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Discretionary stopping of stochastic differential equations with generalised drift

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Abstract

We consider the problem of optimally stopping a general one-dimensional stochastic differential equation (SDE) with generalised drift over an infinite time horizon. First, we derive a complete characterisation of the solution to this problem in terms of variational inequalities. In particular, we prove that the problem's value function is the difference of two convex functions and satisfies an appropriate variational inequality in the sense of distributions. We also establish a verification theorem that is the strongest one possible because it involves only the optimal stopping problem's data. Next, we derive the complete explicit solution to the problem that arises when the state process is a skew geometric Brownian motion and the reward function is the one of a financial call option. In this case, we show that the optimal stopping strategy can take several qualitatively different forms, depending on parameter values. Furthermore, the explicit solution to this special case shows that the so-called "principle of smooth fit" does not hold in general for optimal stopping problems involving solutions to SDEs with generalised drift.

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1 Introduction

We consider the optimal stopping of the one-dimensional SDE with generalised drift

$$X_t = x + \int_{\underline{\iota}}^{\overline{\iota}} L_t^z \, \nu(dz) + \int_0^t \sigma(X_s) \, dW_s, \quad x \in \mathring{\mathcal{I}}, \tag{1.1}$$

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where L^z is the symmetric local time of X at level z, W is a standard one-dimensional Brownian motion and $\mathring{\mathcal{I}} = \,]\underline{\iota}, \overline{\iota}[$ is the interior of a given interval $\mathcal{I} \subseteq [-\infty, \infty]$. We assume that the signed Radon measure ν and the Borel-measurable function $\sigma : \mathring{\mathcal{I}} \to \mathbb{R} \setminus \{0\}$ satisfy suitable conditions ensuring that the SDE (1.1) has a weak solution $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x, W, X\big)$ that is unique in the sense of probability law up to a possible explosion time at which X hits the boundary $\{\underline{\iota}, \overline{\iota}\}$ of $\mathcal{I}$ (see Assumption 2.1 in Section 2). If the boundary point $\underline{\iota}$ (resp., $\overline{\iota}$) is inaccessible, then the interval $\mathcal{I}$ is open from the left (resp., open from the right). On the other hand, if the boundary point $\underline{\iota}$ (resp., $\overline{\iota}$) is not inaccessible, then we assume that it is absorbing and the interval $\mathcal{I}$ is closed from the left (resp., closed from the right). Comprehensive studies of these SDEs as well as relevant literature surveys can be found in Engelbert and Schmidt [14], and Lejay [23].

In the special case when ν is absolutely continuous, namely, when

$$\nu(dx) = \frac{b(x)}{\sigma^2(x)}\, dx,$$

for a Borel-measurable function $b : \mathring{\mathcal{I}} \to \mathbb{R}$ satisfying suitable integrability conditions, an application of the occupation times formula shows that the solution to (1.1) admits the expression

$$X_t = x + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s, \quad x \in \mathring{\mathcal{I}}. \tag{1.2}$$

In view of this simple observation, we can see that usual SDEs are special cases of SDEs with generalised drift. The skew Brownian motion, which is characterised by the choices

$$\nu(dx) = \beta\, \delta_0(dx), \quad \sigma \equiv 1 \quad \text{and} \quad \mathcal{I} = \mathbb{R},$$

where $\beta \in \,]-1, 1[\, \setminus \{0\}$ and $\delta_0(dx)$ is the Dirac probability measure that assigns mass 1 at $\{0\}$, and corresponds to the SDE

$$X_t = x + \beta L_t^0 + W_t, \quad x \in \mathbb{R}, \tag{1.3}$$

is a fundamental example of an SDE with generalised drift (see Itô and McKean [18, Problem 4.2.1], Harrison and Shepp [17], Lejay [23], and several references therein). A further important example is the skew geometric Brownian motion, which is characterised by the choices

$$\nu(dx) = \frac{b}{\sigma^2 x}\, dx + \beta\, \delta_z(dx), \quad \sigma(x) = \sigma x \quad \text{and} \quad \mathcal{I} = \,]0, \infty[,$$

where $b \in \mathbb{R}$, $\beta \in \,]-1, 1[\, \setminus \{0\}$, $\sigma \neq 0$ are constants and $\delta_z(dx)$ is the Dirac probability measure that assigns mass 1 at $\{z\}$, for some $z > 0$, and corresponds to the SDE

$$dX_t = bX_t\, dt + \beta\, dL_t^z + \sigma X_t\, dW_t, \quad X_0 = x > 0. \tag{1.4}$$

Furthermore, an SDE with generalised drift whose dynamics identify with the dynamics of a usual SDE away from a finite number of points at which it exhibits a skew behaviour is discussed in Example 2.5 below. At this point, we emphasise that the processes L^z, $z \in \mathring{\mathcal{I}}$, in SDEs with generalised drift are the local times of the solution X to the corresponding SDEs and not the local times of the driving Brownian motion W (see also Example 2.6 where this observation is considered further in the context of the skew Brownian motion given by (1.3)).

The objective of the optimal stopping problem that we study aims at maximising the performance criterion

$$\mathbb{E}_x\left[\exp\left(-\int_0^\tau r(X_s)\, ds\right) f(X_\tau) \mathbf{1}_{\{\tau < \infty\}}\right] \tag{1.5}$$

over all stopping times τ, where the positive Borel-measurable discounting rate function r satisfies Assumption 2.3 in Section 2, while the positive and possibly unbounded reward function f satisfies Assumption 3.1 in Section 3. To the best of our knowledge, there exist no results in the literature addressing the solvability of such a problem by means of variational inequalities when ν is not absolutely continuous, even in special cases. We derive a complete characterisation of the solution to this problem in terms of variational inequalities. In particular, we prove that the value function v of the optimal stopping problem associated with (1.1) and (1.5) is the difference of two convex functions and satisfies the variational inequality

$$\max\left\{\frac{1}{2}\sigma^2(x)p'_-(x)\left(\frac{v'_-}{p'_-}\right)'(dx) - r(x)v(x)\,dx,\ f(x)-v(x)\right\} = 0, \tag{1.6}$$

in the sense of distributions (see Definition 3.2 and Theorem 3.3.(i)-(ii) in Section 3), where p is the scale function of the diffusion associated with the SDE (1.1). We also establish a verification theorem that is the strongest one possible because it involves only the optimal stopping problem's data. In particular, we derive a simple necessary and sufficient condition for a solution to (1.6) to identify with the problem's value function (see Theorem 3.3.(iii)).

The second main contribution of the paper is to derive the complete explicit solution to the special case that arises if $f(x) = (x-K)^+$, for some constant $K > 0$, and X is a skew geometric Brownian motion. In this case, the SDE (1.4), has a unique non-explosive strong solution. Given such a solution X, which exists on any given filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\big)$ satisfying the usual conditions and supporting a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W, the value function of the discretionary stopping problem that we solve is defined by

$$v(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-r\tau}(X_\tau - K)^+ \mathbf{1}_{\{\tau<\infty\}}\right], \tag{1.7}$$

where $\mathcal{T}$ is the family of all $(\mathcal{F}_t)$-stopping times and $r, K > 0$ are constants (we write $\mathbb{E}$ in place of $\mathbb{E}_x$ because we consider strong rather than weak solutions here). We prove that the optimal stopping strategy can take several qualitatively different forms, depending on parameter values (see Theorems 6.1, 6.3 and 6.4, and Figures 4-13). In contrast to the optimal stopping of an SDE with absolutely continuous drift and reward function such as the one of a financial call option, the optimal stopping region may involve two distinct components, one of which may be an isolated point. Furthermore, the analysis of this problem shows that the so-called "principle of smooth fit" does not hold even in the case of a "right-sided" optimal stopping strategy in the sense that none of the functions

$$v'_-(x) = \lim_{\varepsilon\downarrow 0}\frac{v(x)-v(x-\varepsilon)}{\varepsilon}, \quad \frac{v'_-(x)}{p'_-(x)} = \lim_{\varepsilon\downarrow 0}\frac{v(x)-v(x-\varepsilon)}{p(x)-p(x-\varepsilon)}$$
$$\text{or}\quad \frac{v'_-(x)}{\psi'_-(x)} = \lim_{\varepsilon\downarrow 0}\frac{v(x)-v(x-\varepsilon)}{\psi(x)-\psi(x-\varepsilon)} \tag{1.8}$$

is continuous, where p (resp., ψ) is the scale function (resp., the increasing minimal excessive function) of the diffusion associated with the SDE (1.4) (see Remarks 3.7 and 6.2).

To derive the solution to the optimal stopping problem defined by (1.4) and (1.7), we need to consider a partition of the set $\mathbb{R} \times (\mathbb{R}\setminus\{0\}) \times \,]0,\infty[\, \times \big(]-1,1[\setminus\{0\}\big)$ in which the parameter vector (b, σ, z, β) takes values into four sets (see Cases (I)-(IV) of Lemma 4.2). In Cases (I) and (II), $\beta \in \,]-1,0[$ and ψ is convex. In Cases (III) and (IV), $\beta \in \,]-1,0[$ and $\beta \in \,]0,1[$, respectively, and ψ fails to be convex. The best part of our analysis' complexity (including the whole of Section 5) is due to the facts that (a) the optimal strategy takes several qualitatively different forms, and (b) we establish necessary and sufficient conditions on the problem's data that differentiate between the different possible cases without leaving any "gap" in the parameter space. The solution to the problem in the easier Cases (I), (II) and (IV) has been presented in the PhD thesis Lon [24]. Although the analysis of Case (III) is not in itself harder, linking it with Cases (I) and (II) with necessary and sufficient conditions on the problem's data requires rather tedious analysis, due to the complex structure of the increasing minimal excessive function ψ (see Figure 2). The solution to all cases was announced without proofs in the conference paper Lon, Rodosthenous and Zervos [25].

Variational inequalities take center stage in the continuous time optimal stopping theory because they are efficient for the investigation of specific problems. In the context of this paper, they can be used to easily identify critical parts of the state space $\mathcal{I}$ that are subsets of the so-called waiting region in a systematic way that involves no guesswork. For instance, the regularity of v implies that all points at which the reward function f is discontinuous as well as all "minimal" intervals in which f cannot be expressed as the difference of two convex functions (e.g., intervals in which f has the regularity of a Brownian sample path) should be parts of the closure of the waiting region. For further analysis and discussion in this direction, see Remark 3.6 at the end of Section 3. Beyond its usefulness in identifying optimal stopping strategies, the variational inequality characterisation is also very effective in verifying whether a candidate function identifies with the value function of a specific problem because, in the context of SDEs driven by a Brownian motion, it has a local character in the sense that it involves only derivatives.

The solution to optimal stopping problems using classical solutions to variational inequalities has been extensively studied. Results in Friedman [15, Chapter 16], Bensoussan and Lions [5, Chapter 3], Øksendal [28, Chapter 10] and Peskir and Shiryaev [31], listed in chronological order, typically make strong regularity assumptions on the problem data (e.g., the problem data are assumed to be Lipschitz continuous). To relax such assumptions, Øksendal and Reikvam [29] and Bassan and Ceci [3] have considered viscosity solutions to the variational inequalities associated with the optimal stopping problems that they study. Results with minimal assumptions on the problem data, such as the ones that we derive here for the optimal stopping problem associated with (1.1) and (1.5), have been obtained by Lamberton [21] and Lamberton and Zervos [22] who consider the optimal stopping of the SDE (1.2) over a finite and an infinite time horizon, respectively. Recently, the solution to suitable optimal stopping problems by means of variational inequalities has been used to characterise the boundary of Root's solution to the Skorokhod embedding problem (see Cox and Wang [9], Cox, Obłój and Touzi [8], and references therein). Furthermore, variational inequalities arise most naturally in the study of optimal stopping problems involving controlled stochastic processes (e.g., see Bensoussan and Lions [5, Chapter 4], Krylov [20, Chapters 3, 6], Beneš [4], Davis and Zervos [11], Karatzas and Sudderth [19], listed in chronological order, as well as many more recent contributions).

There exist few references in the literature addressing special cases of a general problem such as the one we study. Peskir [30] considered the validity of the "principle of smooth fit" in terms of derivatives such as the first two ones in (1.8). In particular, failure of the "principle of smooth fit" was exhibited by Examples 2.2 and 3.1 in this reference. These examples involve the optimal stopping of the processes $X = F(B)$, where B is a standard Brownian motion absorbed at the boundaries of $[-1, 1]$ and

$$F(x) = \begin{cases} x^{1/3}, & \text{if } x \in [0,1], \\ -|x|^{1/3}, & \text{if } x \in [-1,0[, \end{cases} \quad \text{or} \quad F(x) = \begin{cases} \sqrt{x}, & \text{if } x \in [0,1], \\ -x^2, & \text{if } x \in [-1,0[. \end{cases}$$

These processes are skew diffusions that cannot be associated with solutions to SDEs with generalised drift because Itô-Tanaka's formula cannot be applied due to the fact that $F'_+(0) = \infty$. In the second paragraph of Section 3.2 in Peskir [30], a further similar example exhibiting failure of the "principle of smooth fit" is briefly discussed. If we make a suitable choice for σ, such as $\sigma \equiv 1$, then we can associate the diffusion of this example with the solution to an SDE with generalised drift by replicating the analysis in Example 2.6 below.

Other references have considered the optimal stopping of a skew Brownian motion, which is given by the strong solution to the SDE (1.3), with the objective to maximise the performance index given by (1.5) for $r > 0$ being a constant and for f being an increasing function associated with "right-sided" optimal stopping strategies. Crocce and Mordecki [10] studied the validity of the "principle of smooth fit" in terms of derivatives such as the ones given by (1.8), and presented two examples with optimal stopping strategies such as the ones in Theorem 6.1 that are illustrated by Figures 5 and 10 below. Alvarez and Salminen [1] derived sufficient conditions on f that are associated with optimal stopping strategies of the same qualitative nature as the ones in Theorems 6.1 and 6.4 that are illustrated by Figures 10, 12 and 13 below. The analysis in these references is based on Dynkin's characterisation of an optimal stopping problem's value function as the minimal excessive majorant of its reward function and the Martin representation theory of excessive functions. In contrast to variational inequalities, which involve only the problem's primary data, this approach has a non-local character or it requires conditions involving the elements of the set

$$\left\{ x \in \mathcal{I} \;\middle|\; \frac{f(x)}{\psi(x)} = \sup_{u \in \mathcal{I}} \frac{f(u)}{\psi(u)} \right\}.$$

As a consequence, its applicability has largely been limited to problems with "one-sided" optimal stopping strategies because, with notable exceptions such as the ones associated with a Brownian motion or a geometric Brownian motion, the minimal excessive functions are not in general expressible in terms of elementary functions.

Beyond its contributions to the optimal stopping theory, the present paper has been motivated by applications to the optimal timing of investment decisions involving an underlying asset price or economic indicator. In mathematical finance and the theory of real options, such time series are typically modelled by SDEs driven by a standard Brownian motion or, more generally, a Lévy process. A skew geometric Brownian motion or, more generally, SDEs such as the ones considered in Example 2.5 can be used to model asset prices and economic indicators that exhibit support and resistance levels[1] (see Hämäläinen [16] for a recent survey of studies focusing on such directional predictability). Indeed, the skew geometric Brownian motion (1.4) behaves like a standard geometric Brownian motion, except that the sign of each excursion from z is chosen using an independent Bernoulli random variable of parameter $p = \frac{1}{2}(\beta + 1)$, namely, $\mathbb{P}(X_t > z \mid t > T_z) = p$, where T_z is the first hitting time of z. Recently, several authors have studied financial models involving SDEs with generalised drift (e.g., see Corns and Satchell [7], Decamps, Goovaerts and Schoutens [12, 13], Rosello [33], and references therein). Alternatively, a skew geometric Brownian motion can be used to capture phenomena of bounces and sinks that are exhibited by financial firms in distress (see Nilsen and Sayit [27]).

The paper is organised as follows. In Section 2, we derive an analytic characterisation of the minimal excessive functions of the one-dimensional diffusion associated with the

[1] "Support is a level or area on the chart under the market where buying interest is sufficiently strong to overcome selling pressure. As a result, a decline is halted and prices turn back again... Resistance is the opposite of support." (Murphy [26]).

SDE (1.1). In Section 3, we establish a complete characterisation of the solution to the general optimal stopping problem given by (1.1) and (1.5) in terms of variational inequalities. Section 4 presents a study of a skew geometric Brownian motion's minimal excessive functions. In Section 5, we prove a couple of technical results that will facilitate the streamlining of the presentation of the solution to the optimal stopping problem defined by (1.4) and (1.7). We present the complete solution to this problem in Section 6. Finally, the proofs of all results stated in Section 6 are collected in Section 7.

2 The SDE (1.1) and its associated minimal excessive functions

We start with the following assumption.[2]

Assumption 2.1. *The function $\sigma : \mathring{\mathcal{I}} \to \mathbb{R}$ is Borel-measurable,*

$$\sigma(x) \neq 0 \textit{ for all } x \in \mathring{\mathcal{I}} \quad \textit{and} \quad \int_{\underline{x}}^{\overline{x}} \sigma^{-2}(u)\, du < \infty \textit{ for all } \underline{x} < \overline{x} \textit{ in } \mathring{\mathcal{I}}.$$

Also, ν is a signed Radon measure on $\big(\mathring{\mathcal{I}}, \mathcal{B}(\mathring{\mathcal{I}})\big)$ such that $\nu\big(\{z\}\big) \in \,]-1, 1[$.

In the presence of this assumption, the SDE (1.1) has a weak solution that is unique in the sense of probability law (see Engelbert and Schmidt [14, Theorems 4.35 and 4.37]). In particular, given any initial point $x \in \mathring{\mathcal{I}}$, there is a collection $\mathbb{S}_x = \big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x, W, X\big)$ such that $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x)$ is a filtered probability space satisfying the usual conditions, W is a standard $(\mathcal{F}_t)$-Brownian motion and X is a continuous $(\mathcal{F}_t)$-adapted stochastic process such that (1.1) holds true in the stochastic interval $[0, T_{\underline{\iota}} \wedge T_{\overline{\iota}}[$, where

$$T_y = \inf\{t \geq 0 \mid \ X_t = y\}, \quad \text{for } y \in [\underline{\iota}, \overline{\iota}], \tag{2.1}$$

with the usual assumption that $\inf \emptyset = \infty$. We assume that either of the endpoints $\underline{\iota}$, $\overline{\iota}$ is either inaccessible or absorbing. Accordingly, if $\underline{\iota}$ (resp., $\overline{\iota}$) is absorbing, then $X_t = \underline{\iota}$ (resp., $X_t = \overline{\iota}$) for all $t \geq T_{\underline{\iota}}$ (resp., $t \geq T_{\overline{\iota}}$).

The scale function of the diffusion associated with the SDE (1.1) is the unique, up to a strictly increasing affine transformation, continuous strictly increasing function $p : \mathcal{I} \to \mathbb{R}$ that satisfies

$$\mathbb{P}_x(T_{\overline{x}} < T_{\underline{x}}) = 1 - \mathbb{P}_x(T_{\underline{x}} < T_{\overline{x}}) = \frac{p(x) - p(\underline{x})}{p(\overline{x}) - p(\underline{x})},$$

for all points $\underline{x} < x < \overline{x}$ in $\mathcal{I}$. The restriction of p to $\mathring{\mathcal{I}}$ is the difference of two convex functions[3] and satisfies the ordinary differential equation (ODE)

$$p''(dx) = -\big[p'_+(x) + p'_-(x)\big]\, \nu(dx) \tag{2.2}$$

in the sense that

$$p'_+(\overline{x}) - p'_+(\underline{x}) = -\int_{]\underline{x}, \overline{x}]} \big[p'_+(z) + p'_-(z)\big]\, \nu(dz) \quad \text{for all } \underline{x} < \overline{x} \text{ in } \mathring{\mathcal{I}}. \tag{2.3}$$

[2] We denote by $\mathcal{B}(\mathring{\mathcal{I}})$ the Borel σ-algebra on $\mathring{\mathcal{I}}$.

