

## On the construction of measure-valued dual processes

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### Abstract

Markov intertwining is an important tool in stochastic processes: it enables to prove equalities in law, to assess convergence to equilibrium in a probabilistic way, to relate apparently distinct random models or to make links with wave equations, see Carmona, Petit and Yor [8], Aldous and Diaconis [2], Borodin and Olshanski [7] and Pal and Shkolnikov [23] for examples of applications in these domains. Unfortunately the basic construction of Diaconis and Fill [10] is not easy to manipulate. An alternative approach, where the underlying coupling is first constructed, is proposed here as an attempt to remedy to this drawback, via random mappings for measure-valued dual processes, first in a discrete time and finite state space setting. This construction is related to the evolving sets of Morris and Peres [22] and to the coupling-from-the-past algorithm of Propp and Wilson [27]. Extensions to continuous frameworks enable to recover, via a coalescing stochastic flow due to Le Jan and Raimond [16], the explicit coupling underlying the intertwining relation between the Brownian motion and the Bessel-3 process due to Pitman [25]. To generalize such a coupling to more general one-dimensional diffusions, new coalescing stochastic flows would be needed and the paper ends with challenging conjectures in this direction.

**Keywords:** Markov intertwining relations; measure-valued dual processes; set-valued dual processes; Diaconis-Fill couplings; random mappings; coalescing stochastic flows; Pitman’s theorem; one-dimensional diffusions.

**AMS MSC 2010:** Primary 60J10, Secondary 37A25; 60J05; 60J25; 60J60; 60G17; 60J65.

Submitted to EJP on November 9, 2018, final version accepted on January 15, 2020.

## 1 Introduction

Consider two Markov processes  $X$  and  $\mathfrak{X}$  on respective state spaces  $V$  and  $\mathfrak{V}$ , whose generators are denoted  $L$  and  $\mathfrak{L}$ . The Markov process  $\mathfrak{X}$  is said to be a dual (by intertwining) of  $X$  when  $\mathfrak{L}$  and  $L$  are linked via a weak conjugation relation  $\mathfrak{L}\Lambda = \Lambda L$ , where  $\Lambda$  is a Markov kernel going from  $\mathfrak{V}$  to  $V$  (so that at least formally, the previous commutation makes sense). In such circumstances, the processes  $X$  and  $\mathfrak{X}$  can usually be nicely coupled, so that useful informations on the behavior of  $X$  can be deduced from  $\mathfrak{X}$ . The

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construction of the coupling was given by Diaconis and Fill [10] in the case of finite state spaces and it is sometimes possible to extend it to more general situations, up to cumbersome technicalities (see e.g. [21], for one-dimensional diffusions). Here we propose a direct construction in the context of measure-valued duals  $\mathfrak{X}$ , namely those for which  $\mathfrak{V}$  is a set of (non-negative and different from 0) measures on  $V$  and  $\Lambda$  corresponds to the canonical Markov kernel from  $\mathfrak{V}$  to  $V$ :

$$\forall \eta \in \mathfrak{V}, \quad \Lambda(\eta, \cdot) := \eta(\cdot)/\eta(V) \tag{1.1}$$

When  $X$  admits an invariant probability measure  $\pi$ , we can also consider set-valued duals: then  $\mathfrak{V}$  is a set of measurable subsets of  $V$  and  $\Lambda$  corresponds to the conditioning of  $\pi$  with respect to the elements of  $\mathfrak{V}$ . Since any measurable subset  $S$  of  $V$  can be seen as  $\mathbb{1}_S\pi$ , the measure admitting the indicator function  $\mathbb{1}_S$  of  $S$  as density with respect to  $\pi$  (i.e. the restriction of  $\pi$  on  $S$ ), set-valued duals are in fact a particular case of measure-valued duals. When  $V$  is finite and  $X$  is irreducible, Chapter 17 of Levin, Peres and Wilmer [18] presents a particular set-valued dual as a Doob transform of the evolving sets of Morris and Peres [22]. We go further in this direction, by constructing measure-valued dual  $\mathfrak{X}$  directly from  $X$  via an approach sharing similarities with the coupling-from-the-past algorithm of Propp and Wilson [27]. Some reverse random mappings play the main role, since after having been conditioned to be compatible with a given trajectory of  $X$ , they are used to make the measure-valued dual evolve (or the set-valued dual spread through the state space). We will reinterpret the classical examples of the discrete Pitman theorem [25] and of the top-to-random shuffle due to Aldous and Diaconis [1] as particular instances of this general construction. To facilitate the exposition, we will mainly consider finite state spaces and discrete time in this introduction, leaving the extensions to continuous time and spaces to the last part of the paper, that will enable us to recover the classical Pitman's theorem [25] (see also Rogers and Pitman [29]). To deal with more general one-dimensional diffusions and even multi-dimensional diffusions (see the preprint [9] for the corresponding definition of  $\mathfrak{L}$ ), we would need some coalescing flows of a new type, whose theory has not yet been developed, despite the works of Le Jan and Raimond, in particular [15, 16]. Their investigation should lead to direct constructions of stochastic perturbations of mean curvature flows, in the spirit of [9], which is the remote motivation for the present study. An alternative approach has recently been proposed by Machida [19] for set-valued duals of diffusions and another one is under construction in [3].

Let  $V$  be a finite space, endowed with a Markov kernel  $P := (P(x, y))_{x, y \in V}$ . A traditionally associated generator  $L$  is  $P - I$ , where  $I$  is the  $V \times V$  identity matrix, to make a connexion with the above paragraph. Instead, we first consider discrete time and keep working with  $P$  instead of  $L$ . When we are given a distribution  $m_0$  on  $V$ , there exists a Markov chain  $X := (X_n)_{n \in \mathbb{Z}_+}$  on  $V$  whose initial position  $X_0$  is distributed according to  $m_0$  and whose transition are dictated by  $P$ . The law  $\mathcal{L}(X)$  of  $X$  is uniquely determined by  $m_0$  and  $P$ . From now on, we assume that  $P$  is irreducible (i.e.  $\exp(P)$  has only positive entries), so that it admits a unique invariant probability  $\pi := (\pi(x))_{x \in V}$ . The entries of  $\pi$  are positive. As usual, measures (respectively functions) are seen as row (resp. column) vectors and the **invariance** of  $\pi$  writes  $\pi P = \pi$ . This terminology comes from the fact that when the initial distribution is chosen to be  $\pi$ , then for any time  $n \in \mathbb{Z}_+$ , the law of  $X_n$  is equal to  $\pi$ . In this situation, it is possible to consider a stationary Markov chain  $X := (X_n)_{n \in \mathbb{Z}}$  defined for all times  $n \in \mathbb{Z}$ . The time-reversed process  $(X_{-n})_{n \in \mathbb{Z}}$  is also a stationary Markov chain, whose transition matrix  $P^* := (P^*(x, y))$  is given by

$$\forall x, y \in V, \quad P^*(x, y) := \frac{\pi(y)}{\pi(x)} P(y, x) \tag{1.2}$$

Let us recall the **evolving set process** of Morris and Peres [22] (see also Chapter 17 of Levin, Peres and Wilmer [18]). It is a Markov chain  $\mathcal{X} := (\mathcal{X}_n)_{n \in \mathbb{Z}_+}$  taking values in  $\mathfrak{S}$ , where  $\mathfrak{S}$  is the set of all subsets of  $V$ . To define its transition matrix  $J$ , fix  $S \in \mathfrak{S}$  and consider a random variable  $U$  uniformly distributed on  $[0, 1]$ . Denote

$$\Phi(S) := \{y \in V : P^*(y, S) \geq U\} \tag{1.3}$$

and

$$\forall S, S' \in \mathfrak{S}, \quad J(S, S') := \mathbb{P}[\Phi(S) = S'] \tag{1.4}$$

Note that  $\mathcal{X}$  is absorbed at  $\emptyset$  and at  $V \in \mathfrak{S}$ . The mapping  $\pi : \mathfrak{S} \ni S \mapsto \pi(S)$  is harmonic for  $J$ , in the sense that  $J\pi = \pi$ . This harmonicity of  $\pi$  is equivalent to the fact that  $(\pi(\mathcal{X}_n))_{n \in \mathbb{Z}_+}$  is a martingale. It leads to consider the **Doob transform** of  $J$  by  $\pi$ , which is the  $\mathfrak{S} \times \mathfrak{S}$  transition matrix  $\mathfrak{P}_J$  given by

$$\forall S, S' \in \mathfrak{S}, \quad \mathfrak{P}_J(S, S') := \frac{\pi(S')}{\pi(S)} J(S, S') \tag{1.5}$$

where  $\mathfrak{S} := \mathfrak{S} \setminus \{\emptyset\}$  is the set of all nonempty subsets of  $V$ . The matrix  $\mathfrak{P}_J$  is the transition kernel of  $\mathcal{X}$  conditioned not to be absorbed at  $\emptyset$ , cf. Chapter 17 of Levin, Peres and Wilmer [18]. Let  $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{Z}_+}$  be a Markov chain whose transitions are dictated by  $\mathfrak{P}_J$ . It is a set-valued dual to  $X$ . Indeed, let  $\Lambda$  be the Markov matrix from  $\mathfrak{S}$  to  $V$  given by

$$\forall S \in \mathfrak{S}, \forall x \in V, \quad \Lambda(S, x) := \frac{\pi(x)}{\pi(S)} \tag{1.6}$$

It is not difficult to check the intertwining relation  $\mathfrak{P}_J \Lambda = \Lambda P$ .

More generally, let  $\mathfrak{P}$  be a transition kernel on a finite set  $\mathfrak{V}$  and  $\Lambda$  be a Markov kernel from  $\mathfrak{V}$  to  $V$  such that

$$\mathfrak{P}\Lambda = \Lambda P \tag{1.7}$$

and let  $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{Z}_+}$  be a corresponding Markov chain. Assuming furthermore

$$\mathcal{L}(\mathfrak{X}_0)\Lambda = \mathcal{L}(X_0) \tag{1.8}$$

Diaconis and Fill [10] constructed a coupling of  $X$  and  $\mathfrak{X}$  such that the two following properties hold:

$$\forall n \in \mathbb{Z}_+, \quad \mathcal{L}(\mathfrak{X}_{[0,n]}|X) = \mathcal{L}(\mathfrak{X}_{[0,n]}|X_{[0,n]}) \tag{1.9}$$

$$\forall n \in \mathbb{Z}_+, \quad \mathcal{L}(X_n|\mathfrak{X}_{[0,n]}) = \Lambda(\mathfrak{X}_n, \cdot) \tag{1.10}$$

In these identities,  $\mathcal{L}(\mathcal{Y}|\mathcal{Z})$  stands for the conditional law of  $\mathcal{Y}$  knowing  $\mathcal{Z}$ , for any (here finite valued) random variables  $\mathcal{Y}, \mathcal{Z}$  defined on a same probability space,  $[0, n] := \{0, 1, \dots, n\}$  and  $Y_{[0,n]}$  is the stopped trajectory  $(Y_m)_{m \in [0,n]}$  for any process  $Y := (Y_n)_{n \in \mathbb{Z}_+}$ .

The first relation (1.9) requires that  $\mathfrak{X}$  can be deduced from  $X$  in an adapted way: for any  $n \in \mathbb{Z}_+$ , only the knowledge of  $X_{[0,n]}$  is needed to construct  $\mathfrak{X}_{[0,n]}$ , maybe with the help of independent randomness. The second relation can be seen as a stochastic prolongation of both  $\mathcal{L}(\mathfrak{X}_0)\Lambda = \mathcal{L}(X_0)$  and  $\mathfrak{P}\Lambda = \Lambda P$ .

Unfortunately, the construction of Diaconis and Fill [10] is not easy to manipulate, that is why we propose an alternative approach, valid for a extension of the evolving set point of view (for the generality of this method, see Remark 2.1).

Our main ingredient is the following object. A random mapping  $\psi : V \rightarrow V$  (defined on some underlying probability space) is said to be **associated to  $P^*$**  when

$$\forall x, x' \in V, \quad \mathbb{P}[\psi(x) = x'] = P^*(x, x')$$

Consider  $\bar{\mathfrak{M}}$  the set of non-negative measures on  $V$ . It is convenient to have at our disposal a random mapping  $\psi_\eta$  for any given  $\eta \in \bar{\mathfrak{M}}$ . Such a family  $(\psi_\eta)_{\eta \in \bar{\mathfrak{M}}}$  is said to be **locally associated to  $P^*$**  if

$$\forall \eta \in \bar{\mathfrak{M}}, \forall x \in V, \forall x' \in \text{supp}(\eta), \quad \mathbb{P}[\psi_\eta(x) = x'] = \frac{P^*(x, x')}{\zeta(\eta)} \quad (1.11)$$

where  $\text{supp}(\eta)$  is the support of  $\eta$  and where  $\zeta : \bar{\mathfrak{M}} \rightarrow (0, +\infty)$  is a given positive mapping on  $\bar{\mathfrak{M}}$  (note that necessarily,  $\zeta(\eta) = 1$  as soon as  $\text{supp}(\eta) = V$ ). From now on, all the families  $(\psi_\eta)_{\eta \in \bar{\mathfrak{M}}}$  we will consider will be implicitly assumed to be locally associated to  $P^*$ .

A family  $(\psi_\eta)_{\eta \in \bar{\mathfrak{M}}}$  enables to define a random mapping  $\Psi$  from  $\bar{\mathfrak{M}}$  to  $\bar{\mathfrak{M}}$  in the following way. For any  $\eta \in \bar{\mathfrak{M}}$ , consider  $\mathfrak{f}$  the density of  $\eta$  with respect to  $\pi$ :

$$\forall x \in V, \quad \mathfrak{f}(x) := \frac{\eta(x)}{\pi(x)}$$

We define

$$\forall \eta \in \bar{\mathfrak{M}}, \quad \Psi(\eta) := (\mathfrak{f} \circ \psi_\eta)\pi \quad (1.12)$$

namely the measure admitting the density  $\mathfrak{f} \circ \psi_\eta$  with respect to  $\pi$ .

Denote  $\mathcal{F}(V)$  the set of real functions defined on  $V$  and recall the following notation for the duality measures-functions:

$$\forall \eta \in \bar{\mathfrak{M}}, \forall f \in \mathcal{F}(V), \quad \eta[f] := \sum_{x \in V} f(x)\eta(x)$$

We compute that

$$\begin{aligned} \forall \eta \in \bar{\mathfrak{M}}, \forall f \in \mathcal{F}(V), \quad \mathbb{E}[\Psi(\eta)[f]] &= \sum_{x \in V} f(x)\mathbb{E}[\mathfrak{f}(\psi_\eta(x))]\pi(x) \\ &= \sum_{x, y \in V} f(x)\mathfrak{f}(y)\mathbb{P}[\psi_\eta(x) = y]\pi(x) \\ &= \sum_{x, y \in V} f(x)\mathfrak{f}(y)\frac{P^*(x, y)}{\zeta(\eta)}\pi(x) \\ &= \frac{\pi[fP^*[f]]}{\zeta(\eta)} \\ &= \frac{\pi[\mathfrak{f}P[f]]}{\zeta(\eta)} \\ &= \frac{\eta[P[f]]}{\zeta(\eta)} \end{aligned} \quad (1.13)$$

where in the third equality, we used that the sum can be restricted to  $y$  belonging to the support of  $\eta$ , i.e. satisfying  $\mathfrak{f}(y) > 0$ .

The above relation will be crucial for our purposes, but before developing them, let us make a link with the evolving sets of Morris and Peres [22].

Consider a family of random mappings  $(\psi_S)_{S \in \bar{\mathfrak{S}}}$ , namely which is rather indexed by the subsets of  $V$ . We will refer to this situation as the **subset case** to distinguish it from the previous **measure case**. These two settings are related, as alluded to in the introductory paragraph: consider the mapping  $\mathfrak{T} : \bar{\mathfrak{S}} \rightarrow \bar{\mathfrak{M}}$  defined by

$$\forall S \in \bar{\mathfrak{S}}, \quad \mathfrak{T}(S) := 1_S\pi \quad (1.14)$$

This embedding enables us to see  $\bar{\mathfrak{S}}$  as a subset of  $\bar{\mathfrak{M}}$ . In particular, by restriction, any measure-indexed family of random mappings leads to a subset-indexed family of random mappings. Conversely subset-indexed family of random mappings  $(\psi_S)_{S \in \bar{\mathfrak{S}}}$  can be extended into a measure-indexed family of random mappings  $(\psi_\eta)_{\eta \in \bar{\mathfrak{M}}}$ , by example via

$$\forall \eta \in \bar{\mathfrak{M}}, \quad \psi_\eta := \begin{cases} \psi_S, & \text{when } \eta = \mathfrak{T}(S) \\ \psi_\emptyset, & \text{otherwise} \end{cases}$$

**Remark 1.1.** Measure (respectively subset)-indexed family of random mappings will lead to measure (resp. subset)-valued dual processes. Measure-valued dual processes are more general than subset-valued processes, while the latter can be interesting in their own right, as they are related to natural objects such as the evolving sets of Morris and Peres [22] or mean curvature flows, see for instance [9]. Nevertheless in hypoelliptic continuous settings, subset-valued dual processes may lack a natural continuity property of their trajectories, cf. [20], and to recover this regularity, it is better to reinterpret them as measure-valued dual processes, as the topologies put on sets of measures are traditionally weaker than topologies put on sets of subsets.

**Remark 1.2.** A family  $(\psi_\eta)_{\eta \in \bar{\mathfrak{M}}}$  is said to be **globally associated** to  $P^*$  when all its random mappings  $\psi_\eta$  are associated to  $P^*$ . All the examples of Section 3 are set-valued and globally associated to  $P^*$ . The interest of the notion of local association only appeared while writing Section 5, when dealing with Polish spaces. A posteriori it seemed a natural point of view that can be useful for some applications, even in the finite setting. Indeed, looking for set-valued intertwining relations with a link  $\Lambda$  as in (1.6) where  $\pi$  is replaced by a probability measure  $\mu$  which is not invariant for the transition kernel  $P$  of the (primal) Markov chain, it is tempting modify  $P$  “far away” from the initial point so that  $\mu$  becomes invariant. Until these “far away” points are attained, the Markov chain “does not know” its transition kernel has been modified, so we can intertwine it using the modified  $\Lambda$ . Here we will not investigate the consequences of this possibility of working locally.

The family  $(\psi_S)_{S \in \bar{\mathfrak{S}}}$  enables to define a random mapping  $\Psi$  from  $\bar{\mathfrak{S}}$  to  $\bar{\mathfrak{S}}$  via

$$\forall S \in \bar{\mathfrak{S}}, \quad \Psi(S) := \{y \in V : \psi_S(y) \in S\} \tag{1.15}$$

It is easy to check that the definitions (1.12) and (1.15) are compatible with the identification map  $\mathfrak{T}$ , in the sense that  $\mathfrak{T} \circ \Psi = \Psi \circ \mathfrak{T}$ . Similarly to (1.13), we compute that

$$\begin{aligned} \forall S \in \bar{\mathfrak{S}}, \forall y \in V, \quad \mathbb{P}[y \in \Psi(S)] &= \mathbb{P}[\psi_S(y) \in S] \\ &= \sum_{y' \in S} \mathbb{P}[\psi_S(y) = y'] \\ &= \sum_{y' \in S} P^*(y, y') / \zeta(S) \\ &= P^*(y, S) / \zeta(S) \end{aligned}$$

Thus when  $\zeta \equiv 1$ , in particular for globally associated to  $P^*$  random mappings, it appears that for any  $y \in V$  and  $S \in \bar{\mathfrak{S}}$ ,  $\mathbb{P}[y \in \Psi(S)] = \mathbb{P}[y \in \Phi(S)]$ , but in general the law of  $\Phi(S)$  and  $\Psi(S)$  are not equal, as it can be seen on the examples of Section 3.

**Remark 1.3.** The subset case is absorbing for the measure case in the following sense. Let  $(\psi_\eta)_{\eta \in \bar{\mathfrak{M}}}$  be a family of random mappings indexed by measures and  $\Psi$  be the associated mapping defined in (1.12). Then we have

$$\Psi(\bar{\mathfrak{S}}) \subset \bar{\mathfrak{S}}$$

where  $\bar{\mathfrak{C}}$  is seen as a subset of  $\bar{\mathfrak{V}}$ , via the identification (1.14). This is due to the fact that  $\bar{\mathfrak{C}}$  corresponds to the measures whose density with respect to  $\pi$  only takes values in  $\{0, 1\}$ . Note from (1.12) that the set of values taken by  $\Psi(\eta)$  is a subset of the set of values taken by  $\eta \in \bar{\mathfrak{V}}$ .

As a consequence, if the algorithm we present below produces at some time a measure belonging to  $\bar{\mathfrak{C}}$ , then all subsequent measures will also belong to  $\bar{\mathfrak{C}}$ . In particular, when we start with a Dirac mass (which is identified with the singleton of  $\bar{\mathfrak{C}}$  where is concentrated the Dirac mass), only  $(\psi_\eta)_{\eta \in \bar{\mathfrak{C}}}$  is needed for the constructions presented below, namely it is sufficient to consider the subset case.

We now come back to the measure situation. Generalize (1.4), by considering the transition matrix  $K$  from  $\bar{\mathfrak{V}}$  to  $\bar{\mathfrak{V}}$  given by

$$\forall \eta, \eta' \in \bar{\mathfrak{V}}, \quad K(\eta, \eta') := \mathbb{P}[\Psi(\eta) = \eta'] \tag{1.16}$$

In order to extend the Doob transform of (1.5), let us define

$$\forall \eta \in \bar{\mathfrak{V}}, \quad \pi(\eta) := \eta(V) \tag{1.17}$$

This definition may seem strange at first view, but note that in the subset case, it is quite natural, as it just asserts that

$$\forall S \in \bar{\mathfrak{C}}, \quad \pi(\mathbb{1}_S \pi) = \pi(S)$$

We can now generalize (1.5) via

$$\forall \eta, \eta' \in \bar{\mathfrak{V}}, \quad \mathfrak{P}(\eta, \eta') := \frac{\pi(\eta')\zeta(\eta)}{\pi(\eta)} K(\eta, \eta') \tag{1.18}$$

where  $\bar{\mathfrak{V}} := \bar{\mathfrak{V}} \setminus \{0\}$  is the set of measures on  $V$  which do not vanish identically. Even in the subset case, in general  $\mathfrak{P}$  does not coincide with  $\mathfrak{P}_J$ . It will be shown in Corollary 1.6 below that  $\mathfrak{P}$  is a Markov transition kernel. It will also be useful to introduce the following conditioned transition: fix  $x, x' \in V$  such that  $P(x, x') > 0$  (i.e.  $P^*(x', x) > 0$ ) and denote for any  $\eta \in \bar{\mathfrak{V}}$  whose support contains  $x$ ,

$$\forall \eta' \in \bar{\mathfrak{V}}, \quad K_{x,x'}(\eta, \eta') := \mathbb{P}[\Psi(\eta) = \eta' | \psi_\eta(x') = x] \tag{1.19}$$

Note that the conditioning is non-degenerate, since  $\mathbb{P}[\psi_\eta(x') = x] = P^*(x', x)/\zeta(S) > 0$ , due to the fact that  $x \in \text{supp}(\eta)$ .

Consider

$$W := \{(x, \eta) \in V \times \bar{\mathfrak{V}} : x \in \text{supp}(\eta)\} \tag{1.20}$$

and let  $\mathcal{A}$  be the set of probability measures  $m$  on  $W$  which can be written under the form

$$\forall (x, \eta) \in W, \quad m(x, \eta) = \mu(\eta)\Lambda(\eta, x) \tag{1.21}$$

where  $\mu$  is the marginal of  $m$  on  $\bar{\mathfrak{V}}$  (i.e. the image of  $m$  by the mapping  $W \ni (x, \eta) \mapsto \eta \in \bar{\mathfrak{V}}$ ). Define a Markov kernel  $Q$  on  $W$  via

$$\forall (x, \eta), (x', \eta') \in W, \quad Q((x, \eta), (x', \eta')) := P(x, x')K_{x,x'}(\eta, \eta') \tag{1.22}$$

**Remark 1.4.** This expression should be compared with the one given by Levin, Peres and Wilmer in Section 17.7 of [18], where they introduce a Markov kernel  $\tilde{Q}$  on  $\tilde{W}$  via

$$\forall (x, S), (x', S') \in W, \quad \tilde{Q}((x, S), (x', S')) := P(x, x')J_{x'}(S, S')$$

where, with the notation of (1.3),

$$\forall S \in \mathfrak{S}, \forall (x', S') \in \widetilde{W}, \quad J_{x'}(S, S') := \mathbb{P}[\Phi(S) = S' | x' \in S']$$

and where  $\widetilde{W}$  corresponds to  $W$  in the subset case:

$$\widetilde{W} := \{(x, S) \in V \times \mathfrak{S} : x \in S\} \tag{1.23}$$

In Subsection 2.3, we will check that  $\widetilde{Q}$  is a particular case of  $Q$ , for an appropriate choice of the family of random mappings  $(\psi_S)_{S \in \mathfrak{S}}$  in the subset case. The generality of the random mapping point of view leads to easy constructions in practice, even if they are not optimal, with a coupling-from-the-past flavor (see Subsection 2.4). In particular, it facilitates the reinterpretation of classical set-valued dual processes (see Subsections 3.1 and 3.2). Another advantage of the random mappings is that they can be transformed into coalescing stochastic flows in the diffusion setting (see Section 6 for the general approach, Subsection 7.1 for an application to the real Brownian motion and Subsection 7.2 for corresponding conjectures about general one-dimensional diffusions).

We can now state the first main result of this paper (continuous space and time extensions will be presented in Theorems 5.2, 5.4 and 6.7 of Sections 5 and 6).

**Theorem 1.5.** *Let  $(X_n, \mathfrak{X}_n)_{n \in \mathbb{Z}_+}$  be a Markov chain on  $W$  whose initial distribution  $\mathcal{L}(X_0, \mathfrak{X}_0)$  belongs to  $\mathcal{A}$  and whose transitions are given by  $Q$ . Then  $X := (X_n)_{n \in \mathbb{N}}$  and  $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{N}}$  are Markov chains whose respective transitions are given by  $P$  and  $\mathfrak{P}$ . Furthermore the conditions (1.9) and (1.10) are fulfilled.*

It follows that the kernel  $\mathfrak{P}$  is Markovian. As another consequence, we get an extension of properties recalled for the evolving sets:

**Corollary 1.6.** *The intertwining relation (1.7) is satisfied. Furthermore when  $\zeta \equiv 1$ , in particular in the case of globally associated to  $P^*$  random mappings,  $\pi$ , as defined in (1.17), is harmonic for  $K$ .*

*Proof.* The last assertion is obtained by summing in (1.18) with respect to  $\eta' \in \mathfrak{S}$ . The intertwining relation can be checked directly, but it also comes from the computation of  $\mathbb{P}[X_{n+1} = x | \mathfrak{X}_n = \eta]$ , for  $n \in \mathbb{Z}_+$ ,  $x \in V$  and  $\eta \in \mathfrak{S}$ , in two different ways:

On one hand, we have

$$\begin{aligned} \mathbb{P}[X_{n+1} = x | \mathfrak{X}_n = \eta] &= \sum_{\eta' \in \mathfrak{S}} \mathbb{P}[X_{n+1} = x | \mathfrak{X}_{n+1} = \eta', \mathfrak{X}_n = \eta] \mathbb{P}[\mathfrak{X}_{n+1} = \eta' | \mathfrak{X}_n = \eta] \\ &= \sum_{\eta' \in \mathfrak{S}} \Lambda(\eta', x) \mathfrak{P}(\eta, \eta') \\ &= \mathfrak{P} \Lambda(\eta, x) \end{aligned}$$

and on the other hand,

$$\begin{aligned} \mathbb{P}[X_{n+1} = x | \mathfrak{X}_n = \eta] &= \sum_{x' \in V} \mathbb{P}[X_{n+1} = x | X_n = x', \mathfrak{X}_n = \eta] \mathbb{P}[X_n = x' | \mathfrak{X}_n = \eta] \\ &= \sum_{x' \in V} \Lambda(\eta, x') \mathbb{P}[X_{n+1} = x | X_n = x', \mathfrak{X}_n = \eta] \\ &= \sum_{x' \in V} \Lambda(\eta, x') P(x', x) \\ &= \Lambda P(\eta, x) \end{aligned}$$

where the last-but-one equality comes by summing over  $\eta'$  in (1.22). □

Theorem 1.5 leads to an algorithm for the construction of  $\mathfrak{X}$ , given  $X$ , when  $\mu_0\Lambda = m_0$ , where  $\mu_0 := \mathcal{L}(\mathfrak{X}_0)$  and  $m_0 := \mathcal{L}(X_0)$  (in practice, one is often only interested in the situation where  $m_0$  is a Dirac mass at some  $x_0 \in V$  and  $\mu_0$  is the Dirac mass at  $\{x_0\}$ , trivially satisfying  $\Lambda(\{x_0\}, x_0) = 1$ ). Assume that a trajectory  $(x_n)_{n \in \mathbb{Z}_+}$  of  $X$  is given. We begin by sampling  $\mathfrak{X}_0$  according to the probability measure  $\mathfrak{V} \ni \eta \mapsto \mu_0(\eta)\Lambda(\eta, x_0)/m_0(x_0)$ . Next, for  $n \in \mathbb{Z}_+$ , assume that  $\mathfrak{X}_n$  has been constructed. We consider a random mapping  $\psi_{\mathfrak{X}_n}$  locally associated to  $P^*$ , whose law may depend on  $\mathfrak{X}_n$  (but not directly on  $(x_m)_{m \in \llbracket 0, n \rrbracket}$ , see Subsection 2.2 for a generalization with a dependence on  $\mathfrak{X}_{\llbracket 0, n \rrbracket}$ ) and whose underlying randomness is independent from all that has been done before (except for the index parameter  $\mathfrak{X}_n$ ). We condition by the fact that  $\psi_{\mathfrak{X}_n}(x_{n+1}) = x_n$  and we sample a corresponding mapping  $\varphi$  (which is no longer associated to  $P^*$ , since in general this property is not preserved by conditioning), to construct  $\mathfrak{X}_{n+1}$  via

$$\forall y \in V, \quad \mathfrak{X}_{n+1}(y) := \mathfrak{X}_n(\varphi(y)) \frac{\pi(y)}{\pi(\varphi(y))} \tag{1.24}$$

In particular, since  $\varphi(x_{n+1}) = x_n$ , we get that  $\mathfrak{X}_{n+1}(x_{n+1}) = \mathfrak{X}_n(x_n)\pi(x_{n+1})/\pi(x_n)$ , and by iteration it appears that  $x_n \in \text{supp}(\mathfrak{X}_n)$  for all  $n \in \mathbb{Z}_+$ .

In the subset case,  $(\mathfrak{X}_n)_{n \in \mathbb{Z}_+}$  is subset-valued and the evolution step (1.24) is replaced by

$$\mathfrak{X}_{n+1} := \{y \in V : \varphi(y) \in \mathfrak{X}_n\}$$

By iteration, we check that  $x_n \in \mathfrak{X}_n$  for all  $n \in \mathbb{Z}_+$ .

This procedure is maybe better illustrated by the explicit constructions of Subsections 2.3, 2.4, 3.1 and 3.2 in the subset case. There, only classical examples are considered, as this paper is primarily concerned with the theoretical aspects of the random mapping point of view. Theorem 1.5 describes a general method of construction of measure-valued dual processes as well as their couplings with the primal processes. The obtained dual processes will be good or bad (with respect to their fast spreading over the primal state space in the set-valued case) depending on the underlying random mappings. In practice, a relevant choice of the latter is thus crucial (the principle alluded to at the end of Subsection 2.3 could serve as a general guide). This is another task, so that more examples will be presented in future works. While in the finite state space setting random mappings are easy to describe, this is no longer true in continuous frameworks, as the underlying coalescing stochastic flows should be investigated further, beginning with dimension one. Nevertheless we also expect applications for elliptic diffusions on manifolds, then even the construction of dual processes is difficult (see the recent preprint [9]) and the coupling apparently out-of-reach by traditional approaches. We believe that the coalescing stochastic flows will provide a constructive existence of the dual processes and of their couplings with the primal processes. As illustrated by Theorem 1.5 and Corollary 1.6, the coupling should be constructed first and the dual process deduced in a second step. Thus our method is in reverse order in comparison to the original work of Diaconis and Fill [10].

**Remark 1.7.** In the first version of this paper, only the subset case was considered. The referee pointed out that there are natural measure-valued dual processes, as illustrated e.g. by the papers of Avena, Castell, Gaudillière and Méléot [4, 5] and he/she was wondering if random mappings could be used in this situation. It led to the present theoretical extension. The question of finding the random mappings indexed by mea-

dures inducing the measure-valued dual processes of Avena et al. [4, 5] is very interesting and should be investigated further, as it would provide an algorithm for the coupling of the primal and dual processes. Again, this is out of the scope of the present paper, whose main goal is to establish the principles behind such coupling constructions.

From Section 3 on, we will only work in the subset case, since in Sections 3 and 4, we consider Dirac masses (i.e. singletons) as starting points for the dual processes, and Sections 5, 6 and 7 should be seen as the first steps toward the coupling of stochastically modified mean curvature flows with their primal diffusion processes.

**Remark 1.8.** The finite state space framework is sufficiently important (as illustrated by the book of Levin, Peres and Wilmer [18]) to prevent us from presenting a more general version of Theorem 1.5 in this introduction. Nevertheless, it can be extended to infinite transition matrices  $P := (P(x, y))_{x, y \in V}$ , where  $V$  is a denumerable state space, under the following conditions:

**Finite degree:** for any  $x \in V$ , there is only a finite number of  $y \in V$  such that  $P(x, y) > 0$ .

**Reversibility:** there exists a measure  $\pi$  giving a positive weight to any  $x \in V$ , such that

$$\forall x, y \in V, \quad \pi(x)P(x, y) = \pi(y)P(y, x)$$

Indeed, in this situation, take  $\bar{\mathfrak{V}}$  to be the set of measures with a finite support in  $V$ , so that the kernel  $\Lambda$  given in (1.6) is still well-defined on  $\mathfrak{V} := \bar{\mathfrak{V}} \setminus \{0\}$ . We look for dual processes taking values in  $\mathfrak{V}$ . By reversibility, we have  $P^* = P$ . Consider  $(\psi_\eta)_{\eta \in \bar{\mathfrak{V}}}$  a family of random mappings locally associated to  $P$ . Note that  $\bar{\mathfrak{V}}$  is left stable by the mapping  $\Psi$  defined in (1.12), due to the finite degree assumption. Theorem 1.5 is still valid, because it is sufficient to work up to some arbitrarily fixed time-horizon  $n \geq 0$  and the Markov chains we are interested in have an initial distribution with finite support (i.e. belongs to  $\mathfrak{V}$ , up to a normalisation). Thus up to time  $n$ , the Markov chain stays in a fixed finite state space  $S \subset V$  (depending on  $n$ ). We can then apply the previous constructions on the finite state space  $S$ . Let  $P_S$  be the restriction of  $P$  to  $S$ , obtained by transferring the probabilities to exit  $S$  to self-loops. The reversibility assumption implies that the renormalization of the restriction of  $\pi$  to  $S$  is invariant for  $P_S$ . This property insures us of the compatibility of these constructions for different times  $n$ .

The latter property is not true for general denumerable Markov chains (even under the finite degree assumption). Furthermore the invariant measure may not be unique (even up to a factor), even when there is one invariant measure which is reversible. Thus, at least locally in time and for finite degree Markov kernels, it should be possible to construct different measure-valued duals, associated to various invariant measures through the corresponding  $\Lambda$ . We did not try to investigate further the opportunities suggested by this observation.

