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Asymptotic behaviour of heavy-tailed branching processes in random environments*

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Abstract

Consider a heavy-tailed branching process (denoted by Z_n) in random environments, under the condition which infers that $\mathbb{E}\log m(\xi_0)=\infty$. We show that (1) there exists no proper c_n such that $\{Z_n/c_n\}$ has a proper, non-degenerate limit; (2) normalized by a sequence of functions, a proper limit can be obtained, i.e., $y_n\left(\bar{\xi}, Z_n(\bar{\xi})\right)$ converges almost surely to a random variable $Y(\bar{\xi})$, where $Y\in(0,1)$ η-a.s.; (3) finally, we give the necessary and sufficient conditions for the almost sure convergence of $\left\{\frac{U(\bar{\xi},Z_n(\bar{\xi}))}{c_n(\bar{\xi})}\right\}$, where $U(\bar{\xi})$ is a slowly varying function that may depend on $\bar{\xi}$.

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1 Introduction

Let $\{Z_n\}$ be a Galton-Watson branching process with $Z_0=1$ and governed by the family size probability generating function $f(s)=\sum_{j=0}^{\infty}p_js^j$, where $p_1\neq 1$. Let $m=\sum_{j=0}^{\infty}jp_j=f^{'}(1-)$ denote the mean of the offspring distribution.

Martingale convergence of branching processes have been investigated extensively. Kesten and Stigum ([9]) showed that the limit of the martingale $\left\{\frac{Z_n}{m^n}\right\}$ is proper if and only if $EZ_1\log Z_1<\infty$. After that, if only the condition $EZ_1<\infty$ is fulfilled, Seneta ([11]) showed that if we use $f_n(s)$ to denote the probability generating function of Z_n, $k_n(s)=-\log f_n(e^{-s})$, $h_n(s)$ is the inverse function of $k_n(s)$, then for every $s\in(0,-\log q)$, $Z_nh_n(s)$ converges in distribution to a proper, non-degenerate law. Heyde ([8]) strengthened this result to almost sure convergence, using a martingale argument.

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When $m = \infty$, the situation is more complicated. In this case, Seneta ([11]) showed that it is never possible to find $\{c_n\}$ such that $\left\{\frac{Z_n}{c_n}\right\}$ converges in distribution to a proper, non-degenerate law. Darling ([6]) and Seneta ([12]) gave sufficient conditions for the existence of a sequence $\{c_n\}$ such that $\left\{\frac{\log(Z_n+1)}{c_n}\right\}$ converges in distribution to a non-degenerate law. Schuh and Barbour ([10]) showed that branching process with infinite mean can be classified as regular or irregular according to the property that whether there exists a sequence of constants $\{c_n\}$ such that $P\left(0 < \lim_{n\to\infty} \frac{Z_n}{c_n} < \infty\right) > 0$. In that paper, they derived necessary and sufficient conditions for the almost sure convergence of $\frac{U(Z_n)}{c_n}$, where U is a slow varying function, moreover, the distribution function of the limit satisfies a $Poincaré$ functional equation.

When this model is extended to a random environment, the corresponding martingale convergence results have been proved by Tanny([14], [15]). The Kesten-Stigum type theorem was proved in Tanny ([15]), $\lim_{n\to\infty} \frac{Z_n}{\pi_n} = W \ w.p.1$, where W is proper and non-degenerate if and only if $E(Z_1 \log^+ Z_1/m(\xi_0)) < \infty$ when the environmental sequence $\bar{\xi} = (\xi_0, \xi_1, \cdots)$ is i.i.d. (Theorem 2, [15]), where $m(\xi_i)$ is the expected number of offspring of particle conditioned on the environment ξ_i, and $\pi_n := \Pi_{i=0}^{n-1} m(\xi_i)$. The Seneta-Heyde type theorem was considered in Tanny ([14]) if the environmental sequence $\bar{\xi} = (\xi_0, \xi_1, \cdots)$ is stationary and ergodic and satisfies $\mathbb{E}|\log m(\xi_0)| < \infty$, then there exists a sequence of random variables $c_n(\bar{\xi})$, depending only on the environment sequence $\bar{\xi}$ such that $\lim_{n\to\infty} \frac{Z_n}{c_n} = W \ w.p.1$ and W is proper and non-degenerate, i.e., $\mathbb{P}(0 < W < \infty|\bar{\xi}) = 1 - q(\bar{\xi})$, where $q(\bar{\xi})$ is the extinction probability conditioned on $\bar{\xi}$.

In the present paper, we are interested in the case $\mathbb{E}|\log m(\xi_0)| = \infty$. We investigate the asymptotic behaviors of branching processes in random environments under the condition **(A2)** (which infers $\mathbb{E}|\log m(\xi_0)| = \infty$). Part of the results in Schuh and Barbour ([10]) will be extended to this random environment situation, in particular, (1) we show that for $a.s.\ \bar{\xi}$, there exists no $\{c_n(\bar{\xi})\}$ such that $\left\{\frac{Z_n(\bar{\xi})}{c_n(\bar{\xi})}\right\}$ converges to a proper random variable; (2) Z_n can be normalized by a sequence of functions, i.e., let $y_n(\bar{\xi}, x) = f_{\xi_0}\left(\cdots\left(f_{\xi_{n-1}}\left(e^{-\frac{1}{x}}\right)\right)\cdots\right)$, then $y_n(\bar{\xi}, Z_n(\bar{\xi}))$ converges almost surely to a proper and non-degenerate random variable $Y(\bar{\xi})$, where $Y \in (0,1)$ η-a.s.; (3) we give the necessary and sufficient conditions for the almost sure convergence of $\left\{\frac{U(\bar{\xi}, Z_n(\bar{\xi}))}{c_n(\bar{\xi})}\right\}$, where $U(\bar{\xi})$ is a slowly varying function that may depend on $\bar{\xi}$.

2 Description of the model and main results

Let $\bar{\xi} = \{\xi_n : n \in \mathbb{Z}\}$ be a sequence of independent and identically distributed probability distributions on nonnegative integers, where

$$\xi_n = \left\{\xi_n^{(0)}, \xi_n^{(1)}, \cdots\right\}, \qquad \xi_n^{(i)} \geqslant 0, \qquad \sum_{i=0}^{\infty} \xi_n^{(i)} = 1.$$

The law of the environment $\bar{\xi}$ is given by η.

Let $Z_0 = 1$, Z_n be the sum of Z_{n-1} independent random variables, each of which has distribution ξ_{n-1}. Then the sequence of random variables $Z_0, Z_1, \cdots$ is called a branching process in the random environment $\bar{\xi}$. We use $P_{\bar{\xi}}$ to denote the probability when the environment $\bar{\xi}$ is fixed. As usual, $P_{\bar{\xi}}$ is called quenched law. The total probability $\mathbb{P}$, which is usually called annealed law, is given by

$$\mathbb{P}(\cdot) := \int P_{\bar{\xi}}(\cdot)\eta(d\bar{\xi}).$$

Assumption 2.1. ***(A1)*** $\eta(\xi_0^{(0)} = 0) = 1.$

Remark 2.2. **(A1)** ensures that each particle produces at least one particle, then this is an increasing branching process in the random environment, i.e., the extinction probability $q(\bar{\xi}) = 0$. We propose this assumption to simplify our statement, but in fact this assumption can be removed by using $Theorem$ 2 in [14] to get the main result in our passage on the non-extinction event.

Some notations:

- $m(\xi_0) := E_{\xi_0}(Z_1) = \sum_{y=0}^{\infty} y\xi_0^{(y)}; \quad f_{\xi_i}(s) := \sum_{k=0}^{\infty} \xi_i^{(k)} s^k$;
- $k_{\xi_i}(s) := -\log f_{\xi_i}(e^{-s}); \quad h_{\xi_i}(s) := -\log f_{\xi_i}^{(-1)}(e^{-s})$, $0 < s < \infty$;
- $k_0(\bar{\xi}, s) := s; \quad h_0(\bar{\xi}, s) := s$;
- $k_n(\bar{\xi}, s) := k_{\xi_0}(k_{\xi_1}(\cdots(k_{\xi_{n-1}}(s))\cdots) = -\log f_{\xi_0}\left(f_{\xi_1}\left(\cdots\left(f_{\xi_{n-1}}(e^{-s})\right)\cdots\right)\right)$ $(n \geq 1)$,
- $h_n(\bar{\xi}, s) := h_{\xi_{n-1}}(\cdots(h_{\xi_0}(s))\cdots) = -\log f_{\xi_{n-1}}^{(-1)}\left(\cdots\left(f_{\xi_0}^{(-1)}\left(e^{-s}\right)\right)\cdots\right)$ $(n \geq 1)$;
- θ is the shift operator, for any $\bar{\xi} = \{\xi_0, \xi_1, \cdots\}$, $\theta\bar{\xi} := \{\xi_1, \xi_2, \cdots\}$;
- $d\left(\bar{\xi}, s\right) := \lim_{n\to\infty} \frac{h_{n+1}(\bar{\xi},s)}{h_n(\theta\bar{\xi},s)} = \lim_{n\to\infty} \frac{h_{\xi_n}(\cdots(h_{\xi_0}(s))\cdots)}{h_{\xi_n}(\cdots(h_{\xi_1}(s))\cdots)}$.

