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Phase singularities in complex arithmetic random waves

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Abstract

Complex arithmetic random waves are stationary Gaussian complex-valued solutions of the Helmholtz equation on the two-dimensional flat torus. We use Wiener-Itô chaotic expansions in order to derive a complete characterization of the second order high-energy behaviour of the total number of phase singularities of these functions. Our main result is that, while such random quantities verify a universal law of large numbers, they also exhibit non-universal and non-central second order fluctuations that are dictated by the arithmetic nature of the underlying spectral measures. Such fluctuations are qualitatively consistent with the cancellation phenomena predicted by Berry (2002) in the case of complex random waves on compact planar domains. Our results extend to the complex setting recent pathbreaking findings by Rudnick and Wigman (2008), Krishnapur, Kurlberg and Wigman (2013) and Marinucci, Peccati, Rossi and Wigman (2016). The exact asymptotic characterization of the variance is based on a fine analysis of the Kac-Rice kernel around the origin, as well as on a novel use of combinatorial moment formulae for controlling long-range weak correlations. As a by-product of our analysis, we also deduce explicit bounds in smooth distances for the second order non-central results evoked above.

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1 Introduction

1.1 Overview and main results

Let $\mathbb{T} := \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional flat torus, and define $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ to be the associated Laplace-Beltrami operator. Our aim in this paper is to characterize the high-energy behaviour of the zero set of complex-valued random eigenfunctions of Δ, that is, of solutions f of the Helmholtz equation

$$\Delta f + Ef = 0, \tag{1.1}$$

for some adequate $E > 0$. In order to accomplish this task, we will extend and generalise the approach initiated in [M-P-R-W], in particular by providing a new set of techniques that allow one to control residual terms arising in Wiener-Itô chaotic expansions (see Section 6 below), as well as to deduce explicit bounds in smooth distances for second order fluctuations (see Theorem 1.5).

In order to understand our setting, recall that the eigenvalues of $-\Delta$ on $\mathbb{T}$ are the positive reals of the form $E_n := 4\pi^2 n$, where $n = a^2 + b^2$ for some $a, b \in \mathbb{Z}$ (that is, n is an integer that can be represented as the sum of two squares). Here, and throughout the paper, we set

$$S := \{n \in \mathbb{N} : a^2 + b^2 = n, \text{ for some } \ a, b \in \mathbb{Z}\},$$

and for $n \in S$ we define

$$\Lambda_n := \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 : \|\lambda\|^2 := \lambda_1^2 + \lambda_2^2 = n\}$$

to be the set of **energy levels** associated with n, while $\mathcal{N}_n := |\Lambda_n|$ denotes its cardinality. An orthonormal basis (in $L^2(\mathbb{T})$) for the eigenspace associated with E_n is given by the set of complex exponentials $\{e_\lambda : \lambda \in \Lambda_n\}$, defined as

$$e_\lambda(x) := \mathrm{e}^{i2\pi\langle \lambda, x\rangle}, \quad x \in \mathbb{T},$$

with $i = \sqrt{-1}$.

For every $n \in S$, the integer $\mathcal{N}_n$ counts the number of distinct ways of representing n as the sum of two squares: it is a standard fact (proved e.g. by using Landau's theorem) that $\mathcal{N}_n$ grows *on average* as $\sqrt{\log n}$, and also that there exists an infinite sequence of prime numbers $p \in S$, $p \equiv 1 \bmod 4$, such that $\mathcal{N}_p = 8$. A classical discussion of the properties of S and $\mathcal{N}_n$ can be found e.g. in [H-W, Section 16.9 and 16.10]. In the present paper, we will systematically consider sequences $\{n_j\} \subset S$ such that $\mathcal{N}_{n_j} \to \infty$ (this is what we refer to as the **high-energy limit**).

The complex waves considered in this paper are natural generalizations of the real-valued arithmetic waves introduced by Rudnick and Wigman in [R-W], and further studied in [K-K-W, M-P-R-W, P-R, O-R-W, R-W2]; as such, they are close relatives of the complex fields considered in the physical literature — see e.g. [B-D, Be3, N, N-V], as well as the discussion provided below. For every $n \in S$, we define the **complex arithmetic random wave of order** n to be the random field

$$\Theta_n(x) := \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} v_\lambda \, e_\lambda(x), \quad x \in \mathbb{T}, \tag{1.2}$$

where the v_λ, $\lambda \in \Lambda_n$, are independent and identically distributed (i.i.d.) complex-valued Gaussian random variables such that, for every $\lambda \in \Lambda_n$, $\mathrm{Re}(v_\lambda)$ and $\mathrm{Im}(v_\lambda)$ are two independent centered Gaussian random variables with mean zero and variance one[1]. We

[1] Considering random variables v_λ with variance 2 (instead of a more usual unit variance) will allow us to slightly simplify the discussion contained in Section 1.2.

will see in Section 1.2 that these assumptions imply that the real and imaginary parts of the random field Θ_n are stochastically independent. The family $\{v_\lambda : \lambda \in \Lambda_n,\, n \in S\}$ is tacitly assumed to be defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with $\mathbb{E}$ indicating expectation with respect to $\mathbb{P}$. It is immediately verified that Θ_n satisfies the equation (1.1), that is, $\Delta\Theta_n + E_n\Theta = 0$, and also that Θ_n is **stationary**, in the sense that, for every $y \in \mathbb{T}$, the translated process $x \mapsto \Theta_n(y + x)$ has the same distribution as Θ_n (this follows from the fact that the distribution of $\{v_\lambda : \lambda \in \Lambda_n\}$ is invariant with respect to unitary transformations; see Section 1.2 for further details on this straightforward but fundamental point).

The principal focus of our investigation are the high-energy fluctuations of the following **zero sets**:

$$\begin{aligned} \mathscr{I}_n &:= \{x \in \mathbb{T} : \Theta_n(x) = 0\} \qquad (1.3)\\ &= \{x \in \mathbb{T} : \mathrm{Re}(\Theta_n(x)) = 0\} \cap \{x \in \mathbb{T} : \mathrm{Im}(\Theta_n(x)) = 0\}, \quad n \in S. \end{aligned}$$

We will show below (Part 1 of Theorem 1.2) that, with probability one, $\mathscr{I}_n$ is a finite collection of isolated points for every $n \in S$; throughout the paper, we will write

$$I_n := |\mathscr{I}_n| = \mathrm{Card}(\mathscr{I}_n), \qquad n \in S. \tag{1.4}$$

In accordance with the title of this work, the points of $\mathscr{I}_n$ are called **phase singularities** for the field Θ_n, in the sense that, for every $x \in \mathscr{I}_n$, the phase of $\Theta_n(x)$ (as a complex-valued random quantity) is not defined.

As for nodal lines of real arithmetic waves [K-K-W, M-P-R-W], our main results crucially involve the following collection of probability measures on the unit circle $S^1 \subset \mathbb{R}^2$:

$$\mu_n(dz) := \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \delta_{\lambda/\sqrt{n}}(dz), \quad n \in S, \tag{1.5}$$

as well as the associated Fourier coefficients

$$\widehat{\mu}_n(k) := \int_{S^1} z^{-k} \mu_n(dz), \quad k \in \mathbb{Z}. \tag{1.6}$$

In view of the definition of Λ_n, the probability measure μ_n defined in (1.5) is trivially invariant with respect to the transformations $z \mapsto \overline{z}$ and $z \mapsto i \cdot z$. The somewhat erratic behaviours of such objects in the high-energy limit are studied in detail in [K-K-W, K-W]. Here, we only record the next Proposition 1.1, implying in particular that the sequences $\{\mu_n : n \in S\}$ and $\{\widehat{\mu}_n(4) : n \in S\}$ *do not* admit limits as $\mathcal{N}_n$ diverges to infinity within the set S. Such a statement is necessary in order to understand the non-universal nature of the forthcoming Theorem 1.2, as well as of Theorem 1.4.

Recall from [K-K-W, K-W] that a measure μ on $(S^1, \mathscr{B})$ (where $\mathscr{B}$ is the Borel σ-field) is said to be **attainable** if there exists a sequence $\{n_j\} \subset S$ such that $\mathcal{N}_{n_j} \to \infty$ and μ_{n_j} converges to μ in the sense of the weak-$\star$ topology.

Proposition 1.1 (See [K-W, K-K-W]). *The class of attainable measures is an infinite strict subset of the collection of all probability measures on S^1 that are invariant with respect to the transformations $z \mapsto \overline{z}$ and $z \mapsto i \cdot z$. Also, for every $\eta \in [0, 1]$ there exists a sequence $\{n_j\} \subset S$ such that $\mathcal{N}_{n_j} \to \infty$ and $|\widehat{\mu}_{n_j}(4)| \to \eta$.*

Note that, if μ_{n_j} converges to μ_∞ in the weak-$\star$ topology, then $\widehat{\mu}_{n_j}(4) \to \widehat{\mu}_\infty(4)$. For instance, one knows from [E-H, K-K-W] that there exists a density one sequence $\{n_j\} \subset S$

such that $\mathcal{N}_{n_j} \to \infty$ and μ_{n_j} converges to the uniform measure on S^1, in which case $\widehat{\mu}_{n_j}(4) \to 0$.

Some conventions. Given two sequences of positive numbers $\{a_m\}$ and $\{b_m\}$, we shall write $a_m \sim b_m$ if $a_m/b_m \to 1$, and $a_m \ll b_m$ or (equivalently and depending on notational convenience) $a_m = O(b_m)$ if a_m/b_m is asymptotically bounded. The notation $a_m = o(b_m)$ means as usual that $a_m/b_m \to 0$. Convergence in distribution for random variables on $(\Omega, \mathscr{F}, \mathbb{P})$ will be denoted by $\overset{\text{law}}{\Longrightarrow}$, whereas equality in distribution will be indicated by the symbol $\overset{\text{law}}{=}$.

The main result of the present work is the following exact characterization of the first and second order behaviours of I_n, as defined by (1.4), in the high-energy limit. As discussed below, it is an extension of the results proved in [K-K-W, M-P-R-W], providing a rigorous description of the **Berry's cancellation phenomenon** [Be3] in the context of phase singularities of complex random waves.

Theorem 1.2. 1. **(Finiteness and mean)** *With probability one, for every $n \in S$ the set $\mathscr{I}_n$ is composed of a finite collection of isolated points, and*

$$\mathbb{E}[I_n] = \frac{E_n}{4\pi} = \pi n. \tag{1.7}$$

2. **(Non-universal variance asymptotics)** *As $\mathcal{N}_n \to \infty$,*

$$\mathrm{Var}(I_n) = d_n \times \frac{E_n^2}{\mathcal{N}_n^2}\,(1 + o(1)) = V_n\,(1 + o(1)), \tag{1.8}$$

where

$$d_n := \frac{3\widehat{\mu}_n(4)^2 + 5}{128\pi^2}, \quad \text{and} \quad V_n := d_n \times \frac{E_n^2}{\mathcal{N}_n^2}. \tag{1.9}$$

3. **(Universal law of large numbers)** *Let $\{n_j\} \subset S$ be a subsequence such that $\mathcal{N}_{n_j} \to +\infty$. Then, for every sequence $\{\epsilon_{n_j}\}$ such that $\epsilon_{n_j}\mathcal{N}_{n_j} \to \infty$, one has that*

$$\mathbb{P}\left[\left|\frac{I_{n_j}}{\pi n_j} - 1\right| > \epsilon_{n_j}\right] \to 0. \tag{1.10}$$

4. **(Non-universal and non-central second order fluctuations)** *Let $\{n_j\} \subset S$ be such that $\mathcal{N}_{n_j} \to +\infty$ and $|\widehat{\mu}_{n_j}(4)| \to \eta \in [0,1]$. Then,*

$$\begin{aligned} \widetilde{I}_{n_j} \quad &:= \quad \frac{I_{n_j} - \mathbb{E}[I_{n_j}]}{V_{n_j}^{1/2}} \\ &\overset{\text{law}}{\Longrightarrow} \quad \frac{1}{2\sqrt{10 + 6\eta^2}}\left(\frac{1+\eta}{2}A + \frac{1-\eta}{2}B - 2(C-2)\right) =: \mathcal{J}_\eta, \end{aligned} \tag{1.11}$$

with A, B, C independent random variables such that $A \overset{\text{law}}{=} B \overset{\text{law}}{=} 2X_1^2 + 2X_2^2 - 4X_3^2$ and $C \overset{\text{law}}{=} X_1^2 + X_2^2$, where (X_1, X_2, X_3) is a standard Gaussian vector of $\mathbb{R}^3$.

Relations (1.8)–(1.9) show that the asymptotic behaviour of the variance of I_n is **non-universal**. Indeed, when $\mathcal{N}_{n_j} \to \infty$, the fluctuations of the sequence d_{n_j} depend on the chosen subsequence $\{n_j\} \subset S$, via the squared Fourier coefficients $\widehat{\mu}_{n_j}(4)^2$: in particular, the possible limit values of the sequence $\{d_{n_j}\}$ correspond to the whole interval $\left[\frac{5}{128\pi^2}, \frac{1}{16\pi^2}\right]$. We also observe that the random variables $\mathcal{J}_\eta$ appearing in (1.11) are clearly non Gaussian, and one can easily check that, if $\eta_1 \neq \eta_2$, then $\mathcal{J}_{\eta_1}$ and $\mathcal{J}_{\eta_2}$ have different distributions.

Remark 1.3. The arguments leading to the proof of (1.7) show also that, for every measurable $A \subset \mathbb{T}$,

$$\mathbb{E}[|\mathscr{I}_n \cap A|] = \mathrm{Leb}(A) \times \pi n, \tag{1.12}$$

where 'Leb' indicates the Lebesgue measure on the torus.

The non-universal nature of the asymptotic relations (1.8) and (1.11) is not surprising, once Theorem 1.2 is compared with analogous findings for real-valued arithmetic random waves — see e.g. [B-M-W, K-K-W, M-P-R-W, O-R-W, R-W, R-W2]. To this end, for every $n \in S$ we define the **(real-valued) arithmetic random wave** of order n, written $f_n = \{f_n(x) : x \in \mathbb{T}\}$, to be the centred Gaussian random field

$$f_n(x) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} b_\lambda e_\lambda(x), \tag{1.13}$$

where

$$\mathbf{B}(n) := \{b_\lambda : \lambda \in \Lambda_n\} \tag{1.14}$$

is a collection of complex random weights verifying the following properties: (i) b_λ is a complex-valued Gaussian random variable whose real and imaginary parts are independent Gaussian random variables with mean zero and variance $1/2$, (ii) if $\lambda \notin \{\sigma, -\sigma\}$, then b_λ and b_σ are stochastically independent, and (iii) $b_\lambda = \overline{b_{-\lambda}}$. Elementary computations show that, for every $n \in S$, the random function f_n is a stationary Gaussian field verifying

$$\mathbb{E}\Big[f_n(x) f_n(y)\Big] = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \cos(2\pi\langle \lambda, x - y\rangle) =: r_n(x - y). \tag{1.15}$$

We recall that, according e.g. to [C] and in view of the stationarity of f_n, with probability one $f_n^{-1}(0)$ is a finite union of disjoint rectifiable closed curves. For $n \in S$, the **nodal length** of f_n is defined as

$$L_n := \mathrm{length}\,(f_n^{-1}(0)).$$

The following statement collects some of the most relevant findings from [R-W] (Point 1), [K-K-W] (Point 2) and [M-P-R-W] (Point 3), and should be compared with Theorem 1.2.

Theorem 1.4 (**See [R-W, K-K-W, M-P-R-W]).** 1. *For every $n \in S$*

$$\mathbb{E}[L_n] = \frac{E_n}{2\sqrt{2}}. \tag{1.16}$$

2. *As $\mathcal{N}_n \to \infty$,*

$$\mathrm{Var}(L_n) = c_n \times \frac{E_n}{\mathcal{N}_n^2}\,(1 + o(1)), \tag{1.17}$$

where

$$c_n := \frac{1 + \widehat{\mu}_n(4)^2}{512}. \tag{1.18}$$

3. *Let $\{n_j\} \subset S$ be such that $\mathcal{N}_{n_j} \to +\infty$ and $|\widehat{\mu}_{n_j}(4)| \to \eta \in [0, 1]$. Then,*

$$\begin{aligned} \widetilde{L}_{n_j} \quad &:= \quad \frac{L_{n_j} - \mathbb{E}[L_{n_j}]}{\sqrt{\mathrm{Var}(L_{n_j})}} \\ &\overset{\text{law}}{\Longrightarrow} \quad \mathcal{M}_\eta := \frac{1}{2\sqrt{1+\eta^2}}(2 - (1+\eta)X_1^2 - (1-\eta)X_2^2), \end{aligned} \tag{1.19}$$

where (X_1, X_2) is a standard Gaussian vector of $\mathbb{R}^2$.

A further important point is that the techniques introduced in the present paper allow one also to obtain quantitative versions of the non-central convergence result at Point 4 of Theorem 1.2. This is the object of the next statement, that one can use in order to deal with subsequences $\{n_j\} \subset S$ such that the numerical sequence $\left\{\widehat{\mu}_{n_j}(4)^2\right\}$ is not necessarily converging.

Theorem 1.5 (Explicit bounds and coupling). *Let $\{n_j\} \subset S$ be such that $\mathcal{N}_{n_j} \to +\infty$, and consider a twice continuously differentiable function $h : \mathbb{R} \to \mathbb{R}$ such that $\|h'\|_\infty, \|h''\|_\infty \leq C < \infty$. Then, as $n_j \to +\infty$,*

$$\left| \mathbb{E}\left[h\Big(\widetilde{I}_{n_j}\Big)\right] - \mathbb{E}\left[h\Big(\mathcal{J}\left(\widehat{\mu}_{n_j}(4)\right)\Big)\right]\right| = O\big(\mathcal{N}_{n_j}^{-1/4}\big), \tag{1.20}$$

where the constants implicitly involved in the $O(\cdot)$ notation only depend on C. This implies that, on some appropriate probability space $(\Omega^*, \mathscr{A}^*, \mathbb{P}^*)$, there exists a collection of random variables $\{U(n_j), V(n_j) : j \geq 1\}$ such that each $(U(n_j), V(n_j))$ is a coupling of $\widetilde{I}_{n_j}$ and $\mathcal{J}\left(\widehat{\mu}_{n_j}(4)\right)$, and moreover

$$\mathbb{E}^*[|U(n_j) - V(n_j)|] \leq \frac{K}{\mathcal{N}_{n_j}^{1/8}}, \tag{1.21}$$

for some absolute constant $K > 0$.

As explained in Section 2.5, the quantitative bound (1.21) is based on a Lemma from [D-P].

In the next section, we will discuss some further connections with the real arithmetic random waves defined in (1.13).

1.2 Complex zeros as nodal intersections

For simplicity, from now on we will write

$$T_n(x) := \mathrm{Re}(\Theta_n(x)), \quad \widehat{T}_n(x) := \mathrm{Im}(\Theta_n(x)), \tag{1.22}$$

for every $x \in \mathbb{T}$ and $n \in S$; in this way, one has that

$$\mathscr{I}_n = T_n^{-1}(0) \cap \widehat{T}_n^{-1}(0) \quad \text{and} \quad I_n = |T_n^{-1}(0) \cap \widehat{T}_n^{-1}(0)|.$$

We will also adopt the shorthand notation

$$\mathbf{T}_n := \{\mathbf{T}_n(x) = (T_n(x), \widehat{T}_n(x)) : x \in \mathbb{T}\}, \quad n \in S.$$

Our next statement (whose elementary proof is omitted) yields a complete characterization of the distribution of the vector-valued process $\mathbf{T}_n$, as a two-dimensional field whose components are independent and identically distributed real arithmetic random waves, in the sense of (1.13).

Proposition 1.6. *Fix $n \in S$. Then, T_n and $\widehat{T}_n$ are two independent copies of the field f_n defined in* (1.13)*, so that in particular*

$$\mathbb{E}\Big[T_n(x)T_n(y)\Big] = \mathbb{E}\Big[\widehat{T}_n(x)\widehat{T}_n(y)\Big] = r_n(x-y). \tag{1.23}$$

As a consequence, there exist two collections of complex random variables

$$\mathbf{A}(n) = \{a_\lambda : \lambda \in \Lambda_n\} \quad \textit{and} \quad \widehat{\mathbf{A}}(n) = \{\widehat{a}_\lambda : \lambda \in \Lambda_n\}, \tag{1.24}$$

such that $\mathbf{A}(n)$ and $\widehat{\mathbf{A}}(n)$ are two independent copies of $\mathbf{B}(n)$, as defined in (1.14)*, and*

$$T_n(x) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_\lambda e_\lambda(x), \quad \textit{and} \quad \widehat{T}_n(x) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} \widehat{a}_\lambda e_\lambda(x), \tag{1.25}$$

for every $x \in \mathbb{T}$.

The fact that r_n only depends on the difference $x - y$ confirms in particular that $\mathbf{T}_n$ is a two-dimensional Gaussian stationary process.

Assumption 1.7. Without loss of generality, for the rest of the paper we will assume that, for $n \neq m$, the two Gaussian families

$$\mathbf{A}(n) \cup \widehat{\mathbf{A}}(n) \quad \text{and} \quad \mathbf{A}(m) \cup \widehat{\mathbf{A}}(m)$$

are stochastically independent; this is the same as assuming that the two vector-valued fields $\mathbf{T}_n$ and $\mathbf{T}_m$ are stochastically independent.

1.3 Comparison with relevant previous work

Random waves and cancellation phenomena. To the best of our knowledge, the first systematic analysis of phase singularities in wave physics appears in the seminal contribution by Nye and Berry [N-B]. Since then, zeros of complex waves have been the object of an intense study in a variety of branches of modern physics, often under different names, such as *nodal points*, *wavefront dislocations*, *screw dislocations*, *optical vortices* and *topological charges*. The reader is referred e.g. to [D-O-P, N, U-R], and the references therein, for detailed surveys on the topic, focussing in particular on optical physics, quantum chaos and quantum statistical physics.

One crucial reference for our analysis is Berry [Be2], where the author studies several statistical quantities involving singularities of random waves on the plane. Such an object, usually called the (complex) **Berry's random wave model** (RWM), is defined as a complex centered Gaussian field, whose real and imaginary parts are independent Gaussian functions on the plane, with covariance

$$r_{\mathrm{RWM}}(x, y) := J_0\left(\sqrt{E}\,\|x - y\|\right), \quad x, y \in \mathbb{R}^2, \tag{1.26}$$

where $E > 0$ is an energy parameter, and J_0 is the standard Bessel function (see also [Be1]). Formula (1.26) implies in particular that Berry's RWM is **stationary and isotropic**, that is: its distribution is invariant both with respect to translations and rotations. As discussed e.g. in [K-K-W, Section 1.6.1], if $\{n_j\} \subset S$ is a sequence such that $\mathcal{N}_{n_j} \to \infty$ and μ_{n_j} converges weakly to the uniform measure on the circle, then, for every $x \in \mathbb{T}$ and using the notation (1.15),

$$r_{n_j}\left(\sqrt{\frac{E}{n_j}} \cdot \frac{x}{2\pi}\right) \longrightarrow r_{\mathrm{RWM}}(x), \tag{1.27}$$

showing that Berry's RWM is indeed the local scaling limit of the arithmetic random waves considered in the present paper.

