mathbf{y}\|}$, where $\overline{\mathrm{co}}(C)$ denotes the closed convex hull in $\tilde{B}$ of the spherical cap C. It follows from this that

$$L^{q-2}(N) = \beta^{q-2}\big(t(N)\big), \qquad (3.21)$$

where

$$t(N) = \min\Big\{t \;\Big|\; \exists\, \mathbf{w} \in \tilde{\mathcal{S}}^{q-2}_{\|\mathbf{y}\|},\, C(\mathbf{w};\, t) \in \tilde{\mathcal{C}}^{q-2}_{\|\mathbf{y}\|},\, N \cap C(\mathbf{w};\, t) = \emptyset\Big\}. \qquad (3.22)$$

Define the unit vectors $\mathbf{z}^q_k \in \tilde{\mathcal{M}}^{q-1}_{\le}$, $k = 1, \dots, q-1$ as follows:

$$\mathbf{z}^q_k := \sqrt{\tfrac{1}{q}}\Big(\underbrace{-\sqrt{\tfrac{q-k}{k}}, \dots, -\sqrt{\tfrac{q-k}{k}}}_{k}, \underbrace{\sqrt{\tfrac{k}{q-k}}, \dots, \sqrt{\tfrac{k}{q-k}}}_{q-k}\Big)'. \qquad (3.23)$$

For $1 \le k < l \le q-1$, the inner product between $\mathbf{z}^q_k$ and $\mathbf{z}^q_l$ is found to be

$$(\mathbf{z}^q_k)'\mathbf{z}^q_l = \sqrt{\tfrac{k(q-l)}{(q-k)l}} > 0. \qquad (3.24)$$

Lemma 3.3. *For nonzero* $\mathbf{y} \in \tilde{\mathcal{M}}^{q-1}_{\le}$*,*

$$t(\Pi(\mathbf{y})) = \tfrac{1}{\|\mathbf{y}\|} \min_{1 \le k \le q-1} \mathbf{y}'\mathbf{z}^q_k, \qquad (3.25)$$

$$L^{q-2}\big(\Pi(\mathbf{y})\big) \le \tfrac{1}{2}. \qquad (3.26)$$

Proof. For (3.25), it follows from (3.3) that if $\mathbf{w} \in \tilde{\mathcal{S}}^{q-2}_{\|\mathbf{y}\|}$ then

$$\Pi(\mathbf{y}) \cap C(\mathbf{w}; t) = \emptyset \iff \max_{P \in \mathcal{P}^q} (P\mathbf{y})'\mathbf{w} \le \|\mathbf{y}\|^2 t. \qquad (3.27)$$

Thus from (3.22) and the Rearrangement Inequality,

$$\begin{aligned} t(\Pi(\mathbf{y})) &= \tfrac{1}{\|\mathbf{y}\|^2} \min_{\mathbf{w} \in \tilde{\mathcal{S}}^{q-2}_{\|\mathbf{y}\|}} \max_{P \in \mathcal{P}^q} (P\mathbf{y})'\mathbf{w} \qquad (3.28) \\ &= \tfrac{1}{\|\mathbf{y}\|^2} \min_{\mathbf{w} \in \tilde{\mathcal{M}}^{q-1}_{\le},\, \|\mathbf{w}\| = \|\mathbf{y}\|} \mathbf{y}'\mathbf{w} \qquad (3.29) \end{aligned}$$

The set $\tilde{\mathcal{M}}^{q-1}_{\le}$ is a pointed convex simplicial cone[4] whose $q-1$ extreme rays are spanned by $\mathbf{z}^q_1, \dots, \mathbf{z}^q_{q-1}$, so $\tilde{\mathcal{M}}^{q-1}_{\le}$ is their nonnegative span. Thus for $\mathbf{w} \in \tilde{\mathcal{M}}^{q-1}_{\le}$ with $\|\mathbf{w}\| = \|\mathbf{y}\|$,

$$\mathbf{w} = \|\mathbf{y}\| \cdot \tfrac{\lambda_1 \mathbf{z}^q_1 + \cdots + \lambda_{q-1}\mathbf{z}^q_{q-1}}{\|\lambda_1 \mathbf{z}^q_1 + \cdots + \lambda_{q-1}\mathbf{z}^q_{q-1}\|},$$

[3] See [1] Section 1.2.

[4] The geometric properties of the polyhedral cone $\tilde{\mathcal{M}}^{q-1}_{\le}$ that we use here stem from its role as a *fundamental region* of the finite reflection group (Coxeter group) of all $q \times q$ permutation matrices acting effectively on $\tilde{\mathcal{M}}^{q-1}$. A readable reference is [12]; also see [10].

for some $\lambda_1 \geq 0, \dots, \lambda_{q-1} \geq 0$ with $\lambda_1 + \cdots + \lambda_{q-1} = 1$. Therefore

$$\mathbf{y}'\mathbf{w} \geq \|\mathbf{y}\| \min_{1 \leq k \leq q-1} \mathbf{y}'\mathbf{z}_k^q,$$

since $\mathbf{y} \in \tilde{\mathcal{M}}_{\leq}^{q-1} \Rightarrow \mathbf{y}'\mathbf{z}_k^q \geq 0$ by (3.24) and $\|\lambda_1 \mathbf{z}_1^q + \cdots + \lambda_{q-1}\mathbf{z}_{q-1}^q\| \leq 1$, hence

$$\tfrac{1}{\|\mathbf{y}\|^2} \min_{\mathbf{w} \in \tilde{\mathcal{M}}_{\leq}^{q-1},\, \|\mathbf{w}\| = \|\mathbf{y}\|} \mathbf{y}'\mathbf{w} \geq \tfrac{1}{\|\mathbf{y}\|} \min_{1 \leq k \leq q-1} \mathbf{y}'\mathbf{z}_k^q. \tag{3.30}$$

However equality must hold in (3.30) because $\mathbf{w}_k := \|\mathbf{y}\|\, \mathbf{z}_k^q \in \tilde{\mathcal{M}}_{\leq}^{q-1}$ and $\mathbf{w}_k = \|\mathbf{y}\|$. This confirms (3.25).

For (3.26), suppose that $L^{q-2}\big(\Pi(\mathbf{y})\big) > \frac{1}{2}$. Then $\Pi(\mathbf{y})$ must be contained in the complement of some closed hemisphere in $\tilde{\mathcal{S}}_{\|\mathbf{y}\|}^{q-2}$, hence there is some $\mathbf{v_0} \in \tilde{\mathcal{S}}_{\|\mathbf{y}\|}^{q-2}$ such that $0 > \mathbf{w}'P\mathbf{y}$ for all $P \in \mathcal{P}^q$. Sum over P to obtain $0 > \mathbf{w}'(\mathbf{e}^q(\mathbf{e}^q)')\mathbf{y} = \mathbf{w}'\mathbf{e}^q((\mathbf{e}^q)'\mathbf{y}) = 0$, hence a contradiction. □

It is noted in [13] Lemma 2.11.6 and [14] Section 1 that if $\{N_n\}$ is a sequence of finite sets in $\tilde{\mathcal{S}}_{\|\mathbf{y}\|}^{q-2}$ (q fixed), then the uniform distribution on N_n converges weakly to $\tilde{\mathbf{U}}_{\|\mathbf{y}\|}^{q-2}$ as $n \to \infty$ iff $\lim_{n\to\infty} D^{q-2}(N_n) = 0$. This motivates the following definition.

Definition 3.4. *A sequence of nonzero vectors* $\{\mathbf{y}^q \in \tilde{\mathcal{M}}_{\leq}^{q-1}\}$ *(q varying) is* asymptotically permutation-uniform (APU) *if*

$$\lim_{q\to\infty} D^{q-2}(\Pi(\mathbf{y}^q)) = 0; \tag{3.31}$$

it is asymptotically permutation-full (APF) *if*

$$\lim_{q\to\infty} L^{q-2}(\Pi(\mathbf{y}^q)) = 0. \quad \square \tag{3.32}$$

By (3.20), APU $\Rightarrow$ APF.

We also require a definition of asymptotic emptiness for a sequence of nonzero vectors $\{\mathbf{y}^q \in \tilde{\mathcal{M}}_{\leq}^{q-1}\}$. Because $\Pi(\mathbf{y}^q)$ is a finite subset of the sphere $\tilde{\mathcal{S}}_{\|\mathbf{y}^q\|}^{q-2}$, it always holds that $\tilde{\mathcal{S}}_{\|\mathbf{y}^q\|}^{q-2} \backslash \Pi(\mathbf{y}^q)$ is an infinite union of very small empty spherical caps, so a more stringent definition of emptiness is required.

Definition 3.5. *A sequence of nonzero vectors* $\{\mathbf{y}^q \in \tilde{\mathcal{M}}_{\leq}^{q-1}\}$ *(q varying) is* asymptotically permutation-empty (APE) *if* $\exists\, \epsilon > 0$ *and, for each* q, $\exists$ *a finite collection* $\{C_i^q \mid i = 1, \dots, n^q\}$ *of (possibly overlapping) empty spherical caps in* $\tilde{\mathcal{S}}_{\|\mathbf{y}^q\|}^{q-2} \backslash \Pi(\mathbf{y}^q)$ *such that* $\tilde{\mathbf{U}}_{\|\mathbf{y}^q\|}^{q-2}(C_i^q) \geq \epsilon$ *and*

$$\lim_{q\to\infty} \tilde{\mathbf{U}}_{\|\mathbf{y}^q\|}^{q-2}(\cup_{i=1}^{n^q} C_i^q) = 1. \quad \square \tag{3.33}$$

If $\{\mathbf{y}^q \in \tilde{\mathcal{M}}_{\leq}^{q-1}\}$ is APE then $\Pi(\mathbf{y}^q)$ is asymptotically small in the sense that $\Pi(\mathbf{y}^q) \subseteq (\cup_{i=1}^{n^q} C_i^q)^c$ with $\tilde{\mathbf{U}}_{\|\mathbf{y}^q\|}^{q-2}\big((\cup_{i=1}^{n^q} C_i^q)^c\big) \to 0$ as $q \to \infty$. That is, $\Pi(\mathbf{y}^q)$ occupies only an increasingly negligible portion of the sphere $\tilde{\mathcal{S}}_{\|\mathbf{y}^q\|}^{q-2}$. Clearly APE $\Rightarrow$ not APF $\Rightarrow$ not APU.

Now modify the definitions of LECD and APF as follows:

Definition 3.6. *The* largest empty cap angular discrepancy (LECAD) *of* N *in* $\tilde{\mathcal{S}}_{\|\mathbf{y}\|}^{q-2}$ *is defined to be*

$$\begin{aligned} A^{q-2}(N) &:= \sup\Big\{ \cos^{-1}(t) \;\Big|\; \exists \mathbf{w} \in \tilde{\mathcal{S}}_{\|\mathbf{y}\|}^{q-2},\, C(\mathbf{w};\, t) \in \tilde{\mathcal{C}}_{\|\mathbf{y}\|}^{q-2},\, N \cap C(\mathbf{w};\, t) = \emptyset \Big\} \\ &= \cos^{-1}(t(N)), \end{aligned} \tag{3.34}$$

where $t(N)$ *is defined in* (3.22). *A sequence of nonzero vectors* $\{\mathbf{y}^q \in \tilde{\mathcal{M}}_{\leq}^{q-1}\}$ *(q varying) is* asymptotically permutation-dense (APD) *if*

$$\lim_{q\to\infty} A^{q-2}(\Pi(\mathbf{y}^q)) = 0. \quad \square \tag{3.35}$$

Note that (3.21) and (3.34) yield the relation

$$L^{q-2}(N) = \beta^{q-2}(\cos(A^{q-2}(N))). \tag{3.36}$$

If we set $t^q = t(\Pi(\mathbf{y}^q))$, it follows from (3.34) and (3.14) with $\lambda = \infty$ that

$$\{\mathbf{y}^q\}\ \text{APD} \iff \lim_{q\to\infty} \cos^{-1}(t^q) = 0 \iff t^q \to 1 \Longrightarrow \beta^{q-2}(t^q) \to 0,$$

hence APD ⇒ APF. However the converse need not hold: it will be shown in Section 5 that the sequence $\{\hat{\mathbf{y}}^q\}$ of *L-minimal configurations* defined in (5.8) is APF but not APD.

Remark 3.7. Consider a sequence of spherical caps $C(\mathbf{w}^q; t^q) \subseteq \tilde{\mathcal{M}}^{q-1}$ such that $t^q \to 0$ while $t^q\sqrt{q} \to \infty$. Then $\cos^{-1}(t^q) \to \frac{\pi}{2}$, while $\beta^{q-2}(t^q) \to 0$ by (3.14) with $\lambda = \infty$, that is, the spherical caps approach hemispheres in terms of their angular measure but their surface areas approach 0. An example can be seen in Section 5 by taking $C(\mathbf{w}^q; t^q)$ to be the largest empty spherical cap for the set $\Pi(\hat{\mathbf{y}}^q)$, then comparing (5.28) with (7.16). □

Question 2 now can be refined further as follows:

Question 3. *For which $\mathbf{y} \in \tilde{\mathcal{M}}^{q-1}_{\leq}$, if any, are $D^{q-2}(\Pi(\mathbf{y}))$, $L^{q-2}(\Pi(\mathbf{y}))$, and/or $A^{q-2}(\Pi(\mathbf{y}))$ small? Which sequences $\{\mathbf{y}^q \in \tilde{\mathcal{M}}^{q-1}_{\leq}\}$, if any, are APU? APF? APD? APE?*

Some answers to these questions will be derived in Sections 4–6 and summarized in Section 7; for example, no APD sequence exists (Proposition 7.1). Some results about the volumes of the corresponding permutohedra with vertices $\Pi(\mathbf{y}^q)$ are presented in section 8.