Assumption 2.3. ***(A2)*** *$\eta(D) = 1$, where $D = \left\{\bar{\xi} : \text{for any } s \in (0,\infty),\ d(\bar{\xi}, s) = 0\right\}$.*

Remark 2.4. Tanny ($Lemma$ 2.4, [14]) proved that for $a.e.\ \bar{\xi}$, $d(\bar{\xi}, s)$ always exists. What's more, if $\mathbb{E}|\log m(\xi_0)| < \infty$, then $0 < d(\bar{\xi}, s) \leqslant 1 \;\; w.p.1$. Thus under the assumption **(A2)** we know $\mathbb{E}|\log m(\xi_0)| = \infty$. Actually, we conjecture **(A2)** is equivalent with $\mathbb{E}|\log m(\xi_0)| = \infty$, but unfortunately we have not proved it yet.

An example is given in Example 4.3, where the Assumption **(A1)** and **(A2)** are fulfilled.

1 No proper limit exists

If $\mathbb{E}|\log m(\xi_0)| < \infty$, Tanny ([14]) proved that there exists a sequence of random variables $c_n(\bar{\xi})$, depending only on the environment sequence $\bar{\xi}$ such that $\lim_{n\to\infty} \frac{Z_n}{c_n} = W$ $w.p.1$ and W is proper and non-degenerate, i.e., $\mathbb{P}\left(0 < W < \infty|\bar{\xi}\right) = 1 - q(\bar{\xi})$, where $q(\bar{\xi})$ is the extinction probability conditioned on $\bar{\xi}$. The key step of the proof is in Tanny ($Lemma$ 2.4, [14]), where showed that if $\mathbb{E}|\log m(\xi_0)| < \infty$, then $0 < d(\bar{\xi}, s) \leqslant 1 \;\; w.p.1$.

We are interested in the other situation when $d(\bar{\xi}, s) = 0 \;\; w.p.1$, $0 < s < \infty$, i.e., Assumption 2.3 (where (A2) infer that $\mathbb{E}|\log m(\xi_0)| = \infty$), we will see that for a.e. $\bar{\xi}$, no $c_n(\bar{\xi})$ exists such that $Z_n(\bar{\xi})/c_n(\bar{\xi})$ has a proper and non-degenerate limit. At first, we show that for any $s \in (0,\infty)$, $h_n(\bar{\xi}, s)$ is not the suitable norming for $Z_n(\bar{\xi})$ as the following,

Theorem 2.5. *For any $s \in (0,\infty)$, $Z_n(\bar{\xi})h_n(\bar{\xi}, s)$ converges to $W(\bar{\xi}, s)$ $w.p.1$. If $\eta(D) > 0$ then $P_{\bar{\xi}}\left(W(\bar{\xi}, s) = \infty\right) > 0,\ P_{\bar{\xi}}\left(W(\bar{\xi}, s) = 0\right) > 0,\ \eta$-a.e..*

Remark 2.6. Note that the condition in Theorem 2.5 is weaker than condition (A2). We conjecture that $\eta(D) = 0$ or 1.

Based on these facts, it is necessary to classify $s \in (0,\infty)$ as two different types of points from the following definition.

Definition 2.7. *A point $s \in (0,\infty)$ is called $\bar{\xi}$-regular if $P_{\bar{\xi}}\left(W(\bar{\xi}, s) \in \{0,\infty\}\right) = 1$, and $\bar{\xi}$-irregular otherwise.*

Definition 2.8. *The branching process $\left\{Z_n(\bar{\xi})\right\}$ is called $\bar{\xi}$-regular if all $0 < s < \infty$ are $\bar{\xi}$-regular and $\bar{\xi}$-irregular otherwise.*

We have the following 0-1 law.

Theorem 2.9. *Let $A = \left\{\bar{\xi} : Z_n(\bar{\xi}) \text{ is } \bar{\xi}\text{-regular}\right\}$, then $\eta(A) = 0$ or 1.*

In section 3.2, we will discuss the limit behavior of $Z_n(\bar{\xi})h_n(\bar{\xi},s)$ in details, and finally conclude that no $c_n(\bar{\xi})$ exists such that $Z_n(\bar{\xi})/c_n(\bar{\xi})$ has a proper and non-degenerate limit.

2 Normalized by a sequence of functions

Since for η-*a.e.* $\bar{\xi}$, $\lim_{n\to\infty} Z_n(\bar{\xi})/c_n(\bar{\xi})$ is never a proper, non-degenerate random variable, we now consider other possibilities for normalizing $Z_n(\bar{\xi})$.

For $0 \leqslant x < \infty$, let

$$y_n\left(\bar{\xi},x\right) = f_{\xi_0}\left(\cdots\left(f_{\xi_{n-1}}\left(e^{-\frac{1}{x}}\right)\right)\cdots\right),\ \ \left(y_n(\bar{\xi},0) = f_n(\bar{\xi},0)\right).$$

Theorem 2.10. *Under Assumption 2.3 (1)* $y_n\left(\bar{\xi},Z_n(\bar{\xi})\right)$ *converges almost surely to a random variable* $Y(\bar{\xi})$. *Furthermore, if* s_r *is a* $\bar{\xi}$*-regular point and* $x_r = e^{-s_r}$, *then* $P_{\bar{\xi}}\left(Y(\bar{\xi}) \leqslant x_r\right) = x_r$ *and* $P_{\bar{\xi}}\left(Y(\bar{\xi}) = x_r\right) = 0$.
(2) $Y \in (0,1)$ η*-a.s.. In particular, if* $\{Z_n\}$ *is a regular branching process, then* Y *is uniformly distributed on* $(0,1)$.

3 Normalized by an increasing slowly varying function

Theorem 2.11. *For* η*-a.e.* $\bar{\xi}$, *let* $U(\bar{\xi},x):[0,\infty)\to[0,\infty)$ *be an increasing slowly varying function with* $U(\bar{\xi},0)=0$, $\lim_{x\to\infty}U(\bar{\xi},x)=\infty$, *and* $\{c_n(\bar{\xi})\}$ *a sequence of positive constants. Then under Assumption 2.3 we have:*
(1) *For* η*-a.e.* $\bar{\xi}$, *if*

$$H(\bar{\xi},s) := \lim_{n\to\infty}\left(U(\bar{\xi},1/h_n(\bar{\xi},s))/c_n(\bar{\xi})\right) \tag{2.1}$$

exists for all but at most countably many $s\in(0,\infty)$, *then for* η*-a.e.* $\bar{\xi}$, $U\left(\bar{\xi},Z_n(\bar{\xi})\right)/c_n(\bar{\xi})$ *converges to* $H\left(\bar{\xi},T(\bar{\xi})\right)$ $\left(T(\bar{\xi}) = \sup\left\{s|0<s<\infty \text{ and } W(\bar{\xi},s)<1\right\}\right)$ *almost surely.*
(2) *On the other hand, if for* η*-a.e.* $\bar{\xi}$, $\left\{U(\bar{\xi},Z_n(\bar{\xi}))/c_n(\bar{\xi})\right\}$ *converges in distribution to a distribution function* $F_{\bar{\xi}}$, *and define*

$$G\left(\bar{\xi},x\right) = \inf\left\{y \mid 0\leqslant y<\infty \text{ and } F_{\bar{\xi}}(y)\geqslant x\right\},\quad 0\leqslant x<\infty. \tag{2.2}$$

Then for η*-a.e.* $\bar{\xi}$,

$$\lim_{n\to\infty}\left(U(\bar{\xi},1/h_n(\bar{\xi},s))/c_n(\bar{\xi})\right) = G(\bar{\xi},e^{-s}) \tag{2.3}$$

exists for all the points $s\in(0,\infty)$ *such that* $G(\bar{\xi})$ *is continues at* e^{-s} *(Since* $G(\bar{\xi})$ *is an increasing function, these are all but at most countably many* $s\in(0,\infty)$*).*
(3) *Furthermore, under the condition of (2), if for* η*-a.e.* $\bar{\xi}$, $U(\bar{\xi},x) = U(\theta\bar{\xi},x)$ *and for* η*-a.e.* $\bar{\xi}$, $F_{\bar{\xi}}$ *satisfies for any* $0<x<\infty$,

$$0<F_{\bar{\xi}}(x)<1,\qquad \lim_{x\to 0}F_{\bar{\xi}}(x)=0,\qquad \lim_{x\to\infty}F_{\bar{\xi}}(x)=1. \tag{2.4}$$

Then,

$$\lim_{n\to\infty}\frac{c_{n-1}(\theta\bar{\xi})}{c_n(\bar{\xi})} = \alpha(\bar{\xi})>0 \tag{2.5}$$

exists, and

$$G\left(\bar{\xi},e^{-s}\right)/G\left(\theta\bar{\xi},e^{-h_{\xi_0}(s)}\right) = \alpha\left(\bar{\xi}\right)\qquad \text{for } s\in(0,\infty). \tag{2.6}$$

What's more, for η*-a.e.* $\bar{\xi}$, *the distribution function* $F_{\bar{\xi}}$ *and* $F_{\theta\bar{\xi}}$ *satisfy the functional equation*

$$F_{\bar{\xi}}\left(\alpha(\bar{\xi})u\right) = f_{\xi_0}\left(F_{\theta\bar{\xi}}(u)\right),\ \ 0\leqslant u<\infty, \tag{2.7}$$

where $\alpha(\bar{\xi})$ *is as in* (2.5).