Reference [Be3], building upon previous findings of Berry and Dennis [B-D], contains the following remarkable results: **(a)** the expected nodal length per unit area of the real RWM equals $\sqrt{E}/(2\sqrt{2})$ [Be3, Section 3.1], **(b)** as $E \to \infty$ the variance of the nodal length at Point **(a)** is proportional to $\log E$ [Be3, Section 3.2], **(c)** the expected number of phase singularities for unit area of the complex RWM is $E/(4\pi)$ [Be3, Section 4.1], and **(d)** as $E \to \infty$ the variance of the number of singularities at Point **(c)** is proportional to $E \log E$ [Be3, Section 4.2]. Point **(a)** and **(c)** are perfectly consistent with (1.16) and (1.7), respectively. Following [Be3], the estimates at Points **(b)** and **(d)** are due to an 'obscure' cancellation phenomenon, according to which the natural leading term in variance (that should be of the order of $\sqrt{E}$ and $E^{3/2}$, respectively) cancels out in the high-energy limit. The content of Point **(b)** has been rigorously confirmed by Wigman [W] in the related model of real *random spherical harmonics*, whose scaling limit is again the real RWM. See also [A-L-W].

As explained in [K-K-W], albeit improving conjectures from [R-W], the order of the variance established in (1.17) differs from that predicted in **(b)**: this discrepancy is likely due to the fact that, differently from random spherical harmonics, the convergence in (1.27) does not take place uniformly over suitable regions. As already discussed, in [M-P-R-W] it was shown that the asymptotic relation (1.17) is generated by a remarkable chaotic cancellation phenomenon, which also explains the non-central limit theorem stated in (1.19).

The main result of the present paper (see Theorem 1.2) confirms that such a chaotic cancellation continues to hold for phase singularities of complex arithmetic waves, and that it generates non-universal and non-central second order fluctuations for such a random quantity. This fact lends further evidence to the natural conjecture that cancellation phenomena analogous to those described in [Be3, W, K-K-W, M-P-R-W, Ro] should hold for global quantities associated with the zero set of Laplace eigenfunctions on more general manifolds displaying spectral multiplicities, as long as such quantities can be expressed in terms of some area/co-area integral formula.

We stress that the fact that the order of the variance stated in (1.8) differs from the one predicted at Point **(d)** above, can once again be explained by the non-uniform nature of the scaling relation (1.27).

Variance estimates and occupation densities. While the present paper can be seen as a natural continuation of the analysis developed in [K-K-W, M-P-R-W], the methods implemented below will substantially diverge from those used in such references on three fundamental points:

- **(i)** whereas the analysis [M-P-R-W] could directly exploit the variance estimates from [K-K-W], in the present paper we have to compute exact asymptotics for the variance of phase singulairities from scratch, by using a new approach based on the use of combinatorial moment formulae;
- **(ii)** differently from [M-P-R-W], where qualitative limit theorems were proved, the techniques developed in the present paper lead to explicit bounds, such as the ones appearing in Theorem 1.5;
- **(iii)** in order to deal with strong correlations between vectors of the type

$$(T_n(x), \partial/\partial_1 T_n(x), \partial/\partial_2 T_n(x)) \text{ and } (T_n(y), \partial/\partial_1 T_n(y), \partial/\partial_2 T_n(y)),\ x \neq y,$$

the authors of [K-K-W] extensively use results from [O-R-W] (see in particular [K-K-W, Section 4.1]) about the fluctuations of the **Leray measure**

$$A_n := \int_{\mathbb{T}} \delta_0(T_n(x))\, dx,$$

which is defined as the limit in $L^2(\mathbb{P})$ of the sequence $k \mapsto \int_{\mathbb{T}} \varphi_k(T_n(x))\, dx$, with $\{\varphi_k\}$ a suitable approximation of the identity, but following such a route in the framework of random phase singularities is impossible, since the formal quantity

$$B_n := \int_{\mathbb{T}} \delta_{(0,0)}(T_n(x), \widehat{T}_n(x))\, dx$$

cannot be defined as an element of $L^2(\mathbb{P})$. In order to circumvent this difficulty, in Section 5 we will perform a novel technical analysis of singular and non-singular cubes in the framework of Wiener-Itô chaotic expansion. Our use of singular and non-singular cubes is strongly inspired by [K-K-W, O-R-W]. We also stress that one of our fundamental tools is the arithmetic estimate presented in Lemma 8.3 of the Appendix, which already plays a crucial role in [K-K-W] (albeit in a slightly weaker form) — see [K-K-W, Theorem 2.2].

We observe that, in the parlance of stochastic calculus, the quantity A_n (resp. B_n) is the **occupation density at zero** of the random field T_n (resp. $\mathbf{T}_n$) — in particular, the fact that A_n is well-defined in $L^2(\mathbb{P})$ and B_n is not — follows from the classical criterion stated in [G-H, Theorem 22.1], as well as from the relations

$$\int_{\mathbb{T}} \frac{dx}{\sqrt{1-r_n^2(x)}} < \infty \quad \text{and} \quad \int_{\mathbb{T}} \frac{dx}{1-r_n^2(x)} = \infty, \tag{1.28}$$

where we have used the fact that, according e.g. to [O-R-W, Lemma 5.3], the mapping $x \mapsto (1-r_n^2(x))^{-1}$ behaves like a multiple of $1/\|x-x_0\|^2$ around any point x_0 such that $r_n(x_0) = \pm 1$.

Nodal intersections of arithmetic random waves with a fixed curve. A natural problem related to the subject of our paper is that of studying the number of nodal intersections with a fixed deterministic curve $\mathcal{C} \subset \mathbb{T}$ whose length equals L, i.e. number of zeroes of T_n that lie on $\mathcal{C}$:

$$\mathcal{Z}_n := T_n^{-1}(0) \cap \mathcal{C}.$$

In [R-W2], the case where $\mathcal{C}$ is a smooth curve with nowhere zero-curvature has been investigated. The expected number of nodal intersections is $\mathbb{E}[|\mathcal{Z}_n|] = (\pi\sqrt{2})^{-1} \times E_n \times L$, hence proportional to the length L of the curve times the wave number, independent of the geometry. The asymptotic behaviour of the nodal intersections variance in the high energy limit is a subtler matter: it depends on both the angular distribution of lattice points lying on the circle with radius corresponding to the given wavenumber, in particular on the sequence of measures $\{\mu_n\}$, and on the geometry of $\mathcal{C}$. The asymptotic distribution of $|\mathcal{Z}_n|$ is analyzed in [Ro-W]. See [Ma] for the case where $\mathcal{C}$ is a segment.

Zeros of random analytic functions. To the best of our expertise, our limit result (1.11) is the first non-central limit theorem for the number of zeros of random complex analytic functions defined on some manifold $\mathcal{M}$. As such, our findings should be contrasted with the works by Sodin and Tsirelson [S-T, N-S], where one can find central limit results for local statistics of zeros of analytic functions corresponding to three different models (elliptic, flat and hyperbolic). As argued in [W, Section 1.6.4], these results are roughly comparable to those one would obtain by studying zeros of complex random spherical harmonics, for which a central high-energy behaviour is therefore likely to be expected. References [S-Z1, S-Z2], by Shiffman and Zelditch, contain central limit result for the volume of the intersection of the zero sets of independent Gaussian sections of high powers of holomorphic line bundles on a Kähler manifold of a fixed dimension.

1.4 Short plan of the paper

In Section 2 we explain the main ideas and steps of the proof of our main result (Theorem 1.2). The remaining sections are devoted to the detailed proofs. In particular, we collect in Section 7 some technical computations and proofs of intermediate results, whereas Section 8 is an Appendix gathering together several ancillary results and definitions that will be needed in the sequel.

2 Structure of the proofs of Theorem 1.2 and Theorem 1.5

After a short discussion of some preliminary technical notion (Section 2.1 and Section 2.2), in Sections 2.3–2.5 we provide a precise description of the strategy we will adopt in order to attack the proof of our main findings.

2.1 Preliminaries on Wiener chaos

Let $\{H_k : k = 0, 1, ...\}$ be the sequence of **Hermite polynomials** on $\mathbb{R}$, recursively defined as follows: $H_0 \equiv 1$, and, for $k \geq 1$,

$$H_k(t) = tH_{k-1}(t) - H'_{k-1}(t), \qquad t \in \mathbb{R}.$$

It is a standard fact that the collection $\mathbb{H} := \{H_k/\sqrt{k!} : k \geq 0\}$ is a complete orthonormal system for

$$L^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), \gamma) := L^2(\gamma),$$

where $\gamma(dt) := \phi(t)dt = \frac{e^{-t^2/2}}{\sqrt{2\pi}}dt$ is the standard Gaussian measure on the real line. By construction, for every $k \geq 0$, one has that

$$H_{2k}(-t) = H_{2k}(t), \quad \text{and} \quad H_{2k+1}(-t) = -H_{2k+1}(t), \quad t \in \mathbb{R}. \tag{2.1}$$

In view of Proposition 1.6 (recall also Assumption 1.7), every random object considered in the present paper is a measurable functional of the family of complex-valued Gaussian random variables

$$\bigcup_{n\in S} \Big(\mathbf{A}(n) \cup \widehat{\mathbf{A}}(n)\Big),$$

where $\mathbf{A}(n)$ and $\widehat{\mathbf{A}}(n)$ are defined in (1.24). Now define the space $\mathbf{A}$ to be the closure in $L^2(\mathbb{P})$ of all real finite linear combinations of random variables ξ of the form

$$\xi = c_1(z\, a_\lambda + \overline{z}\, a_{-\lambda}) + c_2(u\, \widehat{a}_\tau + \overline{u}\, \widehat{a}_{-\tau})$$

where $\lambda, \tau \in \mathbb{Z}^2$, $z, u \in \mathbb{C}$ and $c_1, c_2 \in \mathbb{R}$. The space $\mathbf{A}$ is a real centered Gaussian Hilbert subspace of $L^2(\mathbb{P})$.

Definition 2.1. For a given integer $q \geq 0$, the q-th **Wiener chaos** associated with $\mathbf{A}$, denoted by C_q, is the closure in $L^2(\mathbb{P})$ of all real finite linear combinations of random variables of the type

$$\prod_{j=1}^{k} H_{p_j}(\xi_j),$$

with $k \geq 1$, where the integers $p_1, ..., p_k \geq 0$ verify $p_1 + \cdots + p_k = q$, and $(\xi_1, ..., \xi_k)$ is a centered standard real Gaussian vector contained in $\mathbf{A}$ (so that $C_0 = \mathbb{R}$).

In view of the orthonormality and completeness of $\mathbb{H}$ in $L^2(\gamma)$, it is not difficult to show that $C_q \perp C_{q'}$ (where the orthogonality holds in $L^2(\mathbb{P})$) for every $q \neq q'$, and moreover

$$L^2(\Omega, \sigma(\mathbf{A}), \mathbb{P}) = \bigoplus_{q=0}^{\infty} C_q;$$

the previous relation simply indicates that every real-valued functional F of $\mathbf{A}$ can be uniquely represented in the form

$$F = \sum_{q=0}^{\infty} \mathrm{proj}(F \,|\, C_q) = \sum_{q=0}^{\infty} F[q], \tag{2.2}$$

where $F[q] := \mathrm{proj}(F \,|\, C_q)$ stands for the the projection of F onto C_q, and the series converges in $L^2(\mathbb{P})$. By definition, one has $F[0] = \mathrm{proj}(F \,|\, C_0) = \mathbb{E}[F]$. See e.g. [N-P, Theorem 2.2.4] for further details.

2.2 About gradients

Differentiating both terms in (1.25) yields that, for $j = 1, 2$,

$$\partial_j T_n(x) = \frac{2\pi i}{\sqrt{\mathcal{N}_n}} \sum_{(\lambda_1,\lambda_2)\in\Lambda_n} \lambda_j a_\lambda e_\lambda(x), \text{ and } \partial_j \widehat{T}_n(x) = \frac{2\pi i}{\sqrt{\mathcal{N}_n}} \sum_{(\lambda_1,\lambda_2)\in\Lambda_n} \lambda_j \widehat{a}_\lambda e_\lambda(x) \quad (2.3)$$

(where we used the shorthand notation $\partial_j = \frac{\partial}{\partial x_j}$). It follows that, for every $n \in S$ and every $x \in \mathbb{T}$,

$$T_n(x),\, \partial_1 T_n(x),\, \partial_2 T_n(x), \widehat{T}_n(x),\, \partial_1 \widehat{T}_n(x),\, \partial_2 \widehat{T}_n(x) \in \mathbf{A}. \quad (2.4)$$

Another important fact (that one can check by a direct computation) is that, for fixed $x \in \mathbb{T}$, the six random variables appearing in (2.4) are stochastically independent. Routine computations (see also [R-W, Lemma 2.3]) yield that

$$\mathrm{Var}(\partial_j T_n(x)) = \mathrm{Var}(\partial_j \widehat{T}_n(x)) = \frac{E_n}{2},$$

for any $j = 1, 2$, any n and any $x \in \mathbb{T}$. Accordingly, we will denote by $\widetilde{\partial}_j$ the normalised derivative

$$\widetilde{\partial}_j := \sqrt{\frac{2}{E_n}} \frac{\partial}{\partial x_j},$$

and adopt the following (standard) notation for the gradient and its normalised version:

$$\nabla := \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix}, \qquad \widetilde{\nabla} := \begin{pmatrix} \widetilde{\partial}_1 \\ \widetilde{\partial}_2 \end{pmatrix}.$$

2.3 Chaotic projections and cancellation phenomena

We will start by showing in Lemma 3.1 that I_n can be formally obtained in $L^2(\mathbb{P})$ as

$$I_n = \int_{\mathbb{T}} \delta_{\mathbf{0}}(\mathbf{T}_n(x))\, |J_{\mathbf{T}_n}(x)|\; dx, \quad (2.5)$$

where $\delta_{\mathbf{0}}$ denotes the Dirac mass in $\mathbf{0} = (0, 0)$, $J_{\mathbf{T}_n}$ is the Jacobian matrix

$$J_{\mathbf{T}_n} = \begin{pmatrix} \partial_1 T_n & \partial_2 T_n \\ \partial_1 \widehat{T}_n & \partial_2 \widehat{T}_n \end{pmatrix}$$

and $|J_{\mathbf{T}_n}|$ is shorthand for the absolute value of its determinant. Since I_n is a square-integrable functional of a Gaussian field, according to the general decomposition (2.2) one has that

$$I_n = \sum_{q\geq 0} I_n[q], \quad (2.6)$$

where $I_n[q] = \mathrm{proj}(I_n|C_q)$ denotes the orthogonal projection of I_n onto the q-th Wiener chaos C_q. Since $I_n[0] = \mathbb{E}[I_n]$, the computation of the 0-order chaos projection will allow us to conclude the proof of Part 1 of Theorem 1.2 in Section 3.2.

One crucial point in our analysis is that, as proved in Lemma 3.4, the projections of I_n onto odd-order Wiener chaoses vanish and, more subtly, also the second chaotic component disappears. Namely, we will show that, for every $n \in S$, it holds

$$I_n[q] = 0 \qquad \text{for odd } q \geq 1$$

and moreover

$$I_n[2] = 0. \quad (2.7)$$

Our proof of (2.7) is based on Green's identity and the properties of Laplacian eigenfunctions (see also [Ro, Section 7.3 and p.134]).

2.4 Leading term: fourth chaotic projections

The first non-trivial chaotic projection of I_n to investigate is therefore $I_n[4]$. One of the main achievements of our paper is an explicit computation of its asymptotic variance, as well as a proof that it gives the dominant term in the asymptotic behaviour of the total variance $\mathrm{Var}(I_n) = \sum_{q\ge 2} \mathrm{Var}(I_n[2q])$. The forthcoming Propositions 2.2, 2.3 and 2.4, that we will prove in Section 4, are the key steps in order to achieve our goals.

Proposition 2.2. *Let $\{n_j\}_j \subset S$ be such that $\mathcal{N}_{n_j} \to +\infty$ and $|\widehat{\mu}_{n_j}(4)| \to \eta$. Then*

$$\mathrm{Var}(I_{n_j}[4]) = d(\eta)\,\frac{E_{n_j}^2}{\mathcal{N}_{n_j}^2}(1+o(1)),$$

where

$$d(\eta) = \frac{3\eta^2+5}{128\pi^2}.$$

It is easily seen that Proposition 2.2 coincides with Part 2 of Theorem 1.2, once we replace $I_{n_j}[4]$ with I_{n_j}. Let us now set, for $n \in S$,

$$R_n(4) \quad := \quad \int_{\mathbb{T}} r_n(x)^4\,dx = \frac{|S_n(4)|}{\mathcal{N}_n^4} = \frac{3\mathcal{N}_n(\mathcal{N}_n-1)}{\mathcal{N}_n^4}, \tag{2.8}$$

$$R_n(6) \quad := \quad \int_{\mathbb{T}} r_n(x)^6\,dx = \frac{|S_n(6)|}{\mathcal{N}_n^6}, \tag{2.9}$$

where $S_n(4), S_n(6)$ are the sets of 4- and 6-correlation coefficients defined in Section 8.2 of the Appendix, and we have used Lemma 8.2 in (2.8). The following result (Proposition 2.3), combined with Proposition 2.2 and Lemma 8.3 allows us to conclude that, as $\mathcal{N}_n \to \infty$,

$$\mathrm{Var}(I_n) \sim \mathrm{Var}(I_n[4]), \tag{2.10}$$

thus achieving the proof of Part 2 of Theorem 1.2. Note that, by virtue of Lemma 8.3 and (2.9), as $\mathcal{N}_n \to \infty$ one has that

$$R_n(6) = O\left(\frac{1}{\mathcal{N}_n^{5/2}}\right), \quad \text{yielding} \quad R_n(6) = o\left(R_n(4)\right).$$

Proposition 2.3. *As $\mathcal{N}_n \to +\infty$, we have*

$$\sum_{q\ge 3} \mathrm{Var}\left(I_n[2q]\right) = O\left(E_n^2 R_n(6)\right).$$

Part 3 of Theorem 1.2 follows immediately from the relation

$$\mathbb{P}\left[\left|\frac{I_n}{\pi n} - 1\right| > \epsilon\right] \le \frac{\mathrm{Var}(I_n/(\pi n))^{1/2}}{\epsilon}$$

(which is a consequence of the Markov inequality), as well as from Part 1 and Part 2 of the same Theorem. Finally, the proof of Part 4 of Theorem 1.2 relies on a careful and technical investigation of $I_n[4]$, leading us to the following result, which indeed coincides with (1.11), once replacing $\frac{I_{n_j}[4]}{\sqrt{\mathrm{Var}(I_{n_j}[4])}}$ with $\widetilde{I}_{n_j}$.

Proposition 2.4. *Let $\{n_j\}_j \subset \{n\}$ be a subsequence such that $\mathcal{N}_{n_j} \to +\infty$ and $|\widehat{\mu}_{n_j}(4)| \to \eta$, then*

$$\frac{I_{n_j}[4]}{\sqrt{\mathrm{Var}(I_{n_j}[4])}} \overset{\mathrm{law}}{\Longrightarrow} \mathcal{J}_\eta,$$

where $\mathcal{J}_\eta$ is defined in (1.11).

2.4.1 Controlling the variance of higher-order chaoses

In order to prove Proposition 2.3, we need to carefully control the remainder given by $\sum_{q\geq 3}\mathrm{Var}(I_n[2q])$; our argument (extending the approach developed in [O-R-W, §6.1] and [R-W2, §4.3]) is the following.

We partition the torus into a union of disjoint squares Q of side length $1/M$, where M is proportional to $\sqrt{E_n}$. Of course

$$I_n = \sum_Q I_{n_{|Q}}, \tag{2.11}$$

where $I_{n_{|Q}}$ is the number of zeroes contained in Q. It holds that, for every $q \geq 0$, $I_n[q] = \sum_Q I_{n_{|Q}}[q]$ and hence

$$\mathrm{Var}\left(\sum_{q\geq 3} I_n[2q]\right) = \sum_{Q,Q'} \mathrm{Cov}\,\left(\text{proj}\left(I_{n_{|Q}}|C_{\geq 6}\right), \text{proj}\left(I_{n_{|Q'}}|C_{\geq 6}\right)\right), \tag{2.12}$$

where proj $(\cdot|C_{\geq 6})$ denotes the orthogonal projection onto $\bigoplus_{q\geq 6} C_q$, that is, the orthogonal sum of Wiener chaoses of order larger or equal than six.

We now split the double sum on the RHS of (2.12) into two parts: namely, one over *singular* pairs of cubes and the other one over *non-singular* pairs of cubes. Loosely speaking, for a pair of non-singular cubes (Q, Q'), we have that for every $(z, w) \in Q \times Q'$, the covariance function r_n of the field T_n and all its normalized derivatives up to the order two $\widetilde{\partial}_i r_n, \widetilde{\partial}_{ij} r_n := (E_n/2)^{-1}\partial/\partial_{x_i x_j} r_n$ for $i, j = 1, 2$ are bounded away from 1 and -1, once evaluated in $z - w$ (see Definition 5.1 and Lemma 5.2).

Lemma 2.5 (Contribution of the singular part)**.** *As $\mathcal{N}_n \to +\infty$,*

$$\left|\sum_{(Q,Q')\ \textit{sing.}} \mathrm{Cov}\,\left(\textit{proj}\left(I_{n_{|Q}}|C_{\geq 6}\right), \textit{proj}\left(I_{n_{|Q'}}|C_{\geq 6}\right)\right)\right| \ll E_n^2 R_n(6). \tag{2.13}$$

In order to show Lemma 2.5 (see Section 5), we use the Cauchy-Schwarz inequality and the stationarity of $\mathbf{T}_n$, in order to reduce the problem to the investigation of nodal intersections in a small square Q_0 around the origin: for the LHS of (2.13) we have

$$\begin{aligned}&\left|\sum_{(Q,Q')\ \text{sing.}} \mathrm{Cov}\,\left(\text{proj}\left(I_{n_{|Q}}|C_{\geq 6}\right), \text{proj}\left(I_{n_{|Q'}}|C_{\geq 6}\right)\right)\right| \\ &\ll \sum_{(Q,Q')\ \text{sing.}} \mathbb{E}\left[I_{n_{|Q_0}}\left(I_{n_{|Q_0}} - 1\right)\right] + \mathbb{E}\left[I_{n_{|Q_0}}\right].\end{aligned}$$

Thus, we need to (i) count the number of singular pairs of cubes, (ii) compute the expected number of nodal intersections in Q_0 and finally (iii) calculate the second factorial moment of $I_{n_{|Q_0}}$. Issue (i) will be dealt with by exploiting the definition of singular pairs of cubes and the behavior of the moments of the derivatives of r_n on the torus (see Lemma 5.3), thus obtaining that

$$|\{(Q, Q')\ \text{sing.}\}| \ll E_n^2 R_n(6).$$

Relations (1.12) and (2.11) yield immediately that $\mathbb{E}\left[I_{n_{|Q_0}}\right]$ is bounded by a constant independent of n.

To deal with (iii) is much subtler matter. Indeed, we need first to check the assumptions for Kac-Rice formula (see [A-W2, Theorem 6.3]) to hold in Proposition 8.4. The latter allows us to write the second factorial moment $\mathbb{E}\left[I_{n_{|Q_0}}\left(I_{n_{|Q_0}}-1\right)\right]$ as an integral on $Q_0\times Q_0$ of the so-called **two-point correlation** function K_2, given by

$$K_2(x,y) := p_{(\mathbf{T}_n(x),\mathbf{T}_n(y))}(\mathbf{0},\mathbf{0})\mathbb{E}\left[|J_{\mathbf{T}_n}(x)|\,|J_{\mathbf{T}_n}(y)|\,\Big|\mathbf{T}_n(x)=\mathbf{T}_n(y)=\mathbf{0}\right],$$

where $x,y\in\mathbb{T}$ and $p_{(\mathbf{T}_n(x),\mathbf{T}_n(y))}$ is the density of $(\mathbf{T}_n(x),\mathbf{T}_n(y))$.