Example 3.8. Despite the agreement of the first and second moments of $\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}}$ and $\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}\|}$ (cf. (3.1), (3.2)), $L^{q-2}(\Pi(\mathbf{y}))$ need not be small. For example, take $\mathbf{y} = \mathbf{f}^q_q$ where, for $i = 1, \ldots, q$, $\mathbf{f}^q_i \in \tilde{\mathcal{M}}^{q-1}_{\leq}$ is the unit column vector

$$\mathbf{f}^q_i = \tfrac{1}{\sqrt{q(q-1)}}(\underbrace{-1, \ldots, -1}_{i-1}, q-1, \underbrace{-1 \ldots, -1}_{q-i})'. \tag{3.37}$$

Here $\Pi(\mathbf{f}^q_q) = \{\mathbf{f}^q_1, \ldots, \mathbf{f}^q_q\}$, so $|\Pi(\mathbf{f}^q_q)| = q$ not $q!$. From (3.25), (3.21), and (3.14) with $t^q = \frac{1}{q-1}$ and $\lambda = 0$,

$$t(\Pi(\mathbf{f}^q_q)) = \tfrac{1}{q-1}, \tag{3.38}$$

$$L^{q-2}(\Pi(\mathbf{f}^q_q)) = \beta^{q-2}\big(\tfrac{1}{q-1}\big) \to \tfrac{1}{2} \tag{3.39}$$

as $q \to \infty$. Thus the sequence $\{\mathbf{f}^q_q\}$ is not APF, hence not APU. □

Remark 3.9. For later use, we note that for $i = 1, \ldots, q$,

$$\mathbf{f}^q_i = \tfrac{1}{\sqrt{q(q-1)}}(q\mathbf{e}^q_i - \mathbf{e}^q) = \sqrt{\tfrac{q}{q-1}}\,\Omega_q \mathbf{e}^q_i, \tag{3.40}$$

where $\mathbf{e}^q_i \equiv (0, \ldots, 0, 1, 0, \ldots, 0)'$ denotes the ith coordinate vector in $\mathbb{R}^q$ and $\Omega_q \mathbf{e}^q_i$ is the projection of $\mathbf{e}^q_i$ onto $\tilde{\mathcal{M}}^{q-1}$. Thus $\mathbf{f}^q_1, \ldots, \mathbf{f}^q_q$ form the vertices of a standard simplex in $\tilde{\mathcal{M}}^{q-1}$: an equilateral triangle when $q = 3$, a regular tetrahedron when $q = 4$, etc. □

For any nonzero $\mathbf{y}^q \in \tilde{\mathcal{M}}^{q-1}_{\leq}$, $\frac{\sqrt{q-1}}{\|\mathbf{y}^q\|}\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}^q\|}$ is uniformly distributed on the sphere of radius $\sqrt{q-1}$ in $\tilde{\mathcal{M}}^{q-1}$. It is well known (cf. [8, 18], [9] Proposition 7.5), and also follows from (3.6)–(3.9) and Lemma 3.1, that the marginal distributions from this uniform distribution converge to the standard normal distribution $N(0,1)$ as $q \to \infty$. More precisely, for any sequence of unit vectors $\{\mathbf{u}^q\}$ in $\tilde{\mathcal{M}}^{q-1}$,

$$(\mathbf{u}^q)'\Big(\tfrac{\sqrt{q-1}}{\|\mathbf{y}^q\|}\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}^q\|}\Big) = \tfrac{\sqrt{q-1}}{\|\mathbf{y}^q\|}(\mathbf{u}^q)'\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}^q\|} \xrightarrow{d} N(0,1)$$

as $q \to \infty$. If we take $\mathbf{u}^q = \mathbf{f}_i^q = \sqrt{\frac{q}{q-1}}\Omega_q \mathbf{e}_i^q$ (see (3.40)) for any fixed i, where $\mathbf{e}_i^q \equiv (0,\dots,0,1,0,\dots,0)'$ is the ith coordinate vector in $\mathbb{R}^q$, then

$$\tfrac{\sqrt{q-1}}{\|\mathbf{y}^q\|}(\mathbf{u}^q)'\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}^q\|} = \tfrac{\sqrt{q}}{\|\mathbf{y}^q\|}(\mathbf{e}_i^q)'\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}^q\|} \equiv \tfrac{\sqrt{q}}{\|\mathbf{y}^q\|}(\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}^q\|})_i \xrightarrow{d} N(0,1) \tag{3.41}$$

as $q \to \infty$, where $(\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}^q\|})_i$ denotes the ith component of $\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}^q\|}$.

Proposition 3.10. *A necessary condition that a sequence $\{\mathbf{y}^q \in \tilde{\mathcal{M}}^{q-1}_{\leq}\}$ of nonzero vectors be APU is that for each fixed $i \geq 1$,*

$$\tfrac{\sqrt{q}}{\|\mathbf{y}^q\|}(\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}^q})_i \xrightarrow{d} N(0,1) \tag{3.42}$$

as $q \to \infty$, where $(\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}^q})_i$ denotes the ith component of $\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}^q}$.

Proof. From (3.17) and (3.3)–(3.4),

$$\begin{aligned}
&D^{q-2}(\Pi(\mathbf{y}^q)) \\
&= \sup\left\{\left|\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}^q}(C) - \tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}^q\|}(C)\right| \;\middle|\; C \in \tilde{\mathcal{C}}^{q-2}_{\|\mathbf{y}\|}\right\} \\
&\geq \sup_{-1\leq t<1}\left|\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}^q}(C(\|\mathbf{y}^q\|\mathbf{f}_i^q;t)) - \tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}^q\|}(C(\|\mathbf{y}^q\|\mathbf{f}_i^q;t))\right| \\
&= \sup_{-1\leq t<1}\left|\Pr\big[(\mathbf{f}_i^q)'\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}^q} > \|\mathbf{y}^q\|\,t\big] - \Pr\big[(\mathbf{f}_i^q)'\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}^q\|} > \|\mathbf{y}^q\|\,t\big]\right| \\
&= \sup_{-1\leq t<1}\left|\Pr\Big[\sqrt{\tfrac{q}{q-1}}(\mathbf{e}_i^q)'\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}^q} > \|\mathbf{y}^q\|\,t\Big] - \Pr\Big[\sqrt{\tfrac{q}{q-1}}(\mathbf{e}_i^q)'\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}^q\|} > \|\mathbf{y}^q\|\,t\Big]\right| \\
&= \sup_{-1\leq t<1}\left|\Pr\Big[\tfrac{\sqrt{q}}{\|\mathbf{y}^q\|}(\tilde{\mathbf{U}}^{q-2}_{\mathbf{y}^q})_i > \sqrt{q-1}\,t\Big] - \Pr\Big[\tfrac{\sqrt{q}}{\|\mathbf{y}^q\|}(\tilde{\mathbf{U}}^{q-2}_{\|\mathbf{y}^q\|})_i > \sqrt{q-1}\,t\Big]\right|.
\end{aligned}$$

Because $D^{q-2}(\Pi(\mathbf{y}^q)) \to 0$ if $\{\mathbf{y}^q\}$ is APU, this and (3.41) yield (3.42). □

4 The regular configurations $\bar{\mathbf{x}}^q$, $\bar{\mathbf{y}}^q$ are not spherically uniform

It is seen from (3.26) and (3.39) that $\{\mathbf{f}_q^q\}$ fails to be APF (hence fails to be APU and APD) to the greatest possible extent. Clearly this is due to the fact that the components of $\mathbf{f}_q^q$ comprise only two distinct values -1 and $q-1$. This suggests that the APU, APF, and APD properties are more likely to hold for vectors $\mathbf{y}^q \equiv \Omega_q \mathbf{x}^q \in \tilde{\mathcal{M}}^{q-1}_{\leq}$ whose components are distinct, so that $|\Pi(\mathbf{y}^q)|$, equivalently $|\Pi(\mathbf{x}^q)|$, attains its maximum value $q!$.

At this point, one might conjecture that the APU, APD, and APF properties are most likely to hold for vectors whose components are evenly spaced, that is, for the vectors

$$\bar{\mathbf{x}}^q = (1, 2, \dots, q)', \tag{4.1}$$

$$\bar{\mathbf{y}}^q = \Omega_q \bar{\mathbf{x}}^q = (-\tfrac{q-1}{2}, -\tfrac{q-3}{2}, \dots, \tfrac{q-3}{2}, \tfrac{q-1}{2})'. \tag{4.2}$$

We call $\bar{\mathbf{x}}^q$ and $\bar{\mathbf{y}}^q$ the *regular configurations* in $\mathbb{R}^q_{\leq}$ and $\tilde{\mathcal{M}}^{q-1}_{\leq}$ respectively. This conjecture is supported by the case $q = 2$ with $\bar{\mathbf{y}}_2 = (-\frac{1}{2}, \frac{1}{2})'$, where the two permutations $(-\frac{1}{2}, \frac{1}{2})'$ and $(\frac{1}{2}, -\frac{1}{2})'$ trivially are uniformly distributed on $\tilde{\mathcal{S}}^0_{\|\bar{\mathbf{y}}^2\|}$, and by the case $q = 3$ with $\bar{\mathbf{y}}^3 = (1, 2, 3)'$, where the $3! = 6$ permutations of $\bar{\mathbf{y}}^3$ comprise the vertices of a regular hexagon, the most uniform among all configurations of 6 points on the circle $\tilde{\mathcal{S}}^1_{\|\bar{\mathbf{y}}^3\|}$.

When $q = 4$, however, the $4! = 24$ permutations of $\mathbf{y}^{(4)} \equiv (-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ comprise the vertices of the regular permutohedron $\tilde{\mathfrak{R}}^4$ (see Section 8), a truncated octahedron whose 14 faces consist of 8 regular hexagons and 6 squares, hence is not a regular solid. Thus support for the conjecture begins to waver.

In this section we present two arguments that show this asymptotic spherical uniformity conjecture is strongly invalid for the regular configurations. The first argument (Propositions 4.1 and 4.2) examines the APF and APE properties for $\{\bar{\mathbf{x}}^q\}$ and $\{\bar{\mathbf{y}}^q\}$, the second argument (Proposition 4.4) compares the univariate marginal distributions of $\tilde{\mathbf{U}}^{q-2}_{\bar{\mathbf{y}}^q}$ and $\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$. A third comparison of $\tilde{\mathbf{U}}^{q-2}_{\bar{\mathbf{y}}^q}$ and $\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$ will be presented in Section 8.

Proposition 4.1. *The sequences of regular configurations* $\{\bar{\mathbf{x}}^q \in \mathbb{R}^q_{\leq}\}$ *and* $\{\bar{\mathbf{y}}^q \in \tilde{\mathcal{M}}^{q-1}_{\leq}\}$ *are not APF, hence not APU and not APD.*

Proof. It suffices to consider $\{\bar{\mathbf{y}}^q\}$. Beginning with the relations

$$\|\bar{\mathbf{y}}^q\| = \sqrt{\tfrac{1}{12}q(q^2-1)}, \tag{4.3}$$

$$(\bar{\mathbf{y}}^q)'\mathbf{z}^q_k = (\bar{\mathbf{x}}^q)'\mathbf{z}^q_k = \tfrac{1}{2}\sqrt{qk(q-k)}, \tag{4.4}$$

it follows from (3.21) and (3.25) that the LECD of $\Pi(\bar{\mathbf{y}}^q)$ is given by

$$L^{q-2}\big(\Pi(\bar{\mathbf{y}}^q)\big) = \beta^{q-2}\Big(\tfrac{1}{\|\bar{\mathbf{y}}^q\|}\min_{1\leq k\leq q-1}(\bar{\mathbf{y}}^q)'\mathbf{z}^q_k\Big) \tag{4.5}$$

$$= \beta^{q-2}\Big(\sqrt{\tfrac{3}{q+1}}\Big), \tag{4.6}$$

where the minimum is attained for $k=1$ and $k=q-1$. From Lemma 3.1 with $\lambda=\sqrt{3}$,

$$\lim_{q\to\infty}\beta^{q-2}\Big(\sqrt{\tfrac{3}{q+1}}\Big) = 1-\Phi(\sqrt{3}) \approx .0416 > 0, \tag{4.7}$$

so $\{\bar{\mathbf{y}}^q\}$ is not APF. □

In fact, $\{\bar{\mathbf{x}}^q\}$ and $\{\bar{\mathbf{y}}^q\}$ fail asymptotic uniformity in a stronger sense:

Proposition 4.2. *The regular configurations* $\{\bar{\mathbf{x}}^q\}$ *and* $\{\bar{\mathbf{y}}^q\}$ *are APE.*