[3] A function $g : \mathring{\mathcal{I}} \to \mathbb{R}$ is the difference of two convex functions if and only if it is absolutely continuous with left-hand derivative that is a function of finite variation. Given such a function, we denote by $g'_\pm$ its right-hand and left-hand side first derivatives, which are defined by

$$g'_+(x) = \lim_{\varepsilon \downarrow 0} \frac{g(x+\varepsilon) - g(x)}{\varepsilon} \quad \text{and} \quad g'_-(x) = \lim_{\varepsilon \downarrow 0} \frac{g(x) - g(x-\varepsilon)}{\varepsilon},$$

and by $g''(dx)$ the measure that identifies with its second distributional derivative.

In particular, it is given by

$$p'_+(x) = e^{-2\nu([x_1,x])} \prod_{z\in[x_1,x]} \frac{1-\nu(\{z\})}{1+\nu(\{z\})} e^{2\nu(\{z\})}, \quad \text{if } x \geq x_1,$$

and

$$p'_+(x) = e^{2\nu(]x,x_1[)} \prod_{z\in]x,x_1[} \frac{1+\nu(\{z\})}{1-\nu(\{z\})} e^{-2\nu(\{z\})}, \quad \text{if } x < x_1,$$

where $x_1 \in \mathring{\mathcal{I}}$ is an arbitrary fixed point. All these claims can be found, e.g., in Engelbert and Schmidt [14, Section 4.3]. For future reference, we also note that these expressions imply that

$$p'_+(x) = p'_-(x) \frac{1-\nu(\{x\})}{1+\nu(\{x\})} \quad \text{for all } x \in \mathring{\mathcal{I}}. \tag{2.4}$$

We will need the following real analysis result.[4]

Lemma 2.2. *Let $\widetilde{u} : p(\mathring{\mathcal{I}}) \to \mathbb{R}$ be a difference of two convex functions and define $u(x) = \widetilde{u}\big(p(x)\big)$, for $x \in \mathring{\mathcal{I}}$. The following statements hold true:*

(i) *u is the difference of two convex functions.*

(ii) *The function u'_-/p'_- is of finite variation. Furthermore, the measure on $\big(\mathring{\mathcal{I}}, \mathcal{B}(\mathring{\mathcal{I}})\big)$ that identifies with the first distributional derivative of the function u'_-/p'_- is the image of the measure $\widetilde{u}''$ under the function p^{-1}, namely, $(u'_-/p'_-)'(dx) = (\widetilde{u}'' \circ p)(dx)$. In particular,*

$$\left(\frac{u'_-}{p'_-}\right)'\big([\underline{x}, \overline{x}[\big) \equiv \frac{u'_-}{p'_-}(\overline{x}) - \frac{u'_-}{p'_-}(\underline{x}) = \widetilde{u}''\big([p(\underline{x}), p(\overline{x})[\big) \quad \text{for all } \underline{x} < \overline{x} \text{ in } \mathring{\mathcal{I}}$$

and

$$\widetilde{u}''\big([\underline{q}, \overline{q}[\big) \equiv \widetilde{u}'_-(\overline{q}) - \widetilde{u}'_-(\underline{q}) = \left(\frac{u'_-}{p'_-}\right)'\big([p^{-1}(\underline{q}), p^{-1}(\overline{q})[\big) \quad \text{for all } \underline{q} < \overline{q} \text{ in } p(\mathring{\mathcal{I}}).$$

(iii) *If $\widetilde{u}$ has absolutely continuous first derivative $\widetilde{u}'$ $(= \widetilde{u}'_- = \widetilde{u}'_+)$, then u'_-/p'_- is absolutely continuous and* [5]

$$\left(\frac{u'_-}{p'_-}\right)'(x) = p'_-(x)\widetilde{u}''\big(p(x)\big), \quad \text{for } x \in \mathring{\mathcal{I}}.$$

Proof. We first note that $u = \widetilde{u} \circ p$ is absolutely continuous because it is the composition of absolutely continuous functions and p is increasing. Given any $x \in \mathring{\mathcal{I}}$,

$$\lim_{\varepsilon\downarrow 0} \frac{u(x)-u(x-\varepsilon)}{p(x)-p(x-\varepsilon)} = \lim_{\varepsilon\downarrow 0} \frac{\widetilde{u}\big(p(x)\big) - \widetilde{u}\big(p(x-\varepsilon)\big)}{p(x)-p(x-\varepsilon)} = \widetilde{u}'_-\big(p(x)\big).$$

Combining this observation with the fact that the limit $p'_-(x) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\big[p(x) - p(x-\varepsilon)\big]$ exists, we can see that the limit $u'_-(x) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\big[u(x) - u(x-\varepsilon)\big]$ exists for all $x \in \mathring{\mathcal{I}}$.

[4] We denote by $p(\mathring{\mathcal{I}})$ the interval $]p(\underline{\iota}), p(\overline{\iota})[$ and by $\mathcal{B}\big(p(\mathring{\mathcal{I}})\big)$ the Borel σ-algebra on $p(\mathring{\mathcal{I}})$.

[5] In this part of the lemma, we use the same notation for the signed Radon measure that identifies with the second distributional derivative of $\widetilde{u}$ as well as for the Radon-Nikodym derivative of this measure with respect to the Lebesgue measure, namely, we write $\widetilde{u}''(dq) = \widetilde{u}''(q)\,dq$. We refer to this footnote whenever we make such an abuse of notation; confusion is unlikely to occur.

Given any points $\underline{x} < \overline{x}$ in $\mathring{\mathcal{I}}$, we use the change of variables formula (e.g., see Revuz and Yor [32, Proposition 0.4.10]) to calculate

$$\frac{u'_-}{p'_-}(\overline{x}) - \frac{u'_-}{p'_-}(\underline{x}) = \widetilde{u}'_-\big(p(\overline{x})\big) - \widetilde{u}'_-\big(p(\underline{x})\big) \\ = \int_{[p(\underline{x}),p(\overline{x})[} \widetilde{u}''(dq) = \int_{[\underline{x},\overline{x}[} (\widetilde{u}'' \circ p)(dx),$$

and (ii) follows. Furthermore, (i) follows from the absolute continuity of u and the fact that u'_- is the product of the finite variation functions p'_- and u'_-/p'_-.

Finally, if $\widetilde{u}'$ is absolutely continuous (see also footnote 5), then

$$\frac{u'_-}{p'_-}(\overline{x}) - \frac{u'_-}{p'_-}(\underline{x}) = \int_{p(\underline{x})}^{p(\overline{x})} \widetilde{u}''(q)\, dq = \int_{\underline{x}}^{\overline{x}} \widetilde{u}''\big(p(x)\big)p'_-(x)\, dx \quad \text{for all } \underline{x} < \overline{x} \text{ in } \mathring{\mathcal{I}},$$

and (iii) follows. □

Given a weak solution $\mathbb{S}_x = \big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x, W, X\big)$ to the SDE (1.1), the collection $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x, W, p(X)\big)$ is a weak solution to the SDE

$$d\widetilde{X}_t = \big(\sigma \circ p^{-1}\big)(\widetilde{X}_t)\, \big(p'_- \circ p^{-1}\big)(\widetilde{X}_t)\, d\widetilde{W}_t, \quad \widetilde{X}_0 = p(x) \in p(\mathring{\mathcal{I}}), \tag{2.5}$$

for $\widetilde{W} = W$. Conversely, given a weak solution $\widetilde{\mathbb{S}}_x = \big(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t), \widetilde{\mathbb{P}}_x, \widetilde{W}, \widetilde{X}\big)$ to the SDE (2.5), the collection $\big(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t), \widetilde{\mathbb{P}}_x, \widetilde{W}, p^{-1}(\widetilde{X})\big)$ is a weak solution to the SDE (1.1) for $W = \widetilde{W}$. These results, which are established in Engelbert and Schmidt [14, Proposition 4.29]), will play a fundamental role in our analysis.

To proceed further, we consider the discounting rate function r appearing in (1.5) and we make the following assumption.

Assumption 2.3. *The function $r : \mathring{\mathcal{I}} \to \mathbb{R}_+$ is Borel-measurable, uniformly bounded away from 0, namely, $r(x) \geq r_0$ for all $x \in \mathring{\mathcal{I}}$, for some $r_0 > 0$, and such that*

$$\int_{\underline{x}}^{\overline{x}} \frac{r(u)}{\sigma^2(u)}\, du < \infty \quad \textit{for all } \underline{x} < \overline{x} \textit{ in } \mathring{\mathcal{I}}.$$

The minimal r-excessive functions $\varphi, \psi : \mathring{\mathcal{I}} \to \mathbb{R}_+$ of the diffusion associated with the SDE (1.1) are the unique, modulo multiplicative constants, functions that satisfy

$$\varphi(\overline{x}) = \varphi(\underline{x})\, \mathbb{E}_{\overline{x}}\left[\exp\left(-\int_0^{T_{\underline{x}}} r(X_s)ds\right)\right] \text{ for all } \underline{x} < \overline{x} \text{ in } \mathring{\mathcal{I}} \tag{2.6}$$

and

$$\psi(\underline{x}) = \psi(\overline{x})\, \mathbb{E}_{\underline{x}}\left[\exp\left(-\int_0^{T_{\overline{x}}} r(X_s)ds\right)\right] \text{ for all } \underline{x} < \overline{x} \text{ in } \mathring{\mathcal{I}}, \tag{2.7}$$

where $T_{\underline{x}}$, $T_{\overline{x}}$ are as in (2.1) (see Borodin and Salminen [6, Section II.1]).

Lemma 2.4. *The functions $\varphi, \psi : \mathring{\mathcal{I}} \to \mathbb{R}_+$ given by (2.6)–(2.7) are such that φ (resp., ψ) is strictly decreasing (resp., increasing),*

$$\textit{if } \underline{\iota} \textit{ is absorbing, then } \varphi(\underline{\iota}) := \lim_{x \downarrow \underline{\iota}} \varphi(x) < \infty \textit{ and } \psi(\underline{\iota}) := \lim_{x \downarrow \underline{\iota}} \psi(x) = 0,$$
$$\textit{if } \overline{\iota} \textit{ is absorbing, then } \varphi(\overline{\iota}) := \lim_{x \uparrow \overline{\iota}} \varphi(x) = 0 \textit{ and } \psi(\overline{\iota}) := \lim_{x \uparrow \overline{\iota}} \psi(x) < \infty,$$

and

$$\text{if } \underline{\iota} \text{ (resp., } \overline{\iota}) \text{ is inaccessible, then } \lim_{x\downarrow\underline{\iota}} \varphi(x) = \infty \text{ (resp., } \lim_{x\uparrow\overline{\iota}} \psi(x) = \infty).$$

Both of the functions φ and ψ are absolutely continuous. Furthermore, the functions φ'_-/p'_- and ψ'_-/p'_- are absolutely continuous and the homogeneous ODE [6]

$$\frac{1}{2}\sigma^2(x)p'_-(x)\left(\frac{g'_-}{p'_-}\right)'(x) - r(x)g(x) = 0$$

is satisfied Lebesgue-a.e. in $\mathring{\mathcal{I}}$ for g standing for either φ or ψ.

Proof. In view of the results connecting the solvability of (1.1) with the solvability of (2.5) that we have discussed above, we can see that, given any $\underline{x} < \overline{x}$ in $\mathring{\mathcal{I}}$,

$$\frac{\varphi(\overline{x})}{\varphi(\underline{x})} = \mathbb{E}_{\overline{x}}\left[\exp\left(-\int_0^{\widetilde{T}_{p(\underline{x})}} (r\circ p^{-1})(\widetilde{X}_s)ds\right)\right] = \frac{\widetilde{\varphi}\big(p(\overline{x})\big)}{\widetilde{\varphi}\big(p(\underline{x})\big)} \quad \text{and} \quad \frac{\psi(\overline{x})}{\psi(\underline{x})} = \frac{\widetilde{\psi}\big(p(\overline{x})\big)}{\widetilde{\psi}\big(p(\underline{x})\big)},$$

where

$$\widetilde{T}_y = \inf\{t \geq 0 \mid \; \widetilde{X}_t \equiv p(X_t) = y\}, \quad \text{for } y \in \big[p(\underline{\iota}), p(\overline{\iota})\big],$$

and $\widetilde{\varphi}, \widetilde{\psi} : p(\mathring{\mathcal{I}}) \to \mathbb{R}_+$ are the minimal $(r\circ p^{-1})$-excessive functions of the diffusion associated with the SDE (2.5), given by

$$\widetilde{\varphi}(\overline{q}) = \widetilde{\varphi}(\underline{q})\,\widetilde{\mathbb{E}}_{\overline{q}}\left[\exp\left(-\int_0^{\widetilde{T}_{\underline{q}}} (r\circ p^{-1})(\widetilde{X}_s)\,ds\right)\right] \text{ for all } \underline{q} < \overline{q} \text{ in } p(\mathring{\mathcal{I}}) \tag{2.8}$$

and

$$\widetilde{\psi}(\underline{q}) = \widetilde{\psi}(\overline{q})\,\widetilde{\mathbb{E}}_{\underline{q}}\left[\exp\left(-\int_0^{\widetilde{T}_{\overline{q}}} (r\circ p^{-1})(\widetilde{X}_s)\,ds\right)\right] \text{ for all } \underline{q} < \overline{q} \text{ in } p(\mathring{\mathcal{I}}). \tag{2.9}$$

It follows that

$$\varphi = \widetilde{\varphi}\circ p \quad \text{and} \quad \psi = \widetilde{\psi}\circ p. \tag{2.10}$$

In view of the general theory reviewed, e.g., in Borodin and Salminen [6, Section II.1], the functions $\widetilde{\varphi}$, $\widetilde{\psi}$ are unique modulo multiplicative constants, C^1 with absolutely continuous first derivatives, and such that $\widetilde{\varphi}$ (resp., $\widetilde{\psi}$) is strictly decreasing (resp., increasing). Also, since $\underline{\iota}$ (resp., $\overline{\iota}$) is an absorbing (resp., inaccessible) boundary point for X if and only if $p(\underline{\iota})$ (resp., $p(\overline{\iota})$) is an absorbing (resp., inaccessible) boundary point for $\widetilde{X} \equiv p(X)$,

$$\text{if } \underline{\iota} \text{ is absorbing for } X\text{, then } \widetilde{\varphi}\big(p(\underline{\iota})\big) := \lim_{y\downarrow p(\underline{\iota})} \widetilde{\varphi}(y) < \infty \text{ and } \widetilde{\psi}\big(p(\underline{\iota})\big) := \lim_{y\downarrow p(\underline{\iota})} \widetilde{\psi}(y) = 0,$$
$$\text{if } \overline{\iota} \text{ is absorbing for } X\text{, then } \widetilde{\varphi}\big(p(\overline{\iota})\big) := \lim_{y\uparrow p(\overline{\iota})} \widetilde{\varphi}(y) = 0 \text{ and } \widetilde{\psi}\big(p(\overline{\iota})\big) := \lim_{y\uparrow p(\overline{\iota})} \widetilde{\psi}(y) < \infty,$$

and

$$\text{if } \underline{\iota} \text{ (resp., } \overline{\iota}) \text{ is inaccessible for } X\text{, then } \lim_{y\downarrow p(\underline{\iota})} \widetilde{\varphi}(y) = \infty \text{ (resp., } \lim_{y\uparrow p(\overline{\iota})} \widetilde{\psi}(y) = \infty),$$

[6] As in Lemma 2.2.(iii), we use here $(\varphi'_-/p'_-)'$ and $(\psi'_-/p'_-)'$ to denote the Radon-Nikodym derivatives of the measures that identify with the first distributional derivatives of the functions φ'_-/p'_- and ψ'_-/p'_- with respect to the Lebesgue measure (see also footnote 5).

Furthermore, $\widetilde{\varphi}$ and $\widetilde{\psi}$ satisfy the homogeneous ODE in $\widetilde{g}$

$$\frac{1}{2}\big(\sigma \circ p^{-1}\big)^2(y)\,\big(p'_- \circ p^{-1}\big)^2(y)\,\widetilde{g}''(y) - \big(r \circ p^{-1}\big)(y)\,\widetilde{g}(y) = 0,$$

Lebesgue-a.e. in $p(\mathring{\mathcal{I}})$. This fact and the absolute continuity of p^{-1} imply that the ODE

$$\frac{1}{2}\sigma^2(x)(p'_-)^2(x)\widetilde{g}''\big(p(x)\big) - r(x)\widetilde{g}\big(p(x)\big) = 0$$

is satisfied Lebesgue-a.e. in $\mathring{\mathcal{I}}$ for $\widetilde{g}$ standing for either $\widetilde{\varphi}$ or $\widetilde{\psi}$. Combining these observations with (2.10) and Lemma 2.2.(iii), we obtain all of the required results. □

Example 2.5. Suppose that the measure ν is of the form

$$\nu(dz) = \frac{b(z)}{\sigma^2(z)}\,dz + \sum_{j=1}^{k} \beta_j\,\delta_{z_j}(dz),$$

for some function $b : \mathring{\mathcal{I}} \to \mathbb{R}$ such that

$$\int_{\underline{x}}^{\overline{x}} \frac{\big|b(u)\big|}{\sigma^2(u)}\,du < \infty \quad \text{for all } \underline{\iota} < \underline{x} < \overline{x} < \overline{\iota},$$

some constants $\beta_1, \ldots, \beta_k \in \left]-1,1\right[\setminus \{0\}$ and some distinct points $z_1, \ldots, z_k \in \mathring{\mathcal{I}}$, where $\delta_{z_j}(dz)$ is the Dirac probability measure that assigns unit mass on $\{z_j\}$. Using the occupation times formula, we can see that, in this case, X satisfies the SDE

$$X_t = x + \int_0^t b(X_s)\,ds + \sum_{j=1}^{k} \beta_j L_t^{z_j}(X) + \int_0^t \sigma(X_s)\,dW_s, \quad x \in \mathring{\mathcal{I}}.$$

In view of (2.2), the restriction of the scale function p to $\mathring{\mathcal{I}} \setminus \{z_1, \ldots, z_k\}$ has absolutely continuous first derivative p' $(= p'_- = p'_+)$ that satisfies the ODE

$$\frac{1}{2}\sigma^2(x)p''(x) + b(x)p'(x) = 0, \tag{2.11}$$

Lebesgue-a.e. in $\mathring{\mathcal{I}} \setminus \{z_1, \ldots, z_k\}$ (see also footnote 5 about p''). Furthermore, (2.4) implies that

$$(1+\beta_j)p'_+(z_j) = (1-\beta_j)p'_-(z_j), \quad \text{for } j = 1, \ldots, k. \tag{2.12}$$

Using these observations, we derive the expression

$$p'_+(x) = \exp\left(-\int_{x_1}^{x} \frac{2b(u)}{\sigma^2(u)}\,du\right) \prod_{j=1}^{k} \left(\frac{1-\beta_j}{1+\beta_j}\right)^{\mathbf{1}_{\{x_1 \le z_j \le x\}}}, \quad \text{for } x \ge x_1, \tag{2.13}$$

as well as a similar one for $x < x_1$, where $x_1 \in \mathring{\mathcal{I}}$ is an arbitrary fixed point. If we denote by g either of the excessive functions φ or ψ given by (2.6) and (2.7), then (2.12) and the (absolute) continuity of g'_-/p'_- (see Lemma 2.4) imply that

$$\frac{g'_-}{p'_-}(z_j) = \frac{g'_+}{p'_+}(z_j) \quad \Rightarrow \quad (1+\beta_j)g'_+(z_j) = (1-\beta_j)g'_-(z_j), \quad \text{for } j = 1, \ldots, k. \tag{2.14}$$

Furthermore, Lemma 2.4 and (2.11) imply that g satisfies the ODE

$$\frac{1}{2}\sigma^2(x)g''(x) + b(x)g'(x) - r(x)g(x) = 0, \tag{2.15}$$

Lebesgue-a.e. in $\mathring{\mathcal{I}} \setminus \{z_1, \ldots, z_k\}$ (see also footnotes 5, 6 about g'').