When there is an invariant *probability* measure  $\pi$  for  $P$ , the extension of Theorem 1.5 is simpler, since it can be easily verified that all the computations are still valid. In this situation, we take  $\bar{\mathfrak{V}}$  to be the set of measures on  $V$  whose total weight is finite. Again the kernel  $\Lambda$  given in (1.6) is still well-defined on  $\mathfrak{V} := \bar{\mathfrak{V}} \setminus \{0\}$ . Let  $(\psi_\eta)_{\eta \in \bar{\mathfrak{V}}}$  be a family of random mappings locally associated to  $P^*$ . The only point which has to be checked is that the mapping  $\Psi$  defined in (1.12) a.s. leaves  $\bar{\mathfrak{V}}$  invariant. Consider  $\eta \in \bar{\mathfrak{V}}$ , it is

sufficient to show that  $\mathbb{E}[\Psi(\eta)(V)] < +\infty$ . We compute:

$$\begin{aligned} \mathbb{E}[\Psi(\eta)(V)] &= \sum_{x,y \in V} \eta(y) \frac{\pi(x)}{\pi(y)} \mathbb{P}[\psi_\eta(x) = y] \\ &= \sum_{x,y \in V} \eta(y) \frac{\pi(x)}{\pi(y)} \frac{P^*(x,y)}{\zeta(\eta)} \\ &= \sum_{x,y \in V} \eta(y) \frac{P(y,x)}{\zeta(\eta)} \\ &= \sum_{y \in V} \frac{\eta(y)}{\zeta(\eta)} \\ &= \frac{\eta(V)}{\zeta(\eta)} \\ &< +\infty \end{aligned}$$

The plan of the paper is as follows. Theorem 1.5 is proven in the next section and we will see how random mappings can also be used to deduce non-Markov measure-valued dual processes. The classical set-valued dual processes of the discrete Pitman theorem and of the top-to-random shuffle are treated in Section 3. In Section 4, we generalize the discrete Pitman theorem to restless birth and death chains, i.e. necessarily moving at each time step. In Sections 5 and 6, we provide extensions of the random mapping analysis, respectively to Polish state spaces and to continuous time. From these abstract considerations, Section 7 recovers Pitman’s theorem [25] about the explicit and deterministic coupling associated to the intertwining of the Brownian motion with the Bessel-3 process and proposes some conjectures about general one-dimensional diffusion processes. Our hope is that the underlying challenge of the existence of some needed singular coalescing stochastic flows will motivate a more thorough investigation in their direction. An appendix ends the paper, showing why in dimension 1 it is sufficient to study diffusions whose variance coefficient is 1, via some traditional transformations of the state space  $\mathbb{R}$ .

## 2 Random mappings

The proof of Theorem 1.5 generalizes that of Theorem 17.23 of Levin, Peres and Wilmer [18], itself in the spirit of Diaconis and Fill [10]. The argument will be extended to non-Markov measure-valued dual processes in Subsection 2.2, obtained by slightly relaxing the notion of random mappings, it leads to more general dual processes. In Subsection 2.3, we justify the assertions of Remark 1.4 and in Subsection 2.4 we discuss the link with the coupling-from-the-past algorithm.

### 2.1 Proof of Theorem 1.5

By definition, we have for any  $n \in \mathbb{Z}_+$  and  $(x_0, \eta_0), (x_1, \eta_1), \dots, (x_n, \eta_n) \in W$ ,

$$\begin{aligned} \mathbb{P}[(X_0, \mathfrak{X}_0) = (x_0, \eta_0), (X_1, \mathfrak{X}_1) = (x_1, \eta_1), \dots, (X_n, \mathfrak{X}_n) = (x_n, \eta_n)] \\ = \mu_0(\eta_0) \Lambda(\eta_0, x_0) \prod_{m \in \llbracket 0, n-1 \rrbracket} P(x_m, x_{m+1}) K_{x_m, x_{m+1}}(\eta_m, \eta_{m+1}) \end{aligned} \tag{2.1}$$

where  $\mu_0 := \mathcal{L}(\mathfrak{X}_0)$ . Summing over all  $\eta_0, \eta_1, \dots, \eta_n \in \mathfrak{A}$  (so that  $x_0 \in \text{supp}(\eta_0), x_1 \in \text{supp}(\eta_1), \dots, x_n \in \text{supp}(\eta_n)$ ), we get that for any  $x_0, x_1, \dots, x_n \in V$ ,

$$\mathbb{P}[X_0 = x_0, X_1 = x_1, \dots, X_n = x_n] = m_0(x_0) \prod_{m \in \llbracket 0, n-1 \rrbracket} P(x_m, x_{m+1}) \tag{2.2}$$

where

$$m_0(x_0) := \sum_{\eta_0 \in \mathfrak{X}} \mu_0(\eta_0) \Lambda(\eta_0, x_0)$$

(the r.h.s. sum can be restricted to  $\eta_0$  whose support contains  $x_0$ , since otherwise  $\Lambda(\eta_0, x_0) = 0$ ). It follows that  $(X_m)_{m \in \llbracket 0, n \rrbracket}$  is Markovian with transitions given by  $P$  and initial distribution  $m_0$ . Since this is true for all  $n \in \mathbb{Z}_+$ , we get that  $X$  is Markovian with transitions given by  $P$  and initial distribution  $m_0$ .

For any  $m \in \llbracket 0, n \rrbracket$ , dividing (2.1) by (2.2) and summing over  $\eta_{m+1}, \dots, \eta_n$ , we get

$$\begin{aligned} & \mathbb{P}[\mathfrak{X}_0 = \eta_0, \mathfrak{X}_1 = \eta_1, \dots, \mathfrak{X}_m = \eta_m | X_0 = x_0, X_1 = x_1, \dots, X_n = x_n] \\ &= \frac{\mu_0(\eta_0) \Lambda(\eta_0, x_0)}{m_0(x_0)} \prod_{l \in \llbracket 0, m-1 \rrbracket} K_{x_l, x_{l+1}}(\eta_l, \eta_{l+1}) \\ &= \mathbb{P}[\mathfrak{X}_0 = \eta_0, \mathfrak{X}_1 = \eta_1, \dots, \mathfrak{X}_m = \eta_m | X_0 = x_0, X_1 = x_1, \dots, X_m = x_m] \end{aligned}$$

Fixing  $m \in \mathbb{Z}_+$  and  $\eta_{\llbracket 0, m \rrbracket}$ , note that the process  $(\mathbb{P}[\mathfrak{X}_{\llbracket 0, m \rrbracket} = \eta_{\llbracket 0, m \rrbracket} | X_0, X_1, \dots, X_n])_{n \in \mathbb{Z}_+}$  is a non-negative martingale which is well-known to converge toward  $\mathbb{P}[\mathfrak{X}_{\llbracket 0, m \rrbracket} = \eta_{\llbracket 0, m \rrbracket} | X]$  for  $n$  large. It follows that

$$\mathbb{P}[\mathfrak{X}_{\llbracket 0, m \rrbracket} = \eta_{\llbracket 0, m \rrbracket} | X] = \mathbb{P}[\mathfrak{X}_{\llbracket 0, m \rrbracket} = \eta_{\llbracket 0, m \rrbracket} | X_{\llbracket 0, m \rrbracket}]$$

namely (1.9).

The Markov property of  $\mathfrak{X}$  and (1.10) are less immediate and the argument is based on an iteration with respect to the following statements, for  $n \in \mathbb{Z}_+$ :

The finite stochastic chain  $(\mathfrak{X}_m)_{m \in \llbracket 0, n \rrbracket}$  is Markovian with transitions given by  $\mathfrak{P}$  ( $A_n$ )

$$\mathcal{L}(X_n | \mathfrak{X}_{\llbracket 0, n \rrbracket}) = \Lambda(\mathfrak{X}_n, \cdot) \tag{B_n}$$

For  $n = 0$ , the assertion ( $A_0$ ) is void and ( $B_0$ ) is a rewriting of the assumption  $\mathcal{L}(X_0, \mathfrak{X}_0) \in \mathcal{A}$ .

Next assume that ( $A_n$ ) and ( $B_n$ ) are true for some  $n \in \mathbb{Z}_+$  and let us prove ( $A_{n+1}$ ) and ( $B_{n+1}$ ).

Let  $(x, \eta) \in W$  be given, we compute that

$$\begin{aligned} & \mathbb{P}[X_{n+1} = x, \mathfrak{X}_{n+1} = \eta | \mathfrak{X}_{\llbracket 0, n \rrbracket}] \\ &= \sum_{y \in \text{supp}(\mathfrak{X}_n)} \mathbb{P}[X_{n+1} = x, \mathfrak{X}_{n+1} = \eta | X_n = y, \mathfrak{X}_{\llbracket 0, n \rrbracket}] \mathbb{P}[X_n = y | \mathfrak{X}_{\llbracket 0, n \rrbracket}] \end{aligned}$$

Due to the Markov property of the couple  $(X, \mathfrak{X})$ , we deduce that for any  $y \in \text{supp}(\mathfrak{X}_n)$ ,

$$\begin{aligned} \mathbb{P}[X_{n+1} = x, \mathfrak{X}_{n+1} = \eta | X_n = y, \mathfrak{X}_{\llbracket 0, n \rrbracket}] &= \mathbb{P}[X_{n+1} = x, \mathfrak{X}_{n+1} = \eta | X_n = y, \mathfrak{X}_n] \\ &= P(y, x) K_{y, x}(\mathfrak{X}_n, \eta) \\ &= P(y, x) \frac{\mathbb{P}[\Psi(\mathfrak{X}_n) = \eta, \psi_{\mathfrak{X}_n}(x) = y]}{\mathbb{P}[\psi_{\mathfrak{X}_n}(x) = y]} \\ &= P(y, x) \zeta(\mathfrak{X}_n) \frac{\mathbb{P}[\Psi(\mathfrak{X}_n) = \eta, \psi_{\mathfrak{X}_n}(x) = y]}{P^*(x, y)} \\ &= \frac{\pi(x)}{\pi(y)} \zeta(\mathfrak{X}_n) \mathbb{P}[\Psi(\mathfrak{X}_n) = \eta, \psi_{\mathfrak{X}_n}(x) = y] \tag{2.3} \end{aligned}$$

where the local association with  $P^*$  was used in the fourth equality. On the other hand, ( $B_n$ ) asserts that for  $y \in \text{supp}(\mathfrak{X}_n)$ ,

$$\begin{aligned} \mathbb{P}[X_n = y | \mathfrak{X}_{\llbracket 0, n \rrbracket}] &= \Lambda(\mathfrak{X}_n, y) \\ &= \frac{\mathfrak{X}_n(y)}{\pi(\mathfrak{X}_n)} \end{aligned}$$

and we get that

$$\begin{aligned}
 \mathbb{P}[X_{n+1} = x, \mathfrak{X}_{n+1} = \eta | \mathfrak{X}_{[0,n]}] &= \sum_{y \in \text{supp}(\mathfrak{X}_n)} \frac{\pi(x)}{\pi(\mathfrak{X}_n)} \zeta(\mathfrak{X}_n) \mathbb{P}[\Psi(\mathfrak{X}_n) = \eta, \psi_{\mathfrak{X}_n}(x) = y] \\
 &= \frac{\pi(x)}{\pi(\mathfrak{X}_n)} \zeta(\mathfrak{X}_n) \mathbb{P}[\Psi(\mathfrak{X}_n) = \eta, \psi_{\mathfrak{X}_n}(x) \in \mathfrak{X}_n] \\
 &= \frac{\pi(x)}{\pi(\mathfrak{X}_n)} \zeta(\mathfrak{X}_n) \mathbb{P}[\Psi(\mathfrak{X}_n) = \eta, x \in \eta] \\
 &= \frac{\pi(x)}{\pi(\mathfrak{X}_n)} \zeta(\mathfrak{X}_n) \mathbb{P}[\Psi(\mathfrak{X}_n) = \eta] \\
 &= \frac{\pi(x)}{\pi(\eta)} \mathfrak{P}(\mathfrak{X}_n, \eta)
 \end{aligned}$$

where we used Definitions (1.16) and (1.18).

Summing over  $x \in \text{supp}(\eta)$ , we get

$$\mathbb{P}[\mathfrak{X}_{n+1} = \eta | \mathfrak{X}_{[0,n]}] = \mathfrak{P}(\mathfrak{X}_n, \eta)$$

whose validity for all  $\eta \in \mathfrak{Y}$  is  $(A_{n+1})$ . Using that for all  $(x, \eta) \in W$ ,

$$\begin{aligned}
 \mathbb{P}[X_{n+1} = x | \mathfrak{X}_{n+1} = \eta, \mathfrak{X}_{[0,n]}] &= \frac{\mathbb{P}[X_{n+1} = x, \mathfrak{X}_{n+1} = \eta | \mathfrak{X}_{[0,n]}]}{\mathbb{P}[\mathfrak{X}_{n+1} = \eta | \mathfrak{X}_{[0,n]}]} \\
 &= \frac{\pi(x)}{\pi(\eta)} \\
 &= \Lambda(\eta, x)
 \end{aligned}$$

we conclude to  $(B_{n+1})$ .

## 2.2 Auxiliary measure-valued processes

In the definition of random mappings locally associated to  $P^*$  given in the introduction, we assumed that we had at our disposal a family  $(\psi_\eta)_{\eta \in \mathfrak{Y}}$ . Suppose now that we rather have a family  $(\psi_{\eta_{[0,n]}})_{\eta_{[0,n]} \in \mathfrak{Y}}$ , where

$$\bar{\mathfrak{Y}} := \sqcup_{n \in \mathbb{N}} \bar{\mathfrak{Y}}^n$$

and whose elements are written under the form  $\eta_{[0,n]} := (\eta_0, \eta_1, \dots, \eta_n)$ , for some  $n \in \mathbb{Z}_+$  and  $\eta_0, \eta_1, \dots, \eta_n \in \bar{\mathfrak{Y}}$ . The local association with  $P^*$  of the family  $(\psi_{\eta_{[0,n]}})_{\eta_{[0,n]} \in \bar{\mathfrak{Y}}}$  now means that

$$\forall \eta_{[0,n]} \in \bar{\mathfrak{Y}}, \forall x \in V, \forall x' \in \text{supp}(\eta_n), \quad \mathbb{P}[\psi_{\eta_{[0,n]}}(x) = x'] = \frac{P^*(x, x')}{\zeta(\eta_{[0,n]})}$$

where  $\zeta : \bar{\mathfrak{Y}} \rightarrow (0, +\infty)$  is a positive mapping on  $\bar{\mathfrak{Y}}$  (note that necessarily  $\zeta(\eta_{[0,n]}) = 1$  as soon as  $\text{supp}(\eta_n) = V$ ). Following (1.12), we get a random measure depending on  $\eta_{[0,n]} \in \bar{\mathfrak{Y}}$  and given by:

$$\forall \eta_{[0,n]} \in \bar{\mathfrak{Y}}, \quad \Psi(\eta_{[0,n]}) := (f_n \circ \psi_{\eta_{[0,n]}})\pi$$

where  $f_n$  is the density of  $\eta_n$  with respect to  $\pi$ .

The analogue of (1.16) is a transition kernel from  $\bar{\mathfrak{Y}}$  to  $\bar{\mathfrak{Y}}$

$$\forall \eta_{[0,n]} \in \bar{\mathfrak{Y}}, \forall \eta' \in \bar{\mathfrak{Y}}, \quad K(\eta_{[0,n]}, \eta') := \mathbb{P}[\Psi(\eta_{[0,n]}) = \eta']$$

and (1.18) must be transformed into a kernel from  $\mathfrak{W}$  to  $\mathfrak{V}$ :

$$\forall \eta_{\llbracket 0, n \rrbracket} \in \mathfrak{W}, \forall \eta' \in \mathfrak{V}, \quad \mathfrak{P}(\eta_{\llbracket 0, n \rrbracket}, \eta') := \frac{\pi(\eta') \zeta(\eta_{\llbracket 0, n \rrbracket})}{\pi(\eta_n)} K(\eta_{\llbracket 0, n \rrbracket}, \eta') \quad (2.4)$$

where  $\mathfrak{W}$  is the space

$$\mathfrak{W} := \sqcup_{n \in \mathbb{N}} \mathfrak{V}^n$$

By working as in the introduction, define conditioned transition kernels as in (1.19) via

$$\forall \eta_{\llbracket 0, n \rrbracket} \in \mathfrak{W}, \forall \eta' \in \mathfrak{V}, \quad K_{x, x'}(\eta_{\llbracket 0, n \rrbracket}, \eta') := \mathbb{P}[\Psi(\eta_{\llbracket 0, n \rrbracket}) = \eta' | \psi_{\eta_{\llbracket 0, n \rrbracket}}(x') = x]$$

for any  $x, x' \in V$  with  $x \in \text{supp}(\eta_n)$  and  $P(x, x') > 0$ . As a consequence of the dependence on  $\eta_{\llbracket 0, n \rrbracket} \in \mathfrak{W}$  in (2.4), we cannot expect in Theorem 1.5 that  $\mathfrak{X}$  will still be Markovian.

Nevertheless, the other parts of Theorem 1.5 are satisfied. Indeed, it is sufficient to replace everywhere in the previous subsection  $\psi_{\mathfrak{X}_n}$  by  $\psi_{\mathfrak{X}_{\llbracket 0, n \rrbracket}}$ ,  $\Psi(\mathfrak{X}_n)$  by  $\Psi(\mathfrak{X}_{\llbracket 0, n \rrbracket})$  and  $\mathfrak{P}(\mathfrak{X}_n, \eta)$  by  $\mathfrak{P}(\mathfrak{X}_{\llbracket 0, n \rrbracket}, \eta)$ .

The algorithm explained after the proof of Corollary 1.6 is straightforwardly adapted to this extended situation. The finite sequence  $\mathfrak{X}_{\llbracket 0, n \rrbracket}$  constructed in this way from a given trajectory  $x_{\llbracket 0, n \rrbracket}$  is called an **auxiliary measure-valued process**. It can be used to construct strong stationary times for  $X$  (see Fill and Diaconis [10]): consider

$$\tau := \inf\{n \in \mathbb{Z}_+ : \mathfrak{X}_n = \pi\} \in \mathbb{Z}_+ \sqcup \{+\infty\} \quad (2.5)$$

When  $\tau$  is (a.s.) finite, we have that  $\tau$  and  $X_\tau$  are independent and the distribution of  $X_\tau$  is the invariant measure  $\pi$ . In the subset case, (2.5) has to be replaced by

$$\tau := \inf\{n \in \mathbb{Z}_+ : \mathfrak{X}_n = V\} \in \mathbb{Z}_+ \sqcup \{+\infty\} \quad (2.6)$$

and this is under this form that that strong stationary times are often met in the literature (see the top-to-random shuffle of Aldous and Diaconis [1], recalled in Subsection 3.2 below).

An auxiliary measure-valued process can also be seen as a Markovian non-measure-valued dual process. Indeed, use the traditional trick transforming any process into a Markov process by adding all its history in its present state. More precisely, consider the  $\mathfrak{W}$ -valued process  $\mathfrak{Y} := (\mathfrak{X}_{\llbracket 0, n \rrbracket})_{n \in \mathbb{Z}_+}$ . The process  $\mathfrak{Y}$  is clearly Markovian, its transition kernel  $\Omega$  being given, for any  $\eta_{\llbracket 0, n \rrbracket}, \eta'_{\llbracket 0, n' \rrbracket} \in \mathfrak{W}$ , by

$$\Omega(\eta_{\llbracket 0, n \rrbracket}, \eta'_{\llbracket 0, n' \rrbracket}) := \begin{cases} \mathfrak{P}(\eta_{\llbracket 0, n \rrbracket}, \eta'_{n+1}), & \text{if } n' = n + 1 \text{ and } \eta_{\llbracket 0, n \rrbracket} = \eta'_{\llbracket 0, n \rrbracket} \\ 0, & \text{otherwise} \end{cases}$$

Extend  $\Lambda$  into a Markov kernel from  $\mathfrak{W}$  to  $V$  via

$$\forall \eta_{\llbracket 0, n \rrbracket} \in \mathfrak{W}, \forall x \in V, \quad \Lambda(\eta_{\llbracket 0, n \rrbracket}, x) = \frac{\pi(x)}{\pi(\eta_n)}$$

As in Corollary 1.6, we deduce the intertwining relation

$$\Omega \Lambda = \Lambda P$$

showing that  $\mathfrak{Y}$  is a dual process to  $X$ . Thus from the general theory of Fill and Diaconis [10], we know that a Markov chain with transition kernel  $\Omega$  can be used to construct a strong stationary time, as soon as a.s. it ends up reaching the set  $\{Y \in \mathfrak{W} : \Lambda(Y, \cdot) = \pi\}$ , which corresponds in the above situation to the fact that  $\tau$  defined in (2.6) is a.s. finite.

**Remark 2.1.**

- (a) Measure-valued processes (not necessarily Markovian) are essentially the more general intertwining dual processes that can be associated to  $X$ , up to a deterministic factorization. Indeed, let  $\bar{X} := (\bar{X}_n)_{n \in \mathbb{Z}_+}$  be an intertwining dual process associated to  $X$  through a Markov kernel  $\bar{\Lambda}$  going from the state space of  $\bar{X}$  to the state space of  $X$ , namely such that

$$\forall n \in \mathbb{Z}_+, \quad \begin{cases} \mathcal{L}(\bar{X}_{[0,n]}|X) &= \mathcal{L}(\bar{X}_{[0,n]}|X_{[0,n]}) \\ \mathcal{L}(X_n|\bar{X}_{[0,n]}) &= \bar{\Lambda}(\bar{X}_n, \cdot) \end{cases} \quad (2.7)$$

Consider the (probability) measure-valued process  $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{Z}_+}$  defined as a deterministic function of  $\bar{X}$  via

$$\forall n \in \mathbb{Z}_+, \quad \mathfrak{X}_n := \bar{\Lambda}(\bar{X}_n, \cdot)$$

We deduce from (2.7) that (1.9) and (1.10) are satisfied, with  $\Lambda$  given by (1.1). In general  $\mathfrak{X}$  will not be Markovian, even when  $\bar{X}$  is Markovian.

- (b) Consider  $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{Z}_+}$  a measure-valued process, not necessarily Markovian. We say it is **algebraically intertwined** with  $X$  if the following is true: for any  $n \in \mathbb{Z}_+$ , given  $\mathfrak{X}_0, \mathfrak{X}_1, \dots, \mathfrak{X}_n$ , on one hand sample  $\tilde{\mathfrak{X}}_n$  according to  $\mathfrak{X}_n/\mathfrak{X}_n(V)$ , then sample  $\tilde{\mathfrak{X}}_{n+1}$  according to  $P(\tilde{\mathfrak{X}}_n, \cdot)$ . On the other hand, sample  $\mathfrak{X}_{n+1}$  according to its conditional law knowing  $\mathfrak{X}_0, \mathfrak{X}_1, \dots, \mathfrak{X}_n$  and next sample  $\hat{\mathfrak{X}}_{n+1}$  according to  $\mathfrak{X}_{n+1}/\mathfrak{X}_{n+1}(V)$ . Then  $\tilde{\mathfrak{X}}_{n+1}$  and  $\hat{\mathfrak{X}}_{n+1}$  should have the same law, still conditioned on  $\mathfrak{X}_0, \mathfrak{X}_1, \dots, \mathfrak{X}_n$ . This condition writes down

$$\forall n \in \mathbb{Z}_+, \quad \mathbb{E}\left[\frac{\mathfrak{X}_{n+1}}{\mathfrak{X}_{n+1}(V)} \mid \mathfrak{X}_n, \mathfrak{X}_{n-1}, \dots, \mathfrak{X}_0\right] = \frac{\mathfrak{X}_n}{\mathfrak{X}_n(V)} P \quad (2.8)$$

(of course the equality  $\mathbb{E}[\mathfrak{X}_{n+1}/\mathfrak{X}_{n+1}(V) \mid \mathfrak{X}_n, \mathfrak{X}_{n-1}, \dots, \mathfrak{X}_0] = \mathbb{E}[\mathfrak{X}_{n+1}/\mathfrak{X}_{n+1}(V) \mid \mathfrak{X}_n]$  does not imply that  $\mathfrak{X}$  is Markovian).

Given a family  $(\psi_{\eta_{[0,n]}})_{\eta_{[0,n]} \in \mathfrak{W}}$  as in this Subsection, such a measure-valued process  $\mathfrak{X}$  can be constructed by using the transition kernel defined in (2.4). But it should be observed that not all measure-valued processes algebraically intertwined with  $X$  can be constructed in this way. Indeed, since the set  $V^V$  of mappings from  $V$  to  $V$  is finite, for any given  $n \in \mathbb{Z}_+$  and  $\eta_{[0,n]} \in \mathfrak{W}$ , the distribution  $\mathfrak{P}(\eta_{[0,n]}, \cdot)$  is necessarily a finite sum of Dirac masses, so the same is true for  $\mathcal{L}(\mathfrak{X}_{n+1} \mid \mathfrak{X}_n, \mathfrak{X}_{n-1}, \dots, \mathfrak{X}_0)$  when  $\mathfrak{X}$  is constructed using a family  $(\psi_{\eta_{[0,n']}})_{\eta_{[0,n']} \in \mathfrak{W}}$ . But in general, one can find measure-valued processes satisfying (2.8) such that  $\mathcal{L}(\mathfrak{X}_{n+1} \mid \mathfrak{X}_n, \mathfrak{X}_{n-1}, \dots, \mathfrak{X}_0)$  is a diffuse distribution. Maybe the most general case can be obtained by replacing random mappings by random transition kernels locally associated to  $P^*$ .

- (c) The previous argument cannot be applied to set-valued dual processes, since the set  $\{0, 1\}^V$  of subsets from  $V$  is finite, contrary to the set of measures on  $V$ .

**2.3 The Levin, Peres and Wilmer construction**

Let us come back to the construction of Chapter 17 of Levin, Peres and Wilmer [18] and interpret it in the random mapping setting, in the subset case.

In practice, a random mapping  $\psi$  is often given in the following way: let  $I := (I_{x,y})_{x,y \in V}$  be a family of measurable subsets of  $[0, 1)$  such that

$$\forall x \in V, \quad [0, 1) = \bigsqcup_{y \in V} I_{x,y} \quad (2.9)$$

$$\forall x, y \in V, \quad \lambda(I_{x,y}) = P^*(x, y) \quad (2.10)$$

where  $\lambda$  is the Lebesgue measure. Let  $U$  be a random variable uniformly distributed on  $[0, 1)$ . A random mapping  $\psi$  associated to  $P^*$  can be defined in terms of  $I$  and  $U$  by deciding that

$$\forall x, y \in V, \quad \{\psi(x) = y\} = \{U \in I_{x,y}\}$$

**Remark 2.2.** Conversely, any random mapping associated to  $P^*$  has the same law as a random mapping constructed as above. This is a consequence of the fact that the set of functions from  $V$  to  $V$  is finite and that any probability distribution on a finite number of points can be seen as an image of the restriction of the Lebesgue measure on  $[0, 1)$ . Thus there is no loss of generality in only considering random mappings of the previous form.

Let  $S \in \mathfrak{S}$  be fixed and label the elements of  $V$  as  $1, 2, \dots, |V|$ , where  $|V|$  is the cardinal of  $V$ , in order to insure that  $S = \llbracket 1, |S| \rrbracket$ . Next define a family  $I := (I_{x,y})_{x,y \in V}$  via

$$\forall x, y \in V = \llbracket 1, |V| \rrbracket, \quad I_{x,y} := [P^*(x, \llbracket 1, y-1 \rrbracket), P^*(x, \llbracket 1, y \rrbracket))$$

it is immediate to check both (2.9) and (2.10). Let  $\phi_S$  be the globally associated to  $P^*$  random mapping constructed in this way. With the notations of the introduction, we have

$$\begin{aligned} \Psi(S) &= \{x \in V : \phi_S(x) \in S\} \\ &= \{x \in V : U \in [0, P^*(x, S))\} \\ &= \{x \in V : U \leq P^*(x, S)\} \\ &= \Phi(S) \end{aligned}$$

It follows that the evolving sets of Morris and Peres [22] are a particular case of the construction via the random mappings. The special random mapping  $\phi_S$  has the tendency to put a maximal number of points inside  $\Psi(S)$ , when  $U$  is small, and a minimal number of points inside  $\Psi(S)$ , when  $U$  is close to 1. So it seems that among all random mappings,  $\phi_S$  induces the maximal possible oscillation for  $\pi(\Psi(S))$  (e.g. measured through its variance). By analogy with the result stating that the best way to couple two Brownian motions is the mirror symmetry coupling (see for instance Jacka, Mijatović, and Siraj [13]), a tempting conjecture is that the evolving set construction is the best possible choice for  $\mathfrak{X}$  to grow as fast as possible (property which is important in the construction of strong stationary times). We believe it is true when the underlying geometry is simple (as for birth and death processes, see Section 4, this phenomenon was encountered for one-dimensional diffusions in [21]), but maybe not in the general setting.

**Remark 2.3.** The task of finding a “good” random mapping should be illuminated by characterizations of measure-valued dual processes which are sharp in the sense of Diaconis and Fill [10]. In a diffusion context, here is a conjecture on how to recognize sharp set-valued duals (inspired by results from [21] and [9]): the volume (with respect to the invariant measure) of the dual process conveniently time-changed by the square of the volume of its boundary should be a Bessel-3 process, namely in some sense, the Pitman intertwining relation is a prototype for sharpness. It would be instructive to state and to prove a similar result in a discrete context.

## 2.4 Independent random iterative mappings

A particularly simple instance of globally associated to  $P^*$  random mappings is when they are not allowed to depend on a measure  $\eta \in \mathfrak{M}$ , corresponding to the current state of  $\mathfrak{X}$ . The algorithm of the introduction then takes the following form. Let be given  $n \in \mathbb{Z}_+$  and a trajectory  $x_{[0,n]}$  of  $X$ . In this subsection, we assume for simplicity that  $\mathfrak{X}_0 = \{x_0\}$ , in particular we are in the subset case, as seen in Remark 1.3. Consider  $n$

independent mappings  $\psi_1, \psi_2, \dots, \psi_n$  globally associated to  $P^*$ , which may not have the same law. For any  $m \in \llbracket 1, n \rrbracket$ , condition  $\psi_m$  by  $\psi_m(x_m) = x_{m-1}$ , and to avoid confusion, let us call the new random mapping  $\varphi_m$  (it is no longer associated to  $P^*$ , except when  $P^*(x_m, x_{m-1}) = 1$ ). The construction of  $\mathfrak{X}_n$  is now:

$$\mathfrak{X}_n = \{y \in V : \varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_n(y) = x_0\} \tag{2.11}$$

In particular, the strong stationary time defined by (2.6) is given by the collapsing time

$$\tau = \inf\{n \in \mathbb{Z}_+ : \forall y \in V, \varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_n(y) = x_0\} \tag{2.12}$$

Formulas (2.11) and (2.12) are valid more generally, up to the forward simultaneous definition of the  $(\varphi_1, \mathfrak{X}_1), (\varphi_2, \mathfrak{X}_2), \dots, (\varphi_n, \mathfrak{X}_n)$  presented in the introduction. But when  $\psi_1, \psi_2, \dots, \psi_n$  are independent, a backward construction can also be envisaged. This possibility will also be satisfied by the non-independent random mappings associated to restless birth and death chains in Section 4. Let us describe an algorithm constructing  $\mathfrak{X}_n$ , for any fixed  $n \in \mathbb{Z}_+$ , in this spirit. Label  $V$  as  $\{y_0, y_1, \dots, y_{|V|-1}\}$ , with  $y_0 = x_n$  and define  $\mathfrak{X}_n^{(0)} := \{x_n\}$ . We look iteratively for the first  $m \in \llbracket 1, n \rrbracket$  such that  $\varphi_{n-m+1} \circ \varphi_{n-m+2} \circ \dots \circ \varphi_n(y_1) = x_{n-m}$ . If there is no such  $m$ , we know that  $y_1 \notin \mathfrak{X}_n$  and we let  $\mathfrak{X}_n^{(1)} := \{x_n\}$ . Otherwise we define  $\mathfrak{X}_n^{(1)} := \{x_n, y_1\}$ . Assume that  $\mathfrak{X}_n^{(l)}$  has been constructed for some  $l \in \llbracket 1, |V| - 2 \rrbracket$ . We look iteratively for the first  $m \in \llbracket 1, n \rrbracket$  such that  $\varphi_{n-m+1} \circ \varphi_{n-m+2} \circ \dots \circ \varphi_n(y_{l+1}) \in \mathfrak{X}_n^{(l)}$ . If there is no such  $m$ , we infer that  $y_{l+1} \notin \mathfrak{X}_n$  and let  $\mathfrak{X}_n^{(l+1)} = \mathfrak{X}_n^{(l)}$ . Otherwise we define  $\mathfrak{X}_n^{(l+1)} = \mathfrak{X}_n^{(l)} \sqcup \{y_{l+1}\}$ . At the end, we consider  $\mathfrak{X}_n := \mathfrak{X}_n^{(|V|-1)}$ . This procedure can also be used to test if  $\tau \leq n$ , where  $\tau$  is defined in (2.12): this is equivalent to the fact that for all the above steps for  $l \in \llbracket 0, |V| - 2 \rrbracket$ , there is a  $m \in \llbracket 1, n \rrbracket$  such that  $\varphi_{n-m+1} \circ \varphi_{n-m+2} \circ \dots \circ \varphi_n(y_{l+1}) \in \mathfrak{X}_n^{(l)}$ .

This test can be strongly simplified when  $V$  is endowed with a partial order admitting a unique minimal element  $y_\wedge$  and a unique maximal element  $y_\vee$  and when the independent random mappings  $\psi_1, \psi_2, \dots, \psi_n$  preserve the partial order (of course this is only possible if  $P^*$  is equally preserving the partial order). Then the random mappings  $\varphi_1, \varphi_2, \dots, \varphi_n$  equally preserve the partial order and the validity of  $\tau \leq n$  is equivalent to the existence of  $m \in \llbracket 1, n \rrbracket$  such that

$$\varphi_{n-m+1} \circ \varphi_{n-m+2} \circ \dots \circ \varphi_n(y_\wedge) = \varphi_{n-m+1} \circ \varphi_{n-m+2} \circ \dots \circ \varphi_n(y_\vee)$$

These observations are strongly reminiscent of the coupling-from-the-past algorithm of Propp and Wilson [27] (see also their review in Chapter 22 of Levin, Peres and Wilmer [18]). Recall they consider a family  $(\psi_n)_{n \in -\mathbb{N}}$  of independent identically distributed random mappings associated to  $P$ . For any  $N \in \mathbb{N}$ , they test if  $\psi_{-1} \circ \dots \circ \psi_{-N+1} \circ \psi_{-N}$  sends the whole state space  $V$  into a single point. When this is true, the single point is distributed according to  $\pi$ . Otherwise they consider another integer number  $N' > N$  (usually  $N' = 2N$ ) and start again the above procedure. Their algorithm is equally greatly simplified under the assumptions that  $V$  is endowed with a partial order admitting a unique minimal element  $y_\wedge$  and a unique maximal element  $y_\vee$  and that the independent random mappings  $\psi_{-1}, \psi_{-2}, \dots, \psi_{-N}$  preserve the partial order (this is only possible if  $P$  is equally preserving the partial order).

Of course there are big differences between the two procedures: our initial point is fixed, the final point of coupling-from-the-past is distributed accordingly to the invariant measure, the preservation of the partial order by  $P^*$  and  $P$  are not equivalent, we fix a trajectory and allow (except in this subsection) dependence between the random mappings through the already constructed set-valued dual etc. Nevertheless, it would be interesting to investigate further the links between the two algorithms, e.g.:

- Could the convergence of the coupling-from-the-past algorithm be improved by allowing, for  $N \in \mathbb{N}$  fixed, choices of  $\psi_{-n}$  depending on  $\psi_{-n-1} \circ \dots \circ \psi_{-N+1} \circ \psi_{-N}(V)$ , for  $n \in \llbracket 1, N \rrbracket$ ?
- What happens to the examples treated by the coupling-from-the-past algorithm if we look at them from the intertwining/strong stationary times point of view?

### 3 Classical examples

Up to now, the considerations were both abstract and simple. It is time to give some examples showing how Theorem 1.5 works in practice. As it can be guessed, all the difficulty is to find nice random mappings. The families of random mappings considered here will be globally associated to  $P^*$ , as mentioned in Remark 1.2. We will work in the subset case, since the dual processes are to start from Dirac masses, identified with singletons (recall Remark 1.3).

#### 3.1 The discrete Pitman theorem

For this example due to Pitman [25], the state space is the denumerable set  $\mathbb{Z}$ , but we are in the situation described in the beginning of Remark 1.8. The kernel  $P$  is the transition “matrix” of the simple random walk on  $\mathbb{Z}$ , namely,

$$\forall x, y \in \mathbb{Z}, \quad P(x, y) := \begin{cases} 1/2, & \text{if } |y - x| = 1 \\ 0, & \text{otherwise} \end{cases}$$

which has finite degree 2 at every point. The counting measure  $\pi$  is invariant for  $P$  and is even reversible for  $P$ . It follows that  $P^* = P$ , where  $P^*$  is defined as in (1.2). Following Remark 1.8, we take  $\mathfrak{S}$  to be the set of finite non-empty subsets of  $\mathbb{Z}$ , so that the kernel  $\Lambda$  given in (1.6) is still well-defined.

Let  $X := (X_n)_{n \in \mathbb{Z}_+}$  be a random walk with transition kernel  $P$  and starting from 0. Introduce the process  $X^\vee := (X_n^\vee)_{n \in \mathbb{Z}_+}$  defined by

$$\forall n \in \mathbb{Z}_+, \quad X_n^\vee := 2M_n - X_n$$

where  $M_n := \max\{X_m : m \in \llbracket 0, n \rrbracket\}$ . Finally consider  $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{Z}_+}$  given by

$$\forall n \in \mathbb{Z}_+, \quad \mathfrak{X}_n := \{X_n^\vee - 2m : m \in \llbracket 0, X_n^\vee \rrbracket\} \tag{3.1}$$

Pitman [25] has shown that (1.9) and (1.10) hold with the above processes  $X$  and  $\mathfrak{X}$ , it is in fact the first historical instance of such a coupling.