Proof. Again it suffices to consider $\{\bar{\mathbf{y}}^q\}$. Define $\bar{\mathbf{z}}^q_k = \|\bar{\mathbf{y}}^q\|\mathbf{z}^q_k$, where $\mathbf{z}^q_k$ is the unit vector in (3.23). Because the minimum in (4.5) is attained for $k=1$ and $q-1$, i.e., for $\mathbf{z}^q_1$ and $\mathbf{z}^q_{q-1}$, both $C\big(\bar{\mathbf{z}}^q_1; \sqrt{\tfrac{3}{q+1}}\big)$ and $C\big(\bar{\mathbf{z}}^q_{q-1}; \sqrt{\tfrac{3}{q+1}}\big)$ are (overlapping) largest empty spherical caps for $\Pi(\bar{\mathbf{y}}^q)$ in $\tilde{\mathcal{S}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$. (Note that $\mathbf{z}^q_1 = -\mathbf{f}^q_1$ and $\mathbf{z}^q_{q-1} = \mathbf{f}^q_q$.) Because $P\Pi(\bar{\mathbf{y}}^q) = \Pi(\bar{\mathbf{y}}^q)$ for all $P\in\mathcal{P}^q$, $C\big(P\bar{\mathbf{z}}^q_1; \sqrt{\tfrac{3}{q+1}}\big)$ and $C\big(P\bar{\mathbf{z}}^q_{q-1}; \sqrt{\tfrac{3}{q+1}}\big)$ also are (overlapping) largest empty spherical caps for $\Pi(\bar{\mathbf{y}}^q)$; there are $2q!$ such caps, all congruent. However

$$\{P\bar{\mathbf{z}}^q_1 \mid P\in\mathcal{P}^q\} = \{-\bar{\mathbf{f}}^q_1,\dots,-\bar{\mathbf{f}}^q_q\},$$
$$\{P\bar{\mathbf{z}}^q_{q-1} \mid P\in\mathcal{P}^q\} = \{\bar{\mathbf{f}}^q_1,\dots,\bar{\mathbf{f}}^q_q\},$$

where $\bar{\mathbf{f}}^q_i = \|\bar{\mathbf{y}}^q\|\mathbf{f}^q_i$, so these $2q!$ empty caps reduce to $2q$, namely

$$\Big\{C\big(-\bar{\mathbf{f}}^q_i; \sqrt{\tfrac{3}{q+1}}\big)\ \Big|\ i=1,\dots q\Big\} \cup \Big\{C\big(\bar{\mathbf{f}}^q_i; \sqrt{\tfrac{3}{q+1}}\big)\ \Big|\ i=1,\dots q\Big\}.$$

By (4.5)–(4.7), each of these congruent empty caps remains nonnegligible as $q\to\infty$, so $\{\bar{\mathbf{y}}^q\}$ is APE if

$$\lim_{q\to\infty}\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}(\Upsilon^q) = 1,$$

where

$$\Upsilon^q = \bigcup_{i=1}^q \Big[C\big(-\bar{\mathbf{f}}^q_i; \sqrt{\tfrac{3}{q+1}}\big) \cup C\big(\bar{\mathbf{f}}^q_i; \sqrt{\tfrac{3}{q+1}}\big)\Big].$$

Therefore, because

$$\tilde{\mathcal{S}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}\bigcap\big((\Upsilon_q)^c\big) = \tilde{\mathcal{S}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}\bigcap\big(\cap_{i=1}^q S^q_i\big),$$

where S_i^q is the closed symmetric slab

$$S_i^q = \Big\{ \mathbf{v} \in \tilde{\mathcal{M}}^{q-1} \;\Big|\; |\mathbf{v}'\bar{\mathbf{f}}_i^q| \le \|\bar{\mathbf{y}}^q\|^2 \sqrt{\tfrac{3}{q+1}} \Big\},$$

to show that $\{\bar{\mathbf{y}}^q\}$ is APE it suffices to show that

$$\lim_{q\to\infty} \tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}\big((\Upsilon_q)^c\big) \equiv \lim_{q\to\infty} \tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}\big(\cap_{i=1}^q S_i^q\big) = 0. \tag{4.8}$$

If $S_1^q, \dots, S_q^q$ were mutually geometrically orthogonal, i.e., if $\mathbf{f}_1^q, \dots, \mathbf{f}_q^q$ were orthonormal, then the S_i^q would be subindependent under $\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$ (cf. [3]), that is,

$$\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}(\cap_{i=1}^q S_i^q) \le \prod\nolimits_{i=1}^q \tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}(S_i^q),$$

which would readily yield (4.8). However, $(\mathbf{f}_i^q)'\mathbf{f}_j^q = -\frac{1}{q-1} \ne 0$ if $i \ne j$ so this approach fails.[5] Instead we can apply the cruder one-sided bound

$$\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}(\cap_{i=1}^q S_i^q) \le \tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}(\cap_{i=2}^q H_i^q), \tag{4.9}$$

where H_i^q is the halfspace

$$H_i^q := \big\{ \mathbf{v} \in \tilde{\mathcal{M}}^{q-1} \;\big|\; \mathbf{v}'\bar{\mathbf{f}}_i^q \le \|\bar{\mathbf{y}}^q\|^2 \sqrt{\tfrac{3}{q+1}} \big\}.$$

Again $H_2^q, \dots, H_q^q$ are not mutually geometrically orthogonal, but now this works in our favor: because $(\mathbf{f}_i^q)'\mathbf{f}_j^q < 0$ if $i \ne j$, the extension of Slepian's inequality to spherically symmetric density functions ([7], Lemma 5.1) and a standard approximation argument yields

$$\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}(\cap_{i=2}^q H_i^q) \le \tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}(\cap_{i=2}^q K_i^q), \tag{4.10}$$

where K_i^q is the halfspace

$$K_i^q := \big\{ \mathbf{v} \in \tilde{\mathcal{M}}^{q-1} \;\big|\; \mathbf{v}'\gamma_i^q \le \|\bar{\mathbf{y}}^q\| \sqrt{\tfrac{3}{q+1}} \big\}$$

and $\gamma_2^q, \dots, \gamma_q^q$ are the last $q-1$ columns of the Helmert matrix Γ^q in Section 2, which form an orthonormal basis in $\tilde{\mathcal{M}}^{q-1}$ so $(\gamma_i^q)'\gamma_j^q = 0$. Now Proposition A.1 in the Appendix and the orthogonal invariance of $\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$ imply that

$$\begin{aligned} \tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}(\cap_{i=2}^q K_i^q) &\le \textstyle\prod_{i=2}^q \tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}(K_i^q) && (4.11)\\ &= \big[\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}(K_i^q)\big]^{q-1} && (4.12)\\ &= \Big[1 - \beta^{q-2}\Big(\sqrt{\tfrac{3}{q+1}}\Big)\Big]^{q-1}. && (4.13) \end{aligned}$$

Therefore by (4.7),

$$\limsup_{q\to\infty} \big[\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}(\cap_{i=2}^q K_i^q)\big]^{\frac{1}{q-1}} \le \Phi(\sqrt{3}) \approx .9584, \tag{4.14}$$

hence by (4.8)–(4.10),

$$\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}((\Upsilon^q)^c) \le \tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}(\cap_{i=2}^q K_i^q) \le (.96)^{q-1} \tag{4.15}$$

for sufficiently large q. Thus (4.8) holds, in fact $\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}((\Upsilon^q)^c) \to 0$ at a geometric rate, hence $\{\bar{\mathbf{y}}^q\}$ is APE as asserted. □

[5] In fact, Theorem 2.1 of [7] suggests that $S_1^q, \dots, S_q^q$ may be superdependent under $\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$.

Remark 4.3. The above result can be framed in terms of statistical hypothesis testing. Based on one random observation $\mathbf{Y} \equiv (Y_1, \dots, Y_q)' \in \tilde{\mathcal{S}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$, suppose that it is wished to test the spherical-uniformity hypothesis H_0 that $\mathbf{Y} \stackrel{d}{=} \tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$ against the permutation-uniformity alternative H_1 that $\mathbf{Y} \stackrel{d}{=} \tilde{\mathbf{U}}^{q-2}_{\bar{\mathbf{y}}^q}$. Consider the test that rejects H_0 in favor of H_1 iff $\mathbf{Y} \in (\Upsilon^q)^c$, that is, iff

$$\max_{1 \le i \le q} |Y_i - \bar{Y}| \le \tfrac{q-1}{2},$$

where $\bar{Y} = \frac{1}{q}\sum_{i=1}^q Y_i$. The size of this test is $\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}((\Upsilon^q)^c)$, which by (4.15) rapidly approaches 0 as $q \to \infty$, while its power $= 1$ for every q because $\Pi(\bar{\mathbf{y}}^q) \subset (\Upsilon^q)^c$. □

A second argument for the invalidity of the spherical uniformity conjecture for the regular configuration $\{\bar{\mathbf{y}}^q\}$ (and $\{\bar{\mathbf{x}}^q\}$) stems from Proposition 3.10 and the following fact:

Proposition 4.4. *For each fixed $i \ge 1$, as $q \to \infty$,*

$$\sqrt{\tfrac{12}{q^2-1}}(\tilde{\mathbf{U}}^{q-2}_{\bar{\mathbf{y}}^q})_i \stackrel{d}{\to} \mathrm{Uniform}\big(-\sqrt{3}, \sqrt{3}\big) \tag{4.16}$$

as $q \to \infty$. Thus $\{\bar{\mathbf{y}}^q\}$ does not satisfy (3.42)*, hence is not APU.*

Proof. By (4.2), for each $i = 1, \dots, q$, $(\tilde{\mathbf{U}}^{q-2}_{\bar{\mathbf{y}}^q})_i$ is uniformly distributed over the range $\{-\frac{q-1}{2}, -\frac{q-3}{2}, \dots, \frac{q-3}{2}, \frac{q-1}{2}\}$. Therefore $\sqrt{\frac{12}{q^2-1}}(\tilde{\mathbf{U}}^{q-2}_{\bar{\mathbf{y}}^q})_i$ is uniformly distributed over the range

$$\sqrt{3}\Big\{-\tfrac{q-1}{\sqrt{q^2-1}}, -\tfrac{q-3}{\sqrt{q^2-1}}, \dots, \tfrac{q-3}{\sqrt{q^2-1}}, \tfrac{q-1}{\sqrt{q^2-1}}\Big\}, \tag{4.17}$$

from which (4.16) follows readily. □

5 A more favorable configuration for spherical uniformity

It was shown in Proposition 4.1 that the regular configurations $\bar{\mathbf{x}}^q$ and $\bar{\mathbf{y}}^q$ are not APF, hence not APU or APD, although the components of $\bar{\mathbf{x}}^q$ and $\bar{\mathbf{y}}^q$ are exactly evenly spaced. A possibly more favorable configuration for spherical uniformity of permutations is now constructed, namely, a nonzero vector $\hat{\mathbf{y}}^q$ in $\tilde{\mathcal{S}}^{q-2}_{\|\bar{\mathbf{y}}^q\|} \cap \mathbb{R}^q_{\le}$ that minimizes the LECD $L^{q-2}(\Pi(\mathbf{y}))$ in $\tilde{\mathcal{S}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$; equivalently, that minimizes the LECAD $A^{q-2}(\Pi(\mathbf{y}))$. We call $\hat{\mathbf{y}}^q$ the *L-minimal configuration.*

For any nonzero vector $\mathbf{y}$ in $\tilde{\mathcal{S}}^{q-2}_{\|\bar{\mathbf{y}}^q\|} \cap \mathbb{R}^q_{\le}$, (3.21), (3.25), and (3.36) yield

$$L^{q-2}(\Pi(\mathbf{y})) = \beta^{q-2}(t(\Pi(\mathbf{y}))) = \beta^{q-2}\Big(\tfrac{1}{\|\mathbf{y}\|}\min_{1 \le k \le q-1} \mathbf{y}'\mathbf{z}^q_k\Big), \tag{5.1}$$

$$A^{q-2}(\Pi(\mathbf{y})) = \cos^{-1}(t(\Pi(\mathbf{y}))) = \cos^{-1}\Big(\tfrac{1}{\|\mathbf{y}\|}\min_{1 \le k \le q-1} \mathbf{y}'\mathbf{z}^q_k\Big). \tag{5.2}$$

Thus, because $\beta^{q-2}(\cdot)$ and $\cos^{-1}(\cdot)$ are decreasing and $\frac{\mathbf{y}}{\|\mathbf{y}\|}$ is a unit vector, we seek a unit vector $\mathbf{z} \equiv (z_1, \dots, z_q)' \in \tilde{\mathcal{M}}^{q-1}_{\le}$ that attains the maximum

$$\hat{\Lambda}_q := \max_{\mathbf{z} \in \tilde{\mathcal{M}}^{q-1}_{\le},\, \|\mathbf{z}\|=1} \ \min_{1 \le k \le q-1} \mathbf{z}'\mathbf{z}^q_k. \tag{5.3}$$

For $1 \le k \le q$ define

$$b^q_k = \sqrt{\tfrac{3k(q-k)}{q(q+1)}}, \qquad (b^q_0 = 0), \tag{5.4}$$

$$\hat{a}^q_k = b^q_{k-1} - b^q_k, \tag{5.5}$$

$$\hat{\mathbf{a}}^q = (\hat{a}^q_1, \dots, \hat{a}^q_q)', \tag{5.6}$$

$$\hat{\mathbf{z}}^q \equiv (\hat{z}^q_1, \dots, \hat{z}^q_q)' = \tfrac{\hat{\mathbf{a}}^q}{\|\hat{\mathbf{a}}^q\|}. \tag{5.7}$$

Then $\hat{a}_1^q + \cdots + \hat{a}_q^q = 0$ so $\hat{z}_1^q + \cdots + \hat{z}_q^q = 0$, and it is straightforward to show that $\hat{a}_1^q < \cdots < \hat{a}_q^q$, so $\hat{z}_1^q < \cdots < \hat{z}_q^q$, hence $\hat{\mathbf{z}}^q \in \tilde{\mathcal{M}}_{\leq}^{q-1}$. Trivially, $\|\hat{\mathbf{z}}^q\| = 1$.