Example 2.6. In the case of the skew Brownian motion, which is the unique strong solution to the SDE (1.3), we can see that (2.13) yields the expressions

$$p(x) = \begin{cases} x, & \text{if } x < 0, \\ \frac{1-\beta}{1+\beta} x, & \text{if } x \geq 0, \end{cases} \quad \text{and} \quad p^{-1}(\widetilde{x}) = \begin{cases} \widetilde{x}, & \text{if } \widetilde{x} < 0, \\ \frac{1+\beta}{1-\beta} \widetilde{x}, & \text{if } \widetilde{x} \geq 0, \end{cases}$$

for the scale function p and its inverse p^{-1}. Accordingly, the SDE (2.5) takes the form

$$\begin{aligned} d\widetilde{X}_t &= \big(p'_- \circ p^{-1}\big)(\widetilde{X}_t)\, dW_t \\ &= \left(\mathbf{1}_{]-\infty,0]}(\widetilde{X}_t) + \frac{1-\beta}{1+\beta} \mathbf{1}_{]0,\infty[}(\widetilde{X}_t) \right) dW_t, \quad \widetilde{X}_0 = p(x) \in \mathbb{R}, \end{aligned} \tag{2.16}$$

where W is a standard Brownian motion. This SDE has a unique strong solution and the skew Brownian motion is the process $X = p^{-1}(\widetilde{X})$. To verify that this process indeed satisfies the SDE (1.3), we first use Itô-Tanaka's formula (e.g., see Assing and Schmidt [2, Proposition 1.14]) to obtain

$$\begin{aligned} dX_t &= \frac{1}{2} \left(\frac{1+\beta}{1-\beta} - 1 \right) dL_t^{\widetilde{X},0} + \big(p^{-1}\big)'_-(\widetilde{X}_t) \big(p'_- \circ p^{-1}\big)(\widetilde{X}_t)\, dW_t \\ &= \frac{\beta}{1-\beta}\, dL_t^{\widetilde{X},0} + dW_t, \end{aligned}$$

where $L^{\widetilde{X},0}$ is the local time process of $\widetilde{X}$ at level 0. In view of this result and the connection

$$L^0 = \frac{1}{2} \big[(p^{-1})'_+(0) + (p^{-1})'_-(0) \big] L^{\widetilde{X},0} = \frac{1}{1-\beta} L^{\widetilde{X},0},$$

of the local time L^0 of X with the local time $L^{\widetilde{X},0}$ of $\widetilde{X}$ (e.g., see Assing and Schmidt [2, Lemma 1.18] or Engelbert and Schmidt [14, Proposition 4.29.iii]), we can see that X satisfies the SDE (1.3).

3 The solution to the general optimal stopping problem

The value function of the optimal stopping problem that aims at maximising the performance criterion appearing in (1.5) is defined by

$$v(x) = \sup_{(\mathbb{S}_x, \tau) \in \mathcal{T}_x} \mathbb{E}_x \left[\exp\left(- \int_0^\tau r(X_s)\, ds \right) f(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right], \quad \text{for } x \in \mathcal{I}, \tag{3.1}$$

where the set of all stopping strategies $\mathcal{T}_x$ consists of all pairs $(\mathbb{S}_x, \tau)$ such that $\mathbb{S}_x = \big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x, W, X\big)$ is a weak solution to (1.1) and τ is an associated $(\mathcal{F}_t)$-stopping time. We assume that the discounting rate function r satisfies Assumption 2.3, while the reward function f satisfies the following assumption.

Assumption 3.1. *The positive function $f : \mathcal{I} \to \mathbb{R}_+$ is Borel-measurable and its restriction to $\mathring{\mathcal{I}}$ is upper semicontinuous, namely,*

$$f(x) = \limsup_{y \to x} f(y) \quad \textit{for all } x \in \mathring{\mathcal{I}}.$$

Our main result in this section establishes a complete characterisation of the general optimal stopping problem defined by (1.1), (3.1) in terms of solutions to the variational inequality (1.6) in the sense of distributions, which are introduced by the following definition.

Definition 3.2. *A function $v : \mathcal{I} \to \mathbb{R}_+$ is a solution to the variational inequality (1.6) if it satisfies the following conditions:*

(i) *The restriction of v to $\mathring{\mathcal{I}}$ is the difference of two convex functions.*

(ii) *The signed Radon measure on $\big(\mathring{\mathcal{I}}, \mathcal{B}(\mathring{\mathcal{I}})\big)$ defined by*[7]

$$\mu_v(dx) = -\frac{1}{2}\sigma^2(x)p'_-(x)\left(\frac{v'_-}{p'_-}\right)'(dx) + r(x)v(x)\,dx$$

is positive and such that $\mu_v\big(\big\{x \in \mathring{\mathcal{I}} \mid \ v(x) > f(x)\big\}\big) = 0$.

(iii) *The inequality $v(x) \geq f(x)$ holds true for all $x \in \mathring{\mathcal{I}}$.*

Theorem 3.3. *Consider the optimal stopping problem defined by (1.1) and (3.1), and recall the minimal excessive functions φ and ψ given by (2.6) and (2.7). The following statements hold true:*

(i) *If the problem data is such that*

$$f(x) < \infty \text{ for all } x \in \mathcal{I}, \quad \limsup_{y \downarrow \underline{\iota}} \frac{f(y)}{\varphi(y)} < \infty \quad \text{and} \quad \limsup_{y \uparrow \bar{\iota}} \frac{f(y)}{\psi(y)} < \infty, \tag{3.2}$$

then $v(x) < \infty$ for all $x \in \mathcal{I}$. If any of the inequalities in (3.2) fails, then $v(x) = \infty$ for all $x \in \mathcal{I}$.

(ii) *If the problem data is such that the inequalities in (3.2) all hold true, then the value function v satisfies the variational inequality (1.6) in the sense of Definition 3.2,*

$$\lim_{y \in \mathring{\mathcal{I}},\, y \downarrow \underline{\iota}} \frac{v(y)}{\varphi(y)} = \limsup_{y \downarrow \underline{\iota}} \frac{f(y)}{\varphi(y)}, \quad \lim_{y \in \mathring{\mathcal{I}},\, y \uparrow \bar{\iota}} \frac{v(y)}{\psi(y)} = \limsup_{y \uparrow \bar{\iota}} \frac{f(y)}{\psi(y)} \tag{3.3}$$
$$\text{and} \quad v(\underline{\iota}) = f(\underline{\iota}) \ \big(\text{resp., } v(\bar{\iota}) = f(\bar{\iota})\big) \quad \text{if } \underline{\iota} \text{ (resp., } \bar{\iota}) \text{ is absorbing.}$$

(iii) *In the presence of (3.2), if a positive function $w : \mathcal{I} \to \mathbb{R}_+$ satisfies the variational inequality (1.6) in the sense of Definition 3.2 as well as the growth conditions*

$$\lim_{y \in \mathring{\mathcal{I}},\, y \downarrow \underline{\iota}} \frac{w(y)}{\varphi(y)} = \limsup_{y \downarrow \underline{\iota}} \frac{f(y)}{\varphi(y)}, \quad \lim_{y \in \mathring{\mathcal{I}},\, y \uparrow \bar{\iota}} \frac{w(y)}{\psi(y)} = \limsup_{y \uparrow \bar{\iota}} \frac{f(y)}{\psi(y)} \tag{3.4}$$
$$\text{and} \quad w(\underline{\iota}) = f(\underline{\iota}) \ \big(\text{resp., } w(\bar{\iota}) = f(\bar{\iota})\big) \quad \text{if } \underline{\iota} \text{ (resp., } \bar{\iota}) \text{ is absorbing,}$$

then $w(x) = v(x)$ for all $x \in \mathcal{I}$.

(iv) *If*

$$\limsup_{y \downarrow \underline{\iota}} \frac{f(y)}{\varphi(y)} = 0 \text{ if } \underline{\iota} \text{ is inaccessible}, \quad \limsup_{y \uparrow \bar{\iota}} \frac{f(y)}{\psi(y)} = 0 \text{ if } \bar{\iota} \text{ is inaccessible},$$
$$f(\underline{\iota}) = \limsup_{y \downarrow \underline{\iota}} f(y) \text{ if } \underline{\iota} \text{ is absorbing} \quad \text{and} \quad f(\bar{\iota}) = \limsup_{y \uparrow \bar{\iota}} f(y) \text{ if } \bar{\iota} \text{ is absorbing},$$

then the stopping strategy $(\mathbb{S}_x, \tau^\star)$, where $\mathbb{S}_x$ is a weak solution to (1.1) and

$$\tau^\star = \inf\big\{t \geq 0 \mid \ v(X_t) = f(X_t)\big\},$$

is optimal.

Proof. In view of the results connecting the solvability of (1.1) with the solvability of (2.5) that we have already used in the proof of Lemma 2.4, we can see that

$$v(x) = \sup_{(\mathbb{S}_x, \tau) \in \mathcal{T}_x} \mathbb{E}_x\left[e^{-\int_0^\tau (r \circ p^{-1})(\widetilde{X}_s)\,ds}\widetilde{f}(\widetilde{X}_\tau)\right] =: \widetilde{v}\big(p(x)\big) \quad \text{for all } x \in \mathcal{I}, \tag{3.5}$$

[7] See Lemma 2.2 about the measure $(v'_-/p'_-)'$.

where $\widetilde{f} = f \circ p^{-1}$. Also, we note that

$$\limsup_{y \downarrow \underline{\iota}} \frac{f(y)}{\varphi(y)} = \limsup_{y \downarrow \underline{\iota}} \frac{\widetilde{f}\big(p(y)\big)}{\widetilde{\varphi}\big(p(y)\big)} \quad \text{and} \quad \limsup_{y \uparrow \overline{\iota}} \frac{f(y)}{\psi(y)} = \limsup_{y \uparrow \overline{\iota}} \frac{\widetilde{f}\big(p(y)\big)}{\widetilde{\psi}\big(p(y)\big)}, \tag{3.6}$$

where $\widetilde{\varphi} = \varphi \circ p^{-1}$ and $\widetilde{\psi} = \psi \circ p^{-1}$ are the minimal $(r \circ p^{-1})$-excessive functions of the diffusion associated with the SDE (2.5), given by (2.8) and (2.9).

The identities in (3.6) and Theorem 6.3.(I) in Lamberton and Zervos [22] imply (i). If the inequalities in (3.2) all hold true, then Theorem 6.3 in Lamberton and Zervos [22] asserts that the restriction of the value function $\widetilde{v}$ to $p(\mathring{\mathcal{I}})$ is the difference of two convex functions and satisfies the variational inequality

$$\max\left\{\frac{1}{2}\big(\sigma \circ p^{-1}\big)^2(q)\big(p'_- \circ p^{-1}\big)^2(q)\, \widetilde{v}''(dq) - \big(r \circ p^{-1}\big)(q)\widetilde{v}(q)\, dq,\ \widetilde{f}(q) - \widetilde{v}(q)\right\} = 0$$

in the sense that the Radon measure on $\big(p(\mathring{\mathcal{I}}), \mathcal{B}(p(\mathring{\mathcal{I}}))\big)$ defined by

$$\mu_{\widetilde{v}}(dq) = -\frac{1}{2}\big(\sigma \circ p^{-1}\big)^2(q)\big(p'_- \circ p^{-1}\big)^2(q)\, \widetilde{v}''(dq) + \big(r \circ p^{-1}\big)(q)\widetilde{v}(q)\, dq$$

is positive and such that $\mu_{\widetilde{v}}\big(\{q \in p(\mathring{\mathcal{I}}) \mid \ \widetilde{v}(q) > \widetilde{f}(q)\}\big) = 0$, while $\widetilde{v}(q) \geq \widetilde{f}(q)$ for all $q \in p(\mathring{\mathcal{I}})$.

In view of Lemma 2.2.(i)-(ii), $v = \widetilde{v} \circ p$ is the difference of two convex functions and $\mu_v = \mu_{\widetilde{v}} \circ p$ because $(v'_-/p'_-)' = \widetilde{v}'' \circ p$. Combining these observations with (3.5)–(3.6) and Theorems 6.3, 6.4 in Lamberton and Zervos [22], we obtain all of the required results in (ii)-(iv). □

Remark 3.4. It is worth stressing the precise nature of the boundary conditions appearing in (3.3) and (3.4). The existence of the limits on the left-hand side of (3.3) is a result, while, the existence of the limits on the left-hand side of (3.4) is an assumption. Also, the limits on the left-hand sides of (3.3), (3.4) are taken from inside the interior $\mathring{\mathcal{I}}$ of $\mathcal{I}$. On the other hand, the limsups on the right-hand sides of (3.3), (3.4) are taken from inside $\mathcal{I}$ itself. In view of these observations, we can see that, e.g., if $\underline{\iota}$ is absorbing, then we are faced in (3.3) with either the possibility that

$$v(\underline{\iota}) = f(\underline{\iota}) = \lim_{y \in \mathring{\mathcal{I}},\, y \downarrow \underline{\iota}} v(y) = \limsup_{y \downarrow \underline{\iota}} f(y), \quad \text{if } f(\underline{\iota}) = \limsup_{y \downarrow \underline{\iota}} f(y) \geq \limsup_{y \in \mathring{\mathcal{I}},\, y \downarrow \underline{\iota}} f(y),$$

or the possibility that

$$v(\underline{\iota}) = f(\underline{\iota}) < \lim_{y \in \mathring{\mathcal{I}},\, y \downarrow \underline{\iota}} v(y) = \limsup_{y \downarrow \underline{\iota}} f(y), \quad \text{if } f(\underline{\iota}) < \limsup_{y \downarrow \underline{\iota}} f(y) = \limsup_{y \in \mathring{\mathcal{I}},\, y \downarrow \underline{\iota}} f(y),$$

where we have used the fact that, in this case, $\varphi(\underline{\iota}) := \lim_{x \downarrow \underline{\iota}} \varphi(x) < \infty$ (see Lemma 2.4).

Example 3.5. Suppose that the measure ν is as in Example 2.5. Given $\underline{x} < \overline{x}$ such that $[\underline{x}, \overline{x}[\subseteq \mathring{\mathcal{I}} \setminus \{z_1, \ldots, z_k\}$, we use the integration by parts formula and the fact that the scale function p has absolutely continuous derivative satisfying (2.11) to calculate

$$\frac{v'_-}{p'}(\overline{x}) - \frac{v'_-}{p'}(\underline{x}) = \int_{[\underline{x}, \overline{x}[} \left[\frac{1}{p'(y)} v''(dy) + \frac{2b(y)}{\sigma^2(y)p'(y)} v'_-(y)\, dy\right].$$

Also, we note that the measure μ_v defined in Definition 3.2.(ii) is such that

$$\begin{aligned} \mu_v\big(\{z_j\}\big) &= -\frac{1}{2}\sigma^2(z_j)p'_-(z_j)\left[\frac{v'_+}{p'_+}(z_j) - \frac{v'_-}{p'_-}(z_j)\right] \\ &\overset{(2.4)}{=} -\frac{\sigma^2(z_j)}{2(1-\beta_j)}\big[(1+\beta_j)v'_+(z_j) - (1-\beta_j)v'_-(z_j)\big]. \end{aligned}$$

It follows that, in this case, the variational inequality (1.6) takes the form

$$\max\left\{\frac{1}{2}\sigma^2(x)v''(dx)+b(x)v'_-(x)\,dx-r(x)v(x)\,dx,\;f(x)-v(x)\right\}=0 \tag{3.7}$$

inside $\big(\mathring{\mathcal{I}}\setminus\{z_1,\ldots,z_k\},\mathcal{B}(\mathring{\mathcal{I}}\setminus\{z_1,\ldots,z_k\})\big)$, coupled with the conditions

$$\max\Big\{(1+\beta_j)v'_+(z_j)-(1-\beta_j)v'_-(z_j),\;f(z_j)-v(z_j)\Big\}=0,\quad \text{for } j=1,\ldots,k. \tag{3.8}$$

Furthermore, if v has absolutely continuous first derivative, namely, if $v''(dx)$ is equal to $v''(x)\,dx$ (see also footnote 5), then v should satisfy

$$\max\left\{\frac{1}{2}\sigma^2(x)v''(x)+b(x)v'(x)-r(x)v(x),\;f(x)-v(x)\right\}=0, \tag{3.9}$$

Lebesgue-a.e. in $\mathring{\mathcal{I}}\setminus\{z_1,\ldots,z_k\}$ as well as the conditions (3.8).

Remark 3.6. To appreciate how variational inequalities can be used to systematically identify critical parts of the state space $\mathcal{I}$ that belong to the waiting region, consider the previous example. In this context, we can make the following observations:

(a) Given $x\in\mathring{\mathcal{I}}$, if either of the limits

$$f'_+(x)=\lim_{\varepsilon\downarrow 0}\frac{f(x+\varepsilon)-f(x)}{\varepsilon}\quad\text{and}\quad f'_-(x)=\lim_{\varepsilon\downarrow 0}\frac{f(x)-f(x-\varepsilon)}{\varepsilon},$$

does not exist, then x belongs to the closure of the waiting region.

(b) Given $j=1,\ldots,k$, suppose that both of the derivatives $f'_\pm(z_j)$ exist and

$$(1+\beta_j)f'_+(z_j)-(1-\beta_j)f'_-(z_j)>0. \tag{3.10}$$

In this case, (3.8) implies that z_j belongs to the waiting region. In particular, if f is C^1 at z_j, then z_j belongs to the waiting region if $\beta_j\in\,]0,1[$. To see these claims, we argue by contradiction and we assume that z_j belongs to the stopping region, namely, $f(z_j)=v(z_j)$. Such an assumption and the fact that $f\le v$ imply that

$$f'_+(z_j)=\lim_{\varepsilon\downarrow 0}\frac{f(z_j+\varepsilon)-f(z_j)}{\varepsilon}\le\lim_{\varepsilon\downarrow 0}\frac{v(z_j+\varepsilon)-v(z_j)}{\varepsilon}=v'_+(z_j)$$

and

$$f'_-(z_j)=\lim_{\varepsilon\downarrow 0}\frac{f(z_j)-f(z_j-\varepsilon)}{\varepsilon}\ge\lim_{\varepsilon\downarrow 0}\frac{v(z_j)-v(z_j-\varepsilon)}{\varepsilon}=v'_-(z_j).$$

Combining these observations with (3.10), we obtain the inequalities

$$(1+\beta_j)v'_+(z_j)-(1-\beta_j)v'_-(z_j)\ge(1+\beta_j)f'_+(z_j)-(1-\beta_j)f'_-(z_j)>0,$$

which contradict (3.8).