Let us prove that this result is a consequence of Theorem 1.5. Since (1.9) is obvious, we concentrate our attention on (1.10).

Consider the function  $\psi$  given by

$$\forall S \in \mathfrak{S}, \forall x \in \mathbb{Z}, \forall b \in \{-1, 1\}, \quad \psi(S, x, b) := \begin{cases} x + b, & \text{if } x > \max(S) \\ x - b, & \text{if } x \leq \max(S) \end{cases} \tag{3.2}$$

Consider a Rademacher variable  $B$ , i.e. such that  $\mathbb{P}[B = -1] = \mathbb{P}[B = 1] = 1/2$  and for fixed  $S \in \mathfrak{S}$ , let  $\psi_S$  be the random mapping given by

$$\forall x \in \mathbb{Z}, \quad \psi_S(x) := \psi(S, x, B) \tag{3.3}$$

It is clear that  $\psi_S$  is a random mapping associated to  $P^* = P$ . So let be given a trajectory  $x_{\llbracket 0, n \rrbracket}$  of  $X$  stopped at time  $n \in \mathbb{Z}_+$  and starting with  $x_0 = 0$ . Construct the finite sequence  $\mathfrak{X}_{\llbracket 0, n \rrbracket}$  as in the introduction, starting with  $\mathfrak{X}_0 := \{0\}$ . Denote by  $(\varphi_m)_{m \in \llbracket 1, n \rrbracket}$  the corresponding random mappings used in this construction, conditioned

by the compatibility relations  $\varphi_m(x_m) = x_{m-1}$  for  $m \in \llbracket 1, n \rrbracket$ . Given the stopped trajectory  $x_{\llbracket 0, n \rrbracket}$ , these random mappings are here deterministic:

$$\forall m \in \llbracket 1, n \rrbracket, \quad \varphi_m = \psi(\mathfrak{X}_{m-1}, \cdot, b_m)$$

with

$$b_m := \begin{cases} -1, & \text{if } x_{m-1} = \max(\mathfrak{X}_{m-1}) \text{ and } x_m - x_{m-1} = 1 \\ x_m - x_{m-1}, & \text{otherwise} \end{cases}$$

Since under these mappings the parity of the positions are alternating, we remark that

$$\forall m \in \llbracket 0, n \rrbracket, \quad \mathfrak{X}_m \subset \begin{cases} 2\mathbb{Z}, & \text{if } m \text{ is even} \\ 2\mathbb{Z} + 1, & \text{if } m \text{ is odd} \end{cases}$$

Consider for any  $m \in \llbracket 0, n \rrbracket$ ,  $Y_m := \min(\mathfrak{X}_m)$  and  $Z_m := \max(\mathfrak{X}_m)$ . By a forward iteration on  $m \in \llbracket 0, n \rrbracket$ , we show that  $Z_m = X_n^\vee$ ,  $Y_m = -Z_m$  and that  $\mathfrak{X}_m$  contains all the elements in  $\llbracket Y_m, Z_m \rrbracket$  with the same parity as  $Z_m$ . It proves the validity of (3.1). This is well-illustrated by Figure 1, where for  $m \in \llbracket 0, n - 1 \rrbracket$ , the elements of  $\mathfrak{X}_m$  are represented by full disks, the elements of  $\mathfrak{X}_{m+1}$  by circles, the transition from  $X_m$  to  $X_{m+1}$  by a double line, the dashed lines are the actions of the random mappings (from the right to the left), the green (respectively red) line is the transition from  $Z_m$  to  $Z_{m+1}$  (resp. from  $Y_m$  to  $Y_{m+1}$ ).

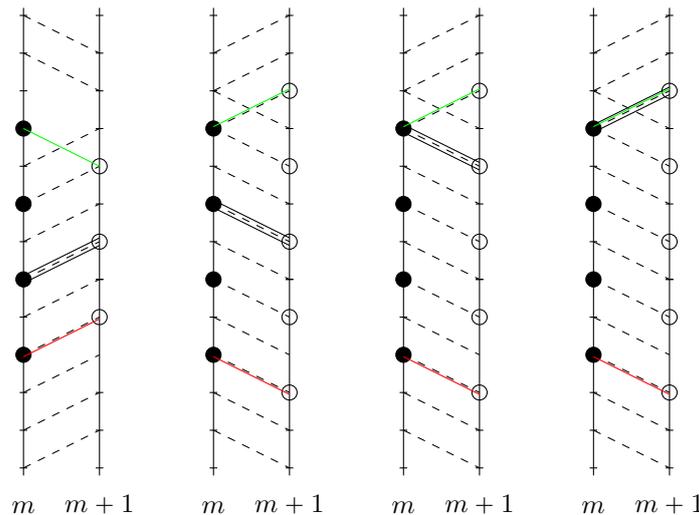


Figure 1: Schematic proof of the discrete Pitman theorem via random mappings

The symmetry with respect 0 leads to another Pitman transformation: rather introduce the process  $X^\wedge := (X_n^\wedge)_{n \in \mathbb{Z}_+}$  defined by

$$\forall n \in \mathbb{Z}_+, \quad X_n^\wedge := X_n - 2M_n^\wedge$$

where  $M_n^\wedge := \min\{X_m : m \in \llbracket 0, n \rrbracket\}$  and consider  $\tilde{\mathfrak{X}} := (\tilde{\mathfrak{X}}_n)_{n \in \mathbb{Z}_+}$  given by

$$\forall n \in \mathbb{Z}_+, \quad \tilde{\mathfrak{X}}_n := \{X_n^\wedge - 2m : m \in \llbracket 0, X_n^\wedge \rrbracket\} \tag{3.4}$$

By symmetry in law of  $X$ , it is clear that (1.9) and (1.10) equally hold for processes  $X$  and  $\tilde{\mathfrak{X}}$ . This can also be obtained by replacing the mapping  $\psi$  of (3.2) by

$$\forall S \in \mathfrak{S}, \forall x \in \mathbb{Z}, \forall b \in \{-1, 1\}, \quad \tilde{\psi}(S, x, b) := \begin{cases} x + b, & \text{if } x \geq \min(S) \\ x - b, & \text{if } x < \min(S) \end{cases} \tag{3.5}$$

More generally, at each time  $n \in \mathbb{Z}_+$ , either  $\psi$  or  $\tilde{\psi}$  can be chosen to construct random mappings and this choice itself may depend on the current state  $\mathfrak{X}_n$  and on independent noise. Of course the description of the deduced set-valued dual  $\mathfrak{X}$  will then be more tricky than (3.1) or (3.4).

One may wonder how to guess that the random mappings described by (3.3) or (3.5) are interesting. It is not mere inverse engineering: the underlying idea is that they strictly satisfy the principle put forward in Subsection 2.3: these random mappings (as well as their above variants) are such that the two points just outside the current discrete segment  $S$  (forgetting the points with the “bad” parity) either both enter  $S$  or both go away from  $S$ . In the spirit and the notations of Subsection 2.3, the above random mappings can be described via labelings of  $\mathbb{Z}$  depending on  $S$ . For instance for (3.3),  $\max(S) + 1$  is named 1,  $\max(S)$  is named 2,  $\max(S) - 1$  is named 3, ..., until  $\min(S) - 1$  is named  $\max(S) - \min(S) + 3$ , then  $\max(S) + 2$  is named  $\max(S) - \min(S) + 4$ ,  $\min(S) - 2$  is named  $\max(S) - \min(S) + 5$ ,  $\max(S) + 3$  is named  $\max(S) - \min(S) + 6$ , etc. ((3.5) corresponds to a “mirror” labeling). One can imagine other labelings (where the first labels are given to the elements of  $[\min(S) - 1, \max(S) + 1]$ , or only to those of  $S$  as in Subsection 2.3, according to any fancy rule), it will not change the law of  $\mathfrak{X}$  (starting from  $\{0\}$ ), only the law of the coupling  $(X, \mathfrak{X})$  will be modified, as in the above cases deduced from (3.2) and (3.5).

Let us present two other examples of dual processes for the usual random walk based on other random couplings following strictly the principle of Subsection 2.3. They are discrete analogues of intertwining couplings of subset-valued processes for diffusions on manifolds, on which we are currently working with Marc Arnaudon and Koléhè Coulibaly-Pasquier [3]. There we use a different approach relying on directly coupling a stochastic variant of mean curvature flow with the primal diffusion process. At least in the discrete context and for the two examples below, this can be easily translated in terms of random mappings.

**Example 3.1.** Looking for dual processes that will stay symmetric with respect to 0, we can consider random mappings not depending on a finite subset  $S$  of  $\mathbb{Z}$  (as mentioned in Subsection 2.4), by breaking their direction exactly at 0 (and not at  $\max(S)$  or  $\min(S)$  as above). More precisely, in analogy with (3.2), define

$$\forall x \in \mathbb{Z}, \forall b \in \{-1, 1\}, \quad \psi(x, b) := \begin{cases} x + b, & \text{if } x > 0 \\ x - b, & \text{if } x \leq 0 \end{cases} \quad (3.6)$$

and the random mappings associated to  $P^* = P$  via

$$\forall x \in \mathbb{Z}, \quad \psi_S(x) := \psi(S, x, B)$$

where  $B$  is a Rademacher variable  $B$ .

Denote  $(L_n)_{n \in \mathbb{Z}_+}$  the local time associated to  $X$  at the transition from 1 to 0:

$$\forall n \in \mathbb{Z}_+, \quad L_n := \sum_{l=1}^n \mathbb{1}_{(X_{l-1}, X_l) = (1, 0)}$$

We let as an exercise of manipulations of graphics similar to Figure 1, to check that the dual process constructed by Theorem 1.5 in this situation is given by

$$\forall n \in \mathbb{Z}_+, \quad \mathfrak{X}_n = \{|X_n| + L_n - 2m : m \in \llbracket 0, |X_n| + L_n \rrbracket\}$$

A continuous equivalent of this dual process will appear in Subsection 7.1.

In (3.6), instead of 0, the break of direction could be chosen at any other point  $k \in \mathbb{Z}$ . The resulting dual process is given, for any  $n \in \mathbb{Z}_+$ , by

$$\mathfrak{X}_n = \begin{cases} \{X_n\}, & \text{if } n < T_k \\ \{k + |X_n - k| + L_n^{(k)} - 2m : m \in \llbracket 0, k + |X_n - k| + L_n^{(k)} \rrbracket\}, & \text{otherwise} \end{cases}$$

where

$$T_k := \inf\{n \in \mathbb{Z}_+ : X_n = k\}$$

$$\forall n \in \mathbb{Z}_+, \quad L_n^{(k)} := \sum_{l=1}^n \mathbf{1}_{(X_{l-1}, X_l) = (k+1, k)}$$

The volume of these subset-valued dual processes for  $k \neq 0$  has an initial slower growth than in the case  $k = 0$ , in the sense of the stochastic domination for the stopping times defined, for any  $A \in \mathbb{N}$ , by

$$\tau_A := \inf\{n \in \mathbb{Z}_+ : \pi(\mathfrak{X}_n) \geq A\}$$

It can be seen as a consequence of the fact that the principle of Subsection 2.3 is not satisfied, until  $X$  reaches  $k$ . Note that the worst case corresponds to letting  $k$  goes to  $+\infty$  or  $-\infty$ : with probability 1/2 one of the two mappings

$$\forall x \in \mathbb{Z}, \quad \psi_-(x) := x - 1$$

$$\forall x \in \mathbb{Z}, \quad \psi_+(x) := x + 1$$

is chosen. Starting with  $\mathfrak{X}_0 = \{X_0\}$ , we get that for all  $n \in \mathbb{Z}_+$ ,  $\mathfrak{X}_n = \{X_n\}$  (more generally, for any initial law of  $\mathfrak{X}_0$  on  $\mathfrak{S}$  satisfying (1.8), we end up with  $\pi(\mathfrak{X}_n) = \pi(\mathfrak{X}_0)$  for all  $n \in \mathbb{Z}_+$ ).

**Example 3.2.** Random mappings are clearly stable by mixture. Thus we can consider the random mapping which, given  $S \in \mathfrak{S}$ , chooses with probability 1/2 the random mapping (3.3) and with the remaining probability 1/2, the random mapping deduced from (3.5). Write  $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{Z}_+}$  the corresponding set-valued dual process. It remains symmetric with respect to 0, so let us write for any  $n \in \mathbb{Z}_+$ ,  $\mathfrak{X}_N =: \{x \in \llbracket -R_n, R_n \rrbracket$  with the same parity as  $R_n\}$ . Using graphics similar to Figure 1 and taking into account the independent Bernoulli variables choosing between (3.3) and (3.5), it is not difficult to check that for any  $n \in \mathbb{N}$ ,  $R_n - R_{n-1}$  is independent from  $X_n - X_{n-1}$  and uniformly distributed on  $\{-1, 1\}$ , except in two cases:

- when  $X_{n-1} = R_{n-1}$  and  $X_n = X_{n-1} + 1$ , then  $R_n = R_{n-1} + 1$
- when  $X_{n-1} = -R_{n-1}$  and  $X_n = X_{n-1} - 1$ , then  $R_n = R_{n-1} + 1$

Namely,  $(R_n)_{n \in \mathbb{Z}_+}$  evolves as a random walk independent from  $X$ , except when  $X$  hits the boundary of  $\mathfrak{X}$  and tends to push it away from 0, in which case  $(R_n)_{n \in \mathbb{Z}_+}$  do the only possible move keeping  $X$  inside  $\mathfrak{X}$ : it also go away from 0 by adding 1 to its previous value.

### 3.2 The top-to-random shuffle

The top-to-random shuffle is a simple model of shuffling cards: at each time, take the top card and put it at a uniform random location in the deck. This stochastic evolution is described mathematically by a Markov chain  $X := (X_n)_{n \in \mathbb{Z}_+}$  on the symmetric group  $V := \mathcal{S}_N$ , with  $N \in \mathbb{N} \setminus \{1\}$ , whose transition matrix  $P$  is given, for any  $\sigma, \sigma' \in \mathcal{S}_N$ , by

$$P(\sigma, \sigma') := \begin{cases} 1/N, & \text{if there exists } l \in \llbracket 1, N \rrbracket \text{ with } \sigma' = (1 \rightarrow l \rightarrow l-1 \rightarrow \dots \rightarrow 2) \circ \sigma \\ 0, & \text{otherwise} \end{cases}$$

where  $(1 \rightarrow l \rightarrow l-1 \rightarrow \dots \rightarrow 2)$  is the cyclic permutation, seen as the function from  $\llbracket 1, N \rrbracket$  to  $\llbracket 1, N \rrbracket$ , transferring the card at position 1 to position  $l$ , the card at position  $l$  to position  $l-1$ , ... and the card at position 2 to position 1.

The transition matrix  $P$  is irreducible and the corresponding invariant measure  $\pi$  is the uniform probability distribution on  $\mathcal{S}_N$ . The Markov chain  $X$  admits a famous dual process defined by Aldous and Diaconis [1] in terms of the position of the last card of  $X_0$  in the deck  $X_n$  at time  $n \in \mathbb{Z}_+$ . More precisely, represent a permutation  $\sigma$  by the sequence of its values  $(\sigma(1), \sigma(2), \dots, \sigma(N))$ . Start  $X$  from the identity:  $X_0 = (1, 2, \dots, N) =: \text{id}$  and at any time  $n \in \mathbb{Z}_+$ , let  $Y_n \in \llbracket 1, N \rrbracket$  be the position of the last card defined by  $X_n(Y_n) = N$ . It is not difficult to check that  $Y := (Y_n)_{n \in \mathbb{Z}_+}$  is a Markov chain. Define  $\tilde{\tau} := \inf\{n \in \mathbb{Z}_+ : Y_n = 1\}$ , the first time the last card arrives at the top of the deck. It is well-known that  $\tilde{\tau} + 1$  is a strong stationary time, it is even the first historical instance of a strong stationary time in a finite context. We modify  $Y$  by imposing that  $Y_n = 0$  for any  $n > \tau$ .

For any  $\sigma \in \mathcal{S}_N$  and  $y \in \llbracket 0, N \rrbracket$ , define

$$A_{\sigma,y} := \{\sigma' \in \mathcal{S}_N : \sigma'(1) = \sigma(1), \dots, \sigma'(y) = \sigma(y)\}$$

with the usual convention that  $A_{\sigma,0} = \mathcal{S}_N$ . Aldous and Diaconis [1] considered the set-valued dual  $\tilde{\mathfrak{X}} := (\tilde{\mathfrak{X}}_n)_{n \in \mathbb{Z}_+}$  deduced from  $(X, Y)$  by defining

$$\forall n \in \mathbb{Z}_+, \quad \tilde{\mathfrak{X}}_n := A_{X_n, Y_n} \tag{3.7}$$

Let us construct a better set-valued dual  $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{Z}_+}$  via random mappings. Note that  $P^*$  is the transition matrix of the random-to-top shuffle and corresponds to taking a card of the deck at a uniform random location and putting it at the top. Consider for any  $x \in \llbracket 1, N \rrbracket$ , the mapping  $\psi^{(x)} : \mathcal{S}_N \rightarrow \mathcal{S}_N$  which acts on any permutation  $\sigma$  by removing the card  $x$  from the deck and putting it at the top. Formally, we have

$$\forall \sigma \in \mathcal{S}_N, \quad \psi^{(x)}(\sigma) = (1 \rightarrow 2 \rightarrow \dots \rightarrow \sigma^{-1}(x)) \circ \sigma \tag{3.8}$$

(note that  $\sigma^{-1}(x)$  is the position of the card  $x$ ). Let  $(U_n)_{n \in \mathbb{N}}$  be a family of independent random variables uniformly distributed on  $\llbracket 1, N \rrbracket$  and for any  $n \in \mathbb{N}$ , denote by  $\psi_n$  the random mapping  $\psi^{(U_n)}$ , which is clearly associated to  $P^*$ . There is no dependence on a subset  $S \in \mathfrak{S}$  and we are in the context of independent random mappings of Subsection 2.4. Let be given a trajectory  $x_{\llbracket 0, n \rrbracket}$ , for some fixed  $n \in \mathbb{Z}_+$ , starting from the identity,  $x_0 = \text{id}$ . For any  $m \in \llbracket 1, n \rrbracket$ , let  $\varphi_m$  be the conditioning of  $\psi_m$  by  $\psi_m(x_m) = x_{m-1}$ . Remark that as in the previous subsection,  $\varphi_m$  is deterministic, as we have  $\varphi_n = \psi^{(x_{n-1}(1))}$ . Starting from  $\mathfrak{X}_0 = \{\text{id}\}$ , we get from (2.11) that

$$\mathfrak{X}_n = \{\sigma \in \mathcal{S}_N : \varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_n(\sigma) = \text{id}\}$$

Let us check that  $\mathfrak{X}$  is better than the set-valued dual  $\tilde{\mathfrak{X}}$  of Aldous and Diaconis [1], in the sense that

$$\forall n \in \mathbb{Z}_+, \quad \tilde{\mathfrak{X}}_n \subset \mathfrak{X}_n \tag{3.9}$$

It is furthermore strictly better, because  $\tilde{\mathfrak{X}}_{\tilde{\tau}} = A_{X_{\tilde{\tau}}, 1}$  is strictly included into  $\mathfrak{X}_{\tilde{\tau}} = \mathcal{S}_N$ , as we will see below. It implies that  $\tau \leq \tilde{\tau} < \tilde{\tau} + 1$ , where  $\tau$  is the strong stationary time associated to  $\mathfrak{X}$  as in (2.12) (recall that  $\tilde{\tau} + 1$  is the strong stationary time associated to  $\tilde{\mathfrak{X}}$ ).

Indeed, to show (3.9), consider  $\sigma \in \tilde{\mathfrak{X}}_n$ . By definition, we have

$$\sigma = (\sigma(1), \sigma(2), \dots, \sigma(Y_n - 1), N, \sigma(Y_n + 1), \dots, \sigma(N))$$

Observe that for  $l \in \llbracket 1, Y_n - 1 \rrbracket$ ,  $\sigma(l) = X_n(l)$  and for  $l \in \llbracket Y_n + 1, N \rrbracket$ , the values  $\sigma(l)$  belongs to  $\{x_m(1) : m \in \llbracket 0, n \rrbracket\}$ , since they have had to be at the top of the deck before time  $n$

to be sent below the last card  $N$ . By iteration on  $m \in \llbracket 1, n \rrbracket$ , it follows that the  $Y_{n-m+1}$  first coordinates of  $\varphi_{n-m+1} \circ \varphi_{n-m+2} \circ \dots \circ \varphi_n(\sigma)$  and  $\varphi_{n-m+1} \circ \varphi_{n-m+2} \circ \dots \circ \varphi_n(X_n)$  coincide. In particular for  $m = 1$ , since  $\varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_n(X_n) = \text{id}$ , we get that  $\sigma \in \mathfrak{X}_n$ , so that (3.9) is proven.

To prove that  $\mathfrak{X}$  is strictly better than  $\tilde{\mathfrak{X}}$ , note that  $\tilde{\mathfrak{X}}_{\tilde{\tau}}$  is the set of permutations  $\sigma \in \mathcal{S}_N$  such that  $\sigma(1) = N$ , in particular  $\tilde{\mathfrak{X}}_{\tilde{\tau}} \neq \mathcal{S}_N$ . Applying a reasoning similar to the proof of (3.9), we get that for any  $\sigma \in \mathcal{S}_N$ ,  $\varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_{\tilde{\tau}}(\sigma) = \varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_{\tilde{\tau}}(X_{\tilde{\tau}})$ , except maybe for the last coordinate  $\sigma(N)$ . This is a consequence of the fact that all the values of  $\llbracket 1, N - 1 \rrbracket$  will have been placed at a same time  $m \in \llbracket 1, \tilde{\tau} \rrbracket$  at the top of the deck by  $\varphi_m \circ \varphi_{m+1} \circ \dots \circ \varphi_{\tilde{\tau}}$ . But if all the coordinates except the last one coincide for two permutations, it means that the permutations are the same. It follows that  $\tilde{\mathfrak{X}}_{\tilde{\tau}} = \mathcal{S}_N$  as announced.

**Corollary 3.3.** *The first time  $\hat{\tau}$  that the card  $N - 1$  comes to the top and is inserted is a strong stationary time. It is a strict improvement over the strong stationary time constructed by Aldous and Diaconis [1], which is the first time  $\tilde{\tau} + 1$  that the card  $N$  comes to the top and is inserted. But this improvement is negligible: we have  $\mathbb{E}[\tilde{\tau}] = \mathbb{E}[\tau] + N$ , while as  $N$  goes to infinity,  $\mathbb{E}[\tilde{\tau}] \sim N \ln(N)$ .*

This result is the content of Exercise 6.2 of the book of Levin, Peres, and Wilmer [18].

*Proof.* Let us show that  $\hat{\tau}$  coincides with the strong stationary time  $\tau$  defined in (2.12). Indeed, as a consequence of the above proof that  $\mathfrak{X}_{\hat{\tau}} = \mathcal{S}_N$ , we see that  $\tau$  is smaller than 1 plus the first time when all the cards except  $N$  have been at the top, namely  $\tau \leq \hat{\tau}$ . Conversely, let  $\sigma$  be a permutation where the card  $N$  is above the card  $N - 1$ . For  $n < \hat{\tau}$ , the same is true for the permutation  $\varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_n(\sigma)$ , since neither  $N$  nor  $N - 1$  have been put at the top. So we get that  $\sigma \notin \mathfrak{X}_n$ , i.e.  $\tau > n$  and it follows that  $\hat{\tau} = \tau$ . Note that  $\hat{\tau}$  is a sum of independent geometric random variables of parameters  $2/N, 3/N, \dots, 1$ , which correspond respectively to the first time a card goes under  $N - 1$ , the inter-time until a second card goes behind  $N - 1$ , etc. Similarly,  $\tilde{\tau}$  is a sum of independent geometric random variables of parameters  $1/N, 2/N, \dots, 1$ . Thus  $\mathbb{E}[\tilde{\tau}] - \mathbb{E}[\tau]$  is equal to the expectation of a geometric random variable of parameter  $1/N$ , namely  $N$ . The last assertion of the corollary is a consequence of the equality

$$\mathbb{E}[\tilde{\tau}] = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N} \quad \square$$

One can wonder if the set-valued dual given in (3.7) has a random mapping representation. It is indeed the case, the subsequent simple construction resorts to random mappings depending on a fix set  $S \in \mathfrak{S}$  (we don't know if it is possible to devise a construction via independent random mappings as in Subsection 2.4). The underlying random mappings  $\tilde{\psi}_S$  are described as follows.

- Assume  $S$  is of the form  $A_{\sigma,k}$  for some  $\sigma \in \mathcal{S}_N$  and  $k \in \llbracket 1, N \rrbracket$  with  $\sigma(k) = N$ . Let  $(U, \tilde{U})$  be a random variable taking values in  $\{0, 1\} \times \llbracket 1, N \rrbracket$  such that  $\mathbb{P}[U = 0] = (k - 1)/N$  and knowing that  $U = 0$  (respectively  $U = 1$ ),  $\tilde{U}$  is uniformly distributed on  $\llbracket 1, k - 1 \rrbracket$  (resp. on  $\llbracket 1, N - k + 1 \rrbracket$ ). Let be given a permutation  $\sigma' \in \mathcal{S}_N$ . When  $U = 0$ , we take  $\tilde{\psi}_S(\sigma') = \hat{\psi}^{(\tilde{U})}(\sigma')$ , where for any  $x \in \llbracket 1, N \rrbracket$ ,

$$\forall \sigma' \in \mathcal{S}_N, \quad \hat{\psi}^{(x)}(\sigma') = (1 \rightarrow 2 \rightarrow \dots \rightarrow x) \circ \sigma'$$

In words, a position is chosen among the first  $k - 1$  ones and the corresponding card is sent to the top. On the contrary, when  $U = 1$ , we choose a label of card among those whose position are in  $\llbracket k, N \rrbracket$  according to the following procedure. Let  $\hat{\sigma}(1) < \hat{\sigma}(2) < \dots < \hat{\sigma}(N - k + 1)$  be the ordering of the set  $\{\sigma'(l) : l \in \llbracket k, N \rrbracket\}$ . We

then take  $\tilde{\psi}_S(\sigma') := \psi^{(\tilde{\sigma}(U))}(\sigma')$ , with the notation introduced in (3.8). The mappings  $\psi^{(x)}$  and  $\tilde{\psi}^{(x)}$  may look similar at first view, but it is the difference between the choice of a position and a label of card that will result in the distinction between the dual processes  $\mathfrak{X}$  and  $\tilde{\mathfrak{X}}$ . Note that  $\tilde{\psi}_S$  depends on subsets  $S$  as above only through  $k$ .

- Assume  $S$  is not of the form  $A_{\sigma,k}$  for some  $\sigma \in \mathcal{S}_N$  and  $k \in \llbracket 1, N \rrbracket$  with  $\sigma(k) = N$ . This situation is not important, since the algorithm will only construct subsets of the previous form (starting with  $\mathfrak{X}_0 = \{\text{id}\}$ ), and it would be possible to restrict  $\mathfrak{S}$  to contain only such sets. Nevertheless, for definiteness, take for instance  $\tilde{\psi}_S := \psi^{(U)}$  where  $U$  is uniformly distributed on  $\llbracket 1, N \rrbracket$ .

It is immediate to check that  $\tilde{\psi}_S$  is associated to  $P^*$  and its interest is encapsulated in the following result.

**Lemma 3.4.** *Let  $S$  be of the form  $A_{\sigma,k}$  for some  $\sigma \in \mathcal{S}_N$  and  $k \in \llbracket 1, N \rrbracket$  with  $\sigma(k) = N$ . Fix  $x \in \llbracket 1, N \rrbracket$  and denote by  $\tilde{\sigma}$  the deck of cards obtained from  $\sigma$  by putting the top card  $\sigma(1)$  at position  $x$ . Condition  $\tilde{\psi}_S$  by the fact that  $\tilde{\psi}_S(\tilde{\sigma}) = \sigma$ , to obtain a random mapping  $\tilde{\varphi}_S$ . The mapping  $\tilde{\varphi}_S$  is in fact deterministic and defining*

$$B := \{\sigma' \in \mathcal{S}_N : \tilde{\varphi}_S(\sigma') \in S\}$$

we have

$$B = A_{\tilde{\sigma}, \tilde{k}}$$

with  $\tilde{k} := (\tilde{\sigma})^{-1}(N)$ .

*Proof.* Let  $(U, \tilde{U})$  be the random variable appearing in the definition of  $\tilde{\psi}_S$ . The value of  $U$  can be deduced by comparing  $x$  and  $k$ : if  $x \in \llbracket 1, k-1 \rrbracket$  then  $U = 0$  and otherwise  $U = 1$ . When  $U = 0$ ,  $\tilde{U}$  is determined by the relation  $\tilde{\sigma}(\tilde{U}) = \sigma(1)$  and when  $U = 1$ ,  $\tilde{U}$  is determined as the rank of  $\sigma(1)$  in  $\{\tilde{\sigma}(k), \tilde{\sigma}(k+1), \dots, \tilde{\sigma}(N)\}$  (which is also the rank of  $\sigma(1)$  in  $\{\sigma(1), \sigma(k+1), \sigma(k+2), \dots, \sigma(N)\}$ ). It follows that  $\tilde{\varphi}_S$  is determined, since its randomness only comes from  $(U, \tilde{U})$ .

For the assertion concerning  $B$ , let be given  $\sigma'' \in S$  and  $\sigma' \in \mathcal{S}_N$  such that  $\tilde{\varphi}_S(\sigma') = \sigma''$  and consider separately two alternatives.

- The case  $U = 0$ , which is equivalent to the identity  $\tilde{k} = k$ . Necessarily we have  $\sigma'(l) = \sigma''(l)$  for  $l \in \llbracket \tilde{U} + 1, N \rrbracket$  and

$$(\sigma'(\tilde{U}), \sigma'(1), \dots, \sigma'(\tilde{U} - 1)) = (\sigma''(1), \sigma''(2), \dots, \sigma''(\tilde{U}))$$

Furthermore, note that

$$\begin{aligned} (\sigma''(1), \sigma''(2), \dots, \sigma''(\tilde{U})) &= (\sigma(1), \sigma(2), \dots, \sigma(\tilde{U})) \\ &= (\tilde{\sigma}(\tilde{U}), \tilde{\sigma}(1), \dots, \tilde{\sigma}(\tilde{U} - 1)) \end{aligned}$$

It follows that the set of  $\sigma'$  obtained when  $\sigma''$  runs through  $S$  is just  $A_{\tilde{\sigma}, \tilde{k}}$ , as announced.

- The case  $U = 1$ , which is equivalent to the identity  $\tilde{k} = k - 1$ . We get that

$$\begin{aligned} (\sigma'(1), \sigma'(2), \dots, \sigma'(k-1)) &= (\sigma''(2), \sigma''(3), \dots, \sigma''(k)) \\ (\sigma'(k), \sigma'(k+1), \dots, \sigma'(N))_{\sigma''(1)} &= (\sigma''(k+1), \sigma''(k+2), \dots, \sigma''(N)) \end{aligned}$$

where  $(\sigma'(k), \sigma'(k+1), \dots, \sigma'(N))_{\sigma''(1)}$  stands for the finite sequence  $(\sigma'(k), \sigma'(k+1), \dots, \sigma'(N))$  where  $\sigma''(1)$  has been deleted. It appears that contrary to the case  $U = 0$ , the permutation  $\sigma'$  is not determined by  $\sigma''$ , as we have  $N - k + 1$  possibilities for the insertion of  $\sigma''(1)$  in  $(\sigma'(k), \sigma'(k+1), \dots, \sigma'(N))_{\sigma''(1)}$ . Nevertheless, the set of  $\sigma'$  obtained when  $\sigma''$  run through  $S$  is again  $A_{\tilde{\sigma}, \tilde{k}}$ .  $\square$

Construct, as described in the introduction, a set-valued dual  $\tilde{\mathfrak{X}} := (\tilde{\mathfrak{X}}_n)_{n \in \mathbb{Z}_+}$  starting from  $\tilde{\mathfrak{X}}_0 := \{\text{id}\}$  and using the random mappings modeled after the family  $(\psi_S)_{S \in \mathfrak{S}}$ . By applying iteratively Lemma 3.4, we get that  $\tilde{\mathfrak{X}}$  is given by (3.7), showing in particular that it is taking values in subsets of the form  $A_{\sigma,k}$  for some  $\sigma \in S_N$  and  $k \in \llbracket 1, N \rrbracket$  with  $\sigma(k) = N$ , as forecasted.

### 4 Birth and death chains

The construction of the random mappings used to recover the discrete Pitman theorem is extended here to “restless” birth and death chains. It is a discrete analogue of the results we are looking for in the context of one-dimensional diffusions (see Subsection 7.2). As in the previous section, we restrict our attention to the subset case (the dual processes will start from singletons) and only globally associated to  $P^*$  random mappings will be considered here: *associated* will mean *globally associated*.

Let be given  $(p(x))_{x \in \mathbb{Z}}$  a family of elements of  $(0, 1)$ . We are interested in the irreducible transition kernel  $P$  given by

$$\forall x, y \in \mathbb{Z}, \quad P(x, y) := \begin{cases} p(x), & \text{if } y = x + 1 \\ 1 - p(x), & \text{if } y = x - 1 \\ 0, & \text{otherwise} \end{cases} \quad (4.1)$$

An associated Markov chain  $X := (X_n)_{n \in \mathbb{Z}_+}$  is said to be a **restless** birth and death chain, since at each time  $n \in \mathbb{Z}_+$  it chooses to go up or down of one unity and cannot stay at the same position. Up to a factor, a corresponding invariant measure  $\pi$  is given by

$$\forall x \in \mathbb{Z}, \quad \pi(x) := \begin{cases} \frac{P(0,1)P(1,2)\dots P(x-1,x)}{P(x,x-1)P(x-1,x-2)\dots P(1,0)}, & \text{if } x \geq 1 \\ 1, & \text{if } x = 0 \\ \frac{P(0,-1)P(-1,-2)\dots P(x+1,x)}{P(x,x+1)P(x+1,x+2)\dots P(-1,0)}, & \text{if } x \leq -1 \end{cases}$$

Depending on the family of coefficients  $(p(x))_{x \in \mathbb{Z}}$ , the measure  $\pi$  may be finite or not. Whatever the case, as in Subsection 3.1, the measure  $\pi$  is reversible for  $P$ , in the sense that  $P^* = P$ , where  $P^*$  is defined as in (1.2). The kernel  $\Lambda$  given in (1.6) is also well-defined, as soon as we restrict  $\mathfrak{S}$  to be the set of finite non-empty subsets of  $\mathbb{Z}$ .

A random mapping associated to  $P^* = P$  can be constructed by mimicking the definition given in Subsection 3.1. Define for  $x \in \mathbb{Z}$  and  $u \in [0, 1)$ ,

$$\begin{aligned} \psi_+(x, u) &= \begin{cases} x - 1, & \text{if } u \in [0, 1 - p(x)) \\ x + 1, & \text{if } u \in [1 - p(x), 1) \end{cases} \\ \psi_-(x, u) &= \psi_+(x, 1 - u) \end{aligned}$$

and introduce the mapping

$$\forall S \in \mathfrak{S}, \forall x \in \mathbb{Z}, \forall u \in [0, 1), \quad \psi(S, x, u) := \begin{cases} \psi_+(x, u), & \text{if } x > \max(S) \\ \psi_-(x, u), & \text{if } x \leq \max(S) \end{cases} \quad (4.2)$$

A random mapping  $\psi_S$  is obtained by considering  $\psi(S, \cdot, U)$ , where  $U$  is uniformly distributed on  $[0, 1)$ . It leads to the construction of a (conditioned) set-valued dual  $\mathfrak{X}$  (starting from a singleton) as prescribed after the proof of Corollary 1.6. More precisely, fix a finite trajectory  $x_{\llbracket 0, n \rrbracket}$  of  $X$  with  $n \in \mathbb{Z}_+$  and let  $U_1, U_2, \dots, U_n$  be independent random variables uniformly distributed on  $[0, 1)$ , so that  $\mathfrak{X}_m$  is constructed recursively on  $m \in \llbracket 1, n \rrbracket$  as follows. Condition the random mappings  $\psi(\mathfrak{X}_0, \cdot, U_1), \psi(\mathfrak{X}_1, \cdot, U_2), \dots, \psi(\mathfrak{X}_{m-1}, \cdot, U_m)$  by  $\psi(\mathfrak{X}_0, x_1, U_1) = x_0, \psi(\mathfrak{X}_1, x_2, U_2) = x_1, \dots, \psi(\mathfrak{X}_{m-1}, x_m, U_m) = x_{m-1}$  and call  $\varphi_1, \varphi_2, \dots, \varphi_m$  the induced random mappings. We define  $\mathfrak{X}_0 := \{x_0\}$  and iteratively

$$\forall m \in \llbracket 1, n \rrbracket, \quad \mathfrak{X}_m := \{y \in \mathbb{Z} : \varphi_m(y) \in \mathfrak{X}_{m-1}\} \quad (4.3)$$

Our goal in this section is to simplify the description of  $\mathfrak{X}_{\llbracket 0, n \rrbracket}$  via a backward construction that will be useful for one-dimensional diffusion processes.