Proposition 5.1. *The unit vector $\hat{\mathbf{z}}^q$ uniquely attains the maximum $\hat{\Lambda}_q$. Thus in the original scale,*

$$\hat{\mathbf{y}}^q := \|\bar{\mathbf{y}}^q\|\, \hat{\mathbf{z}}^q = \sqrt{\tfrac{q(q^2-1)}{12}}\, \tfrac{\hat{\mathbf{a}}^q}{\|\hat{\mathbf{a}}^q\|} \tag{5.8}$$

is the unique L-minimal configuration. Furthermore, $\hat{\mathbf{y}}^q \neq \bar{\mathbf{y}}^q$ when $q \geq 4$, and the minimum LECD and LECAD are, respectively,

$$L^{q-2}(\Pi(\hat{\mathbf{y}}^q)) = \beta^{q-2}\Big(\tfrac{1}{\|\hat{\mathbf{a}}^q\|}\sqrt{\tfrac{3}{q+1}}\Big), \tag{5.9}$$

$$A^{q-2}(\Pi(\hat{\mathbf{y}}^q)) = \cos^{-1}\Big(\tfrac{1}{\|\hat{\mathbf{a}}^q\|}\sqrt{\tfrac{3}{q+1}}\Big). \tag{5.10}$$

Proof. For any unit vector $\mathbf{z} \equiv (z_1, \ldots, z_q)' \in \tilde{\mathcal{M}}_{\leq}^{q-1}$, $z_1 + \cdots + z_q = 0$, so after some algebra we find that

$$\mathbf{z}'\mathbf{z}_k^q = \sqrt{\tfrac{q}{k(q-k)}}\,(z_{k+1} + \cdots + z_q), \tag{5.11}$$

hence

$$\hat{\Lambda}_q = \max_{\mathbf{z} \in \tilde{\mathcal{M}}_{\leq}^{q-1},\, \|\mathbf{z}\|=1}\ \min_{1 \leq k \leq q-1} \sqrt{\tfrac{q}{k(q-k)}}\,(z_{k+1} + \cdots + z_q). \tag{5.12}$$

We now show that the maximum in (5.12) is uniquely attained when $\mathbf{z} = \hat{\mathbf{z}}^q$.

Because $\hat{z}_{k+1}^q + \cdots + \hat{z}_q^q = \frac{b_k^q}{\|a^q\|}$,

$$\sqrt{\tfrac{q}{k(q-k)}}\,(\hat{z}_{k+1}^q + \cdots + \hat{z}_q^q) = \tfrac{1}{\|\hat{\mathbf{a}}^q\|}\sqrt{\tfrac{3}{q+1}} \tag{5.13}$$

for each $k = 1, \ldots, q-1$. Thus we must show that

$$\min_{1 \leq k \leq q-1} \sqrt{\tfrac{q}{k(q-k)}}\,(z_{k+1} + \cdots + z_q) < \tfrac{1}{\|\hat{\mathbf{a}}^q\|}\sqrt{\tfrac{3}{q+1}} \tag{5.14}$$

for every $\mathbf{z} \neq \hat{\mathbf{z}}^q$ such that $z_1 + \cdots + z_q = 0$, $\|\mathbf{z}\| = 1$, $z_1 \leq \cdots \leq z_q$. Suppose that there is such a $\mathbf{z}$ that satisfies

$$\min_{1 \leq k \leq q-1} \sqrt{\tfrac{q}{k(q-k)}}\,(z_{k+1} + \cdots + z_q) \geq \tfrac{1}{\|\hat{\mathbf{a}}^q\|}\sqrt{\tfrac{3}{q+1}}. \tag{5.15}$$

Therefore if $1 \leq k \leq q-1$ then

$$z_{k+1} + \cdots + z_q \geq \tfrac{b_k^q}{\|\hat{\mathbf{a}}^q\|} = \hat{z}_{k+1}^q + \cdots + \hat{z}_q^q,$$

with equality for $k = 0$, so $\mathbf{z}$ *majorizes* $\hat{\mathbf{z}}^q$ ([15]). Because $\|\mathbf{z}\|^2$ is symmetric and strictly convex in $(z_1, \ldots, z_q)$ and $\mathbf{z} \neq \hat{\mathbf{z}}^q$, this implies that

$$1 = \|\mathbf{z}\|^2 > \|\hat{\mathbf{z}}^q\|^2 = 1, \tag{5.16}$$

a contradiction. Thus the maximum value $\hat{\Lambda}^q$ is uniquely achieved when $\mathbf{z} = \hat{\mathbf{z}}^q$ as asserted. It is easy to verify that $\hat{a}_1^q, \ldots, \hat{a}_q^q$ are not evenly spaced when $q \geq 4$, hence $\hat{\mathbf{y}}^q \neq \bar{\mathbf{y}}^q$. Lastly, (5.9) and (5.10) follow from (5.13). □

The vectors $\hat{\mathbf{y}}^q$ and $\hat{\mathbf{x}}^q \equiv \hat{\mathbf{y}}^q + \frac{q+1}{2}\mathbf{e}^q$ are called the *L-minimal configurations* in $\tilde{\mathcal{M}}_{\leq}^{q-1}$ and $\mathbb{R}_{\leq}^q$ respectively. It is now obvious to ask whether or not the sequences $\{\hat{\mathbf{y}}^q\}$ and $\{\hat{\mathbf{x}}^q\}$ are APF, and if so, are APU. These questions will be answered in Propositions 5.3 and 5.5.

Because the LECD of $\Pi(\hat{\mathbf{y}}^q)$ given by (5.9) depends on $\|\hat{\mathbf{a}}^q\|$, bounds for $\|\hat{\mathbf{a}}^q\|$ are needed. Since $\hat{\mathbf{y}}^q \neq \bar{\mathbf{y}}^q$, necessarily $\|\hat{\mathbf{a}}^q\| < 1$ by the uniqueness of $\hat{\mathbf{y}}^q$, but sharper bounds will be required.

Lemma 5.2.

$$\sqrt{\tfrac{3[\log(2q+1)-2]}{2(q+1)}} < \|\hat{\mathbf{a}}^q\| < \sqrt{\tfrac{3[2\log(2q-1)+1]}{2(q+1)}}. \tag{5.17}$$

Therefore

$$\|\hat{\mathbf{a}}^q\| = O\Big(\sqrt{\tfrac{\log q}{q+1}}\Big) \ \text{ as } q \to \infty. \tag{5.18}$$

Proof. For $k = 1, \ldots, q$ set

$$\begin{aligned} c_k^q &= \big[\sqrt{(k-1(q-k+1)} - \sqrt{k(q-k)}\big]^2, \\ d_k^q &= \sqrt{k(k-1)(q-k)(q-k+1)}, \\ \bar{q} &= \tfrac{q+1}{2}, \end{aligned} \tag{5.19}$$

then verify that

$$c_k^q = 2\big[\tfrac{q^2-1}{4} - (k-\bar{q})^2 - d_k^q\big]. \tag{5.20}$$

From (5.4)–(5.5) and (5.19)–(5.20) we find that

$$\begin{aligned} \|\hat{\mathbf{a}}^q\|^2 &= \tfrac{3}{q(q+1)} \textstyle\sum_{k=1}^q c_k^q \\ &= (q-1) - \tfrac{6}{q(q+1)} \displaystyle\sum_{k=1}^q d_k^q. \end{aligned} \tag{5.21}$$

For the upper bound, use the harmonic mean-geometric mean inequality:

$$\begin{aligned} \|\hat{\mathbf{a}}^q\|^2 &< (q-1) - \tfrac{6}{q(q+1)} \textstyle\sum_{k=1}^q \frac{k(k-1)(q-k)(q-k+1)}{(k-\frac{1}{2})(q-k+\frac{1}{2})} \\ &= (q-1) - \tfrac{6}{q(q+1)} \textstyle\sum_{k=1}^q \frac{[(k-\frac{1}{2})^2-\frac{1}{4}][(q-k+\frac{1}{2})^2-\frac{1}{4}]}{(k-\frac{1}{2})(q-k+\frac{1}{2})} \\ &< (q-1) - \tfrac{6}{q(q+1)} \textstyle\sum_{k=1}^q \Big\{(k-\frac{1}{2})(q-k+\frac{1}{2}) - \frac{k-\frac{1}{2}}{4(q-k+\frac{1}{2})} - \frac{q-k+\frac{1}{2}}{4(k-\frac{1}{2})}\Big\} \\ &= (q-1) - \tfrac{6}{q(q+1)}\Big\{ \textstyle\sum_{k=1}^q (k-\frac{1}{2})(q-k+\frac{1}{2}) - \sum_{k=1}^q \frac{q-k+\frac{1}{2}}{2(k-\frac{1}{2})}\Big\} \\ &= (q-1) - \tfrac{6}{q(q+1)}\Big\{ \textstyle\sum_{k=1}^q (k-\frac{1}{2})(q-k+\frac{1}{2}) - \frac{q}{2}\sum_{k=1}^q \frac{1}{k-\frac{1}{2}} + \frac{q}{2}\Big\} \\ &= \tfrac{3}{q+1}\Big\{ \textstyle\sum_{k=1}^q \frac{1}{k-\frac{1}{2}} - \frac{3}{2}\Big\} \\ &< \tfrac{3[2\log(2q-1)+1]}{2(q+1)}; \end{aligned} \tag{5.22}$$

the final inequality follows from (7) of [16].

Similarly, the geometric mean-arithmetic mean inequality yields the non-logarithmic lower bound $\frac{3(3q-2)}{2q(q+1)}$. However, the asserted logarithmic lower bound, which is sharper, can be obtained as follows. We will show that

$$c_k^q \equiv \big[\sqrt{k(q-k)} - \sqrt{(k-1)(q-k+1)}\,\big]^2 \geq \tfrac{(k-\bar{q})^2}{(k-\frac{1}{2})(q-k+\frac{1}{2})}, \tag{5.23}$$

for $k = 1, \ldots, q$, where $\bar{q} = \frac{q+1}{2}$. Thus from (5.21),

$$\begin{aligned} \|\hat{\mathbf{a}}^q\|^2 &\geq \tfrac{3}{q(q+1)} \textstyle\sum_{k=1}^q \frac{(k-\bar{q})^2}{(k-\frac{1}{2})(q-k+\frac{1}{2})} \\ &= \tfrac{3}{q^2(q+1)} \textstyle\sum_{k=1}^q \Big[\frac{(k-\bar{q})^2}{k-\frac{1}{2}} + \frac{(k-\bar{q})^2}{q-k+\frac{1}{2}}\Big] \\ &= \tfrac{3}{q^2(q+1)} \textstyle\sum_{k=1}^q \Big[\frac{(k-\frac{1}{2})^2 - 2(k-\frac{1}{2})(\frac{q}{2}) + (\frac{q}{2})^2}{k-\frac{1}{2}} + \frac{(q-k+\frac{1}{2})^2 - 2(q-k+\frac{1}{2})(\frac{q}{2}) + (\frac{q}{2})^2}{q-k+\frac{1}{2}}\Big] \\ &= \tfrac{3}{q+1}\Big[\textstyle\sum_{k=1}^q \frac{1}{4}\big(\frac{1}{k-\frac{1}{2}} + \frac{1}{q-k+\frac{1}{2}}\big) - 1\Big] \\ &= \tfrac{3}{q+1}\big[\textstyle\sum_{k=1}^q \frac{1}{2k-1} - 1\big] \\ &> \tfrac{3[\log(2q+1)-2]}{2(q+1)}, \end{aligned} \tag{5.24}$$

where the inequality used in (5.24) also follows from (7) of [16].