The paper is arranged as the following. All the above results will be proved in section 3. Based on Theorem 2.5 (which will be proved in section 3.1), we will give the classification for $s \in (0,\infty)$ as the regular and irregular point in section 3.2, and some facts for the regular and irregular point will be pointed out, but the proof will omit as it is similar as those in Schuh and Barbour ([10]). In section 3.3, a proper limit will be obtained when Z_n is normalized by a sequence of functions, i.e., Theorem 2.10 will be proved. In section 3.4, we will discuss the necessary and sufficient conditions for the almost sure convergence of $\left\{\frac{U(\bar{\xi},Z_n(\bar{\xi}))}{c_n(\bar{\xi})}\right\}$, where $U(\bar{\xi})$ is a slowly varying function that may depends on $\bar{\xi}$, i.e., Theorem 2.11 will be proved. In section 4, we will give a sufficient criteria for regular process, and finally we give an Example 4.3 to illustrate our results in this paper.

3 Proofs

3.1 Proof of Theorem 2.5

Proof. (i) Denote by $\mathscr{F}_n(\bar{\xi})$ the σ-field generated by $Z_0, Z_1, \cdots, Z_n$ and $\bar{\xi}$, let

$$X_n(\bar{\xi},s) = e^{-Z_n(\bar{\xi})h_n(\bar{\xi},s)}. \tag{3.1}$$

Then it is easy to check that $\left\{X_n(\bar{\xi},s), \mathscr{F}_n(\bar{\xi})\right\}_{n=0}^{\infty}$ is a martingale bounded between 0 and 1, by the Martingale Convergence Theorem,

$$X(\bar{\xi},s) := \lim_{n\to\infty} X_n(\bar{\xi},s) \quad \text{ exists w.p.1.}$$

By the property of martingale,

$$\mathbb{E}(X_n(\bar{\xi},s)|\bar{\xi}) = \mathbb{E}(X_0(\bar{\xi},s)|\bar{\xi}) = e^{-s}$$

$$\mathbb{E}(X(\bar{\xi},s)|\bar{\xi}) = \lim_{n\to\infty} \mathbb{E}(X_n(\bar{\xi},s)|\bar{\xi}) = e^{-s} \tag{3.2}$$

Therefore for any $s \in (0,\infty)$, $Z_n(\bar{\xi})h_n(\bar{\xi},s)$ converges to $W(\bar{\xi},s) := -\log X(\bar{\xi},s)$ $w.p.1.$

(ii) Let

$$W_n(\bar{\xi},s) = Z_n(\bar{\xi})h_n(\bar{\xi},s),$$

then $X_n(\bar{\xi},s)^u = e^{-uW_n(\bar{\xi},s)} = e^{-uZ_n(\bar{\xi})h_n(\bar{\xi},s)}$ and

$$\begin{aligned} E_{\bar{\xi}}\left[X_n(\bar{\xi},s)^u\right] &= E_{\bar{\xi}}\left[e^{-uZ_n(\bar{\xi})h_n(\bar{\xi},s)}\right] \\ &= f_{\xi_0}\left(\cdots\left(f_{\xi_{n-1}}\left(e^{-uh_n(\bar{\xi},s)}\right)\right)\cdots\right) \\ &= f_{\xi_0}\left(f_{\xi_1}\left(\cdots\left(f_{\xi_{n-1}}\left(e^{-uh_{n-1}(\theta\bar{\xi},h_{\xi_0}(s))}\right)\right)\cdots\right)\right) \\ &= f_{\xi_0}\left(E_{\theta\bar{\xi}}\left(X_{n-1}(\theta\bar{\xi},h_{\xi_0}(s))^u\right)\right). \end{aligned} \tag{3.3}$$

Let $\chi(u;\bar{\xi},s) = E_{\bar{\xi}}(X(\bar{\xi},s)^u)$, then n goes to infinity in (3.3) yields

$$\chi(u;\bar{\xi},s) = f_{\xi_0}\left(\chi(u;\theta\bar{\xi},h_{\xi_0}(s))\right). \tag{3.4}$$

It is easily seen that

$$\lim_{u\downarrow 0}\chi(u;\bar{\xi},s) = \lim_{u\downarrow 0} E_{\bar{\xi}}\left(e^{-uW(\bar{\xi},s)}\right) = P_{\bar{\xi}}\left(W(\bar{\xi},s)<\infty\right). \tag{3.5}$$

Using the functional relation (3.4) with equation (3.5) yields:

$$P_{\bar{\xi}}\left(W(\bar{\xi},s)<\infty\right) = f_{\xi_0}\left(P_{\theta\bar{\xi}}(W(\theta\bar{\xi},h_{\xi_0}(s))<\infty)\right). \tag{3.6}$$

Let

$$A = \left\{\bar{\xi} : \text{there exists } s \text{ such that } P_{\bar{\xi}}(W(\bar{\xi}, s) < \infty) = 1\right\}.$$

Combined with (3.6) and the property of $f_{\xi_0}(s)$, we have $\theta A = A$, $i.e.$ A is a θ-invariant set, since θ is ergodic,

$$\eta(A) = 0 \quad \text{or} \quad 1. \tag{3.7}$$

Then we only need to prove $\eta(A) \neq 1$.

Since

$$E_{\bar{\xi}}\left[X_n(\bar{\xi}, s)^u\right] = f_{\xi_0}\left(f_{\xi_1}\left(\cdots\left(f_{\xi_{n-1}}\left(e^{-uh_{n-1}(\theta\bar{\xi},s)\frac{h_n(\bar{\xi},s)}{h_{n-1}(\theta\bar{\xi},s)}}\right)\right)\cdots\right)\right), \tag{3.8}$$

let $n \to \infty$, we have

$$\chi(u; \bar{\xi}, s) = f_{\xi_0}\left(\chi(ud(\bar{\xi}, s); \theta\bar{\xi}, s)\right). \tag{3.9}$$

Consequently for any $\bar{\xi} \in D,\ 0 < s < \infty$, $u > 0$, we can get that

$$\chi(u; \bar{\xi}, s) = f_{\xi_0}\left(\chi(0; \theta\bar{\xi}, s)\right).$$

Using the fact that

$$\lim_{u\uparrow\infty} \chi(u; \bar{\xi}, s) = \lim_{u\uparrow\infty} E_{\bar{\xi}}\left(e^{-uW(\bar{\xi},s)}\right) = P_{\bar{\xi}}\left(W(\bar{\xi}, s) = 0\right),$$

we have

$$\text{for any } \bar{\xi} \in D,\quad P_{\bar{\xi}}\left(W(\bar{\xi}, s) = 0\right) = f_{\xi_0}\left(\chi(0; \theta\bar{\xi}, s)\right). \tag{3.10}$$

Note that $E_{\bar{\xi}}\left(e^{-W(\bar{\xi},s)}\right) = e^{-s}$, which implies $P_{\bar{\xi}}\left(W(\bar{\xi}, s) = 0\right) < 1$. From (3.10) we have $\chi\left(0; \theta\bar{\xi}, s\right) < 1$, thus

$$\text{for any } \bar{\xi} \in D,\quad P_{\theta\bar{\xi}}\left(W(\theta\bar{\xi}, s) = \infty\right) > 0.$$

This means for any $\bar{\xi} \in D,\ \theta\bar{\xi} \in A^c$. Hence

$$\eta(A) = 1 - \eta(A^c) \leqslant 1 - \eta(\theta D) = 1 - \eta(D) < 1,$$

recall (3.7), we have $\eta(A) = 0,\ i.e.$,

$$\text{for any } s \in (0, \infty),\quad P_{\bar{\xi}}\left(W(\bar{\xi}, s) = \infty\right) > 0,\quad \eta\text{-a.e.}.$$

(iii) Let

$$B = \left\{\bar{\xi} : \text{there exists } s \text{ such that } P_{\bar{\xi}}(W(\bar{\xi}, s) = 0) = 0\right\}.$$

Similar to (ii) we can get that

$$P_{\bar{\xi}}\left(W(\bar{\xi}, s) = 0\right) = f_{\xi_0}\left(P_{\theta\bar{\xi}}(W(\theta\bar{\xi}, h_{\xi_0}(s)) = 0)\right), \tag{3.11}$$

$$\chi\left(\frac{u}{d(\bar{\xi}, s)}; \bar{\xi}, s\right) = f_{\xi_0}\left(\chi(u; \theta\bar{\xi}, s)\right). \tag{3.12}$$

From (3.11), we know that $\theta B = B$, then $\eta(B) = 0$ *or* 1.