For any  $m \in \llbracket 0, n \rrbracket$ , consider

$$\begin{aligned} R_m &:= \max(\mathfrak{X}_m) \\ R_m^\wedge &:= \min(\mathfrak{X}_m) \end{aligned}$$

These numbers determine  $\mathfrak{X}_m$ :

**Lemma 4.1.** *For any  $m \in \llbracket 0, n \rrbracket$ ,  $R_m$  and  $R_m^\wedge$  have the same parity and  $\mathfrak{X}_m$  is the set of integers between  $R_m$  and  $R_m^\wedge$  with the same parity as  $R_m$ .*

*Proof.* All the mappings  $\varphi_1, \varphi_2, \dots, \varphi_m$  change the parity of their argument. Since  $\mathfrak{X}_0 = \{x_0\}$ , we deduce that for all  $m \in \llbracket 0, n \rrbracket$ , the parity of all the elements of  $\mathfrak{X}_m$  is the same as that of  $x_0$  when  $m$  is even and is the other one when  $m$  is odd.

The second assertion of the lemma is proven by iteration on  $m \in \llbracket 1, n \rrbracket$ , based on the fact that if  $x$  and  $x + 2$  belong to  $\mathfrak{X}_{m-1}$ , then necessarily  $x + 1 \in \mathfrak{X}_m$ , by restlessness.  $\square$

Before proceeding toward a simplified presentation, let us be more specific about our conditioning operations. For any  $m \in \llbracket 1, n \rrbracket$  and any  $r \in \mathbb{Z}$ , consider

$$a(x_{m-1}, x_m, r) := \begin{cases} p(x_m), & \text{if } x_{m-1} < x_m \leq r \\ 0, & \text{otherwise} \end{cases} \quad (4.4)$$

Define another number  $b(x_{m-1}, x_m)$  via

$$b(x_{m-1}, x_m) := \begin{cases} 1 - p(x_m), & \text{if } x_{m-1} < x_m \\ p(x_m), & \text{if } x_{m-1} > x_m \end{cases} \quad (4.5)$$

The interest of these numbers is:

**Lemma 4.2.** *The conditioning  $\psi(\mathfrak{X}_{m-1}, x_m, U_m) = x_{m-1}$  a.s. amounts to the conditioning  $U_m \in [a(x_{m-1}, x_m, R_{m-1}), a(x_{m-1}, x_m, R_{m-1}) + b(x_{m-1}, x_m)]$ .*

*Proof.* We consider several cases:

- When  $x_m > x_{m-1}$  and  $x_m > R_{m-1}$ : then  $\psi(\mathfrak{X}_{m-1}, x_m, U_m) = \psi_+(x_m, U_m)$ , so for this term to be equal to  $x_{m-1} = x_m - 1$ , we must have  $U_m \in [0, 1 - p(x_m)] = [a(x_{m-1}, x_m, R_{m-1}), a(x_{m-1}, x_m, R_{m-1}) + b(x_{m-1}, x_m)]$ , with  $a(x_{m-1}, x_m, R_{m-1}) = 0$  and  $b(x_{m-1}, x_m) = 1 - p(x_m)$ .
- When  $x_m > x_{m-1}$  and  $x_m \leq R_{m-1}$ : then  $\psi(\mathfrak{X}_{m-1}, x_m, U_m) = \psi_-(x_m, U_m) = \psi_+(x_m, 1 - U_m)$  so for this term to be equal to  $x_{m-1} = x_m - 1$ , we must have  $1 - U_m \in [0, 1 - p(x_m)]$ , namely  $U_m \in (p(x_m), 1]$ , which a.s. corresponds to  $U_m \in [a(x_{m-1}, x_m, R_{m-1}), a(x_{m-1}, x_m, R_{m-1}) + b(x_{m-1}, x_m)]$ , with  $a(x_{m-1}, x_m, R_{m-1}) = p(x_m)$  and  $b(x_{m-1}, x_m) = 1 - p(x_m)$ .
- When  $x_m < x_{m-1}$ : since  $x_m < x_{m-1} \leq R_{m-1}$ , we have  $\psi(\mathfrak{X}_{m-1}, x_m, U_m) = \psi_-(x_m, U_m) = \psi_+(x_m, 1 - U_m)$ , so for this term to be equal to  $x_{m-1} = x_m + 1$ , we must have  $1 - U_m \in [1 - p(x_m), 1]$ , namely  $U_m \in (0, p(x_m)]$ , which a.s. corresponds to  $U_m \in [a(x_{m-1}, x_m, R_{m-1}), a(x_{m-1}, x_m, R_{m-1}) + b(x_{m-1}, x_m)]$ , with  $a(x_{m-1}, x_m, R_{m-1}) = 0$  and  $b(x_{m-1}, x_m) = p(x_m)$ .  $\square$

It follows from Lemma 4.2 there exist  $\tilde{U}_1, \tilde{U}_2, \dots, \tilde{U}_n$  independent random variables uniformly distributed on  $[0, 1)$  so that

$$\forall m \in \llbracket 1, n \rrbracket, \quad \varphi_m(\cdot) = \psi(\mathfrak{X}_{m-1}, \cdot, a(x_{m-1}, x_m, R_{m-1}) + b(x_{m-1}, x_m)\tilde{U}_m) \quad (4.6)$$

where we recall that  $\psi$  is defined in (4.2).

Let us now show how  $(R_m)_{m \in \llbracket 0, n \rrbracket}$  can be constructed backwardly in the spirit of Subsection 2.4. Similarly to  $\varphi_1, \varphi_2, \dots, \varphi_m$ , we would like to consider the random mappings  $\phi_1, \phi_2, \dots, \phi_m$  obtained by conditioning the random mappings  $\psi(\{x_0, x_1\}, \cdot, U_1), \psi(\{x_1, x_2\}, \cdot, U_2), \dots, \psi(\{x_{m-1}, x_m\}, \cdot, U_m)$  by  $\psi(\{x_0, x_1\}, x_1, U_1) = x_0, \psi(\{x_1, x_2\}, x_2, U_2) = x_1, \dots, \psi(\{x_{m-1}, x_m\}, x_m, U_m) = x_{m-1}$ . Since we want the construction of  $\phi_1, \phi_2, \dots, \phi_m$  to be coupled with  $\varphi_1, \varphi_2, \dots, \varphi_m$ , we define

$$\forall m \in \llbracket 1, n \rrbracket, \quad \begin{aligned} \phi_m(\cdot) &:= \psi(\{x_{m-1}, x_m\}, \cdot, a(x_{m-1}, x_m, x_{m-1} \vee x_m) + b(x_{m-1}, x_m) \tilde{U}_m) \\ &= \psi(\{x_{m-1}, x_m\}, \cdot, \tilde{a}(x_{m-1}, x_m) + b(x_{m-1}, x_m) \tilde{U}_m) \end{aligned} \quad (4.7)$$

where

$$\tilde{a}(x_{m-1}, x_m) := \begin{cases} p(x_m), & \text{if } x_{m-1} < x_m \\ 0, & \text{otherwise} \end{cases}$$

Remark that the above definition of the  $\phi_m$ , for  $m \in \llbracket 1, n \rrbracket$ , is using the finite trajectory  $\mathfrak{X}_{\llbracket 0, m-1 \rrbracket}$ , only through  $x_m$  and  $x_{m-1}$ , and is not recursive ( $\phi_n$  does not need the knowledge of  $\phi_{n-1}, \dots, \phi_1$ ). More precisely, this definition coincides with (4.6) if we had  $\mathfrak{X}_{m-1} = \{x_{m-1}, x_m\}$ . Denote

$$\forall m \in \llbracket 1, n \rrbracket, \quad \phi_{m,n} := \phi_{m+1} \circ \phi_{m+2} \circ \dots \circ \phi_n$$

with the usual convention that  $\phi_{n,n}$  is the identity mapping.

**Proposition 4.3.** *With the above notations, we have*

$$R_n = \max\{x \geq x_n : \exists m \in \llbracket 0, n \rrbracket \text{ with } \phi_{m,n}(x) = x_m\}$$

*Proof.* Let us prove by iteration on  $m \in \llbracket 1, n \rrbracket$  that

$$R_m = \tilde{R}_m := \max\{x \geq x_m : \exists l \in \llbracket 0, m \rrbracket \text{ with } \phi_{l,m}(x) = x_l\}$$

For  $m = 1$ , by definition

$$\begin{aligned} \tilde{R}_1 &= \max\{x \geq x_1 : \exists l \in \llbracket 0, 1 \rrbracket \text{ with } \phi_{l,1}(x) = x_l\} \\ &= \max\{x \geq x_1 : x = x_1 \text{ or } \phi_1(x) = x_0\} \\ &= \max\{x \in \mathbb{Z} : \phi_1(x) = x_0\} \\ &= \max\{x \in \mathbb{Z} : \varphi_1(x) = x_0\} \\ &= \max(\mathfrak{X}_1) \\ &= R_1 \end{aligned}$$

where in the third equality we used that  $\phi_1(x_1) = x_0$  and in the fourth, that  $\varphi_1 = \phi_1$ , when  $x_1 < x_0$ , and that  $\varphi_1(x_1) = \phi_1(x_1) = x_0$  (as well as  $\varphi_1(x_1 + 2) > x_0$  and  $\phi_1(x_1 + 2) > x_0$ ), when  $x_1 > x_0$ .

Assume that  $R_m = \tilde{R}_m$  for some  $m \in \llbracket 1, n - 1 \rrbracket$  and let us show that  $R_{m+1} = \tilde{R}_{m+1}$ .

We have

$$\begin{aligned} \tilde{R}_{m+1} &= \max\{x \geq x_{m+1} : \exists l \in \llbracket 0, m + 1 \rrbracket \text{ with } \phi_{l,m+1}(x) = x_l\} \\ &= \max\{x \geq x_{m+1} : x = x_{m+1} \text{ or } \exists l \in \llbracket 0, m \rrbracket \text{ with } \phi_{l,m}(\phi_{m+1}(x)) = x_l\} \\ &= \max(x_{m+1}, \max\{x \geq x_{m+1} : \phi_{m+1}(x) \in A_m\}) \end{aligned}$$

where

$$A_m := \{y \in \mathbb{Z} : \exists l \in \llbracket 0, m \rrbracket \text{ with } \phi_{l,m}(y) = x_l\}$$

In particular  $x_m \in A_m$ , so that  $x_{m+1} \in \{x \geq x_{m+1} : \phi_{m+1}(x) \in A_m\}$ , since  $\phi_{m+1}(x_{m+1}) = x_m$ . We deduce that

$$\tilde{R}_{m+1} = \max\{x \geq x_{m+1} : \phi_{m+1}(x) \in A_m\} \tag{4.8}$$

Due to restlessness, the parity of the elements of  $A_m$  is the parity of  $x_m$  and the parity of the elements of  $\{x \geq x_{m+1} : \phi_{m+1}(x) \in A_m\}$  is the same as the parity of  $x_{m+1}$ . Furthermore, on the set of odd (respectively even) integers, the mapping  $\phi_{m+1}$  is non-decreasing, thus

$$\{x \geq x_{m+1} : \phi_{m+1}(x) \in A_m\} = \{x \geq x_{m+1} : \phi_{m+1}(x) \in B_m\}$$

where

$$B_m := \{y \geq x_m : \exists l \in \llbracket 0, m \rrbracket \text{ with } \phi_{l,m}(y) = x_l\}$$

It follows by our iteration assumption that  $\max(B_m) = \tilde{R}_m = R_m$ .

Note also that  $B_m$  is exactly equal to the subset of elements from  $\llbracket x_m, R_m \rrbracket$  which have the same parity as  $x_m$ . This comes from restlessness, which implies that two trajectories  $(\phi_{l,m}(y))_{l \in \llbracket 0, m \rrbracket}$  and  $(\phi_{l,m}(y'))_{l \in \llbracket 0, m \rrbracket}$ , where  $y$  and  $y'$  are integers with the same parity, either stay one above the other or end up coalescing. So for any  $y \in \llbracket x_m, R_m \rrbracket$  which has the same parity as  $x_m$ , the trajectory  $(\phi_{l,m}(y))_{l \in \llbracket 0, m \rrbracket}$  is sandwiched between  $(\phi_{l,m}(x_m))_{l \in \llbracket 0, m \rrbracket} = (x_l)_{l \in \llbracket 0, m \rrbracket}$  and  $(\phi_{l,m}(R_m))_{l \in \llbracket 0, m \rrbracket}$ , thus ends up coalescing with  $(x_l)_{l \in \llbracket 0, m \rrbracket}$ .

It follows from this description of  $B_m$  and (4.8) that  $\tilde{R}_{m+1} = R_m - 1$ , except if  $\phi_{m+1}(R_m + 1) = R_m$ , in which case  $\tilde{R}_{m+1} = R_m + 1$ . Let us show that

$$\phi_{m+1}(R_m + 1) = \varphi_{m+1}(R_m + 1) \tag{4.9}$$

We consider two cases:

- When  $R_m + 1 > x_m \vee x_{m+1}$ : we have

$$\phi_{m+1}(R_m + 1) = \psi_+(R_m + 1, \tilde{a}(x_m, x_{m+1}) + b(x_m, x_{m+1})\tilde{U}_{m+1})$$

Since  $R_m + 1 > R_m$ , we also get

$$\varphi_{m+1}(R_m + 1) = \psi_+(R_m + 1, a(x_m, x_{m+1}, R_m) + b(x_m, x_{m+1})\tilde{U}_{m+1})$$

Thus to deduce (4.9), it is sufficient to see that  $\tilde{a}(x_m, x_{m+1}) = a(x_m, x_{m+1}, R_m)$ . This is always true when  $x_{m+1} > x_m$  and when  $x_{m+1} < x_m$ , it requires that  $x_{m+1} \leq R_m$ , which is implied by  $R_m + 1 > x_{m+1}$  here.

- When  $R_m + 1 \leq x_m \vee x_{m+1}$ : since  $x_m \leq R_m$ , we get  $R_m = x_m < x_{m+1} = R_m + 1$ . It follows that  $\phi_{m+1}(R_m + 1) = \phi_m(x_{m+1}) = x_m = \varphi_{m+1}(R_m + 1)$ .

It follows that

$$\tilde{R}_{m+1} = \begin{cases} R_m + 1, & \text{if } \varphi_{m+1}(R_m + 1) = R_m \\ R_m - 1, & \text{otherwise} \end{cases}$$

It is easy to check by similar arguments that the r.h.s. is the iteration defining  $R_{m+1}$ , showing that  $\tilde{R}_{m+1} = R_{m+1}$ . □

A first guess would be that a similar formula holds for the minima of  $\mathfrak{X}_n$ , namely that

$$R_n^\wedge = \tilde{R}_n^\wedge := \min\{x \leq x_n : \exists m \in \llbracket 0, n \rrbracket \text{ with } \phi_{m,n}(x) = x_m\}$$

but this is wrong! Indeed, in the discrete Pitman example of Subsection 3.1, we get that  $\tilde{R}_n^\wedge = x_n$  for all  $n \in \mathbb{Z}_+$ , see Picture 2 showing that  $x_{n+1} - \tilde{R}_{n+1}^\wedge = x_n - \tilde{R}_n^\wedge$  for all  $n \in \mathbb{Z}_+$  (the dotted lines correspond to the action of the mapping  $\phi_{n+1}$ , to be read from the right to the left, remember it is constructed by pretending that  $\mathfrak{X}_n = \{x_n, x_{n+1}\}$ ): whatever the motion from  $x_n$  to  $x_{n+1}$ ,  $\tilde{R}_n^\wedge$  follows a parallel motion to go to  $\tilde{R}_{n+1}^\wedge$ . Since  $\tilde{R}_0^\wedge = x_0$ , we deduce that  $\tilde{R}_n^\wedge = x_n$  for all  $n \in \mathbb{Z}_+$ .

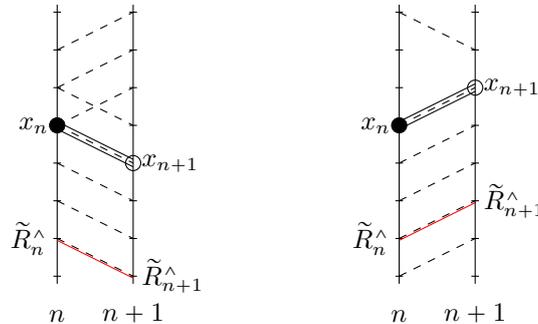


Figure 2: Action of  $\phi_{n+1}$

To get a correct backward formula, we must replace  $\phi_m$ , for  $m \in \llbracket 1, n \rrbracket$ , by the random mapping  $\varphi_m$ , using the acquired knowledge of  $R_{\llbracket 1, n \rrbracket}$ . Indeed, from (4.6), the mapping  $\varphi_m$  is completely determined by  $x_{m-1}, x_m, R_{m-1}$  and the random variable  $\tilde{U}_m$ . Denote

$$\forall m \in \llbracket 1, n \rrbracket, \quad \varphi_{m,n} := \varphi_{m+1} \circ \varphi_{m+2} \circ \dots \circ \varphi_n$$

**Proposition 4.4.** *With the above notations, we have*

$$R_n^\wedge = \min\{x \leq x_n : \exists m \in \llbracket 0, n \rrbracket \text{ with } \varphi_{m,n}(x) = x_m\}$$

*Proof.* The arguments are similar to those of the proof of Proposition 4.3, but simpler since we just play with the family of random mappings  $(\varphi_m)_{m \in \llbracket 1, n \rrbracket}$ . So let us prove by iteration on  $m \in \llbracket 0, n \rrbracket$  that

$$R_m^\wedge = \tilde{R}_m^\wedge := \min\{x \leq x_m : \exists l \in \llbracket 0, m \rrbracket \text{ with } \varphi_{l,m}(x) = x_l\}$$

For  $m = 0$ , by definition

$$\begin{aligned} \tilde{R}_0^\wedge &= \min\{x \leq x_0 : \varphi_{0,0}(x) = x_0\} \\ &= \min\{x \leq x_0 : x = x_0\} \\ &= x_0 \\ &= R_0^\wedge \end{aligned}$$

Assume that  $R_m^\wedge = \tilde{R}_m^\wedge$  for some  $m \in \llbracket 0, n-1 \rrbracket$  and let us show that  $R_{m+1}^\wedge = \tilde{R}_{m+1}^\wedge$ . We have

$$\begin{aligned} \tilde{R}_{m+1}^\wedge &= \min\{x \leq x_{m+1} : \exists l \in \llbracket 0, m+1 \rrbracket \text{ with } \varphi_{l,m+1}(x) = x_l\} \\ &= \min\{x \leq x_{m+1} : x = x_{m+1} \text{ or } \exists l \in \llbracket 0, m \rrbracket \text{ with } \varphi_{l,m}(\varphi_{m+1}(x)) = x_l\} \\ &= \min(x_{m+1}, \min\{x \leq x_{m+1} : \varphi_{m+1}(x) \in A_m^\wedge\}) \end{aligned}$$

where

$$A_m^\wedge := \{y \in \mathbb{Z} : \exists l \in \llbracket 0, m \rrbracket \text{ with } \varphi_{l,m}(y) = x_l\}$$

In particular  $x_m \in A_m^\wedge$ , so that  $x_{m+1} \in \{x \leq x_{m+1} : \varphi_m(x) \in A_m^\wedge\}$ , since  $\varphi_m(x_{m+1}) = x_m$ . We deduce that

$$\tilde{R}_{m+1}^\wedge = \min\{x \leq x_{m+1} : \varphi_m(x) \in A_m^\wedge\} \tag{4.10}$$

Due to restlessness, the parity of the elements of  $A_m^\wedge$  is the parity of  $x_m$  and the parity of the elements of  $\{x \leq x_{m+1} : \varphi_m(x) \in A_m^\wedge\}$  is the same as the parity of  $x_{m+1}$ . Furthermore, on the set of odd (respectively even) integers, the mapping  $\varphi_m$  is non-decreasing, thus

$$\{x \leq x_{m+1} : \varphi_m(x) \in A_m^\wedge\} = \{x \leq x_{m+1} : \varphi_m(x) \in B_m^\wedge\}$$

where

$$B_m^\wedge := \{y \leq x_m : \exists l \in \llbracket 0, m \rrbracket \text{ with } \varphi_{l,m}(y) = x_l\}$$

It follows by our iteration assumption that  $\min(B_m^\wedge) = \tilde{R}_m^\wedge = R_m^\wedge$ .

As in proof of Proposition 4.3, remark that  $B_m^\wedge$  is equal to the subset of elements from  $\llbracket R_m^\wedge, x_m \rrbracket$  which have the same parity as  $x_m$ . It follows from (4.10) that  $\tilde{R}_{m+1}^\wedge = R_m^\wedge + 1$ , except if  $\varphi_m(R_m^\wedge - 1) = R_m^\wedge$ , in which case  $\tilde{R}_{m+1}^\wedge = R_m^\wedge - 1$ . Thus we have

$$\tilde{R}_{m+1}^\wedge = \begin{cases} R_m^\wedge - 1, & \text{if } \varphi_m(R_m^\wedge - 1) = R_m^\wedge \\ R_m^\wedge + 1, & \text{otherwise} \end{cases}$$

It is easy to check by similar arguments that the r.h.s. is the iteration defining  $R_{m+1}^\wedge$ , showing that  $\tilde{R}_{m+1}^\wedge = R_{m+1}^\wedge$ . □

To facilitate the analogy with the last section, let us summarize the procedure followed here to construct  $\mathfrak{X}_n$  for any fixed  $n \in \mathbb{Z}_+$ , given  $x_{\llbracket 0, n \rrbracket}$ . First we sample  $\tilde{U}_1, \tilde{U}_2, \dots, \tilde{U}_n$  independent random variables uniformly distributed on  $[0, 1)$ . Next we construct the family  $(\phi_m)_{m \in \llbracket 1, n \rrbracket}$  via (4.7). Proposition 4.3 enables to deduce the family  $R_{\llbracket 0, n \rrbracket}$ . From the latter, we construct the family  $(\varphi_m)_{m \in \llbracket 1, n \rrbracket}$  via (4.6). Then Proposition 4.4 enables to deduce the family  $R_{\llbracket 0, n \rrbracket}^\wedge$ . Finally, Lemma 4.1 leads to the construction of  $\mathfrak{X}_n$ .

**Remark 4.5.** In the discrete Pitman example of Subsection 3.1, the random variables  $\tilde{U}_1, \tilde{U}_2, \dots, \tilde{U}_n$  are not needed. Here this extra randomness is necessary to construct the family  $R_{\llbracket 0, n \rrbracket}$ . But the two families  $\tilde{U}_{\llbracket 0, n \rrbracket}$  and  $R_{\llbracket 0, n \rrbracket}$  are sufficient to deduce  $R_{\llbracket 0, n \rrbracket}^\wedge$ , namely no additional randomness is required.

## 5 Markov chains on Polish spaces

The construction of set-valued intertwining dual processes presented in the introduction for finite state spaces can be extended in several directions. Here, while keeping the time discrete, we let the state space be a general Polish space.

Let  $V$  be a Polish space endowed with a Markov kernel  $P$ . We assume that  $P$  admits an invariant probability  $\pi$ , so that  $P$  can be extended as a bounded operator on  $\mathbb{L}^2(\pi)$ . Let  $P^*$  be its adjoint operator. It is an abstract Markov operator:  $P^*$  preserves non-negativity as well as  $\mathbb{1}_V$ . The probability measure  $\pi$  is invariant for  $P^*$  in the sense that  $\pi[P^*[f]] = \pi[f]$ , for any  $f \in \mathcal{B}(V)$ , the space of bounded and measurable functions defined on  $V$ . We used a traditional notation for integration:  $\pi[f] := \int f d\pi$ . The motivation for the Polish assumption on  $V$  is that  $P^*$  can also be seen as a Markov kernel. More precisely, consider on  $V \times V$  the coordinate mappings  $X_0$  and  $X_1$  and the probability measure  $\pi(dx_0)P(x_0, dx_1)$ . Note that by the invariance of  $\pi$ , the law of  $X_1$  is  $\pi$ . Using that  $V \times V$  and  $V$  are Polish spaces we get that the conditional law of  $X_0$  given  $X_1$  can be described by a Markov kernel  $M$  from  $V$  to  $V$  (see for instance Section V.8 of

Parthasarathy [24]). It follows that  $\pi(dx_0)P(x_0, dx_1) = \pi(dx_1)M(x_1, dx_0)$ . It is now easy to see that  $P^*$  is the extension of  $M$  as an operator on  $\mathbb{L}^2(\pi)$ . From now on,  $M$  will be denoted  $P^*$ . Let us extend the definitions of the introduction to the present setting.

A random mapping  $\psi$  from  $V$  to  $V$  is a measurable mapping

$$\psi : \Omega \times V \rightarrow V$$

where  $(\Omega, \mathcal{F}, \mathbb{P})$  is an auxiliary probability space. Since we want to insure the existence of conditional distributions, let us furthermore impose that  $(\Omega, \mathcal{F})$  is the Borel  $\sigma$ -field associated to a Polish space. The random mapping  $\psi$  is said to be **associated with  $P^*$** , when for any  $x \in V$ , the law of  $\psi(x)$  is  $P^*(x, \cdot)$ . As in the finite state space case, we want to allow for the dependence of  $\psi$  on some subsets  $S \subset V$ . Let  $\mathfrak{S}$  be a set of measurable subsets  $S \subset V$  such that  $\pi(V) > 0$ . We assume that  $\mathfrak{S}$  is endowed with a Polish space topology and that the mapping  $V \times \mathfrak{S} \ni (x, S) \mapsto \mathbb{1}_S(x)$  is measurable with respect to the corresponding product Borelian  $\sigma$ -field. This hypothesis will be called **(H1)**. Define  $\Lambda$  the mapping associating to any  $S \in \mathfrak{S}$  the probability measure  $\Lambda(S, \cdot)$  which is the conditioning of  $\pi$  on  $S$ . This is an elementary conditioning, since  $\pi(V) > 0$ . It is straightforward to deduce from the measurability part of Fubini's theorem and from the above assumption on  $\mathfrak{S}$  that for any  $f \in \mathcal{B}(V)$ , the mapping  $\mathfrak{S} \ni S \mapsto \Lambda(S, f)$  is measurable. It follows that  $\Lambda$  can be seen as a Markov kernel from  $\mathfrak{S}$  to  $V$ .

A measurable mapping

$$\psi : \Omega \times \mathfrak{S} \times V \ni (\omega, S, x) \mapsto \psi_S(\omega, x)$$

is called a  **$\mathfrak{S}$ -random mapping** and is said to be **locally associated to  $P^*$**  when for any fixed  $S \in \mathfrak{S}$ , any measurable  $A \subset S$  and any  $x \in V$ , we have

$$\mathbb{P}[\psi_S(x) \in A] = \frac{P^*(x, A)}{\zeta(S)} \tag{5.1}$$

where  $\zeta : \mathfrak{S} \rightarrow (0, +\infty)$  is a measurable and positive mapping on  $\mathfrak{S}$  (as in the finite case, we must have  $\zeta(V) = 1$ ). As it is customary, the dependence on  $\omega \in \Omega$  will often not be written explicitly. When for any fixed  $S \in \mathfrak{S}$ ,  $\psi_S(\cdot)$  is a random mapping associated to  $P^*$ ,  $\psi$  is said to be **globally associated to  $P^*$** .

As in (1.12), to a  $\mathfrak{S}$ -random mapping  $\psi$ , we associate

$$\forall \omega \in \Omega, \forall S \in \mathfrak{S}, \quad \Psi(\omega, S) := \{y \in V : \psi_S(\omega, y) \in S\}$$

A priori  $\Psi(S)$  is a measurable subset of  $V$  for any  $S \in \mathfrak{S}$ . We furthermore make the assumption, subsequently called **(H2)**, that  $\mathfrak{S}$ , its topology and  $\psi$  have been chosen so that  $\Psi$  is a random mapping from  $\mathfrak{S}$  to  $\mathfrak{S}$ . We would like to extend the definition given in (1.19), but the conditioning by  $\psi_S(x') = x$  is no longer an elementary one. Nevertheless, our topological hypotheses make it possible, for any given  $x' \in V$  and  $S \in \mathfrak{S}$ , to find a Markov kernel  $K_{x',S}$  from  $V$  to  $\mathfrak{S}$  such that for any  $(x, x', S) \in V \times V \times \mathfrak{S}$ ,  $K_{x',S}(x, \cdot)$  is a regular version of the conditional law  $\mathbb{P}[\Psi(S) \in \cdot | \psi_S(x') = x]$ . Introduce Assumption **(H3)**, asserting that  $K(x, x', S, \cdot) := K_{x',S}(x, \cdot)$  is a Markov kernel from  $V \times V \times \mathfrak{S}$  to  $\mathfrak{S}$ , i.e. we are furthermore requiring that for any measurable  $A \subset \mathfrak{S}$ , the mapping  $V^2 \times \mathfrak{S} \ni (x, x', S) \mapsto K_{x',S}(x, A)$  is measurable. This technical assumption is needed to be able to use  $K(x, x', S, \cdot)$  to construct associated Markov chains, via Ionescu-Tulcea's theorem. It is automatically satisfied when  $V$  is denumerable. When the measurability assumptions (H1) (H2) and (H3) are satisfied, we say that the  $\mathfrak{S}$ -random mapping  $\psi$  locally associated to  $P^*$  is **standard**.

**Remark 5.1.**

- (i) Let  $\mathcal{M}_b(V)$  be the set of signed measures on  $V$  with finite total absolute weight. From (H1) and the measurability part of Fubini's theorem, we have that for any  $\mu \in \mathcal{M}_b(V)$ , the mapping

$$\forall S \in \mathfrak{S}, \quad F_\mu(S) := \int_S d\mu$$

is measurable.

If we have that the Borel  $\sigma$ -field on  $\mathfrak{S}$  is generated by these mappings, then (H2) follows from (H1). Indeed, it is then sufficient to check that for any  $\mu \in \mathcal{M}_b(V)$ , the mapping  $\Omega \times \mathfrak{S} \ni (\omega, S) \mapsto F_\mu(\Psi(\omega, S))$  is measurable. This is again a consequence of (H1) and of the measurability part of Fubini's theorem, since the mapping  $\Omega \times \mathfrak{S} \times V \ni (\omega, S, x) \mapsto \mathbb{1}_S(\psi(\omega, S, x))$  is measurable.

We believe (H3) should equally follow from (H1), under the same assumption on the Borel  $\sigma$ -field on  $\mathfrak{S}$ , but we prefer not entering such measurability questions here.

- (ii) A priori (H2) and (H3) depend on  $\mathfrak{S}$  and  $\psi$ , but we rather see these conditions as relative to the topology of  $\mathfrak{S}$ , in the sense they should be satisfied for all  $\mathfrak{S}$ -random mappings  $\psi$ . In this situation,  $\mathfrak{S}$  is said to be a **standard subset topological space**.

Similarly, (1.22) is replaced by

$$\forall (x, S) \in V \times \mathfrak{S}, \quad Q((x, S), d(x', S')) := P(x, dx')K(x, x', S, dS') \quad (5.2)$$

As in (1.21), we are interested in the set  $\mathcal{A}$  of probability measures  $m$  on  $V \times \mathfrak{S}$  which can be decomposed under the form

$$m(dx, dS) = \mu(dS)\Lambda(S, dx) \quad (5.3)$$

where  $\mu$  is the marginal law of  $m$  on  $\mathfrak{S}$ . When considering Markov chains starting from initial distributions in  $\mathcal{A}$  and evolving according to  $Q$ , it is possible to reduce the state space  $V \times \mathfrak{S}$  to

$$W := \{(x, S) \in V \times \mathfrak{S} : x \in S\} \quad (5.4)$$

as in (1.20), since for  $x \in S$ ,  $K(x, x', S, \cdot)$  should be supported by  $\{S' \in \mathfrak{S} : x' \in S'\}$ . But in the definition of the regular version of a conditional expectation, one has to be careful with negligible subsets, that is why the justification of this restriction will only be given below, in the proof of Theorem 5.2. Note that  $W$  is a Borelian subset, according to (H1).

Finally, extend (1.16) and (1.18) by defining the kernels  $K$  and  $\mathfrak{P}$  on  $\mathfrak{S}$  via:

$$\forall S \in \mathfrak{S}, \quad \begin{cases} K(S, dS') & := \mathbb{P}[\Psi(S) \in dS'] \\ \mathfrak{P}(S, dS') & := \frac{\pi(S')\zeta(S)}{\pi(S)} K(S, dS') \end{cases} \quad (5.5)$$

Both  $K$  and  $\mathfrak{P}$  are Markovian: it is obvious for  $K$  and it is a consequence of the following generalization of Theorem 1.5 to the present framework:

**Theorem 5.2.** *Assume that we are given a standard  $\mathfrak{S}$ -random mapping  $\psi$  locally associated to  $P^*$ . Let  $(X_n, \mathfrak{X}_n)_{n \in \mathbb{Z}_+}$  be a Markov chain on  $V \times \mathfrak{S}$  whose initial distribution  $\mathcal{L}(X_0, \mathfrak{X}_0)$  belongs to  $\mathcal{A}$  and whose transitions are given by  $Q$ , constructed as in (5.2). Then  $X := (X_n)_{n \in \mathbb{N}}$  and  $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{N}}$  are Markov chains whose respective transitions are given by  $P$  and  $\mathfrak{P}$ . Furthermore the conditions (1.9) and (1.10) are fulfilled and a.s. for all  $n \in \mathbb{Z}_+$ ,  $(X_n, \mathfrak{X}_n) \in W$ .*

*Proof.* The arguments are essentially the same as those of the proof of Theorem 1.5, to make them rigorous we just have to resort to conditional expectations.

The first part of the proof, namely that  $X := (X_n)_{n \in \mathbb{N}}$  is a Markov chain whose transitions are given by  $P$  and the validity of (1.9), is very simple, as well as checking  $(A_0)$  and  $(B_0)$ , with the notation from the proof of Theorem 1.5. Thus we concentrate our attention to the deduction of  $(A_{n+1})$  and  $(B_{n+1})$  from  $(A_n)$  and  $(B_n)$ , for some given  $n \in \mathbb{Z}_+$ .