The stationarity of $\mathbf{T}_n$ then reduces the problem to investigating $K_2(x):=K_2(x,0)$ around the origin. Cauchy-Schwartz inequality and the independence and equidistribution of the random fields T_n and $\widehat{T}_n$ yield the following estimation

$$K_2(x)\le 2\frac{|\Omega_n(x)|}{1-r_n(x)^2}=:2\Psi_n(x), \tag{2.14}$$

where $|\Omega_n(x)|$ stands for the absolute value of the determinant of the matrix $\Omega_n(x)$, defined as the covariance matrix of the vector $\nabla T_n(0)$, conditionally on $T_n(x)=T_n(0)=0$. An explicit Taylor expansion at 0 for Ψ_n (made particularly arduous by the diverging integral in (1.28) — see Lemma 7.1) will allow us to prove that $\mathbb{E}\left[I_{n_{|Q_0}}\left(I_{n_{|Q_0}}-1\right)\right]$ is also bounded by a constant independent of n. This concludes the proof of Lemma 2.5.

To achieve the proof of Theorem 1.2, we will eventually show the following result, whose proof relies on Proposition 8.1 in the Appendix, on the definition of non-singular cubes, as well as on the behavior of even moments of derivatives of the covariance function r_n.

Lemma 2.6 (Contribution of the non-singular part)**.** *As $\mathcal{N}_n\to+\infty$, we have*

$$\left|\sum_{(Q,Q')\text{ non sing.}}\operatorname{Cov}\left(\textit{proj}\left(I_{n_{|Q}}|C_{\ge 6}\right),\textit{proj}\left(I_{n_{|Q'}}|C_{\ge 6}\right)\right)\right| = O\left(E_n^2R_n(6)\right).$$

2.5 Proof of Theorem 1.5

Fix $\{n_j\}$ and h as in the statement. We may assume without loss of generality that $C=1$. Exploiting the Lipschitz property of h, as well as the triangle and Cauchy-Schwarz inequalities, yields that

$$\begin{aligned}
&\left|\mathbb{E}\left[h\left(\widetilde{I}_{n_j}\right)\right]-\mathbb{E}\left[h\Big(\mathcal{J}\left(\widehat{\mu}_{n_j}(4)\right)\Big)\right]\right|\\
&\le\left|\mathbb{E}\left[h\left(\frac{I_{n_j}[4]}{V_{n_j}^{1/2}}\right)\right]-\mathbb{E}\left[h\Big(\mathcal{J}\left(\widehat{\mu}_{n_j}(4)\right)\Big)\right]\right|+\sqrt{\frac{\sum_{q\ge3}\operatorname{Var}\left(I_n[2q]\right)}{V_n}}\\
&=\left|\mathbb{E}\left[h\left(\frac{I_{n_j}[4]}{V_{n_j}^{1/2}}\right)\right]-\mathbb{E}\left[h\Big(\mathcal{J}\left(\widehat{\mu}_{n_j}(4)\right)\Big)\right]\right|+O(\mathcal{N}_{n_j}^{-1/4}),
\end{aligned}$$

where the last equality follows from Proposition 2.3. The proof of (1.20) is concluded by applying the following quantitative results.

Proposition 2.7. *Let $\{n_j\}\subset S$ be such that $\mathcal{N}_{n_j}\to+\infty$, and consider a twice continuously differentiable function $h:\mathbb{R}\to\mathbb{R}$ such that $\|h'\|_\infty,\|h''\|_\infty\le1$. Then, as $n_j\to+\infty$,*

$$\left|\mathbb{E}\left[h\left(\frac{I_{n_j}[4]}{V_{n_j}^{1/2}}\right)\right]-\mathbb{E}\left[h\Big(\mathcal{J}\left(\widehat{\mu}_{n_j}(4)\right)\Big)\right]\right|\ll\mathcal{N}_{n_j}^{-1/2}. \tag{2.15}$$

In order to deal with (1.21), we recall that the **1-Wassertein distance** between the distribution of two integrable random variables X,Y is defined as

$$d_W(X,Y):=\inf\mathbb{E}[|U-V|],$$

where the infimum is taken over all couplings (U, V) of X and Y; write $D(X, Y)$ to indicate the supremum of the quantity $|\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]|$, where h runs over the class of all twice differentiable function such that $\|h'\|_\infty, \|h''\|_\infty \leq 1$. According to [D-P, Lemma 1.4], one has that, whenever $D(X, Y) \leq 1$,

$$d_W(X, Y) \leq 2\sqrt{D(X, Y)},$$

so that relation (2.15) follows immediately from (1.20) and standard measure-theoretical results about the existence of infinite product measures – see e.g. [D, Theorem 8.2.2].

The rest of the paper contains the formal proofs of all the statements discussed in the present section.

3 Phase singularities and Wiener chaos

3.1 Chaotic expansions for phase singularities

In this part we find the chaotic expansion (2.6) for I_n. The first achievement in this direction is the following approximation result.

3.1.1 An integral expression for the number of zeros

For $\varepsilon > 0$ and $n \in S$, we consider the ε-approximating random variable

$$I_n(\varepsilon) := \frac{1}{4\varepsilon^2} \int_{\mathbb{T}} \mathbf{1}_{[-\varepsilon,\varepsilon]^2}(\mathbf{T}_n(x)) |J_{\mathbf{T}_n}(x)|\, dx, \tag{3.1}$$

where $\mathbf{1}_{[-\varepsilon,\varepsilon]^2}$ denotes the indicator function of the square $[-\varepsilon, \varepsilon]^2$. The following result makes the formal equality in (2.5) rigorous.

Lemma 3.1. *For $n \in S$, with probability one, I_n is composed of a finite number of isolated points and, as $\varepsilon \to 0$,*

$$I_n(\varepsilon) \to I_n, \tag{3.2}$$

both a.s. and in the $L^p(\mathbb{P})$-sense, for every $p \geq 1$.

Proof. Fix $n \in S$. In order to directly apply some statements taken from [A-W2], we will canonically identify the random field $(x_1, x_2) \mapsto \mathbf{T}_n(x_1, x_2)$ with a random mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ that is 1-periodic in each of the coordinates x_1, x_2. In what follows, for $x \in \mathbb{R}^2$ we will write $\mathbf{T}_n(x, \omega)$ to emphasize the dependence of $\mathbf{T}_n(x)$ on $\omega \in \Omega$. We subdivide the proof into several steps, numbered from (i) to (vi).

(i) First of all, since $\mathbf{T}_n$ is an infinitely differentiable stationary Gaussian field such that, for every $x \in \mathbb{R}^2$, the vector $\mathbf{T}_n(x)$ has a standard Gaussian distribution, one can directly apply [A-W2, Proposition 6.5] to infer that there exists a measurable set $\Omega_0 \subset \Omega$ with the following properties: $\mathbb{P}(\Omega_0) = 1$ and, for every $\omega \in \Omega_0$ and every $x \in \mathbb{R}^2$ such that $\mathbf{T}_n(x, \omega) = 0$, one has necessarily that the Jacobian matrix $J_{\mathbf{T}_n}(x, \omega)$ is invertible.

(ii) A standard application of the inverse function theorem (see e.g. [A-T, p. 136]) implies that, for every $\omega \in \Omega_0$, any bounded set $B \subset \mathbb{R}^2$ only contains a finite number of points x such that $\mathbf{T}_n(x, \omega) = 0$. This implies in particular that, with probability one, $\mathscr{I}_n$ (as defined in (1.3)) is composed of a finite number of isolated points and $I_n < +\infty$.

(iii) Sard's Lemma yields that, for every $\omega \in \Omega_0$, there exists a set $U_\omega \subset \mathbb{R}^2$ such that U_ω^c has Lebesgue measure 0 and, for every $u \in U_\omega$ there is no $x \in \mathbb{R}^2$ such that $\mathbf{T}_n(x, \omega) = u$ and $J_{\mathbf{T}_n}(x, \omega)$ is not invertible. Note that, by definition, one has that $0 \in U_\omega$ for every $\omega \in \Omega_0$.

(iv) Define $B := \{x = (x_1, x_2) \in \mathbb{R}^2 : 0 \le x_i < 1/L\},\ i = 1, 2\}$, where L is any positive integer such that $L > \sqrt{n}$. For every $u \in \mathbb{R}^2$, we set $I_{n,u}(B)$ to be the cardinality of the set composed of those $x \in B$ such that $\mathbf{T}_n(x) = u$; the quantity $I_{n,u}(\mathbb{T})$ is similarly defined, in such a way that $I_{n,0}(\mathbb{T}) = I_n$. Two facts will be crucial in order to conclude the proof: (a) for every $\omega \in \Omega_0$ and every $u = (u_1, u_2) \in U_\omega$, by virtue of Lemma 8.5 as applied to the pair (P, Q) given by

$$P(x) = T_n(x, \omega) - u_1 \quad \text{and} \quad Q(x) = \widehat{T}_n(x, \omega) - u_2,$$

as well as of the fact that $B \subset W$, one has that $I_{n,u}(B)(\omega) \le \alpha(n)$, and (b) as a consequence of the inverse function theorem, for every $\omega \in \Omega_0$ there exists $\eta_\omega \in (0, \infty)$ such that the equality $I_n(\omega) = I_{n,u}(\mathbb{T})(\omega)$ holds for every u such that $\|u\| \le \eta_\omega$. Indeed, reasoning as in [A-T, Proof of Theorem 11.2.3] if this was not the case, then there would exist a sequence $u_k \to 0$, $u_k \ne 0$, and a point $x \in \mathbb{T}$ such that: (1) $\mathbf{T}_n(x, \omega) = 0$, and (2) for every neighborhood V of x (in the topology of $\mathbb{T}$) there exist $k \ge 1$ and $x_0, x_1 \in V$ such that $x_0 \ne x_1$ and $\mathbf{T}_n(x_0) = \mathbf{T}_n(x_1) = u_k$ — which is in contradiction with the inverse function theorem.

(v) By the area formula (see e.g. [A-W2, Proposition 6.1 and formula (6.2)]), one has that, for every $\omega \in \Omega_0$,

$$\begin{aligned}\frac{1}{4\varepsilon^2} &\int_{\mathbb{T}} \mathbf{1}_{[-\varepsilon,\varepsilon]^2}(\mathbf{T}_n(x, \omega)) |J_{\mathbf{T}_n}(x, \omega)| dx \\ &= \frac{1}{4\varepsilon^2} \int_{[-\varepsilon,\varepsilon]^2} I_{n,u}(\mathbb{T})(\omega)\, du = \frac{1}{4\varepsilon^2} \int_{[-\varepsilon,\varepsilon]^2 \cap U_\omega} I_{n,u}(\mathbb{T})(\omega)\, du,\end{aligned} \tag{3.3}$$

where we used the property that the complement of U_ω has Lebesgue measure 0. Since the integral on the right-hand side of (3.3) equals I_n whenever $\varepsilon \le \eta_\omega / \sqrt{2}$, we conclude that (3.2) holds $\mathbb{P}$-a.s.

(vi) According to the discussion at Point (iv)-(a) above and using stationarity, one has that

$$\mathbb{P}[I_n \le L^2 \alpha(n)] = \mathbb{P}\left[\frac{1}{4\varepsilon^2} \int_{[-\varepsilon,\varepsilon]^2} I_{n,u}(\mathbb{T})\, du \le L^2 \alpha(n)\right] = 1.$$

The fact that (3.2) holds also in $L^p(\mathbb{P})$ now follows from Point (v) and dominated convergence. □

3.1.2 Chaotic expansions

Let us consider the collections of coefficients $\{\beta_l : l \ge 0\}$ and $\{\alpha_{a,b,c,d} : a, b, c, d \ge 0\}$ defined as follows. For $l \ge 0$

$$\beta_{2l+1} := 0, \qquad \beta_{2l} := \frac{1}{\sqrt{2\pi}} H_{2l}(0), \tag{3.4}$$

where (as before) H_{2l} is the $2l$-th Hermite polynomial. For instance,

$$\beta_0 = \frac{1}{\sqrt{2\pi}}, \qquad \beta_2 = -\frac{1}{\sqrt{2\pi}}, \qquad \beta_4 = \frac{3}{\sqrt{2\pi}}. \tag{3.5}$$

Also, we set

$$\alpha_{a,b,c,d} := \mathbb{E}[|XW - YV| H_a(X) H_b(Y) H_c(V) H_d(W)], \tag{3.6}$$

with (X, Y, V, W) a standard real four-dimensional Gaussian vector. Note that on the right-hand side of (3.6), $|XW - YV|$ is indeed the absolute value of the determinant of the matrix

$$\begin{pmatrix} X & Y \\ V & W \end{pmatrix}.$$

Lemma 3.2. *If a, b, c, d do not have the same parity, then*

$$\alpha_{a,b,c,d} = 0.$$

Proof. Let us assume without loss of generality (by symmetry) that a is odd and that at least one integer among b, c and d is even. We will exploit (2.1). If b is even, then, using that $(X, Y, V, W) \stackrel{\text{law}}{=} (-X, -Y, V, W)$, one can write that

$$\begin{aligned}
\alpha_{a,b,c,d} &= \mathbb{E}[|XW - YV|H_a(X)H_b(Y)H_c(V)H_d(W)] \\
&= \mathbb{E}[|YV - XW|H_a(-X)H_b(-Y)H_c(V)H_d(W)] \\
&= -\mathbb{E}[|XW - YV|H_a(X)H_b(Y)H_c(V)H_d(W)] = -\alpha_{a,b,c,d},
\end{aligned}$$

leading to $\alpha_{a,b,c,d} = 0$. If c (resp. d) is even, the same reasoning based on $(X, Y, V, W) \stackrel{\text{law}}{=} (-X, Y, -V, W)$ (resp. $(X, Y, V, W) \stackrel{\text{law}}{=} (-X, Y, V, -W)$) leads to the same conclusion. □

We will not need the explicit values of $\alpha_{a,b,c,d}$, unless $a + b + c + d \in \{0, 2, 4\}$. The following technical result will be proved in Section 7.

Lemma 3.3. *It holds that*

$$\begin{aligned}
&\alpha_{0,0,0,0} = 1, \\
&\alpha_{2,0,0,0} = \alpha_{0,2,0,0} = \alpha_{0,0,2,0} = \alpha_{0,0,0,2} = \frac{1}{2}, \\
&\alpha_{4,0,0,0} = \alpha_{0,4,0,0} = \alpha_{0,0,4,0} = \alpha_{0,0,0,4} = -\frac{3}{8}, \\
&\alpha_{2,2,0,0} = \alpha_{0,0,2,2} = \alpha_{2,2,0,0} = -\frac{1}{8}, \\
&\alpha_{2,0,2,0} = \alpha_{0,2,0,2} - \frac{1}{8}, \\
&\alpha_{2,0,0,2} = \alpha_{0,2,2,0} = \frac{5}{8}, \\
&\alpha_{1,1,1,1} = -\frac{3}{8}.
\end{aligned}$$

Lemma 3.4 (Chaotic expansion of I_n)**.** *For $n \in S$ and $q \geq 0$, we have*

$$I_n[2q+1] = 0, \tag{3.7}$$

and

$$\begin{aligned}
I_n[2q] =& \frac{E_n}{2} \sum_{i_1+i_2+i_3+j_1+j_2+j_3=2q} \frac{\beta_{i_1}\beta_{j_1}}{i_1!j_1!} \frac{\alpha_{i_2 i_3 j_2 j_3}}{i_2!i_3!j_2!j_3!} \times \\
&\times \int_{\mathbb{T}} H_{i_1}(T_n(x))H_{j_1}(\widehat{T}_n(x))H_{i_2}(\widetilde{\partial}_1 T_n(x))H_{i_3}(\widetilde{\partial}_2 T_n(x))H_{j_2}(\widetilde{\partial}_1 \widehat{T}_n(x))H_{j_3}(\widetilde{\partial}_2 \widehat{T}_n(x))\,dx,
\end{aligned} \tag{3.8}$$

where the sum can be restricted to the set of those indices $(i_1, j_1, i_2, i_3, j_2, j_3)$ such that i_1, j_1 are even and i_2, i_3, j_2, j_3 have the same parity. In particular,

$$I_n[2] = 0. \tag{3.9}$$

The chaotic expansion for I_n is hence

$$\begin{aligned}
I_n =& I_n[0] + \sum_{q\geq 2} \frac{E_n}{2} \sum_{i_1+i_2+i_3+j_1+j_2+j_3=2q} \frac{\beta_{i_1}\beta_{j_1}}{i_1!j_1!} \frac{\alpha_{i_2 i_3 j_2 j_3}}{i_2!i_3!j_2!j_3!} \times \\
&\times \int_{\mathbb{T}} H_{i_1}(T_n(x))H_{j_1}(\widehat{T}_n(x))H_{i_2}(\widetilde{\partial}_1 T_n(x))H_{i_3}(\widetilde{\partial}_2 T_n(x))H_{j_2}(\widetilde{\partial}_1 \widehat{T}_n(x))H_{j_3}(\widetilde{\partial}_2 \widehat{T}_n(x))\,dx,
\end{aligned} \tag{3.10}$$

where the sum runs over the set of those indices $(i_1, j_1, i_2, i_3, j_2, j_3)$ such that i_1, j_1 are even and i_2, i_3, j_2, j_3 have the same parity.

Proof. The main idea is to deduce the chaotic expansion for I_n from the chaotic expansion for (3.1) and Lemma 3.1. Let us first rewrite (3.1) as

$$I_n(\varepsilon) = \frac{E_n}{8\varepsilon^2} \int_{\mathbb{T}} 1_{[-\varepsilon,\varepsilon]^2}(\mathbf{T}_n(x)) \left| \widetilde{\partial}_1 T_n(x) \widetilde{\partial}_2 \widehat{T}_n(x) - \widetilde{\partial}_1 \widehat{T}_n(x) \widetilde{\partial}_2 T_n(x) \right| dx. \tag{3.11}$$

We recall the chaos decomposition of the indicator function (see e.g. [M-P-R-W, Lemma 3.4]):

$$\frac{1}{2\varepsilon} 1_{[-\varepsilon,\varepsilon]}(\cdot) = \sum_{l=0}^{+\infty} \frac{1}{l!} \beta_l^\varepsilon \, H_l(\cdot),$$

where, for $l \geq 0$

$$\beta_0^\varepsilon = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \phi(t)\, dt, \qquad \beta_{2l+1}^\varepsilon = 0, \qquad \beta_{2l+2}^\varepsilon = -\frac{1}{\varepsilon} \phi\left(\varepsilon\right) H_{2l+1}\left(\varepsilon\right), \tag{3.12}$$

and ϕ is still denoting the standard Gaussian density. For the indicator function of $[-\varepsilon, \varepsilon]^2$ appearing in (3.11), we thus have

$$\frac{1}{4\varepsilon^2} 1_{[-\varepsilon,\varepsilon]^2}(x, y) = \sum_{l=0}^{\infty} \sum_{q=0}^{l} \frac{\beta_{2q}^\varepsilon \beta_{2l-2q}^\varepsilon}{(2q)!(2l-2q)!} H_{2q}(x) H_{2l-2q}(y). \tag{3.13}$$

The chaotic expansion for the absolute value of the Jacobian determinant appearing in (3.11) is, thanks to Lemma 3.2,

$$\begin{aligned} &\left| \widetilde{\partial}_1 T_n(x) \widetilde{\partial}_2 \widehat{T}_n(x) - \widetilde{\partial}_1 \widehat{T}_n(x) \widetilde{\partial}_2 T_n(x) \right| \\ &= \sum_{q \geq 0} \sum_{\substack{a+b+c+d=2q \\ (a,b,c,d \text{ the same parity})}}^{\infty} \frac{\alpha_{a,b,c,d}}{a!b!c!d!} H_a(\widetilde{\partial}_1 T_n(x)) H_b(\widetilde{\partial}_2 T_n(x)) H_c(\widetilde{\partial}_1 \widehat{T}_n(x)) H_d(\widetilde{\partial}_2 \widehat{T}_n(x)), \end{aligned} \tag{3.14}$$

where $\alpha_{a,b,c,d}$ are given in (3.6). In particular, observe that Lemma 3.2 ensures that the odd chaoses vanish in the chaotic expansion for the Jacobian.

It hence follows from (3.13) and (3.14) that the chaotic expansion for $I_n(\varepsilon)$ in (3.11) is (taking sums over even i_1, j_1 and i_2, i_3, j_2, j_3 with the same parity)

$$\begin{aligned} I_n(\varepsilon) = &\frac{E_n}{2} \sum_{q \geq 0} \sum_{i_1+i_2+i_3+j_1+j_2+j_3=2q} \frac{\beta_{i_1}^\varepsilon \beta_{j_1}^\varepsilon}{i_1! j_1!} \frac{\alpha_{i_2 i_3 j_2 j_3}}{i_2! i_3! j_2! j_3!} \times \\ &\times \int_{\mathbb{T}} H_{i_1}(T_n(x)) H_{j_1}(\widehat{T}_n(x)) H_{i_2}(\widetilde{\partial}_1 T_n(x)) H_{i_3}(\widetilde{\partial}_2 T_n(x)) H_{j_2}(\widetilde{\partial}_1 \widehat{T}_n(x)) H_{j_3}(\widetilde{\partial}_2 \widehat{T}_n(x))\, dx. \end{aligned} \tag{3.15}$$

Noting that, as $\varepsilon \to 0$,

$$\beta_l^\varepsilon \to \beta_l,$$

where β_l are given in (3.4) and using Lemma 3.1, we prove both (3.7) and (3.8).

Let us now prove (3.9) that allows to conclude the proof. Equation (3.8) with $q = 1$ together with Equation (3.5) and Lemma 3.3, imply that the projection of I_n on the second Wiener chaos equals the quantity

$$\begin{aligned} I_n[2] \quad := \quad & 2\pi^2 n\beta_0\beta_2\alpha_{0,0,0,0} \int_{\mathbb{T}} H_2(T_n(x))dx + 2\pi^2 n\beta_2\beta_0\alpha_{0,0,0,0} \int_{\mathbb{T}} H_2(\widehat{T}_n(x))dx \\ & + 2\pi^2 n\beta_0^2\alpha_{2,0,0,0} \int_{\mathbb{T}} H_2(\widetilde{\partial}_1 T_n(x))dx + 2\pi^2 n\beta_0^2\alpha_{0,2,0,0} \int_{\mathbb{T}} H_2(\widetilde{\partial}_2 T_n(x))dx \\ & + 2\pi^2 n\beta_0^2\alpha_{0,0,2,0} \int_{\mathbb{T}} H_2(\widetilde{\partial}_1 \widehat{T}_n(x))dx + 2\pi^2 n\beta_0^2\alpha_{0,0,0,2} \int_{\mathbb{T}} + H_2(\widetilde{\partial}_2 \widehat{T}_n(x))dx \\ = \quad & \frac{\pi n}{2} \left\{ \int_{\mathbb{T}} \left[H_2(\widetilde{\partial}_1 T_n(x)) + H_2(\widetilde{\partial}_2 T_n(x)) + H_2(\widetilde{\partial}_1 \widehat{T}_n(x)) + H_2(\widetilde{\partial}_2 \widehat{T}_n(x)) \right] dx \right. \\ & \left. \qquad -2 \int_{\mathbb{T}} \left[H_2(T_n(x)) + H_2(\widehat{T}_n(x)) \right] dx \right\}. \end{aligned}$$

According to Green's first identity (see e.g. [L, p. 44]),

$$\int_{\mathbb{T}} \nabla v \cdot \nabla w \; dx = - \int_{\mathbb{T}} w \, \Delta v \; dx.$$

Using the facts that $H_2(t) = t^2 - 1$ and that T_n and $\widehat{T}_n$ are eigenfunctions of Δ, we eventually infer that

$$\begin{aligned} I_n[2] &= \frac{1}{4\pi} \int_{\mathbb{T}} \left[\|\nabla T_n(x)\|^2 + \|\nabla \widehat{T}_n(x)\|^2 \right] dx - n\pi \int_{\mathbb{T}} \left[T_n(x)^2 + \widehat{T}_n(x)^2 \right] dx \\ &= -\frac{1}{4\pi} \int_{\mathbb{T}} \left[T_n(x)\, \Delta T_n(x) + \widehat{T}_n(x) \Delta \widehat{T}_n(x) \right] dx - n\pi \int_{\mathbb{T}} \left[T_n(x)^2 + \widehat{T}_n(x)^2 \right] dx \\ &= n\pi \int_{\mathbb{T}} \left[T_n(x)^2 + \widehat{T}_n(x)^2 \right] dx - n\pi \int_{\mathbb{T}} \left[T_n(x)^2 + \widehat{T}_n(x)^2 \right] dx = 0. \end{aligned}$$ □

3.2 Proof of Part 1 of Theorem 1.2

According to Lemma 3.3 and Equation (3.5), for every $n \in S$ one has that

$$I_n[0] = \mathbb{E}[I_n] = 2\pi^2 n\, \beta_0^2\, \alpha_{0,0,0,0} = \pi n = \frac{E_n}{4\pi},$$

thus yielding the desired conclusion.