(c) Suppose that the restriction of f to an interval $]\underline{\iota},\overline{\iota}[\,\subseteq\mathcal{I}$ is C^2. The validity of (3.7) implies that

$$\left\{x\in\,]\underline{\iota},\overline{\iota}[\;\middle|\;\frac{1}{2}\sigma^2(x)f''(x)+b(x)f'(x)-r(x)f(x)>0\right\}$$

is a subset of the waiting region. On the other hand, the intersection of the stopping region with $]\underline{\iota},\overline{\iota}[$ is a (usually strict) subset of the complement of this set.

Remark 3.7. In the context of Example 3.5, we can make the following observations relative to the so-called "principle of smooth fit":

(a) If f is C^1 then (3.7) implies that the restriction of the value function v to $\mathring{\mathcal{I}} \setminus \{z_1, \ldots, z_k\}$ is C^1 (see Lamberton and Zervos [22, Corollary 7.5]).

(b) Given $j = 1, \ldots, k$, if z_j belongs to the stopping region, namely, $f(z_j) = v(z_j)$, then (2.12) and (3.8) imply that

$$\frac{v'_+(z_j)}{p'_+(z_j)} - \frac{v'_-(z_j)}{p'_-(z_j)} = \frac{1}{(1-\beta_j)p'_-(z_j)} \big[(1+\beta_j)v'_+(z_j) - (1-\beta_j)v'_-(z_j)\big] \leq 0. \tag{3.11}$$

In general, this inequality can be strict: see Remark 6.2 in Section 6.
(c) Given $j = 1, \ldots, k$, if z_j belongs to the waiting region, namely, $f(z_j) < v(z_j)$, then (2.12) and (3.8) imply that

$$\frac{v'_+(z_j)}{p'_+(z_j)} - \frac{v'_-(z_j)}{p'_-(z_j)} = 0.$$

4 The minimal excessive functions of a skew geometric Brownian motion

From this point onwards, we fix a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\big)$ satisfying the usual conditions and supporting a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. In such a setting, we denote by X the unique non-explosive strong solution to the SDE (1.4).

The conditions (2.14) in Example 2.5 reduce to

$$(1+\beta)g'_+(z) = (1-\beta)g'_-(z), \tag{4.1}$$

while, given a constant $r > 0$, the ODE (2.15) in Example 2.5 reduces to the Euler ODE

$$\frac{1}{2}\sigma^2 x^2 g''(x) + bxg'(x) - rg(x) = 0. \tag{4.2}$$

It is well-known that every solution to (4.2) is given by

$$g(x) = Ax^n + Bx^m,$$

for some constants $A, B \in \mathbb{R}$, where $m < 0 < n$ are the solutions to the quadratic equation

$$\frac{1}{2}\sigma^2 k^2 + \left(b - \frac{1}{2}\sigma^2\right) k - r = 0,$$

given by

$$m, n = \frac{-\left(b - \frac{1}{2}\sigma^2\right) \mp \sqrt{\left(b - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}}{\sigma^2}.$$

It is straightforward to verify that

$$r > b \;\Leftrightarrow\; n > 1, \quad n + m - 1 = -\frac{2b}{\sigma^2}, \quad nm = -\frac{2r}{\sigma^2}, \tag{4.3}$$

$$r - bm = \frac{1}{2}\sigma^2 m(m-1) > 0 \quad \text{and} \quad r - bn = \frac{1}{2}\sigma^2 n(n-1) > 0. \tag{4.4}$$

The excessive functions $\psi = \psi(\cdot; z)$ and $\varphi = \varphi(\cdot; z)$ that satisfy the ODE (4.2) inside $]0, z[\cup]z, \infty[$ as well as the condition (4.1) are given by

$$\psi(x; z) = \begin{cases} x^n, & \text{if } x < z, \\ Ax^n + B(z)x^m, & \text{if } x \geq z, \end{cases} \tag{4.5}$$

and

$$\varphi(x; z) = \begin{cases} C(z)x^n + Dx^m, & \text{if } x < z, \\ x^m, & \text{if } x \geq z, \end{cases}$$

for

$$A = \frac{n(1-\beta) - m(1+\beta)}{(n-m)(1+\beta)} \begin{cases} > 1, & \text{if } \beta \in]-1, 0[, \\ \in]0, 1[, & \text{if } \beta \in]0, 1[, \end{cases} \tag{4.6}$$

$$B(z) = \frac{2n\beta z^{n-m}}{(n-m)(1+\beta)} \begin{cases} < 0, & \text{if } \beta \in]-1, 0[, \\ > 0, & \text{if } \beta \in]0, 1[, \end{cases} \tag{4.7}$$

$$C(z) = \frac{2n\beta z^{n-m}}{(n-m)(1-\beta)} \begin{cases} < 0, & \text{if } \beta \in]-1, 0[, \\ > 0, & \text{if } \beta \in]0, 1[, \end{cases}$$

and

$$D = \frac{n(1-\beta) - m(1+\beta)}{(n-m)(1-\beta)} \begin{cases} \in]0, 1[, & \text{if } \beta \in]-1, 0[, \\ > 1, & \text{if } \beta \in]0, 1[. \end{cases}$$

It is straightforward to verify that

$$\psi'(z-; z) \equiv nz^{n-1} < nAz^{n-1} + mB(z)z^{m-1} \equiv \psi'(z+; z) \quad \Leftrightarrow \quad \beta < 0. \tag{4.8}$$

Here, as well as in the rest of the paper, we adopt the notation $\psi'(x; z) = \frac{\partial \psi}{\partial x}(x; z)$ and $\psi''(x; z) = \frac{\partial^2 \psi}{\partial x^2}(x; z)$.

In the rest of the paper, we make the following assumption, which is sufficient for the value function of the optimal stopping problem defined by (1.4) and (1.7) to be real-valued.

Assumption 4.1. $r > b \overset{(4.3)}{\Leftrightarrow} n > 1$.

Indeed, if $r < b \Leftrightarrow n < 1$, then Theorem 3.3.(i) implies that the value function given by (1.7) is identically equal to ∞.

In the presence of Assumption 4.1, we can verify that

$$\frac{r}{r-b} = \frac{nm}{(n-1)(m-1)} < \frac{n}{n-1} \tag{4.9}$$

and

$$\beta_c := \frac{n-1}{n+2m-1} \in \begin{cases}]1, \infty[, & \text{if } n + 2m - 1 > 0, \\ ]-\infty, -1], & \text{if } n + 2m - 1 < 0 \text{ and } b \leq 0, \\ ]-1, 0[, & \text{if } n + 2m - 1 < 0 \text{ and } b > 0. \end{cases} \tag{4.10}$$

Here, deriving the possible values of β_c involves the observation that, if $n + 2m - 1 < 0$, then

$$\beta_c \in]-1, 0[\quad \Leftrightarrow \quad n + m - 1 < 0 \quad \overset{(4.3)}{\Leftrightarrow} \quad b > 0. \tag{4.11}$$

Combining the range of values of the point β_{c} given by (4.10), with the observation that

$$(n-1)(1-\beta) - 2m\beta < 0 \quad \Leftrightarrow \quad \beta \begin{cases} > \beta_{\mathrm{c}}, & \text{if } n+2m-1 > 0, \\ < \beta_{\mathrm{c}}, & \text{if } n+2m-1 < 0, \end{cases}$$

we can see that

$$\begin{aligned} &\text{given any } \beta \in \left]-1,1\right[, \\ &\qquad (n-1)(1-\beta) - 2m\beta < 0 \quad \Leftrightarrow \quad \Big(b > 0 \text{ and } \beta \in \left]-1, \beta_{\mathrm{c}}\right[\Big). \end{aligned} \tag{4.12}$$

Furthermore,

$$\begin{aligned} &\text{given any } \beta \in \left]-1,0\right[, \\ &\qquad (n-1)(1-\beta) - 2m\beta < 0 \quad \Leftrightarrow \quad \frac{n}{n-\frac{1+\beta}{1-\beta}} < \frac{r}{r-b}. \end{aligned} \tag{4.13}$$

In the following result, we concentrate on the increasing function ψ because only this is involved in the solution to the optimal stopping problem we consider in this main section. In particular, the critical points defined by

$$\mathfrak{z}_{\mathrm{c}} = \frac{rK}{r-b}, \quad \mathfrak{z}_{\beta} = \frac{nK}{n-\frac{1+\beta}{1-\beta}}, \text{ if } n \neq \frac{1+\beta}{1-\beta}, \quad \text{and} \quad \mathfrak{z}_0 = \frac{nK}{n-1} \tag{4.14}$$

play a critical role in differentiating the different qualitative forms of the optimal strategy.

Lemma 4.2. *Suppose that Assumption 4.1 holds true. The function $\psi(\cdot\,; z)$ defined by (4.5)–(4.7) is such that the following statements hold true:*

(I) *If $b \leq 0$ and $\beta \in \left]-1,0\right[$, then $\psi(\cdot\,; z)$ is convex.*

(II) *If $b > 0$ and $\beta \in \left[\beta_{\mathrm{c}}, 0\right[$, where $\beta_{\mathrm{c}} \in \left]-1,0\right[$ is defined by (4.10), then $\psi(\cdot\,; z)$ is convex.*

(III) *If $b > 0$ and $\beta \in \left]-1, \beta_{\mathrm{c}}\right[$, where $\beta_{\mathrm{c}} \in \left]-1,0\right[$ is defined by (4.10), then the restrictions of $\psi(\cdot\,; z)$ to $[0, z]$ as well as to $\left[\mathfrak{C}^{-1}z, \infty\right[$ are convex, while the restriction of $\psi(\cdot\,; z)$ to $[z, \mathfrak{C}^{-1}z]$ is concave, where*

$$\mathfrak{C} = \left(-\frac{(n-1)\big[n(1-\beta) - m(1+\beta)\big]}{2m(m-1)\beta}\right)^{\frac{1}{n-m}} \in \left]0,1\right[. \tag{4.15}$$

(IV) *If $\beta \in \left]0,1\right[$, then the restrictions of $\psi(\cdot\,; z)$ to $[0, z]$ as well as to $\left[z, \infty\right[$ are convex but $\psi(\cdot\,; z)$ is not convex in its entire domain.*

Cases (I) and (II) are illustrated by Figure 1, while Cases (III) and (IV) are illustrated by Figures 2 and 3, respectively. Furthermore, the critical points $\mathfrak{z}_{\mathrm{c}}$, $\mathfrak{z}_{\beta}$, $\mathfrak{z}_0$ defined by (4.14) are such that,

$$\textit{in Cases (I), (II)}, \quad (n-1)(1-\beta) - 2m\beta \geq 0 \ \textit{ and } \ 0 < \mathfrak{z}_{\mathrm{c}} \leq \mathfrak{z}_{\beta} < \mathfrak{z}_0, \tag{4.16}$$

$$\textit{in Case (III)}, \quad (n-1)(1-\beta) - 2m\beta < 0 \ \textit{ and } \ 0 < \mathfrak{z}_{\beta} < \mathfrak{z}_{\mathrm{c}} < \mathfrak{z}_0, \tag{4.17}$$

$$\begin{aligned} \textit{and in Case (IV)}, \quad & (n-1)(1-\beta) - 2m\beta < 0, \quad 0 < \mathfrak{z}_{\mathrm{c}} < \mathfrak{z}_0, \\ & \frac{1+\beta}{1-\beta} < n \ \Rightarrow \ 0 < \mathfrak{z}_0 < \mathfrak{z}_{\beta} \ \textit{ and } \ n < \frac{1+\beta}{1-\beta} \ \Rightarrow \ \mathfrak{z}_{\beta} < 0 < \mathfrak{z}_0, \end{aligned} \tag{4.18}$$

with equalities (resp., strict inequalities) in place of weak inequalities in (4.16) if $b > 0$ and $\beta = \beta_{\mathrm{c}}$ (resp., otherwise).

Proof. We first note that $\psi(\cdot; z)$ is always convex in $[0, z]$. On the other hand, the inequality $n > 1$ and the calculation

$$\psi''(x; z) \quad = \big[n(n-1)Ax^{n-m} + m(m-1)B(z)\big]x^{m-2}, \quad \text{for } x > z,$$

imply the equivalences

$$\psi''(x; z) > 0 \text{ for all } x > z \quad \Leftrightarrow \quad \psi''(z+; z) \equiv \frac{n\big[(n-1)(1-\beta) - 2m\beta\big]z^{n-2}}{1+\beta} \geq 0$$
$$\Leftrightarrow \quad (n-1)(1-\beta) - 2m\beta \geq 0.$$

Combining these observations with (4.8)–(4.14), we obtain the required results. □

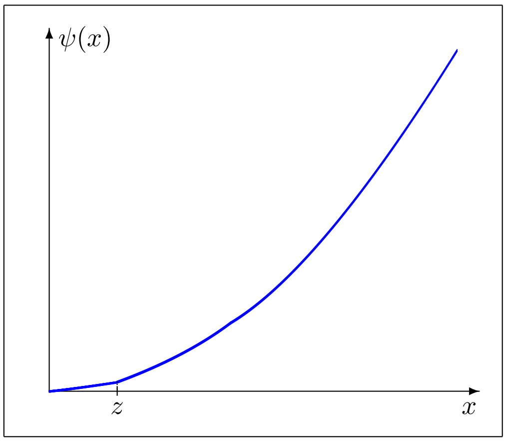

Figure 1. Graph of the function ψ in Cases (I) and (II) of Lemma 4.2.

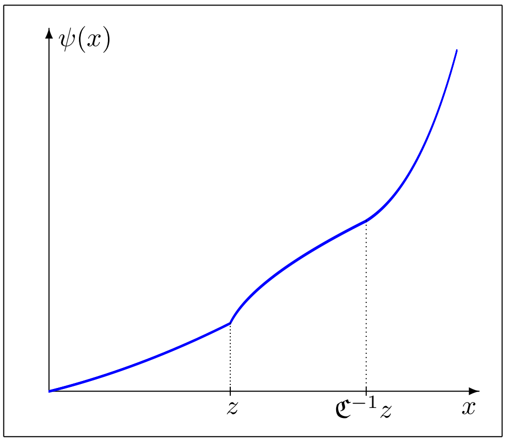

Figure 2. Graph of the function ψ in Case (III) of Lemma 4.2.

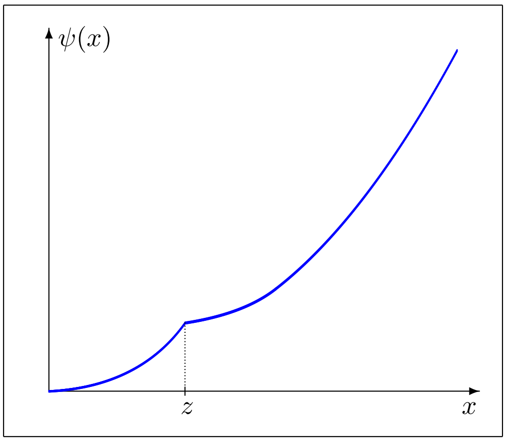

Figure 3. Graph of the function ψ in Case (IV) of Lemma 4.2.

5 Preliminary analytic results for the solution to the optimal stopping problem defined by (1.4) and (1.7)

We now establish a pair of technical analytic results that we will need for the solution of the optimal stopping problem defined by (1.4) and (1.7), which we derive in the next section. (This section can easily be skipped at a first reading.) Given any $z > 0$ fixed, we consider the equation

$$F(x; z) = 0, \tag{5.1}$$

for $x > z$, where F is the function defined by

$$F(x;z) = \big[(n-1)x - nK\big]Ax^{n-m} + \big[(m-1)x - mK\big]B(z), \quad \text{for } x > 0, \tag{5.2}$$

which admits the expression

$$F(x;z) = \big[(x-K)\psi'(x;z) - \psi(x;z)\big]x^{-m+1}, \quad \text{for } x > z. \tag{5.3}$$

The following result involves the critical points $\mathfrak{z}_c$, $\mathfrak{z}_\beta$, $\mathfrak{z}_0$ defined by (4.14) and is structured based on the four cases of Lemma 4.2.

Lemma 5.1. *In the presence of Assumption 4.1, the following statements hold true:*

(i) *If the problem's parameters are as in Cases (I) or (II) of Lemma 4.2, then equation (5.1) defines uniquely a strictly decreasing C^1 function $\alpha :]0, \mathfrak{z}_\beta[\to]\mathfrak{z}_\beta, \mathfrak{z}_0[$ such that*

$$\lim_{z\to 0} \alpha(z) = \mathfrak{z}_0, \quad \lim_{z\to \mathfrak{z}_\beta} \alpha(z) = \mathfrak{z}_\beta, \tag{5.4}$$

$$F(x;z) \begin{cases} < 0 & \text{for all } x \in [z \vee K, \alpha(z)[, \\ > 0 & \text{for all } x > \alpha(z), \end{cases} \quad \text{and} \quad \frac{\partial F}{\partial x}(x;z) > 0 \text{ for all } x \geq \alpha(z). \tag{5.5}$$

(ii) *If the problem's parameters are as in Case (III) of Lemma 4.2, then $K < \mathfrak{C}\mathfrak{z}_c$, where $\mathfrak{C} \in]0,1[$ is given by (4.15), and equation (5.1) defines uniquely a strictly decreasing C^1 function $\alpha :]0, \mathfrak{C}\mathfrak{z}_c[\to]\mathfrak{z}_c, \mathfrak{z}_0[$ such that*

$$\lim_{z\to 0} \alpha(z) = \mathfrak{z}_0, \quad \lim_{z\to \mathfrak{C}\mathfrak{z}_c} \alpha(z) = \mathfrak{z}_c, \tag{5.6}$$

$$F(x;z) \begin{cases} < 0 & \text{for all } x \in [\mathfrak{z}_c, \alpha(z)[, \\ > 0 & \text{for all } x > \alpha(z), \end{cases} \quad \text{and} \quad \frac{\partial F}{\partial x}(x;z) > 0 \text{ for all } x \geq \alpha(z). \tag{5.7}$$

Furthermore,

$$F(x;z) > 0 \quad \text{for all } z \in [\mathfrak{C}\mathfrak{z}_c, \mathfrak{z}_c[\text{ and } x > \mathfrak{z}_c. \tag{5.8}$$

(iii) *If $\beta \in]0,1[$ (Case (IV) of Lemma 4.2), then equation (5.1) defines uniquely a C^1 function $\alpha :]0,\infty[\to]\mathfrak{z}_0, \infty[$ such that*

$$\alpha(z) \in \begin{cases}]z \vee \mathfrak{z}_0, \infty[\ \text{for all } z \in]0,\infty[, & \text{if } n \leq \frac{1+\beta}{1-\beta}, \\ ]z \vee \mathfrak{z}_0, \mathfrak{z}_\beta[\ \text{for all } z \in]0, \mathfrak{z}_\beta[, & \text{if } n > \frac{1+\beta}{1-\beta}, \\ ]\mathfrak{z}_\beta, z[\ \text{for all } z \in]\mathfrak{z}_\beta, \infty[, & \text{if } n > \frac{1+\beta}{1-\beta}, \end{cases} \tag{5.9}$$

$$F(x;z) \begin{cases} < 0 & \text{for all } x \in]z \wedge \mathfrak{z}_0, \alpha(z)[, \\ > 0 & \text{for all } x > \alpha(z), \end{cases} \quad \text{and} \quad \frac{\partial F}{\partial x}(x;z) > 0 \text{ for all } x \geq \alpha(z). \tag{5.10}$$