Let  $G : V \times \mathfrak{S} \rightarrow \mathbb{R}$  be a bounded and measurable test function. We have

$$\begin{aligned} \mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | \mathfrak{X}_{[0,n]}] &= \int_V \mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | X_n = y, \mathfrak{X}_{[0,n]}] \mathbb{P}[X_n \in dy | \mathfrak{X}_{[0,n]}] \\ &= \int_{\mathfrak{X}_n} \mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | X_n = y, \mathfrak{X}_{[0,n]}] \mathbb{P}[X_n \in dy | \mathfrak{X}_{[0,n]}] \end{aligned}$$

since due to  $(B_n)$ ,  $\mathbb{P}[X_n \in \cdot | \mathfrak{X}_{[0,n]}]$  is supported by  $\mathfrak{X}_n$ . The reformulation of (2.3) in the present context is that for any bounded measurable test function  $h : V \rightarrow \mathbb{R}$ , we have

$$\begin{aligned} \int_{\mathfrak{X}_n} \mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | X_n = y, \mathfrak{X}_{[0,n]}] h(y) \pi(dy) \\ = \zeta(\mathfrak{X}_n) \int_V \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) h(\psi_{\mathfrak{X}_n}(x))] \pi(dx) \end{aligned} \tag{5.6}$$

where the exponent of the expectation of the r.h.s. indicates that the integration is only with respect to the randomness of the random mapping  $\psi_{\mathfrak{X}_n}$ , while  $\mathfrak{X}_n$  is fixed, as told by its presence as an index. In this equality,  $h$  can depend on  $\mathfrak{X}_n$  (even on  $\mathfrak{X}_{[0,n]}$ ), it will be written  $h_{\mathfrak{X}_n}$  in the following computations. Let us prove (5.6). By the Markov property, we have that the l.h.s. is equal to

$$\begin{aligned} \int_{\mathfrak{X}_n} \mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | X_n = y, \mathfrak{X}_n] h_{\mathfrak{X}_n}(y) \pi(dy) \\ = \int_{\mathfrak{X}_n \times V \times \mathfrak{S}} \pi(dy) P(y, dx) K(y, x, \mathfrak{X}_n, dS') G(x, S') h_{\mathfrak{X}_n}(y) \\ = \int_{\mathfrak{X}_n \times V} \pi(dy) P(y, dx) \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) | \psi_{\mathfrak{X}_n}(x) = y] h_{\mathfrak{X}_n}(y) \\ = \int_{V \times \mathfrak{X}_n} \pi(dx) P^*(x, dy) \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) | \psi_{\mathfrak{X}_n}(x) = y] h_{\mathfrak{X}_n}(y) \end{aligned} \tag{5.7}$$

Recall that by local association,  $P^*(x, \cdot) / \zeta(\mathfrak{X}_n)$  and the law of  $\psi_{\mathfrak{X}_n}(x)$  coincide when they are restricted on  $\mathfrak{X}_n$ , so that

$$\begin{aligned} \frac{1}{\zeta(\mathfrak{X}_n)} \int_{\mathfrak{X}_n} P^*(x, dy) \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) | \psi_{\mathfrak{X}_n}(x) = y] h_{\mathfrak{X}_n}(y) \\ = \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [\mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) | \psi_{\mathfrak{X}_n}(x)] h_{\mathfrak{X}_n}(\psi_{\mathfrak{X}_n}(x))] \\ = \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) h_{\mathfrak{X}_n}(\psi_{\mathfrak{X}_n}(x))] \end{aligned} \tag{5.8}$$

The announced equality (5.6) follows. Taking into account  $(B_n)$ , asserting that

$$\begin{aligned} \mathbb{P}[X_n \in dy | \mathfrak{X}_{[0,n]}] &= \Lambda(\mathfrak{X}_n, dy) \\ &= \mathbb{1}_{\mathfrak{X}_n}(y) \frac{\pi(dy)}{\pi(\mathfrak{X}_n)} \end{aligned}$$

we deduce from (5.6), with  $h = h_{\mathfrak{X}_n} := \mathbb{1}_{\mathfrak{X}_n} / \pi(\mathfrak{X}_n)$ ,

$$\begin{aligned}
 \mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | \mathfrak{X}_{[0,n]}] &= \int_{\mathfrak{X}_n} \mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | X_n = y, \mathfrak{X}_{[0,n]}] \frac{\pi(dy)}{\pi(\mathfrak{X}_n)} \\
 &= \int_{\mathfrak{X}_n} \mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | X_n = y, \mathfrak{X}_{[0,n]}] \frac{\mathbb{1}_{\mathfrak{X}_n}(y)}{\pi(\mathfrak{X}_n)} \pi(dy) \\
 &= \zeta(\mathfrak{X}_n) \int_V \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) \mathbb{1}_{\mathfrak{X}_n}(\psi_{\mathfrak{X}_n}(x))] / \pi(\mathfrak{X}_n) \pi(dx) \\
 &= \zeta(\mathfrak{X}_n) \int_V \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) \mathbb{1}_{\Psi(\mathfrak{X}_n)}(x)] / \pi(\mathfrak{X}_n) \pi(dx) \\
 &= \zeta(\mathfrak{X}_n) \int_{V \times \mathfrak{S}} \pi(dx) K(\mathfrak{X}_n, dS) G(x, S) \mathbb{1}_S(x) / \pi(\mathfrak{X}_n) \\
 &= \int_{V \times \mathfrak{S}} \pi(dx) \mathfrak{P}(\mathfrak{X}_n, dS) G(x, S) \mathbb{1}_S(x) / \pi(S) \tag{5.9}
 \end{aligned}$$

where we used the definitions from (5.5). When  $G$  does not depend on the first variable, i.e. is of the form

$$\forall (x, S) \in V \times \mathfrak{S}, \quad G(x, S) := g(S)$$

for a bounded and measurable test function  $g : V \rightarrow \mathbb{R}$ , we get

$$\mathbb{E}[g(\mathfrak{X}_{n+1}) | \mathfrak{X}_{[0,n]}] = \int_{\mathfrak{S}} \mathfrak{P}(\mathfrak{X}_n, dS) g(S)$$

which amounts to  $(A_{n+1})$ .

Next consider  $G$  of product form:

$$\forall (x, S) \in V \times \mathfrak{S}, \quad G(x, S) := h(x)g(S)$$

where  $g : V \rightarrow \mathbb{R}$  and  $h : \mathfrak{S} \rightarrow \mathbb{R}$  are bounded and measurable test functions. We compute that

$$\int_{V \times \mathfrak{S}} \pi(dx) \mathfrak{P}(\mathfrak{X}_n, dS) G(x, S) \mathbb{1}_S(x) / \pi(S) = \int_{\mathfrak{S}} \mathfrak{P}(\mathfrak{X}_n, dS) g(S) \Lambda[h](S)$$

Let  $F : \mathfrak{S}^{[0,n]} \rightarrow \mathbb{R}$  be another bounded and measurable test function. From the above consideration, we get

$$\begin{aligned}
 \mathbb{E}[F(\mathfrak{X}_{[0,n]}) g(\mathfrak{X}_{n+1}) h(X_{n+1})] &= \mathbb{E}[F(\mathfrak{X}_{[0,n]}) \mathbb{E}[g(\mathfrak{X}_{n+1}) h(X_{n+1}) | \mathfrak{X}_{[0,n]}]] \\
 &= \mathbb{E}[F(\mathfrak{X}_{[0,n]}) \mathfrak{P}[g \Lambda[h]](\mathfrak{X}_n)] \\
 &= \mathbb{E}[F(\mathfrak{X}_{[0,n]}) g(\mathfrak{X}_{n+1}) \Lambda[h](\mathfrak{X}_{n+1})]
 \end{aligned}$$

Since  $F$  and  $g$  are arbitrary bounded and measurable functions, it follows that

$$\mathbb{E}[h(X_{n+1}) | \mathfrak{X}_{[0,n+1]}] = \Lambda[h](\mathfrak{X}_{n+1})$$

namely  $(B_{n+1})$ , due to the fact that  $h$  is equally an arbitrary bounded and measurable function.

It remains to show that a.s. for all  $n \in \mathbb{Z}_+$ ,  $(X_n, \mathfrak{X}_n) \in W$ . For  $n = 0$ , this is an immediate consequence of the belonging of the initial distribution to  $\mathcal{A}$ . For  $n \in \mathbb{Z}_+$ , successively apply (5.9) with the mappings  $G : V \times \mathfrak{S} \ni (x, S) \mapsto \mathbb{1}_S(x)$  and  $G := \mathbb{1}_{V \times \mathfrak{S}}$

to get

$$\begin{aligned}
 \mathbb{P}[X_{n+1} \in \mathfrak{X}_{n+1} | \mathfrak{X}_{[0,n]}] &= \int_{V \times \mathfrak{S}} \pi(dx) \mathfrak{P}(\mathfrak{X}_n, dS) \mathbb{1}_S(x) \mathbb{1}_S(x) / \pi(S) \\
 &= \int_{V \times \mathfrak{S}} \pi(dx) \mathfrak{P}(\mathfrak{X}_n, dS) \mathbb{1}_S(x) / \pi(S) \\
 &= \mathbb{E}[\mathbb{1}_{V \times \mathfrak{S}}(X_{n+1}, \mathfrak{X}_{n+1}) | \mathfrak{X}_{[0,n]}] \\
 &= 1 \quad \square
 \end{aligned}$$

It is straightforward to generalize Corollary 1.6 to the present framework. In the statement of this result,  $\pi$  is seen as the measurable mapping  $\mathfrak{S} \ni S \mapsto \pi(S) \in (0, 1]$ .

In the applications we have in mind, the state space  $\mathfrak{S}$  is too small since we would like that it contains the singletons, which in practice are often negligible with respect to  $\pi$  when  $V$  is not denumerable. Consider for instance  $\tilde{X} := (\tilde{X}_t)_{t \geq 0}$  an elliptic diffusion on a compact Riemannian manifold  $V$ . For fixed  $\epsilon > 0$ , we are interested in the time-skeleton Markov chain  $X := (\tilde{X}_{\epsilon n})_{n \in \mathbb{Z}_+}$ . Its invariant probability measure  $\pi$  is also the invariant probability measure of  $\tilde{X}$ , which gives zero mass to every singleton, since it admits a density with respect to the Riemannian measure. Nevertheless, we are looking for a set-valued dual  $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{Z}_+}$  starting from  $\mathfrak{X}_0 := \{X_0\}$ , constructed via some random mappings (e.g. to escape the difficulties encountered in [21] and [9] to get singletons as starting points). In discrete time, the problem is only in the initial step, since for  $n \in \mathbb{N}$ ,  $\mathfrak{X}_n$  should be a “nice” compact subset of  $V$  with  $\pi(\mathfrak{X}_n) > 0$ . So to end this section, we show a way to enlarge  $\mathfrak{S}$  to include the singletons. Again, random mappings are very helpful in this respect, but they will no longer be locally associated to  $P^*$  and we need a slight generalization of this notion, see Remark 5.3(a) below. Unfortunately, this approach will not receive here the full treatment it deserves and this lack of development will prevent it from being useful in the next section.

We come back to the general setting of this section and begin by presenting some straightforward modifications of the definitions. Consider

$$\bar{\mathfrak{S}} := \mathfrak{S} \cup \{\{x\} : x \in V\} \tag{5.10}$$

(here the notation differs substantially from that of the introduction, where  $\bar{\mathfrak{S}}$  was the set of all subsets of  $V$  and  $\mathfrak{S} := \bar{\mathfrak{S}} \setminus \{\emptyset\}$ ). Assume that  $\bar{\mathfrak{S}}$  is endowed with a Polish topology such that  $\mathfrak{S}$  is a measurable subset of  $\bar{\mathfrak{S}}$  and such that the mapping  $V \times \bar{\mathfrak{S}} \ni (x, S) \mapsto \mathbb{1}_S(x)$  is measurable.

Extend  $\Lambda$  into a Markov kernel  $\bar{\Lambda}$  from  $\bar{\mathfrak{S}}$  to  $V$ , via the convention that when  $S$  is the singleton  $\{x\}$ , then  $\Lambda(\{x\}, \cdot) := \delta_x$ , the Dirac mass at  $x$  (this definition is coherent with the conditioning when  $\pi(\{x\}) > 0$ ). The state space  $W$  has to be enlarged into

$$\begin{aligned}
 \bar{W} &:= \{(x, S) \in V \times \bar{\mathfrak{S}} : x \in S\} \\
 &= W \cup \{(x, \{x\}) : x \in S\}
 \end{aligned}$$

The set of initial distribution we are interested in is  $\bar{\mathcal{A}}$ , the set of probability measure on  $\bar{W}$  which can be decomposed as in (5.3).

Consider a measurable mapping

$$\bar{\psi} : \Omega \times \bar{\mathfrak{S}} \times V \ni (\omega, S, x) \mapsto \bar{\psi}_S(\omega, x)$$

Assume that the restriction  $\psi$  of  $\bar{\psi}$  to  $\Omega \times \mathfrak{S} \times V$  is a standard  $\mathfrak{S}$ -random mapping locally associated with  $P^*$ . Replacing  $\mathfrak{S}$  by  $\bar{\mathfrak{S}}$ , we could define similarly a notion of a  $\bar{\mathfrak{S}}$ -random mapping  $\bar{\psi}$  locally associated with  $P^*$ , nevertheless the condition (5.1) on singletons  $\{x_0\} \in \bar{\mathfrak{S}}$  would just mean

$$\mathbb{P}[\bar{\psi}_{\{x_0\}}(x) = x_0] = P^*(x, \{x_0\}) / \zeta(\{x_0\}) \tag{5.11}$$

where  $\zeta$  has been extended on  $\bar{\mathfrak{S}}$  as a measurable and positive function. In our present context, the r.h.s. often vanishes and the requirement that  $\mathbb{P}[\bar{\psi}_{\{x_0\}}(x) = x_0] = 0$  is not appropriate for our purposes. In some sense, we need a density equivalent of (5.11), which leads us to strengthen our assumption on the Markov kernel  $P$ . So introduce Hypothesis **(H4)** asking for the existence of a measurable function  $p : V \times V \rightarrow \mathbb{R}_+$  such that

$$\forall x, y \in V, \quad P(x, dy) = p(x, y)\pi(dy)$$

This density assumption implies immediately that  $P^*$  is given by

$$\forall x, y \in V, \quad P^*(x, dy) = p(y, x)\pi(dy)$$

Condition (5.1) has to be amended with

$$\forall x_0, x \in V, \quad \mathbb{P}[\bar{\psi}_{\{x_0\}}(x) = x_0] = p(x_0, x)/\zeta(\{x_0\}) \tag{5.12}$$

where  $\zeta : \bar{\mathfrak{S}} \rightarrow (0, +\infty)$  is a measurable function. When (5.12) is satisfied (in addition to  $\psi$  being a standard  $\mathfrak{S}$ -random mapping locally associated with  $P^*$ ), we say that  $\bar{\psi}$  is  **$\bar{\mathfrak{S}}$ -random mapping locally associated with  $P^*$** .

**Remark 5.3.**

- (a) As observed above and strictly speaking,  $\bar{\psi}_{\{x_0\}}$  is not locally associated to  $P^*$ , since it does not satisfy (5.11) in general. Nevertheless, Equation (5.12) can be seen as a limit of (5.1), when  $S$  converges to  $\{x_0\}$  and  $\zeta(S)/\pi(S)$  converges to  $\zeta(\{x_0\})$ . This point of view inspired the notion of local association to  $P^*$ . It also explains the expression obtained in (5.14) below.
- (b) The measurability of  $\zeta$  in (5.12) plays no role when we start with  $X_0 = x_0$  and  $\mathfrak{X}_0 = \{x_0\}$ , for some fixed  $x_0 \in V$ . Anyway, it seems quite natural to assume that  $V \ni x \mapsto \{x\} \in \bar{\mathfrak{S}}$  is a measurable bijection, as well as its inverse (i.e.  $V$  can be identified as a measurable space to  $\{\{x\} : x \in V\}$ ), so that one can go from the measurability of  $\bar{\mathfrak{S}} \ni S \mapsto \zeta(S)$  to the measurability of  $\bar{\mathfrak{S}} \ni S \mapsto \zeta(S)$  via the additional condition that  $V \ni x \mapsto \zeta(\{x\})$  is measurable.

The mapping  $\bar{\psi}$  will be said to be **standard**, when the following extensions ( $\bar{\text{H2}}$ ) and ( $\bar{\text{H3}}$ ) of Assumptions (H2) and (H3) hold:

- ( $\bar{\text{H2}}$ ) requires that the mapping

$$\forall S \in \bar{\mathfrak{S}}, \quad \bar{\Psi}(S) := \{y \in V : \bar{\psi}_S(y) \in S\}$$

is measurable from  $\bar{\mathfrak{S}}$  to  $\mathfrak{S}$ . In particular,  $\bar{\Psi}(\{x_0\})$  must have positive mass for any  $x_0 \in V$ .

- ( $\bar{\text{H3}}$ ) provides us with a Markov kernel  $\bar{K}$  from  $V \times V \times \bar{\mathfrak{S}}$  to  $\mathfrak{S}$ , which, as in (5.2), enables us to define a Markov kernel  $\bar{Q}$  from  $V \times \bar{\mathfrak{S}}$  to  $V \times \mathfrak{S}$  via

$$\forall (x, S) \in V \times \bar{\mathfrak{S}}, \quad \bar{Q}((x, S), d(x', S')) := P(x, dx')\bar{K}(x, x', S, dS') \tag{5.13}$$

Next let us come to the analogue of (5.5). The kernel  $\bar{K}$  is defined similarly to  $K$ :

$$\forall S \in \bar{\mathfrak{S}}, \quad \bar{K}(S, dS') := \mathbb{P}[\bar{\Psi}(S) \in dS']$$

Due to ( $\bar{\text{H2}}$ ),  $\bar{K}$  is a Markov kernel from  $\bar{\mathfrak{S}}$  to  $\mathfrak{S}$ . One has to be more careful with the definition of  $\bar{\mathfrak{P}}$ :

$$\forall S \in \bar{\mathfrak{S}}, \quad \bar{\mathfrak{P}}(S, dS') := \begin{cases} \frac{\pi(S')\zeta(S)}{\pi(S)}\bar{K}(S, dS'), & \text{if } \pi(S) > 0 \\ \pi(S')\zeta(\{x_0\})\bar{K}(S, dS'), & \text{if } S = \{x_0\} \text{ is a singleton} \end{cases} \tag{5.14}$$

Now we have all the ingredients necessary for stating the extension of Theorem 5.2:

**Theorem 5.4.** Assume that we are given a standard  $\bar{\mathfrak{S}}$ -random mapping  $\bar{\psi}$  locally associated to  $P^*$ . Let  $(\bar{X}_n, \bar{\mathfrak{X}}_n)_{n \in \mathbb{Z}_+}$  be a Markov chain on  $V \times \bar{\mathfrak{S}}$  whose initial distribution  $\mathcal{L}(\bar{X}_0, \bar{\mathfrak{X}}_0)$  belongs to  $\bar{\mathcal{A}}$  and whose transitions are given by  $\bar{Q}$ , constructed in (5.13). Then  $\bar{X} := (\bar{X}_n)_{n \in \mathbb{N}}$  and  $\bar{\mathfrak{X}} := (\bar{\mathfrak{X}}_n)_{n \in \mathbb{N}}$  are Markov chains whose respective transitions are given by  $P$  and  $\bar{\mathfrak{P}}$ . Furthermore the conditions (1.9) and (1.10) are fulfilled and a.s. for all  $n \in \mathbb{Z}_+$ ,  $(\bar{X}_n, \bar{\mathfrak{X}}_n) \in \bar{W}$ .

As in Corollary 1.6, we deduce the following consequences.

**Corollary 5.5.** The kernel  $\bar{\mathfrak{P}}$  is Markovian and the intertwining relation (1.7) is satisfied. When  $\zeta \equiv 1$  (in particular for globally associated to  $P^*$  random mappings),  $\pi$  is harmonic for  $\bar{K}$ . Furthermore,  $\bar{\mathfrak{S}} \setminus \mathfrak{S}$  is an entrance boundary for  $\bar{\mathfrak{X}}$ , in the sense that for any  $n \geq 1$ ,  $\bar{\mathfrak{X}}_n \in \mathfrak{S}$ , namely after time 1,  $(\bar{X}_n, \bar{\mathfrak{X}}_n)_{n \in \mathbb{N}}$  is a Markov chain on  $V \times \mathfrak{S}$  of the same type as those considered in Theorem 5.2.

*Proof of Theorem 5.4.* Let us come back to the proof of Theorem 5.2 and review the changes to be made. They correspond to the situation where  $\bar{\mathfrak{X}}_n$  is a singleton and it is sufficient to consider the time  $n = 0$ . Fix some  $x_0 \in V$  and assume that  $\bar{\mathfrak{X}}_0 = \{x_0\}$ . Equation (5.6) is to be replaced by

$$\mathbb{E}[G(\bar{X}_1, \bar{\mathfrak{X}}_1) | \bar{X}_0 = x_0, \bar{\mathfrak{X}}_0 = \{x_0\}] = \int_V \mathbb{E}^{\bar{\psi}_{\{x_0\}}} [G(x, \bar{\Psi}(\{x_0\})) \mathbb{1}_{\bar{\psi}_{\{x_0\}}(x) = x_0}] \pi(dx)$$

where  $G : V \times \bar{\mathfrak{S}} \rightarrow \mathbb{R}$  is a bounded and measurable test function. Its proof, justifying Condition (H4), is the following modification of (5.7):

$$\begin{aligned} & \mathbb{E}[G(X_1, \bar{\mathfrak{X}}_1) | \bar{X}_0 = x_0, \bar{\mathfrak{X}}_0 = \{x_0\}] \\ &= P(x_0, dx) \bar{K}(x_0, x, \{x_0\}, dS') G(x, S') \\ &= \int_V P(x_0, dx) \mathbb{E}^{\bar{\psi}_{\{x_0\}}} [G(x, \bar{\Psi}(\{x_0\})) | \bar{\psi}_{\{x_0\}}(x) = x_0] \\ &= \int_V p(x_0, x) \mathbb{E}^{\bar{\psi}_{\{x_0\}}} [G(x, \bar{\Psi}(\{x_0\})) | \bar{\psi}_{\{x_0\}}(x) = x_0] \pi(dx) \\ &= \zeta(\{x_0\}) \int_V \mathbb{E}^{\bar{\psi}_{\{x_0\}}} [G(x, \bar{\Psi}(\{x_0\})) \mathbb{1}_{\bar{\psi}_{\{x_0\}}(x) = x_0}] \pi(dx) \\ &= \zeta(\{x_0\}) \int_V \mathbb{E}^{\bar{\psi}_{\{x_0\}}} [G(x, \bar{\Psi}(\{x_0\})) \mathbb{1}_{\bar{\Psi}_{\{x_0\}}(x)}] \pi(dx) \end{aligned}$$

The definition of  $\bar{\mathfrak{P}}$  is dictated by the analogue of (5.9), which now writes:

$$\begin{aligned} \mathbb{E}[G(\bar{X}_1, \bar{\mathfrak{X}}_1) | \bar{\mathfrak{X}}_0] &= \mathbb{E}[G(\bar{X}_1, \bar{\mathfrak{X}}_1) | \bar{X}_0 = x_0, \bar{\mathfrak{X}}_0 = \{x_0\}] \\ &= \zeta(\{x_0\}) \int_V \mathbb{E}^{\bar{\psi}_{\{x_0\}}} [G(x, \bar{\Psi}(\{x_0\})) \mathbb{1}_{\bar{\Psi}_{\{x_0\}}(x)}] \pi(dx) \\ &= \zeta(\{x_0\}) \int_{V \times \mathfrak{S}} \pi(dx) \bar{K}(\{x_0\}, dS) G(x, S) \mathbb{1}_S(x) \\ &= \int_{V \times \mathfrak{S}} \pi(dx) \bar{\mathfrak{P}}(\{x_0\}, dS) G(x, S) \mathbb{1}_S(x) / \pi(S) \end{aligned}$$

The end of the proof readily follows the arguments given in the proof of Theorem 5.2.  $\square$

**Remark 5.6.** As in the finite situation, the law of the random mapping enabling to construct  $\bar{\mathfrak{X}}_{n+1}$  (or  $\mathfrak{X}_{n+1}$  in Theorem 5.2) from  $(X_n, \bar{\mathfrak{X}}_n, X_{n+1})$  may depend on the time  $n \in \mathbb{Z}_+$ . Indeed the proofs of Theorems 5.2 and 5.4 are only concerned with a transition from  $n$  to  $n + 1$ . One can even go further, by considering different state spaces  $V_n$  at each time  $n \in \mathbb{Z}_+$ . The invariant probability  $\pi$  has then to be replaced by a family

$(\pi_n)_{n \in \mathbb{Z}_+}$  of probability measures which are related by the underlying Markov kernels:  $\pi_n P_n = \pi_{n+1}$ , for any  $n \in \mathbb{Z}_+$ , where  $P_n$  is the transition kernel between times  $n$  and  $n + 1$ . Corresponding Markov kernels  $(\Lambda_n)_{n \in \mathbb{N}}$  have to be considered.

## 6 Markov processes

Here we leave the discrete-time setting for a continuous-time framework, with the purpose of extending the construction of set-valued intertwining duals given by Theorem 5.2 to diffusion processes. The full development of this theory is out of the scope of the present paper and we hope to provide more details in future works. Nevertheless, below we outline the principles underlying such extensions. An illustration will be given in the next section, where we will also discuss further applications, up to the availability of convenient stochastic flows.

Let  $V$  be a Polish space endowed with a probability distribution  $\pi$ . Let  $\mathfrak{S}$  be a set of certain measurable subsets of  $V$  which are given a positive weight by  $\pi$ . For any  $S \in \mathfrak{S}$ , let  $\Lambda(S, \cdot)$  be the elementary conditioning of  $\pi$  on  $S$ . Assume that  $\mathfrak{S}$  is endowed with a Polish topology such that the mapping  $V \times \mathfrak{S} \ni (x, S) \mapsto \mathbb{1}_S(x)$  is measurable. It follows that  $\Lambda$ , the mapping associating to any  $S \in \mathfrak{S}$  the probability  $\Lambda(S, \cdot)$ , is a Markov kernel from  $\mathfrak{S}$  to  $V$ .

Let  $X := (X_t)_{t \geq 0}$  be a time-homogeneous  $V$ -valued **diffusion** (i.e. a Markov process with continuous paths), whose semigroup  $P := (P_t)_{t \geq 0}$  admits  $\pi$  as an invariant probability measure. Denote  $P^* := (P_t^*)_{t \geq 0}$  the adjoint Markov semi-group in  $L^2(\pi)$ . As in Section 5, our topological assumptions insure that the semi-group is given by Markov kernels. By our hypotheses below, the Markov processes associated to  $P^*$ ,  $X^* := (X_t^*)_{t \geq 0}$ , will admit versions that are diffusions. For any  $x \in V$ ,  $X^*(x)$  will stand for such a process starting from  $x$ .

We want to consider stochastic flows on  $V$  extending the random mappings of the previous sections. We will need a notion of stochastic flow more general than that considered in a series of papers by Le Jan and Raimond [14, 15, 16, 17] (see also Tsirelson [30]), since typically, due to the possible dependence on subsets of  $\mathfrak{S}$ , we would like the increments of the flow to be non-stationary. It is even worse, since once the time has been returned, the subset on which the construction depends is in the future of the flow, fortunately there is an important independence property helping us, see Lemma 6.3 below. In some sense, there is the same difference between the stochastic flows of Le Jan and Raimond and those we would like to construct as the coupling-from-the-past technique of Propp and Wilson [27] and the random mappings considered in the introduction. So a lot remains to be investigated in this direction.

It is convenient to be quite explicit about the underlying probability space, so we are led to the following definitions.

We assume that we are given  $\mathcal{R}(V)$  a vector space of measurable functions from  $V$  to  $V$  endowed with a Polish topology such that the mapping  $V \times \mathcal{R}(V) \ni (x, \psi) \mapsto \psi(x)$  is measurable (all product spaces are endowed with the product measurable structure). Let  $\Delta$  stand for  $\{(s, t) \in \mathbb{R}_+^2 : s \leq t\}$  and more generally for any  $I \subset \mathbb{R}_+$ , we define  $\Delta_I := \{(u, v) \in I^2 : u \leq v\}$ . Define the space  $\Omega^{(5)}$  as the set of all measurable mappings

$$\psi : \Delta \times V \ni (s, t, x) \mapsto \psi_{s,t}(x) \in V \tag{6.1}$$

such that for any fixed  $(s, t) \in \Delta$ , the restricted mapping  $\psi_{s,t}$  belongs to  $\mathcal{R}(V)$ . The notation  $\psi = (\psi_{s,t}(x))_{0 \leq s \leq t, x \in V}$  will designate the canonical coordinates on  $\Omega^{(5)}$ . The space  $\Omega^{(5)}$  is endowed with the sigma-field generated by the canonical coordinates.

**Remark 6.1.** The space  $\Omega^{(5)}$  is too large to be endowed with a Polish structure. It would

be nicer to define  $\Omega^{(5)}$  as the space of mappings of the form (6.1) such that

$$\Delta \ni (s, t) \mapsto \psi_{s,t} \in \mathcal{R}(V)$$

is continuous. Unfortunately this assumption is for the moment too strong and would not enable us to rely on the results of Le Jan and Raimond [17] in Subsection 7.1 below, where  $V = \mathbb{R}$  and  $\mathcal{R}(\mathbb{R})$  will be the space of non-decreasing càdlàg mappings from  $\mathbb{R}$  to  $\mathbb{R}$ , endowed with the Skorohod topology.

The above presentation differs from that of Le Jan and Raimond [15] by the introduction of the space  $\mathcal{R}(V)$  (in the same spirit as that of  $\mathfrak{S}$ ) to avoid the handling of measurable representations.

A **stochastic flow** on the state space  $V$  is a probability distribution  $\mathbb{P}^{(5)}$  on  $\Omega^{(5)}$  such that for all  $0 \leq s \leq t \leq u$ , a.s.  $\psi_{s,s}$  is the identity operator and  $\psi_{s,t} \circ \psi_{t,u} = \psi_{s,u}$ . Given such a probability  $\mathbb{P}^{(5)}$ , the sigma-field of  $\Omega^{(5)}$  is completed with all its negligible subsets. By a slight abuse of terminology, we will also say that the canonical  $\psi$  on  $\Omega^{(5)}$  is a **stochastic flow** (implicitly under  $\mathbb{P}^{(5)}$ ). The stochastic flow is said to be **associated to  $P^*$**  if for any fixed  $t \geq 0$  and  $x \in V$ ,  $(\psi_{t-s,t}(x))_{s \in [0,t]}$  has the same finite-dimensional marginal laws, over the time domain  $[0, t]$ , as a diffusion associated to  $P^*$  and starting from  $x$ . Namely  $\psi$  (or  $\mathbb{P}^{(5)}$ ) provides a coupling of the  $X_{[0,t]}^*(x)$ , for any  $x \in V$  and  $t \geq 0$ . Let us ask more, since we are rather interested in the notion of global association with  $P^*$ , where there is an underlying dependence on subsets of  $\mathfrak{S}$ . To proceed toward its definition, let us extend (H2) into **(H5)**: for any  $0 \leq s \leq t$ , the mapping

$$\Omega^{(5)} \times \mathfrak{S} \ni (\psi, S) \mapsto \psi_{s,t}^{-1}(S)$$

takes values in  $\mathfrak{S} \sqcup \{\emptyset\}$  and is measurable (where  $\mathfrak{S} \sqcup \{\emptyset\}$  is endowed with the sigma-field generated by that of  $\mathfrak{S}$  and  $\{\emptyset\}$ ).

Enlarge  $\Omega^{(5)}$  into  $\Omega^{(4)} := \mathfrak{S} \times \Omega^{(5)}$ , the canonical coordinate on  $\mathfrak{S}$  will be denoted  $\mathfrak{X}_0$ . A probability distribution  $\mathbb{P}^{(4)}$  on  $\Omega^{(4)}$  will be said to be a **stochastic flow** when the coordinates  $\mathfrak{X}_0$  and  $\psi$  are independent and the distribution of  $\psi$  is a stochastic flow. On  $(\Omega^{(4)}, \mathbb{P}^{(4)})$ , we define

$$\forall t \geq 0, \quad \mathfrak{X}_t := \psi_{0,t}^{-1}(\mathfrak{X}_0) \tag{6.2}$$

It follows from (H5) that  $\mathfrak{X} := (\mathfrak{X}_t)_{t \geq 0}$  is a  $\mathfrak{S}$ -valued stochastic process whose initial variable  $\mathfrak{X}_0$  is independent from the stochastic flow  $\psi$ . Note that no regularity is assumed with respect to the time (to go into this direction, one should improve the time regularity of the flow, for instance by considering the condition mentioned in Remark 6.1),  $\mathfrak{X}$  is only a collection of random variables indexed by the time.

For any  $A \subset \Delta \times V$ , denote  $\psi_A := (\psi_{u,v}(x))_{(u,v,x) \in A}$  and for any  $t \geq 0$ , let  $\mathcal{G}_t$  be the sigma-field generated by  $\psi_{\Delta_{[0,t]} \times V}$  and  $\mathfrak{X}_0$ . The stochastic flow  $\psi$  is said to be a  **$\mathfrak{S}$ -stochastic flow** when

$$\forall t \geq 0, \forall x \in V, \quad \mathcal{L}[\psi_{\Delta_{[t,+\infty)} \times V} | \mathcal{G}_t] = \mathcal{L}[\psi_{\Delta_{[t,+\infty)} \times V} | \mathfrak{X}_t] \tag{6.3}$$

Since  $\Omega^{(5)}$  is not endowed with a Polish topology, we have to be more careful about the meaning of the above identity: the signification is that any corresponding conditional expectations are a.s. equal. With the same convention, a  $\mathfrak{S}$ -stochastic flow is said to be **globally associated to  $P^*$** , when

$$\forall 0 \leq s \leq t, \forall x \in V, \quad \mathbb{P}^{(4)}[\psi_{s,t}(x) \in dy | \mathcal{G}_s] = P_{t-s}^*(x, dy) \tag{6.4}$$

In particular such a flow  $\psi$  is associated to  $P^*$ .

**Remark 6.2.** By analogy with the definitions of the previous sections, a  $\mathfrak{S}$ -stochastic flow  $\psi$  is said to be **locally associated to  $P^*$** , when for any  $0 \leq s \leq t$ , any  $x \in V$  and any measurable  $A \subset \mathfrak{X}_s$ , we have

$$\mathbb{P}^{(4)}[\psi_{s,t}(x) \in A | \mathcal{G}_s] = P_{t-s}^*(x, A) / \zeta(s, t, \mathfrak{X}_s)$$

where  $\zeta : \Delta \times (\mathfrak{S} \sqcup \{\emptyset\}) \rightarrow (0, +\infty)$  is a measurable mapping. But to avoid technicalities and since we will not need it in the next section, we will not investigate this promising notion in the time-continuous setting.

Enlarge the probability space  $(\Omega^{(4)}, \mathbb{P}^{(4)})$  into  $(\Omega^{(3)}, \mathbb{P}^{(3)})$ , with

$$\begin{aligned} \Omega^{(3)} &:= \Omega^{(4)} \times V \\ \mathbb{P}^{(3)} &:= \mathbb{P}^{(4)} \otimes \pi \end{aligned} \tag{6.5}$$

and denote  $Z$  for the canonical coordinate on  $V$ .

An important consequence of global association to  $P^*$  is:

**Lemma 6.3.** For any  $(s, t) \in \Delta$ ,  $\mathfrak{X}_s$  and  $\psi_{s,t}(Z)$  are independent under  $\mathbb{P}^{(3)}$  as soon as  $\mathbb{P}^{(4)}$  is a  $\mathfrak{S}$ -stochastic flow globally associated to  $P^*$ . It follows that  $\psi_{\Delta_{[0,s]} \times V}$  and  $\psi_{s,t}(Z)$  are independent.

*Proof.* Let  $F$  and  $G$  be bounded and measurable functions defined respectively on  $\mathfrak{S}$  and  $V$ . By definition, we compute that

$$\begin{aligned} \mathbb{E}^{(3)}[F(\mathfrak{X}_s)G(\psi_{s,t}(Z))] &= \int_V \pi(dz) \mathbb{E}^{(4)}[F(\mathfrak{X}_s)G(\psi_{s,t}(z))] \\ &= \int_V \pi(dz) \mathbb{E}^{(4)}[F(\mathfrak{X}_s) \mathbb{E}^{(4)}[G(\psi_{s,t}(z)) | \mathfrak{X}_s]] \\ &= \int_V \pi(dz) \mathbb{E}^{(4)}[F(\mathfrak{X}_s) P_{t-s}^*[G](z)] \\ &= \mathbb{E}^{(4)}[F(\mathfrak{X}_s)] \int_V \pi(dz) P_{t-s}^*[G](z) \\ &= \mathbb{E}^{(3)}[F(\mathfrak{X}_s)] \pi[G] \end{aligned}$$

where we used (6.4) in the third equality and the invariance of  $\pi$  for the semi-group  $P^*$  in the last equality. Considering  $F \equiv 1$  in the above computation, we also get that  $\pi[G] = \mathbb{E}^{(3)}[G(\psi_{s,t}(Z))]$ , so that finally

$$\mathbb{E}^{(3)}[F(\mathfrak{X}_s)G(\psi_{s,t}(Z))] = \mathbb{E}^{(3)}[F(\mathfrak{X}_s)] \mathbb{E}^{(3)}[G(\psi_{s,t}(Z))]$$

and the wanted independence of  $\mathfrak{X}_s$  and  $\psi_{s,t}(Z)$ .