To establish (5.23), rewrite it in the equivalent form

$$\left(\sqrt{(\bar{q}+u)(\bar{q}-u-1)}-\sqrt{(\bar{q}+u-1)(\bar{q}-u)}\,\right)^2 \geq \tfrac{u^2}{\tilde{q}^2-u^2}, \tag{5.25}$$

where $u \equiv k-\bar{q} \in \{-\frac{q-1}{2},\dots,\frac{q-1}{2}\}$ and $\tilde{q}=\bar{q}-\frac{1}{2}=\frac{q}{2}$. Now set $v=\frac{u}{\tilde{q}}$, so $|v| \leq \frac{q-1}{q} < 1$. Then (5.25) can be written in the equivalent forms

$$\begin{aligned}
\left(\sqrt{(\bar{q}+\tilde{q}v)(\bar{q}-\tilde{q}v-1)}-\sqrt{(\bar{q}+\tilde{q}v-1)(\bar{q}-\tilde{q}v)}\,\right)^2 &\geq \tfrac{v^2}{1-v^2},\\
\left(\sqrt{(\bar{q}^2-\tilde{q}^2v^2)-(\bar{q}+\tilde{q}v)}-\sqrt{(\bar{q}^2-\tilde{q}^2v^2)-(\bar{q}-\tilde{q}v)}\,\right)^2 &\geq \tfrac{v^2}{1-v^2},\\
2\mu(v)-2\sqrt{(\mu(v)-\tilde{q}v)(\mu(v)+\tilde{q}v)} &\geq \tfrac{v^2}{1-v^2},\\
2\mu(v)-2\sqrt{\mu(v)^2-\tilde{q}^2v^2} &\geq \tfrac{v^2}{1-v^2},
\end{aligned} \tag{5.26}$$

where

$$\mu(v)=\bar{q}^2-\tilde{q}^2v^2-\bar{q}=\tfrac{1}{4}[q^2(1-v^2)-1].$$

It will be shown that for $|v| \leq \frac{q-1}{q}$,

$$2\mu(v)-\tfrac{v^2}{1-v^2} \geq 0, \tag{5.27}$$

so (5.26) is equivalent to each of the following inequalities:

$$\begin{aligned}
[2\mu(v)-\tfrac{v^2}{1-v^2}]^2 &\geq 4\mu(v)^2-q^2v^2,\\
q^2v^2-4\mu(v)\tfrac{v^2}{1-v^2}+\tfrac{v^4}{(1-v^2)^2} &\geq 0,\\
q^2v^2-[q^2(1-v^2)-1]\tfrac{v^2}{1-v^2}+\tfrac{v^4}{(1-v^2)^2} &\geq 0,\\
q^2v^2(1-v^2)-[q^2(1-v^2)-1]v^2+\tfrac{v^4}{1-v^2} &\geq 0,\\
v^2(1+\tfrac{v^2}{1-v^2}) &\geq 0,
\end{aligned}$$

which clearly is true. Thus (5.23) will be established once (5.27) is verified.

For this set $x=v^2$, so (5.27) can be expressed equivalently as

$$h(x) \equiv (1-x)[q^2(1-x)-1]-2x \geq 0,$$

where $0 \leq x \leq (\frac{q-1}{q})^2$. The quadratic function $h(x)$ satisfies

$$h(0)=q^2-1 > h\left[\left(\tfrac{q-1}{q}\right)^2\right]=2-\tfrac{2}{q} > 0 > h(1)=-2,$$

hence $h(x) > 0$ for $0 \leq x \leq (\frac{q-1}{q})^2$, as required. □

Proposition 5.3. *The L-minimal configurations $\{\hat{\mathbf{y}}^q\}$ and $\{\hat{\mathbf{x}}^q\}$ are APF.*

Proof. Set $t^q=\frac{1}{\|\hat{\mathbf{a}}^q\|}\sqrt{\frac{3}{q+1}}$, so that (5.17) yields

$$\sqrt{\tfrac{q}{2\log(2q-1)+1}} < t^q\sqrt{q} < \sqrt{\tfrac{2q}{\log(2q+1)-2}}.$$

Then by Lemma 3.1 with $\lambda=\infty$,

$$\lim_{q\to\infty} L^{q-2}(\Pi(\hat{\mathbf{y}}^q))=0, \tag{5.28}$$

hence $\{\hat{\mathbf{y}}^q\}$ (and $\{\hat{\mathbf{x}}^q\}$) is APF. □

It follows from (3.41), (4.3), and (4.16) that for each fixed $i \geq 1$, $\{\bar{\mathbf{y}}^q\}$ satisfies

$$W_q := \tfrac{12}{q^2-1}\big[(\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|})_i\big]^2 \xrightarrow{d} \chi_1^2, \tag{5.29}$$

$$\bar{W}_q := \tfrac{12}{q^2-1}\big[(\tilde{\mathbf{U}}^{q-2}_{\bar{\mathbf{y}}^q})_i\big]^2 \xrightarrow{d} 3\,\mathrm{Beta}(\tfrac{1}{2},1), \tag{5.30}$$

as $q \to \infty$. The bounds for $\|\hat{\mathbf{a}}^q\|$ in (5.17) yield a corresponding result for the L-minimal configurations $\{\hat{\mathbf{y}}^q\}$:

Proposition 5.4. *For each fixed $i \geq 1$,*

$$\tfrac{Z_q}{\log(2q-1)+2} <_{\mathrm{st}} \hat{W}_q := \tfrac{12}{q^2-1}\big[(\tilde{\mathbf{U}}^{q-2}_{\hat{\mathbf{y}}^q})_i\big]^2 <_{\mathrm{st}} \tfrac{2Z_q+1}{\log(2q+1)-2}, \tag{5.31}$$

where $\{Z_q\}$ is a sequence of positive random variables such that

$$Z_q \xrightarrow{d} F_{1,2} \tag{5.32}$$

as $q \to \infty$. Here $<_{\mathrm{st}}$ denotes stochastic ordering and $F_{1,2}$ denotes the F distribution with 1 and 2 degrees of freedom. Therefore

$$\hat{W}_q = O_p\Big(\tfrac{F_{1,2}}{\log q}\Big) \xrightarrow{p} 0, \tag{5.33}$$

$$\sqrt{\tfrac{12}{q^2-1}}(\tilde{\mathbf{U}}^{q-2}_{\hat{\mathbf{y}}^q})_i = O_p\big(\tfrac{t_2}{\sqrt{\log q}}\big) \xrightarrow{p} 0, \tag{5.34}$$

where t_2 denotes Student's t-distribution with 2 degrees of freedom.

Proof. From (5.4)–(5.8) and (5.19), $\hat{W}_q$ is uniformly distributed over the set

$$\big\{\tfrac{12}{q^2-1}\,\tfrac{q(q^2-1)}{12}\,\tfrac{3}{\|\hat{\mathbf{a}}^q\|^2 q(q+1)}\,c_k^q \;\big|\; k=1,\dots,q\big\} = \big\{\tfrac{3c_k^q}{(q+1)\|\hat{\mathbf{a}}^q\|^2} \;\big|\; k=1,\dots,q\big\}, \tag{5.35}$$

so by (5.24), $\hat{W}_q$ is stochastically smaller than the uniform distribution on

$$\Big\{\tfrac{2c_k^q}{\log(2q+1)-2} \;\Big|\; k=1,\dots,q\Big\} = \Big\{\tfrac{4[\frac{q^2-1}{4}-(k-\bar{q})^2-d_k^q]}{\log(2q+1)-2} \;\Big|\; k=1,\dots,q\Big\}.$$

Now apply the harmonic mean-geometric mean inequality to d_k^q to obtain

$$\begin{aligned}
\tfrac{q^2-1}{4} - (k-\bar{q})^2 - d_k^q &< \tfrac{q^2-1}{4} - (k-\bar{q})^2 - \tfrac{k(k-1)(q-k)(q-k+1)}{(k-\frac12)(q-k+\frac12)} \\
&= \tfrac{q^2-1}{4} - (k-\bar{q})^2 - \tfrac{[(k-\frac12)^2-\frac14][(q-k+\frac12)^2-\frac14]}{(k-\frac12)(q-k+\frac12)} \\
&= \tfrac{q^2-1}{4} - (k-\bar{q})^2 - (k-\tfrac12)(q-k+\tfrac12) \\
&\quad + \tfrac{q-k+\frac12}{4(k-\frac12)} + \tfrac{k-\frac12}{4(q-k+\frac12)} - \tfrac{1}{16}\,\tfrac{1}{(k-\frac12)(q-k+\frac12)} \\
&= \tfrac14\Big[\tfrac{q^2-\frac14}{(k-\frac12)(q-k+\frac12)} - 3\Big] \\
&= \tfrac14\Big[\tfrac{q^2-\frac14}{\frac{q^2}{4}-(k-\bar{q})^2} - 3\Big] \\
&< \tfrac{1}{1-4\left(\frac{k-\bar{q}}{q}\right)^2} - \tfrac34,
\end{aligned}$$

where we have twice used the relation

$$(k-\bar{q})^2 + (k-\tfrac12)(q-k+\tfrac12) = \tfrac{q^2}{4}. \tag{5.36}$$

Therefore $\hat{W}_q$ is stochastically smaller than

$$\tfrac{4}{\log(2q+1)-2}\big(\tfrac{1}{1-4V_q^2} - \tfrac34\big) \equiv \tfrac{4Y_q+1}{\log(2q+1)-2},$$

where

$$V_q \stackrel{d}{=} \text{Uniform}\{\tfrac{k-\bar{q}}{q} \mid k=1,\dots,q\},$$
$$Y_q = \tfrac{4V_q^2}{1-4V_q^2}.$$

Because $\frac{k-\bar{q}}{q} = \frac{2k-q-1}{2q}$, clearly

$$V_q \xrightarrow{d} V \stackrel{d}{=} \text{Uniform}(-\tfrac{1}{2}, \tfrac{1}{2}),$$
$$4V_q^2 \xrightarrow{d} 4V^2 \stackrel{d}{=} \text{Beta}(\tfrac{1}{2}, 1),$$

as $q \to \infty$, from which it follows that $Y_q \xrightarrow{d} \frac{1}{2}F_{1,2}$. Now set $Z_q = 2Y_q$.

Similarly from (5.22), (5.23), (5.35), and (5.36), $\hat{W}_q$ is stochastically larger than the uniform distribution on

$$\Big\{\tfrac{2}{\log(2q+1)-2}\Big[\tfrac{(k-\bar{q})^2}{\frac{q^2}{4}-(k-\bar{q})^2}\Big] \;\Big|\; k=1,\dots,q\Big\},$$

so $\hat{W}_q$ is stochastically larger than

$$\tfrac{2}{\log(2q+1)-2}\big(\tfrac{4V_q^2}{1-4V_q^2}\big) \equiv \tfrac{Z_q}{\log(2q+1)-2},$$

as asserted. □

Proposition 5.5. *The sequences of L-minimal configurations $\{\hat{\mathbf{y}}^q\}$ and $\{\hat{\mathbf{x}}^q\}$ are not APU.*

Proof. It follows from (5.34) that for any fixed i,

$$\tfrac{\sqrt{q}}{\|\hat{\mathbf{y}}^q\|}(\tilde{\mathbf{U}}^{q-2}_{\hat{\mathbf{y}}^q})_i = \sqrt{\tfrac{12}{q^2-1}}(\tilde{\mathbf{U}}^{q-2}_{\hat{\mathbf{y}}^q})_i = O_p\big(\tfrac{1}{\sqrt{\log q}}\big) \xrightarrow{p} 0, \tag{5.37}$$

hence by Proposition 3.10 $\{\hat{\mathbf{y}}^q\}$ and $\{\hat{\mathbf{x}}^q\}$ cannot be APU. □

6 The normal configuration

The sequence $\{\hat{\mathbf{y}}^q\}$, like $\{\bar{\mathbf{y}}^q\}$, fails to satisfy the necessary condition (3.42) for APU, yet $\{\hat{\mathbf{y}}^q\}$ uniquely minimizes the LECD and LECAD, so it seems reasonable to conjecture that no APU sequence exists. However, it is easy to find a sequence $\{\check{\mathbf{y}}^q \in \tilde{\mathcal{M}}^{q-1}_{\leq}\}$ that does satisfy (3.42). Define

$$\check{\mathbf{a}}^q \equiv (\check{a}^q_1, \dots, \check{a}^q_q) = \big(\Phi^{-1}(\tfrac{1}{q+1}), \Phi^{-1}(\tfrac{2}{q+1}), \dots, \Phi^{-1}(\tfrac{q}{q+1})\big)', \tag{6.1}$$

the $\frac{k}{q+1}$-quantiles of the $N(0,1)$ distribution, then in the original scale let

$$\check{\mathbf{y}}^q = \|\bar{\mathbf{y}}^q\|\tfrac{\check{\mathbf{a}}^q}{\|\check{\mathbf{a}}^q\|}. \tag{6.2}$$

Clearly $\check{a}^q_1 < \cdots < \check{a}^q_q$ while $\check{a}^q_1 + \cdots + \check{a}^q_q = 0$ by the symmetry of $N(0,1)$, hence $\check{\mathbf{y}}^q \in \tilde{\mathcal{M}}^{q-1}_{\leq}$. The vector $\check{\mathbf{y}}^q$ is called the *normal configuration*.