Proof. Throughout the proof, we use repeatedly the expressions and signs of A, B, given by (4.6), (4.7), as well as the results in (4.3), (4.4) and (4.9) without special mention. We first note that

$$\lim_{x\to 0} F(x;z) = -mKB(z) \begin{cases} < 0, & \text{if } \beta < 0, \\ > 0, & \text{if } \beta > 0, \end{cases} \tag{5.11}$$

$$F(\mathfrak{z}_0;z) = -\frac{(n-m)K}{n-1}B(z) \begin{cases} > 0, & \text{if } \beta < 0, \\ < 0, & \text{if } \beta > 0, \end{cases} \quad \text{and} \quad \lim_{x\to\infty} F(x;z) = \infty. \tag{5.12}$$

We also calculate

$$\frac{\partial F}{\partial x}(x;z) = (n-1)\Big[(n-m+1)x - (n-m)\mathfrak{z}_0\Big] A x^{n-m-1} + (m-1)B(z), \tag{5.13}$$

$$\frac{\partial^2 F}{\partial x^2}(x;z) = (n-1)(n-m)\Big[(n-m+1)x - (n-m-1)\mathfrak{z}_0\Big] A x^{n-m-2}, \tag{5.14}$$

$$\frac{\partial F}{\partial z}(x;z) = \frac{2n\big[(m-1)x - mK\big]\beta}{1+\beta} z^{n-m-1}, \tag{5.15}$$

and

$$F(z;z) = \frac{1-\beta}{1+\beta}\left[\left(n - \frac{1+\beta}{1-\beta}\right) z - nK\right] z^{n-m}. \tag{5.16}$$

The calculation (5.14) implies that

$$\frac{\partial F}{\partial x}(\cdot;z) \text{ is strictly} \begin{cases} \text{decreasing in }]0, x_\dagger[, \\ \text{increasing in }]x_\dagger, \infty[, \end{cases}$$

where

$$x_\dagger = \frac{(n-m-1)\mathfrak{z}_0}{n-m+1} \in \,]0, \mathfrak{z}_0[. \tag{5.17}$$

Combining this observation with the limits

$$\lim_{x\to 0} \frac{\partial F}{\partial x}(x;z) = (m-1)B(z) \begin{cases} >0, & \text{if } \beta < 0, \\ <0, & \text{if } \beta > 0, \end{cases} \quad \text{and} \quad \lim_{x\to\infty} \frac{\partial F}{\partial x}(x;z) = \infty,$$

which follow from (5.13), we can see that

$$\text{if } \beta < 0 \text{ and } \frac{\partial F}{\partial x}(x_\dagger;z) \geq 0, \quad \text{then } \frac{\partial F}{\partial x}(x;z) \geq 0 \text{ for all } x > 0,$$

or there exist strictly positive constants $\underline{x}_\dagger(z) < x_\dagger < \overline{x}_\dagger(z)$ such that

$$\begin{aligned} &\text{if } \beta < 0 \text{ and } \frac{\partial F}{\partial x}(x_\dagger;z) < 0, \\ &\qquad\qquad \text{then } \frac{\partial F}{\partial x}(x;z) \begin{cases} >0, & \text{for all } x \in \,]0, \underline{x}_\dagger(z)[\,\cup\,]\overline{x}_\dagger(z), \infty[, \\ <0, & \text{for all } x \in \,]\underline{x}_\dagger(z), \overline{x}_\dagger(z)[, \end{cases} \end{aligned} \tag{5.18}$$

or there exists a constant $x_\ddagger(z) > x_\dagger$ such that

$$\text{if } \beta > 0, \quad \text{then } \frac{\partial F}{\partial x}(x;z) \begin{cases} <0, & \text{for all } x \in \,]0, x_\ddagger(z)[, \\ >0, & \text{for all } x \in \,]x_\ddagger(z), \infty[. \end{cases} \tag{5.19}$$

Keeping in mind that $n > 1$, and $\mathfrak{z}_\beta > 0$ if and only if $n > \frac{1+\beta}{1-\beta}$, we can see that

$$\begin{aligned} F(\mathfrak{z}_\beta; z) &= \frac{K}{\left(n - \frac{1+\beta}{1-\beta}\right)(1-\beta)} \Big[2n\beta A \mathfrak{z}_\beta^{n-m} - \big[n(1-\beta) - m(1+\beta)\big] B(z)\Big] \\ &= \frac{2\big[n(1-\beta) - m(1+\beta)\big]\beta}{(n-m)(1-\beta)(1+\beta)} \mathfrak{z}_\beta \big[\mathfrak{z}_\beta^{n-m} - z^{n-m}\big] \\ &\begin{cases} <0, & \text{if } (\beta < 0 \text{ and } z < \mathfrak{z}_\beta) \text{ or } (\beta > 0 \text{ and } z > \mathfrak{z}_\beta), \\ >0, & \text{if } (\beta < 0 \text{ and } z > \mathfrak{z}_\beta) \text{ or } (\beta > 0 \text{ and } z < \mathfrak{z}_\beta). \end{cases} \end{aligned} \tag{5.20}$$

Furthermore, we can use the definition (5.2) of F and (5.13) to see that, given any $\beta \in \left]-1, 0\right[$,

$$\begin{aligned}\frac{\partial F}{\partial x}(K; z) &= -(m-1)K^{-1}F(K; z)\\ &= \frac{(m-1)K^{n-m}}{(n-m)(1+\beta)}\left[n(1-\beta) - m(1+\beta) + 2n\beta\left(\frac{z}{K}\right)^{n-m}\right]\\ &< 0 \qquad \text{for all } z < \widetilde{z},\end{aligned} \tag{5.21}$$

where

$$\widetilde{z} = \left(-\frac{n(1-\beta) - m(1+\beta)}{2n\beta}\right)^{\frac{1}{n-m}} K > K.$$

<u>*Proof of (i).*</u> Given any $z \in \left]0, \widetilde{z} \wedge \mathfrak{z}_\beta\right[$, the calculations in (5.21) imply that (5.18) is true with $\underline{x}_\dagger(z) < K < \overline{x}_\dagger(z)$ as well as that $F(K; z) < 0$. Also, (5.16) implies that $F(z, z) < 0$. Combining these observations with (4.16), (5.12) and the relevant inequality in (5.20), we can see that there exists a unique $\alpha(z) \in \left]\mathfrak{z}_\beta, \mathfrak{z}_0\right[$ such that (5.5) holds true.

If $\widetilde{z} < \mathfrak{z}_\beta$ and $z \in \left[\widetilde{z}, \mathfrak{z}_\beta\right[$, then (5.21) implies that $F(K; z) \geq 0$ and $\frac{\partial F}{\partial x}(K; z) \geq 0$. In this case, the inequality $F(z, z) < 0$, which follows from (5.16), implies that (5.18) is true with $\underline{x}_\dagger(z) < z$. This observation, (5.12) and the relevant inequality in (5.20) imply that there exists a unique $\alpha(z) \in \left]\mathfrak{z}_\beta, \mathfrak{z}_0\right[$ such that (5.5) holds true.

Differentiating the identity $F\big(\alpha(z); z\big) = 0$ with respect to z, and using (5.15), the inequalities $\frac{mK}{m-1} < \mathfrak{z}_\mathrm{c} \leq \mathfrak{z}_\beta < \alpha(z)$ (see also (4.16)) and (5.5), we obtain

$$\alpha'(z) = -\frac{\frac{\partial F}{\partial z}\big(\alpha(z); z\big)}{\frac{\partial F}{\partial x}\big(\alpha(z); z\big)} < 0 \quad \text{for all } z \in \left]0, \mathfrak{z}_\beta\right[,$$

which proves that α is strictly decreasing. Furthermore, the first limit in (5.4) follows from the calculation

$$0 = \lim_{z \to 0} F\big(\alpha(z); z\big) = \lim_{z \to 0}\big[(n-1)\alpha(z) - nK\big]A\alpha^{n-m}(z), \tag{5.22}$$

while, the second limit in (5.4) follows from (5.5) and (5.20).

<u>*Proof of (ii).*</u> We first note that, in this case,

$$\frac{mK}{m-1} < K < x_\dagger < \mathfrak{z}_\mathrm{c}, \tag{5.23}$$

where $x_\dagger$ is given by (5.17) (see (4.9), (4.11) and the statement of Lemma 4.2.(III)). Combining this observation with (5.18) and the identities $F(K; \widetilde{z}) = \frac{\partial F}{\partial x}(K; \widetilde{z}) = 0$, which follow from (5.21), we can see that

$$K = \underline{x}_\dagger(\widetilde{z}) < x_\dagger \quad \text{and} \quad F(x_\dagger; \widetilde{z}) < 0. \tag{5.24}$$

Using the definition (5.2) of F and (5.13), we calculate

$$F(\mathfrak{z}_\mathrm{c}; z) = -\frac{rK}{r - bm}\left[A\mathfrak{z}_\mathrm{c}^{n-m} + \frac{m(m-1)}{n(n-1)}B(z)\right],$$

and

$$\begin{aligned}\frac{\partial F}{\partial x}(\mathfrak{z}_\mathrm{c}; z) &= -\frac{n(r-bn)}{r}\left[A\mathfrak{z}_\mathrm{c}^{n-m} + \frac{m(m-1)}{n(n-1)}B(z)\right]\\ &= \frac{n(r-bn)(r-bm)}{r^2 K}F(\mathfrak{z}_\mathrm{c}; z).\end{aligned}$$

It follows that

$$F(\mathfrak{z}_{\mathrm{c}};z),\ \frac{\partial F}{\partial x}(\mathfrak{z}_{\mathrm{c}};z)\begin{cases}<0 & \text{for all } z\in\,]0,\mathfrak{C}\mathfrak{z}_{\mathrm{c}}[,\\ >0 & \text{for all } z\in\,]\mathfrak{C}\mathfrak{z}_{\mathrm{c}},\mathfrak{z}_{\mathrm{c}}[,\end{cases} \tag{5.25}$$

where $\mathfrak{C}$ is defined by (4.15). These inequalities and (5.23) imply that

$$x_{\dagger}<\overline{x}_{\dagger}(\mathfrak{C}\mathfrak{z}_{\mathrm{c}})=\mathfrak{z}_{\mathrm{c}}\quad\text{and}\quad F(x_{\dagger};\mathfrak{C}\mathfrak{z}_{\mathrm{c}})>0.$$

Combining this result with (5.24) and the fact that $\frac{\partial F}{\partial z}(x_{\dagger};z)>0$, which follows from (5.15) and (5.23), we can see that $\widetilde{z}<\mathfrak{C}\mathfrak{z}_{\mathrm{c}}$, which implies that $K<\mathfrak{C}\mathfrak{z}_{\mathrm{c}}$.

Given any $z\in\,]0,\mathfrak{C}\mathfrak{z}_{\mathrm{c}}[$, the inequalities in (5.25) imply that (5.18) is true with $\underline{x}_{\dagger}(z)<\mathfrak{z}_{\mathrm{c}}<\overline{x}_{\dagger}(z)$. It follows that, given any $z\in\,]0,\mathfrak{C}\mathfrak{z}_{\mathrm{c}}[$, there exists a unique $\alpha(z)\in\,]\mathfrak{z}_{\mathrm{c}},\mathfrak{z}_0[$ such that (5.7) holds true. Furthermore, (5.7), (5.22) and (5.25) imply the limits in (5.6). Taking into account the inequalities $\frac{mK}{m-1}<\mathfrak{z}_{\mathrm{c}}<\alpha(z)$, we can show that α is strictly decreasing in the same way as in Part (i). The inequality (5.8) follows from (5.15), the inequality $\frac{mK}{m-1}<\mathfrak{z}_{\mathrm{c}}$ and the fact that

$$F(x;\mathfrak{C}\mathfrak{z}_{\mathrm{c}})>0\quad\text{for all } x>\mathfrak{z}_{\mathrm{c}}$$

(see also (5.6) and (5.7)).

Proof of (iii). If $n\le\frac{1+\beta}{1-\beta}$, then (5.11), (5.12) and (5.16) imply that

$$\lim_{x\to0}F(x;z)>0,\quad F(z;z)\le0,\quad F(\mathfrak{z}_0;z)<0\quad\text{and}\quad\lim_{x\to\infty}F(x;z)=\infty.$$

If $n>\frac{1+\beta}{1-\beta}$, which implies that $\mathfrak{z}_0<\mathfrak{z}_\beta$ (see (4.18)), then (5.11), (5.12), (5.16) and (5.20) imply that

$$\lim_{x\to0}F(x;z)>0,\quad F(\mathfrak{z}_0;z)<0,\quad\lim_{x\to\infty}F(x;z)=\infty,$$

$$F(z;z)\begin{cases}<0, & \text{if } z\in\,]0,\mathfrak{z}_\beta[,\\ >0, & \text{if } z\in\,]\mathfrak{z}_\beta,\infty[,\end{cases}\quad\text{and}\quad F(\mathfrak{z}_\beta;z)\begin{cases}>0, & \text{if } z\in\,]0,\mathfrak{z}_\beta[,\\ <0, & \text{if } z\in\,]\mathfrak{z}_\beta,\infty[.\end{cases}$$

Combining these observations with (5.19), we can see that, given any $z>0$, there exists a unique $\alpha(z)>z\vee\mathfrak{z}_0$ such that (5.9) and (5.10) both hold true. □

The next result addresses the inequality

$$g(x,z):=\frac{\alpha(z)-K}{\psi\big(\alpha(z);z\big)}-\frac{x-K}{\psi(x;z)}\ge0,\quad\text{for } x\in\,]0,\alpha(z)], \tag{5.26}$$

which will play an important role in our analysis in the next section.

Lemma 5.2. *Suppose that Assumption 4.1 holds true, and let the function α be as in Lemma 5.1. The following statements hold true:*

(i) *Given any $z\in\,]0,\mathfrak{z}_\beta[$, the inequality (5.26) holds true for all $x\in\,]0,\alpha(z)]$ if the problem's parameters are as in Cases (I) or (II) of Lemma 4.2.*

(ii) *If the problem's parameters are as in Case (III) of Lemma 4.2, then there exists a unique point $z_\ominus\in\,]K,\mathfrak{C}\mathfrak{z}_{\mathrm{c}}[$ such that*

$$g(z,z)=\frac{\alpha(z)-K}{\psi\big(\alpha(z);z\big)}-\frac{z-K}{z^n}\begin{cases}>0, & \textit{if } z\in\,]0,z_\ominus[,\\ <0, & \textit{if } z\in\,]z_\ominus,\mathfrak{C}\mathfrak{z}_{\mathrm{c}}[.\end{cases} \tag{5.27}$$

Given any $z\in\,]0,z_\ominus]$, the inequality (5.26) holds true for all $x\in\,]0,\alpha(z)]$. Furthermore, there exists a function $\mathfrak{z}:[z_\ominus,\mathfrak{C}\mathfrak{z}_{\mathrm{c}}[\,\to[z_\ominus,\mathfrak{z}_{\mathrm{c}}[$ such that

$$\mathfrak{z}(z_\ominus)=z_\ominus,\quad z<\mathfrak{z}(z)\quad\textit{and}\quad g\big(\mathfrak{z}(z),z\big)=0\quad\textit{for all } z\in\,]z_\ominus,\mathfrak{C}\mathfrak{z}_{\mathrm{c}}[. \tag{5.28}$$

(iii) *If* $\beta \in \left]0,1\right[$ *(Case (IV) of Lemma 4.2), then there exists a unique point* $z_\oplus \in \left]\mathfrak{z}_0, \infty\right[$ *such that*

$$g(\mathfrak{z}_0, z) = \frac{\alpha(z) - K}{\psi\big(\alpha(z); z\big)} - \frac{\mathfrak{z}_0 - K}{\psi(\mathfrak{z}_0; z)} \begin{cases} > 0, & \text{if } z \in \left]0, z_\oplus\right[, \\ < 0, & \text{if } z \in \left]z_\oplus, \infty\right[. \end{cases} \tag{5.29}$$

Furthermore, given any $z \in \left]0, z_\oplus\right]$*, the inequality (5.26) holds true for all* $x \in \left]0, \alpha(z)\right]$.

Proof. We first calculate

$$g\big(\alpha(z), z\big) = 0 \quad \text{and} \quad \frac{\partial g}{\partial x}(x, z) = \begin{cases} (n-1)(x - \mathfrak{z}_0)x^{-1-n}, & \text{if } x \in \left]0, z\right[, \\ x^{m-1}\psi^{-2}(x; z)F(x; z), & \text{if } x \in \left]z, \alpha(z)\right[, \end{cases} \tag{5.30}$$

for all z in the domain of α. (Note that $\frac{\partial g}{\partial x}(z,z)$ does not exist, the corresponding left and right partial derivatives of $g(\cdot, z)$ are discontinuous at z.)

Proof of (i). This case follows immediately from (4.16), (5.5), (5.30) and the fact that $\alpha(z) > K$.

Proof of (ii). We first recall that, in this case, $K < \mathfrak{C}\mathfrak{z}_c$ (see Lemma 5.1.(ii)). In view of (5.3) and Lemma 4.2.(III), we can see that

$$\begin{aligned} \frac{\partial}{\partial x}\big(x^{m-1}F(x; z)\big) &= (x - K)\psi''(x; z) \\ &\begin{cases} < 0, & \text{if } z < K \text{ and } x \in \left]K, K \vee \mathfrak{C}^{-1}z\right[, \\ > 0, & \text{if } z < K \text{ and } x \in \left]K \vee \mathfrak{C}^{-1}z, \infty\right[, \\ < 0, & \text{if } z \geq K \text{ and } x \in \left]z, \mathfrak{C}^{-1}z\right[, \\ > 0, & \text{if } z \geq K \text{ and } x \in \left]\mathfrak{C}^{-1}z, \infty\right[\end{cases} \end{aligned} \tag{5.31}$$

(in the inequalities here, we list only the cases we will use). Also, given any $z \in \left]0, \mathfrak{C}\mathfrak{z}_c\right[$, we use the identities in (5.30) to calculate

$$\begin{aligned} g(x, z) &= g\big(\alpha(z), z\big) - \int_x^{\alpha(z)} \frac{\partial g}{\partial x}(y, z)\, dy \\ &= -\int_x^{\alpha(z)} \frac{y^{m-1}F(y; z)}{\psi^2(y; z)}\, dy, \quad \text{for } x \in [z, \alpha(z)]. \end{aligned} \tag{5.32}$$

Defining

$$\alpha(\mathfrak{C}\mathfrak{z}_c) := \lim_{z \to \mathfrak{C}\mathfrak{z}_c} \alpha(z) \overset{(5.6)}{=} \mathfrak{z}_c,$$

we note that (5.6), (5.7) imply that $F(\mathfrak{z}_c; \mathfrak{C}\mathfrak{z}_c) = 0$. This observation, the fact that $K < \mathfrak{C}\mathfrak{z}_c$ and the second pair of inequalities in (5.31) imply that

$$F(x; \mathfrak{C}\mathfrak{z}_c) > 0 \quad \text{for all } x \in \left]\mathfrak{C}\mathfrak{z}_c, \mathfrak{z}_c\right[.$$

This inequality and (5.32) imply that

$$g(\mathfrak{C}\mathfrak{z}_c, \mathfrak{C}\mathfrak{z}_c) = -\int_{\mathfrak{C}\mathfrak{z}_c}^{\mathfrak{z}_c} \frac{y^{m-1}F(y; z)}{\psi^2(y; z)}\, dy < 0. \tag{5.33}$$

Combining this result with the observation that $g(z, z) > 0$ for all $z \leq K$, which follows from the definition (5.26) of g and the fact that $\alpha(z) > \mathfrak{z}_c > K$ for all $z < \mathfrak{C}\mathfrak{z}_c$, we can see that

$$z_\ominus = \inf\big\{z \in \left]0, \mathfrak{C}\mathfrak{z}_c\right] \mid\ g(z, z) \leq 0\big\} \in \left]K, \mathfrak{C}\mathfrak{z}_c\right[.$$