4 Investigation of the fourth chaotic components

In this section we shall investigate fourth chaotic components. In particular, we shall prove Proposition 2.2 and Proposition 2.4.

4.1 Preliminary results

For $n \in S$, from (3.8) with $q = 2$ we deduce that

$$\begin{aligned} I_n[4] = \frac{E_n}{2} & \sum_{i_1+i_2+i_3+j_1+j_2+j_3=4} \frac{\beta_{i_1}\beta_{j_1}}{i_1! j_1!} \frac{\alpha_{i_2 i_3 j_2 j_3}}{i_2! i_3! j_2! j_3!} \times \\ & \times \int_{\mathbb{T}} H_{i_1}(T_n(x)) H_{j_1}(\widehat{T}_n(x)) H_{i_2}(\widetilde{\partial}_1 T_n(x)) H_{i_3}(\widetilde{\partial}_2 T_n(x)) H_{j_2}(\widetilde{\partial}_1 \widehat{T}_n(x)) H_{j_3}(\widetilde{\partial}_2 \widehat{T}_n(x))\, dx, \end{aligned} \tag{4.1}$$

where the sum only considers integers i_1, j_1 even and i_2, i_3, j_2, j_3 with the same parity. In order to compute an expression for $I_n[4]$ that is more amenable to analysis, let us introduce, for $n \in S$, the following family of random variables:

$$\begin{aligned}
W(n) &= \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} (|a_\lambda|^2 - 1),\\
\widehat{W}(n) &= \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} (|\widehat{a}_\lambda|^2 - 1),\\
W_j(n) &= \frac{1}{n\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} \lambda_j^2(|a_\lambda|^2 - 1),\\
\widehat{W}_j(n) &= \frac{1}{n\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} \lambda_j^2(|\widehat{a}_\lambda|^2 - 1), \quad j = 1, 2,\\
W_{1,2}(n) &= \frac{1}{n\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} \lambda_1\lambda_2\, |a_\lambda|^2,\\
\widehat{W}_{1,2}(n) &= \frac{1}{n\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} \lambda_1\lambda_2\, |\widehat{a}_\lambda|^2,\\
M(n) &= \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} a_\lambda\overline{\widehat{a}_\lambda},\\
M_j(n) &= \frac{i}{\sqrt{n\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} \lambda_j a_\lambda\overline{\widehat{a}_\lambda}, \quad j = 1, 2,\\
M_{\ell,j}(n) &= \frac{1}{n\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} \lambda_\ell\lambda_j\; a_\lambda\overline{\widehat{a}_\lambda} \qquad j, \ell = 1, 2.
\end{aligned}$$

Note that

$$W_{1,2}(n) = \frac{1}{n\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} \lambda_1\lambda_2\,(|a_\lambda|^2 - 1), \quad \text{and} \quad \widehat{W}_{1,2}(n) = \frac{1}{n\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} \lambda_1\lambda_2\,(|\widehat{a}_\lambda|^2 - 1),$$

since $\sum_{\lambda\in\Lambda_n} \lambda_1\lambda_2 = 0$, and also that M_j is real-valued for $j = 1, 2$.

Now, let us express each summand appearing on the right-hand side of (4.1) in terms of $W(n)$, $W_1(n)$, $W_2(n)$, $W_{1,2}(n)$, $\widehat{W}(n)$, $\widehat{W}_1(n)$, $\widehat{W}_2(n)$, $\widehat{W}_{1,2}(n)$, $M(n)$, $M_1(n)$, $M_2(n)$, $M_{1,1}(n)$, $M_{2,2}(n)$ and/or $M_{1,2}(n)$. The proof of the following result will be given in Section 7. In what follows, the symbol $O_{\mathbb{P}}(\mathcal{N}_{n_j}^{-1/2})$ indicates a generic sequence of random variables $\{X_{n_j}\}$ (whose exact definition can vary from item to item) converging to zero in $L^2(\mathbb{P})$ (and therefore in probability) in such a way that, as $\mathcal{N}_{n_j} \to \infty$,

$$\mathbb{E}[X_{n_j}^2]^{1/2} \ll \mathcal{N}_{n_j}^{-1/2}.$$

Lemma 4.1. *Let $\{n_j\} \subset S$ be such that $\mathcal{N}_{n_j} \to +\infty$. Then*

(i) $\int_{\mathbb{T}} H_4(T_{n_j}(x))\,dx = \frac{3}{\mathcal{N}_{n_j}}\big(W(n_j)^2 - 2 + O_{\mathbb{P}}(\mathcal{N}_{n_j}^{-1/2})\big)$;

(ii) $\int_{\mathbb{T}} H_4(\widetilde{\partial}_k T_{n_j}(x))\,dx = \frac{3}{\mathcal{N}_{n_j}}\big(4W_k(n_j)^2 - 3 - \widehat{\mu}_{n_j}(4) + O_{\mathbb{P}}(\mathcal{N}_{n_j}^{-1/2})\big)$, $k = 1, 2$;

(iii) $\int_{\mathbb{T}} H_2(T_{n_j}(x))\big(H_2(\widetilde{\partial}_1 T_{n_j})(x)) + H_2(\widetilde{\partial}_2 T_{n_j})(x))\big)\,dx = \frac{2}{\mathcal{N}_{n_j}}\big(W(n_j)^2 - 2 + O_{\mathbb{P}}(\mathcal{N}_{n_j}^{-1/2})\big)$;

(iv) $\int_{\mathbb{T}} H_2(\widetilde{\partial}_1 T_{n_j}(x))H_2(\widetilde{\partial}_2 T_{n_j}(x))\,dx = \frac{1}{\mathcal{N}_{n_j}}\big(4W_1(n_j)W_2(n_j) + 8W_{1,2}(n_j)^2 - 3 + 3\widehat{\mu}_{n_j}(4) + O_{\mathbb{P}}(\mathcal{N}_{n_j}^{-1/2})\big)$;

(v) $\int_{\mathbb{T}} H_2(T_{n_j})(x))H_2(\widehat{T}_{n_j})(x))\,dx = \frac{1}{\mathcal{N}_{n_j}}\big(W(n_j)\widehat{W}(n_j) + 2M(n_j)^2 - 2 + O_{\mathbb{P}}(\mathcal{N}_{n_j}^{-1/2})\big)$;

(vi) $\int_{\mathbb{T}} H_2(T_{n_j})(x))\big(H_2(\widetilde{\partial}_1 \widehat{T}_{n_j}(x)) + H_2(\widetilde{\partial}_2 \widehat{T}_{n_j})(x))\big)\,dx = \frac{2}{\mathcal{N}_{n_j}}\big(W(n_j)\widehat{W}(n_j) + M_1(n_j)^2 + M_2(n_j)^2 - 1 + O_{\mathbb{P}}(\mathcal{N}_{n_j}^{-1/2})\big)$;

(vii) $\int_{\mathbb{T}} H_2(\widetilde{\partial}_\ell T_{n_j}(x))H_2(\widetilde{\partial}_k \widehat{T}_{n_j}(x))\,dx = \frac{1}{\mathcal{N}_{n_j}}\big(4W_\ell(n_j)\widehat{W}_j(n_j) + 8M_{\ell,k}(n_j)^2 - (3 + \widehat{\mu}_{n_j}(4))\mathbf{1}_{\{\ell=k\}} - (1-\widehat{\mu}_{n_j}(4))\mathbf{1}_{\{\ell\neq k\}} + O_{\mathbb{P}}(\mathcal{N}_{n_j}^{-1/2})\big)$, $\ell,k=1,2$;

(viii) $\int_{\mathbb{T}} \widetilde{\partial}_1 T_{n_j}(x)\widetilde{\partial}_2 T_{n_j}(x)\widetilde{\partial}_1 \widehat{T}_{n_j}(x)\widetilde{\partial}_2 \widehat{T}_{n_j}(x)\,dx = \frac{1}{\mathcal{N}_{n_j}}\big(4W_{1,2}(n_j)\widehat{W}_{1,2}(n_j) + 4M_{1,1}(n_j) \times M_{2,2}(n_j) + 4M_{1,2}(n_j)^2 - 1 + \widehat{\mu}_{n_j}(4) + O_{\mathbb{P}}(\mathcal{N}_{n_j}^{-1/2})\big)$.

We are now able to give an explicit expression for $I_n[4]$ in (4.1).

Lemma 4.2. *Let $\{n_j\} \subset S$ such that $\mathcal{N}_{n_j} \to +\infty$ and $\widehat{\mu}_{n_j}(4) \to \eta \in [-1,1]$, then*

$$\begin{aligned} I_{n_j}[4] = \frac{n_j\pi}{8\mathcal{N}_{n_j}}\Big(&\frac{1}{2}W(n_j)^2 + \frac{1}{2}\widehat{W}(n_j)^2 - 3W(n_j)\widehat{W}(n_j) - W_1(n_j)^2 - W_2(n_j)^2 - \widehat{W}_1(n_j)^2 \\ &- \widehat{W}_2(n_j)^2 + 6W_1(n_j)\widehat{W}_2(n_j) + 6\widehat{W}_1(n_j)W_2(n_j) - 2W_{1,2}(n_j)^2 - 2\widehat{W}_{1,2}(n_j)^2 \\ &- 12W_{1,2}(n_j)\widehat{W}_{1,2}(n_j) - 4M_1(n_j)^2 - 4M_2(n_j)^2 + 4M(n_j)^2 - 2M_{1,1}(n_j)^2 \\ &- 2M_{2,2}(n_j)^2 - 12M_{1,1}(n_j)M_{2,2}(n_j) + 8M_{1,2}(n_j)^2 + 4 + O_{\mathbb{P}}(\mathcal{N}_{n_j}^{-1/2})\Big). \end{aligned} \tag{4.2}$$

Proof. From Lemma 3.3 and (4.1), we find that

$$\begin{aligned} I_n[4] = \frac{n\pi}{64}\Big(&8\int_{\mathbb{T}} H_4(T_n(x))\,dx - 8\int_{\mathbb{T}} H_2(T_n(x))H_2(\widetilde{\partial}_1 T_n(x))\,dx \\ &-8\int_{\mathbb{T}} H_2(T_n(x))H_2(\widetilde{\partial}_2 T_n(x))\,dx - 2\int_{\mathbb{T}} H_2(\widetilde{\partial}_1 T_n(x))H_2(\widetilde{\partial}_2 T_n(x))\,dx \\ &-\int_{\mathbb{T}} H_4(\widetilde{\partial}_1 T_n(x))\,dx - \int_{\mathbb{T}} H_4(\widetilde{\partial}_2 T_n(x))\,dx \\ &+8\int_{\mathbb{T}} H_4(\widehat{T}_n(x))\,dx - 8\int_{\mathbb{T}} H_2(\widehat{T}_n(x))H_2(\widetilde{\partial}_1 \widehat{T}_n(x))\,dx \\ -8\int_{\mathbb{T}} &H_2(\widehat{T}_n(x))H_2(\widetilde{\partial}_2 \widehat{T}_n(x))\,dx - 2\int_{\mathbb{T}} H_2(\widetilde{\partial}_1 \widehat{T}_n(x))H_2(\widetilde{\partial}_2 \widehat{T}_n(x))\,dx \\ -\int_{\mathbb{T}} &H_4(\widetilde{\partial}_1 \widehat{T}_n(x))\,dx - \int_{\mathbb{T}} H_4(\widetilde{\partial}_2 \widehat{T}_n(x))\,dx \\ +16\int_{\mathbb{T}} &H_2(T_n(x))H_2(\widehat{T}_n(x))\,dx - 8\int_{\mathbb{T}} H_2(T_n(x))\big(H_2(\widetilde{\partial}_1 \widehat{T}_n(x)) + H_2(\widetilde{\partial}_2 \widehat{T}_n(x))\big)\,dx \\ -8\int_{\mathbb{T}} &H_2(\widehat{T}_n(x))\big(H_2(\widetilde{\partial}_1 T_n(x)) + H_2(\widetilde{\partial}_2 T_n(x))\big)dx \\ -2\int_{\mathbb{T}} &H_2(\widetilde{\partial}_1 T_n(x))H_2(\widetilde{\partial}_1 \widehat{T}_n(x))\,dx - 2\int_{\mathbb{T}} H_2(\widetilde{\partial}_2 T_n(x))H_2(\widetilde{\partial}_2 \widehat{T}_n(x))\,dx \\ +10\int_{\mathbb{T}} &H_2(\widetilde{\partial}_1 T_n(x))H_2(\widetilde{\partial}_2 \widehat{T}_n(x))\,dx + 10\int_{\mathbb{T}} H_2(\widetilde{\partial}_2 T_n(x))H_2(\widetilde{\partial}_1 \widehat{T}_n(x))\,dx \\ -24\int_{\mathbb{T}} &\widetilde{\partial}_1 T_n(x)\widetilde{\partial}_2 T_n(x)\widetilde{\partial}_1 \widehat{T}_n(x)\widetilde{\partial}_2 \widehat{T}_n(x)\,dx\Big). \end{aligned} \tag{4.3}$$

Using the previous identities (i)-(viii) in Lemma 4.1 in (4.3), and also using that $W_1(n_j) + W_2(n_j) = W(n_j)$ and $\widehat{W}_1(n_j) + \widehat{W}_2(n_j) = \widehat{W}(n_j)$, one concludes the proof. □

4.2 Proofs of Proposition 2.2 and Proposition 2.4

Let us first study the asymptotic distribution of the centered random vector, defined for $n \in S$ as follows

$$\begin{aligned} \mathbf{W}(n) := (&W(n), W_1(n), W_2(n), W_{1,2}(n), \widehat{W}(n), \widehat{W}_1(n), \widehat{W}_2(n), \widehat{W}_{1,2}(n), \\ &M(n), M_1(n), M_2(n), M_{1,1}(n), M_{2,2}(n), M_{1,2}(n)) \in \mathbb{R}^{14}. \end{aligned}$$

Lemma 4.3. *Let $\{n_j\} \subset S$ be such that $\mathcal{N}_{n_j} \to +\infty$ and $\widehat{\mu}_{n_j}(4) \to \eta \in [-1,1]$. Then, as $\mathcal{N}_{n_j} \to \infty$,*

$$\mathbf{W}(n_j) \stackrel{\text{law}}{\Longrightarrow} \mathbf{G},$$

where $\mathbf{G} = (G_1, \ldots, G_{14})$ denotes a Gaussian real centered vector with covariance matrix given by

$$\mathbf{M}(\eta) = \begin{pmatrix} \mathbf{A}(\eta) & 0 & 0 \\ 0 & \mathbf{A}(\eta) & 0 \\ 0 & 0 & \mathbf{B}(\eta) \end{pmatrix}, \tag{4.4}$$

where

$$\mathbf{A}(\eta) := \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & \frac{3+\eta}{4} & \frac{1-\eta}{4} & 0 \\ 1 & \frac{1-\eta}{4} & \frac{3+\eta}{4} & 0 \\ 0 & 0 & 0 & \frac{1-\eta}{4} \end{pmatrix},$$

and

$$\mathbf{B}(\eta) := \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{3+\eta}{8} & \frac{1-\eta}{8} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1-\eta}{8} & \frac{3+\eta}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-\eta}{8} \end{pmatrix}.$$

Proof. First, for reasons related to independence it is easy to check that the covariance matrix of $\mathbf{W}(n)$ takes the form

$$\Sigma_n = \begin{pmatrix} \mathbf{A}_n & 0 & 0 \\ 0 & \mathbf{A}_n & 0 \\ 0 & 0 & \mathbf{B}_n \end{pmatrix}, \tag{4.5}$$

where $\mathbf{A}_n$ and $\mathbf{B}_n$ denote the covariance matrices of $(W(n), W_1(n), W_2(n), W_{1,2}(n))$ and $(M(n), M_1(n), M_2(n), M_{1,1}(n), M_{2,2}(n), M_{1,2}(n))$ respectively. Let us first compute $\mathbf{A}_n$. Since $\mathbb{E}[(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)] = 1$ if $\lambda = \pm\lambda'$ and is zero otherwise, one has

$$\mathbb{E}(W(n)^2) = \frac{1}{\mathcal{N}_n} \sum_{\lambda,\lambda' \in \Lambda_n} \mathbb{E}[(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)] = 2.$$

Similarly,

$$\mathbb{E}(W(n)W_j(n)) = \frac{1}{n\mathcal{N}_n} \sum_{\lambda,\lambda' \in \Lambda_n} \lambda_j^2 \mathbb{E}[(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)] = \frac{2}{n\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_j^2 = 1,$$

whereas

$$\mathbb{E}(W(n)W_{1,2}(n)) = \frac{1}{n\mathcal{N}_n} \sum_{\lambda,\lambda' \in \Lambda_n} \lambda_1\lambda_2 \mathbb{E}[(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)] = \frac{2}{n\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_1\lambda_2 = 0.$$

We also have

$$\mathbb{E}(W_j(n)^2) = \frac{2}{n^2\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_j^4.$$

To express $\mathbb{E}(W_j(n)^2)$ in a more suitable way, let us rely on $\widehat{\mu}_n(4)$:

$$\begin{aligned} \widehat{\mu}_n(4) &= \int_{\mathcal{S}^1} z^4 d\mu_n(z) = \frac{1}{n^2\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} (\lambda_1 + i\lambda_2)^4 = \frac{1}{n^2\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} (\lambda_1^4 - 6\lambda_1^2\lambda_2^2 + \lambda_2^4) \\ &= \frac{1}{n^2\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} (\lambda_1^2+\lambda_2^2)^2 - \frac{8}{n^2\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_1^2\lambda_2^2 = 1 - \frac{8}{n^2\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_1^2\lambda_2^2. \end{aligned}$$

As a result,

$$\sum_{\lambda\in\Lambda_n}\lambda_1^2\lambda_2^2=\frac{n^2\mathcal{N}_n}{8}(1-\widehat{\mu}_n(4)),$$

leading to

$$\mathbb{E}(W_j(n)^2)=\frac{1}{n^2\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}(\lambda_1^4+\lambda_2^4)=\widehat{\mu}_n(4)+\frac{6}{n^2\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\lambda_1^2\lambda_2^2=\frac{1}{4}(3+\widehat{\mu}_n(4)).$$

Similarly,

$$\mathbb{E}(W_{1,2}(n)^2)=\frac{1}{4}(1-\widehat{\mu}_n(4)),$$

as well as

$$\mathbb{E}(W_1(n)W_2(n))=\frac{1}{4}(1-\widehat{\mu}_n(4)),$$

and

$$\mathbb{E}(W_j(n)W_{1,2}(n))=0.$$

Taking all these facts into consideration, we deduce that

$$\mathbf{A}_n=\begin{pmatrix}2&1&1&0\\1&\frac{3+\widehat{\mu}_n(4)}{4}&\frac{1-\widehat{\mu}_n(4)}{4}&0\\1&\frac{1-\widehat{\mu}_n(4)}{4}&\frac{3+\widehat{\mu}_n(4)}{4}&0\\0&0&0&\frac{1-\widehat{\mu}_n(4)}{4}\end{pmatrix}.$$

Now, let us turn to the expression of $\mathbf{B}_n$. Using that $\mathbb{E}[a_\lambda a_{\lambda'}]=1$ if $\lambda'=-\lambda$ and is zero otherwise, we obtain

$$\mathbb{E}(M(n)^2)=\frac{1}{\mathcal{N}_n}\sum_{\lambda,\lambda'\in\Lambda_n}\mathbb{E}[a_\lambda a_{\lambda'}]\mathbb{E}[\overline{\widehat{a}_\lambda\widehat{a}_{\lambda'}}]=1.$$

Similarly,

$$\mathbb{E}(M_j(n)^2)=-\frac{1}{n\mathcal{N}_n}\sum_{\lambda,\lambda'\in\Lambda_n}\lambda_j\lambda_j'\mathbb{E}[a_\lambda a_{\lambda'}]\mathbb{E}[\overline{\widehat{a}_\lambda\widehat{a}_{\lambda'}}]=\frac{1}{n\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\lambda_j^2=\frac{1}{2},$$

as well as

$$\mathbb{E}(M_{j,j}(n)^2)=\frac{1}{8}(3+\widehat{\mu}_n(4)),$$

and

$$\mathbb{E}(M_{1,2}(n)^2)=\frac{1}{8}(1-\widehat{\mu}_n(4)).$$

Besides, it is immediate to check that, for any l,j,

$$\mathbb{E}(M(n)M_j(n))=\mathbb{E}(M(n)M_{12}(n))=\mathbb{E}(M_j(n)M_{l,j}(n))=\mathbb{E}(M_{j,j}(n)M_{1,2}(n))=0.$$

Finally,

$$\mathbb{E}(M(n)M_{j,j}(n))=\frac{1}{2},$$

whereas

$$\mathbb{E}(M_{1,1}(n)M_{2,2}(n))=\frac{1}{8}(1-\widehat{\mu}_n(4)).$$

Putting everything together, we arrive at the following expression for $\mathbf{B}_n$

$$\mathbf{B}_n = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{3+\widehat{\mu}_n(4)}{8} & \frac{1-\widehat{\mu}_n(4)}{8} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1-\widehat{\mu}_n(4)}{8} & \frac{3+\widehat{\mu}_n(4)}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-\widehat{\mu}_n(4)}{8} \end{pmatrix}.$$

Now, let us prove that each component of $\mathbf{W}_{n_j}$ is asymptotically Gaussian as $\mathcal{N}_{n_j} \to +\infty$. Since all components of $\mathbf{W}_{n_j}$ belong to the same Wiener chaos (the second one) and have a converging variance (see indeed the diagonal part of $\mathbf{B}_n$ just above), according to the Fourth Moment Theorem (see, e.g., [N-P, Theorem 5.2.7]) it suffices to show that the fourth cumulant of each component of $\mathbf{W}_{n_j}$ goes to zero as $\mathcal{N}_{n_j} \to +\infty$. Since we are dealing with sum of independent random variables, checking such a property is straightforward. For sake of illustration, let us only consider the case of $W_2(n_j)$ which is representative of the difficulty. We recall that, given a real-valued random variable Z with mean zero, the fourth cumulant of Z is defined by $\kappa_4(Z) := \mathbb{E}[Z^4] - 3\mathbb{E}[Z^2]$. Since the a_λ are independent except for the relation $\overline{a_\lambda} = a_{-\lambda}$, we can write, setting $\Lambda_n^+ = \{\lambda \in \Lambda_n : \lambda_2 > 0\}$,

$$\begin{aligned} \kappa_4(W_2(n)) &= \kappa_4\left(\frac{2}{n\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n^+} \lambda_2^2(|a_\lambda|^2 - 1)\right) = \frac{16\,\kappa_4(|N_{\mathbb{C}}(0,1)|^2)}{n^4\mathcal{N}_n^2} \sum_{\lambda\in\Lambda_n^+} \lambda_2^8 \\ &\le \frac{8\,\kappa_4(|N_{\mathbb{C}}(0,1)|^2)}{\mathcal{N}_n}; \end{aligned} \tag{4.6}$$

to obtain the last inequality, we have used that $\lambda_2^2 \le \lambda_1^2 + \lambda_2^2 = n$. As a result, $\kappa_4(W_2(n_j)) \to 0$ as $\mathcal{N}_{n_j} \to +\infty$ and it follows from the Fourth Moment Theorem that $W_2(n_j)$ is asymptotically Gaussian. It is not difficult to apply a similar strategy in order to prove that, actually, each component of $\mathbf{W}_{n_j}$ is asymptotically Gaussian, with a fourth cumulant converging to zero at a rate $O(\mathcal{N}_n^{-1})$; standard details are left to the reader.