For any $\bar{\xi} \in D,\ 0 < s < \infty$, from (3.12), we get for any $u > 0$,

$$f_{\xi_0}\left(\chi(u;\theta\bar{\xi},s)\right) = \chi\left(\infty;\bar{\xi},s\right).$$

Let u goes to 0, we have

$$f_{\xi_0}\left(P_{\theta\bar{\xi}}(W(\theta\bar{\xi},s) < \infty)\right) = \chi\left(\infty;\bar{\xi},s\right). \tag{3.13}$$

We note that $E_{\theta\bar{\xi}}\left(e^{-W(\theta\bar{\xi},s)}\right) = e^{-s}$ implies $P_{\theta\bar{\xi}}(W(\theta\bar{\xi},s) < \infty) > 0$. Combined with (3.13) we have $\chi(\infty;\bar{\xi},s) > 0$, thus

$$\text{for any } \bar{\xi} \in D,\ 0 < s < \infty,\ \ P_{\bar{\xi}}\left(W(\bar{\xi},s) = 0\right) > 0.$$

This means for any $\bar{\xi} \in D,\ \bar{\xi} \in B^c$. Then

$$\eta(B) \leqslant 1 - \eta(D) < 1.$$

Accordingly $\eta(B) = 0,\ i.e.$,

$$\text{for any } s \in (0,\infty), \quad P_{\bar{\xi}}\left(W(\bar{\xi},s) = 0\right) > 0, \quad \eta\text{-a.e.}. \qquad \square$$

3.2 $\bar{\xi}$-regular and $\bar{\xi}$-irregular points

From Theorem 2.5, we know that for any $s \in (0,\infty)$,

$$P_{\bar{\xi}}\left(\lim_{n\to\infty} Z_n(\bar{\xi})h_n(\bar{\xi},s) = \infty\right) > 0,\ \ P_{\bar{\xi}}\left(\lim_{n\to\infty} Z_n(\bar{\xi})h_n(\bar{\xi},s) = 0\right) > 0 \quad \eta\text{-a.e.},$$

then it is necessary to distinguish between two types of points. (Recall Definition 2.7)

We can get the following theorem which gives a necessary and sufficient condition for a point to be regular. The proof is almost the same as that of Theorem 1.1.2 in [10], we omit the details.

Theorem 3.1. *$s \in (0,\infty)$ is $\bar{\xi}$-regular if and only if $\lim_{n\to\infty} \frac{h_n(\bar{\xi},t)}{h_n(\bar{\xi},s)} = 0$ for all $0 < t < s$, (or equivalently $\lim_{n\to\infty} \frac{h_n(\bar{\xi},t)}{h_n(\bar{\xi},s)} = \infty$ for all $s < t < \infty$).* $\square$

Remark 3.2. From Theorem 3.1 and the fact that

$$h_n(\bar{\xi},s) = h_{n-k}\left(\theta^k\bar{\xi}, h_k(\bar{\xi},s)\right),$$

we know that if $s \in (0,\infty)$ is $\bar{\xi}$-regular (irregular), then $h_k(\bar{\xi},s)$ is $\theta^k\bar{\xi}$-regular (irregular).

Lemma 3.3. *The set of the irregular points is open. More precisely, if s_i is $\bar{\xi}$-irregular, then an open interval $I(\bar{\xi},s_i) = (s_1,s_2)$ of maximal length exists, such that $s_i \in I(\bar{\xi},s_i)$ and all $s \in I(\bar{\xi},s_i)$ are $\bar{\xi}$-irregular.*

If we define $l(\bar{\xi},s) = \lim_{n\to\infty}\left(h_n(\bar{\xi},s)/h_n(\bar{\xi},s_i)\right)$ for all $s \in [s_1,s_2]$ then $l(\bar{\xi},s)$ is a continuous, strictly increasing function with $l(\bar{\xi},s_1) = 0$ and $l(\bar{\xi},s_2) = \infty$. s_1 and s_2 are both $\bar{\xi}$-regular.

The proof is similar as Lemma 1.1.5 in [10], we omit the details. $\square$

Lemma 3.4. *Under condition (A2) ($i.e.$, for any $\bar{\xi}$ that satisfies for any $s \in (0,\infty)$, $d(\bar{\xi},s) = 0$), every interval $[h_{\xi_0}(s), s],\ 0 < s < \infty$, contains at least one $\theta\bar{\xi}$-regular point s_r.*

Proof. Define

$$s_r = \sup\left\{t \middle|\ h_{\xi_0}(s) \leqslant t \leqslant s \text{ and } \lim_{n\to\infty}\frac{h_n(\theta\bar{\xi},t)}{h_n(\theta\bar{\xi},s)} = 0\right\},$$

$d(\bar{\xi}, s) = 0$ tells us

$$d(\bar{\xi}, s) = \lim_{n\to\infty} \frac{h_{n+1}(\bar{\xi}, s)}{h_n(\theta\bar{\xi}, s)} = \lim_{n\to\infty} \frac{h_n\left(\theta\bar{\xi}, h_{\xi_0}(s)\right)}{h_n(\theta\bar{\xi}, s)} = 0.$$

Then s_r exists and $h_{\xi_0}(s) \leqslant s_r \leqslant s$.

If $\lim_{n\to\infty} \frac{h_n(\theta\bar{\xi}, s_r)}{h_n(\theta\bar{\xi}, s)} = 0$, then $\lim_{n\to\infty} \frac{h_n(\theta\bar{\xi}, s_r)}{h_n(\theta\bar{\xi}, t)} = 0$ for all $t > s_r$, since $\lim_{n\to\infty} \frac{h_n(\theta\bar{\xi}, t)}{h_n(\theta\bar{\xi}, s)} > 0$; and if $\lim_{n\to\infty} \frac{h_n(\theta\bar{\xi}, s_r)}{h_n(\theta\bar{\xi}, s)} > 0$, then $\lim_{n\to\infty} \frac{h_n(\theta\bar{\xi}, t)}{h_n(\theta\bar{\xi}, s_r)} = 0$ for all $t < s_r$, since $\lim_{n\to\infty} \frac{h_n(\theta\bar{\xi}, t)}{h_n(\theta\bar{\xi}, s)} = 0$. In both cases s_r is $\theta\bar{\xi}$-regular by Theorem 3.1. □

We also distinguish the branching process $\{Z_n(\bar{\xi})\}$ between two types (see Definition 2.8). And we have the following 0-1 law.

Theorem 3.5. *Let $A = \{\bar{\xi} : Z_n(\bar{\xi})$ is $\bar{\xi}$-regular$\}$, then $\eta(A) = 0$ or 1.*

Proof. From Definition 2.8 and Theorem 3.1 we know that for any $\bar{\xi} \in A$, $0 < s < \infty$, if $0 < t < s$, then

$$\lim_{n\to\infty} \frac{h_n(\bar{\xi}, t)}{h_n(\bar{\xi}, s)} = 0.$$

On the other hand, for any $\bar{\xi} \in A$, if $\{Z_n(\theta\bar{\xi})\}$ is $\theta\bar{\xi}$-irregular, then there exists $s,\ 0 < t < s$, such that $\lim_{n\to\infty} \frac{h_n(\theta\bar{\xi}, t)}{h_n(\theta\bar{\xi}, s)} > 0, i.e.,$

$$\lim_{n\to\infty} \frac{h_{\xi_n}\left(\cdots h_{\xi_1}(h_{\xi_0}(k_{\xi_0}(t)))\cdots\right)}{h_{\xi_n}\left(\cdots h_{\xi_1}(h_{\xi_0}(k_{\xi_0}(s)))\cdots\right)} > 0. \tag{3.14}$$

Combining with the monotonicity of k_{ξ_0}, we have $k_{\xi_0}(t) < k_{\xi_0}(s)$. (3.14) means $k_{\xi_0}(s)$ is a $\bar{\xi}$-irregular point, as a consequence $\{Z_n(\bar{\xi})\}$ is $\bar{\xi}$-irregular, which contradicts to the fact that $\bar{\xi} \in A$. Thus, for any $\bar{\xi} \in A,\ \theta\bar{\xi} \in A$. In a similar way we see that for any $\theta\bar{\xi} \in A,\ \bar{\xi} \in A$. So $\theta A = A$, i.e., A is a θ-invariant set. Since θ is ergodic, we infer that $\eta(A) = 0$ or 1. □

Thus we can make the following definition classifying the processes.

Definition 3.6. *The branching process in random environment $\{Z_n\}$ is called regular branching process if $\eta\left(\{\bar{\xi} : Z_n(\bar{\xi})\text{ is }\bar{\xi}\text{-regular}\}\right) = 1$, otherwise irregular.*

The following results can also be proved similar as that of Theorem 1.1.7 in [10], we omit the details.