We will establish (5.27) if we show that $g(z,z) < 0$ for all $z \in \,]z_\ominus, \mathfrak{C}\mathfrak{z}_c[$. To this end, we differentiate the expression of the function $]0, \mathfrak{C}\mathfrak{z}_c[\, \ni z \to \overline{g}(z) := g(z,z)$ given by (5.27) and we use the identities

$$F\big(\alpha(z); z\big) = 0 \quad \overset{(5.3)}{\Rightarrow} \quad \big(\alpha(z) - K\big)\psi'\big(\alpha(z); z\big) = \psi\big(\alpha(z); z\big)$$

to calculate

$$\begin{aligned}\overline{g}'(z) &= -\frac{\alpha^{m-1}(z)F\big(\alpha(z);z\big)}{\psi^2\big(\alpha(z);z\big)}\alpha'(z) - \frac{\big(\alpha(z)-K\big)\frac{\partial\psi}{\partial z}\big(\alpha(z);z\big)}{\psi^2\big(\alpha(z);z\big)} - (n-1)(\mathfrak{z}_0 - z)z^{-n-1} \\ &= -\frac{2n\beta\big(\alpha(z)-K\big)z^{n-m-1}\alpha^m(z)}{(1+\beta)\psi^2\big(\alpha(z);z\big)} - (n-1)(\mathfrak{z}_0 - z)z^{-n-1}\end{aligned}$$

and

$$\begin{aligned}\overline{g}''(z) = &-\frac{2n(m-1)\beta z^{n-m-1}\alpha^{m-1}(z)\alpha'(z)}{(1+\beta)\psi^2\big(\alpha(z);z\big)}\left[\alpha(z) - \frac{mK}{m-1}\right] \\ &- \frac{2n\beta z^{n-m-2}\big(\alpha(z)-K\big)\alpha^m(z)}{(1+\beta)\psi^3\big(\alpha(z);z\big)}\left[(n-m-1)\psi\big(\alpha(z);z\big) - \frac{4n\beta z^{n-m}\alpha^m(z)}{1+\beta}\right] \\ &+ n(n-1)\left[\frac{n+1}{n}\mathfrak{z}_0 - z\right]z^{-n-2}.\end{aligned}$$

In view of the fact that $\alpha : \,]0, \mathfrak{C}\mathfrak{z}_c[\, \to \,]\mathfrak{z}_c, \mathfrak{z}_0[$ is strictly decreasing and the inequalities $\frac{mK}{m-1} < K < z_\ominus < \mathfrak{C}\mathfrak{z}_c < \mathfrak{z}_c < \mathfrak{z}_0$ (see (4.17) in Lemma 4.2 and Lemma 5.1.(ii)), the latter expression implies that the function $]0, \mathfrak{C}\mathfrak{z}_c[\, \ni z \to \overline{g}(z)$ is strictly convex. Combining this observation with the identity $\overline{g}(z_\ominus) = 0$ and the inequality $\overline{g}(\mathfrak{C}\mathfrak{z}_c) < 0$ (see (5.33)), we can see that $\overline{g}(z) \equiv g(z,z) < 0$ for all $z \in \,]z_\ominus, \mathfrak{C}\mathfrak{z}_c]$, as required.

To proceed further, we note that, if $z < K$, then the expression (5.3) of F implies that $F(x,z) < 0$ for all $x \in [z, K]$. Combining this observation with the identity $F\big(\alpha(z); z\big) = 0$ and the first pair of inequalities in (5.31), we can see that

$$\text{given any } z \in \,]0, K[, \quad F(x;z) < 0 \quad \text{for all } x \in [z, \alpha(z)[.$$

On the other hand, the second pair of inequalities in (5.31) and the identity $F\big(\alpha(z); z\big) = 0$ imply that, given any $z \in [K, \mathfrak{C}\mathfrak{z}_c[$, either

$$F(x;z) < 0 \quad \text{for all } x \in \,]z, \alpha(z)[,$$

or

$$\text{there exists } x_\star(z) \in \,]z, \mathfrak{z}_c[\text{ such that } F(x;z) \begin{cases} > 0, & \text{if } x \in \,]z, x_\star(z)[, \\ < 0, & \text{if } x \in \,]x_\star(z), \alpha(z)[. \end{cases}$$

These observations and (5.30) imply that either

$$\frac{\partial g}{\partial x}(x,z) < 0 \text{ for all } x \in \,]0, \alpha(z)[, \tag{5.34}$$

or

$$\frac{\partial g}{\partial x}(x,z) \begin{cases} < 0, & \text{if } x \in \,]0, z[\, \cup \,]x_\star(z), \alpha(z)[, \\ > 0, & \text{if } x \in \,]z, x_\star(z)[. \end{cases} \tag{5.35}$$

Given any $z \in \,]0, z_\ominus]$, the inequality $g(z,z) > 0$ (see (5.27)), the identity $g\big(\alpha(z), z\big) = 0$ and (5.34)–(5.35) imply that (5.26) holds true for all $x \in [0, \alpha(z)]$. On the other hand, given any $z \in [z_\ominus, \mathfrak{C}\mathfrak{Z}_{\mathrm{c}}[$, the inequality $g(z,z) < 0$ (see (5.27)), the identity $g\big(\alpha(z), z\big) = 0$ and (5.35) imply that there exists a unique $\mathfrak{z}(z) \in [z_\ominus, \mathfrak{Z}_{\mathrm{c}}[$ such that (5.28) holds true.

Proof of (iii). In this case, (4.18) and (5.9) imply that

$$\lim_{z\to\infty} \alpha(z) = \infty, \text{ if } n \leq \frac{1+\beta}{1-\beta}, \quad \text{and} \quad \alpha(\mathfrak{Z}_\beta) = \mathfrak{Z}_\beta > \mathfrak{Z}_0, \text{ if } n > \frac{1+\beta}{1-\beta}.$$

It follows that

$$\lim_{z\to\infty} g(\mathfrak{Z}_0, z) = -\frac{\mathfrak{Z}_0 - K}{\mathfrak{Z}_0^n} < 0, \quad \text{if } n \leq \frac{1+\beta}{1-\beta},$$

and

$$g(\mathfrak{Z}_0, \mathfrak{Z}_\beta) = \frac{\mathfrak{Z}_\beta - K}{\mathfrak{Z}_\beta^n} - \frac{\mathfrak{Z}_0 - K}{\mathfrak{Z}_0^n} < 0, \quad \text{if } n > \frac{1+\beta}{1-\beta}.$$

On the other hand, (5.10) and (5.30) imply that

$$g(\mathfrak{Z}_0, \mathfrak{Z}_0) = -\int_{\mathfrak{Z}_0}^{\alpha(\mathfrak{Z}_0)} \frac{\partial g}{\partial x}(y, \mathfrak{Z}_0)\, dy = -\int_{\mathfrak{Z}_0}^{\alpha(\mathfrak{Z}_0)} \frac{y^{m-1} F(y; \mathfrak{Z}_0)}{\psi^2(y; \mathfrak{Z}_0)}\, dy > 0.$$

Combining these observations with the calculation

$$\begin{aligned}
\frac{\partial}{\partial z} g(\mathfrak{Z}_0, z) &= -\frac{\alpha^{m-1}(z) F\big(\alpha(z); z\big)}{\psi^2\big(\alpha(z); z\big)} \alpha'(z) - \frac{\big(\alpha(z) - K\big) \frac{\partial \psi}{\partial z}\big(\alpha(z); z\big)}{\psi^2\big(\alpha(z); z\big)} + \frac{(\mathfrak{Z}_0 - K) \frac{\partial \psi}{\partial z}(\mathfrak{Z}_0; z)}{\psi^2(\mathfrak{Z}_0; z)} \\
&= -\frac{2n\beta\big(\alpha(z) - K\big) z^{n-m-1} \alpha^m(z)}{(1+\beta)\psi^2\big(\alpha(z); z\big)} < 0, \quad \text{for } z \in \,]\mathfrak{Z}_0, \infty[,
\end{aligned}$$

we can see that there exists a unique $z_\oplus \in \,]\mathfrak{Z}_0, \infty[$ such that (5.29) holds true.

Finally, we fix any $z \in \,]0, z_\oplus]$. In view of the inequality $g(\mathfrak{Z}_0, z) \geq 0$, the identity $g\big(\alpha(z), z\big) = 0$ and the observation that

$$\frac{\partial g}{\partial x}(x, z) \begin{cases} > 0, & \text{if } x \in \,]\mathfrak{Z}_0, z[, \\ < 0, & \text{if } x \in \,]0, \mathfrak{Z}_0[\, \cup \,]z, \alpha(z)[, \end{cases}$$

which follows from (5.10) and (5.30), we can see that the inequality (5.26) holds true for all $x \in \,]0, \alpha(z)[$. ☐

Remark 5.3. Our analysis in the next sections will make use of the following observation. Suppose that the problem's parameters are as in Case (III) of Lemma 4.2 and fix any $z \in [z_\ominus, \mathfrak{C}\mathfrak{Z}_{\mathrm{c}}[$, where $z_\ominus$ is as in Lemma 5.2.(ii). The function $u(\cdot; z) : [z, \infty[\to \mathbb{R}$ defined by

$$u(x; z) = \Gamma(z)\psi(x; z) - (x - K) = A\Gamma(z)x^n + B(z)\Gamma(z)x^m - (x - K),$$

where $\Gamma(z) = \big(\alpha(z) - K\big)/\psi\big(\alpha(z); z\big)$, is such that

$$u\big(\mathfrak{z}(z); z\big) = 0 \quad \text{and} \quad u\big(\alpha(z); z\big) = \frac{\partial u}{\partial x}\big(\alpha(z); z\big) = 0.$$

The first of these identities follows immediately from (5.28) and the fact that $u(x; z) = g(x, z)\psi(x; z)$ for all $x \geq z$. On the other hand, the identities for $x = \alpha(z)$ hold true because they are equivalent to the identity $F\big(\alpha(z); z\big) = 0$.

6 The solution to the optimal stopping problem defined by (1.4) and (1.7)

We expect that the value function v of the discretionary problem defined by (1.4) and (1.7) should be strictly positive. Combining this observation with the fact that the restriction of the function $x \mapsto (x-K)^+$ to $\mathbb{R}_+ \setminus \{K\}$ is C^∞ and the so-called "principle of smooth fit" (see also Remark 3.7), we expect that the restriction of v to $]0,\infty[\setminus \{z\}$ should be C^1 with absolutely continuous first derivative. In view of (3.8), (3.9) in Example 3.5, we therefore expect that v should identify with a function w satisfying

$$\max\left\{\frac{1}{2}\sigma^2 x^2 w''(x) + bxw'(x) - rw(x),\ (x-K)^+ - w(x)\right\} = 0 \text{ inside }]0,\infty[\setminus \{z\}, \quad (6.1)$$

and

$$\max\left\{(1+\beta)w'_+(z) - (1-\beta)w'_-(z),\ (z-K)^+ - w(z)\right\} = 0. \quad (6.2)$$

Furthermore, the strict positivity of v and Remark 3.6 imply that the waiting region includes the interval $\big]0, K \vee rK/(r-b)\big[$ and, if $\beta \in]0,1[$, then z also belongs to the waiting region.

We now solve the optimal stopping problem we consider in this section by constructing an appropriate solution to the variational inequality (6.1)–(6.2). In its simplest form, the required solution has the same qualitative form as the solution to the optimal stopping problem associated with the usual perpetual American call option that involves a standard geometric Brownian motion ($\beta = 0$). Accordingly, the value function v identifies with the function

$$w(x) = w(x;z) = \begin{cases} \Gamma(z)\psi(x;z), & \text{if } x \le a, \\ x-K, & \text{if } x > a, \end{cases} \quad (6.3)$$

for some constants $a = a(z) > 0$ and $\Gamma(z) > 0$, while

$$\tau^\star = \inf\big\{t \ge 0 \mid\ X_t \ge a\big\} \quad (6.4)$$

is an optimal stopping time. It turns out that this is indeed the case for a wide range of parameter values (see Figures 4-10). To determine the constant $\Gamma(z)$ and the free-boundary point a, we first appeal to the continuity of the value function, which yields the expression

$$\Gamma(z) = (a-K)\psi^{-1}(a;z). \quad (6.5)$$

With the exception of the possibilities depicted by Figures 5 and 8, the value function is C^1 at a, which gives rise to the equation $\Gamma(z)\psi'(a;z) = 1$.[8] This equation and (6.5) imply that a should satisfy equation (5.1) if $z < a$ (see Figures 4, 7 and 10) and should be given by

[8] Recall that we have adopted the notation $\psi'(x;z) = \frac{\partial \psi}{\partial x}(x;z)$.

$$a = \frac{nK}{n-1} \overset{(4.14)}{=:} \mathfrak{z}_0 > 0$$

if $a < z$ (see Figures 6 and 9).

The following result, which we prove in Section 7, involves the parameters $\mathfrak{z}_c$, $\mathfrak{z}_\beta$, $\mathfrak{z}_0$ and $z_\ominus$, $z_\oplus$ that are as in (4.14) and Lemma 5.2.(ii)-(iii), respectively.

Theorem 6.1. *Consider the optimal stopping problem defined by (1.4), (1.7) and suppose that Assumption 4.1 holds true. If the problem parameters are as in Cases (I) or (II) of Lemma 4.2, define*

$$a = \begin{cases} \alpha(z), & \text{if } z \in \,]0, \mathfrak{z}_\beta[, \\ z \wedge \mathfrak{z}_0, & \text{if } z \in [\mathfrak{z}_\beta, \infty[, \end{cases} \tag{6.6}$$

where the function α is as in Lemma 5.1.(i). If the problem parameters are as in Case (III) of Lemma 4.2, suppose that $z \in \,]0, z_\ominus] \cup [\mathfrak{z}_c, \infty[$ and define

$$a = \begin{cases} \alpha(z), & \text{if } z \in \,]0, z_\ominus], \\ z \wedge \mathfrak{z}_0, & \text{if } z \in [\mathfrak{z}_c, \infty[, \end{cases} \tag{6.7}$$

where the function α is as in Lemma 5.1.(ii) and $z_\ominus$ is as in Lemma 5.2.(ii). If $\beta \in \,]0, 1[$ (Case (IV) of Lemma 4.2), suppose that $z \in \,]0, z_\oplus]$ and define

$$a = \alpha(z), \quad \text{for } z \in \,]0, z_\oplus], \tag{6.8}$$

where the function α is as in Lemma 5.1.(iii) and $z_\oplus$ is as in Lemma 5.2.(iii). For such choices of a and for $\Gamma(z) > 0$ given by (6.5), the function w defined by (6.3) identifies with the value function v of the discretionary stopping problem and the stopping time given by (6.4) is optimal.

In the context of (6.6), we can see that Figure 4 transforms "continuously" into Figure 5 and then into Figure 6 as z increases from 0 to ∞, thanks to the second limit in (5.4).

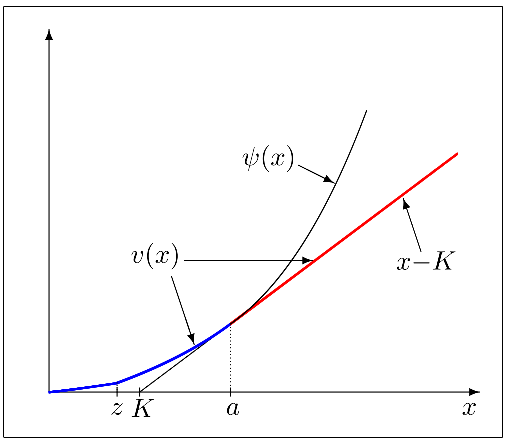

Figure 4. The value function v if the problem parameters are as in Cases (I) or (II) of Lemma 4.2 and $z \in \,]0, \mathfrak{z}_\beta[$.

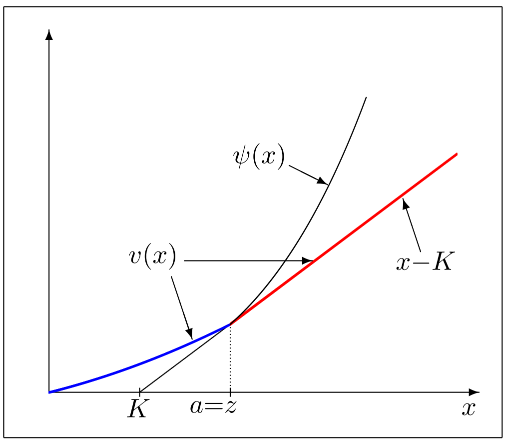

Figure 5. The value function v if the problem parameters are as in Cases (I) or (II) of Lemma 4.2 and $z \in [\mathfrak{z}_\beta, \mathfrak{z}_0]$.

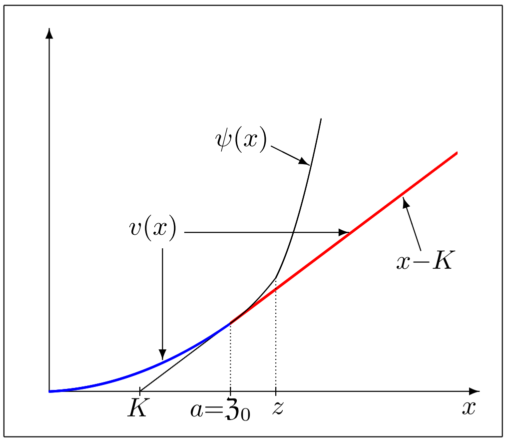

Figure 6. The value function v if the problem parameters are as in Cases (I) or (II) of Lemma 4.2 and $z \in]\mathfrak{z}_0, \infty[$.

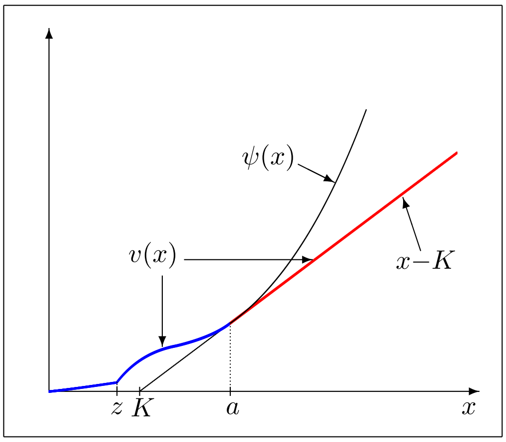

Figure 7. The value function v if the problem parameters are as in Case (III) of Lemma 4.2 and $z \in]0, z_\ominus[$.

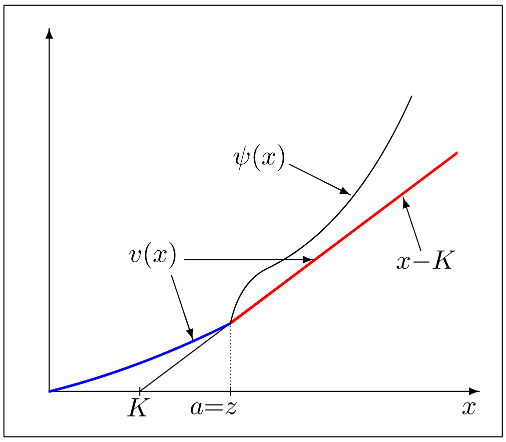

Figure 8. The value function v if the problem parameters are as in Case (III) of Lemma 4.2 and $z \in [\mathfrak{z}_c, \mathfrak{z}_0]$.