Note that the construction of  $\psi_{s,t}(Z)$  depends only on  $Z$  and  $\psi_{\Delta_{[s,+\infty)} \times V}$ , so it follows from (6.3) that  $\psi_{s,t}(Z)$  is in fact also independent from  $\mathcal{G}_s$ .  $\square$

A more interesting enlargement of  $\Omega^{(4)}$  is

$$\Omega'' := \Omega^{(4)} \times V^{[0,+\infty)}$$

The canonical coordinates on  $\mathcal{C}([0, +\infty), V)$  are denoted by the process  $Y := (Y_t)_{t \geq 0}$ . For fixed  $t \geq 0$ , let  $\mathcal{H}_t$  be the sigma-field generated by  $\mathcal{G}_t$  and  $Y_{[0,t]}$ . We consider the probability  $\mathbb{P}_t''$  on  $(\Omega'', \mathcal{H}_t)$  which is the image of  $\mathbb{P}^{(3)}$  by the mapping

$$\Omega^{(3)} \ni (\mathfrak{X}_0, \psi, Z) \mapsto (\mathfrak{X}_0, \psi, (\psi_{s,t}(Z))_{s \in [0,t]})$$

Lemma 6.3 insures that the probability spaces  $(\Omega'', \mathcal{H}_t, \mathbb{P}_t'')$ , for  $t \geq 0$ , satisfy the Kolmogorov compatibility criterion. We get there exists a probability  $\mathbb{P}''$  on  $\Omega''$  endowed

with its natural sigma-field, so that  $\mathbb{P}''$  coincides with  $\mathbb{P}''_t$  on  $\mathcal{H}_t$ , for any  $t \geq 0$ . Note furthermore that under  $\mathbb{P}''$ , the process  $Y$  is a stationary diffusion associated to the semi-group  $P$  starting with  $Y_0$  distributed according to  $\pi$  and that we have

$$\forall (s, t) \in \Delta, \quad \mathbb{P}''[Y_s = \psi_{s,t}(Y_t)] = 1 \tag{6.6}$$

For any measurable functional  $F$  defined on  $\Omega''$ , which is either bounded or non-negative, we are interested in the conditional expectation,  $\mathbb{E}''[F|\mathfrak{X}_s, Y_{[s,t]}]$ , of  $F$  knowing the sigma-field generated by  $\mathfrak{X}_s$  and the  $Y_u$ , for  $u \in [s, t]$ . We denote  $\mathbb{E}''_{S, x_{[s,t]}}[F] := \mathbb{E}''[F|\mathfrak{X}_s = S, Y_{[s,t]} = x_{[s,t]}]$ , keeping in mind that it is only defined a.s. with respect to a set  $S \in \mathfrak{S}$  and a trajectory  $x_{[s,t]} \in V^{[s,t]}$ , distributed according to the law of  $(\mathfrak{X}_s, Y_{[s,t]})$ . Recall from Lemma 6.3 that  $\mathfrak{X}_s$  and  $Y_{[s,t]}$  are independent and that the law of the latter is that of a stationary diffusion. Since neither  $\Omega''$  nor  $V^{[s,t]}$  are Polish spaces, we cannot represent the above conditional expectation via integration w.r.t. a Markov kernel from  $\mathfrak{S} \times V^{[s,t]}$  to  $\Omega''$ .

Here is another compatibility consequence of global association that will be important in the sequel.

**Lemma 6.4.** *Assume the  $\mathfrak{S}$ -stochastic flow  $\psi$  is globally associated to  $P^*$ . Then for any  $0 \leq s < t$ , any  $u \geq 0$  and any bounded and  $\mathcal{G}_t$ -measurable functional  $F$ , we have*

$$\mathbb{E}''_{S, x_{[s,t+u]}}[F] = \mathbb{E}''_{S, x_{[s,t]}}[F]$$

where the equality holds a.s. with respect to  $S \in \mathfrak{S}$  and  $x_{[s,t+u]} \in V^{[s,t+u]}$  independently distributed according to the law of  $\mathfrak{X}_s$  and to a stationary  $X_{[s,t+u]}$ .

*Proof.* To get the above a.s. identity, it is sufficient to show that for any bounded and  $\mathcal{G}_t$ -measurable functional  $F$ , for any bounded and measurable functions  $G_1 : \mathfrak{S} \rightarrow \mathbb{R}$ ,  $G_2 : V^{[s,t]} \rightarrow \mathbb{R}$  and  $G_3 : V^{[t,t+u]} \rightarrow \mathbb{R}$ , we have

$$\begin{aligned} & \mathbb{E}''[FG_1(\mathfrak{X}_s)G_2(Y_{[s,t]})G_3(Y_{[t,t+u]})] \\ &= \mathbb{E}''[\mathbb{E}''_{\mathfrak{X}_s, Y_{[s,t]}}[F]G_1(\mathfrak{X}_s)G_2(Y_{[s,t]})G_3(Y_{[t,t+u]})] \end{aligned} \tag{6.7}$$

So let us start with the l.h.s. and condition it by  $\mathcal{G}_t$ :

$$\begin{aligned} & \mathbb{E}''[FG_1(\mathfrak{X}_s)G_2(Y_{[s,t]})G_3(Y_{[t,t+u]})] \\ &= \mathbb{E}''[FG_1(\mathfrak{X}_s)\mathbb{E}''[G_2(Y_{[s,t]})G_3(Y_{[t,t+u]})|\mathcal{G}_t]] \\ &= \mathbb{E}''[FG_1(\mathfrak{X}_s)\mathbb{E}''[G_2((\psi_{v,t}(Y_t))_{v \in [s,t]})G_3(Y_{[t,t+u]})|\mathcal{G}_t]] \end{aligned}$$

Note that under the conditioning by  $\mathcal{G}_t$ , the mappings  $(\psi_{v,t}(\cdot))_{v \in [s,t]}$  are fixed and that  $(Y_{t+v})_{v \in [0,u]}$  is a stationary diffusion associated to  $P$ . It follows that

$$\begin{aligned} & \mathbb{E}''[G_2((\psi_{v,t}(Y_t))_{v \in [s,t]})G_3(Y_{[t,t+u]})|\mathcal{G}_t] \\ &= \int G_2((\psi_{v,t}(x))_{v \in [s,t]})\mathbb{E}[G_3((X_{v-t}(x))_{v \in [t,t+u]})]\pi(dx) \end{aligned}$$

where  $X(x)$  is a diffusion associated to  $P$  starting from  $x$ .

It leads us to introduce the measurable mapping

$$\begin{aligned} H : V^{[s,t]} & \rightarrow \mathbb{R} \\ y_{[s,t]} & \mapsto G_2(y_{[s,t]})\mathbb{E}[G_3((X_{v-t}(y_t))_{v \in [t,t+u]})] \end{aligned}$$

since we can write, once again taking into account the independence property of Lemma 6.3,

$$\begin{aligned} \mathbb{E}''[FG_1(\mathfrak{X}_s)G_2(Y_{[s,t]})G_3(Y_{[t,t+u]})] &= \int \mathbb{E}''[FG_1(\mathfrak{X}_s)H((\psi_{v,t}(x))_{v \in [s,t]})] \pi(dx) \\ &= \mathbb{E}''[FG_1(\mathfrak{X}_s)H((\psi_{v,t}(Y_t))_{v \in [s,t]})] \\ &= \mathbb{E}''[FG_1(\mathfrak{X}_s)H(Y_{[s,t]})] \\ &= \mathbb{E}''[\mathbb{E}''_{\mathfrak{X}_s, Y_{[s,t]}}[F]G_1(\mathfrak{X}_s)H(Y_{[s,t]})] \end{aligned}$$

To get (6.7), it remains to reverse the above computations, or more precisely, to apply them with  $F \equiv 1$  and the mapping  $\mathfrak{S} \times V^{[s,t]} \ni (S, y_{[s,t]}) \mapsto G_1(S)G_2(y_{[s,t]})$  replaced by

$$\mathfrak{S} \times V^{[s,t]} \ni (S, y_{[s,t]}) \mapsto G_1(S)G_2(y_{[s,t]})\mathbb{E}''_{S, y_{[s,t]}}[F] \quad \square$$

Given a full trajectory  $x_{[0,+\infty)} \in V^{[0,+\infty)}$  and a set  $S \in \mathfrak{S}$ , we can consider conditional expectations  $\mathbb{E}_{S, x_{[0,+\infty)}}[\cdot]$  as before Lemma 6.4, corresponding to the conditioning by  $\mathfrak{X}_0 = S$  and  $Y = x_{[0,+\infty)}$ . The conditional expectations  $\mathbb{E}_{S, x_{[0,+\infty)}}[\cdot]$  are only defined a.s. with respect to  $(S, x_{[0,+\infty)})$  distributed according to the tensor product of the law of  $\mathfrak{X}_0$  and the stationary law of  $X$ . These conditionings are the time-continuous version of the conditioning of the random mappings by the Markov chain  $(X_n)_{n \in \mathbb{Z}_+}$  encountered in the previous sections. As we have done before, when working under  $\mathbb{E}_{S, x_{[0,+\infty)}}[\cdot]$  or under its integrated version  $\mathbb{P}$  defined in the next paragraph, the flow  $\psi := (\psi_{s,t})_{0 \leq s \leq t}$  will be denoted  $\varphi := (\varphi_{s,t})_{0 \leq s \leq t}$  (to avoid the error of thinking that  $\varphi$  is globally associated to  $P^*$ ).

Assume that  $\mathfrak{X}_0$  is deterministic and let  $X := (X_t)_{t \geq 0}$  be a diffusion process associated to  $P$  starting from  $\Lambda(\mathfrak{X}_0, \cdot)$ , say defined on a probability space  $(\Omega', \mathbb{P}')$ . We endow  $\Omega := \Omega' \times \Omega''$  with the probability measure  $\mathbb{P}$  whose marginal distribution on  $\Omega'$  is  $\mathbb{P}'$  and whose conditional distribution on  $\Omega''$  knowing the coordinate  $X$  on  $\Omega'$  is  $\mathbb{P}_{\mathfrak{X}_0, X}[\cdot]$ . This is well-defined through expectations, since the law of  $X$  is then absolutely continuous with respect to the stationary law of a diffusion associated to  $P$  (the Radon-Nikodym density being  $\mathbb{1}_{\mathfrak{X}_0}(X_0)/\pi(\mathfrak{X}_0)$ ). More generally, this construction has to be integrated with respect to the law of  $\mathfrak{X}_0$ . This is possible when the initial law of  $X$  has the form  $\int_{\mathfrak{S}} \Lambda(S, \cdot) \mu(dS)$ , where  $\mu$  is a probability measure on  $\mathfrak{S}$ . In this case the law of  $(X_0, \mathfrak{X}_0)$  belongs to  $\mathcal{A}$ , namely is of the form described in (5.3), or equivalently, we have

$$\mathcal{L}(X_0 | \mathfrak{X}_0) = \Lambda(\mathfrak{X}_0, \cdot) \tag{6.8}$$

When  $\mathbb{P}$  is constructed as above starting with  $\mathbb{P}^{(5)}$ , a stochastic flow globally associated to  $P^*$ , we say that  $\mathbb{P}$  is a  **$(P^*, \mathfrak{S})$ -conditioned stochastic flow**. We deduce from Lemma 6.4 that under such a probability, the analogue of (1.9) is satisfied:

**Lemma 6.5.** *Under a  $(P^*, \mathfrak{S})$ -conditioned stochastic flow  $\mathbb{P}$ , we have*

$$\forall t \geq 0, \quad \mathcal{L}(\mathfrak{X}_{[0,t]} | X) = \mathcal{L}(\mathfrak{X}_{[0,t]} | X_{[0,t]})$$

*Proof.* Let us first compute the conditional expectation under  $\mathbb{P}$  knowing  $X$ . Consider a bounded and  $\mathcal{G}_\infty$ -measurable functional  $F$  (i.e.  $F$  is measurable with respect to  $\mathfrak{X}_0$  and  $\varphi$ ), as well as a bounded and measurable mapping  $G : V^{[0,+\infty)} \rightarrow \mathbb{R}$ . Denote  $\mu$  the law of  $\mathfrak{X}_0$ . By definition, we have

$$\mathbb{E}[FG(X)] = \int_{\mathfrak{S} \times V} \mu(dS) \Lambda(S, dx) \mathbb{E}'_x[G(X) \mathbb{E}''_{S, X}[F]]$$

where under  $\mathbb{P}'_x$ ,  $X$  starts from  $x \in V$  and is associated to the semi-group  $P$ . The previous r.h.s. can be written under the following form, with  $\mathfrak{S}_x := \{S \in \mathfrak{S} : x \in S\}$

$$\int_V \pi(dx) \int_{\mathfrak{S}_x} \mu(dS) \frac{\mathbb{E}'_x[G(X) \mathbb{E}''_{S, X}[F]]}{\pi(S)}$$

Since the distribution of  $X_0$  admits

$$V \ni x \mapsto \int_{\mathfrak{S}_x} \mu(dS) \frac{1}{\pi(S)}$$

as density with respect to  $\pi$ , we get that the conditional expectation of  $F$  knowing  $X = x_{[0,+\infty)} \in V^{[0,+\infty)}$  is given by

$$\mathbb{E}[F|X = x_{[0,+\infty)}] = \frac{\int_{\mathfrak{S}_{x_0}} \mu(dS) \mathbb{E}''_{S, x_{[0,+\infty)}}[F]}{\int_{\mathfrak{S}_{x_0}} \mu(dS) \frac{1}{\pi(S)}} \tag{6.9}$$

Note that by the martingale convergence theorem, we have

$$\lim_{s \rightarrow +\infty} \mathbb{E}''_{S, X_{[0,s]}}[F] = \mathbb{E}''_{S, X_{[0,+\infty)}}[F]$$

(a priori a.s. with respect to a stationary  $X$ , but equally under  $\mathbb{P}$ , by absolute continuity). Fix  $t \geq 0$  and assume now that  $F$  is furthermore  $\mathcal{G}_t$ -measurable. We deduce from Lemma 6.4 that

$$\mathbb{E}''_{S, X_{[0,+\infty)}}[F] = \mathbb{E}''_{S, X_{[0,t]}}[F]$$

so that

$$\mathbb{E}[F|X = x_{[0,+\infty)}] = \frac{\int_{\mathfrak{S}_{x_0}} \mu(dS) \mathbb{E}''_{S, x_{[0,t]}}[F]}{\int_{\mathfrak{S}_{x_0}} \mu(dS) \frac{1}{\pi(S)}}$$

In particular, the l.h.s. only depends on  $x_{[0,t]}$ . By the tower property of conditional expectation, we get that

$$\mathbb{E}[F|X_{[0,\infty)}] = \mathbb{E}[F|X_{[0,t]}]$$

Since this is true for any  $\mathcal{G}_t$ -measurable  $F$ , we deduce the wanted equality. □

To go further in the description of  $\mathbb{P}$ , especially to show that the analogue of (1.10) equally holds, we try to come back to the setting of the previous section by considering discrete-time skeletons.

More precisely, for  $\epsilon > 0$ , consider the Markov kernel  $P_\epsilon$  from  $V$  to  $V$ . Associated Markov chains are the  $\epsilon$ -skeleton  $X^{(\epsilon)} := (X_{\epsilon n})_{n \in \mathbb{Z}_+}$ . For  $n \in \mathbb{Z}_+$ , let  $\psi_{n, \mathfrak{x}_n}^{(\epsilon)}$  be  $\psi_{\epsilon n, \epsilon(n+1)}$  conditioned by  $\mathcal{G}_{\epsilon n}$ . The family  $(\psi_{n, S}^{(\epsilon)})_{S \in \mathfrak{S}}$  has to be understood in the following sense: let  $F$  be a bounded or positive measurable function on  $\mathfrak{X}$ , by definition, the expectation of  $F(\psi_{n, S}^{(\epsilon)})$  is equal to  $\mathbb{E}''[F(\psi_{\epsilon n, \epsilon(n+1)}) | \mathfrak{X}_{\epsilon n} = S]$ , a.s. in  $S$  distributed according to the law of  $\mathfrak{X}_{\epsilon n}$ . Here our notations may be slightly confusing, so let us review the construction considered in Section 5, taking into account Remark 5.6, since the family  $(\psi_{n, S}^{(\epsilon)})_{n \in \mathbb{Z}_+, S \in \mathfrak{S}}$  is inhomogeneous with respect to the time  $n \in \mathbb{Z}_+$ . Starting from  $\mathfrak{x}_0^{(\epsilon)} := \mathfrak{x}_0$ , we define

$$\begin{aligned} \mathfrak{x}_1^{(\epsilon)} &:= (\psi_{0, \mathfrak{x}_0}^{(\epsilon)})^{-1}(\mathfrak{x}_0) \\ &= (\psi_{0, \epsilon}^{(\epsilon)})^{-1}(\mathfrak{x}_0) \\ &= \mathfrak{x}_\epsilon \end{aligned}$$

What is important is that the law of  $\mathfrak{x}_1^{(\epsilon)}$  knowing  $\mathfrak{x}_0^{(\epsilon)}$  is exactly the law of  $\mathfrak{x}_\epsilon$  knowing  $\mathfrak{x}_0$ , so that the law of  $(\mathfrak{x}_0^{(\epsilon)}, \mathfrak{x}_1^{(\epsilon)})$  is equal to the law of  $(\mathfrak{x}_0, \mathfrak{x}_\epsilon)$ . Construct  $\mathfrak{x}^{(\epsilon)} := (\mathfrak{x}_n^{(\epsilon)})_{n \in \mathbb{Z}_+}$  via the induction

$$\forall n \in \mathbb{Z}_+, \quad \mathfrak{x}_{n+1}^{(\epsilon)} := (\psi_{n, \mathfrak{x}_n}^{(\epsilon)})^{-1}(\mathfrak{x}_n^{(\epsilon)})$$

By iteration of the above argument and taking into account (6.3), we end up with  $\mathfrak{X}^{(\epsilon)}$  having the same law as  $(\mathfrak{X}_{\epsilon n})_{n \in \mathbb{Z}_+}$ . We will identify  $\mathfrak{X}^{(\epsilon)}$  with  $(\mathfrak{X}_{\epsilon n})_{n \in \mathbb{Z}_+}$ , since above all we are interested in their law, to be conditioned below. Besides, our previous assumptions on  $\psi$  insure that for any  $n \in \mathbb{Z}_+$ , the random mapping  $\psi_{n,S}^{(\epsilon)}$  is, a.s. in  $S$  with respect to the law of  $\mathfrak{X}_{\epsilon n}$ , a  $\mathfrak{S}$ -random mapping globally associated to  $P_\epsilon^*$ . Assumption (H1) was assumed at the beginning of this section and (H5) implies (H2) for the discrete-time random mappings  $\psi_{n,S}^{(\epsilon)}$  for  $n \in \mathbb{Z}_+$  and  $S \in \mathfrak{S}$ . To go in the direction of (H3), let us define, for any  $0 \leq s < t$ ,  $x, x' \in V$  and  $S \in \mathfrak{S}$ ,  $\mathbb{E}''_{S;s,x;t,x'}[\cdot]$  the conditional expectation under  $\mathbb{P}''$  knowing that  $\mathfrak{X}_s = S$ ,  $Y_s = x$  and  $Y_t = x'$ . Next, let  $K((S; s, x; t, x'), \cdot)$  be the image of  $\mathbb{P}''_{S;s,x;t,x'}[\cdot]$  by the mapping  $\psi \mapsto \psi_{s,t}^{-1}(S)$ . For fixed  $0 \leq s < t$ , we can see it as a Markov kernel from  $\mathfrak{S} \times V \times V$  to  $\mathfrak{S}$ , since it corresponds to the conditioning of  $\mathfrak{X}_t$  by  $(\mathfrak{X}_s, X_s, X_t)$  and we can work on the Polish space  $\mathfrak{S} \times V \times V \times \mathfrak{S}$  endowed with the law of  $(\mathfrak{X}_s, X_s, X_t, \mathfrak{X}_t)$ . When  $s = \epsilon n$  and  $t = \epsilon(n + 1)$ ,  $K((S; \epsilon n, x; \epsilon(n + 1), x'), \cdot)$  can almost play the role of  $K(x, x', S, \cdot)$  in Condition (H3) for the discrete-time random mapping  $\psi_{n,S}^{(\epsilon)}$ , except that  $K((S; \epsilon n, x; \epsilon(n + 1), x'), \cdot)$  is maybe not a Markov kernel corresponding to the conditional distribution of  $(\psi_{n,S}^{(\epsilon)})^{-1}(S)$  knowing  $\psi_{n,S}^{(\epsilon)}(x') = x$ , for all fixed  $S \in \mathfrak{S}$  and  $x' \in S$  (and a.s. in  $x$  with respect to the law of  $\psi_{n,S}^{(\epsilon)}(x')$ ), but *only* a.s. for  $S$  distributed as  $\mathfrak{X}_{\epsilon n}$  and for  $(x, x')$  distributed (independently) according to  $\pi(dx)P_\epsilon(x, dx') = \pi(dx')P_\epsilon^*(x', dx)$ . Nevertheless, this extension of Condition (H3) is sufficient for the validity of Theorem 5.2, as it is checked by a direct examination of its proof. Indeed, (5.7) and (5.8) have now to be understood a.s., respectively w.r.t. the law of  $\mathfrak{X}_{\epsilon n}$  and in  $x$  w.r.t.  $\pi$ .

**Remark 6.6.** The above construction of the process  $(Y_t)_{t \geq 0}$  could also be performed in the discrete-time setting of Section 5 for globally associated to  $P^*$  random mappings to get a stationary chain  $(Y_n)_{n \in \mathbb{Z}_+}$  associated to the transition kernel  $P$  and satisfying the analogue of Property (6.6). In the restricted setting of global association, it leads to a variant of Theorem 5.2 where Condition (H3) has been removed and replaced by the hypothesis that the random mappings belongs to some Polish functional space  $\mathfrak{R}$ . It would be interesting to get similar constructions in the context of local association.

These considerations lead to the following continuous-time extension of (1.10):

**Theorem 6.7.** *Under a  $(P^*, \mathfrak{S})$ -conditioned stochastic flow  $\mathbb{P}$ , we have*

$$\forall t \geq 0, \quad \mathcal{L}(X_t | \mathfrak{X}_{[0,t]}) = \Lambda(\mathfrak{X}_t, \cdot) \tag{6.10}$$

*Proof.* According to the above observations, we are in position to apply Theorem 5.2 to the time-inhomogeneous random mappings induced by the family  $(\psi_{n,S}^{(\epsilon)})_{n \in \mathbb{Z}_+, S \in \mathfrak{S}}$ . More precisely, let us come back to the kernel described by  $K((S; \epsilon n, x; \epsilon(n + 1), x'), \cdot)$ , for  $S \in \mathfrak{S}$ ,  $n \in \mathbb{Z}_+$  and  $x, x' \in V$ . The tower property of conditional expectation implies that

$$\begin{aligned} & \mathbb{E}''_{S;\epsilon n,x;\epsilon(n+1),x'}[\cdot] \\ &= \int_{\mathcal{C}([\epsilon n, \epsilon(n+1)], V)} \mathbb{E}''_{S,x[\epsilon n, \epsilon(n+1)]}[\cdot] \mathbb{P}_{\epsilon n,x;\epsilon(n+1),x'}(dx_{[\epsilon n, \epsilon(n+1)]}) \end{aligned} \tag{6.11}$$

where  $\mathbb{P}_{\epsilon n,x;\epsilon(n+1),x'}$  is the law of the bridge  $X_{[\epsilon n, \epsilon(n+1)]}$  associated to  $P$  and conditioned by  $X_{\epsilon n} = x$  and  $X_{\epsilon(n+1)} = x'$ . Here we don't need the results of Fitzsimmons, Pitman and Yor [12] for the existence for such bridge laws, since we just require their existence  $\pi(dx)P_\epsilon(x, dx')$ -a.s. and not for every  $x, x' \in V$ .

Let  $\tilde{X}^{(\epsilon)} := (\tilde{X}_n^{(\epsilon)})_{n \in \mathbb{Z}_+}$  be a Markov chain with transition kernel  $P_\epsilon$  and starting from  $\Lambda(\mathfrak{X}_0, \cdot)$ , first assuming that  $\mathfrak{X}_0$  is deterministic. As in Section 5, let  $\tilde{\mathfrak{X}}^{(\epsilon)} := (\tilde{\mathfrak{X}}_n^{(\epsilon)})_{n \in \mathbb{Z}_+}$  be obtained from  $(\psi_{n,S}^{(\epsilon)})_{n \in \mathbb{Z}_+, S \in \mathfrak{S}}$  through its iterative conditionings:

$$\forall n \in \mathbb{Z}_+, \quad \psi_{n, \tilde{\mathfrak{X}}_n^{(\epsilon)}}^{(\epsilon)}(\tilde{X}_{n+1}^{(\epsilon)}) = \tilde{X}_n^{(\epsilon)}$$

The law of  $(\tilde{X}^{(\epsilon)}, \tilde{\mathfrak{X}}^{(\epsilon)})$  coincides with the law of  $(X^{(\epsilon)}, \mathfrak{X}^{(\epsilon)})$ . Indeed, by iteration, it is sufficient to see that for any  $n \in \mathbb{Z}_+$ , the conditional law of  $(\tilde{X}_{n+1}^{(\epsilon)}, \tilde{\mathfrak{X}}_{n+1}^{(\epsilon)})$  knowing  $(\tilde{X}_m^{(\epsilon)}, \tilde{\mathfrak{X}}_m^{(\epsilon)})_{m \in [0, n]}$  is equal to the law of  $(X_{\epsilon(n+1)}, \mathfrak{X}_{\epsilon(n+1)})$  knowing  $(X_{\epsilon m}, \mathfrak{X}_{\epsilon m})_{m \in [0, n]}$ . On one hand, by definition, by taking into account (6.11) and with the kernel  $K$  defined above Remark 6.6, we have

$$\begin{aligned} & \mathbb{P}[(\tilde{X}_{n+1}^{(\epsilon)}, \tilde{\mathfrak{X}}_{n+1}^{(\epsilon)}) \in (dx, dS) | (\tilde{X}_m^{(\epsilon)}, \tilde{\mathfrak{X}}_m^{(\epsilon)})_{m \in [0, n]}] \\ &= P_\epsilon(\tilde{X}_n^{(\epsilon)}, dx) K(\tilde{\mathfrak{X}}_n^{(\epsilon)}; \epsilon n, \tilde{X}_n^{(\epsilon)}; \epsilon(n+1), x), dS) \\ &= P_\epsilon(\tilde{X}_n^{(\epsilon)}, dx) \int_{\mathcal{C}([[\epsilon n, \epsilon(n+1)], V])} K((\tilde{X}_n^{(\epsilon)}, x_{[\epsilon n, \epsilon(n+1)]}), dS) \mathbb{P}_{\epsilon n, \tilde{X}_n^{(\epsilon)}; \epsilon(n+1), x}(dx_{[\epsilon n, \epsilon(n+1)]}) \\ &= \int_{\mathcal{C}([[\epsilon n, \epsilon(n+1)], V])} K((\tilde{X}_n^{(\epsilon)}, x_{[\epsilon n, \epsilon(n+1)]}), dS) \delta_{x_{\epsilon(n+1)}}(dx) \mathbb{P}_{\epsilon n, \tilde{X}_n^{(\epsilon)}}(dx_{[\epsilon n, \epsilon(n+1)]}) \end{aligned}$$

where

- $K((S; x_{[\epsilon n, \epsilon(n+1)]}), \cdot)$  is the image of  $\mathbb{P}''_{S; x_{[\epsilon n, \epsilon(n+1)]}}[\cdot]$  by the mapping  $\psi \mapsto \psi^{-1}_{\epsilon n, \epsilon(n+1)}(S)$ ,
- $\mathbb{P}_{\epsilon n, x'}(\cdot)$  is the law of  $X_{[\epsilon n, \epsilon(n+1)]}$  associated to  $P$ , starting at time  $\epsilon n$  from  $x' \in V$ .

On the other hand, with arguments similar to those of Lemmas 6.4 and 6.5, we check that

$$\begin{aligned} & \mathbb{P}[(X_{\epsilon(n+1)}, \mathfrak{X}_{\epsilon(n+1)}) \in (dx, dS) | X_{[0, \epsilon n]}, \mathfrak{X}_{[0, \epsilon n]}] \tag{6.12} \\ &= \int_{\mathcal{C}([[\epsilon n, \epsilon(n+1)], V])} K((X_{[0, \epsilon n]}, x_{[\epsilon n, \epsilon(n+1)]}), dS) \delta_{x_{\epsilon(n+1)}}(dx) \mathbb{P}_{\epsilon n, X_{\epsilon n}}(dx_{[\epsilon n, \epsilon(n+1)]}) \end{aligned}$$

It follows that

$$\begin{aligned} & \mathbb{P}[(X_{\epsilon(n+1)}, \mathfrak{X}_{\epsilon(n+1)}) \in (dx, dS) | (X_{\epsilon m}, \mathfrak{X}_{\epsilon m})_{m \in [0, n]}] \\ &= \int_{\mathcal{C}([[\epsilon n, \epsilon(n+1)], V])} K((X_{[0, \epsilon n]}, x_{[\epsilon n, \epsilon(n+1)]}), dS) \delta_{x_{\epsilon(n+1)}}(dx) \mathbb{P}_{\epsilon n, X_{\epsilon n}}(dx_{[\epsilon n, \epsilon(n+1)]}) \end{aligned}$$

and this ends the proof of the equality in law of  $(\tilde{X}^{(\epsilon)}, \tilde{\mathfrak{X}}^{(\epsilon)})$  and  $(X^{(\epsilon)}, \mathfrak{X}^{(\epsilon)})$ . At least when  $\mathfrak{X}_0$  is deterministic, but this identity in law is next extended by integration with respect to  $\mathcal{L}(\mathfrak{X}_0)$ .

Taking into account Remark 5.6, Theorem 5.2 now implies that

$$\forall n \in \mathbb{Z}_+, \quad \mathcal{L}(X_{\epsilon n} | \mathfrak{X}_0, \mathfrak{X}_\epsilon, \dots, \mathfrak{X}_{\epsilon n}) = \Lambda(\mathfrak{X}_{\epsilon n}, \cdot) \tag{6.13}$$

If we had some time regularity for the process  $\mathfrak{X}$ , the announced result would follow by usual approximations. To go further, let us remark that in the above arguments, the time mesh  $0, \epsilon, 2\epsilon, \dots, n\epsilon$  can be replaced by any finite sequence  $t_0 \leq t_1 \leq t_2 \leq \dots \leq t_n$ . The Markov chain  $(X_{t_0}, X_{t_1}, X_{t_2}, \dots, X_{t_n})$  is no longer time homogeneous, but as pointed out in Remark 5.6, this is not crucial in the deduction that

$$\mathcal{L}(X_{t_n} | \mathfrak{X}_{t_0}, \mathfrak{X}_{t_1}, \mathfrak{X}_{t_2}, \dots, \mathfrak{X}_{t_n}) = \Lambda(\mathfrak{X}_{t_n}, \cdot) \tag{6.14}$$

Now fix  $t \geq 0$  and consider two bounded and measurable functions  $F : V \rightarrow \mathbb{R}$  and  $G : \mathfrak{S}^{[0, t]} \rightarrow \mathbb{R}$ . We want to show that

$$\mathbb{E}[F(X_t)G(\mathfrak{X}_{[0, t]})] = \mathbb{E}[\Lambda[F](\mathfrak{X}_t)G(\mathfrak{X}_{[0, t]})] \tag{6.15}$$

Due to the product measurable structure of  $\mathfrak{S}^{[0, t]}$ , there exists a sequence  $(s_n)_{n \in \mathbb{Z}_+}$  of distinct elements from  $[0, t]$  such that  $G(\mathfrak{X}_{[0, t]})$  only depends on the values  $(\mathfrak{X}_{s_n})_{n \in \mathbb{Z}_+}$ .

For  $N \in \mathbb{N}$ , consider

$$G_N((\mathfrak{X}_{s_n})_{n \in [0, N]}) := \mathbb{E}[G(\mathfrak{X}_{[0, t]}) | (\mathfrak{X}_{s_n})_{n \in [0, N]}]$$

The martingale convergence theorem implies that  $G_N((\mathfrak{X}_{s_n})_{n \in [0, N]})$  converges for large  $N$  toward  $G(\mathfrak{X}_{[0, t]})$  in  $L^2(\mathbb{P})$ , so to prove (6.15), it is sufficient to see that

$$\mathbb{E}[F(X_t)G_N((\mathfrak{X}_{s_n})_{n \in [0, N]})] = \mathbb{E}[\Lambda[F](\mathfrak{X}_t)G_N((\mathfrak{X}_{s_n})_{n \in [0, N]})] \tag{6.16}$$

Let  $t_0 \leq t_1 \leq t_2 \leq t_3 \leq \dots \leq t_{N+1}$  be the ordering of the elements  $s_0, s_1, s_2, \dots, s_N, t$ , (6.14) implies (6.16) via the conditioning by the sigma-field generated by  $(\mathfrak{X}_{s_n})_{n \in [0, N]}$ .  $\square$

Other parts of Theorem 5.2 can be extended to the continuous-time framework: of course under  $\mathbb{P}$ ,  $X$  remains a diffusion process associated to the semi-group  $P$ . The process  $\mathfrak{X} := (\mathfrak{X}_t)_{t \geq 0}$  is Markovian, but in general it will no longer be time-homogenous. The associated semi-group  $\mathfrak{P} := (\mathfrak{P}_{s, t})_{0 \leq s \leq t}$  is given by

$$\forall 0 \leq s \leq t, \forall S \in \mathfrak{S}, \quad \begin{cases} \mathfrak{P}_{s, t}(S, dS') & := \frac{\pi(S')}{\pi(S)} K_{s, t}(S, dS') \\ K_{s, t}(S, dS') & := \mathbb{P}^{(4)}[\mathfrak{X}_t \in dS' | \mathfrak{X}_s = S] \end{cases}$$

Furthermore, the process  $(X_t, \mathfrak{X}_t)_{t \geq 0}$  is Markovian, indeed, a version of this property was used in (6.12). Concerning the set  $W$  defined in (5.4), we get that for any sequence  $(t_n)_{n \in \mathbb{Z}_+}$  of non-negative times, we have

$$\mathbb{P}[\forall n \in \mathbb{Z}_+, (X_{t_n}, \mathfrak{X}_{t_n}) \in W] = 1 \tag{6.17}$$

To go further and deduce that  $W$  can be taken as state space of  $(X, \mathfrak{X})$  under  $\mathbb{P}$ , we would need further regularity conditions, e.g. that there is version of  $\mathfrak{X}$  which is continuous and  $W$  is closed.

As in Theorem 5.2, the initial  $\mathfrak{X}_0$  in Theorem 6.7 have positive weights with respect to  $\pi$ . We cannot go around this drawback by following the approach leading to Theorem 5.4, since we did not investigate the local association in the continuous time setting. Instead of trying to develop an alternative general approach to get a set-valued dual process starting from a singleton, we refer to the particular case presented in Subsection 7.1.

## 7 One-dimensional diffusion processes

As already alluded to, we would like to apply the analysis of the previous section to diffusion processes. Unfortunately and despite the works of Le Jan and Raimond [14, 15, 16, 17], the theory of stochastic flows has not been developed in the direction needed by our purposes. The next subsection presents a treatment of the Brownian case starting from 0 based on a stochastic flow due to Le Jan and Raimond [16], it leads to a segment-valued dual process directly coupled with the primal Brownian motion through its local time at 0. It turns out to be equivalent to the classical Pitman’s theorem [25], which is thus recovered in this way. In the second subsection, we show how to extend this analysis to simple one-dimensional diffusions, if we had at our disposal nice stochastic flows. Subsection 7.2 ends by a conjecture about the existence of the strange stochastic flows we would like to use, as a first step for one-dimensional processes. In Appendix A, we will take advantage of classical transformations of the state space to transfer the considerations of Subsection 7.2 to more general elliptic one-dimensional diffusions.

**7.1 The Brownian motion case**

Here we give a first illustration of how to put in practice the abstract considerations of the previous section.

Let  $X := (X_s)_{s \geq 0}$  be a Brownian motion and let  $P := (P_s)_{s \geq 0}$  be the associated semi-group, whose generator is half the Laplacian  $\partial^2/2$ . Note that the Lebesgue measure  $\lambda$  is reversible for  $P$ , we also interpret  $P$  as a self-adjoint semi-group on  $\mathbb{L}^2(\lambda)$ , so that  $P^* = P$ .

A priori we are not in the framework of Section 6, since  $\lambda$  cannot be renormalized into a probability measure. Nevertheless Theorem 6.7 can be extended to this situation when the elements of  $\mathfrak{S}$  are furthermore assumed to have a finite weight under  $\lambda$ . Indeed, in (6.5),  $\mathbb{P}^{(3)}$  must be replaced by the sigma-finite measure  $\mathbb{P}^{(4)} \otimes \lambda$  and the following expectations have to be changed into integrations with respect to the corresponding measures (the test functions have to be assumed to be integrable or non-negative). We end up with a probability measure  $\mathbb{P}$ , since in its construction,  $\mathbb{P}''$  is conditioned by the event  $\{X_0 \in \mathfrak{X}_0\}$ , which has the finite weight  $\lambda(\mathfrak{X}_0) \in (0, +\infty)$ .

Let  $B := (B_s)_{s \geq 0}$  be another Brownian motion, it is not important to specify its initial condition, since only the associated white noise will be needed. For fixed  $t \geq 0$ , we consider the following system of equations, for all  $y \in \mathbb{R}$ ,

$$\begin{cases} dY_s^{(t)}(y) &= -\text{sgn}(Y_s^{(t)}(y))dB_s^{(t)}, \quad \forall s \in [0, t] \\ Y_0^{(t)}(y) &= y \end{cases} \tag{7.1}$$

where  $\text{sgn}$  is the sign function on  $\mathbb{R}$  taking the value  $-1$  on  $(-\infty, 0]$  and  $1$  on  $(0, +\infty)$  and where  $B^{(t)} := (B_s^{(t)})_{s \in [0, t]} := (B_{t-s})_{s \in [0, t]}$  is the time-reversed process associated to  $B$  at time  $t \geq 0$ .