Theorem 3.7. (1) *Let $c_n(\bar{\xi})$ be a sequence of positive constants, such that $Z_n(\bar{\xi})/c_n(\bar{\xi})$ converges in distribution, and let $F_{\bar{\xi}}$ denote the distribution function of the limit. Then there are four cases:*

(a) $F_{\bar{\xi}}(0) = 1$ $\Longrightarrow$ $\lim_{n\to\infty} h_n(\bar{\xi}, s)c_n(\bar{\xi}) = \infty$ *for all* $0 < s < \infty$;

(b) $F_{\bar{\xi}}(0) = F_{\bar{\xi}}(\infty) = 0$ $\Longrightarrow$ $\lim_{n\to\infty} h_n(\bar{\xi}, s)c_n(\bar{\xi}) = 0$ *for all* $0 < s < \infty$;

(c) $1 > F_{\bar{\xi}}(0) = F_{\bar{\xi}}(\infty) > 0$ $\Longrightarrow$ *a $\bar{\xi}$-regular point s_r exists such that*

$$\lim_{n\to\infty} h_n(\bar{\xi}, t)c_n(\bar{\xi}) = \begin{cases} 0 & \text{if } 0 < t < s_r \\ \infty & \text{if } s_r < t < \infty; \end{cases}$$

(d) $F_{\bar{\xi}}(0) < F_{\bar{\xi}}(\infty)$ $\Longrightarrow$ *a $\bar{\xi}$-irregular point s_i exists such that*

$$\lim_{n\to\infty} h_n(\bar{\xi}, s_i)c_n(\bar{\xi}) = 1.$$

(2) *On the other hand, in (1) if one of the conditions on the right-hand side is satisfied, then* $Z_n(\bar{\xi})/c_n(\bar{\xi})$ *converges almost surely to a (possibly infinite-valued) random variable* $W(\bar{\xi})$*; more specifically there are four cases:*

$$(a)\ \lim_{n\to\infty} h_n(\bar{\xi},s)c_n(\bar{\xi}) = \infty \quad \textit{for all} \quad 0<s<\infty \qquad \Longrightarrow \qquad \lim_{n\to\infty}\frac{Z_n}{c_n} = 0 \ \ P_{\bar{\xi}}\text{-a.s.};$$

$$(b)\ \lim_{n\to\infty} h_n(\bar{\xi},s)c_n(\bar{\xi}) = 0 \quad \textit{for all} \quad 0<s<\infty \qquad \Longrightarrow \qquad \lim_{n\to\infty}\frac{Z_n}{c_n} = \lim_{n\to\infty} Z_n(\bar{\xi}) \ \ P_{\bar{\xi}}\text{-a.s.};$$

(c) *a* $\bar{\xi}$*-regular point* s_r *exists such that*

$$\lim_{n\to\infty} h_n(\bar{\xi},t)c_n(\bar{\xi}) = \begin{cases} 0 & \textit{if } 0<t<s_r \\ \infty & \textit{if } s_r<t<\infty \end{cases} \qquad \Longrightarrow \qquad \lim_{n\to\infty}\frac{Z_n}{c_n} = W(\bar{\xi},s_r) \ \ P_{\bar{\xi}}\text{-a.s.};$$

(d) *a* $\bar{\xi}$*-irregular point* s_i *exists such that*

$$\lim_{n\to\infty} h_n(\bar{\xi},s_i)c_n(\bar{\xi}) = 1 \qquad \Longrightarrow \qquad \lim_{n\to\infty}\frac{Z_n}{c_n} = W(\bar{\xi},s_i) \ \ P_{\bar{\xi}}\text{-a.s.}. \quad \square$$

Remark 3.8. Combining with Theorem 2.5, Theorem 3.7 implies that in our case, for a.e. $\bar{\xi}$, no $c_n(\bar{\xi})$ exists that $Z_n(\bar{\xi})/c_n(\bar{\xi})$ has a proper and non-degenerate limit. Morever, if $\{Z_n\}$ is a regular branching process, then the growth of Z_n can not be measured by a sequence of positive constants.

3.3 Normalized by a sequence of functions

Since for η-a.e. $\bar{\xi}$, $\lim_{n\to\infty} Z_n(\bar{\xi})/c_n(\bar{\xi})$ is never a proper, non-degenerate random variable, we now consider other possibilities for normalizing $Z_n(\bar{\xi})$.

For $0 \leqslant x < \infty$, let

$$y_n\left(\bar{\xi},x\right) = f_{\xi_0}\left(\cdots\left(f_{\xi_{n-1}}\left(e^{-\frac{1}{x}}\right)\right)\cdots\right),\ y_n(\bar{\xi},0) = f_n(\bar{\xi},0).$$

Proof of Theorem 2.10. (1) The proof of the first part is similar to the discussion of Theorem 2.1.1 in [10], we rewrite it briefly as follows. For any $x \in (0,1)$,

$$\begin{aligned}\left\{y_n(\bar{\xi},Z_n(\bar{\xi})) < x \ \text{ eventually}\right\} &= \left\{Z_n(\bar{\xi}) < \left(-\log f_n^{(-1)}(\bar{\xi},x)\right)^{-1} \ \text{ eventually}\right\} \\ &\supseteq \left\{W(\bar{\xi},-\log x) < 1\right\},\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\left\{y_n(\bar{\xi},Z_n(\bar{\xi})) > x \ \text{ eventually}\right\} &= \left\{Z_n(\bar{\xi}) > \left(-\log f_n^{(-1)}(\bar{\xi},x)\right)^{-1} \ \text{ eventually}\right\} \\ &\supseteq \left\{W(\bar{\xi},-\log x) > 1\right\}.\end{aligned} \tag{3.16}$$

From (3.15) , (3.16), similar to the discussion of Theorem 2.1.1 in [10], we have $y_n(\bar{\xi},Z_n(\bar{\xi}))$ converges almost surely to a random variable $Y(\bar{\xi})$.

If $s_r = -\log x_r$ is a $\bar{\xi}$-regular point, from the property of regular points and (3.15) , (3.16) we have

$$\begin{aligned} P_{\bar{\xi}}\left(Y(\bar{\xi}) < e^{-s_r}\right) \geqslant P_{\bar{\xi}}\left(W(\bar{\xi},s_r) < 1\right) &= P_{\bar{\xi}}\left(W(\bar{\xi},s_r) = 0\right) = E_{\bar{\xi}}\left(X(\bar{\xi},s_r)\right) = e^{-s_r}, \\ P_{\bar{\xi}}\left(Y(\bar{\xi}) > e^{-s_r}\right) \geqslant P_{\bar{\xi}}\left(W(\bar{\xi},s_r) > 1\right) &= P_{\bar{\xi}}\left(W(\bar{\xi},s_r) = \infty\right) = 1 - e^{-s_r}. \end{aligned}$$

Thus

$$P_{\bar{\xi}}\left(Y(\bar{\xi}) = x_r\right) = 0 \quad \text{and} \quad P_{\bar{\xi}}\left(Y(\bar{\xi}) \leqslant x_r\right) = x_r.$$

(2)We only need to prove $\mathbb{P}(Y=1)=0,\ \mathbb{P}(Y=0)=0$. Let

$$D=\left\{\bar{\xi}: \text{for any } s\in(0,\infty),\ d(\bar{\xi},s)=0\right\},$$

from Lemma 3.4 we know that for any $\bar{\xi}\in D,\ s\in(0,\infty)$, there exists at least one $\theta\bar{\xi}$-regular point in $[h_{\xi_0}(s),s]$. If we assume that

$$P_{\bar{\xi}}\left(Y(\bar{\xi})=1\right)=\delta>0,$$

we claim that for any $s\in(0,-\log(1-\delta))$, s is a $\bar{\xi}$-irregular point. Otherwise, if there exists $s\in(0,-\log(1-\delta))$ a $\bar{\xi}$-regular point, then

$$P_{\bar{\xi}}(Y(\bar{\xi})<e^{-s})=e^{-s}>1-\delta,$$

that contradicts to $P_{\bar{\xi}}(Y(\bar{\xi})=1)=\delta>0$.

Then for any

$$s\in(0,h_{\xi_0}(-\log(1-\delta))),$$

s is a $\theta\bar{\xi}$-irregular point. But since $\bar{\xi}\in D$, we already know (Lemma 3.4) that for any

$$0<s_0<h_{\xi_0}\left(-\log(1-\delta)\right),$$

there exists at least one $\theta\bar{\xi}$-regular point in $[h_{\xi_0}(s_0),s_0]$, since

$$[h_{\xi_0}(s_0),s_0]\subseteq(0,h_{\xi_0}\left(-\log(1-\delta)\right)),$$

thus the assumption is not valid, i.e.,

$$P_{\bar{\xi}}\left(Y(\bar{\xi})=1\right)=0\quad\text{ for any } \bar{\xi}\in D.$$

In a similar way we can prove that

$$P_{\bar{\xi}}\left(Y(\bar{\xi})=0\right)=0\quad\text{ for any } \bar{\xi}\in D.$$

Accordingly $\mathbb{P}\left(Y\in(0,1)\right)=1$ because under the assumption (A2), $\mathbb{P}(D)=1$.

In particular, if $\{Z_n\}$ is a regular branching process, for η-a.e. $\bar{\xi}$, any $e^{-s}\in(0,1)$, s is a $\bar{\xi}$-regular point, $P_{\bar{\xi}}\left(Y(\bar{\xi})\leqslant e^{-s}\right)=e^{-s}$, obviously, Y is uniformly distributed on $(0,1)$. □

Remark 3.9. The proof of the first part is similar to that of Theorem 2.1.1 in [10]. But to conclude that $Y\in(0,1)$ η-a.s. can not followed there, because if $s\in(0,\infty)$ is $\bar{\xi}$-regular (irregular), we only know that $h_k(\bar{\xi},s)$ is $\theta^k\bar{\xi}$-regular (irregular) from Remark 3.2.