Finally, we make use of [N-P, Theorem 6.2.3] to conclude the proof of Lemma 4.3. Indeed, (i) all components of $\mathbf{W}_n$ belong to the same Wiener chaos (the second one), (ii) each component of $\mathbf{W}_{n_j}$ is asymptotically Gaussian (as $\mathcal{N}_{n_j} \to +\infty$), and finally (iii) $\Sigma_{k,l}(n_j) \to \mathbf{M}_{k,l}(\eta)$ for each pair of indices (k,l). □

Proofs of Proposition 2.2 and Proposition 2.4. For each subsequence $\{n_j'\} \subset \{n_j\}$, there exists a subsubsequence $\{n_j''\} \subset \{n_j'\}$ such that it holds either (i) $\widehat{\mu}_{n_j''}(4) \to \eta$ or (ii) $\widehat{\mu}_{n_j''}(4) \to -\eta$.

Combining Lemma 4.2 with Lemma 4.3, we have, as $j \to +\infty$,

$$\begin{aligned} \frac{8\mathcal{N}_{n_j''}}{n_j''\pi} I_{n_j''}[4] \Rightarrow & \frac{1}{2}G_1^2 + \frac{1}{2}G_5^2 - 3G_1G_5 - G_2^2 - G_3^2 - G_6^2 - G_7^2 + 6G_2G_7 + 6G_6G_3 - 2G_4^2 - 2G_8^2 \\ & - 12G_4G_8 - 4G_{10}^2 - 4G_{11}^2 + 4G_9^2 - 2G_{12}^2 - 2G_{13}^2 + 8G_{14}^2 - 12G_{12}G_{13}, \end{aligned} \tag{4.7}$$

where $(G_1, \ldots, G_{14})$ denotes a Gaussian centered vector with covariance matrix (4.4).

Since $\left\{\frac{8N_{n_j''}}{n_j''\pi} I_{n_j''}[4]\right\}$ is a sequence of random variables belonging to a fixed Wiener chaos and converging in distribution, by standard arguments based on uniform integrability, we also have

$$\mathrm{Var}\left(\frac{8N_{n_j''}}{n_j''\pi} I_{n_j''}[4]\right) \to \mathrm{Var}\Big(\frac{1}{2}G_1^2 + \frac{1}{2}G_5^2 - 3G_1G_5 - G_2^2 - G_3^2 - G_6^2 - G_7^2 + 6G_2G_7 + 6G_6G_3 - 2G_4^2 - 2G_8^2 - 12G_4G_8 - 4G_{10}^2 - 4G_{11}^2 + 4G_9^2 - 2G_{12}^2 - 2G_{13}^2 + 8G_{14}^2 - 12G_{12}G_{13}\Big);$$

the proof of Proposition 2.2 is then concluded, once computing

$$\mathrm{Var}\Big(\frac{1}{2}G_1^2 + \frac{1}{2}G_5^2 - 3G_1G_5 - G_2^2 - G_3^2 - G_6^2 - G_7^2 + 6G_2G_7 + 6G_6G_3 - 2G_4^2 - 2G_8^2 - 12G_4G_8 - 4G_{10}^2 - 4G_{11}^2 + 4G_9^2 - 2G_{12}^2 - 2G_{13}^2 + 8G_{14}^2 - 12G_{12}G_{13}\Big) = 8(3\eta^2+5),$$

and noting that the latter variance is the same in both cases (i) and (ii).

Let us now prove Proposition 2.4. Let $(Z_1,\dots,Z_{11}) \sim N_{11}(0,I)$ be a standard Gaussian vector of $\mathbb{R}^{11}$. Then one can check that the vector

$$\begin{pmatrix} \sqrt{2}\,Z_5 \\ \frac{1}{\sqrt{2}}Z_5 + \frac{1}{2}\sqrt{\eta+1}\,Z_3 \\ \frac{1}{\sqrt{2}}Z_5 - \frac{1}{2}\sqrt{\eta+1}\,Z_3 \\ \frac{1}{2}\sqrt{1-\eta}\,Z_8 \\ \sqrt{2}\,Z_6 \\ \frac{1}{\sqrt{2}}Z_6 + \frac{1}{2}\sqrt{\eta+1}\,Z_4 \\ \frac{1}{\sqrt{2}}Z_6 - \frac{1}{2}\sqrt{\eta+1}\,Z_4 \\ \frac{1}{2}\sqrt{1-\eta}\,Z_9 \\ Z_2 \\ \frac{1}{\sqrt{2}}Z_{10} \\ \frac{1}{\sqrt{2}}Z_{11} \\ \frac{1}{2}Z_2 + \sqrt{\frac{1}{8}(\eta+1)}\,Z_1 \\ \frac{1}{2}Z_2 - \sqrt{\frac{1}{8}(\eta+1)}\,Z_1 \\ \sqrt{\frac{1}{8}(1-\eta)}\,Z_7 \end{pmatrix}$$

admits $\mathbf{M}(\eta)$ for covariance matrix as well. Expressing (4.7) in terms of $(Z_1,\dots,Z_{11})$ leads to the fact that (4.7) has the same law as the random variable

$$\frac{1+\eta}{2}A + \frac{1-\eta}{2}B - 2(C-2),$$

with A,B,C independent and $A \overset{\text{law}}{=} B \overset{\text{law}}{=} 2Z_1^2 - Z_2^2 - Z_3^2 - 6Z_2Z_3$ and $C \overset{\text{law}}{=} Z_1^2 + Z_2^2$.

Finally, noting that the law of the random variable $\frac{1+\eta}{2}A + \frac{1-\eta}{2}B - 2(C-2)$ is the same for case (i) and case (ii) and using that $(Z_1,Z_2,Z_3) \overset{\text{law}}{=} (Z_1, \frac{1}{\sqrt{2}}(Z_2-Z_3), \frac{1}{\sqrt{2}}(Z_2+Z_3))$, we get the desired conclusion. □

5 The variance of higher order chaoses

In this section we shall prove Proposition 2.3. Let us decompose the torus $\mathbb{T}$ as a disjoint union of squares Q_k of side length $1/M$, where

$$M = \lceil d\sqrt{E_n}\rceil \quad (d \in \mathbb{R}_{>0} \text{ to be chosen later}), \tag{5.1}$$

obtained by translating along directions k/M, $k \in \mathbb{Z}^2$, the square $Q_0 := [0,1/M)\times[0,1/M)$ containing the origin. By construction, the south-west corner of each square is therefore situated at the point k/M.

5.1 Singular points and cubes

Let us first give some definitions, inspired by [O-R-W, §6.1] and [R-W2, §4.3]. Let us denote by $0 < \varepsilon_1 < \frac{1}{10^{10}}$ a very small number that will be fixed until the end. Let us now choose d in (5.1) such that $d \geq \frac{16\pi^2}{\varepsilon_1}$.

From now on, we shall use the simpler notation $r_j := \partial_j r_n$, and $r_{ij} := \partial_{ij} r_n$ for $i, j = 1, 2$.

Definition 5.1 (Singular pairs of points and cubes)**.**

i) A pair of points $(x, y) \in \mathbb{T} \times \mathbb{T}$ is called singular if either $|r(x - y)| > \varepsilon_1$ or $|r_1(x - y)| > \varepsilon_1 \sqrt{n}$ or $|r_2(x - y)| > \varepsilon_1 \sqrt{n}$ or $|r_{12}(x - y)| > \varepsilon_1 n$ or $|r_{11}(x - y)| > \varepsilon_1 n$ or $|r_{22}(x - y)| > \varepsilon_1 n$.

ii) A pair of cubes (Q, Q') is called singular if the product $Q \times Q'$ contains a singular pair of points.

For instance, $(0, 0)$ is a singular pair of points and hence (Q_0, Q_0) is a singular pair of cubes. In what follows we will often drop the dependence of k from Q_k.

Lemma 5.2. *Let (Q, Q') be a singular pair of cubes, then for every $(z, w) \in Q \times Q'$ either $|r(z - w)| > \frac{1}{2}\varepsilon_1$ or $|r_1(z - w))| > \frac{1}{2} \varepsilon_1 \sqrt{n}$ or $|r_2(z - w))| > \frac{1}{2}\varepsilon_1 \sqrt{n}$ or $|r_{12}(z - w))| > \frac{1}{2}\varepsilon_1 n$ or $|r_{11}(z - w))| > \frac{1}{2}\varepsilon_1 n$ or $|r_{22}(z - w))| > \frac{1}{2}\varepsilon_1 n$.*

Proof. First note that the function $\mathbb{T} \ni s \mapsto r(s/\sqrt{n})$ and its derivatives up to the order two are Lipschitz with a universal Lipschitz constant $c = 8\pi^3$ (in particular, independent of n). Let us denote by (x, y) the singular pair of points contained in $Q \times Q'$ and suppose that $r(x - y) > \varepsilon_1$. For every $(z, w) \in Q \times Q'$,

$$|r(z - w) - r(x - y)| = \left| r\left(\frac{(z - w) \cdot \sqrt{n}}{\sqrt{n}}\right) - r\left(\frac{(x - y) \cdot \sqrt{n}}{\sqrt{n}}\right) \right|$$
$$\leq c\sqrt{n}|(z - x) - (w - y)| \leq 2c\sqrt{n}\frac{1}{M}.$$

Since $d \geq \frac{16\pi^2}{\varepsilon_1}$ in $M = \lceil d\sqrt{E_n} \rceil$, then

$$r(z - w) \geq r(x - y) - \varepsilon_1/2 > \varepsilon_1/2.$$

The case $r(x - y) < -\varepsilon_1$ in indeed analogous. The rest of the proof for derivatives follows the same argument. $\square$

Let us now denote by B_Q the union of all squares Q' such that (Q, Q') is a singular pair. The number of such cubes Q' is $M^2 \mathrm{Leb}(B_Q)$, the area of each cube being $1/M^2$.

Lemma 5.3. *It holds that Leb$(B_Q) \ll \int_{\mathbb{T}} r(x)^6 \, dx$.*

Proof. Let us first note that

$$B_Q \subset B_Q^0 \cup B_Q^1 \cup B_Q^2 \cup B_Q^{12} \cup B_Q^{11} \cup B_Q^{22},$$

where B_Q^0 is the union of all cubes Q' such that there exists $(x, y) \in Q \times Q'$ enjoying $|r(x - y)| > \frac{1}{2} \varepsilon_1$ and for $i, j = 1, 2$, B_Q^i is the union of all cubes Q' such that there exists $(x, y) \in Q \times Q'$ enjoying $|r_i(x - y)| > \frac{1}{2} \varepsilon_1 \sqrt{n}$ and finally B_Q^{ij} is the union of all cubes Q' such that there exists $(x, y) \in Q \times Q'$ enjoying $|r_{ij}(x - y)| > \frac{1}{2} \varepsilon_1 n$. We can hence write

$$\mathrm{Leb}(B_Q) \leq \mathrm{Leb}(B_Q^0) + \mathrm{Leb}(B_Q^1) + \mathrm{Leb}(B_Q^2) + \mathrm{Leb}(B_Q^{12}) + \mathrm{Leb}(B_Q^{11}) + \mathrm{Leb}(B_Q^{22}).$$

Let us now fix $z \in Q$; then Lemma 5.2 yields

$$\mathrm{Leb}(B_Q^0) = \int_{B_Q^0} \frac{|r(z - w)|^6}{|r(z - w)|^6} \, dw \leq \left(\frac{\varepsilon_1}{2}\right)^{-6} \int_{B_Q^0} |r(z - w)|^6 \, dw \leq \left(\frac{\varepsilon_1}{2}\right)^{-6} \int_{\mathbb{T}} |r(x)|^6 \, dx.$$

Moreover, for $i = 1, 2$,

$$\mathrm{Leb}(B_Q^i) = \int_{B_Q^i} \frac{|\widetilde{r}_i(z-w)|^6}{|\widetilde{r}_i(z-w)|^6}\, dw \le \left(\frac{\varepsilon_1}{2}\right)^{-6} \int_{B_Q^i} |\widetilde{r}_i(z-w)|^6\, dw \le \left(\frac{\varepsilon_1}{2}\right)^{-6} \int_{\mathbb{T}} |\widetilde{r}_i(x)|^6\, dx,$$

where $\widetilde{r}_i := r_i/\sqrt{n}$ are the normalized derivatives. Since

$$\begin{aligned}\int_{\mathbb{T}} \widetilde{r}_i(x)^6\, dx &= \frac{1}{\mathcal{N}_n^6} \sum_{\lambda, \lambda', \dots, \lambda^v} \frac{\lambda_i}{\sqrt{n}} \frac{\lambda_i'}{\sqrt{n}} \cdots \frac{\lambda_i^v}{\sqrt{n}} \int_{\mathbb{T}} \mathrm{e}^{i2\pi\langle \lambda - \lambda' + \dots - \lambda^v, x\rangle}\, dx \\ &= \frac{1}{\mathcal{N}_n^6} \sum_{\lambda - \lambda' + \dots - \lambda^v = 0} \frac{\lambda_i}{\sqrt{n}} \frac{\lambda_i'}{\sqrt{n}} \cdots \frac{\lambda_i^v}{\sqrt{n}} \\ &\le \frac{|S_6(n)|}{\mathcal{N}_n^6} = \int_{\mathbb{T}} r(x)^6\, dx,\end{aligned}$$

we have

$$\mathrm{Leb}(B_Q^i) \ll \int_{\mathbb{T}} r(x)^6\, dx.$$

An analogous argument applied to B_Q^{ij} for $i, j = 1, 2$ allows to conclude the proof. □

The number of cubes Q' such that the pair (Q, Q') is singular is hence negligible with respect to $E_n\, R_n(6)$.

5.2 Variance and cubes

We write the total number I_n of nodal intersections as the sum of the number $I_{n_{|_Q}}$ of nodal intersections restricted to each square Q, i.e.

$$I_n = \sum_Q I_{n_{|_Q}}.$$

We have

$$\mathrm{proj}\,(I_n | C_{\ge 6}) = \sum_Q \mathrm{proj}\left(I_{n_{|_Q}} | C_{\ge 6}\right),$$

so that

$$\mathrm{Var}\,(\mathrm{proj}(I_n | C_{\ge 6})) = \sum_{Q, Q'} \mathrm{Cov}\,\left(\mathrm{proj}\left(I_{n_{|_Q}} | C_{\ge 6}\right), \mathrm{proj}\left(I_{n_{|_{Q'}}} | C_{\ge 6}\right)\right).$$

We are going to separately investigate the contribution of the singular pairs and the non-singular pairs of cubes:

$$\begin{aligned}\mathrm{Var}(\mathrm{proj}(I_n | C_{\ge 6})) = &\sum_{(Q,Q')\text{ sing.}} \mathrm{Cov}\,(\mathrm{proj}(I_{n_{|_Q}} | C_{\ge 6}), \mathrm{proj}(I_{n_{|_{Q'}}} | C_{\ge 6})) \\ &+ \sum_{(Q,Q')\text{ non sing.}} \mathrm{Cov}\,(\mathrm{proj}(I_{n_{|_Q}} | C_{\ge 6}), \mathrm{proj}(I_{n_{|_{Q'}}} | C_{\ge 6})).\end{aligned}$$

5.2.1 The contribution of singular pairs of cubes

Proof of Lemma 2.5. By Cauchy-Schwarz inequality and the stationarity of $\mathbf{T}_n$, recalling moreover Lemma 5.3, we have

$$\begin{aligned}
&\left| \sum_{(Q,Q')\text{ sing.}} \operatorname{Cov}\left(\operatorname{proj}\left(I_{n_{|Q}}|C_{\geq 6}\right), \operatorname{proj}\left(I_{n_{|Q'}}|C_{\geq 6}\right)\right)\right| \\
&\leq \sum_{(Q,Q')\text{ sing.}} \left|\operatorname{Cov}\left(\operatorname{proj}\left(I_{n_{|Q}}|C_{\geq 6}\right), \operatorname{proj}\left(I_{n_{|Q'}}|C_{\geq 6}\right)\right)\right| \\
&\leq \sum_{(Q,Q')\text{ sing.}} \sqrt{\operatorname{Var}\left(\operatorname{proj}\left(I_{n_{|Q}}|C_{\geq 6}\right)\right)\operatorname{Var}\left(\operatorname{proj}\left(I_{n_{|Q'}}|C_{\geq 6}\right)\right)} \\
&\ll E_n^2 R_n(6) \operatorname{Var}\left(\operatorname{proj}\left(I_{n_{|Q_0}}|C_{\geq 6}\right)\right),
\end{aligned}$$

where, from now on, Q_0 denotes the square containing the origin. Now,

$$\operatorname{Var}\left(\operatorname{proj}\left(I_{n_{|Q_0}}|C_{\geq 6}\right)\right) \leq \mathbb{E}\left[I^2_{n_{|Q_0}}\right] = \underbrace{\mathbb{E}\left[I^2_{n_{|Q_0}}\right] - \mathbb{E}\left[I_{n_{|Q_0}}\right]}_{=:A} + \mathbb{E}\left[I_{n_{|Q_0}}\right].$$

It is immediate to check that

$$\mathbb{E}\left[I_{n_{|Q_0}}\right] = \frac{2\pi n}{M^2},$$

in particular $\mathbb{E}\left[I_{n_{|Q_0}}\right] = O(1)$. Note that A is the 2-th factorial moment of $I_{n_{|Q_0}}$:

$$A = \mathbb{E}\left[I_{n_{|Q_0}}\left(I_{n_{|Q_0}} - 1\right)\right].$$

Applying [A-W2, Theorem 6.3], we can write

$$A = \mathbb{E}\left[I_{n_{|Q_0}}\left(I_{n_{|Q_0}} - 1\right)\right] = \int_{Q_0}\int_{Q_0} K_2(x,y)\,dxdy, \tag{5.2}$$

where

$$K_2(x,y) := p_{(\mathbf{T}_n(x),\mathbf{T}_n(y))}(0,0)\,\mathbb{E}\left[|J_{\mathbf{T}_n}(x)|\cdot|J_{\mathbf{T}_n}(y)|\,\Big|\,\mathbf{T}_n(x) = \mathbf{T}_n(y) = 0\right]$$

is the so-called 2-point correlation function. Indeed, Proposition 8.4 ensures that for $(x,y) \in Q_0 \times Q_0$, the vector $(\mathbf{T}_n(x), \mathbf{T}_n(y))$ is non-degenerate except on the diagonal $x = y$.

Note that, by stationarity of the model, we can write (5.2) as

$$\mathbb{E}\left[I_{n_{|Q_0}}\left(I_{n_{|Q_0}} - 1\right)\right] = \operatorname{Leb}(Q_0)\int_{\widetilde{Q}_0} K_2(x)\,dx,$$

where $K_2(x) := K_2(x,0)$ and $\widetilde{Q}_0 := Q_0 - Q_0$.

Let us first check that the function $x \mapsto K_2(x)$ is integrable around the origin. Note that, by Cauchy-Schwarz inequality,

$$\begin{aligned}
K_2(x) &= \frac{1}{1-r^2(x)}\,\mathbb{E}\left[|J_{\mathbf{T}_n}(x)|\cdot|J_{\mathbf{T}_n}(0)|\,\Big|\,\mathbf{T}_n(x) = \mathbf{T}_n(0) = 0\right] \\
&\leq \frac{1}{1-r^2(x)}\,\mathbb{E}\left[|J_{\mathbf{T}_n}(0)|^2\,\Big|\,\mathbf{T}_n(x) = \mathbf{T}_n(0) = 0\right].
\end{aligned} \tag{5.3}$$

It is immediate that

$$\begin{aligned} |J_{\mathbf{T}_n(0)}|^2 =&(\partial_1 T_n(0))^2(\partial_2 \widehat{T}_n(0))^2 + (\partial_1 \widehat{T}_n(0))^2(\partial_2 T_n(0))^2 \\ &- 2\partial_1 T_n(0)\partial_1 \widehat{T}_n(0)\partial_2 T_n(0)\widehat{\partial}_2 T_n(0). \end{aligned} \tag{5.4}$$

Bearing in mind that T_n and $\widehat{T}_n$ are independent and equally distributed random field, from (5.4) straightforward computations lead to

$$\begin{aligned} &\mathbb{E}\left[|J_{\mathbf{T}_n}(0)|^2 \middle| \mathbf{T}_n(x) = \mathbf{T}_n(0) = 0\right] \\ =&\mathbb{E}\Big[(\partial_1 T_n(0))^2(\partial_2 \widehat{T}_n(0))^2 + (\partial_1 \widehat{T}_n(0))^2(\partial_2 T_n(0))^2 \\ &\quad - 2\partial_1 T_n(0)\partial_1 \widehat{T}_n(0)\partial_2 T_n(0)\widehat{\partial}_2 T_n(0)|\mathbf{T}_n(x) = \mathbf{T}_n(0) = 0\Big] \\ =&2|\Omega_n(x)|, \end{aligned} \tag{5.5}$$

where $\Omega_n(x)$ denotes the covariance matrix of $\nabla T_n(0)$ conditioned to $T_n(x) = T_n(0) = 0$ (see (7.1) for a precise expression). Substituting (5.5) into (5.3) we get

$$K_2(x) \le 2\frac{|\Omega_n(x)|}{1 - r^2(x)}.$$

Now Lemma 7.1 gives that, as $\|x\| \to 0$,

$$\frac{|\Omega_n(x)|}{1 - r^2(x)} = cE_n^2 + E_n^3 O(\|x\|^2),$$

for some constant $c > 0$, where the constants involving in the 'O' notation do not depend on n, so that

$$\mathbb{E}\left[I_{n_{|Q_0}}\left(I_{n_{|Q_0}} - 1\right)\right] = \text{Leb}(Q_0)\int_{\widetilde{Q}_0} K_2(x)\, dx \ll \frac{E_n^2}{M^4},$$

which is the result we looked for. □

5.2.2 The contribution of non-singular pairs of cubes

Proof of Lemma 2.6. For any square Q, we can write

$$\begin{aligned} &\text{proj}\left(I_{n_{|Q}}|C_{\ge 6}\right) \\ &= \frac{E_n}{2}\sum_{q\ge 3}\sum_{i_1+i_2+i_3+j_1+j_2+j_3=2q} \frac{\beta_{i_1}\beta_{j_1}\alpha_{i_2,i_3,j_2,j_3}}{i_1!i_2!i_3!j_1!j_2!j_3!} \times \\ &\quad \times \int_Q H_{i_1}(T_n(x))H_{i_2}(\widetilde{\partial}_1 T_n(x))H_{i_3}(\widetilde{\partial}_2 T_n(x))H_{j_1}(\widehat{T}_n(x))H_{j_2}(\widetilde{\partial}_1 \widehat{T}_n(x))H_{j_3}(\widetilde{\partial}_2 \widehat{T}_n(x))\, dx, \end{aligned}$$

for even i_1, j_1 and i_2, i_3, j_2, j_3 with the same parity. Recall that $\beta_l = 0$ for odd l, and that as $l \to \infty$,

$$\frac{\beta_{2l}^2}{(2l)!} \approx \frac{1}{\sqrt{l}}.$$

Indeed, it suffices to apply Stirling's formula to the r.h.s. of the following $\beta_{2l}^2/(2l)! = (2\pi)^{-1}(2l)!/\left((l!)^2 2^{2l}\right)$.