For each $i = 1,\dots,q$, $(\tilde{\mathbf{U}}^{q-2}_{\check{\mathbf{a}}^q})_i \stackrel{d}{=} \Phi^{-1}(U_q)$, where

$$U_q \stackrel{d}{=} \text{Uniform}\big(\{\tfrac{1}{q+1}, \dots, \tfrac{q}{q+1}\}\big) \xrightarrow{d} \text{Uniform}(0,1), \tag{6.3}$$

hence

$$(\tilde{\mathbf{U}}^{q-2}_{\check{\mathbf{a}}^q})_i \xrightarrow{d} \Phi^{-1}(\text{Uniform}(0,1)) = N(0,1)$$

as $q \to \infty$. Furthermore,

$$\frac{\|\breve{\mathbf{a}}^q\|^2}{q+1} + \frac{1}{q+1}\left[\Phi^{-1}\left(\frac{q}{q+1}\right)\right]^2 \equiv \frac{1}{q+1}\sum_{k=1}^q \left[\Phi^{-1}\left(\frac{k}{q+1}\right)\right]^2 + \frac{1}{q+1}\left[\Phi^{-1}\left(\frac{q}{q+1}\right)\right]^2$$

is an approximating Riemann sum for

$$\int_0^1 [\Phi^{-1}(u)]^2 du = \int_{-\infty}^{\infty} x^2\phi(x)dx = 1,$$

while

$$\Phi^{-1}\left(\frac{q}{q+1}\right) = \sqrt{2\log(q+1)}(1+o(1)) \tag{6.4}$$

as $q \to \infty$ (e.g. [11] p. 1092). Thus

$$\frac{\|\breve{\mathbf{a}}^q\|^2}{q+1} = 1 - \frac{2\log(q+1)}{q+1} + o(1), \tag{6.5}$$

$$\|\breve{\mathbf{a}}^q\| \sim \sqrt{q+1}, \tag{6.6}$$

hence $\{\breve{\mathbf{y}}^q\}$ satisfies (3.42):

$$\frac{\sqrt{q}}{\|\breve{\mathbf{y}}^q\|}(\tilde{\mathbf{U}}^{q-2}_{\breve{\mathbf{y}}^q})_i = \frac{\sqrt{q}}{\|\breve{\mathbf{a}}^q\|}(\tilde{\mathbf{U}}^{q-2}_{\breve{\mathbf{a}}^q})_i \xrightarrow{d} N(0,1) \tag{6.7}$$

as $q \to \infty$. However, it is now shown that the LECD of $\{\breve{\mathbf{y}}^q\}$, necessarily greater than that of $\{\hat{\mathbf{y}}^q\}$, does not approach 0.

Proposition 6.1. $\{\breve{\mathbf{y}}^q\}$ *is not APF, hence is not APU.*

Proof. By (3.21)–(3.25) and (6.1)–(6.2),

$$L^{q-2}\big(\Pi(\breve{\mathbf{y}}^q)\big) = \beta^{q-2}(\breve{t}^q), \tag{6.8}$$

$$\breve{t}^q := \frac{1}{\|\breve{\mathbf{y}}^q\|} \min_{1\le k\le q-1} (\breve{\mathbf{y}}^q)'\mathbf{z}^q_k \tag{6.9}$$

$$\le \frac{1}{\|\breve{\mathbf{y}}^q\|}(\breve{\mathbf{y}}^q)'\mathbf{z}^q_{q-1}$$

$$= \frac{1}{\|\breve{\mathbf{a}}^q\|}(\breve{\mathbf{a}}^q)'\mathbf{z}^q_{q-1} \tag{6.10}$$

$$= \frac{1}{\|\breve{\mathbf{a}}^q\|}\sqrt{\frac{q}{q-1}}\Phi^{-1}\big(\frac{q}{q+1}\big) \tag{6.11}$$

$$< \frac{1}{\|\breve{\mathbf{a}}^q\|}\sqrt{\frac{q}{q-1}}\frac{\phi(\Phi^{-1}(\frac{q}{q+1}))}{1-\Phi(\Phi^{-1}(\frac{q}{q+1}))}$$

$$= \frac{q+1}{\sqrt{2\pi}\|\breve{\mathbf{a}}^q\|}\sqrt{\frac{q}{q-1}}e^{-\frac{1}{2}[\Phi^{-1}(\frac{q}{q+1})]^2}.$$

It follows from [11] p. 1092 that

$$\Phi^{-1}\big(\frac{q}{q+1}\big) = \sqrt{2\log\left((q+1)\sqrt{4\pi\log(q+1)}\right)}\left(1+\Delta_q\right) \tag{6.12}$$

where $\Delta_q = O\left(\frac{\log(\log(q+1))}{(\log(q+1))^2}\right)$, hence

$$e^{-\frac{1}{2}[\Phi^{-1}(\frac{q}{q+1})]^2} = \frac{1}{q+1}\frac{1}{\sqrt{4\pi\log(q+1)}}\big(\frac{1}{q+1}\big)^{\Delta_q+\Delta_q^2}\big(\frac{1}{\sqrt{4\pi\log(q+1)}}\big)^{\Delta_q+\Delta_q^2}$$

$$= \frac{1}{q+1}o(1)(1+o(1))(1+o(1))$$

$$= \frac{1}{q+1}o(1).$$

Therefore by (6.6),

$$\breve{t}^q\sqrt{q} < \frac{1}{\sqrt{2\pi}\|\breve{\mathbf{a}}^q\|}\frac{q}{\sqrt{q-1}}o(1) = o(1), \tag{6.13}$$

hence $\lim_{q\to\infty}\breve{t}^q\sqrt{q} = 0$, so

$$\lim_{q\to\infty} L^{q-2}\big(\Pi(\breve{\mathbf{y}}^q)\big) = \frac{1}{2} \tag{6.14}$$

by Lemma 3.1 with $\lambda = 0$. This completes the proof. □

Remark 6.2. It should be noted that the convergences in (5.37) and (6.14) occur at very slow, sub-logarithmic rates. □

Proposition 6.3. $\{\breve{\mathbf{y}}^q\}$ *is APE.*

Proof. The proof is similar to that of Proposition 4.2. Again define $\bar{\mathbf{z}}_k^q = \|\bar{\mathbf{y}}^q\|\mathbf{z}_k^q$, where $\mathbf{z}_k^q$ is the unit vector in (3.23), and define

$$\breve{s}^q = \tfrac{1}{\|\breve{\mathbf{a}}^q\|}(\breve{\mathbf{a}}^q)'\mathbf{z}_{q-1}^q,$$

cf. (6.10). As in (6.10)–(6.13),

$$\breve{s}^q\sqrt{q} < \tfrac{1}{\sqrt{2\pi}\|\breve{\mathbf{a}}^q\|}\tfrac{q}{\sqrt{q-1}}o(1) = o(1), \tag{6.15}$$

hence from (3.6)–(3.9) and Lemma 3.1 with $\lambda = 0$,

$$\lim_{q\to\infty} \tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}\big(C\big(\bar{\mathbf{z}}_{q-1}^q; \breve{s}^q\big)\big) = \lim_{q\to\infty} \beta^{q-2}(\breve{s}^q) = \tfrac{1}{2}. \tag{6.16}$$

Furthermore, from (3.27) and the Rearrangement Inequality,

$$\begin{aligned} \Pi(\breve{\mathbf{y}}^q) \cap C\big(\bar{\mathbf{z}}_{q-1}^q; \breve{s}^q\big) = \emptyset \iff & \max_{P\in\mathcal{P}^q}(P\breve{\mathbf{y}}^q)'\bar{\mathbf{z}}_{q-1}^q \le \|\bar{\mathbf{y}}^q\|^2\breve{s}^q \\ \iff & (\breve{\mathbf{y}}^q)'\bar{\mathbf{z}}_{q-1}^q \le \|\bar{\mathbf{y}}^q\|^2\breve{s}^q \\ \iff & \tfrac{1}{\|\breve{\mathbf{a}}^q\|}(\breve{\mathbf{a}}^q)'\mathbf{z}_{q-1}^q \le \breve{s}^q, \end{aligned}$$

hence $C\big(\bar{\mathbf{z}}_{q-1}^q; \breve{s}^q\big)$ is an empty spherical cap for $\Pi(\breve{\mathbf{y}}^q)$.

Because $P\Pi(\breve{\mathbf{y}}^q) = \Pi(\breve{\mathbf{y}}^q)$ for all $P \in \mathcal{P}^q$, each $C\big(P\bar{\mathbf{z}}_{q-1}^q; \breve{s}^q\big)$ is an empty spherical cap for $\Pi(\breve{\mathbf{y}}^q)$ in $\tilde{\mathcal{S}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$; there are $q!$ such congruent caps. However

$$\{P\bar{\mathbf{z}}_{q-1}^q \mid P \in \mathcal{P}^q\} = \{\bar{\mathbf{f}}_1^q, \dots, \bar{\mathbf{f}}_q^q\},$$

where $\bar{\mathbf{f}}_i^q = \|\bar{\mathbf{y}}^q\|\mathbf{f}_i^q$, so these $q!$ empty caps reduce to q congruent ones, namely

$$\big\{C\big(\bar{\mathbf{f}}_i^q; \breve{s}^q\big) \;\big|\; i = 1, \dots q\big\}.$$

By (6.16) each of these congruent caps remains nonnegligible as $q \to \infty$, so to show that $\{\bar{\mathbf{y}}^q\}$ is APE it suffices to show that

$$\lim_{q\to\infty} \tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}\big(\breve{\Upsilon}^q\big) = 1, \tag{6.17}$$

where

$$\breve{\Upsilon}^q = \textstyle\bigcup_{i=1}^q \Big[C\big(\bar{\mathbf{f}}_i^q; \breve{s}^q\big)\Big].$$

Clearly

$$\tilde{\mathcal{S}}^{q-2}_{\|\bar{\mathbf{y}}^q\|} \bigcap \big((\breve{\Upsilon}_q)^c\big) \subseteq \tilde{\mathcal{S}}^{q-2}_{\|\bar{\mathbf{y}}^q\|} \bigcap \big(\cap_{i=2}^q \breve{H}_i^q\big), \tag{6.18}$$

where $\breve{H}_i^q$ is the halfspace

$$\breve{H}_i^q := \big\{\mathbf{v} \in \tilde{\mathcal{M}}^{q-1} \;\big|\; \mathbf{v}'\bar{\mathbf{f}}_i^q \le \|\bar{\mathbf{y}}^q\|^2\,\breve{s}^q\big\}.$$

As in the proof of Proposition 4.2, $(\mathbf{f}_i^q)'\mathbf{f}_j^q = -\frac{1}{q-1} < 0$ if $i \ne j$ so

$$\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}\big(\cap_{i=2}^q \breve{H}_i^q\big) \le \tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}\big(\cap_{i=2}^q \breve{K}_i^q\big), \tag{6.19}$$

where $\breve{K}_i^q$ is the halfspace

$$\breve{K}_i^q := \{\mathbf{v} \in \tilde{\mathcal{M}}^{q-1} \mid \mathbf{v}'\gamma_i^q \le \|\bar{\mathbf{y}}^q\| \, \breve{s}^q\}.$$

Again apply Proposition A.1 and the orthogonal invariance of $\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$ to obtain

$$\begin{aligned}\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}\big(\cap_{i=2}^q \breve{K}_i^q\big) &\le \textstyle\prod_{i=2}^q \tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}(\breve{K}_i^q) &(6.20)\\ &= \big[\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}(\breve{K}_i^q)\big]^{q-1}\\ &= \big[1-\beta^{q-2}\big(\breve{s}^q\big)\big]^{q-1}. &(6.21)\end{aligned}$$

Therefore by (6.16),

$$\limsup_{q\to\infty} \big[\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}\big(\cap_{i=2}^q \breve{K}_i^q\big)\big]^{\frac{1}{q-1}} \le \tfrac{1}{2}, \qquad (6.22)$$

hence by (6.18)–(6.22), for any $\epsilon > 0$

$$\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}\big((\Upsilon^q)^c\big) \le \tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}\big(\cap_{i=2}^q \breve{K}_i^q\big) \le (\tfrac{1}{2}+\epsilon)^{q-1} \qquad (6.23)$$

for sufficiently large q. Thus (6.17) holds, in fact $\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}((\breve{\Upsilon}^q)^c) \to 0$ at a geometric rate, hence $\{\breve{\mathbf{y}}^q\}$ is APE as asserted. □

7 Comparisons among the configurations

Based on the results in Sections 4–6, comparisons among the three uniform distributions $\tilde{\mathbf{U}}^{q-2}_{\bar{\mathbf{y}}^q}$, $\tilde{\mathbf{U}}^{q-2}_{\hat{\mathbf{y}}^q}$, $\tilde{\mathbf{U}}^{q-2}_{\breve{\mathbf{y}}^q}$ on permutations and the uniform distribution $\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$ on the sphere $\tilde{\mathcal{S}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$ are now summarized.

The LECDs of $\Pi(\bar{\mathbf{y}}^q)$, $\Pi(\hat{\mathbf{y}}^q)$, and $\Pi(\breve{\mathbf{y}}^q)$ are as follows:

$$L^{q-2}(\Pi(\bar{\mathbf{y}}^q)) = \beta^{q-2}\Big(\sqrt{\tfrac{3}{q+1}}\Big); \qquad (7.1)$$

$$L^{q-2}(\Pi(\hat{\mathbf{y}}^q)) = \beta^{q-2}\Big(\tfrac{1}{\|\hat{\mathbf{a}}^q\|}\sqrt{\tfrac{3}{q+1}}\Big); \qquad (7.2)$$

$$L^{q-2}\big(\Pi(\breve{\mathbf{y}}^q)\big) = \beta^{q-2}(\breve{t}^q). \qquad (7.3)$$

Here $\|\hat{\mathbf{a}}^q\|$ is given by (5.6) and approximated in (5.17), while $\breve{t}^q$ is given by (6.9) and bounded above by (6.11) together with (6.5). Some explicit bounds and asymptotic comparisons among these LECDs are collected here.