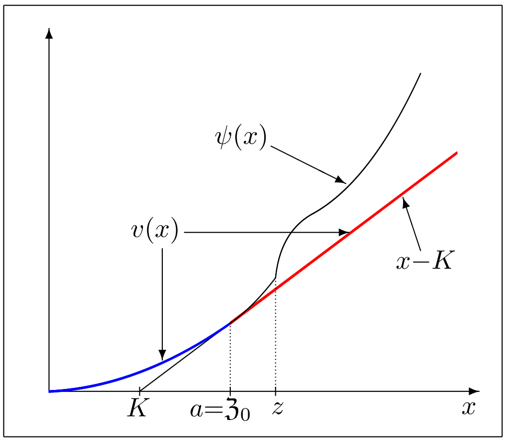

Figure 9. The value function v if the problem parameters are as in Case (III) of Lemma 4.2 and $z \in]\mathfrak{z}_0, \infty[$.

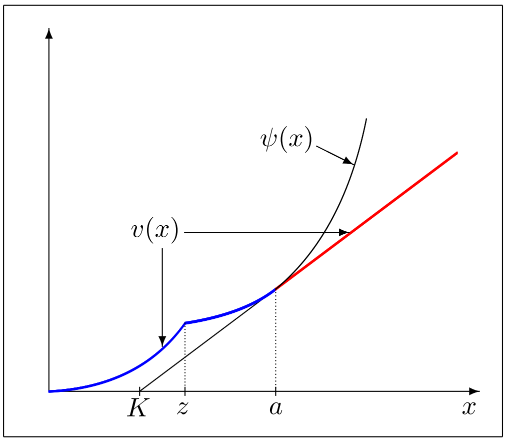

Figure 10. The value function v if the problem parameters are as in Case (IV) of Lemma 4.2 and $z \in]0, z_\oplus[$.

Remark 6.2. Suppose that the problem parameters are as in Cases (I) or (II) of Lemma 4.2. In view of the identity in (3.11) and Theorem 6.1, we can see that, given any

$z \in [\mathfrak{z}_\beta, \mathfrak{z}_0]$,

$$\begin{aligned}\frac{v'_+(z)}{p'_+(z)} - \frac{v'_-(z)}{p'_-(z)} &= \frac{1}{(1-\beta)p'_-(z)}\big[1+\beta-(1-\beta)v'_-(z)\big] \\ &= -\frac{n-\frac{1+\beta}{1-\beta}}{zp'_-(z)}(z-\mathfrak{z}_\beta) \in \left[\frac{2n\beta K}{(n-1)(1-\beta)\mathfrak{z}_0 p'_-(\mathfrak{z}_0)},\ 0\right],\end{aligned}$$

while

$$\frac{v'_+(z)}{\psi'(z+;z)} - \frac{v'_-(z)}{\psi'(z-;z)} = -\frac{n-\frac{1+\beta}{1-\beta}}{nz^n}(z-\mathfrak{z}_\beta) \in \left[\frac{2\beta K}{(n-1)(1-\beta)\mathfrak{z}_0^n},\ 0\right].$$

We are thus faced with an example of “right-sided” optimal stopping of a skew geometric Brownian motion in which the “principle of smooth fit” does not hold in the sense that none of v'_-, v'_-/p'_- or v'_-/ψ'_- is continuous.

If the problem parameters are as in Case (III) of Lemma 4.2, then the function $w = w(\cdot; z)$ given by (6.3), (6.5) is such that

$$w(z_\ominus; z_\ominus) \equiv \Gamma(z_\ominus)\psi(z_\ominus; z_\ominus) = z_\ominus - K \tag{6.9}$$

and

$$w\big(\alpha(z_\ominus); z_\ominus\big) \equiv \Gamma(z_\ominus)\psi\big(\alpha(z_\ominus); z_\ominus\big) = \alpha(z_\ominus) - K \tag{6.10}$$

(see Lemma 5.2.(ii)). This observation and the “singularity” associated with z give rise to the following possibility. For $z \geq z_\ominus$, the stopping time

$$\underline{\tau}^\star = \inf\big\{t \geq 0 \ \mid \ X_t \in \{z\} \cup [\xi, \infty[\big\}, \tag{6.11}$$

where $\xi = \xi(z) > z$ is a constant, may be optimal. In such a context, the value function v should identify with the function

$$\underline{w}(x) = \underline{w}(x; z) = \begin{cases}(z-K)z^{-n}x^n, & \text{if } x \leq z,\\ C(z)x^n + D(z)x^m, & \text{if } x \in \,]z, \xi[,\\ x-K, & \text{if } x \geq \xi,\end{cases} \tag{6.12}$$

for some $C(z), D(z) \in \mathbb{R}$ (see Figure 11). To determine the constants $C(z)$, $D(z)$ and the free-boundary point $\xi = \xi(z)$, we require that $\underline{w}$ should be C^1 at ξ, which is suggested by the “principle of smooth fit”, as well as continuous at z. The system of equations arising from these requirements is equivalent to the expressions

$$C(z) = -\frac{1}{n-m}\big[(m-1)\xi - mK\big]\xi^{-n}, \quad D(z) = \frac{1}{n-m}\big[(n-1)\xi - nK\big]\xi^{-m}, \tag{6.13}$$

and the algebraic equation

$$J(\xi; z) = 0, \tag{6.14}$$

where

$$\begin{aligned}J(x;z) &= \big[(n-1)x - nK\big]Ax^{-m} - \big[(m-1)x - mK\big]Az^{n-m}x^{-n} - (n-m)A(z-K)z^{-m} \\ &= x^{-n}F(x;z) - \big[(m-1)x - mK\big]z^{n-m}x^{-n} - (n-m)A(z-K)z^{-m}.\end{aligned} \tag{6.15}$$

To establish the second identity here, we have used the definitions (4.6), (4.7) of A, B, as well as the definition (5.2) of F.

We prove the following result in Section 7.

Theorem 6.3. *Consider the optimal stopping problem defined by (1.4), (1.7) and suppose that Assumption 4.1 holds true. Also, suppose that the problem parameters are as in Case (III) of Lemma 4.2. Equation (6.14) defines uniquely a strictly decreasing function* $\mathfrak{x} :]0, \mathfrak{z}_c[\to]\mathfrak{z}_c, \mathfrak{z}_0[$ *such that*

$$\lim_{z \to 0} \mathfrak{x}(z) = \mathfrak{z}_0, \quad \mathfrak{x}(z_\ominus) = \mathfrak{a}(z_\ominus) \quad \textit{and} \quad \lim_{z \to \mathfrak{z}_c} \mathfrak{x}(z) = \mathfrak{z}_c, \tag{6.16}$$

where $\mathfrak{a}$ *is as in Lemma 5.1.(ii) and* $z_\ominus$ *is as in Lemma 5.2.(ii). Given any* $z \in]z_\ominus, \mathfrak{z}_c[$*, the function* $\underline{w}$ *defined by (6.12)–(6.13) for* $\xi = \mathfrak{x}(z)$ *identifies with the value function* v *of the discretionary stopping problem and the* $(\mathcal{F}_t)$*-stopping time* $\underline{\tau}^\star$ *defined by (6.11) is optimal.*

In the context of the previous result and the part of Theorem 6.1 addressing the case when the problem parameters are as in Case (III) of Lemma 4.2 (see (6.7) in particular), we can see that Figure 7 transforms "continuously" into Figure 11, then into Figure 8 and then into Figure 9 as z increases from 0 to ∞, thanks to the identities

$$\underline{w}(\cdot; z_\ominus) = w(\cdot; z_\ominus), \quad \mathfrak{x}(z_\ominus) = \mathfrak{a}(z_\ominus) \quad \text{and} \quad \lim_{z \to \mathfrak{z}_c} \mathfrak{x}(z) = \mathfrak{z}_c < \mathfrak{z}_0 \tag{6.17}$$

(see (6.9), (6.10) and (6.16)).

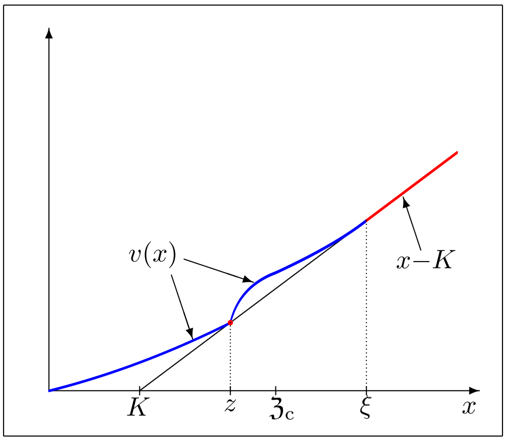

Figure 11. The value function v if the problem parameters are as in Case (III) of Lemma 4.2 and $z \in [z_\ominus, \mathfrak{z}_c[$.

If the problem parameters are as in Case (IV) of Lemma 4.2, then the function $w = w(\cdot; z)$ given by (6.3), (6.5) is such that

$$w(\mathfrak{z}_0; z_\oplus) \equiv \Gamma(z_\oplus)\psi(\mathfrak{z}_0; z_\oplus) = \mathfrak{z}_0 - K \tag{6.18}$$

and

$$w\big(\mathfrak{a}(z_\oplus); z_\oplus\big) \equiv \Gamma(z_\oplus)\psi\big(\mathfrak{a}(z_\oplus); z_\oplus\big) = \mathfrak{a}(z_\oplus) - K \tag{6.19}$$

(see Lemma 5.2.(iii) and Figure 12). This observation suggests the possibility for the stopping time

$$\overline{\tau}^\star = \inf\big\{t \geq 0 \mid \ X_t \in [\mathfrak{z}_0, \gamma] \cup [\zeta, \infty[\big\}, \tag{6.20}$$

where $\gamma = \gamma(z) < \zeta = \zeta(z)$ are constants, to be optimal. In such a context, the value function v should identify with the function

$$\overline{w}(x) = \overline{w}(x; z) = \begin{cases} \frac{1}{n}\mathfrak{z}_0^{-n+1} x^n, & \text{if } x \in]0, \mathfrak{z}_0[, \\ C_l x^n + D_l x^m, & \text{if } x \in]\gamma, z], \\ C_r x^n + D_r x^m, & \text{if } x \in]z, \zeta[, \\ x - K, & \text{if } x \in [\mathfrak{z}_0, \gamma] \cup [\zeta, \infty[, \end{cases} \tag{6.21}$$

for some $C_{\rm l}, D_{\rm l}, C_{\rm r}, D_{\rm r} \in \mathbb{R}$ (see Figure 13). We suppress the actual dependence of $C_{\rm l}$, $D_{\rm l}$, $C_{\rm r}$, $D_{\rm r}$ on z because we will not use variational arguments in the analysis of this case. To determine the constants $C_{\rm l}$, $D_{\rm l}$, $C_{\rm r}$, $D_{\rm r}$ and the free-boundary points γ, ξ, we first require that $\overline{w}$ should be C^1 at γ and ζ, which is suggested by the "principle of smooth fit". This requirement yields the expressions

$$C_{\rm l} = -\frac{1}{n-m}\big[(m-1)\gamma - mK\big]\gamma^{-n}, \quad C_{\rm r} = -\frac{1}{n-m}\big[(m-1)\zeta - mK\big]\zeta^{-n}, \tag{6.22}$$

$$D_{\rm l} = \frac{1}{n-m}\big[(n-1)\gamma - nK\big]\gamma^{-m} \quad \text{and} \quad D_{\rm r} = \frac{1}{n-m}\big[(n-1)\zeta - nK\big]\zeta^{-m}. \tag{6.23}$$

We next require that $\overline{w}$ should be continuous at z and satisfy the identity

$$(1+\beta)\overline{w}'_+(z) = (1-\beta)\overline{w}'_-(z)$$

that is associated with (6.2). These requirements yield the identities

$$C_{\rm r} = \frac{n(1-\beta) - m(1+\beta)}{(n-m)(1+\beta)} C_{\rm l} - \frac{2m\beta}{(n-m)(1+\beta)} D_{\rm l} z^{-(n-m)}$$

and

$$D_{\rm r} = \frac{2n\beta}{(n-m)(1+\beta)} C_{\rm l} z^{n-m} + \frac{n(1+\beta) - m(1-\beta)}{(n-m)(1+\beta)} D_{\rm l}.$$

Using the expressions in (6.22), (6.23) to substitute for $C_{\rm l}$, $D_{\rm l}$, $C_{\rm r}$, $D_{\rm r}$, we obtain the system of equations

$$\begin{aligned}\big[(m-1)\zeta - mK\big] z^n \zeta^{-n} - \frac{n(1-\beta) - m(1+\beta)}{(n-m)(1+\beta)} \big[(m-1)\gamma - mK\big] z^n \gamma^{-n} \\ - \frac{2m\beta}{(n-m)(1+\beta)} \big[(n-1)\gamma - nK\big] z^m \gamma^{-m} = 0\end{aligned} \tag{6.24}$$

and

$$\begin{aligned}\big[(n-1)\zeta - nK\big] z^m \zeta^{-m} + \frac{2n\beta}{(n-m)(1+\beta)} \big[(m-1)\gamma - mK\big] z^n \gamma^{-n} \\ - \frac{n(1+\beta) - m(1-\beta)}{(n-m)(1+\beta)} \big[(n-1)\gamma - nK\big] z^m \gamma^{-m} = 0.\end{aligned} \tag{6.25}$$

By (a) subtracting (6.24) from (6.25) and (b) solving (6.25) for $\big[(m-1)\gamma - mK\big] z^n \gamma^{-n}$ and substituting for the resulting expression in (6.24), we obtain the system of equations

$$G(\gamma, \zeta; z) = 0 \quad \text{and} \quad H(\gamma, \zeta; z) = 0, \tag{6.26}$$

which is equivalent to (6.24) and (6.25), where

$$\begin{aligned}G(x, y; z) = \big[(n-1)y - nK\big] z^m y^{-m} - \big[(m-1)y - mK\big] z^n y^{-n} \\ - \big[(n-1)x - nK\big] z^m x^{-m} + \big[(m-1)x - mK\big] z^n x^{-n}\end{aligned} \tag{6.27}$$

and

$$H(x, y; z) = y^{-n} F(y; z) - \frac{1-\beta}{1+\beta} \big[(n-1)x - nK\big] x^{-m}, \tag{6.28}$$

in which expressions, F is the function defined by (5.2).

We prove the final result of the paper in Section 7.

Theorem 6.4. *Consider the optimal stopping problem defined by (1.4), (1.7) and suppose that Assumption 4.1 holds true. Also, suppose that* $\beta \in]0,1[$ *(Case (IV) of Lemma 4.2). The following statements hold true:*

(i) *The system of equations (6.26) has a unique solution* (γ, ζ) *such that* $\mathfrak{z}_0 < \gamma < z < \zeta$ *if and only if* $z > z_\oplus$*, where* $z_\oplus$ *is as in Lemma 5.2.(iii).*

(ii) *Given any* $z > z_\oplus$ *and the associated solution* (γ, ζ) *to the system of equations (6.26), the function* $\overline{w}$ *defined by (6.21), for* $C_{\mathrm{l}}, D_{\mathrm{l}}, C_{\mathrm{r}}, D_{\mathrm{r}} > 0$ *given by (6.22)–(6.23) identifies with the value function* v *of the discretionary stopping problem and the* $(\mathcal{F}_t)$*-stopping time* $\overline{\tau}^\star$ *defined by* (6.20) *is optimal.*

In the context of the previous result and the part of Theorem 6.1 addressing the case when the problem parameters are as in Case (IV) of Lemma 4.2 (see (6.8) in particular), we can see that Figure 10 transforms "continuously" into Figure 12 and then into Figure 13 as z increases from 0 to ∞, thanks to the identity $\overline{w}(\cdot; z_\oplus) = w(\cdot; z_\oplus)$, which follows from (6.18) and (6.19).

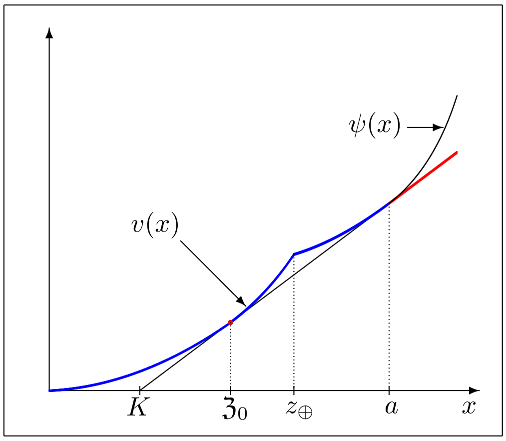

Figure 12. The value function v if the problem parameters are as in Case (IV) of Lemma 4.2 and $z = z_\oplus$.

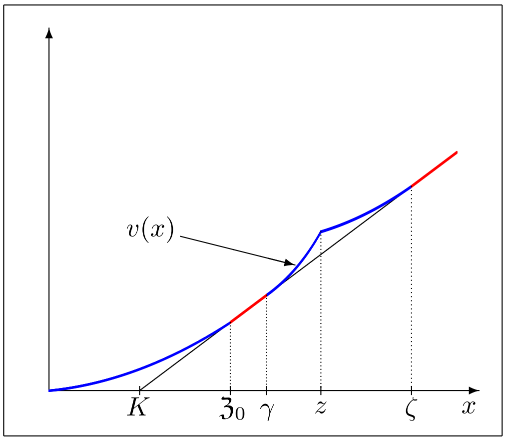

Figure 13. The value function v if the problem parameters are as in Case (IV) of Lemma 4.2 and $z \in]z_\oplus, \infty[$.

7 Proofs of Theorems 6.1, 6.3 and 6.4

If we denote by g any of the functions w, $\underline{w}$ or $\overline{w}$, defined by (6.3), (6.12) and (6.21), respectively, then

$$\lim_{y \to 0} \frac{g(y)}{\varphi(y)} = \lim_{y \to 0} \frac{(y-K)^+}{\varphi(y)} = \lim_{y \to \infty} \frac{(y-K)^+}{\psi(y)} = \lim_{y \to \infty} \frac{g(y)}{\psi(y)} = 0.$$

In view of this observation and Theorem 3.3.(III)-(IV), we can see that we will prove Theorems 6.1, 6.3 and 6.4 if we establish the claims made on the solvability of their associated free-boundary problems as well as show that the corresponding functions w, $\underline{w}$ and $\overline{w}$ satisfy the variational inequality (6.1)–(6.2).