We have

$$\Bigg|\sum_{(Q,Q')\text{ non sing.}} \mathrm{Cov}\left(\mathrm{proj}\left(I_{n_{|Q}}|C_{\geq 6}\right), \mathrm{proj}\left(I_{n_{|Q'}}|C_{\geq 6}\right)\right)\Bigg|$$
$$\leq E_n^2 \sum_{q\geq 3} \sum_{i_1+i_2+i_3+j_1+j_2+j_3=2q} \sum_{a_1+a_2+a_3+b_1+b_2+b_3=2q} \left|\frac{\beta_{i_1}\beta_{j_1}\alpha_{i_2,i_3,j_2,j_3}}{i_1!i_2!i_3!j_1!j_2!j_3!}\right| \cdot \left|\frac{\beta_{a_1}\beta_{b_1}\alpha_{a_2,a_3,b_2,b_3}}{a_1!a_2!a_3!b_1!b_2!b_3!}\right|$$
$$\times \Bigg| \sum_{(Q,Q')\text{ non sing.}}$$
$$\int_Q\int_{Q'} \mathbb{E}\Big[H_{i_1}(T_n(x))H_{i_2}(\widetilde{\partial}_1 T_n(x))H_{i_3}(\widetilde{\partial}_2 T_n(x))H_{j_1}(\widehat{T}_n(x))H_{j_2}(\widetilde{\partial}_1\widehat{T}_n(x))H_{j_3}(\widetilde{\partial}_2\widehat{T}_n(x))$$
$$\times H_{a_1}(T_n(y))H_{a_2}(\widetilde{\partial}_1 T_n(y))H_{a_3}(\widetilde{\partial}_2 T_n(y))H_{b_1}(\widehat{T}_n(y))H_{b_2}(\widetilde{\partial}_1\widehat{T}_n(y))H_{b_3}(\widetilde{\partial}_2\widehat{T}_n(y))\Big]\,dxdy\Bigg|. \tag{5.6}$$

Let us now adopt the same notation as in Proposition 8.1. For $n \in S$ we set

$$\begin{aligned}&(X_0(x), X_1(x), X_2(x), Y_0(x), Y_1(x), Y_2(x))\\ &\quad := (T_n(x),\, \widetilde{\partial}_1 T_n(x),\, \widetilde{\partial}_2 T_n(x), \widehat{T}_n(x),\, \widetilde{\partial}_1\widehat{T}_n(x),\, \widetilde{\partial}_2\widehat{T}_n(x)), \quad x \in \mathbb{T}.\end{aligned}$$

From Proposition 8.1 and (5.6), we have

$$\Bigg|\sum_{(Q,Q')\text{ non sing.}} \mathrm{Cov}\left(\mathrm{proj}\left(I_{n_{|Q}}|C_{\geq 6}\right), \mathrm{proj}\left(I_{n_{|Q'}}|C_{\geq 6}\right)\right)\Bigg| \tag{5.7}$$
$$\leq E_n^2 \sum_{q\geq 3} \sum_{i_1+i_2+i_3+j_1+j_2+j_3=2q} \sum_{a_1+a_2+a_3+b_1+b_2+b_3=2q} \left|\frac{\beta_{i_1}\beta_{j_1}\alpha_{i_2,i_3,j_2,j_3}}{i_1!i_2!i_3!j_1!j_2!j_3!}\right| \cdot \left|\frac{\beta_{a_1}\beta_{b_1}\alpha_{a_2,a_3,b_2,b_3}}{a_1!a_2!a_3!b_1!b_2!b_3!}\right|$$
$$\times \mathbf{1}_{\{i_1+i_2+i_3=a_1+a_2+a_3\}}\mathbf{1}_{\{j_1+j_2+j_3=b_1+b_2+b_3\}}\Big|V(i_1,i_2,i_3;j_1,j_2,j_3;a_1,a_2,a_3;b_1,b_2,b_3)\Big|,$$
$$:= E_n^2 \times Z, \tag{5.8}$$

where each of the terms $V = V(i_1,i_2,i_3;j_1,j_2,j_3;a_1,a_2,a_3;b_1,b_2,b_3)$ is the sum of no more than $(2q)!$ terms of the type

$$v = \sum_{(Q,Q')\text{ non sing.}} \int_Q\int_{Q'} \prod_{u=1}^{2q} R_{l_u,k_u}(x-y)\,dxdy, \tag{5.9}$$

where $k_u, l_u \in \{0,1,2\}$ and where, for $l,k = 0,1,2$ and $x,y \in \mathbb{T}$, we set

$$R_{l,k}(x-y) := \mathbb{E}\left[X_l(x)X_k(y)\right] = \mathbb{E}\left[Y_l(x)Y_k(y)\right].$$

Note that, for any even $p \in \mathbb{N}$, we have

$$\int_{\mathbb{T}} R_{l,k}(x)^p\,dx \leq \int_{\mathbb{T}} r_n(x)^p\,dx =: R_n(p) \tag{5.10}$$

and recall moreover that, for $x,y \in \mathbb{T}$, $|R_{l,k}(x-y)| \leq 1$, and, for $(x,y) \in Q\times Q'$,

$$|R_{l,k}(x-y)| < \varepsilon_1. \tag{5.11}$$

Using the definition of a non-singular pair of cubes, as well as the fact that the sum defining Z in (5.8) involves indices $q \geq 3$, one deduces that, for v as in (5.9),

$$\begin{aligned}|v| &\leq \varepsilon_1^{2q-6} \sum_{(Q,Q')\text{ non sing.}} \int_Q\int_{Q'} \prod_{u=1}^{6} |R_{l_u,k_u}(x-y)|\,dxdy\\ &\leq \varepsilon_1^{2q-6} \int_{\mathbb{T}} \prod_{u=1}^{6} |R_{l_u,k_u}(x)|\,dx \leq \varepsilon_1^{2q-6} R_n(6),\end{aligned}$$

where we have applied a generalized Hölder inequality together with (5.10) in order to deduce the last estimate. This bound implies that each of the terms V contributing to Z can be bounded as follows:

$$\begin{aligned}&\Big|V(i_1,i_2,i_3;j_1,j_2,j_3;a_1,a_2,a_3;b_1,b_2,b_3)\Big|\\&\quad\le(2q)!\frac{R_n(6)}{\varepsilon_1^6}\varepsilon_1^{2q}=(2q)!\frac{R_n(6)}{\varepsilon_1^6}(\sqrt{\varepsilon_1})^{i_1+\cdots+j_3}(\sqrt{\varepsilon_1})^{a_1+\cdots+b_3}.\end{aligned}$$

One therefore infers that

$$\begin{aligned}Z\le\frac{R_n(6)}{\varepsilon_1^6}\sum_{q\ge3}(2q)!\sum_{i_1+i_2+i_3+j_1+j_2+j_3=2q}\ \sum_{a_1+a_2+a_3+b_1+b_2+b_3=2q}\left|\frac{\beta_{i_1}\beta_{j_1}\alpha_{i_2,i_3,j_2,j_3}}{i_1!i_2!i_3!j_1!j_2!j_3!}\right|\times\\ \left|\frac{\beta_{a_1}\beta_{b_1}\alpha_{a_2,a_3,b_2,b_3}}{a_1!a_2!a_3!b_1!b_2!b_3!}\right|\times(\sqrt{\varepsilon_1})^{i_1+\cdots+j_3}(\sqrt{\varepsilon_1})^{a_1+\cdots+b_3}=:\frac{R_n(6)}{\varepsilon_1^6}\times S.\end{aligned}$$

In order to show that S is finite, we write

$$\begin{aligned}S&=\sum_{q\ge3}(2q)!\sum_{i_1+i_2+i_3+j_1+j_2+j_3=2q}\ \sum_{a_1+a_2+a_3+b_1+b_2+b_3=2q}\left|\frac{\beta_{i_1}\beta_{j_1}\alpha_{i_2,i_3,j_2,j_3}}{i_1!i_2!i_3!j_1!j_2!j_3!}\right|\times\\&\qquad\left|\frac{\beta_{a_1}\beta_{b_1}\alpha_{a_2,a_3,b_2,b_3}}{a_1!a_2!a_3!b_1!b_2!b_3!}\right|\times(\sqrt{\epsilon_1})^{i_1+\cdots+j_3}(\sqrt{\epsilon_1})^{a_1+\cdots+b_3}\\&\le\sum_{q\ge0}\ \sum_{i_1+i_2+i_3+j_1+j_2+j_3=2q}\ \sum_{a_1+a_2+a_3+b_1+b_2+b_3=2q}\left|\frac{\beta_{i_1}\beta_{j_1}\alpha_{i_2,i_3,j_2,j_3}}{i_1!i_2!i_3!j_1!j_2!j_3!}\right|\times\\&\qquad\sqrt{(i_1+\cdots+j_3)!}\sqrt{(a_1+\cdots+b_3)!}\left|\frac{\beta_{a_1}\beta_{b_1}\alpha_{a_2,a_3,b_2,b_3}}{a_1!a_2!a_3!b_1!b_2!b_3!}\right|\times(\sqrt{\epsilon_1})^{i_1+\cdots+j_3+a_1+\cdots+b_3}\\&\le\sum_{i_1,\dots,j_3,a_1\dots,b_3}\left|\frac{\beta_{i_1}\beta_{j_1}\alpha_{i_2,i_3,j_2,j_3}}{i_1!i_2!i_3!j_1!j_2!j_3!}\right|\times\\&\qquad\sqrt{(i_1+\cdots+j_3)!}\sqrt{(a_1+\cdots+b_3)!}\left|\frac{\beta_{a_1}\beta_{b_1}\alpha_{a_2,a_3,b_2,b_3}}{a_1!a_2!a_3!b_1!b_2!b_3!}\right|\times(\sqrt{\epsilon_1})^{i_1+\cdots+j_3+a_1+\cdots+b_3}\\&\le\left(\sum_{i_1,\dots,j_3,a_1\dots,b_3}\left|\frac{\beta_{i_1}\beta_{j_1}\alpha_{i_2,i_3,j_2,j_3}}{i_1!i_2!i_3!j_1!j_2!j_3!}\right|^2(i_1+\cdots+j_3)!(\sqrt{\epsilon_1})^{i_1+\cdots+j_3+a_1+\cdots+b_3}\right)^{1/2}\times\\&\qquad\times\left(\sum_{i_1,\dots,j_3,a_1\dots,b_3}\left|\frac{\beta_{a_1}\beta_{b_1}\alpha_{a_2,a_3,b_2,b_3}}{a_1!a_2!a_3!b_1!b_2!b_3!}\right|^2(a_1+\cdots+b_3)!(\sqrt{\epsilon_1})^{i_1+\cdots+j_3+a_1+\cdots+b_3}\right)^{1/2}\\&=\sum_{i_1,\dots,j_3,a_1\dots,b_3}\left|\frac{\beta_{i_1}\beta_{j_1}\alpha_{i_2,i_3,j_2,j_3}}{i_1!i_2!i_3!j_1!j_2!j_3!}\right|^2(i_1+\cdots+j_3)!(\sqrt{\epsilon_1})^{i_1+\cdots+j_3+a_1+\cdots+b_3}<\infty,\end{aligned}$$

where: (a) the third inequality follows by applying the Cauchy-Schwarz inequality to the symmetric finite measure μ on $\mathbb{N}^{12}$ such that

$$\mu\{(k_1,\dots,k_{12})\}=(\sqrt{\epsilon_1})^{k_1+\cdots+k_{12}},$$

and, (b) writing $m=m(i_1,\dots,j_3):=i_1+\cdots+j_3$ for every $i_1,\dots,j_3$, the finiteness of the last sum is a consequence of the standard estimate

$$\frac{(i_1+\cdots+j_3)!}{i_1!i_2!i_3!j_1!j_2!j_3!}\le\sum_{\substack{k_1,\dots,k_6\ge0\\k_1+\cdots+k_6=m}}\frac{m!}{k_1!\cdots k_6!}=6^m=6^{i_1+\cdots+j_3},$$

as well as of the fact that the mapping

$$(i_1,\dots,j_3)\mapsto\frac{\beta_{i_1}^2\beta_{j_1}^2\alpha_{i_2,i_3,j_2,j_3}^2}{i_1!i_2!i_3!j_1!j_2!j_3!}$$

is bounded, and $6\sqrt{\varepsilon_1} < 1$ by assumption. This concludes the proof. $\square$

6 End of the Proofs of Theorem 1.2 and Theorem 1.5

6.1 Proof of Part 2 of Theorem 1.2

From Lemma 3.4, for $n \in S$ the chaotic expansion for I_n is

$$I_n = \mathbb{E}[I_n] + \sum_{q\ge 2} I_n[2q],$$

where $I_n[2q]$ is given in (3.8). Proposition 2.2, Proposition 2.3 together with Lemma 8.3 immediately conclude the proof, once we recall that, by orthogonality of different Wiener chaoses

$$\mathrm{Var}(I_n) = \mathrm{Var}(I_n[4]) + \sum_{q\ge 3} \mathrm{Var}(I_n[2q]).$$

6.2 Proof of Part 4 of Theorem 1.2

Part 2 of Theorem 1.2 yields that, as $\mathcal{N}_{n_j} \to +\infty$,

$$\frac{I_{n_j} - \mathbb{E}[I_{n_j}]}{\sqrt{\mathrm{Var}(I_{n_j})}} = \frac{I_{n_j}[4]}{\sqrt{\mathrm{Var}(I_{n_j}[4])}} + o_{\mathbb{P}}(1).$$

Proposition 2.4 hence allows to conclude the proof.

6.3 Proof of Proposition 2.7

First note that we can rewrite (4.2) as

$$\frac{I_{n_j}[4]}{V_{n_j}^{1/2}} = \frac{n_j \pi}{8\mathcal{N}_{n_j}} \frac{p(\mathbf{W}(n_j)) + O_{\mathbb{P}}(\mathcal{N}_{n_j}^{-1/2})}{V_{n_j}^{1/2}} = \frac{p(\mathbf{W}(n_j)) + O_{\mathbb{P}}(\mathcal{N}_{n_j}^{-1/2})}{c\sqrt{3\widehat{\mu}_{n_j}(4)^2+5}}, \tag{6.1}$$

for a certain second degree polynomial p and a constant $c > 0$ (that can be found explicitly), as well as for (1.11)

$$\mathcal{J}(\widehat{\mu}_{n_j}(4)) = \frac{p(\mathbf{G})}{c\sqrt{3\widehat{\mu}_{n_j}(4)^2+5}}, \tag{6.2}$$

where $\mathbf{G}$ denotes a Gaussian real centered vector with covariance matrix given by $\mathbf{M}(\widehat{\mu_{n_j}}(4))$ (see (4.4)). Hence from (6.1) and (6.2) we can write

$$\begin{aligned}
&\left|\mathbb{E}\left[h\left(I_{n_j}[4]/V_{n_j}^{1/2}\right)\right] - \mathbb{E}\left[h\left(\mathcal{J}(\widehat{\mu_{n_j}}(4))\right)\right]\right| \\
&= \left|\mathbb{E}\left[h\left(\frac{p(\mathbf{W}(n_j)) + O_{\mathbb{P}}(\mathcal{N}_{n_j}^{-1/2})}{c\sqrt{3\widehat{\mu}_{n_j}(4)^2+5}}\right)\right] - \mathbb{E}\left[h\left(\frac{p(\mathbf{G})}{c\sqrt{3\widehat{\mu}_{n_j}(4)^2+5}}\right)\right]\right| \\
&\ll \mathcal{N}_{n_j}^{-1/2} + \left|\mathbb{E}\left[h\circ\widetilde{p}(\mathbf{W}(n_j))\right] - \mathbb{E}\left[h\circ\widetilde{p}(\mathbf{G})\right]\right|,
\end{aligned} \tag{6.3}$$

where $\widetilde{p} := p/c\sqrt{3\widehat{\mu}_{n_j}(4)^2+5}$.

At this point, one can apply a standard interpolation and integration by parts procedure, such as the one in [N-P, Proof of Theorem 6.1.2 and Theorem 6.1.1], leading to

$$\begin{aligned}
&\left|\mathbb{E}\left[h\circ\widetilde{p}(\mathbf{W}(n_j))\right] - \mathbb{E}\left[h\circ\widetilde{p}(\mathbf{G})\right]\right| \\
&\le \underbrace{\sum_{i,k=1}^{3}\sqrt{\mathbb{E}[|\partial^2_{i,k} h\circ\widetilde{p}(\mathbf{W}(n_j))|^2]\,\mathbb{E}[|\mathbf{M}_{ik}(\widehat{\mu}_{n_j}(4)) - \langle DW_k(n_j)), -DL^{-1}W_i(n_j)\rangle|^2]}}_{:=I},
\end{aligned}$$

where $\partial^2_{i,k} := \partial^2/\partial x_i \partial x_k$, D denotes the Malliavin derivative (see [N-P, Definition 1.1.8]), L^{-1} the inverse of the infinitesimal generator of the Ornstein-Uhlenbeck semigroup (see [N-P, §1.3]) and $\langle \cdot, \cdot \rangle$ stands for the inner product of an appropriate real separable Hilbert space $\mathcal{H}$ (whose precise definition is not relevant for the present proof).

To eventually deal with I, one can use the upper bound in [N-P, formula (6.2.6)]: indeed, since each $W_i(n_j)$ belongs to the second Wiener chaos, from (4.6) we infer the estimate

$$I \ll \sqrt{\frac{1}{\mathcal{N}_{n_j}}}. \tag{6.4}$$

Substituting (6.4) into (6.3) allows us to conclude the proof.

7 Some technical computations

7.1 Technical proofs

Recall the formulas for the first Hermite polynomials: $H_0(t) = 1$, $H_1(t) = t$, $H_2(t) = t^2 - 1$, $H_4(t) = t^4 - 6t^2 + 3$.

Proof of Lemma 3.3. We have

$$\begin{aligned} \alpha_{0,0,0,0} &= \mathbb{E}[|XW - YV|] \\ &= \frac{1}{(2\pi)^2} \left(\int_0^\infty \rho^2 e^{-\rho^2/2} d\rho \right)^2 \int_0^{2\pi} \int_0^{2\pi} |\sin\theta \cos\theta' - \sin\theta' \cos\theta| d\theta d\theta' \\ &= \frac{1}{(2\pi)^2} \left(\int_0^\infty \rho^2 e^{-\rho^2/2} d\rho \right)^2 \int_0^{2\pi} \int_0^{2\pi} |\sin(\theta - \theta')| d\theta d\theta' = 1. \end{aligned}$$

Setting Z to be any of the variables X, Y, V, W and $\varphi_Z(u)$ to be $\cos(u)$ if $Z = X, V$, or $\sin(u)$ if $Z = Y, W$, we have that

$$\begin{aligned} &\mathbb{E}[|XW - YV| H_2(Z)] \\ = \; &\frac{1}{(2\pi)^2} \int_0^\infty \rho^2 e^{-\rho^2/2} d\rho \int_0^\infty \gamma^4 e^{-\gamma^2/2} d\gamma \int_0^{2\pi} \int_0^{2\pi} |\sin(\theta - \theta')| \varphi_Z(\theta)^2 d\theta d\theta' - 1 = \frac{1}{2}. \end{aligned}$$

As a result, we deduce that

$$\alpha_{2,0,0,0} = \alpha_{0,2,0,0} = \alpha_{0,0,2,0} = \alpha_{0,0,0,2} = \frac{1}{2}.$$

Let us now concentrate on $\alpha_{4,0,0,0}$. We have

$$\begin{aligned} \alpha_{4,0,0,0} &= \mathbb{E}[|XW - YV| H_4(X)] \\ &= \mathbb{E}[|XW - YV| X^4] - 6 \underbrace{\mathbb{E}[|XW - YV| X^2]}_{= \frac{3}{2} \text{ from above}} + 3 \underbrace{\mathbb{E}[|XW - YV|]}_{=1}. \end{aligned}$$

Thus, it remains to calculate

$$\begin{aligned} &\mathbb{E}[|XW - YV| X^4] \\ = \; &\frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} |xw - yv| x^4 \mathrm{e}^{-x^2/2} \mathrm{e}^{-y^2/2} \mathrm{e}^{-v^2/2} \mathrm{e}^{-w^2/2} \, dx dy dv dw \\ = \; &\frac{1}{2\pi} \underbrace{\int_0^{2\pi} \cos^4\theta d\theta}_{= \frac{3\pi}{4}} \underbrace{\int_0^{2\pi} |\sin(\theta' - \theta)| d\theta'}_{=4} \underbrace{\frac{1}{\sqrt{2\pi}} \int_0^\infty (\rho')^2 \mathrm{e}^{-(\rho')^2/2} d\rho'}_{= \frac{1}{2}} \underbrace{\frac{1}{\sqrt{2\pi}} \int_0^\infty \rho^6 \mathrm{e}^{-\rho^2/2} \, d\rho}_{= \frac{15}{2}} \\ = \; &\frac{45}{8}. \end{aligned}$$

Plugging into the previous expression, we deduce

$$\alpha_{4,0,0,0} = -\frac{3}{8}.$$

Since $\int_0^{2\pi} \cos^4 \theta d\theta = \int_0^{2\pi} \sin^4 \theta d\theta$, it is immediate to check that

$$\alpha_{4,0,0,0} = \alpha_{0,4,0,0} = \alpha_{0,0,4,0} = \alpha_{0,0,0,4}.$$

Let us now compute $\alpha_{2,2,0,0}$. We have

$$\begin{aligned} &\alpha_{2,2,0,0} = \mathbb{E}[|XW - YV| H_2(X) H_2(Y)] \\ = \quad & \mathbb{E}[|XW - YV| X^2 Y^2] - \underbrace{\mathbb{E}[|XW - YV| X^2]}_{=\frac{3}{2}} - \underbrace{\mathbb{E}[|XW - YV| Y^2]}_{=\frac{3}{2}} + \underbrace{\mathbb{E}[|XW - YV|]}_{=1}, \end{aligned}$$

whereas

$$\mathbb{E}[|XW - YV| X^2 Y^2] = \frac{15}{8} \quad \text{and} \quad \alpha_{2,2,0,0} = -\frac{1}{8}.$$

Similarly,

$$\alpha_{0,0,2,2} = \alpha_{2,2,0,0} = -\frac{1}{8}.$$

Now, let us compute $\alpha_{2,0,2,0}$. We can write

$$\begin{aligned} \alpha_{2,0,2,0} &= \mathbb{E}[|XW - YV| H_2(X) H_2(V)] \\ &= \mathbb{E}[|XW - YV| X^2 V^2] - \underbrace{\mathbb{E}[|XW - YV| X^2]}_{=\frac{3}{2}} - \underbrace{\mathbb{E}[|XW - YV| V^2]}_{=\frac{3}{2}} \\ &\quad + \underbrace{\mathbb{E}[|XW - YV|]}_{=1}, \end{aligned}$$

whereas

$$\mathbb{E}[|XW - YV| X^2 V^2] = \frac{15}{8} \quad \text{and} \quad \alpha_{2,0,2,0} = \alpha_{0,2,0,2} - \frac{1}{8}.$$

We also compute

$$\begin{aligned} \alpha_{2,0,0,2} &= \mathbb{E}[|XW - YV| H_2(X) H_2(W)] \\ &= \mathbb{E}[|XW - YV| X^2 W^2] - \underbrace{\mathbb{E}[|XW - YV| X^2]}_{=\frac{3}{2}} - \underbrace{\mathbb{E}[|XW - YV| W^2]}_{=\frac{3}{2}} \\ &\quad + \underbrace{\mathbb{E}[|XW - YV|]}_{=1}. \end{aligned}$$

We have consequently

$$\mathbb{E}[|XW - YV| X^2 W^2] = \frac{21}{8} \quad \text{and} \quad \alpha_{2,0,0,2} = \alpha_{0,2,2,0} = \frac{5}{8}.$$

Finally, in the case where $a = b = c = d = 1$, we have

$$\alpha_{1,1,1,1} = \mathbb{E}[|XW - YV| XYVW] = -\frac{3}{8}. \qquad \square$$

7.2 Proof of Lemma 4.1

Proof of (i). We have

$$\begin{aligned}
&\int_{\mathbb{T}} H_4(T_n(x))\,dx = \int_{\mathbb{T}} (T_n(x)^4 - 6T_n(x)^2 + 3)\,dx \\
= \;& \frac{1}{\mathcal{N}_n^2} \sum_{\lambda,\lambda',\lambda'',\lambda'''\in\Lambda_n} a_\lambda \overline{a_{\lambda'}}\, a_{\lambda''} \overline{a_{\lambda'''}} \int \mathrm{e}_{\lambda-\lambda'+\lambda''-\lambda'''}(x)\,dx - \frac{6}{\mathcal{N}_n} \sum_{\lambda,\lambda'\in\Lambda_n} a_\lambda \overline{a_{\lambda'}} \int \mathrm{e}_{\lambda-\lambda'}(x)\,dx + 3 \\
= \;& \frac{1}{\mathcal{N}_n^2} \sum_{\lambda,\lambda''} |a_\lambda|^2 |a_{\lambda''}|^2 + \frac{1}{\mathcal{N}_n^2} \sum_{\lambda} |a_\lambda|^4 + \frac{2}{\mathcal{N}_n^2} \sum_{\lambda\neq\pm\lambda'} |a_\lambda|^2 |a_{\lambda'}|^2 - \frac{6}{\mathcal{N}_n} \sum_{\lambda} |a_\lambda|^2 + 3 \\
= \;& \frac{3}{\mathcal{N}_n^2} \sum_{\lambda,\lambda''} (|a_\lambda|^2 - 1)(|a_{\lambda''}|^2 - 1) - \frac{3}{\mathcal{N}_n^2} \sum_{\lambda} |a_\lambda|^4 = \frac{3}{\mathcal{N}_n} W(n)^2 - \frac{3}{\mathcal{N}_n^2} \sum_{\lambda} |a_\lambda|^4.
\end{aligned}$$

Since $\mathbb{E}[(\frac{1}{\mathcal{N}_n}\sum_\lambda (|a_\lambda|^4 - 2))^2] = O(\mathcal{N}_n^{-1})$, the claim (i) follows.