Le Jan and Raimond [16] provide a coalescing stochastic flow solution to (7.1), but it is non-Wiener, meaning that extra-randomness, in addition to the Brownian motion  $B$ , is necessary to its construction. For the general meaning of a (Wiener) coalescing stochastic flow solution, we refer to Le Jan and Raimond [15]. Define  $\psi := (\psi_{s,t}(y))_{(s,t,y) \in \Delta \times \mathbb{R}}$  via

$$\forall x \in \mathbb{R}, \forall 0 \leq s \leq t, \quad \psi_{s,t}(y) := Y_{t-s}^{(t)}(y) \tag{7.2}$$

(be careful that our convention for the direction of the time is reversed with respect to that of Le Jan and Raimond [15]:  $\psi_{s,t}$  should be seen as acting on the state space at time  $t$  toward the state space at time  $s$ ). The stochastic flow  $\psi$  is associated to  $P$ , as an immediate consequence of (7.1) and of Lévy’s characterization of the Brownian motion. Since the state space is the real line, the regularity of  $\psi$  can be made more precise, bringing us back to the assumptions of Section 6. Let  $\mathcal{R}(\mathbb{R})$  stands for the set of mapping from  $\mathbb{R}$  to  $\mathbb{R}$  which are non-decreasing and càdlàg.

**Lemma 7.1.** *There exists a version of  $\psi$  which is such that a.s., for any  $(s, t) \in \Delta$ ,  $\psi_{s,t} \in \mathcal{R}(\mathbb{R})$ .*

*Proof.* Due to the temporal continuity and coalescing property of the solution  $Y := (Y_s^{(t)}(x))_{(s,t) \in \Delta, x \in \mathbb{R}}$  of (7.1), we get that for any given  $x \leq y \in \mathbb{R}$ , we have  $Y_s^{(t)}(x) \leq Y_s^{(t)}(y)$  a.s. for any  $(s, t) \in \Delta$ . So we can extend  $(Y_s^{(t)}(x))_{(s,t) \in \Delta, x \in \mathbb{Q}}$  into  $\tilde{Y} := (\tilde{Y}_s^{(t)}(x))_{(s,t) \in \Delta, x \in \mathbb{R}}$  by

$$\forall (s, t) \in \Delta, \forall x \in \mathbb{R}, \quad \tilde{Y}_s^{(t)}(x) := \lim_{y \rightarrow x, y > x, y \in \mathbb{Q}} Y_s^{(t)}(y)$$

Let  $\tilde{\psi}$  be obtained from  $\tilde{Y}$  as in (7.2). By construction,  $\tilde{Y}$  is such that a.s. for all  $(s, t) \in \Delta$ ,  $\tilde{\psi}_{s,t} \in \mathcal{R}(\mathbb{R})$ . Furthermore, according to the point (e) in the Definition 1.6 of Le Jan and Raimond [15], we have that for any  $(s, t) \in \Delta$  and any  $x \in \mathbb{R}$ , a.s.  $\tilde{Y}_s^{(t)}(x) = Y_s^{(t)}(x)$ . It follows that  $\tilde{Y}$  is also a solution of (7.1) and thus  $\tilde{\psi}$  provides the wanted version of  $\psi$ .  $\square$

**Remark 7.2.** The validity of Lemma 7.1 is not related to the choice of  $\text{sgn}(0)$ . With a similar proof, one could deduce a version of  $\psi$  which is a.s. càglàd. In fact the definition of  $\text{sgn}(0)$  is irrelevant, as  $\mathbb{E}[\int_0^t \mathbb{1}_{\{0\}}(Y_s^{(t)}(y)) ds] = 0$ , for all  $y \in \mathbb{R}$ .

From now on, we will only consider a version of  $\psi$  as in Lemma 7.1. As announced in Remark 6.1,  $\mathcal{R}(\mathbb{R})$  is endowed with the Skorohod topology, which insures the measurability of the mapping  $\mathbb{R} \times \mathcal{R}(\mathbb{R}) \ni (x, \psi) \mapsto \psi(x)$ . To apply the results of the previous section, we must equally specify the space of nice subdomains  $\mathfrak{S}$ . As it will become apparent later, it is convenient to consider:

$$\mathfrak{S} := \{[a, b) : a < b \in \mathbb{R}\} \tag{7.3}$$

This set  $\mathfrak{S}$  is endowed with the topology inherited from  $\mathbb{R}^2$  and satisfies the properties required in Section 6. Indeed, let us check the following property, relating Lemma 7.1 to our choice of  $\mathfrak{S}$ .

**Lemma 7.3.** For any  $(s, t) \in \Delta$  the mapping

$$\mathfrak{S} \ni S \mapsto \psi_{s,t}^{-1}(S) \in \mathfrak{S} \sqcup \{\emptyset\} \tag{7.4}$$

is measurable.

*Proof.* First, for given  $(s, t) \in \Delta$ , we verify that for  $[a, b) \in \mathfrak{S}$ , the set  $\psi_{s,t}^{-1}([a, b))$  belongs to  $\mathfrak{S} \sqcup \{\emptyset\}$ . This is a consequence of  $\psi_{s,t} \in \mathcal{R}(\mathbb{R})$ : to see that  $\psi_{s,t}^{-1}([a, b))$  is a segment, let  $x \leq y$  belong to  $[a, b)$ . For  $z \in [x, y]$ , we have  $a \leq \psi_{s,t}(x) \leq \psi_{s,t}(z) \leq \psi_{s,t}(y) < b$ , so that  $z \in \psi_{s,t}^{-1}([a, b))$ . Next let  $(x_n)_{n \in \mathbb{Z}_+}$  be a decreasing family of elements from  $\psi_{s,t}^{-1}([a, b))$  converging toward some  $x \in \mathbb{R}$ . Then we have  $\lim_{n \rightarrow \infty} \psi_{s,t}(x_n) = \psi_{s,t}(x)$  and since the l.h.s. belongs to  $[a, b)$ , we deduce that  $x \in \psi_{s,t}^{-1}([a, b))$  and that the segment  $\psi_{s,t}^{-1}([a, b))$  is closed on the left side. To see that it is open on the right side, consider  $x \in \psi_{s,t}^{-1}([a, b))$ . As  $y > x$  decreases toward  $x$ , we have that  $\psi_{s,t}(y)$  converges toward  $\psi_{s,t}(x)$ , so that for  $y$  sufficiently close to  $x$ , we have  $\psi_{s,t}(x) \leq \psi_{s,t}(y) < b$ , i.e.  $y \in \psi_{s,t}^{-1}([a, b))$ .

It follows from these observations that for any  $x < y \in \mathbb{R}$ , we have  $\psi_{s,t}^{-1}([x, y)) = [a, b)$ , with

$$\begin{aligned} a &:= \inf\{u \in \mathbb{R} : \psi_{s,t}(u) \geq x\} \\ b &:= \inf\{u \in \mathbb{R} : \psi_{s,t}(u) \geq y\} \end{aligned}$$

Fix  $u < v \in \mathbb{R}$  and consider  $A := \{[a, b) : a \leq u, b > v\} \subset \mathfrak{S}$ . We compute that

$$\{S \in \mathfrak{S} : \psi_{s,t}^{-1}(S) \in A\} = \{[x, y) \in \mathfrak{S} : x \leq \psi_{s,t}(u) \text{ and } y > \psi_{s,t}(v)\}$$

It follows without difficulty that the mapping defined in (7.4) is measurable. □

Note that the Lebesgue measure  $\lambda$ , invariant for the Brownian semi-group  $P$ , takes positive and finite values on  $\mathfrak{S}$ . As in Subsection 3.1, we get that the corresponding conditioning kernel  $\Lambda$  can be seen as a Markov kernel from  $\mathfrak{S}$  to  $\mathbb{R}$ .

Fix  $t \geq 0$  and a Brownian trajectory  $X_{[0,t]}$ . Conditioning  $\psi$  by the event

$$\forall s \in [0, t], \quad \psi_{s,t}(X_t) = X_s \tag{7.5}$$

implies in particular that

$$\forall s \in [0, t], \quad dX_s^{(t)} = -\text{sgn}(X_s^{(t)})dB_s^{(t)} \tag{7.6}$$

but it is not clear what happens to the extra-randomness, since this Tanaka's stochastic differential equation does not admit a strong solution either (see for instance Exercise 1.19 of Chapter 9 from Revuz and Yor [28]). Nevertheless we deduce that

$$\forall s \in [0, t], \quad dB_s^{(t)} = -\text{sgn}(X_s^{(t)})dX_s^{(t)} \tag{7.7}$$

and it follows that the conditioned flow, denoted  $\varphi := (\varphi_{s,t}(y))_{(s,t,y) \in \Delta \times \mathbb{R}}$  in Section 6, is given by

$$\forall 0 \leq s \leq t, \forall z \in \mathbb{R}, \quad \varphi_{s,t}(z) := Z_{t-s}^{(t)}(z) \tag{7.8}$$

where

$$\begin{cases} dZ_s^{(t)}(z) &= \operatorname{sgn}(Z_s^{(t)}(z))\operatorname{sgn}(X_s^{(t)})dX_s^{(t)} \\ Z_0^{(t)}(z) &= z \end{cases} \tag{7.9}$$

This system is the same as (7.1), once we have replaced  $B^{(t)}$  by  $(-\int_0^s \operatorname{sgn}(X_v^{(t)}) dX_v^{(t)})_{s \in [0,t]}$  which is a standard Brownian motion. In particular the resolution of (7.9) also requires some extra-randomness, but this is not a shortcoming for our present purpose since we just need a solution. Indeed, for any given  $r > 0$ , assume that the initial law of  $X$  is  $\nu_r$ , the uniform distribution on  $[-r, r)$ . Define for any  $t \geq 0$ ,

$$\begin{cases} R_t^{\vee,r} &:= \inf \{z \in \mathbb{R} : Z_0^{(t)}(z) \leq r\} = \sup \{z \in \mathbb{R} : Z_0^{(t)}(z) < r\} \\ R_t^{\wedge,r} &:= \inf \{z \in \mathbb{R} : Z_0^{(t)}(z) \geq -r\} \\ \mathfrak{X}_t^{(r)} &:= [R^{\wedge,r}, R^{\vee,r}) = \varphi_{0,t}^{-1}([-r, r)) \end{cases} \tag{7.10}$$

where for the last equality we took into account the proof of Lemma 7.3. Remark that  $\mathfrak{X}^{(r)} := (\mathfrak{X}_t^{(r)})_{t \geq 0}$  remains non-trivial, namely that it never collapses to a singleton or to the empty set: this is a consequence of the fact that  $X_t \in \mathfrak{X}_t$  and of the structure of the elements of  $\mathfrak{S}$ . We are now in position to apply Theorem 6.7 to deduce:

**Theorem 7.4.** *For any  $r > 0$ , the process  $\mathfrak{X}^{(r)}$  is a set-valued dual for the Brownian motion  $X$  starting from  $\nu_r$ .*

We would like to let  $r$  go to zero to be able to get  $X$  starting from 0. It will also provide a more explicit set-valued dual. Indeed, assume that  $X$  starts from 0 and consider the coalescing flow  $\varphi$  defined by (7.8) and (7.9). By analogy with (7.10), define the process  $\mathfrak{X} := (\mathfrak{X}(t))_{t \geq 0}$  via

$$\forall t \geq 0, \quad \mathfrak{X}(t) := \varphi_{0,t}^{-1}(\{0\}) \tag{7.11}$$

We still have that for any  $t \geq 0$ ,  $\mathfrak{X}(t)$  is a segment closed on the left. We will see in the proof of Proposition 7.5 that for any  $t > 0$ , the right boundary of  $\mathfrak{X}(t)$  is open (but it is closed at time 0, since  $\mathfrak{X}(0) = \{0\}$ ). Anyway, the closure of  $\mathfrak{X}(t)$  is  $[R^\wedge, R^\vee]$  with

$$\begin{aligned} R_t^\vee &:= \sup\{z \in \mathbb{R} : Z_0^{(t)}(z) = 0\} \\ R_t^\wedge &:= \inf\{z \in \mathbb{R} : Z_0^{(t)}(z) = 0\} \end{aligned}$$

and these quantities can be described explicitly:

**Proposition 7.5.** *We have for any  $t \geq 0$ ,*

$$\begin{aligned} R_t^\vee &= L_t^0(X) + |X_t| \\ R_t^\wedge &= -(L_t^0(X) + |X_t|) \end{aligned}$$

where  $L^0(X) := (L_t^0(X))_{t \geq 0}$  is the local time of  $X$  at 0.

*Proof.* Due to the fact that  $\varphi$  is a coalescing flow and that  $X_0 = 0$  we have

$$\begin{aligned} R_t^\vee &:= \sup\{z \in \mathbb{R} : \exists s \in [0, t] \text{ with } Z_s^{(t)}(z) = X_s^{(t)}\} \\ R_t^\wedge &:= \inf\{z \in \mathbb{R} : \exists s \in [0, t] \text{ with } Z_s^{(t)}(z) = X_s^{(t)}\} \end{aligned}$$

From (7.9), we have for any  $z \in \mathbb{R}$ ,

$$\forall s \in [0, t], \quad \text{sgn}(Z_s^{(t)}(z))dZ_s^{(t)}(z) = \text{sgn}(X_s^{(t)})dX_s^{(t)} \quad (7.12)$$

and Tanaka's formula (see e.g. Chapter 6 of Revuz and Yor [28]) implies that

$$\forall s \in [0, t], \quad d|Z_s^{(t)}(z)| - dL_s^0(Z^{(t)}(z)) = d|X_s^{(t)}| - dL_s^0(X^{(t)}) \quad (7.13)$$

where  $L^0(Z^{(t)}(z)) := (L_s^0(Z^{(t)}(z)))_{s \in [0, t]}$  and  $L^0(X^{(t)}) := (L_s^0(X^{(t)}))_{s \in [0, t]}$  are respectively the local times of  $Z^{(t)}(z)$  and  $X^{(t)}$  at 0.

Fix  $z > x := X_0^{(t)} = X_t$ , assume that  $x \geq 0$  and define

$$\tau_z := \inf\{s \in [0, t] : Z_s^{(t)}(z) = 0\}$$

(with the convention that  $\inf(\emptyset) = +\infty$ ). Consider the case  $\tau_z < +\infty$  and let us show that  $X_{\tau_z}^{(t)} = 0$ . The argument is by contradiction, assuming that  $X_{\tau_z}^{(t)} \neq 0$ . Define

$$\gamma := \sup\{s \in [0, \tau_z] : X_s^{(t)} = 0\}$$

with the convention that  $\gamma = -\infty$  if the set in the r.h.s. is empty. For  $s \in [0 \vee \gamma, \tau_z)$ , we have  $Z_s^{(t)}(z) > 0$ ,  $L_s^0(Z^{(t)}(z)) = 0$  and  $L_s^0(X^{(t)}) = L_{0 \vee \gamma}^0(X^{(t)})$ . It follows from (7.13) that

$$\begin{aligned} Z_{0 \vee \gamma}^{(t)}(z) - |X_{0 \vee \gamma}^{(t)}| &= Z_{\tau_z}^{(t)}(z) - |X_{\tau_z}^{(t)}| \\ &= -|X_{\tau_z}^{(t)}| \\ &< 0 \end{aligned}$$

i.e.  $Z_{0 \vee \gamma}^{(t)}(z) < |X_{0 \vee \gamma}^{(t)}|$ . Note that depending on  $\gamma \in [0, t]$  or  $\gamma = -\infty$ , we have  $X_{0 \vee \gamma}^{(t)} = 0$  or  $X_{0 \vee \gamma}^{(t)} > 0$  (since when  $\gamma = -\infty$ ,  $X^{(t)}$  keeps the same sign on  $[0, \tau_z]$ , which is the sign of  $x$ ). In any case, we end up with  $Z_{0 \vee \gamma}^{(t)}(z) < X_{0 \vee \gamma}^{(t)}$ , which is in contradiction with the fact that  $Z^{(t)}(x)$  remains above  $X^{(t)}$  by the coalescing property.

Define

$$\sigma_z := \inf\{s \in [0, t] : Z_s^{(t)} = X_s^{(t)}\}$$

Due to the fact that  $X_{\tau_z}^{(t)} = 0$  when  $\tau_z < +\infty$ , we deduce that  $\sigma_z \leq \tau_z$ . Integrating (7.13) between the times 0 and  $\sigma_z$ , we thus get

$$\begin{aligned} \sigma_z < +\infty &\Rightarrow |Z_{\sigma_z}^{(t)}| - z = |X_{\sigma_z}^{(t)}| - x - L_{\sigma_z}^0(X^{(t)}) \\ &\Rightarrow z = x + L_{\sigma_z}^0(X^{(t)}) \end{aligned} \quad (7.14)$$

Since

$$R_t^\vee = \max\{z \geq x : \sigma_z < +\infty\}$$

we get that

$$R_t^\vee = x + L_{\sigma_{R_t^\vee}}^0(X^{(t)}) \quad (7.15)$$

Let us show that  $\sigma := \sigma_{R_t^\vee}$  is equal to  $t$ . Again the argument is by contradiction: assume that  $\sigma < t$ . Define

$$\tau := \inf\{s \in [\sigma, t] : X_s^{(t)} = 0\}$$

Since 0 is an accumulation point of  $\{s \in [0, t] : X_s = 0\}$ , we get that  $\tau < t$ . Let us show that  $\mathfrak{X}_{t-\tau}$  contains a right neighborhood of 0, namely that for  $z$  sufficiently close to  $0_+$ ,

we have that  $Z^{(t-\tau)}(z)$  coalesces with  $X^{(t-\tau)}$ . Indeed, if this was not true, according to the first part of the proof above, we would have that  $Z^{(t-\tau)}(z)$  does not touch 0 for any  $z > 0$ . From (7.9), we deduce that

$$\forall z \in (0, +\infty), \forall s \in [0, \tau], \quad Z_s^{(t-\tau)}(z) = z + \beta_s$$

where  $\beta := (\beta_s)_{s \in [0, \tau]}$  is the Brownian motion defined by

$$\forall s \in [0, \tau], \quad \beta_s := \int_0^s \text{sign}(X_u^{(t-\tau)}) dX_u^{(t-\tau)}$$

This shows that  $Z^{(t-\tau)}(z)$  touches 0 for any  $z \in (0, -\inf_{s \in [0, \tau]} \beta_s]$ , a contradiction.

Using the right continuity of our stochastic flows, we get

$$\begin{aligned} \lim_{z \rightarrow R_t^\vee} Z_\tau^{(t)}(z) &= Z_\tau^{(t)}(R_t^\vee) \\ &= X_\tau^{(t)} \\ &= 0 \end{aligned}$$

Thus for  $z$  in a right neighborhood of  $R_t^\vee$ , we get that  $Z_\tau^{(t)}(z)$  belongs to  $\mathfrak{X}_{t-\tau}$  and by consequence that  $z \in \mathfrak{X}_t$ . This is in contradiction with the definition of  $R_t^\vee$  and we finally conclude that  $\sigma = t$ .

Coming back to (7.15), we get that

$$\begin{aligned} R_t^\vee &= x + L_t^0(X^{(t)}) \\ &= X_t + L_t^0(X) \end{aligned} \tag{7.16}$$

under the previous assumption that  $X_t = x \geq 0$ .

When  $X_t < 0$ , consider

$$\varsigma := \inf\{s \in [0, t] : X_s^{(t)} = 0\}$$

We deduce from (7.12) that

$$\forall s \in [0, \varsigma], \quad Z_s^{(t)}(-x) = -X_s^{(t)}(x)$$

In particular, we get

$$R_t^\vee = \sup\{z \geq -x : Z_\varsigma^{(t)} \in \mathfrak{X}_{t-\varsigma}\}$$

Note that for all  $z \geq -x$ , we have

$$\forall s \in [0, \varsigma], \quad Z_s^{(t)}(z) - z = Z_s^{(t)}(-x) - (-x)$$

so that

$$\begin{aligned} R_t^\vee &= -x + R_{t-\varsigma}^\vee \\ &= |X_t| + R_{t-\varsigma}^\vee \end{aligned} \tag{7.17}$$

Using (7.16) with  $t$  replaced by  $t - \varsigma$ , we get

$$\begin{aligned} R_{t-\varsigma}^\vee &= X_{t-\varsigma} + L_{t-\varsigma}^0(X) \\ &= 0 + L_t^0(X) \end{aligned}$$

Recalling (7.17), we conclude to the validity, in all cases, of

$$R_t^\vee = |X_t| + L_t^0(X)$$

By symmetry, the expression for  $R_t^\wedge$  is easily deduced:

$$\begin{aligned} R_t^\wedge &:= -\sup\{-z \in \mathbb{R} : \exists s \in [0, t] \text{ with } Z_s^{(t)}(z) = X_s^{(t)}\} \\ &= -\sup\{z \in \mathbb{R} : \exists s \in [0, t] \text{ with } Z_s^{(t)}(-z) = X_s^{(t)}\} \\ &= -\sup\{z \in \mathbb{R} : \exists s \in [0, t] \text{ with } Z_s^{(t)}(z) = -X_s^{(t)}\} \\ &= -(|-X_t| + L_t^0(-X)) \\ &= -(|X_t| + L_t^0(X)) \quad \square \end{aligned}$$

In particular, we get that for any  $t > 0$ ,  $\mathfrak{X}_t \in \mathfrak{S}$  and is not reduced to a singleton. This property is in fact sufficient to deduce the following variant of Pitman's theorem.

**Theorem 7.6.** *The process  $\mathfrak{X} = ([-(L_t^0(X) + |X_t|), L_t^0(X) + |X_t|])_{t \geq 0}$  is a set-valued dual for the Brownian motion  $X$  starting from 0.*

*Proof.* The property

$$\forall t \geq 0, \quad \mathcal{L}(\mathfrak{X}_{[0,t]}|X) = \mathcal{L}(\mathfrak{X}_{[0,t]}|X_{[0,t]})$$

is obvious from the explicit expression for  $\mathfrak{X}$ .

The important point is to prove that

$$\forall t \geq 0, \quad \mathcal{L}(X_t|\mathfrak{X}_{[0,t]}) = \Lambda(\mathfrak{X}_t, \cdot) \tag{7.18}$$

To take advantage from Theorem 7.4, enlarge the underlying probability space for  $X$  so that it contains a random variable  $U$  independent from  $X$  and uniformly distributed on  $[0, 1]$ . For any  $n \in \mathbb{Z}_+$ , let  $t_n := 1/(n + 1)$  and define the random variable  $U_n \in (|X_{t_n}|, +\infty)$  by

$$\int_{|X_{t_n}|}^{U_n} u \exp\left(\frac{X_{t_n}^2 - u^2}{2t_n}\right) \frac{du}{t_n} = U \tag{7.19}$$

The interest of  $U_n$  is that, denoting  $\mathfrak{X}_{t_n}^{(n)} := [-U_n, U_n] \in \mathfrak{S}$ , the law of  $(X_{t_n}, \mathfrak{X}_{t_n}^{(n)})$  on  $\mathbb{R} \times \mathfrak{S}$  is given by

$$\mathbb{P}[X_{t_n} \in dx, \mathfrak{X}_{t_n}^{(n)} \in dS] = \nu^{(n)}(dS)\Lambda(S, dx) \tag{7.20}$$

where  $\nu^{(n)}$  is the probability distribution on  $\mathfrak{S}_{\text{sym}} := \{-u, u : u \geq 0\} \subset \mathfrak{S}$  described by

$$\nu^{(n)}(d[-u, u]) := 2u^2 \exp(-u^2/(2t_n))du/\sqrt{2\pi t_n^3}$$

Indeed, we compute that

$$\begin{aligned} \int_{\mathfrak{S}_{\text{sym}}} \nu^{(n)}(dS)\Lambda(S, dx) &= \exp(-x^2/(2t_n))dx/\sqrt{2\pi t_n} \\ &= \mathbb{P}[X_{t_n} \in dx] \end{aligned}$$

and (7.20) is a consequence of Bayes' formula.

Shifting the origin of time to  $t_n$ , we apply the considerations preceding Theorem 7.4 to the trajectory  $X_{[t_n, +\infty)}$  and to the initial set-valued variable  $\mathfrak{X}_{t_n}^{(n)}$  to construct

$$\begin{aligned} \forall t \geq t_n, \quad \mathfrak{X}_t^{(n)} &:= \varphi_{t_n, t}^{-1}(\mathfrak{X}_{t_n}^{(n)}) \\ &= \{z \in \mathbb{R} : \varphi_{t_n, t}(z) \in [-U_n, U_n]\} \end{aligned}$$

According to Theorem 7.4, we have

$$\forall t \geq t_n, \quad \mathcal{L}(X_t|\mathfrak{X}_{[t_n, t]}^{(n)}) = \Lambda(\mathfrak{X}_{t_n}^{(n)}, \cdot) \tag{7.21}$$

For any  $t > 0$  and  $n \in \mathbb{Z}_+$ , denote

$$[-U_-^{(n)}(t), U_+^{(n)}(t)] := \mathfrak{X}_t^{(n)}$$

According to Lemma 7.7 below, we have a.s.

$$\begin{cases} \lim_{n \rightarrow \infty} U_-^{(n)}(t) = R_t^\vee \\ \lim_{n \rightarrow \infty} U_+^{(n)}(t) = R_t^\vee \end{cases} \tag{7.22}$$

Consider  $p \in \mathbb{N}$ ,  $0 < s_1 < s_2 < \dots < s_p = t$  and  $g_1, g_2, \dots, g_p$  some continuous and bounded functions from  $\mathbb{R}^2$  to  $\mathbb{R}$ . We associate to these ingredients the mapping

$$G : \mathfrak{S}^{(0,t]} \ni ([a_s, b_s])_{s \in (0,t]} \mapsto g_1(a_{s_1}, b_{s_1})g_2(a_{s_2}, b_{s_2}) \cdots g_p(a_{s_p}, b_{s_p})$$

Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a measurable and bounded function. From (7.22), considered with  $t$  replaced by the  $s_1, s_2, \dots, s_p$ , we get the a.s. convergence of  $G(\mathfrak{X}_{[t_n,t]}^{(n)})$  toward  $G(\mathfrak{X}_{[0,t]})$  for  $n$  large. We deduce that

$$\lim_{n \rightarrow \infty} \mathbb{E}[h(X_t)G(\mathfrak{X}_{[t_n,t]}^{(n)})] = \mathbb{E}[h(X_t)G(\mathfrak{X}_{[0,t]})]$$

Next, taking into account that  $\mathfrak{X}_t \in \mathfrak{S}$ , we also deduce from (7.22) the a.s. convergence of  $\Lambda[h](\mathfrak{X}_t^{(n)})$  toward  $\Lambda[h](\mathfrak{X}_t)$  and consequently

$$\lim_{n \rightarrow \infty} \mathbb{E}[\Lambda[h](\mathfrak{X}_t^{(n)})G(\mathfrak{X}_{[t_n,t]}^{(n)})] = \mathbb{E}[\Lambda[h](X_t)G(\mathfrak{X}_{[0,t]})]$$

It follows from (7.21) that

$$\mathbb{E}[h(X_t)G(\mathfrak{X}_{[0,t]})] = \mathbb{E}[\Lambda[h](X_t)G(\mathfrak{X}_{[0,t]})]$$

Since this is true for all  $h$  and  $G$  as above and that  $\mathfrak{X}_0 = \{0\}$  is deterministic, we get (7.18). □

**Lemma 7.7.** *For any  $t > 0$ , the convergences (7.22) are satisfied a.s.*

*Proof.* Since  $\mathfrak{X}_t^{(n)} = [-U_-^{(n)}(t), U_+^{(n)}(t)]$  and  $\mathfrak{X} = [-R_t^\vee, R_t^\vee)$ , it is sufficient to see that for any  $z \in \mathbb{R}$

$$z \notin \mathfrak{X}_t \Rightarrow \lim_{n \rightarrow \infty} \mathbb{1}_{\mathfrak{X}_t^{(n)}}(z) = 0 \tag{7.23}$$

$$z \in (-R_t^\vee, R_t^\vee) \Rightarrow \lim_{n \rightarrow \infty} \mathbb{1}_{\mathfrak{X}_t^{(n)}}(z) = 1 \tag{7.24}$$

Let us come back to (7.19), which is equivalent to

$$\exp(-U_n^2/(2t_n)) = (1 - U) \exp(-(X_{t_n})^2/(2t_n))$$

namely

$$U_n^2 = (X_{t_n})^2 - 2t_n \ln(1 - U)$$

where we see that  $\lim_{n \rightarrow \infty} U_n = 0$ .

We begin by showing (7.23): when  $z \notin \mathfrak{X}_t$ , we have  $\varphi_{0,t}(z) \neq 0$ . So since  $\lim_{n \rightarrow \infty} \varphi_{t_n,t}(z) = \varphi_{0,t}(z)$ , it appears that for  $n \in \mathbb{Z}_+$  large enough, we cannot have  $\varphi_{t_n,t}(z) \in [-U_n, U_n]$ , i.e.  $z$  ends up not belonging to  $\mathfrak{X}_t^{(n)}$ .

We now come to (7.24). From (7.14) we deduce that for  $z$  in the open set  $(-R_t^\vee, R_t^\vee)$ , we have  $\sigma_z < t$ , meaning that the trajectory  $[0, t] \ni s \mapsto \varphi_{s,t}(z)$  is equal to  $X_s$  for  $s$  small enough. Thus for  $n$  large enough, we have

$$\begin{aligned} |\varphi_{t_n,t}(z)| &= |X_{t_n}| \\ &< \sqrt{(X_{t_n})^2 - 2t_n \ln(1 - U)} \\ &= U_n \end{aligned}$$

so that  $z$  belongs to  $\mathfrak{X}_t^{(n)}$ . □

**Remark 7.8.** Define

$$\forall t \geq 0, \quad M_t := \max\{X_s : s \in [0, t]\}$$

The classical Pitman's theorem [25] states that the process  $\tilde{\mathfrak{X}} = ([-(2M_t - X_t), 2M_t - X_t])_{t \geq 0}$  is a set-valued dual for the Brownian motion  $X$  starting from 0 and that  $(2M_t - X_t)_{t \geq 0}$  is a Bessel-3 process. The process  $(L_t^0(X) + |X_t|)_{t \geq 0}$  is also a Bessel-3 process, as a consequence of Lévy's theorem (see e.g. Theorem 2.3 of Chapter 6 of Revuz and Yor [28]).

A segment-valued process  $(\mathfrak{Q}_t)_{t \geq 0}$  is said to be more  $\lambda$ -**expansive** than another segment-valued process  $(\hat{\mathfrak{Q}}_t)_{t \geq 0}$ , when for any fixed  $t \geq 0$ , the volume  $\lambda(\hat{\mathfrak{Q}}_t)$  is stochastically dominated by  $\lambda(\mathfrak{Q}_t)$ . Thus the processes  $\mathfrak{X}$  and  $\tilde{\mathfrak{X}}$  are iso- $\lambda$ -expansive. It can be proven, by extending the arguments of the Remark 2.39 of Diaconis and Fill [10] (see also Fill and Lyzinski [11] or [21]), that if  $(\mathfrak{Q}_t)_{t \geq 0}$  is a set-valued dual for the Brownian motion  $X$  starting from 0, then it is less expansive than  $\tilde{\mathfrak{X}}$ , i.e.  $\mathfrak{X}$  and  $\tilde{\mathfrak{X}}$  are two examples of the most expansive set-valued duals for  $X$ .

Let us deduce more precisely the classical Pitman's theorem [25] from Theorem 7.6. With the notations of the above remark, Lévy's theorem asserts the identity in law

$$(M_t - X_t, M_t)_{t \geq 0} \stackrel{\mathcal{L}}{=} (|X_t|, L_t^0(X))_{t \geq 0}$$

We infer for any  $t \geq 0$  the identity in law

$$\begin{aligned} ((L^0(X) + |X|)_{[0,t]}, |X_t|) &\stackrel{\mathcal{L}}{=} ((2M - X)_{[0,t]}, M_t - X_t) \\ &= ((2M - X)_{[0,t]}, (2M_t - X_t)/2 - X_t/2) \end{aligned}$$

From Theorem 7.6, the distribution of  $|X_t|$  knowing  $(L^0(X) + |X|)_{[0,t]}$  is the uniform distribution over  $[0, L_t^0(X) + |X_t|]$ . It follows that the distribution of  $(2M_t - X_t)/2 - X_t/2$  knowing  $(2M - X)_{[0,t]}$  is the uniform distribution over  $[0, 2M_t - X_t]$ . To get Pitman's theorem, it remains to note that for any constant  $r \geq 0$ , when  $U$  is a random variable such that  $r/2 - U/2$  is uniformly distributed on  $[0, r]$ , then  $U$  is uniformly distributed on  $[-r, r]$ .

These arguments can be reversed to conversely deduce Theorem 7.6 from Pitman's theorem. This implication is succinctly mentioned by Yor [32] (at the end of page 4).

**Remark 7.9.** Despite the set-valued dual processes  $\mathfrak{X} := ([-(L_t^0(X) + |X_t|), L_t^0(X) + |X_t|])_{t \geq 0}$  and  $\tilde{\mathfrak{X}} := ([-(2M_t - X_t), 2M_t - X_t])_{t \geq 0}$  are iso- $\lambda$ -expansive, some of their features are quite different: the Brownian motion  $X$  never hits the boundary of  $\mathfrak{X}$  except at time 0 (since for  $t > 0$ , we have  $L_t^0(X) > 0$ ), while  $X$  recurrently hits the upper boundary of  $\tilde{\mathfrak{X}}$ .

An potential advantage of the stochastic flow approach to set-valued dual processes is its expected flexibility in the choice of the stochastic flow. In the next section, we will present a conjectural but promising stochastic flow associated to the Brownian motion, which should enable to directly recover the classical Pitman theorem by mimicking the random mapping proof considered in the finite setting. But one can imagine a lot of other examples, here is another one, which is a Wiener solution to a system of coalescing stochastic flow equations.

Let  $B := (B_s)_{s \geq 0}$  and  $W := (W_s)_{s \geq 0}$  be two independent Brownian motions. For fixed  $t \geq 0$ , we consider the following system of equations, for any  $y \in \mathbb{R}$ ,

$$\begin{cases} dY_s^{(t)}(y) &= -\text{sgn}(Y_s^{(t)}(y) + W_s^{(t)})dB_s^{(t)}, \quad \forall s \in [0, t] \\ Y_0^{(t)}(y) &= y \end{cases} \quad (7.25)$$

where  $W^{(t)} := (W_s^{(t)})_{s \in [0,t]} := (W_{t-s})_{s \in [0,t]}$  and  $B^{(t)} := (B_s^{(t)})_{s \in [0,t]} := (B_{t-s})_{s \in [0,t]}$  are time-reversed processes. Again define  $\psi := (\psi_{s,t}(y))_{(s,t,y) \in \Delta \times \mathbb{R}}$  via

$$\forall x \in \mathbb{R}, \forall 0 \leq s \leq t, \quad \psi_{s,t}(y) := Y_{t-s}^{(t)}(y) \tag{7.26}$$

With the help of Le Jan and Raimond [17], we get

**Proposition 7.10.** *There exists a Wiener solution of (7.25) such that  $\psi$  is coalescing stochastic flow associated to  $P$  and such that a.s., for any  $(s, t) \in \Delta$ ,  $\psi_{s,t} \in \mathcal{R}(\mathbb{R})$ .*

To be a **Wiener** solution of (7.25) means that the filtration generated by the stochastic flow is included into the filtration generated by the white noises associated to  $B$  and  $W$ .