Proof of Theorem 6.1. By construction, w is C^2 inside $]0,\infty[\setminus \{a, z\}$ and C^1 at a if $a \neq z$. It is straightforward to verify that w satisfies (6.2). In view of its structure, we will verify that w satisfies (6.1) if we prove that

$$x - K \le w(x) \quad \text{for all } x < a, \tag{7.1}$$

and

$$\frac{1}{2}\sigma^2 x^2 w''(x) + bxw'(x) - rw(x) \le 0 \quad \text{for all } x > a. \tag{7.2}$$

In view of (6.3) and (6.5), we can see that (7.1) is equivalent to

$$\frac{x-K}{\psi(x)} \le \frac{a-K}{\psi(a)} \quad \text{for all } x < a.$$

In the context of (6.6) with $z < \mathfrak{z}_\beta$ or (6.7) with $z \le z_\ominus$ or (6.8) with $z \le z_\oplus$, this inequality is equivalent to (5.26), which is true thanks to Lemma 5.2. In the context of (6.6) with $z \ge \mathfrak{z}_\beta$ or (6.7) with $z \ge \mathfrak{z}_\mathrm{c}$, this inequality follows immediately from the fact that the function $x \mapsto (x-K)x^{-n}$ is strictly increasing in $]0, \mathfrak{z}_0[$. On the other hand, (7.2) is equivalent to $bx - r(x-K) \le 0$ for all $x > a$, which is true because, in all cases, $a > \mathfrak{z}_\mathrm{c} = \frac{rK}{r-b}$. □

Proof of Theorem 6.3. Fix any $z \in \,]0, \mathfrak{z}_\mathrm{c}[$. Using the identity in (4.9), we calculate

$$\begin{aligned} \frac{\partial J}{\partial x}(x;z) &= -(n-1)(m-1)Az^{n-m}x^{-n-1}(x-\mathfrak{z}_\mathrm{c})\left[\left(\frac{x}{z}\right)^{n-m} - 1\right] \\ &\begin{cases} < 0, & \text{if } x \in \,]z, \mathfrak{z}_\mathrm{c}[, \\ > 0, & \text{if } x \in \,]\mathfrak{z}_\mathrm{c}, \infty[. \end{cases} \end{aligned} \tag{7.3}$$

Combining this result with the observations that

$$J(z;z) = 0 \quad \text{and} \quad \lim_{x\to\infty} J(x;z) = \infty,$$

we can see that there exists a unique $\xi(z) \in \,]\mathfrak{z}_\mathrm{c}, \infty[$ such that

$$J(x;z) \begin{cases} < 0 & \text{for all } x \in \,]z, \xi(z)[, \\ > 0 & \text{for all } x \in \,]\xi(z), \infty[, \end{cases} \quad \text{and} \quad \frac{\partial J}{\partial x}(x;z) > 0 \quad \text{for all } x \ge \xi(z). \tag{7.4}$$

We next consider the function h defined by

$$h(x;z) = C(z)x^n + D(z)x^m - (x-K), \quad (x,z) \in \,]0,\infty[\times]0, \mathfrak{z}_\mathrm{c}[,$$

where C and D are given by (6.13) for $\xi = \xi(z)$, and we note that

$$h(z;z) = 0 \quad \text{and} \quad h\big(\xi(z);z\big) = \frac{\partial h}{\partial x}\big(\xi(z);z\big) = 0. \tag{7.5}$$

The calculation

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\left(x\frac{\partial h}{\partial x}(x;z)\right) + bx\frac{\partial}{\partial x}\left(x\frac{\partial h}{\partial x}(x;z)\right) - rx\frac{\partial h}{\partial x}(x;z) = (r-b)x > 0$$

and the maximum principle imply that the function $x \mapsto x\frac{\partial h}{\partial x}(x;z)$ has no positive maximum. Combining this observation with the fact that $\frac{\partial h}{\partial x}(\xi(z);z) = 0$, we can see that, if we define

$$\overline{x}(z) = \inf\left\{x \in [z, \xi(z)] \;\middle|\; \frac{\partial h}{\partial x}(x;z) = 0\right\},$$

then $\frac{\partial h}{\partial x}(x;z) \le 0$ for all $x \in [\overline{x}(z), \xi(z)]$. Since $h(z;z) = h\big(\xi(z);z\big) = 0$ and $h(\cdot;z)$ is not constant, it is not possible that either $\frac{\partial h}{\partial x}(x;z) \le 0$ for all $x \in [z, \xi(z)]$ or $\frac{\partial h}{\partial x}(x;z) \ge 0$ for all $x \in [z, \xi(z)]$. Therefore,

$$\overline{x}(z) \in \,]z, \xi(z)[\quad \text{and} \quad \frac{\partial h}{\partial x}(x;z) \begin{cases} > 0 & \text{for all } x \in [z, \overline{x}(z)[, \\ \le 0 & \text{for all } x \in \,]\overline{x}(z), \xi(z)[. \end{cases} \tag{7.6}$$

Furthermore, the function $\underline{w}$ defined by (6.12) for $\xi = \mathfrak{x}(z)$ is such that

$$\underline{w}(x) > x - K \quad \text{for all } x \in \,]z, \mathfrak{x}(z)[. \tag{7.7}$$

To derive the monotonicity of $\mathfrak{x}$, we first note that the inequality $z \frac{\partial h}{\partial x}(z; z) > 0$, which follows from (7.6), the identity $h(z; z) = 0$ and the expression for D given by (6.13) with $\xi = \mathfrak{x}(z)$, imply that

$$\big[(n-1)\mathfrak{x}(z) - nK\big]\mathfrak{x}^{-m}(z) < \big[(n-1)z - nK\big]z^{-m}.$$

In view of (6.14) and the first expression in (6.15), we can see that this inequality is equivalent to

$$\big[(m-1)\mathfrak{x}(z) - mK\big]z^{n-m}\mathfrak{x}^{-n}(z) < \big[(m-1)z - mK\big]z^{-m}.$$

In view of (6.14), it follows that

$$\begin{aligned}\frac{\partial J}{\partial z}\big(\mathfrak{x}(z); z\big) &= -(n-m)Az^{-1}\Big(\big[(m-1)\mathfrak{x}(z) - mK\big]z^{n-m}\mathfrak{x}^{-n}(z) - \big[(m-1)z - mK\big]z^{-m}\Big) \\ &> 0.\end{aligned}$$

Differentiating the identity $J\big(\mathfrak{x}(z); z\big) = 0$ with respect to z and using this inequality, along with (7.4), we obtain

$$\mathfrak{x}'(z) = -\frac{\frac{\partial J}{\partial z}\big(\mathfrak{x}(z); z\big)}{\frac{\partial J}{\partial x}\big(\mathfrak{x}(z); z\big)} < 0,$$

which proves that $\mathfrak{x}$ is strictly decreasing. Furthermore, the limits in (6.16) hold true thanks to (7.3) and the fact that

$$0 = \lim_{z \downarrow 0} J\big(\mathfrak{x}(z); z\big) = A \lim_{z \downarrow 0}\big[(n-1)\mathfrak{x}(z) - nK\big]\mathfrak{x}^{-m}(z),$$

which follows from the first expression in (6.15) and the fact that $\mathfrak{x}(z) > \mathfrak{z}_{\mathrm{c}}$ for all $z < \mathfrak{z}_{\mathrm{c}}$.

To complete the proof, we fix any $z \in \,]z_{\ominus}, \mathfrak{z}_{\mathrm{c}}[$. By construction, the function $\underline{w}$ defined by (6.12) for $\xi = \mathfrak{x}(z)$ is continuous, C^1 inside $]0, \infty[\setminus \{z\}$ and C^2 inside $]0, \infty[\setminus \{z, \xi\}$. In view of its structure, we will verify that it satisfies the variational inequality (6.1)–(6.2) if we prove that

$$(1+\beta)\underline{w}'_+(z) \le (1-\beta)\underline{w}'_-(z), \tag{7.8}$$

$$x - K \le \underline{w}(x) \quad \text{for all } x < \mathfrak{x}(z), \tag{7.9}$$

and

$$\frac{1}{2}\sigma^2 x^2 \underline{w}''(x) + bx\underline{w}'(x) - r\underline{w}(x) \le 0 \quad \text{for all } x > \mathfrak{x}(z). \tag{7.10}$$

In view of the definition (6.12) of $\underline{w}$ and the identity $h(z, z) = 0$, we can see that (7.8) is equivalent to

$$(n-m)(1+\beta)C(z)z^n + m(1+\beta)(z-K) \le n(1-\beta)(z-K).$$

Furthermore, using the expressions for A and $C(z)$ given by (4.6) and (6.13), we can see that this inequality is equivalent to

$$-\big[(m-1)\mathfrak{x}(z) - mK\big]z^n\mathfrak{x}^{-n}(z) - (n-m)A(z-K) \le 0 \overset{(6.14)}{\Leftrightarrow} F\big(\mathfrak{x}(z); z\big) \ge 0. \tag{7.11}$$

If $z \in [\mathfrak{C}\mathfrak{Z}_c, \mathfrak{Z}_c[$, then this inequality follows immediately from (5.8) and the fact that $\xi(z) > \mathfrak{Z}_c$. If $z \in]z_\ominus, \mathfrak{C}\mathfrak{Z}_c[$, then the conclusions in Remark 5.3 and (7.5) imply that

$$C\big(\mathfrak{z}(z)\big) = A\Gamma(z), \quad D\big(\mathfrak{z}(z)\big) = B(z)\Gamma(z) \quad \text{and} \quad \xi\big(\mathfrak{z}(z)\big) = \alpha(z),$$

where $\alpha(z)$ and $\mathfrak{z}(z) \in]z, \mathfrak{Z}_c[$ are as in Lemma 5.1.(ii) and Lemma 5.2.(ii), respectively. The last of these identities and the fact that ξ is strictly decreasing imply that $\xi(z) > \alpha(z)$, and (7.11) follows from (5.7) in Lemma 5.1.(ii).

Recalling that $\mathfrak{Z}_c < \mathfrak{Z}_0 = \frac{nK}{n-1}$ (see (4.17) in Lemma 4.2)), we can see that

$$\frac{d}{dx}\big[\underline{w}(x) - (x-K)\big] < (n-1)(z - \mathfrak{Z}_0)z^{-1} < 0 \quad \text{for all } x < z \in]z_\ominus, \mathfrak{Z}_c[.$$

This result, the identity $\underline{w}(z) - (z-K) = 0$ and (7.7) imply that (7.9) holds true. Finally, (7.10) is equivalent to $bx - r(x-K) \le 0$ for all $x > \xi(z)$, which is true because $\xi(z) > \mathfrak{Z}_c = \frac{rK}{r-b}$. □

Proof of Theorem 6.4. In view of the inequality $\mathfrak{Z}_c < \mathfrak{Z}_0$ (see (4.18)) and the definition (6.27) of G, we can see that

$$G(x, x; z) = 0, \qquad \lim_{y \to \infty} G(x, y; z) = \infty$$

and

$$\frac{\partial G}{\partial y}(x, y; z) = -(n-1)(m-1)(y - \mathfrak{Z}_c)\left[\left(\frac{y}{z}\right)^{n-m} - 1\right] z^n y^{-n-1}$$
$$\begin{cases} < 0, & \text{for all } y \in \big]\mathfrak{Z}_0, z\big[, \\ > 0, & \text{for all } y > z. \end{cases}$$

It follows that, given any $z > \mathfrak{Z}_0$, there exists a unique function $L(\cdot; z) : [\mathfrak{Z}_0, z[\to]z, \infty[$, such that

$$G\big(x, L(x; z); z\big) = 0 \quad \text{and} \quad z < L(x; z) \quad \text{for all } x \in [\mathfrak{Z}_0, z[. \tag{7.12}$$

In particular, this function is such that

$$G(x, y; z) \begin{cases} < 0, & \text{for all } \mathfrak{Z}_0 \le x < y < L(x; z), \\ > 0, & \text{for all } \mathfrak{Z}_0 \le x < L(x; z) < y, \end{cases} \quad \text{and} \quad \lim_{x \to z} L(x; z) = z. \tag{7.13}$$

Furthermore, differentiating the identity $G\big(x, L(x; z); z\big) = 0$ with respect to x, we obtain

$$\frac{\partial L}{\partial x}(x; z) = -\frac{(x - \mathfrak{Z}_c)\left[1 - \left(\frac{x}{z}\right)^{n-m}\right] x^{-n-1}}{\big(L(x; z) - \mathfrak{Z}_c\big)\left[\left(\frac{L(x;z)}{z}\right)^{n-m} - 1\right] L^{-n-1}(x; z)}$$
$$< 0 \;\; \text{for all } x \in]\mathfrak{Z}_0, z[. \tag{7.14}$$

In view of (5.16), the definition (6.28) of H and the limit in (7.13), we can see that

$$\lim_{x \to z} H\big(x, L(x; z); z\big) = H(z, z; z) = -\frac{2\beta}{1+\beta} z^{-m+1} < 0.$$

On the other hand, we use (5.13), (7.12), (7.14) and the inequality $\mathfrak{z}_{\mathrm{c}} < \mathfrak{z}_0$ to obtain

$$\begin{aligned}
&\frac{\partial}{\partial x} H\big(x, L(x;z); z\big) \\
&\quad = (n-1)(m-1)\frac{1-\beta}{1+\beta}(x - \mathfrak{z}_{\mathrm{c}})x^{-m-1} \\
&\qquad \times \left[\frac{\big[n(1-\beta) - m(1+\beta)\big] L^{n-m}(x;z) + 2n\beta z^{n-m}}{(n-m)(1-\beta)} \frac{1 - \left(\frac{x}{z}\right)^{n-m}}{\left(\frac{L(x;z)}{z}\right)^{n-m} - 1} x^{-n+m} + 1 \right] \\
&\quad < 0 \qquad \text{for all } x \in \left]\mathfrak{z}_0, z\right[.
\end{aligned}$$

These calculations imply that

$$\text{there exists a unique } x^* \in \left]\mathfrak{z}_0, z\right[\text{ such that } H\big(x^*, L(x^*;z); z\big) = 0 \tag{7.15}$$

if and only if

$$H\big(\mathfrak{z}_0, L(\mathfrak{z}_0;z); z\big) = L^{-n}(\mathfrak{z}_0;z) F\big(L(\mathfrak{z}_0;z); z\big) > 0. \tag{7.16}$$

The analysis leading to (7.12) and (7.15)–(7.16) imply that the system of equations (6.26) has a unique solution (γ, ζ) such that $\mathfrak{z}_0 < \gamma < z < \zeta$ if and only if (7.16) holds true, in which case, $(\gamma, \zeta) = \big(x^*, L(x^*;z)\big)$. We can show that (7.16) is equivalent to $z > z_{\oplus}$, where $z_{\oplus}$ is as in Lemma 5.2.(iii), as follows.

If the problem's parameters are such that $n > \frac{1+\beta}{1-\beta}$ and $z \geq \mathfrak{z}_\beta$, then (4.18) in Lemma 4.2, (5.9)–(5.10) in Lemma 5.1.(iii) and (7.12) imply that (7.16) holds true. On the other hand, if the problem's parameters are such that either $n > \frac{1+\beta}{1-\beta}$ and $z \in \left]\mathfrak{z}_0, \mathfrak{z}_\beta\right[$ or $n \leq \frac{1+\beta}{1-\beta}$ and $z > \mathfrak{z}_0$, then (5.9)–(5.10) imply that (7.16) holds true if and only if $L(\mathfrak{z}_0;z) > \alpha(z)$, where $\alpha(z) > z$ is the unique solution to the equation $F(x;z) = 0$ derived in Lemma 5.1.(iii). In view of (7.13), we can see that the inequality $L(\mathfrak{z}_0;z) > \alpha(z)$ is equivalent to

$$\begin{aligned}
G\big(\mathfrak{z}_0, \alpha(z); z\big) &= \big[(n-1)\alpha(z) - nK\big] z^m \alpha^{-m}(z) \\
&\qquad - \big[(m-1)\alpha(z) - mK\big] z^n \alpha^{-n}(z) - \frac{n-m}{n} z^n \mathfrak{z}_0^{-n+1} \\
&< 0.
\end{aligned} \tag{7.17}$$

Using the identity $F\big(\alpha(z); z\big) = 0$ to eliminate the term $\big[(m-1)a - mK\big]$ and the identity $B(z)z^m + Az^n = z^n$ to simplify, we derive the expression

$$G\big(\mathfrak{z}_0, \alpha(z); z\big) B(z)\alpha^m(z) = \big[(n-1)\alpha(z) - nK\big] z^n - \frac{n-m}{n} z^n \mathfrak{z}_0^{-n+1} B(z)\alpha^m(z).$$

Similarly, we can eliminate the term $\big[(n-1)\alpha(z) - nK\big]$ to obtain

$$G\big(\mathfrak{z}_0, \alpha(z); z\big) A\alpha^n(z) = -\big[(m-1)\alpha(z) - mK\big] z^n - \frac{n-m}{n} z^n \mathfrak{z}_0^{-n+1} A\alpha^n(z).$$

In the case that we consider here (either $n > \frac{1+\beta}{1-\beta}$ and $z \in \left]\mathfrak{z}_0, \mathfrak{z}_\beta\right[$ or $n \leq \frac{1+\beta}{1-\beta}$ and $z > \mathfrak{z}_0$), the fact that $z < \alpha(z)$ (see (5.9) in Lemma 5.1) implies that

$$\psi\big(\alpha(z); z\big) = A\alpha^n(z) + B(z)\alpha^m(z),$$

while, the facts that $\mathfrak{z}_0 = \frac{nK}{n-1} < z$ imply that

$$\frac{\mathfrak{z}_0 - K}{\psi(\mathfrak{z}_0; z)} = \frac{\mathfrak{z}_0 - K}{\mathfrak{z}_0^n} = \frac{1}{n}\mathfrak{z}_0^{-n+1}.$$

Therefore, adding up the last two expressions for G yields

$$G\big(\mathfrak{z}_0, \alpha(z); z\big) = (n-m)z^n \left[\frac{\alpha(z) - K}{\psi\big(\alpha(z); z\big)} - \frac{\mathfrak{z}_0 - K}{\psi(\mathfrak{z}_0; z)}\right].$$

It follows that (7.17) is equivalent to $z > z_\oplus$, as required, thanks to Lemma 5.2.(iii).

By construction, the function $\overline{w}$ defined by (6.21) is continuous, C^1 inside $]0, \infty[\setminus \{z\}$ and C^2 inside $]0, \infty[\setminus \{z, \xi\}$. In view of its structure, we will verify that $\overline{w}$ satisfies the variational inequality (6.1)–(6.2) if we prove that

$$x - K \le \overline{w}(x) \quad \text{for all } x \in \,]0, \mathfrak{z}_0[\cup]\gamma, \zeta[, \tag{7.18}$$

and

$$\frac{1}{2}\sigma^2 x^2 \overline{w}''(x) + bx\overline{w}'(x) - r\overline{w}(x) \le 0 \quad \text{for all } x \in \,]\mathfrak{z}_0, \gamma[\cup]\zeta, \infty[. \tag{7.19}$$

To establish (7.18), we first note that

$$\frac{d}{dx}\big[\overline{w}(x) - (x - K)\big] = \mathfrak{z}_0^{-n+1} x^{n-1} - 1 > 0 \quad \text{for all } x \in \,]0, \mathfrak{z}_0[.$$

Combining this observation with the fact that $\overline{w}(\mathfrak{z}_0) - (\mathfrak{z}_0 - K) = 0$, we can see that (7.18) holds true for all $x \in \,]0, \mathfrak{z}_0[$. On the other hand, the inequalities

$$\frac{mK}{m-1} < \frac{nK}{n-1} = \mathfrak{z}_0 < \gamma < \zeta$$

and the expressions (6.22), (6.23) of C_ℓ, D_ℓ, C_r, D_r imply that these constants are all strictly positive. Therefore, the restrictions of the function $x \mapsto \overline{w}(x) - (x - K)$ to the intervals $]\gamma, z[$ and $]z, \zeta[$ are both strictly convex. Combining this observation with the facts that

$$\overline{w}(\gamma) - (\gamma - K) = \frac{d}{dx}\big[\overline{w}(x) - (x - K)\big]\Big|_{x=\gamma} = 0$$

and

$$\overline{w}(\zeta) - (\zeta - K)\big] = \frac{d}{dx}\big[\overline{w}(x) - (x - K)\big]\Big|_{x=\zeta} = 0,$$

we can see that (7.18) also holds true for $x \in \,]\gamma, \zeta[$. On the other hand, (7.19) is is equivalent to $bx - r(x - K) \le 0$ for all $x \in \,]\mathfrak{z}_0, \gamma[\cup]\zeta, \infty[$, which is true because $\mathfrak{z}_0 > \mathfrak{z}_{\mathrm{c}} = \frac{rK}{r-b}$. □

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