Proof of (ii). We have

$$\begin{aligned}
&\int_{\mathbb{T}} H_4(\widetilde{\partial}_j T_n(x))\,dx = \int_{\mathbb{T}} (\widetilde{\partial}_j T_n(x)^4 - 6\,\widetilde{\partial}_j T_n(x)^2 + 3)\,dx \\
= \;& \frac{4}{n^2\mathcal{N}_n^2} \sum_{\lambda,\lambda',\lambda'',\lambda'''\in\Lambda_n} \lambda_j \lambda_j' \lambda_j'' \lambda_j'''\, a_\lambda \overline{a_{\lambda'}}\, a_{\lambda''} \overline{a_{\lambda'''}} \int \mathrm{e}_{\lambda-\lambda'+\lambda''-\lambda'''}(x)\,dx \\
& -\frac{12}{n\mathcal{N}_n} \sum_{\lambda,\lambda'\in\Lambda_n} \lambda_j \lambda_j'\, a_\lambda \overline{a_{\lambda'}} \int \mathrm{e}_{\lambda-\lambda'}(x)\,dx + 3 \\
= \;& \frac{4}{n^2\mathcal{N}_n^2} \sum_{\lambda,\lambda''} \lambda_j^2 {\lambda_j''}^2 |a_\lambda|^2 |a_{\lambda''}|^2 + \frac{4}{n^2\mathcal{N}_n^2} \sum_{\lambda} \lambda_j^4 |a_\lambda|^4 + \frac{8}{n^2\mathcal{N}_n^2} \sum_{\lambda\neq\pm\lambda'} \lambda_j^2 {\lambda_j'}^2 |a_\lambda|^2 |a_{\lambda'}|^2 \\
& -\frac{12}{n\mathcal{N}_n} \sum_{\lambda} \lambda_j^2 |a_\lambda|^2 + 3
\end{aligned}$$

$$\begin{aligned}
= \;& \frac{12}{n^2\mathcal{N}_n^2} \sum_{\lambda,\lambda''} \lambda_j^2 {\lambda_j''}^2 |a_\lambda|^2 |a_{\lambda''}|^2 - \frac{12}{n^2\mathcal{N}_n^2} \sum_{\lambda} \lambda_j^4 |a_\lambda|^4 - \frac{12}{n\mathcal{N}_n} \sum_{\lambda} \lambda_j^2 |a_\lambda|^2 + 3 \\
= \;& \frac{12}{n^2\mathcal{N}_n^2} \sum_{\lambda,\lambda''} \lambda_j^2 {\lambda_j''}^2 (|a_\lambda|^2 - 1)(|a_{\lambda''}|^2 - 1) - \frac{12}{n^2\mathcal{N}_n^2} \sum_{\lambda} \lambda_j^4 |a_\lambda|^4 \\
= \;& \frac{12}{\mathcal{N}_n} W_j(n)^2 - \frac{12}{n^2\mathcal{N}_n^2} \sum_{\lambda} \lambda_j^4 |a_\lambda|^4.
\end{aligned}$$

Since $\mathbb{E}\left[\frac{1}{n^2\mathcal{N}_n}\sum_\lambda \lambda_j^4 |a_\lambda|^4\right] = \frac{1}{4}(3 + \widehat{\mu}_n(4))$ and $\mathbb{E}\left[(\frac{1}{n^2\mathcal{N}_n}\sum_\lambda \lambda_j^4(|a_\lambda|^4 - 2))^2\right] = O(\mathcal{N}_n^{-1})$, the claim (ii) follows.

Proof of (iii). We have

$$\begin{aligned}
&\int_{\mathbb{T}} H_2(T_n(x))\big(H_2(\widetilde{\partial}_1 T_n(x) + H_2(\widetilde{\partial}_2 T_n(x))\big)\,dx \\
= &\int_{\mathbb{T}} \big(T_n(x)^2 \widetilde{\partial}_1 T_n(x)^2 + T_n(x)^2 \widetilde{\partial}_2 T_n(x)^2 - 2T_n(x)^2 - \widetilde{\partial}_1 T_n(x)^2 - \widetilde{\partial}_2 T_n(x)^2 + 2\big)\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{n\mathcal{N}_n^2} \sum_{\lambda,\lambda',\lambda'',\lambda'''\in\Lambda_n} (\lambda_1''\lambda_1''' + \lambda_2''\lambda_2''')\, a_\lambda \overline{a_{\lambda'}}\, a_{\lambda''} \overline{a_{\lambda'''}} \int \mathrm{e}_{\lambda-\lambda'+\lambda''-\lambda'''}(x)\, dx \\
&\quad - \frac{2}{\mathcal{N}_n} \sum_{\lambda,\lambda'\in\Lambda_n} a_\lambda \overline{a_{\lambda'}} \int \mathrm{e}_{\lambda-\lambda'}(x)\, dx - \frac{2}{n\mathcal{N}_n} \sum_{\lambda,\lambda'\in\Lambda_n} (\lambda_1\lambda_1' + \lambda_2\lambda_2') a_\lambda \overline{a_{\lambda'}} \int \mathrm{e}_{\lambda-\lambda'}(x)\, dx + 2 \\
&= \frac{2}{\mathcal{N}_n^2} \sum_{\lambda,\lambda''} (|a_\lambda|^2 - 1)(|a_{\lambda''}|^2 - 1) - \frac{2}{\mathcal{N}_n^2} \sum_\lambda |a_\lambda|^4 = \frac{2}{\mathcal{N}_n} \left\{ W(n)^2 - \frac{1}{\mathcal{N}_n} \sum_\lambda |a_\lambda|^4 \right\}.
\end{aligned}
$$

Since $\mathbb{E}\left[(\frac{1}{\mathcal{N}_n} \sum_\lambda (|a_\lambda|^4 - 2))^2 \right] = O(\mathcal{N}_n^{-1})$, the claim (iii) follows.

Proof of (iv). We have

$$
\begin{aligned}
&\int_{\mathbb{T}} H_2(\widetilde{\partial}_1 T_n(x)) H_2(\widetilde{\partial}_2 T_n(x))\, dx = \int_{\mathbb{T}} \left(\widetilde{\partial}_1 T_n(x)^2 \widetilde{\partial}_2 T_n(x)^2 - \widetilde{\partial}_1 T_n(x)^2 - \widetilde{\partial}_2 T_n(x)^2 + 1 \right) dx \\
&= \frac{4}{n^2\mathcal{N}_n^2} \sum_{\lambda,\lambda',\lambda'',\lambda'''\in\Lambda_n} \lambda_1\lambda_1'\lambda_2''\lambda_2'''\, a_\lambda \overline{a_{\lambda'}}\, a_{\lambda''} \overline{a_{\lambda'''}} \int \mathrm{e}_{\lambda-\lambda'+\lambda''-\lambda'''}(x)\, dx \\
&\quad - \frac{2}{n\mathcal{N}_n} \sum_{\lambda,\lambda'\in\Lambda_n} \lambda_1\lambda_1' a_\lambda \overline{a_{\lambda'}} \int \mathrm{e}_{\lambda-\lambda'}(x)\, dx - \frac{2}{n\mathcal{N}_n} \sum_{\lambda,\lambda'\in\Lambda_n} \lambda_2\lambda_2' a_\lambda \overline{a_{\lambda'}} \int \mathrm{e}_{\lambda-\lambda'}(x)\, dx + 1 \\
&= \frac{4}{n^2\mathcal{N}_n^2} \sum_{\lambda,\lambda''} \lambda_1^2 \lambda_2''^2 (|a_\lambda|^2 - 1)(|a_{\lambda''}|^2 - 1) - \frac{12}{n^2\mathcal{N}_n^2} \sum_\lambda \lambda_1^2\lambda_2^2 |a_\lambda|^4 + \frac{8}{n^2\mathcal{N}_n^2} \left(\sum_\lambda \lambda_1\lambda_2 |a_\lambda|^2 \right)^2 \\
&= \frac{4}{\mathcal{N}_n} \left\{ W_1(n) W_2(n) + 2W_{1,2}(n)^2 - \frac{3}{n^2\mathcal{N}_n} \sum_\lambda \lambda_1^2\lambda_2^2 |a_\lambda|^4 \right\}.
\end{aligned}
$$

Since $\mathbb{E}\left[\frac{1}{n^2\mathcal{N}_n} \sum_\lambda \lambda_1^2\lambda_2^2 |a_\lambda|^4 \right] = \frac{1}{4}(1 - \widehat{\mu}_n(4))$ and $\mathbb{E}\left[(\frac{1}{n^2\mathcal{N}_n} \sum_\lambda \lambda_1^2\lambda_2^2 (|a_\lambda|^4 - 2))^2 \right] = O(\mathcal{N}_n^{-1})$, the claim (iv) follows.

Proof of (v). We have

$$
\begin{aligned}
&\int_{\mathbb{T}} H_2(T_n(x)) H_2(\widehat{T}_n(x))\, dx = \int_{\mathbb{T}} \left(T_n(x)^2 \widehat{T}_n(x)^2 - T_n(x)^2 - \widehat{T}_n(x)^2 + 1 \right) dx \\
&= \frac{1}{\mathcal{N}_n^2} \sum_{\lambda,\lambda',\lambda'',\lambda'''\in\Lambda_n} a_\lambda \overline{a_{\lambda'}}\, \widehat{a}_{\lambda''} \overline{\widehat{a}_{\lambda'''}} \int \mathrm{e}_{\lambda-\lambda'+\lambda''-\lambda'''}(x)\, dx - \frac{1}{\mathcal{N}_n} \sum_{\lambda,\lambda'\in\Lambda_n} a_\lambda \overline{a_{\lambda'}} \int \mathrm{e}_{\lambda-\lambda'}(x)\, dx \\
&\quad - \frac{1}{\mathcal{N}_n} \sum_{\lambda,\lambda'\in\Lambda_n} \widehat{a}_\lambda \overline{\widehat{a}_{\lambda'}} \int \mathrm{e}_{\lambda-\lambda'}(x)\, dx + 1 \\
&= \frac{1}{\mathcal{N}_n^2} \sum_{\lambda,\lambda'} (|a_\lambda|^2 - 1)(|\widehat{a}_{\lambda'}|^2 - 1) - \frac{1}{\mathcal{N}_n^2} \sum_\lambda a_\lambda^2\, \overline{\widehat{a}_\lambda}^2 + \frac{2}{\mathcal{N}_n^2} \left(\sum_\lambda a_\lambda\, \overline{\widehat{a}_\lambda} \right)^2 - \frac{2}{\mathcal{N}_n^2} \sum_\lambda |a_\lambda|^2 |\widehat{a}_\lambda|^2 \\
&= \frac{1}{\mathcal{N}_n} \left\{ W(n)\widehat{W}(n) + 2M(n)^2 - \frac{1}{\mathcal{N}_n} \sum_\lambda a_\lambda^2\, \overline{\widehat{a}_\lambda}^2 - \frac{2}{\mathcal{N}_n} \sum_\lambda |a_\lambda|^2 |\widehat{a}_\lambda|^2 \right\}.
\end{aligned}
$$

Since $\mathbb{E}\left[\frac{1}{\mathcal{N}_n} \sum_\lambda a_\lambda^2\, \overline{\widehat{a}_\lambda}^2 \right] = 0$ and $\mathbb{E}\left[\frac{1}{\mathcal{N}_n} \sum_\lambda |a_\lambda|^2 |\widehat{a}_\lambda|^2 \right] = 1$ and $\mathbb{E}\left[(\frac{1}{\mathcal{N}_n} \sum_\lambda a_\lambda^2\, \overline{\widehat{a}_\lambda}^2)^2 \right] = O(\mathcal{N}_n^{-1})$ and $\mathbb{E}\left[(\frac{1}{\mathcal{N}_n} \sum_\lambda (|a_\lambda|^2 |\widehat{a}_\lambda|^2 - 1))^2 \right] = O(\mathcal{N}_n^{-1})$, the claim (v) follows.

Proof of (vi). We have

$$\begin{aligned}
&\int_{\mathbb{T}} H_2(T_n(x))H_2(\widetilde{\partial}_j\widehat{T}_n(x))\,dx = \int_{\mathbb{T}} \big(T_n(x)^2\widetilde{\partial}_j\widehat{T}_n(x)^2 \;- T_n(x)^2 \;- \widetilde{\partial}_j\widehat{T}_n(x)^2 \;+1\big)dx\\
=\;& \frac{2}{n\mathcal{N}_n^2}\sum_{\lambda,\lambda',\lambda'',\lambda'''}\lambda''_j\lambda'''_j a_\lambda\overline{a_{\lambda'}}\,\widehat{a}_{\lambda''}\overline{\widehat{a}_{\lambda'''}}\int \mathrm{e}_{\lambda-\lambda'+\lambda''-\lambda'''}(x)\,dx - \frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\\
&-\frac{2}{n\mathcal{N}_n}\sum_\lambda \lambda_j^2|\widehat{a}_\lambda|^2+1\\
=\;& \frac{2}{n\mathcal{N}_n^2}\sum_{\lambda,\lambda''}\lambda''^2_j|a_\lambda|^2|\widehat{a}_{\lambda''}|^2 - \frac{2}{n\mathcal{N}_n^2}\sum_\lambda \lambda_j^2 a_\lambda^2\widehat{a}_\lambda^2 + \frac{2}{n\mathcal{N}_n^2}\sum_{\lambda\neq\pm\lambda'}\lambda_j\lambda'_j a_\lambda\overline{a_{\lambda'}}\,\widehat{a}_{\lambda'}\overline{\widehat{a}_\lambda}\\
&-\frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2 - \frac{2}{n\mathcal{N}_n}\sum_\lambda \lambda_j^2|\widehat{a}_\lambda|^2+1.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{T}} H_2(T_n(x))\big(H_2(\widetilde{\partial}_1\widehat{T}_n(x))+H_2(\widetilde{\partial}_2\widehat{T}_n(x))\big)\,dx\\
=\;& \frac{2}{\mathcal{N}_n^2}\sum_{\lambda,\lambda''}(|a_\lambda|^2-1)(|\widehat{a}_{\lambda''}|^2-1) - \frac{2}{\mathcal{N}_n^2}\sum_\lambda a_\lambda^2\widehat{a}_\lambda^2 + \frac{2}{n\mathcal{N}_n^2}\sum_{\lambda\neq\pm\lambda'}(\lambda_1\lambda'_1+\lambda_2\lambda'_2)a_\lambda\overline{a_{\lambda'}}\,\widehat{a}_{\lambda'}\overline{\widehat{a}_\lambda}\\
=\;& \frac{2}{\mathcal{N}_n^2}\sum_{\lambda,\lambda''}(|a_\lambda|^2-1)(|\widehat{a}_{\lambda''}|^2-1) - \frac{2}{\mathcal{N}_n^2}\sum_\lambda |a_\lambda|^2\,|\widehat{a}_\lambda|^2 + \frac{2}{n\mathcal{N}_n^2}\sum_{\lambda,\lambda'}(\lambda_1\lambda'_1+\lambda_2\lambda'_2)a_\lambda\overline{a_{\lambda'}}\,\widehat{a}_{\lambda'}\overline{\widehat{a}_\lambda}\\
=\;& \frac{2}{\mathcal{N}_n}\left\{W(n)\widehat{W}(n)+M_1(n)^2+M_2(n)^2-\frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\,|\widehat{a}_\lambda|^2\right\}.
\end{aligned}$$

Since $\mathbb{E}\left[\frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2|\widehat{a}_\lambda|^2\right]=1$ and $\mathbb{E}\left[(\frac{1}{\mathcal{N}_n}\sum_\lambda(|a_\lambda|^2|\widehat{a}_\lambda|^2-1))^2\right]=O(\mathcal{N}_n^{-1})$, the claim (vi) follows.

Proof of (vii). We have

$$\begin{aligned}
&\int_{\mathbb{T}} H_2(\widetilde{\partial}_\ell T_n(x))H_2(\widetilde{\partial}_j\widehat{T}_n(x))\,dx\\
=\;& \frac{4}{n^2\mathcal{N}_n^2}\sum_{\lambda,\lambda',\lambda'',\lambda'''}\lambda_\ell\lambda'_\ell\lambda''_j\lambda'''_j a_\lambda\overline{a_{\lambda'}}\,\widehat{a}_{\lambda''}\overline{\widehat{a}_{\lambda'''}}\int \mathrm{e}_{\lambda-\lambda'+\lambda''-\lambda'''}(x)\,dx\\
&-\frac{2}{n\mathcal{N}_n}\sum_\lambda \lambda_\ell^2|a_\lambda|^2 - \frac{2}{n\mathcal{N}_n}\sum_\lambda \lambda_j^2|\widehat{a}_\lambda|^2+1\\
=\;& \frac{4}{n^2\mathcal{N}_n^2}\sum_{\lambda,\lambda''}\lambda_\ell^2\lambda''^2_j(|a_\lambda|^2-1)\,(|\widehat{a}_{\lambda''}|^2-1) - \frac{4}{n^2\mathcal{N}_n^2}\sum_\lambda \lambda_\ell^2\lambda_j^2 a_\lambda^2\overline{\widehat{a}_\lambda}^2\\
&+\frac{8}{n^2\mathcal{N}_n^2}\sum_{\lambda,\lambda'}\lambda_\ell\lambda'_\ell\lambda_j\lambda'_j a_\lambda\overline{a_{\lambda'}}\,\overline{\widehat{a}_\lambda}\widehat{a}_{\lambda'} - \frac{8}{n^2\mathcal{N}_n^2}\sum_\lambda \lambda_\ell^2\lambda_j^2|a_\lambda|^2\,|\widehat{a}_\lambda|^2\\
=\;& \frac{4}{\mathcal{N}_n}\left\{W_l(n)\widehat{W}_j(n)+2M_{\ell,j}(n)^2+\frac{1}{n^2\mathcal{N}_n}\sum_\lambda \lambda_\ell^2\lambda_j^2\,a_\lambda^2\,\widehat{a}_\lambda^2 - \frac{2}{n^2\mathcal{N}_n}\sum_\lambda \lambda_\ell^2\lambda_j^2|a_\lambda|^2\,|\widehat{a}_\lambda|^2\right\}.
\end{aligned}$$

Since $\mathbb{E}\left[\frac{1}{n^2\mathcal{N}_n}\sum_\lambda \lambda_\ell^2\lambda_j^2\,a_\lambda^2\,\widehat{a}_\lambda^2\right]=0$ and

$$\mathbb{E}\left[\frac{1}{n^2\mathcal{N}_n}\sum_\lambda \lambda_\ell^2\lambda_j^2|a_\lambda|^2\,|\widehat{a}_\lambda|^2\right]=\frac{1}{8}(1-\widehat{\mu}_n(4))\mathbf{1}_{\{l\neq j\}}+\frac{1}{8}(3+\widehat{\mu}_n(4))\mathbf{1}_{\{l=j\}},$$

moreover $\mathbb{E}\Big[(\frac{1}{n^2\mathcal{N}_n}\sum_\lambda \lambda_\ell^2\lambda_j^2\, a_\lambda^2\, \widehat{a}_\lambda^2)^2\Big] = O(\mathcal{N}_n^{-1})$ and $\mathbb{E}\Big[(\frac{1}{n^2\mathcal{N}_n}\sum_\lambda \lambda_\ell^2\lambda_j^2(|a_\lambda|^2\, |\widehat{a}_\lambda|^2 - 1))^2\Big] = O(\mathcal{N}_n^{-1})$, the claim (vii) follows.