First, from (3.11) and (3.13),

$$\tfrac{1}{2} - \sqrt{\tfrac{q-2}{q-4}}\Big[\Phi\Big(\sqrt{\tfrac{3(q-4)}{q+1}}\Big) - \tfrac{1}{2}\Big] < L^{q-2}(\Pi(\bar{\mathbf{y}}^q)) < \Big[\tfrac{q-2}{q+1}\Big]^{\frac{q-2}{2}} \sqrt{\tfrac{q+1}{6\pi(q-2)}}. \qquad (7.4)$$

Asymptotically,

$$\lim_{q\to\infty} L^{q-2}(\Pi(\bar{\mathbf{y}}^q)) = 1 - \Phi(\sqrt{3}) \approx 0416. \qquad (7.5)$$

Second, from (5.17),

$$\beta^{q-2}\Big(\sqrt{\tfrac{2}{\log(2q+1)-2}}\Big) < L^{q-2}(\Pi(\hat{\mathbf{y}}^q)) < \beta^{q-2}\Big(\sqrt{\tfrac{2}{2\log(2q-1)+1}}\Big),$$

which, combined with (3.11) and (3.13), yields the explicit bounds

$$\begin{aligned}\tfrac{1}{2} - \sqrt{\tfrac{q-2}{q-4}}\Big[\Phi\Big(\sqrt{\tfrac{2(q-4)}{\log(2q+1)-2}}\Big) - \tfrac{1}{2}\Big] &< L^{q-2}(\Pi(\hat{\mathbf{y}}^q)) &(7.6)\\ &< \Big[\tfrac{2\log(2q-1)-1}{2\log(2q-1)+1}\Big]^{\frac{q-2}{2}} \sqrt{\tfrac{2\log(2q-1)+1}{4\pi(q-2)}}. &(7.7)\end{aligned}$$

Asymptotically,

$$\lim_{q\to\infty} L^{q-2}(\Pi(\hat{\mathbf{y}}^q)) = 0. \tag{7.8}$$

Third, from (3.13) and (6.11),

$$\tfrac{1}{2} - \sqrt{\tfrac{q-2}{q-4}}\Big[\Phi\Big(\tfrac{1}{\|\check{\mathbf{a}}^q\|}\sqrt{\tfrac{q(q-4)}{q-1}}\Phi^{-1}\big(\tfrac{q}{q+1}\big)\Big) - \tfrac{1}{2}\Big] < L^{q-2}(\Pi(\check{\mathbf{y}}^q)).$$

Asymptotically,

$$\lim_{q\to\infty} L^{q-2}\big(\Pi(\check{\mathbf{y}}^q)\big) = \tfrac{1}{2}. \tag{7.9}$$

The LECADs of $\Pi(\bar{\mathbf{y}}^q)$, $\Pi(\hat{\mathbf{y}}^q)$, and $\Pi(\check{\mathbf{y}}^q)$ are as follows:

$$A^{q-2}(\Pi(\bar{\mathbf{y}}^q)) = \cos^{-1}\left(\sqrt{\tfrac{3}{q+1}}\right); \tag{7.10}$$

$$A^{q-2}(\Pi(\hat{\mathbf{y}}^q)) = \cos^{-1}\left(\tfrac{1}{\|\hat{\mathbf{a}}^q\|}\sqrt{\tfrac{3}{q+1}}\right); \tag{7.11}$$

$$A^{q-2}\big(\Pi(\check{\mathbf{y}}^q)\big) = \cos^{-1}(\check{t}^q). \tag{7.12}$$

These yield some explicit expressions and bounds for the LECADs:

$$A^{q-2}(\Pi(\bar{\mathbf{y}}^q)) = \cos^{-1}\left(\sqrt{\tfrac{3}{q+1}}\right); \tag{7.13}$$

$$\cos^{-1}\left(\sqrt{\tfrac{2}{\log(2q+1)-2}}\right) < A^{q-2}(\Pi(\hat{\mathbf{y}}^q)) < \cos^{-1}\left(\sqrt{\tfrac{2}{2\log(2q-1)+1}}\right); \tag{7.14}$$

$$\cos^{-1}\left(\tfrac{1}{\|\check{\mathbf{a}}^q\|}\sqrt{\tfrac{q}{q-1}}\Phi^{-1}\big(\tfrac{q}{q+1}\big)\right) < A^{q-2}(\Pi(\check{\mathbf{y}}^q)). \tag{7.15}$$

Asymptotic comparisons among the LECADs are extremely simple:

Proposition 7.1. *For any sequence of nonzero vectors* $\{\mathbf{y}^q \in \tilde{\mathcal{M}}^{q-1}_{\le}\}$*,*

$$\lim_{q\to\infty} A^{q-2}(\Pi(\mathbf{y}^q)) = \cos^{-1}(0) = \tfrac{\pi}{2}, \tag{7.16}$$

that is, the largest empty cap for $\Pi(\mathbf{y}^q)$ *approaches a hemisphere in terms of its angular measure. Therefore no APD sequence exists.*

Proof. From the lower bound in (7.14) we see that (7.16) holds for the L-minimal configurations $\{\hat{\mathbf{y}}^q\}$. Because $\hat{\mathbf{y}}^q$ minimizes the largest empty cap, (7.16) holds for all nonzero sequences $\{\hat{\mathbf{y}}^q\}$. □

Lastly, the standardized limits of the univariate marginal distributions are as follows: for each fixed $i \ge 1$,

$$\sqrt{\tfrac{12}{q^2-1}}(\tilde{\mathbf{U}}^{q-2}_{\bar{\mathbf{y}}^q})_i \xrightarrow{d} \text{Uniform}\big(-\sqrt{3}, \sqrt{3}\big); \tag{7.17}$$

$$\sqrt{\tfrac{12}{q^2-1}}(\tilde{\mathbf{U}}^{q-2}_{\hat{\mathbf{y}}^q})_i = O_p\big(\tfrac{t_2}{\sqrt{\log q}}\big) \xrightarrow{p} 0; \tag{7.18}$$

$$\sqrt{\tfrac{12}{q^2-1}}(\tilde{\mathbf{U}}^{q-2}_{\check{\mathbf{y}}^q})_i \xrightarrow{d} N(0,1); \tag{7.19}$$

$$\sqrt{\tfrac{12}{q^2-1}}(\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|})_i \xrightarrow{d} N(0,1). \tag{7.20}$$

Our asymptotic results for the LECDs, LECADs, and univariate marginal distributions of the regular, L-minimal, and normal configurations are summarized in Table 1. Neither the regular nor normal sequence is APU, nor is the L-minimal sequence APU even though it is APF. Thus we conjecture, albeit somewhat weakly, that the answer to the following question is no:

Table 1: The first three rows refer to the discrete uniform distribution on the permutations in $\Pi(\mathbf{y}^q)$. The fourth row refers to the continuous uniform distribution on the sphere $\tilde{\mathcal{S}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$, where the "entries" hold trivially. The second and third columns show the limiting LECDs and LECADs, respectively. The final column indicates whether or not the univariate marginal distributions converge to $N(0,1)$, a necessary condition for APU.

$\mathbf{y}^q$	$\lim_{q\to\infty} L^{q-2}(\Pi(\mathbf{y}^q))$	$\lim_{q\to\infty} A^{q-2}(\Pi(\mathbf{y}^q))$	APF	APU	APE	$N(0,1)$
$\bar{\mathbf{y}}^q$ regular	$1-\Phi(\sqrt{3})$	$\pi/2$	no	no	yes	no
$\hat{\mathbf{y}}^q$ L-minimal	0	$\pi/2$	yes	no	no	no
$\check{\mathbf{y}}^q$ normal	$1/2$	$\pi/2$	no	no	yes	yes
$\|\bar{\mathbf{y}}^q\|$ spherical	"0"	"$\pi/2$"	"yes"	"yes"	"no"	"yes"

Table 2: The regular, L-minimal, and normal configurations for $q = 3, 4, 5, 6$. The q components of each vector $\mathbf{y}^q$ are symmetric about 0 so only the nonnegative components are shown.

q	$\bar{\mathbf{y}}^q$	$\hat{\mathbf{y}}^q$	$\check{\mathbf{y}}^q$
3	$(0, 1)$	$(0, 1)$	$(0, 1)$
4	$(.5, 1.5)$	$(.242, 1.56)$	$(.459, 1.51)$
5	$(0, 1, 2)$	$(0, .490, 2.18)$	$(0, .909, 2.04)$
6	$(.5, 1.5, 2.5)$	$(.219, .756, 2.85)$	$(.436, 1.37, 2.59)$

Question 4. *Does any APU sequence* $\{\mathbf{y}^q \in \tilde{\mathcal{M}}^{q-1}_{\leq}\}$ *exist?*

Some exact values of $\bar{\mathbf{y}}^q$, $\hat{\mathbf{y}}^q$, and $\check{\mathbf{y}}^q$, are shown in Table 2. For $q = 3$, $\bar{\mathbf{y}}^3 = \hat{\mathbf{y}}^3 = \check{\mathbf{y}}^3$, while for $q \geq 4$ the components of $\hat{\mathbf{y}}^q$ disperse more rapidly than those of $\bar{\mathbf{y}}^q$ and $\check{\mathbf{y}}^q$ as q increases. This is also seen from the following asymptotic comparisons of the magnitudes of the ranges of the univariate marginal distributions: for each $i = 1, \dots, q$,

$$\begin{aligned}
\big|\mathrm{range}[(\tilde{\mathbf{U}}^{q-2}_{\bar{\mathbf{y}}^q})_i]\big| &= \big|[-\tfrac{q-1}{2},\ \tfrac{q-1}{2}]\big| && = q && = O(q)\\
\big|\mathrm{range}[(\tilde{\mathbf{U}}^{q-2}_{\hat{\mathbf{y}}^q})_i]\big| &= \big|\big[-\tfrac{\|\bar{\mathbf{y}}^q\|}{\|\hat{\mathbf{a}}^q\|}\hat{a}^q_q,\ \tfrac{\|\bar{\mathbf{y}}^q\|}{\|\hat{\mathbf{a}}^q\|}\hat{a}^q_q\big]\big| && = \tfrac{q-1}{\|\hat{\mathbf{a}}^q\|} && = O\Big(\tfrac{q^{\frac{3}{2}}}{\sqrt{\log q}}\Big)\\
\big|\mathrm{range}[(\tilde{\mathbf{U}}^{q-2}_{\check{\mathbf{y}}^q})_i]\big| &= \big|\big[-\tfrac{\|\bar{\mathbf{y}}^q\|}{\|\check{\mathbf{a}}^q\|}\check{a}^q_q,\ \tfrac{\|\bar{\mathbf{y}}^q\|}{\|\check{\mathbf{a}}^q\|}\check{a}^q_q\big]\big| && \sim \sqrt{\tfrac{q^2-1}{3}}\,\Phi^{-1}(\tfrac{q}{q+1}) && = O\Big(q\sqrt{\log q}\Big)\\
\big|\mathrm{range}[(\tilde{\mathbf{U}}^{q-2}_{\|\bar{\mathbf{y}}^q\|})_i]\big| &= \big|[-\|\bar{\mathbf{y}}^q\|,\ \|\bar{\mathbf{y}}^q\|]\big| && = \sqrt{\tfrac{q(q^2-1)}{3}} && = O\big(q^{\frac{3}{2}}\big).
\end{aligned}$$

The four ranges satisfy

$$\text{regular} \ll \text{normal} \ll L\text{-minimal} \ll \text{spherical}, \tag{7.21}$$

where "$\ll$" indicates $o(\cdot)$, whereas the limiting distributions of the univariate marginals in (7.17)–(7.20) satisfy

$$L\text{-minimal} \ll_p \text{regular} \approx_p \text{normal} \approx_p \text{spherical}, \tag{7.22}$$

where "$\ll_p$" indicates $o_p(\cdot)$ and "$\approx_p$" indicates $O_p(\cdot)$. The ordering (7.22) is somewhat unexpected since the L-minimal configuration is the only one of the three uniform permutation distributions that is APF.

8 The regular, L-minimal, and normal permutohedra

The *regular permutohedron*[6] $\mathfrak{R}^q$ is defined to be the convex hull of $\Pi(\bar{\mathbf{x}}^q)$, the set of all $q!$ permutations of the regular configuration $\bar{\mathbf{x}}^q \equiv (1,2,\ldots,q)'$. It is a convex polyhedron in $\mathcal{M}^{q-1}_{\bar{\mathbf{x}}^q}$ (cf. (2.3)) of affine dimension $q-1$. Equivalently we shall consider the congruent polyhedron $\tilde{\mathfrak{R}}^q \equiv \Omega_q\,\mathfrak{R}^q$, the translation of $\mathfrak{R}^q$ into $\tilde{\mathcal{M}}^{q-1}$, so $\tilde{\mathfrak{R}}^q$ is the convex hull of $\Pi(\bar{\mathbf{y}}^q)$ (cf. (4.2)). Thus the uniform distribution $\tilde{\mathbf{U}}^{q-2}_{\bar{\mathbf{y}}^q}$ is the uniform distribution on the vertices of $\tilde{\mathfrak{R}}^q$.