We define the random variable

$$T(\bar{\xi})=\sup\left\{s|0<s<\infty \text{ and } W(\bar{\xi},s)<1\right\}.$$

Then for any $\bar{\xi}\in D$ $\left(D=\left\{\bar{\xi}: \text{for any } s\in(0,\infty),\ d(\bar{\xi},s)=0\right\}\right)$,

$$1\geqslant P_{\bar{\xi}}\left(W(\bar{\xi},s)=0\right)\geqslant P_{\bar{\xi}}\left(W\left(\bar{\xi},k_{\xi_0}(s)\right)<\infty\right)\geqslant e^{-k_{\xi_0}(s)}\overset{s\to 0}{\longrightarrow}1,$$

and

$$1\geqslant P_{\bar{\xi}}\left(W(\bar{\xi},s)=\infty\right)\geqslant P_{\bar{\xi}}\left(W(\bar{\xi},h_{\xi_0}(s))>0\right)\geqslant 1-e^{-h_{\xi_0}(s)}\overset{s\to\infty}{\longrightarrow}1.$$

This implies that for any $\bar\xi \in D$, $P_{\bar\xi}$-a.s. $W\left(\bar\xi, s\right) = 0$ if s is close to 0 and $W\left(\bar\xi, s\right) = \infty$ if s is large enough. Therefore by Lemma 3.3 for $\bar\xi \in D$, either

$$T(\bar\xi) \text{ is a } \bar\xi\text{-regular point with } W\left(\bar\xi, s\right) = \begin{cases} 0 & \text{if } 0 < s < T(\bar\xi) \\ \infty & \text{if } T\left(\bar\xi\right) < s < \infty, \end{cases} \tag{3.17}$$

or

$$T\left(\bar\xi\right) \text{ is a } \bar\xi\text{-irregular point with } W\left(\bar\xi, T(\bar\xi)\right) = 1 \text{ and } W\left(\bar\xi, s\right) \begin{cases} < 1 & \text{for } s < T(\bar\xi) \\ > 1 & \text{for } s > T(\bar\xi). \end{cases} \tag{3.18}$$

Corollary 3.10. *Under Assumption 2.3, for η-a.s. $\bar\xi$,*

$$W\left(\bar\xi, s\right) < 1 < W\left(\bar\xi, t\right) \quad \textit{for } 0 < s < T(\bar\xi) < t < \infty. \tag{3.19}$$

$$T(\bar\xi) = -\log Y(\bar\xi), \tag{3.20}$$

and therefore $T(\bar\xi) \in (0, \infty)$, for any $\bar\xi$-regular point s_r,

$$P_{\bar\xi}\left(T(\bar\xi) \geqslant s_r\right) = e^{-s_r} \quad \textit{and} \quad P_{\bar\xi}\left(T(\bar\xi) = s_r\right) = 0. \tag{3.21}$$

Proof. (3.19) follows immediately from (3.17) and (3.18). From (3.15) and (3.16) we have

$$e^{-T(\bar\xi)} = \inf\left\{x | W(\bar\xi, -\log x) < 1\right\} \geqslant Y(\bar\xi) \geqslant \sup\left\{x | W(\bar\xi, -\log x) > 1\right\},$$

and by (3.19)

$$\inf\left\{x | W(\bar\xi, -\log x) < 1\right\} = \sup\left\{x | W(\bar\xi, -\log x) > 1\right\},$$

hence, $T(\bar\xi) = -\log Y(\bar\xi)$. Other properties are easy corollaries from Theorem 2.10. □

3.4 Proof of Theorem 2.11

In order to prove Theorem 2.11, we need the following two lemmas, which are similar as Lemma 2.2.4 and Lemma 2.2.5 in [10], we omit the details of the proof.

Lemma 3.11. *Let $U(\bar\xi)$ be as in Theorem 2.11 and*

$$V(\bar\xi, x) := \inf\left\{y | y \geqslant 0 \textit{ and } U(\bar\xi, y) \geqslant x\right\}, \quad 0 \leqslant x < \infty.$$

Then

$$U\left(\bar\xi, V(\bar\xi, x)\right) \sim x \tag{3.22}$$

□

Lemma 3.12. *Let $F_{\bar\xi}$ be as in Theorem 2.11 (2). If $F_{\bar\xi}$ is continuous at $x \in (0, \infty)$, then either $F_{\bar\xi}(x) = 0$ and*

$$\lim_{n\to\infty} V\left(\bar\xi, c_n(\bar\xi)x\right) h_n(\bar\xi, t) = 0 \qquad \textit{for } 0 < t < \infty,$$

or $F_{\bar\xi}(x) = 1$ and

$$\lim_{n\to\infty} V\left(\bar\xi, c_n(\bar\xi)x\right) h_n(\bar\xi, t) = \infty \qquad \textit{for } 0 < t < \infty,$$

or $s := -\log F_{\bar\xi}(x)$ is a $\bar\xi$-regular point and

$$\lim_{n\to\infty} V\left(\bar\xi, c_n(\bar\xi)x\right) h_n(\bar\xi, t) = \begin{cases} 0 & \textit{if } \ 0 < t < s \\ \infty & \textit{if } \ s < t < \infty. \end{cases}$$

□

Proof of Theorem 2.11. (1) If s_i is a $\bar{\xi}$-irregular point, and if

$$H(\bar{\xi}, s_i) = \lim_{n\to\infty} \left(U\left(\bar{\xi}, \frac{1}{h_n(\bar{\xi}, s_i)}\right) / c_n(\bar{\xi}) \right)$$

exists, then by Lemma 3.3, the limit $H(\bar{\xi}, s)$ exists for all $I(\bar{\xi}, s_i)$ and is equal to $H(\bar{\xi}, s_i)$, since $U(\bar{\xi})$ varies slowly. Thus $H(\bar{\xi})$ is continuous at s_i.

Since now we assume that (2.1) holds, then the points where the limit $H(\bar{\xi})$ does not exist can only be $\bar{\xi}$-regular points, and there are at most countably many such points. Further $H(\bar{\xi})$ is a monotonic function, and therefore at most countably many points in $(0, \infty)$ exist, where $H(\bar{\xi})$ is not continuous. Since $\bar{\xi} \in D$, by Corollary 3.10, for any $\bar{\xi}$-regular point s_r, $P_{\bar{\xi}}(T(\bar{\xi}) = s_r) = 0$, and $T(\bar{\xi}) \in (0, \infty)$ $P_{\bar{\xi}}$-a.s..

Take $0 < s < T(\bar{\xi}) < t < \infty$, then by (3.19),

$$1/h_n(\bar{\xi}, t) < Z_n(\bar{\xi}) < 1/h_n(\bar{\xi}, s) \qquad \text{eventually.}$$

Therefore,

$$U\left(\bar{\xi}, 1/h_n(\bar{\xi}, t)\right) / c_n(\bar{\xi}) \leqslant U\left(\bar{\xi}, Z_n(\bar{\xi})\right) / c_n(\bar{\xi}) \leqslant U\left(\bar{\xi}, 1/h_n(\bar{\xi}, s)\right) / c_n(\bar{\xi}) \quad \text{eventually.}$$

Since $H(\bar{\xi})$ is continuous at $T(\bar{\xi})$, and $H(\bar{\xi})$ exists at s and t arbitrarily close to $T(\bar{\xi})$, we have

$$\lim_{n\to\infty} U\left(\bar{\xi}, Z_n(\bar{\xi})\right) / c_n(\bar{\xi}) = H\left(\bar{\xi}, T(\bar{\xi})\right) \quad \text{almost surely.} \tag{3.23}$$

(2) We define

$$G\left(\bar{\xi}, x\right) = \inf\left\{y \mid 0 \leqslant y < \infty \text{ and } F_{\bar{\xi}}(y) \geqslant x\right\}, \quad 0 \leqslant x < \infty.$$

Let $G(\bar{\xi})$ be continuous at $y \in (0, 1)$, and $x = G\left(\bar{\xi}, y\right)$. Since $F_{\bar{\xi}}$ is right-continuous as a distribution function, we always have

$$F_{\bar{\xi}}\left(G(\bar{\xi}, y)\right) = F_{\bar{\xi}}\left(\inf\{z | F_{\bar{\xi}}(z) \geqslant y\}\right) \geqslant y. \tag{3.24}$$

If $F_{\bar{\xi}}(x) = y$, then $F_{\bar{\xi}}$ is strictly increasing at x, since $G(\bar{\xi})$ is continuous at y. We can choose $x_1 < x < x_2$ arbitrarily close to x, such that $F_{\bar{\xi}}$ is continuous at x_1 and x_2. Lemma 3.12 implies

$$V\left(\bar{\xi}, c_n(\bar{\xi})x_1\right) < \left(1/h_n(\bar{\xi}, s)\right) < V\left(\bar{\xi}, c_n(\bar{\xi})x_2\right) \tag{3.25}$$

eventually for $s = -\log F_{\bar{\xi}}(x) = -\log y$.

Therefore by Lemma 3.11, when $n \to \infty$,

$$\begin{aligned} \frac{U\left(\bar{\xi}, V(\bar{\xi}, c_n(\bar{\xi})x_1)\right)}{c_n(\bar{\xi})} &\longrightarrow x_1, \\ \frac{U\left(\bar{\xi}, V(\bar{\xi}, c_n(\bar{\xi})x_2)\right)}{c_n(\bar{\xi})} &\longrightarrow x_2. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} U\left(\bar{\xi}, 1/h_n(\bar{\xi}, s)\right) / c_n\left(\bar{\xi}\right) = x = G\left(\bar{\xi}, e^{-s}\right)$$

exists.