*Proof.* Consider  $Z_- := (Z_-(s))_{s \geq 0}$  and  $Z_+ := (Z_+(s))_{s \geq 0}$  two independent Brownian motions (not assumed to be standard). Le Jan and Raimond [17] show there is coalescing stochastic flow  $\xi := (\xi_{u,v}(x))_{(u,v,x) \in \Delta \times \mathbb{R}}$  solution to the following system of equations, for any  $x \in \mathbb{R}$  and  $0 \leq u \leq v$ ,

$$\begin{cases} d_v \xi_{u,v}(x) &= \mathbb{1}_{\xi_{u,v} < 0} dZ_-(v) + \mathbb{1}_{\xi_{u,v} \geq 0} dZ_+(v) \\ \xi_{u,u}(x) &= x \end{cases} \tag{7.27}$$

Furthermore the coalescing stochastic flow  $\xi$  is a.s. unique and a Wiener solution: it is constructed without resorting to extra-randomness outside  $Z_-$  and  $Z_+$ . Fix  $t \geq 0$  and consider the two independent Brownian motions  $Z_-^{(t)}$  and  $Z_+^{(t)}$  over the time interval  $[0, t]$  given by

$$\forall s \in [0, t], \quad \begin{cases} Z_+^{(t)}(s) &= \frac{W_s^{(t)} - B_s^{(t)}}{\sqrt{2}} \\ Z_-^{(t)}(s) &= \frac{W_s^{(t)} + B_s^{(t)}}{\sqrt{2}} \end{cases} \tag{7.28}$$

Let  $(\xi_{u,v}^{(t)}(x))_{(u,v,x) \in \Delta_{[0,t]} \times \mathbb{R}}$  be the corresponding solution of (7.27) and define

$$\forall y \in \mathbb{R}, \forall s \in [0, t], \quad Y_s^{(t)}(y) := \sqrt{2} \xi_{0,s}^{(t)} \left( \frac{y + W_t}{\sqrt{2}} \right) - W_s^{(t)}$$

Via immediate substitution, we check that  $(Y_s^{(t)}(y))_{s \in [0,t], y \in \mathbb{R}}$  is a solution of (7.25).

Let  $t \geq 0$  be a free variable again. From (7.26), we deduce a family of random variables  $\psi := (\psi_{s,t}(y))_{(s,t,y) \in \Delta \times \mathbb{R}}$ . To check that it is a stochastic flow, let us first remark that

$$\forall x \in \mathbb{R}, \forall t \geq 0, \forall s \in [0, t], \forall v \in [0, s], \quad \xi_{0,s-v}^{(s)}(\xi_{0,t-s}^{(t)}(x)) = \xi_{0,t-v}^{(t)}(x) \tag{7.29}$$

Indeed, consider for any fixed  $x \in \mathbb{R}$ ,  $t \geq 0$  and  $s \in [0, t]$ ,

$$\forall u \in [0, s], \quad \begin{cases} \chi_u &:= \xi_{0,u}^{(s)}(\xi_{0,t-s}^{(t)}(x)) \\ \zeta_u &:= \xi_{0,t-s+u}^{(t)}(x) \end{cases}$$

By definition, we have

$$\begin{aligned} d\chi_u &= \mathbb{1}_{\chi_u < 0} dZ_-^{(s)}(u) + \mathbb{1}_{\chi_u \geq 0} dZ_+^{(s)}(u) \\ &= \mathbb{1}_{\chi_u < 0} dZ_-(s-u) + \mathbb{1}_{\chi_u \geq 0} dZ_+(s-u) \\ &= \mathbb{1}_{\chi_u < 0} dZ_-^{(t)}(t-s+u) + \mathbb{1}_{\chi_u \geq 0} dZ_+^{(t)}(t-s+u) \end{aligned}$$

where  $Z_{\pm}$  are defined as in (7.28), with the exponents  $(t)$  removed. Thus  $(\chi_u)_{u \in [0,s]}$  and  $(\zeta_u)_{u \in [0,s]}$  satisfy the same evolution equation. Since we also have  $\chi_0 = \xi_{0,t-s}^{(t)}(x) = \zeta_0$ , we get (7.29) from the uniqueness result of Le Jan and Raimond [17] (see also Prokaj [26]).

Let us deduce from (7.29) the flow property of  $\psi$ . Indeed, we have for  $y \in \mathbb{R}$ ,  $t \geq 0$ ,  $s \in [0, t]$  and  $v \in [0, s]$ ,

$$\begin{aligned} \psi_{v,s}(\psi_{s,t}(y)) &= Y_{s-v}^{(s)}(Y_{t-s}^{(t)}(y)) \\ &= \sqrt{2}\xi_{0,s-v}^{(s)}\left(\frac{Y_{t-s}^{(t)}(y) + W_s}{\sqrt{2}}\right) - W_{s-v}^{(s)} \\ &= \sqrt{2}\xi_{0,s-v}^{(s)}\left(\frac{\sqrt{2}\xi_{0,t-s}^{(t)}((y + W_t)/\sqrt{2}) - W_{t-s}^{(t)} + W_s}{\sqrt{2}}\right) - W_v \\ &= \sqrt{2}\xi_{0,s-v}^{(s)}\left(\xi_{0,t-s}^{(t)}\left(\frac{y + W_t}{\sqrt{2}}\right)\right) - W_v \\ &= \sqrt{2}\xi_{0,t-v}^{(t)}\left(\frac{y + W_t}{\sqrt{2}}\right) - W_{t-v}^{(t)} \\ &= Y_{t-v}^{(t)}(y) \\ &= \psi_{v,t}(y) \end{aligned}$$

Finally, the association of  $\psi$  to  $P$  is an immediate consequence of (7.25) and of Lévy's characterization of the Brownian motion and the last assertion of the lemma is proven as in Lemma 7.1.  $\square$

Most of the previous arguments for the stochastic flow defined in (7.2) can now be extended to the stochastic flow considered in (7.26). E.g. fix  $t \geq 0$  and a Brownian trajectory  $X_{[0,t]}$ . Conditioning  $\psi$  by the event described in (7.5) amounts to asking that  $B_{[0,t]}$  satisfies

$$\forall s \in [0, t], \quad dX_s^{(t)} = -\text{sgn}(X_s^{(t)} + W_s^{(t)})dB_s^{(t)}$$

Indeed, Prokaj [26] has shown (taking into account a transformation similar to that used in the proof of Proposition 7.10) that this equation in  $X^{(t)}$  admits a unique strong solution. We get that

$$\forall s \in [0, t], \quad dB_s^{(t)} = -\text{sgn}(X_s^{(t)} + W_s^{(t)})dX_s^{(t)}$$

and it follows that the conditioned flow  $\varphi := (\varphi_{s,t}(y))_{(s,t,y) \in \Delta \times \mathbb{R}}$  is given by

$$\forall 0 \leq s \leq t, \forall z \in \mathbb{R}, \quad \varphi_{s,t}(z) := Z_{t-s}^{(t)}(z)$$

where

$$\begin{cases} dZ_s^{(t)}(z) &= \text{sgn}(Z_s^{(t)}(z) + W_s^{(t)})\text{sgn}(X_s^{(t)} + W_s^{(t)})dX_s^{(t)} \\ Z_0^{(t)}(z) &= z \end{cases}$$

Then Theorem 7.4 is still valid, with  $\mathfrak{X}^{(r)}$  defined as in (7.10), for any  $r > 0$ . The troubles begin with the process  $\widehat{\mathfrak{X}}$  defined as in (7.11), for which we did not find an explicit expression. Nevertheless we believe that this process  $\widehat{\mathfrak{X}}$  is still a set-valued dual for the Brownian motion  $X$  starting from 0 and not reduced to a singleton, for all positive times. Furthermore, we think that this  $\widehat{\mathfrak{X}}$  is strictly less  $\lambda$ -expansive (in the sense of Remark 7.8) than the process  $\mathfrak{X}$  described in Theorem 7.6.

More generally, for any  $\eta > 0$  it should be possible to solve the coalescing stochastic flow equation

$$\forall y \in \mathbb{R}, \quad \begin{cases} dY_s^{(t)}(y) &= -\text{sgn}(Y_s^{(t)}(y) + \eta W_s^{(t)})dB_s^{(t)} \\ Y_0^{(t)}(y) &= y \end{cases} \tag{7.30}$$

and to show that the corresponding segment-valued are less and less  $\lambda$ -expansive as  $\eta$  increases (see [21] for certain families of segment-valued dual processes satisfying this monotonicity property, with  $\lambda$  replaced by the underlying invariant probability). Unfortunately and despite the work of Prokaj [26] giving us a strong solution  $Y^{(t)}(y)$  for any fixed  $y \in \mathbb{R}$ , we did not find a reference insuring the existence of a coalescing stochastic flow solution for (7.30), for  $\eta \in (0, +\infty) \setminus \{1\}$ .

**7.2 Conjectures about one-dimensional diffusions processes**

Here we propose an extension of Pitman’s theorem [25] (see also Rogers and Pitman [29]) to simple one-dimensional diffusion processes. The considerations of this subsection remain hypothetical since they assume the existence of convenient coalescing stochastic flows, see in particular Conjecture 7.14.

Let be given a smooth and bounded mapping  $b : \mathbb{R} \rightarrow \mathbb{R}$ . We consider the stochastic differential equation

$$dX_t = dW_t + b(X_t)dt \tag{7.31}$$

where  $W := (W_t)_{t \geq 0}$  is a standard Brownian motion.

The process  $X$  is reversible with respect to the measure  $\pi$ , whose density, still denoted  $\pi$ , with respect to the Lebesgue measure on  $\mathbb{R}$  is given by

$$\forall x \in \mathbb{R}, \quad \pi(x) := \exp\left(2 \int_0^x b(y) dy\right) \tag{7.32}$$

Again the measure  $\pi$  gives an infinite weight to  $\mathbb{R}$  and we must take the same precautions as in the previous subsection.

Let  $P := (P_t)_{t \geq 0}$  be the semi-group associated to  $X$ , it is self-adjoint in  $\mathbb{L}^2(\pi)$ , so that  $P^* = P$ . We renew Definition (7.3) for the set of nice subsets. The kernel  $\Lambda$  is still Markovian from  $\mathfrak{G}$  to  $\mathbb{R}$ , corresponding to the elementary conditioning operation under  $\pi$ .

Let  $B := (B_s)_{s \geq 0}$  be another Brownian motion. For fixed  $x_0 \in \mathbb{R}$  and  $t \geq 0$ , we consider the following system of equations, for all  $y \in \mathbb{R}$ ,

$$\begin{cases} dY_s^{(t)}(y) = -\text{sgn}(Y_s^{(t)}(y) - x_0)dB_s^{(t)} + b(Y_s^{(t)}(y))ds, & \forall s \in [0, t] \\ Y_0^{(t)}(y) = y \end{cases} \tag{7.33}$$

with the same conventions as in Subsection 7.1, in particular for the sign. Let us assume we have at our disposal a coalescing flow solution to (7.33). Lemma 7.1 enables us to get a version which is such that for any  $s \in [0, t]$ , the mapping  $\mathbb{R} \ni y \mapsto Y_s^{(t)}(y)$  is non-decreasing and càdlàg. Again to such a version we associate  $\psi := (\psi_{s,t}(y))_{(s,t,y) \in \Delta \times \mathbb{R}}$  via

$$\forall 0 \leq s \leq t, \forall y \in \mathbb{R}, \quad \psi_{s,t}(y) := Y_{t-s}^{(t)}(y) \tag{7.34}$$

Subsequently, we can partially adapt the strategy of the previous subsection. Fix  $t \geq 0$  and a trajectory  $X_{[0,t]}$  associated to  $P$  and whose initial distribution is  $\Lambda(\mathfrak{X}_0, \cdot)$ . Conditioning  $\psi$  by the event (7.5) implies that we have

$$\forall s \in [0, t], \quad dX_s^{(t)} = -\text{sgn}(X_s^{(t)} - x_0)dB_s^{(t)} + b(X_s^{(t)})ds$$

We deduce that

$$\forall s \in [0, t], \quad dB_s^{(t)} = -\text{sgn}(X_s^{(t)} - x_0)(dX_s^{(t)} - b(X_s^{(t)})ds) \tag{7.35}$$

and it follows that the conditioned flow, denoted  $\varphi := (\varphi_{s,t}(y))_{(s,t,y) \in \Delta \times \mathbb{R}}$  in Section 6, is given by

$$\forall 0 \leq s \leq t, \forall z \in \mathbb{R}, \quad \varphi_{s,t}(z) := Z_{t-s}^{(t)}(z) \tag{7.36}$$

where

$$\begin{cases} dZ_s^{(t)}(z) &= \operatorname{sgn}(Z_s^{(t)}(z) - x_0) \operatorname{sgn}(X_s^{(t)} - x_0) (dX_s^{(t)} - b(X_s^{(t)})ds) + b(Z_s^{(t)}(z))ds \\ Z_0^{(t)}(z) &= z \end{cases} \tag{7.37}$$

The observations mentioned after (7.9) are still valid. For given  $r^\wedge, r^\vee > 0$ , assume that the initial law of  $X$  is  $\Lambda([r^\wedge, r^\vee], \cdot)$ . Define for any  $t \geq 0$ ,

$$\begin{cases} R_t^\vee &:= \inf \{z \in \mathbb{R} : Z_0^{(t)}(z) \leq r^\vee\} = \sup \{z \in \mathbb{R} : Z_0^{(t)}(z) < r^\vee\} \\ R_t^\wedge &:= \inf \{z \in \mathbb{R} : Z_0^{(t)}(z) \geq r^\wedge\} \\ \mathfrak{X}_t &:= [R_t^\wedge, R_t^\vee) = \varphi_{0,t}^{-1}([r^\wedge, r^\vee)) \end{cases}$$

where for the last equality we took into account the proof of Lemma 7.3, which can also be applied here. Thus the process  $\mathfrak{X} := (\mathfrak{X}_t)_{t \geq 0}$  remains non-trivial and as in the previous subsection, Theorem 6.7 would lead to the following result, if we had at our disposal a solution to (7.33):

**Conjecture 7.11.** *The process  $\mathfrak{X}$  is a set-valued dual for the diffusion process  $X$  starting from the distribution  $\Lambda([r^\wedge, r^\vee], \cdot)$ .*

The above conjecture should hold for any  $x_0 \in \mathbb{R}$ , but we would like the dual process  $\mathfrak{X}$  to be quite large and this requires a good choice of  $x_0$ . In Subsection 7, we could have replaced  $\operatorname{sgn}(\cdot)$  by  $\operatorname{sgn}(\cdot - x_0)$ , but the corresponding dual process  $\mathfrak{X}$  would have stayed trivial until the first time  $\tau$  that  $X$ , the Brownian motion starting from 0, hits  $x_0$ :

$$\forall t \in [0, \tau], \quad \mathfrak{X}_t = \{X_t\}$$

When, as in Theorem 7.4, the Brownian motion starts from the uniform distribution on  $[-r, r]$  for some  $r > 0$ , the set valued dual is a translation of  $\mathfrak{X}_0 = [-r, r)$  until the closure of  $\mathfrak{X}_t$  contain  $x_0$ , in particular the volume of  $\mathfrak{X}_t$  remains constant until the occurrence of this event.

In the setting of this subsection, we expect similar behaviors. In particular if we want  $\mathfrak{X}_t$  to be quite large for small times  $t \geq 0$ , it seems preferable to take  $x_0 \in \mathfrak{X}_0$ . Conjecture 7.14 below can be seen as an attempt to keep  $x_0$  in the closure of  $\mathfrak{X}$ , by letting  $x_0$  evolve and be the supremum of  $\mathfrak{X}$ .

We are equally wondering about letting  $X$  start from a deterministic point. From the above observation, it seems wise to take  $x_0$  equal to this initial point. So let us assume that (7.33) admits a solution when  $\mathfrak{X}_0 = \{x_0\}$ . As in Subsection 7.1, we are led to introduce

$$\begin{aligned} R_t^\vee &:= \sup\{z \in \mathbb{R} : Z_0^{(t)}(z) = x_0\} \\ R_t^\wedge &:= \inf\{z \in \mathbb{R} : Z_0^{(t)}(z) = x_0\} \\ \mathfrak{X}_t &:= [R_t^\vee, R_t^\wedge) \end{aligned}$$

Unfortunately, we did not found a nice explicit expression of  $\mathfrak{X}$  in terms of  $X$ , as in Proposition 7.5, this prevents us from to conclude, as in Proposition 7.6 that  $\mathfrak{X}$  is a non-trivial set-dual associated to  $X$  starting from  $x_0$ . Nevertheless, here is an indication going in this direction.

**Lemma 7.12.** For any  $t > 0$ ,  $\mathfrak{X}_t$  is a right neighborhood of  $X_t$ .

*Proof.* Fix  $t > 0$  as well as  $\epsilon \in (\min\{X_s : s \in [0, t]\}, x_0)$ . Define

$$\begin{aligned} \tau &:= \inf\{s > 0 : X_s = \epsilon\} \in (0, t) \\ \gamma &:= \sup\{s \in [0, \tau] : X_s = x_0\} \end{aligned}$$

Consider the flow  $(F_s^{(t)}(z))_{z \in \mathbb{R}, s \in [0, t]}$  solution of the system

$$\begin{cases} dF_s^{(t)}(z) &= -(dX_s^{(t)} - b(X_s^{(t)})ds) + b(F_s^{(t)}(z))ds \\ F_0^{(t)}(z) &= z \end{cases} \tag{7.38}$$

It is a regular flow: for any  $s \in [0, t]$ , the mapping  $\mathbb{R} \ni x \mapsto F_s^{(t)}(x)$  is a diffeomorphism of  $\mathbb{R}$ . This is also true for the random times  $\tau$  and  $\gamma$ : the mapping  $\mathbb{R} \ni x \mapsto F_{\tau-\gamma}^{(\tau)}(x)$  is a diffeomorphism. Consider the unique  $\xi \in \mathbb{R}$  such that  $F_{\tau-\gamma}^{(\tau)}(\xi) = x_0$ .

Let us show that for  $s > 0$  sufficiently small,  $F_{\tau-\gamma-s}^{(\tau)}(\xi) > x_0$ . Indeed, we have

$$\begin{aligned} F_{\tau-\gamma-s}^{(\tau)}(\xi) - x_0 &= -(F_{\tau-\gamma}^{(\tau)}(\xi) - F_{\tau-\gamma-s}^{(\tau)}(\xi)) \\ &= \int_{\tau-\gamma-s}^{\tau-\gamma} dX_u^{(t)} - \int_{\tau-\gamma-s}^{\tau-\gamma} b(X_u^{(t)}) + b(F_u^{(t)}(z)) du \\ &= x_0 - X_{\tau-\gamma-s}^{(\tau)} - \int_{\tau-\gamma-s}^{\tau-\gamma} b(X_u^{(t)}) + b(F_u^{(t)}(z)) du \end{aligned} \tag{7.39}$$

If  $X_{[0, t]}$  was a Brownian motion, the process  $x_0 - X_{[\gamma, \tau]}$  would be a Bessel-3 process starting from 0 and stopped when it reaches  $-\epsilon$ , according to Williams' Brownian path decomposition (see e.g. Theorem 4.9 of Chapter 7 of Revuz and Yor [28]). From Wichura [31], we would then be able to get that a.s.

$$\liminf_{s \rightarrow 0^+} \frac{x_0 - X_{\gamma+s}}{\sqrt{s/\ln(\ln(1/s))}} > 0$$

This behavior is shared by the diffusion process  $X_{[0, t]}$ , since Girsanov theorem insures that its law is equivalent to that of the Brownian motion on the time interval  $[0, t]$ , see also Bass and Erickson [6]. Furthermore, we have that

$$\left| \int_{\tau-\gamma-s}^{\tau-\gamma} b(X_u^{(t)}) + b(F_u^{(t)}(z)) du \right| \leq 2 \|b\|_\infty s$$

thus (7.39) implies that for  $s > 0$  small,

$$\begin{aligned} F_{\tau-\gamma-s}^{(\tau)}(\xi) - x_0 &\sim x_0 - X_{\gamma+s} \\ &> 0 \end{aligned}$$

As a consequence, consider  $\sigma \in (0, \tau - \gamma)$  such that

$$\forall s \in (0, \sigma], \quad \begin{cases} X_{\tau-\gamma-s}^{(\tau)}(\xi) < x_0 \\ F_{\tau-\gamma-s}^{(\tau)}(\xi) > x_0 \end{cases}$$

From (7.37), we deduce that

$$\forall s \in (0, \sigma), \quad F_{\tau-\gamma-s}^{(\tau)}(\xi) = Z_{\tau-\gamma-s}^{(\tau)}(\xi)$$

Define

$$\zeta := F_{\tau-\gamma-\sigma}^{(\tau)}(\xi)$$

so that

$$\begin{aligned} Z_\sigma^{(\gamma+\sigma)}(\zeta) &= F_\sigma^{(\gamma+\sigma)}(\zeta) \\ &= x_0 \\ &= X_\sigma^{(\gamma+\sigma)} \end{aligned}$$

We deduce that  $\zeta \in \mathfrak{X}_{\gamma+\sigma}$ . Note that  $\zeta > x_0 > X_{\gamma+\sigma}$ , it follows that  $\mathfrak{X}_{\gamma+\sigma}$  is a right neighborhood of  $X_{\gamma+\sigma}$ . Finally, taking into account the right continuity of  $\mathbb{R} \ni z \mapsto Z_{t-\gamma-\sigma}^{(t)}(z)$ , we obtain that  $\mathfrak{X}_t$  is a right neighborhood of  $X_t$ .  $\square$

**Remark 7.13.**

- (a) The above proof can be adapted to show that for any  $t > 0$ , there exists  $s \in (0, t)$  such that  $\mathfrak{X}_s$  is a neighborhood of  $X_s$ . But this is not sufficient to get that  $\mathfrak{X}_t$  is a neighborhood of  $X_t$ . Nevertheless, we believe it is true. This is specific to the set-valued duals to be constructed with flows of the form (7.33), it is not true for the classical Pitman’s dual, recall Remark 7.9, and cannot be expected for set-valued duals to be constructed with flows of the form (7.40) below.
- (b) Proposition 7.5 is still valid in the particular case where  $b$  is given by the non-regular drift

$$\forall x \in \mathbb{R}, \quad b(x) := c \operatorname{sign}(x)$$

where  $c \in \mathbb{R}$  is a fixed constant. Indeed, in general (7.12) should be replaced by

$$\forall s \in [0, t], \quad \operatorname{sgn}(Z_s^{(t)}(z))(dZ_s^{(t)}(z) - b(Z_s^{(t)}(z))ds) = \operatorname{sgn}(X_s^{(t)})(dX_s^{(t)} - b(X_s^{(t)})ds)$$

and one exactly recovers (7.12) with  $b$  of the above form.

The existence of a convenient coalescent flow solution to the system (7.33) seems a reasonable conjecture and hopefully could be worked out using the techniques of Le Jan and Raimond [15, 16, 17], but is out of the scope of this paper. What we really would like to do is to solve the following more challenging one. It will probably require a fixed point approach and thus a global topological structure on an adequate set of flows, in the spirit of Remark 6.1. The conjecture below is a direct transposition of the approach presented in Section 4 for birth and death chains and it would enable to construct set-valued dual processes for one-dimensional diffusions in a similar way.

**Conjecture 7.14.** *As at the beginning of this subsection, let  $B := (B_s)_{s \geq 0}$  be a Brownian motion and fix some  $\mathfrak{X}_0 \in \mathfrak{S}$ . We are interested in the following system of equations, for all  $t \geq 0$  and  $y \in \mathbb{R}$ ,*

$$\begin{cases} dY_s^{(t)}(y) = \operatorname{sgn}(R_{t-s}^\vee - Y_s^{(t)}(y))dB_s^{(t)} + b(Y_s^{(t)}(y))ds, & \forall s \in [0, t] \\ Y_0^{(t)}(y) = y \\ R_{t-s}^\vee := \sup\{y \in \mathbb{R} : Y_{t-s}^{(t-s)}(y) \leq \max(\mathfrak{X}_0)\} \end{cases} \tag{7.40}$$

*Obviously there is a measurability problem in (7.40), since  $R_{t-s}^\vee$  belongs to the future at time  $s$  of the filtration generated by  $B^{(t)}$ . Nevertheless, we believe that for any  $s \geq 0$ ,  $R_s$  will be independent from the white noise of  $B$  after time  $s$ , namely from the sigma-field generated by the  $B_v - B_u$ , for  $v \geq u \geq s$ . In particular, the solution will be such that the process  $(\int_0^s \operatorname{sgn}(Y_u^{(t)}(y) - R_{t-u}^\vee)dB_u^{(t)})_{s \in [0, t]}$  is a Brownian motion in the filtration generated by  $B^{(t)}$ , so that for any  $y \in \mathbb{R}$ ,  $Y^{(t)}(y) := (Y_s^{(t)}(y))_{s \in [0, t]}$  is a diffusion associated to  $P$ . This solution should be sufficiently regular and in particular the process  $R^\vee := (R_t^\vee)_{t \geq 0}$  should be a continuous semi-martingale.*

### A Reduction of one-dimensional diffusion processes

The purpose of this appendix is to show that it is not very restrictive to only consider diffusions of the form (7.31).

Consider the second order operator  $L := a\partial^2 + b\partial$  on  $C^\infty(I)$ , where  $a > 0$  and  $b$  are smooth functions on the open interval  $I := (\iota^-, \iota^+) \subset \mathbb{R}$ . Up to performing a shift, assume that  $0 \in I$ . The corresponding **scale and speed functions**  $\nu$  and  $\mu$  are defined by

$$\forall x \in I, \quad \begin{cases} \nu(x) & := \exp\left(-\int_0^x \frac{b}{a}(y) dy\right) \\ \mu(x) & := \frac{1}{2a(x)\nu(x)} \end{cases} \tag{A.1}$$

By a usual abuse of notation,  $\nu$  and  $\mu$  will also stand for the the **scale and speed measures** which admit respectively the scale and speed functions as densities with respect to the Lebesgue measure restricted to  $I$ .

Let  $(X_t)_{t \in [0, \tau)}$  be a diffusion whose generator is  $L$  (in the sense of martingale problems), where  $\tau > 0$  is its (random) explosion time, which is such that

$$\lim_{t \rightarrow \tau^-} X_t = \iota^- \quad \text{or} \quad \lim_{t \rightarrow \tau^-} X_t = \iota^+$$

To simplify the notation, we extend the above diffusion to all times via

$$\forall t \in [\tau, +\infty), \quad X_t := \lim_{s \rightarrow \tau^-} X_s$$

so that  $X := (X_t)_{t \geq 0}$  is  $V$ -valued with  $V := [\iota^-, \iota^+] \subset \overline{\mathbb{R}}$ . The process  $X$  is absorbed at  $\iota^-$  and  $\iota^+$  (if it reaches them). It is sometimes called a minimal Markov process associated to  $L$  and its law is determined by its initial law  $\mathcal{L}(X_0)$ .

Denote

$$\begin{aligned} \mathfrak{S} &:= \{[z^-, z^+] : z^-, z^+ \in V \text{ and } z^- < z^+\} \\ \bar{\mathfrak{S}} &:= \mathfrak{S} \sqcup \{z\} : z \in V \cap \mathbb{R} \end{aligned}$$

and as usual, define the Markov kernel  $\Lambda$  from  $\bar{\mathfrak{S}}$  to  $V$  via

$$\forall z := [z^-, z^+] \in \bar{\mathfrak{S}}, \quad \Lambda([z^-, z^+], \cdot) := \begin{cases} \delta_x, & \text{if } z^+ = z^- \\ \frac{\mu([z^-, z^+] \cap \cdot)}{\mu(z)}, & \text{otherwise} \end{cases}$$

We are looking for a  **$\Lambda$ -spreading** for  $X$ , namely a set-valued dual process  $\mathfrak{X} := (\mathfrak{X}_t)_{t \geq 0}$  such that for any  $t > 0$ , we have  $\mathfrak{X}_t \in \mathfrak{S}$ . We show below how to come back to the setting of Subsection 7.2.

First, let us check that the problem of finding a  $\Lambda$ -spreading is invariant by diffeomorphisms. More precisely, let  $\phi$  be a smooth function from  $I$  to  $\mathbb{R}$ , whose derivative is positive (in particular it never vanishes). There will be no loss of generality in assuming that  $\phi(0) = 0$ . Let  $\tilde{I} := (\tilde{\iota}_-, \tilde{\iota}_+) := \phi(I)$  and also interpret  $\phi$  as an operator  $\Phi$  from  $C^\infty(\tilde{I})$  to  $C^\infty(I)$  via

$$\forall f \in C^\infty(\tilde{I}), \quad \Phi[f] := f \circ \phi \tag{A.2}$$

Consider the operator  $\tilde{L} := \Phi^{-1} \circ L \circ \Phi$  on  $C^\infty(\tilde{I})$ . It is not difficult to check that if  $X$  is a diffusion associated to  $L$ , then  $\tilde{X} := (\phi(X_t))_{t \geq 0}$  is a diffusion associated to  $\tilde{L}$  (where  $\phi$  has been extended to  $\tilde{V}$  by  $\phi(\iota^-) := \tilde{\iota}^-$  and  $\phi(\iota^+) := \tilde{\iota}^+$ ).

**Lemma A.1.** *Assume that  $\mathfrak{X}$  is a  $\Lambda$ -spreading for  $X$ . Then  $\tilde{\mathfrak{X}} := (\phi(\mathfrak{X}_t))_{t \geq 0}$  is a  $\tilde{\Lambda}$ -spreading for  $\tilde{X}$ .*

*Proof.* Since  $\phi$  is an homeomorphism between  $I$  and  $\tilde{I}$ , all the defining properties of a  $\Lambda$ -spreading are immediate to obtain, except (6.10), which requires the knowledge of  $\tilde{\Lambda}$ . So let us compute  $\tilde{L}$ . Recall that we have

$$\forall f \in \mathcal{C}^\infty(I), \forall F \in \mathcal{C}^\infty(f(I)), \quad L[F \circ f] = F'[f]L[f] + F''[f]\Gamma[f]$$

(this property is equivalent to the fact that  $L$  generates processes with continuous trajectories), where the carré du champ operator  $\Gamma$  is defined by

$$\begin{aligned} \forall f \in \mathcal{C}^\infty(I), \quad \Gamma[f] &:= \frac{1}{2}(L[f^2] - 2fL[f]) \\ &= a(\partial f)^2 \end{aligned}$$

It follows that for any  $F \in \mathcal{C}^\infty(\tilde{I})$ , we have the **change of coordinate formula**:

$$L[F \circ \phi] = \Gamma[\phi]F''[\phi] + L[\phi]F'[\phi]$$

so that

$$\tilde{L}[F] = (\Gamma[\phi] \circ \phi^{-1})F'' + (L[\phi] \circ \phi^{-1})F'$$

namely

$$\tilde{a} = \Gamma[\phi] \circ \phi^{-1} = (a(\phi')^2) \circ \phi^{-1} \tag{A.3}$$

$$\tilde{b} = L[\phi] \circ \phi^{-1} = (a\phi'') \circ \phi^{-1} + (b\phi') \circ \phi^{-1} \tag{A.4}$$

We deduce that

$$\begin{aligned} \forall x \in \tilde{I}, \quad \tilde{\nu}(x) &= \exp\left(-\int_0^x \frac{\tilde{b}(y)}{\tilde{a}(y)} dy\right) \\ &= \exp\left(-\int_0^{\phi^{-1}(x)} \frac{\tilde{b}(\phi(y))}{\tilde{a}(\phi(y))} \phi'(y) dy\right) \\ &= \exp\left(-\int_0^{\phi^{-1}(x)} \left(\frac{\phi''}{(\phi')^2}(y) + \frac{b}{a\phi'}(y)\right) \phi'(y) dy\right) \\ &= \exp\left(-\int_0^{\phi^{-1}(x)} \frac{\phi''}{\phi'}(y) + \frac{b(y)}{a(y)} dy\right) \\ &= \frac{\phi'(0)}{\phi'(\phi^{-1}(x))} \nu(\phi^{-1}(x)) \\ &= \phi'(0) \nu(\phi^{-1}(x)) (\phi^{-1})'(x) \end{aligned}$$

Similarly, we get

$$\begin{aligned} \forall x \in \tilde{I}, \quad \tilde{\mu}(x) &= \frac{1}{2\tilde{a}(x)\tilde{\nu}(x)} \\ &= \frac{1}{2(a(\phi')^2) \circ \phi^{-1} \phi'(0) \nu(\phi^{-1}(x)) (\phi^{-1})'(x)} \\ &= \mu(\phi^{-1}(x)) (\phi^{-1})'(x) / \phi'(0) \end{aligned}$$

It just means that  $\tilde{\nu}$  and  $\tilde{\mu}$  are the images of  $\nu$  and  $\mu$  by  $\phi$ , up to the factors  $\phi'(0)$  and  $1/\phi'(0)$ . It follows that

$$\forall z := [z^-, z^+] \in \tilde{\mathfrak{S}}, \quad \tilde{\Lambda}([z^-, z^+], \cdot) = \Lambda((\phi^{-1}(z^-), \phi^{-1}(z^+)), \phi^{-1}(\cdot)) \tag{A.5}$$

Denote also by  $\phi$  the mapping  $\tilde{\mathfrak{S}} \ni [z^-, z^+] \mapsto (\phi(z^-), \phi(z^+)) \in \tilde{\mathfrak{S}}$  and by  $\Phi$  the corresponding functional operator, as in (A.2), transforming measurable mappings on  $\tilde{\mathfrak{S}}$  into measurable mappings on  $\mathfrak{S}$ . At the operator level, the relation (A.5) translates into the intertwining relation

$$\Phi\tilde{\Lambda} = \Lambda\Phi \tag{A.6}$$

From these invariance relations and from the fact that the  $\sigma$ -field generated by  $\tilde{\mathfrak{X}}_{[0,t]}$  contains the same events as the one generated by  $\mathfrak{X}_{[0,t]}$ , we deduce that for any  $t \geq 0$ ,

$$\begin{aligned} \mathcal{L}(\tilde{X}_t | \tilde{\mathfrak{X}}_{[0,t]}) &= \mathcal{L}(\phi(X_t) | \mathfrak{X}_{[0,t]}) \\ &= \Phi(\mathcal{L}(X_t) | \mathfrak{X}_{[0,t]}) \\ &= \Phi(\Lambda(X_t, \cdot)) \\ &= \tilde{\Lambda}(\phi(X_t), \cdot) \\ &= \tilde{\Lambda}(\tilde{X}_t, \cdot) \end{aligned}$$

where in the second and third lines,  $\Phi$  stands for the natural action induced on measures by the mapping  $\phi$ , obtained by duality from the action of  $\Phi$  on the functions, and the fourth equality corresponds to (A.6). □

As a consequence, we get:

**Corollary A.2.** *To know how to find a  $\Lambda$ -spreading for all initial distributions  $\mathcal{L}(X_0)$  and for all generators  $L$  as above is equivalent to know how to solve this problem when  $a \equiv 1/2$ .*

*Proof.* Note that conversely in Lemma A.1, the diffeomorphism  $\phi^{-1}$  from  $\tilde{I}$  to  $I$  enables to go from  $(\tilde{X}, \tilde{\mathfrak{X}})$  to  $(X, \mathfrak{X})$ . Thus, taking into account (A.3), the proof of the above corollary is reduced to the finding of increasing diffeomorphism  $\phi$  on  $I$  such that  $\tilde{a} \equiv 1/2$ , namely  $\phi' = 1/\sqrt{2a}$ . This is solved by considering

$$\forall x \in I, \quad \phi(x) := \int_0^x \frac{1}{\sqrt{2a(y)}} dy$$

From a geometric point of view, this amounts to changing the usual metric on  $I$  so that  $2a\partial^2$  corresponds to the second order terms of the Laplacian for the new metric. □

Similarly, we can also removed the drift:

**Corollary A.3.** *To know how to find a  $\Lambda$ -spreading for all initial distributions  $\mathcal{L}(X_0)$  and for all generators  $L$  as above is equivalent to know how to solve this problem when  $b \equiv 0$ .*

*Proof.* By the same reasoning as in the proof of Corollary A.2, it is sufficient to find an increasing diffeomorphism  $\phi$  on  $I$  such that  $\tilde{b} \equiv 0$ . From (A.4), this amounts to  $L[\phi] = 0$ . This is solved by considering

$$\forall x \in I, \quad \phi(x) := \nu([0, x]) \tag{□}$$

Except when  $L[1/\sqrt{a}] = 0$ , it is not possible to perform the two operations of the proofs of Corollary A.2 and A.3 simultaneously, to end up with the generator of the Brownian motion on  $I$  (absorbed at the boundary when it is reached). Namely, not every search for  $\Lambda$ -spreadings can be reduced to Pitman's theorem [25]. Nevertheless, by Corollary A.2, the one-dimensional processes considered at the beginning of this appendix can be reduced to the case of Subsection 7.2, up to relaxing the hypotheses on the drift  $b$  there and to assume here that  $\iota^-$  and  $\iota^+$  are natural boundaries.

**Remark A.4.** In Subsection 7.2 we preferred to work with the reduction of Corollary A.2. We also tried the reduction of Corollary A.3 but it did not led to more explicit set-valued dual processes, that is why the computations are not presented in this paper. The reductions of one-dimensional diffusions deduced in Corollaries A.2 and A.3 are the simplest ones, but other ones can be imagined and maybe among them there is one leading to more natural couplings.

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**Acknowledgments.** I’m very grateful to Marc Arnaudon and Koléhè Coulibaly-Pasquier for the discussions we had about explicit couplings of primal and dual processes by intertwining. In [3], we are developing another approach and Examples 3.1 and 3.2 comes from this joint work in progress, after adaptation to the discrete setting and translation in terms of random mappings.