Proof of (viii). We have

$$\begin{aligned}
&\int_{\mathbb{T}} \widetilde{\partial}_1 T_n(x)\widetilde{\partial}_2 T_n(x)\widetilde{\partial}_1 \widehat{T}_n(x)\widetilde{\partial}_2 \widehat{T}_n(x)\, dx \\
=\;& \frac{4}{n^2\mathcal{N}_n^2}\sum_{\lambda,\lambda',\lambda'',\lambda'''} \lambda_1\lambda_2'\lambda_1''\lambda_2''' a_\lambda \overline{a_{\lambda'}}\, \widehat{a}_{\lambda''}\overline{\widehat{a}_{\lambda'''}} \int \mathrm{e}_{\lambda-\lambda'+\lambda''-\lambda'''}(x)\, dx \\
=\;& \frac{4}{n^2\mathcal{N}_n^2}\sum_{\lambda,\lambda''} \lambda_1\lambda_2\lambda_1''\lambda_2''|a_\lambda|^2|\widehat{a}_{\lambda''}|^2 + \frac{4}{n^2\mathcal{N}_n^2}\sum_\lambda \lambda_1^2\lambda_2^2 a_\lambda^2 \overline{\widehat{a}_\lambda}^2 + \frac{4}{n^2\mathcal{N}_n^2}\sum_{\lambda\neq\pm\lambda'} \lambda_1^2{\lambda_2'}^2 a_\lambda\overline{a_{\lambda'}}\,\widehat{a}_{\lambda'}\overline{\widehat{a}_\lambda} \\
&+\frac{4}{n^2\mathcal{N}_n^2}\sum_{\lambda\neq\pm\lambda'} \lambda_1\lambda_2\lambda_1'\lambda_2' a_\lambda\overline{a_{\lambda'}}\,\widehat{a}_{\lambda'}\overline{\widehat{a}_\lambda} \\
=\;& \frac{4}{\mathcal{N}_n}\Bigg\{ W_{1,2}(n)\widehat{W}_{1,2}(n) + M_{11}(n)M_{22}(n) + M_{12}(n)^2 - \frac{2}{n^2\mathcal{N}_n}\sum_\lambda \lambda_1^2\lambda_2^2|a_\lambda|^2\, |\widehat{a}_\lambda|^2 \\
&-\frac{1}{n^2\mathcal{N}_n}\sum_\lambda \lambda_1^2\lambda_2^2 a_\lambda^2\overline{\widehat{a}_\lambda}^2 \Bigg\}.
\end{aligned}$$

Since $\mathbb{E}\left[\frac{1}{n^2\mathcal{N}_n}\sum_\lambda \lambda_1^2\lambda_2^2\, a_\lambda^2\, \widehat{a}_\lambda^2\right] = 0$ and $\mathbb{E}\left[\frac{1}{n^2\mathcal{N}_n}\sum_\lambda \lambda_1^2\lambda_2^2|a_\lambda|^2\, |\widehat{a}_\lambda|^2\right] = \frac{1}{8}(1-\widehat{\mu}_n(4))$ and moreover $\mathbb{E}\Big[(\frac{1}{n^2\mathcal{N}_n}\sum_\lambda \lambda_1^2\lambda_2^2\, a_\lambda^2\, \widehat{a}_\lambda^2)^2\Big] = O(\mathcal{N}_n^{-1})$ and $\mathbb{E}\Big[(\frac{1}{n^2\mathcal{N}_n}\sum_\lambda \lambda_1^2\lambda_2^2(|a_\lambda|^2\, |\widehat{a}_\lambda|^2 - 1))^2\Big] = O(\mathcal{N}_n^{-1})$, the claim $(viii)$ follows.

7.3 Taylor expansions for the two-point correlation function

Let us set $\mathbf{x} := (x_1, x_2) \in \mathbb{T}$. The matrix

$$\Omega_n(\mathbf{x}) := \begin{pmatrix} \frac{E_n}{2} - \frac{(\partial_1 r_n(\mathbf{x}))^2}{1-r_n(\mathbf{x})^2} & -\frac{\partial_1 r_n(\mathbf{x})\partial_2 r_n(\mathbf{x})}{1-r_n(\mathbf{x})^2} \\ -\frac{\partial_1 r_n(\mathbf{x})\partial_2 r_n(\mathbf{x})}{1-r_n(\mathbf{x})^2} & \frac{E_n}{2} - \frac{(\partial_2 r_n(\mathbf{x}))^2}{1-r_n(\mathbf{x})^2} \end{pmatrix} \tag{7.1}$$

is the covariance matrix of the random vector $\nabla T_n(\mathbf{x})$ conditioned to $T_n(\mathbf{x}) = T_n(0) = 0$ (see [K-K-W, (24)]).

We will sometimes write $\Omega = \Omega_n(\mathbf{x})$, $E = E_n$ and $r = r_n(\mathbf{x})$ for brevity. The determinant of Ω is

$$\det\Omega = \frac{E}{2}\left(\frac{E}{2} - \frac{(\partial_1 r)^2 + (\partial_2 r)^2}{1-r^2}\right). \tag{7.2}$$

It is easy to show that

$$\lim_{\|\mathbf{x}\|\to 0} \det\Omega_n(\mathbf{x}) = 0;$$

we need however the speed of convergence to 0 of $\det\Omega$. Hence, we will use a Taylor expansion argument around 0.

Lemma 7.1. *As $\|\mathbf{x}\| \to 0$, we have*

$$\det\Omega_n(\mathbf{x}) = cE_n^3\|\mathbf{x}\|^2 + E_n^4\, O(\|\mathbf{x}\|^4),$$

and hence

$$\Psi_n(\mathbf{x}) := \frac{|\Omega_n(\mathbf{x})|}{1-r_n^2(\mathbf{x})} = cE_n^2 + E_n^3 O\left(\|\mathbf{x}\|^2\right),$$

(see (2.14)) where both $c > 0$ and the constants involved in the 'O' notation do not depend on n.

Proof. Let us start by Taylor expanding r around $(0,0)$. The symmetries of Λ_n lead to

$$\begin{aligned} r(\mathbf{x}) =& r(0,0) + \frac{1}{2}\langle \mathrm{Hess}_r(0,0)\mathbf{x},\mathbf{x}\rangle + \frac{1}{4!}\partial_{1111}r(0,0)x_1^4 \\ &+ \frac{1}{4!}\partial_{2222}r(0,0)x_2^4 + \frac{1}{2!2!}\partial_{1122}r(0,0)x_1^2x_2^2 + R_n^r(\mathbf{x}), \end{aligned} \tag{7.3}$$

where by $\mathrm{Hess}_r(0,0)$ we denote the Hessian matrix of r in $(0,0)$, for $i,j,k,l = 1,2$

$$\partial_{ijkl}r := \frac{\partial^4}{\partial x_i \partial x_j \partial x_k \partial x_l} r,$$

and $R_n^r(\mathbf{x})$ is the remainder. The following estimate holds

$$R_n^r(\mathbf{x}) = O\left(\sup \|\partial^6 r_n\| \, \|\mathbf{x}\|^6\right),$$

once defining

$$\sup \|\partial^6 r_n\| := \sup_{i,j,k,l,p,q=1,2} \left\| \frac{\partial^6}{\partial_{x_i}\partial_{x_j}\partial_{x_k}\partial_{x_l}\partial_{x_p}\partial_{x_q}} r_n \right\|.$$

It is immediate that

$$\sup \|\partial^6 r_n\| \le E_n^3,$$

and hence

$$R_n^r(\mathbf{x}) = E_n^3 \, O\left(\|\mathbf{x}\|^6\right),$$

where the constants involved in the 'O' notation do not depend on n. It is crucial to recall that

$$E_n \|\mathbf{x}\|^2 = O(1),$$

which holds true for $\|\mathbf{x}\| \le 1/M$.

Some straightforward computations give

$$\mathrm{Hess}_r(0,0) = \begin{pmatrix} -\frac{E}{2} & 0 \\ 0 & -\frac{E}{2} \end{pmatrix} \tag{7.4}$$

and moreover

$$\begin{aligned} \partial_{1111}r(0,0) &= (2\pi)^4 n^2 \psi_n \\ \partial_{2222}r(0,0) &= (2\pi)^4 n^2 \psi_n \\ \partial_{1122}r(0,0) &= (2\pi)^4 n^2 \left(1/2 - \psi_n\right), \end{aligned} \tag{7.5}$$

where

$$\psi_n := \frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_1^4 = \frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_2^4.$$

Substituting (7.4) and (7.5) into (7.3) we get

$$\begin{aligned} r(\mathbf{x}) = 1 &- \frac{E}{4}(x_1^2 + x_2^2) + \frac{1}{4!}(2\pi)^4 n^2 \psi_n x_1^4 + \frac{1}{4!}(2\pi)^4 n^2 \psi_n x_2^4 \\ &+ \frac{1}{2!2!}(2\pi)^4 n^2 (1/2 - \psi_n) x_1^2 x_2^2 + E_n^3 \, O\left(\|\mathbf{x}\|^6\right), \end{aligned} \tag{7.6}$$

where the constants involved in the 'O' notation still do not depend on n. Analogously, we find that

$$\begin{aligned} \partial r_1(\mathbf{x}) &= -\tfrac{E}{2}x_1 + \tfrac{1}{3!}(2\pi)^4 n^2 \psi_n x_1^3 + \tfrac{1}{2}(2\pi)^4 n^2 (1/2 - \psi_n) x_1 x_2^2 + R_n^1(\mathbf{x}) \\ \partial r_2(\mathbf{x}) &= -\tfrac{E}{2}x_2 + \tfrac{1}{3!}(2\pi)^4 n^2 \psi_n x_2^3 + \tfrac{1}{2}(2\pi)^4 n^2 (1/2 - \psi_n) x_1^2 x_2 + R_n^2(\mathbf{x}), \end{aligned} \tag{7.7}$$

where for both the remainders $R_n^1(\mathbf{x})$ and $R_n^2(\mathbf{x})$ it holds

$$R_n^j = \sup \|\partial^6 r_n\| \cdot O(\|\mathbf{x}\|^5),$$

the constants involved in the ‘O’ notation not depending on n. Squaring (7.6) we get

$$\begin{aligned} r^2(\mathbf{x}) =& 1 + \left(\frac{E}{4}\right)^2 (x_1^2+x_2^2)^2 - \frac{E}{2}(x_1^2+x_2^2) \\ &+ 2\left(\frac{1}{4!}(2\pi)^4 n^2 \psi_n x_1^4 + \frac{1}{4!}(2\pi)^4 n^2 \psi_n x_2^4 + \frac{1}{2!2!}(2\pi)^4 n^2 (1/2-\psi_n) x_1^2 x_2^2\right) + o_n(\|\mathbf{x}\|^4) \\ =& 1 - \frac{E}{2}(x_1^2+x_2^2) + f_n(x_1,x_2) + E_n^3 \cdot O(\|\mathbf{x}\|^6), \end{aligned} \tag{7.8}$$

where $f_n(x_1,x_2)$ is defined as

$$\begin{aligned} f_n(x_1,x_2) :=& \left(\frac{E}{4}\right)^2 (x_1^2+x_2^2)^2 \\ &+ 2\left(\frac{1}{4!}(2\pi)^4 n^2 \psi_n x_1^4 + \frac{1}{4!}(2\pi)^4 n^2 \psi_n x_2^4 + \frac{1}{2!2!}(2\pi)^4 n^2 (1/2-\psi_n) x_1^2 x_2^2\right). \end{aligned}$$

From (7.8), we can write

$$1 - r^2 = \frac{E}{2}(x_1^2+x_2^2) - f_n(x_1,x_2) + E_n^3 \cdot O(\|\mathbf{x}\|^6). \tag{7.9}$$

Let us now investigate the squared derivatives $(\partial_i r)^2$, $i=1,2$. From (7.7), firstly

$$\begin{aligned} (\partial_1 r)^2 =& \left(\frac{E}{2}\right)^2 x_1^2 - E x_1 \left(\frac{1}{3!}(2\pi)^4 n^2 \psi_n x_1^3 + \frac{1}{2}(2\pi)^4 n^2 (1/2-\psi_n) x_1 x_2^2\right) \\ &+ E_n \cdot E_n^3 O(\|\mathbf{x}\|^6), \end{aligned} \tag{7.10}$$

where the constants involved in the ‘O’ notation still do not depend on n. Secondly,

$$\begin{aligned} (\partial_2 r)^2 =& \left(\frac{E}{2}\right)^2 x_2^2 - E y \left(\frac{1}{3!}(2\pi)^4 n^2 \psi_n x_2^3 + \frac{1}{2}(2\pi)^4 n^2 (1/2-\psi_n) x_1^2 x_2\right) \\ &+ E_n \cdot E_n^3 O(\|\mathbf{x}\|^6). \end{aligned} \tag{7.11}$$

For brevity, let us denote

$$a_n(x_1,x_2) := -E x_1 \left(\frac{1}{3!}(2\pi)^4 n^2 \psi_n x_1^3 + \frac{1}{2}(2\pi)^4 n^2 (1/2-\psi_n) x_1 x_2^2\right)$$

and

$$b_n(x_1,x_2) := -E x_2 \left(\frac{1}{3!}(2\pi)^4 n^2 \psi_n x_2^3 + \frac{1}{2}(2\pi)^4 n^2 (1/2-\psi_n) x_1^2 x_2\right),$$

so that we can rewrite (7.10) as

$$(\partial_1 r)^2 = \left(\frac{E}{2}\right)^2 x_1^2 + a_n(x_1,x_2) + E_n \cdot E_n^3 O(\|\mathbf{x}\|^6), \tag{7.12}$$

and (7.11) as

$$(\partial_2 r)^2 = \left(\frac{E}{2}\right)^2 x_2^2 + b_n(x_1,x_2) + E_n \cdot E_n^3 O(\|\mathbf{x}\|^6). \tag{7.13}$$

Substituting (7.9), (7.12) and (7.13) into (7.2), we have the following: for fixed n, as $\|\mathbf{x}\| \to 0$

$$
\begin{aligned}
\det\Omega &= \frac{E}{2}\left(\frac{E}{2} - \frac{(\partial_1 r)^2 + (\partial_2 r)^2}{1-r^2}\right)\\
&= \frac{E}{2}\left(\frac{E}{2} - \frac{\left(\frac{E}{2}\right)^2 (x_1^2+x_2^2) + a_n(x_1,x_2) + b_n(x_1,x_2) + E_n \cdot E_n^3 O(\|\mathbf{x}\|^6)}{\frac{E}{2}(x_1^2+x_2^2) - f_n(x_1,x_2) + E_n^3 O(\|\mathbf{x}\|^6)}\right)\\
&= \frac{E}{2}\left(\frac{E}{2} - \frac{E}{2}\frac{1 + \left(\frac{2}{E}\right)^2\left(\frac{a_n(x_1,x_2)+b_n(x_1,x_2)}{x_1^2+x_2^2} + E_n \cdot E_n^3 O(\|\mathbf{x}\|^4)\right)}{1 - \frac{2}{E}\left(\frac{f_n(x_1,x_2)}{x_1^2+x_2^2} + E_n^3 O(\|\mathbf{x}\|^4)\right)}\right)\\
&= \left(\frac{E}{2}\right)^2\left(1 - \frac{1 + \left(\frac{2}{E}\right)^2\left(\frac{a_n(x_1,x_2)+b_n(x_1,x_2)}{x_1^2+x_2^2} + E_n^4 O(\|\mathbf{x}\|^4)\right)}{1 - \frac{2}{E}\left(\frac{f_n(x_1,x_2)}{x_1^2+x_2^2} + E_n^3 O(\|\mathbf{x}\|^4)\right)}\right)\\
&= cE_n^3\|\mathbf{x}\|^2 + E_n^4\, O(\|\mathbf{x}\|^4),
\end{aligned}
$$

which concludes the proof. □

8 Appendix: ancillary results used throughout the paper

This appendix contains some useful results connected to combinatorial moment formulae and arithmetic estimates.

8.1 Leonov-Shiryaev formulae

In the proof of our variance estimates, we crucially use the following special case of the so-called **Leonov-Shiryaev combinatorial formulae** for computing joint moments. It follows immediately e.g. from [P-T, Proposition 3.2.1], by taking into account the independence of T_n and $\widehat{T}_n$, the independence of the six random variables appearing in (2.4), as well as the specific covariance structure of Hermite polynomials (see e.g. [N-P, Proposition 2.2.1]).

Proposition 8.1. *Fix $n \in S$ and write*

$$
\begin{aligned}
&(X_0(x), X_1(x), X_2(x), Y_0(x), Y_1(x), Y_2(x))\\
&\quad := (T_n(x),\, \widetilde{\partial}_1 T_n(x),\, \widetilde{\partial}_2 T_n(x), \widehat{T}_n(x),\, \widetilde{\partial}_1 \widehat{T}_n(x),\, \widetilde{\partial}_2 \widehat{T}_n(x)), \quad x \in \mathbb{T}.
\end{aligned}
$$

Consider integers $p_0, p_1, p_2, q_0, q_1, q_2 \geq 0$ and $a_0, a_1, a_2, b_0, b_1, b_2 \geq 0$, and write

$$
\begin{aligned}
(X_1^\star(x), ..., X_{p_0+p_1+p_2}^\star(x)) &:= (\underbrace{X_0(x), ..., X_0(x)}_{p_0 \text{ times}}, \underbrace{X_1(x), ..., X_1(x)}_{p_1 \text{ times}}, \underbrace{X_2(x), ..., X_2(x)}_{p_2 \text{ times}})\\
(X_1^{\star\star}(y), ..., X_{a_0+a_1+a_2}^{\star\star}(y)) &:= (\underbrace{X_0(y), ..., X_0(y)}_{a_0 \text{ times}}, \underbrace{X_1(y), ..., X_1(y)}_{a_1 \text{ times}}, \underbrace{X_2(y), ..., X_2(y)}_{a_2 \text{ times}})\\
(Y_1^\star(x), ..., Y_{q_0+q_1+q_2}^\star(x)) &:= (\underbrace{Y_0(x), ..., Y_0(x)}_{q_0 \text{ times}}, \underbrace{Y_1(x), ..., Y_1(x)}_{q_1 \text{ times}}, \underbrace{Y_2(x), ..., Y_2(x)}_{q_2 \text{ times}})\\
(Y_1^{\star\star}(y), ..., Y_{b_0+b_1+b_2}^{\star\star}(y)) &:= (\underbrace{Y_0(y), ..., Y_0(y)}_{b_0 \text{ times}}, \underbrace{Y_1(y), ..., Y_1(y)}_{b_1 \text{ times}}, \underbrace{Y_2(y), ..., Y_2(y)}_{b_2 \text{ times}}).
\end{aligned}
$$

Then, for every $x, y \in \mathbb{T}$*,*

$$\mathbb{E}\left[\prod_{j=0}^{2} H_{p_j}(X_j(x))H_{a_j}(X_j(y))\prod_{k=0}^{2} H_{q_k}(Y_k(x))H_{b_k}(Y_k(y))\right] \tag{8.1}$$
$$= \mathbf{1}_{\{p_0+p_1+p_2=a_0+a_1+a_2\}}\mathbf{1}_{\{q_0+q_1+q_2=b_0+b_1+b_2\}} \times$$
$$\times \sum_{\sigma,\pi}\left(\prod_{j=1}^{p_0+p_1+p_2}\mathbb{E}[X_j^\star(x)X_{\sigma(j)}^{\star\star}(y)]\right)\left(\prod_{k=1}^{q_0+q_1+q_2}\mathbb{E}[Y_k^\star(x)Y_{\pi(k)}^{\star\star}(y)]\right),$$

where the sum runs over all permutations σ, π *of* $\{1, ..., p_0 + p_1 + p_2\}$ *and of* $\{1, ..., q_0 + q_1 + q_2\}$*, respectively.*

8.2 Arithmetic facts

We will now present three results having an arithmetic flavour, that are extensively used in the proofs of our main findings. To this end, for every $n \in S$ we introduce the 4- and 6-**correlation set of frequencies**

$$\begin{aligned} S_4(n) &:= \{\boldsymbol{\lambda} = (\lambda, \lambda', \lambda'', \lambda''') \in \Lambda_n^4 : \lambda - \lambda' + \lambda'' - \lambda''' = 0\}, \\ S_6(n) &:= \{\boldsymbol{\lambda} = (\lambda, \lambda', \lambda'', \lambda''', \lambda'''', \lambda^v) \in \Lambda_n^6 : \lambda - \lambda' + \lambda'' - \lambda''' + \lambda^{iv} - \lambda^v = 0\}. \end{aligned}$$

The first statement exploited in our proofs yields an exact value for $|S_4(n)|$; the proof (omitted) is based on an elegant geometric argument due to Zygmund [Zy].

Lemma 8.2. *For every* $n \in S$*,* $|S_4(n)| = 3\mathcal{N}_n(\mathcal{N}_n - 1)$*.*

The second estimate involves 6-correlations, and follows from a deep result by Bombieri and Bourgain [B-B, Theorem 1] — see also [K-K-W, Theorem 2.2].

Lemma 8.3 (See [B-B]). *As* $\mathcal{N}_n \to \infty$*,*

$$|S_6(n)| = O\left(\mathcal{N}_n^{7/2}\right).$$

In our analysis of singular points we also use the following elementary fact about the behaviour of the correlation function r_n, as defined in (1.15), in a small square containing the origin.

Proposition 8.4. *Let* $n \in S$*, with* $n \geq 1$*, let* $c_n = (1000\sqrt{n})^{-1}$*, and* $Q_n := \{(x, y) \in \mathbb{R}^2 : |x|, |y| \leq c_n\}$*. Assume that* $z = (x, y) \in Q_n$ *is such that* $r_n(z) = \pm 1$*; then,* $z = 0$*.*

Proof. Assume first that $r_n(z) = 1$. Then, for every $(\lambda_1, \lambda_2) \in \Lambda_n$, one has necessarily that there exist $j, k, l \in \mathbb{Z}$ such that (i) $\lambda_1 x + \lambda_2 y = j$, (ii) $-\lambda_1 x + \lambda_2 y = k$, and (iii) $\lambda_1 y + \lambda_2 x = l$. Assume first that $\lambda_1 = 0$ (and therefore $|\lambda_2| = \sqrt{n}$): equation (i) implies that $|y| = |j|/\sqrt{n}$, and such an expression is $> c_n$ unless $j = y = 0$; similarly, equation (iii) implies that $|x| > c_n$, unless $x = l = 0$. The case where $\lambda_2 = 0$ is dealt with analogously. Now assume that $\lambda_1, \lambda_2 \neq 0$: equations (i) and (ii) imply therefore that $y = (j + k)/2\lambda_2$ and $x = (j - k)/2\lambda_1$; since $|\lambda_i| \leq \sqrt{n}$, for $i = 1, 2$, we infer that $|x|, |y| \leq c_n$ if and only if $j + k = 0 = j - k$, and therefore $x = y = j = k = 0$. If $r_n(z) = -1$, then, for every $(\lambda_1, \lambda_2) \in \Lambda_n$, one has necessarily that there exist $j, k, l \in \mathbb{Z} + \frac{1}{2}$ such that equations (i), (ii), (iii) above are verified: reasoning exactly as in the first part of the proof, we conclude that $\max\{|x|, |y|\} > c_n$, and consequently z cannot be an element of Q_n. □

Finally, we also use the following result, corresponding to a special case of [Ko, Corollary 9, p. 80]: it yields a local ersatz of Bézout theorem for systems of equations involving trigonometric polynomials. We recall that, given a smooth mapping $U : \mathbb{R}^2 \to \mathbb{R}^2$ and a point $x \in \mathbb{R}^2$ such that $U(x) = (0, 0)$, one says that x is *nondegenerate* if the Jacobian matrix of U at x is invertible.

Lemma 8.5 (See [Ko]). *Fix $n \in S$, and consider two trigonometric polynomials on $\mathbb{R}^2$:*

$$P(x) = c + \sum_{\lambda \in \Lambda_n} a_\lambda e_\lambda(x), \quad \textit{and} \quad Q(x) = c' + \sum_{\lambda \in \Lambda_n} b_\lambda e_\lambda(x),$$

where $c, c' \in \mathbb{R}$ and the complex numbers $\{a_\lambda, b_\lambda\}$ verify the following:

- *for every $\lambda \in \Lambda_n$, one has that $a_\lambda = \overline{a_{-\lambda}}$ and $b_\lambda = \overline{b_{-\lambda}}$;*
- *every solution of the system $(P(x), Q(x)) = (0, 0)$ such that $\|x\| < \pi/\sqrt{n}$ is nondegenerate.*

Then, the number of solutions of the system $(P(x), Q(x)) = (0, 0)$ contained in the open window $W := \{x \in \mathbb{R}^2 : \|x\| < \pi/\sqrt{n}\}$ is bounded by a universal constant $\alpha(n) \in (0, \infty)$ depending uniquely on $\mathcal{N}_n = |\Lambda_n|$.

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