If on the other hand $F_{\bar{\xi}}(x) > y$, then

$$a := F_{\bar{\xi}}(x-) < y < F_{\bar{\xi}}(x) =: b.$$

Choose $x_1 < x < x_2$ arbitrarily close to x such that $F_{\bar{\xi}}$ is continuous at x_1 and x_2, and hence

$$-\log F_{\bar{\xi}}(x_2) \leqslant -\log b < s = -\log y < -\log a \leqslant -\log F_{\bar{\xi}}(x_1).$$

Again Lemma 3.12 implies (3.25) and therefore $\lim_{n\to\infty} U\left(\bar{\xi}, 1/h_n(\bar{\xi}, s)\right)/c_n\left(\bar{\xi}\right) = G\left(\bar{\xi}, e^{-s}\right).$

(3) Since both $G(\bar{\xi})$ and $G(\theta\bar{\xi})$ are continuous at all but at most countably many $s \in (0,\infty)$, there exists at least one $s_0 \in (0,\infty)$ that $G(\bar{\xi})$ and $G(\theta\bar{\xi})$ are both continuous at e^{-s_0}, $e^{-h_{\xi_0}(s_0)}$. Then from (2.3), for η-a.s. $\bar{\xi}$,

$$\lim_{n\to\infty} U\left(\bar{\xi}, \frac{1}{h_n(\bar{\xi}, s_0)}\right)/c_n\left(\bar{\xi}\right) = G\left(\bar{\xi}, e^{-s_0}\right) \quad \text{exists,} \tag{3.26}$$

$$\lim_{n\to\infty} U\left(\theta\bar{\xi}, \frac{1}{h_{n-1}(\theta\bar{\xi}, h_{\xi_0}(s_0))}\right)/c_{n-1}\left(\theta\bar{\xi}\right) = G\left(\theta\bar{\xi}, e^{-h_{\xi_0}(s_0)}\right) \quad \text{exists.} \tag{3.27}$$

Since from our assumption we know that $U\left(\bar{\xi}, \frac{1}{h_n\left(\bar{\xi}, s_0\right)}\right) = U\left(\theta\bar{\xi}, \frac{1}{h_{n-1}\left(\theta\bar{\xi}, h_{\xi_0}(s_0)\right)}\right)$, moreover, assumptions on $F_{\bar{\xi}}$ ensures that for $0<y<1$, $0 < G(\bar{\xi}, y) < \infty$ for η-a.s. $\bar{\xi}$. Then combine with (3.26), (3.27), we have

$$\lim_{n\to\infty} \frac{c_{n-1}(\theta\bar{\xi})}{c_n(\bar{\xi})} = \frac{G\left(\bar{\xi}, e^{-s_0}\right)}{G\left(\theta\bar{\xi}, e^{-h_{\xi_0}(s_0)}\right)} := \alpha(\bar{\xi}) > 0. \tag{3.28}$$

Then (2.3), (3.28) and the assumption $U(\bar{\xi}) = U(\theta\bar{\xi})$ imply

$$\frac{G\left(\bar{\xi}, e^{-s}\right)}{G\left(\theta\bar{\xi}, e^{-h_{\xi_0}(s)}\right)} = \lim_{n\to\infty} \frac{c_{n-1}\left(\theta\bar{\xi}\right)}{c_n\left(\bar{\xi}\right)} = \alpha\left(\bar{\xi}\right) \tag{3.29}$$

for all s for which $G(\bar{\xi})$ is continuous at e^{-s}. In particular $G(\bar{\xi})$ is continuous at e^{-s} if and only if $G(\theta\bar{\xi})$ is continuous at $e^{-h_{\xi_0}(s)}$.

Since $G(\bar{\xi})$ is left-continuous, (3.29) is true for all $s \in (0,\infty)$.

Now for every $0 < u < \infty$,

$$\left\{y | G(\theta\bar{\xi}, y) \leqslant u\right\} = \left\{y | y \leqslant F_{\theta\bar{\xi}}(u)\right\}, \tag{3.30}$$

since $F_{\theta\bar{\xi}}(x) < y$ for all $x < G\left(\theta\bar{\xi}, y\right)$ by definition, and because of (3.24).

Our assumption ensures that $-\log F_{\theta\bar{\xi}}(u) \in (0,\infty)$ for $0<u<\infty$. Since $F_{\theta\bar{\xi}}$ is right-continuous there exists a sequence of points $u_n > u$, such that $\lim_{n\to\infty} F_{\theta\bar{\xi}}(u_n) = F_{\theta\bar{\xi}}(u)$, and $F_{\theta\bar{\xi}}$ is continuous at every u_n. Lemma 3.12 implies that $-\log F_{\theta\bar{\xi}}(u_n)$ are $\theta\bar{\xi}$-regular points, and therefore for every $0<u<\infty$,

$$-\log F_{\theta\bar{\xi}}(u) \text{ is a } \theta\bar{\xi}\text{-regular point,}$$

since $\{s | s \text{ is } \theta\bar{\xi}\text{-regular}\} \cup \{0,\infty\}$ is a closed set by Lemma 3.3.

Now since

$$k_{\xi_0}\left(-\log F_{\theta\bar{\xi}}(u)\right) = -\log f_{\xi_0}\left(F_{\theta\bar{\xi}}(u)\right),$$

$-\log f_{\xi_0}\left(F_{\theta\bar{\xi}}(u)\right)$ is a $\bar{\xi}$-regular point.

Theorem 2.11 (1),(2) tells us that $\lim_{n\to\infty} U\left(\bar{\xi}, Z_n(\bar{\xi})\right)/c_n\left(\bar{\xi}\right) = G\left(\bar{\xi}, Y(\bar{\xi})\right) \quad P_{\bar{\xi}}$-a.s.. This combines with (2.6), (3.30) imply

$$\begin{aligned} F_{\bar{\xi}}\left(\alpha(\bar{\xi})u\right) &= P_{\bar{\xi}}\left(G\left(\bar{\xi}, Y(\bar{\xi})\right) \leqslant \alpha(\bar{\xi})u\right) = P_{\bar{\xi}}\left(G\left(\bar{\xi}, e^{-T(\bar{\xi})}\right) \leqslant \frac{G\left(\bar{\xi}, e^{-T(\bar{\xi})}\right)}{G\left(\theta\bar{\xi}, e^{-h_{\xi_0}(T(\bar{\xi}))}\right)} u\right) \\ &= P_{\bar{\xi}}\left(G\left(\theta\bar{\xi}, e^{-h_{\xi_0}(T(\bar{\xi}))} \leqslant u\right)\right) = P_{\bar{\xi}}\left(e^{-h_{\xi_0}(T(\bar{\xi}))} \leqslant F_{\theta\bar{\xi}}(u)\right) \\ &= P_{\bar{\xi}}\left(f_{\xi_0}^{(-1)}\left(e^{-T(\bar{\xi})}\right) \leqslant F_{\theta\bar{\xi}}(u)\right) = P_{\bar{\xi}}\left(Y\left(\bar{\xi}\right) \leqslant f_{\xi_0}\left(F_{\theta\bar{\xi}}(u)\right)\right) \\ &= f_{\xi_0}\left(F_{\theta\bar{\xi}}(u)\right) \end{aligned}$$

for $0 < u < \infty$, the last equality is due to the fact that $-\log f_{\xi_0}(F_{\theta\bar{\xi}}(u))$ is a $\bar{\xi}$-regular point and Theorem 2.10. Since

$$F_{\bar{\xi}}\left(\alpha(\bar{\xi}) \cdot 0\right) = F_{\bar{\xi}}(0) = 0 = f_{\xi_0}\left(F_{\theta\bar{\xi}}(0)\right),$$

(2.7) is true. ☐

Remark 3.13. Since $T\left(\bar{\xi}\right) \in (0,\infty)$, Theorem 2.11 (1) tells us that if we can find suitable $U(\bar{\xi}, x)$, $c_n(\bar{\xi})$ that makes $H\left(\bar{\xi}, s\right) := \lim_{n\to\infty} U\left(\bar{\xi}, 1/h_n(\bar{\xi}, s)\right)/c_n\left(\bar{\xi}\right)$ exists for all but at most countably many $s \in (0,\infty)$ and $0 < H\left(\bar{\xi}, s\right) < \infty$ for $s \in (0,\infty)$, then $U\left(\bar{\xi}, Z_n(\bar{\xi})\right)/c_n\left(\bar{\xi}\right)$ has a non-degenerate and proper limit. What's more, (1) combines with (2) show that if $U\left(\bar{\xi}, Z_n(\bar{\xi})\right)/c_n\left(\bar{\xi}\right)$ converges in distribution, then it must converge almost surely.