Proposition 4.2 shows that $\Pi(\bar{\mathbf{y}}^q)$ occupies a vanishingly small portion of the sphere $\tilde{\mathcal{S}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$ as $q \to \infty$. As a complement to Proposition 4.2, it will now be shown that $\tilde{\mathfrak{R}}^q$ occupies a vanishingly small portion of the corresponding ball $\tilde{\mathfrak{B}}^q := \tilde{\mathfrak{B}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$ in which $\tilde{\mathfrak{R}}^q$ is inscribed.

Proposition 8.1. *As* $q \to \infty$, $\frac{\mathrm{Vol}(\tilde{\mathfrak{R}}^q)}{\mathrm{Vol}(\tilde{\mathfrak{B}}^q)} \to 0$ *at a geometric rate.*

Proof. From Proposition 2.11 of [2] with $d = q-1$, the volume of $\tilde{\mathfrak{R}}^q$ is $q^{q-\frac{3}{2}}$, while the volume of $\tilde{\mathfrak{B}}^q$ is

$$\frac{\pi^{\frac{q-1}{2}}\|\mathbf{y}^q\|^{q-1}}{\Gamma(\frac{q+1}{2})} = \frac{\pi^{\frac{q-1}{2}}}{\Gamma(\frac{q+1}{2})}\left[\frac{q(q^2-1)}{12}\right]^{\frac{q-1}{2}}.$$

Therefore, using Stirling's formula, the ratio of the volumes is given by

$$\begin{aligned}\frac{\mathrm{Vol}(\tilde{\mathfrak{R}}^q)}{\mathrm{Vol}(\tilde{\mathfrak{B}}^q)} &= \left(\tfrac{12}{\pi}\right)^{\frac{q-1}{2}}\frac{q^{q-\frac{3}{2}}\,\Gamma(\frac{q+1}{2})}{[q(q^2-1)]^{\frac{q-1}{2}}}\\ &\sim \left(\tfrac{\pi}{e}\right)^{\frac{1}{2}}\left(\tfrac{6}{\pi e}\right)^{\frac{q-1}{2}}\\ &\approx 1.0750\,\big(0.7026\big)^{\frac{q-1}{2}}\end{aligned}\tag{8.1}$$

as $q \to \infty$, which converges to zero at a geometric rate.

Remark 8.2. By comparison, the cube $\tilde{\mathfrak{C}}^q$ inscribed in $\tilde{\mathfrak{B}}^q$ has vertices

$$\left(\pm\tfrac{\|y^q\|}{\sqrt{q}},\ldots,\pm\tfrac{\|y^q\|}{\sqrt{q}}\right) = \left(\pm\sqrt{\tfrac{q^2-1}{12}},\ldots,\pm\sqrt{\tfrac{q^2-1}{12}}\right),$$

so

$$\begin{aligned}\frac{\mathrm{Vol}(\tilde{\mathfrak{C}}^q)}{\mathrm{Vol}(\tilde{\mathfrak{R}}^q)} &= \left(\tfrac{q^2-1}{3}\right)^{\frac{q-1}{2}} q^{\frac{3}{2}-q}\\ &\sim \frac{q^{\frac{1}{2}}}{3^{\frac{q-1}{2}}}\\ &\approx q^{\frac{1}{2}}\,(0.3333)^{\frac{q-1}{2}}\end{aligned}\tag{8.2}$$

as $q \to \infty$, which also converges to zero at a geometric rate. Therefore

$$\mathrm{Vol}(\tilde{\mathfrak{C}}^q) \ll \mathrm{Vol}(\tilde{\mathfrak{R}}^q) \ll \mathrm{Vol}(\tilde{\mathfrak{B}}^q)\tag{8.3}$$

for large q. □

Next, define the *L-minimal permutohedron* $\tilde{\mathfrak{M}}^q$ (*normal permutohedron* $\tilde{\mathfrak{N}}^q$) to be the convex hull of $\Pi(\hat{\mathbf{y}}^q)$ ($\Pi(\check{\mathbf{y}}^q)$), the set of all $q!$ permutations of the L-minimal configuration $\hat{\mathbf{y}}^q$ (normal configuration $\check{\mathbf{y}}^q$). Like the regular permutohedron $\tilde{\mathfrak{R}}^q$ defined above, $\tilde{\mathfrak{M}}^q$ and $\tilde{\mathfrak{N}}^q$ are convex polyhedrons in $\tilde{\mathcal{M}}^{q-1}$ (cf. (2.3)) of affine dimension $q-1$. Thus the uniform distribution $\tilde{\mathbf{U}}^{q-2}_{\hat{\mathbf{y}}^q}$ ($\tilde{\mathbf{U}}^{q-2}_{\check{\mathbf{y}}^q}$) is the uniform distribution on the vertices of $\tilde{\mathfrak{M}}^q$ ($\tilde{\mathfrak{N}}^q$). The following question is suggested:

[6] A.k.a. permutahedron.

Question 5. *What are the volumes of* $\tilde{\mathfrak{M}}^q$ *and* $\tilde{\mathfrak{N}}^q$*? As in Proposition 8.1 and Remark 8.2, compare* $\mathrm{Vol}(\tilde{\mathfrak{R}}^q)$*,* $\mathrm{Vol}(\tilde{\mathfrak{M}}^q)$*,* $\mathrm{Vol}(\tilde{\mathfrak{N}}^q)$*, and* $\mathrm{Vol}(\tilde{\mathfrak{B}}^q)$*.*

We conjecture, again somewhat weakly, that as $q \to \infty$,

$$\mathrm{Vol}(\tilde{\mathfrak{R}}^q) \ll \mathrm{Vol}(\tilde{\mathfrak{M}}^q) \ll \mathrm{Vol}(\tilde{\mathfrak{B}}^q), \tag{8.4}$$

more precisely, that $\frac{\mathrm{Vol}(\tilde{\mathfrak{R}}^q)}{\mathrm{Vol}(\tilde{\mathfrak{M}}^q)} \to 0$ at a geometric rate and $\frac{\mathrm{Vol}(\tilde{\mathfrak{M}}^q)}{\mathrm{Vol}(\tilde{\mathfrak{B}}^q)} \to 0$ at a slower rate. Similar results are expected if $\tilde{\mathfrak{M}}^q$ is replaced by $\tilde{\mathfrak{N}}^q$.

9 Final remarks

We conclude with two further questions.

Question 6. *Regarding Question 4 about the existence or non-existence of an APU sequence, can one find a* D*-minimal configuration? That is, a nonzero vector* $\check{\mathbf{y}}^q$ *in* $\tilde{\mathcal{S}}^{q-2}_{\|\bar{\mathbf{y}}^q\|} \cap \mathbb{R}^q_{\leq}$ *that minimizes the NSCD* $D^{q-2}(\Pi(\mathbf{y}))$ *in* $\tilde{\mathcal{S}}^{q-2}_{\|\bar{\mathbf{y}}^q\|}$*.*

Question 7. *Suppose that the permutation group is replaced by some other finite subgroup* G *of orthogonal transformations on* $\mathbb{R}^q$*. For nonzero* $\mathbf{y}^q \in \mathbb{R}^q$*, how close to spherical uniformity is the* G*-orbit* $\Pi_G(\mathbf{y}^q) \equiv \{g\mathbf{y}^q \mid g \in G\}$ *on the smallest sphere containing* $\Pi_G(\mathbf{y}^q)$*?*

For Question 7, finite reflection groups (Coxeter groups) acting on $\mathbb{R}^q$ for all $q \geq 2$ are of particular interest, cf. [10, 12]. These include, and in fact are limited to, the permutation (= symmetric) group, the alternating group, and the group generated by all permutations and sign changes of coordinates.

A Subindependence of coordinate halfspaces

The following inequality was used in the proof of Proposition 4.2:

Proposition A.1. *Let* $\mathbf{U}_n \equiv (U_1, \ldots, U_n)'$ *be uniformly distributed on the unit* $(n-1)$*-sphere* $\mathcal{S}^{n-1}$ *in* $\mathbf{R}^n$*. For any positive real numbers* $t_1, \ldots, t_n$*,*

$$\Pr[\cap_{i=1}^n \{U_i \leq t_i\}] \leq \prod_{i=1}^n \Pr[U_i \leq t_i]. \tag{A.1}$$

Proof. The proof is modelled on that of Proposition 2.10 in [4]. We shall show more generally that for $1 \leq r < n$,

$$\Pr[\cap_{i=1}^n \{U_i \leq t_i\}] \leq \Pr[\cap_{i=1}^r \{U_i \leq t_i\}] \Pr[\cap_{i=r+1}^n \{U_i \leq t_i\}]. \tag{A.2}$$

Because $\mathbf{U}_n$ is the unique orthogonally invariant distribution on $\mathcal{S}^{n-1}$,

$$\mathbf{U}_n \equiv \begin{pmatrix} \mathbf{U}_r \\ \mathbf{U}_{-r} \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \Psi & 0 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} \mathbf{U}_r \\ \mathbf{U}_{-r} \end{pmatrix} = \begin{pmatrix} \psi \mathbf{U}_r \\ \mathbf{U}_{-r} \end{pmatrix}$$

for every orthogonal $r \times r$ matrix Ψ, where

$$\mathbf{U}_r = (U_1, \ldots, U_r)',$$
$$\mathbf{U}_{-r} = (U_{r+1}, \ldots, U_n)'.$$

Therefore

$$\begin{pmatrix} \mathbf{W_r} \\ \mathbf{U}_{-r} \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \psi \mathbf{W_r} \\ \mathbf{U}_{-r} \end{pmatrix},$$

where

$$\mathbf{W}_r \equiv (W_1, \ldots, W_r)' = \tfrac{\mathbf{U}_r}{\|\mathbf{U}_r\|} \in \mathcal{S}^{r-1},$$
$$\mathbf{W}_{-r} \equiv (W_{r+1}, \ldots, W_n)' = \tfrac{\mathbf{U}_{-r}}{\|\mathbf{U}_{-r}\|} \in \mathcal{S}^{n-r+1}.$$

Thus the conditional distribution of $\mathbf{W}_r|\mathbf{U}_{-r}$ is the same as that of $\Psi\mathbf{W}_r|\mathbf{U}_{-r}$ so, by uniqueness, is the uniform distribution on $\mathcal{S}^{r-1}$. Therefore $\mathbf{W}_r$ is independent of $\mathbf{U}_{-r}$, hence $\mathbf{W}_r$ is independent of $(\mathbf{W}_{-r}, \|\mathbf{U}_{-r}\|)$. Similarly, $\mathbf{W}_{-r}$ is independent of $(\mathbf{W}_r, \|\mathbf{U}_r\|)$. However, $\|\mathbf{U}_r\|$ and $\|\mathbf{U}_{-r}\|$ are statistically equivalent because $\|\mathbf{U}_r\|^2 + \|\mathbf{U}_{-r}\|^2 = \|\mathbf{U}_n\|^2 = 1$, hence $\mathbf{W}_{-r}$ is independent of $\|\mathbf{U}_{-r}\|$, so $\mathbf{W}_r$, $\mathbf{W}_{-r}$, and $\|\mathbf{U}_{-r}\|$ are mutually independent. Thus $\mathbf{W}_r$, $\mathbf{W}_{-r}$, and $\|\mathbf{U}_r\|$ are mutually independent, so

$$\begin{aligned}
&\Pr[\cap_{i=1}^n \{U_i \le t_i\}] \\
&\quad = \mathrm{E}\Big\{ \Pr[\cap_{i=1}^r \{W_i \le t_i \|\mathbf{U}_r\|^{-1}\} \mid \|\mathbf{U}_r\|] \\
&\qquad \cdot \Pr[\cap_{i=r+1}^n \{W_i \le t_i (1 - \|\mathbf{U}_r\|^2)^{-1/2}\} \mid \|\mathbf{U}_r\|] \Big\} \\
&\quad \le \mathrm{E}\big\{ \Pr[\cap_{i=1}^r \{W_i \le t_i \|\mathbf{U}_r\|^{-1}\} \mid \|\mathbf{U}_r\|] \big\} \\
&\qquad \cdot \mathrm{E}\big\{ \Pr[\cap_{i=r+1}^n \{W_i \le t_i (1 - \|\mathbf{U}_r\|^2)^{-1/2}\} \mid \|\mathbf{U}_r\|] \big\} \\
&\quad = \Pr[\cap_{i=1}^r \{U_i \le t_i\}] \cdot \Pr[\cap_{i=r+1}^n \{U_i \le t_i\}].
\end{aligned}$$

The inequality holds because

$$\Pr[\cap_{i=1}^r \{W_i \le t_i \|\mathbf{U}_r\|^{-1}\} \mid \|\mathbf{U}_r\|]$$

is decreasing in $\|\mathbf{U}_r\|$ while

$$\Pr[\cap_{i=r+1}^n \{U_i \le t_i (1 - \|\mathbf{U}_r\|^2)^{-1/2}\} \mid \|\mathbf{U}_r\|]$$

is increasing in $\|\mathbf{U}_r\|$. □

Remark A.2. The inequality (A.1) is a one-sided version for coordinate halfspaces of a two-sided inequality for symmetric coordinate slabs, where $|U_i|$ appears in place of U_i; see [4] pp. 329–330 and the references cited therein. As in [4], it is straightforward to extend Proposition A.1 to distributions on the unit sphere in ℓ_p for $1 \le p < \infty$. □

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