4 Sufficient criteria for regular process

From Definition 3.6 we know that for a regular process, for $a.e.\ \bar{\xi}$, any $s < t$, $\lim_{n\to\infty} \frac{h_n(\bar{\xi},s)}{h_n(\bar{\xi},t)} = 0$, then $d\left(\bar{\xi}, s\right) = 0$ since

$$d(\bar{\xi}, s) = \lim_{n\to\infty} \frac{h_n\left(\bar{\xi}, s\right)}{h_{n-1}\left(\theta\bar{\xi}, s\right)} = \lim_{n\to\infty} \frac{h_n\left(\bar{\xi}, s\right)}{h_n\left(\bar{\xi}, k_{\xi_0}(s)\right)},$$

where $k_{\xi_0}(s) > s$. Thus, every regular process satisfies Assumption **(A2)**. In this section, we will derive some sufficient conditions for a process to be regular.

Let $Q_{\xi_i} : [0,1) \longrightarrow [0,1)$ defined by

$$Q_{\xi_i}(s) = \frac{f'_{\xi_i}(s)(1-s)}{1 - f_{\xi_i}(s)}.$$

Since $f_{\xi_i}(x)$ is strictly convex and $f_{\xi_i}(1) = 1$, then $f'_{\xi_i}(s) < \frac{1-f_{\xi_i}(s)}{1-s}$, i.e.

$$0 \leqslant Q_{\xi_i}(s) < 1.$$

Theorem 4.1. *$s \in (0,\infty)$ is $\bar{\xi}$-regular if and only if* $\prod_{n=0}^{\infty} Q_{\xi_n}\left(f_{n+1}^{(-1)}\left(\bar{\xi}, e^{-s}\right)\right) = 0$, *where* $f_n(\bar{\xi}, s) = f_{\xi_0}\left(\cdots(f_{\xi_{n-1}}(s))\cdots\right)$.

Proof. From the proof of Theorem 3.1(for details see [10] Theorem 1.1.2) we know that s is $\bar{\xi}$-regular if and only if

$$\lim_{n\to\infty} k_n\left(\bar{\xi}, h_n(\bar{\xi}, s)x\right) = s \quad \text{for all } 0 < x < \infty.$$

Since $k_n\left(\bar{\xi}, h_n(\bar{\xi},s)x\right)$ is a concave function of x, $k_n\left(\bar{\xi}, h_n(\bar{\xi},s)\cdot 1\right)=s$ and $k_n\left(\bar{\xi}, h_n(\bar{\xi},s)x\right)\leqslant s$ for all $x\leqslant 1 (\geqslant s$ for all $x\geqslant 1)$, this is equivalent to

$$\gamma_n\left(\bar{\xi},s\right)=\frac{d}{dx}\big(k_n(\bar{\xi},h_n(\bar{\xi},s)\cdot x)\big)\Big|_{x=1}\overset{n\to\infty}{\longrightarrow}0. \tag{4.1}$$

We can calculate that

$$\gamma_n\left(\bar{\xi},s\right)=e^s\cdot f_n^{'}\left(\bar{\xi},f_n^{(-1)}\left(\bar{\xi},e^{-s}\right)\right)f_n^{(-1)}\left(\bar{\xi},e^{-s}\right)\left(-\log f_n^{(-1)}\left(\bar{\xi},e^{-s}\right)\right),$$

and since

$$\lim_{n\to\infty}\left(-\log f_n^{(-1)}\left(\bar{\xi},e^{-s}\right)\right)\left(1-f_n^{(-1)}\left(\bar{\xi},e^{-s}\right)\right)^{-1}\to 1,$$

(4.1) is equivalent to

$$f_n^{'}\left(\bar{\xi},f_n^{(-1)}\left(\bar{\xi},e^{-s}\right)\right)\left(1-f_n^{(-1)}\left(\bar{\xi},e^{-s}\right)\right)\to 0.$$

Since

$$f_n^{'}\left(\bar{\xi},f_n^{(-1)}\left(\bar{\xi},e^{-s}\right)\right)=\prod_{j=0}^{n-1}f_{\xi_j}^{'}\left(f_{j+1}^{(-1)}\left(\bar{\xi},e^{-s}\right)\right),$$

thus (4.1) is equivalent to $\displaystyle\prod_{n=0}^{\infty}Q_{\xi_n}\left(f_{n+1}^{(-1)}\left(\bar{\xi},e^{-s}\right)\right)=0.$ □

Corollary 4.2. *If* $\mathbb{P}\left(\{\xi_0:\sup_{0<s<1}Q_{\xi_0}(s)\leqslant c<1\}\right)>0$, *then* $\{Z_n\}$ *is a regular branching process.*

Proof.

$$\begin{aligned}\mathbb{E}\prod_{i=0}^{\infty}Q_{\xi_i}\left(f_{i+1}^{(-1)}(e^{-s})\right) &= \mathbb{E}e^{\log\prod_{i=0}^{\infty}Q_{\xi_i}\left(f_{i+1}^{(-1)}(e^{-s})\right)}\\ &= \mathbb{E}e^{\sum_{i=0}^{\infty}\log Q_{\xi_i}\left(f_{i+1}^{(-1)}(e^{-s})\right)}\\ &\leqslant \mathbb{E}e^{\sum_{i=0}^{\infty}\log\left(\sup_{0<s<1}Q_{\xi_i}(s)\right)}. \end{aligned}\tag{4.2}$$

If

$$\mathbb{P}\left(\left\{\xi_0:\sup_{0<s<1}Q_{\xi_0}(s)\leqslant c<1\right\}\right)>0, \tag{4.3}$$

since $\{\xi_i\}$ is a sequence of independent and identically distributed random variables, we know that $\mathbb{P}$-a.e.,

$$\frac{\sum_{i=0}^{n}\log\left(\sup_{0<s<1}Q_{\xi_i}(s)\right)}{n}\overset{n\to\infty}{\longrightarrow}\mathbb{E}\log\left(\sup_{0<s<1}Q_{\xi_0}(s)\right).$$

Since $\mathbb{P}(\sup Q_{\xi_0}(s)\leqslant 1)=1$, (4.3) ensures that on a set with positive probability $\sup_{0<s<1}Q_{\xi_0}(s)<1$, thus $\mathbb{E}\log(\sup_{0<s<1}Q_{\xi_0}(s))<0$, that is to say, for η-a.e. $\bar{\xi}$,

$$\sum_{i=0}^{\infty}\log\left(\sup_{0<s<1}Q_{\xi_i}(s)\right)=-\infty\quad P_{\bar{\xi}}\text{-a.e.}.$$

As a result,

$$\mathbb{E}\prod_{i=0}^{\infty} Q_{\xi_i}\left(f_{i+1}^{(-1)}(e^{-s})\right) = 0,$$

for any $0 < s < \infty$.

From Theorem 4.1 we know that for η-a.e. $\bar{\xi}$, every $s \in (0,\infty)$ is a $\bar{\xi}$-regular point, then $\{Z_n(\bar{\xi})\}$ is $\bar{\xi}$ regular η-a.e., and $\{Z_n\}$ is a regular branching process by Definition 3.6. □

Example 4.3. Let $f_{\xi_i}(s) = 1-(1-s)^{\alpha_{\xi_i}}$, where $\{\alpha_{\xi_i}\}$ is a collection of independent and identically distributed random variables, taking values in $(0, 1-\epsilon)$ ($0<\epsilon<1$ is a constant). Then this process is a regular process which satisfies our assumption and $U(\bar{\xi},x) = \log x,\ c_n(\bar{\xi}) = \frac{1}{\alpha_{\xi_0}}\cdots\frac{1}{\alpha_{\xi_{n-1}}}$ is a suitable choice for the normalization of $\{Z_n\}$.

Proof. (1) Since $f_{\xi_0}(s) = 1-(1-s)^{\alpha_{\xi_0}}$, we can calculate that $f'_{\xi_0}(s) = \alpha_{\xi_0}(1-s)^{\alpha_{\xi_0}-1}$. Since $0<\alpha_{\xi_0}<1-\epsilon, \mathbb{E}\log m(\xi_0) = \infty$.

Let $r_n(\bar{\xi},s) := 1 - f_{\xi_{n-1}}^{(-1)}\left(\cdots\left(f_{\xi_0}^{(-1)}(1-s)\right)\cdots\right)$ for all $0\leqslant s\leqslant 1$, then it is clear that for all $0<s<1$,

$$r_n(\bar{\xi},s) \sim h_n\left(\bar{\xi}, -\log(1-s)\right).$$

In our example, it is easy to calculate that $r_n(\bar{\xi},s) = s^{\frac{1}{\alpha_{\xi_0}}\frac{1}{\alpha_{\xi_1}}\cdots\frac{1}{\alpha_{\xi_{n-1}}}}$. Then for any $0<s<1$,

$$d\left(\bar{\xi}, -\log(1-s)\right) = \lim_{n\to\infty}\frac{h_{n+1}\left(\bar{\xi},-\log(1-s)\right)}{h_n\left(\theta\bar{\xi},-\log(1-s)\right)} = \lim_{n\to\infty}\frac{r_{n+1}\left(\bar{\xi},s\right)}{r_n\left(\theta\bar{\xi},s\right)} = \lim_{n\to\infty}\frac{s^{\frac{1}{\alpha_{\xi_0}}\frac{1}{\alpha_{\xi_1}}\cdots\frac{1}{\alpha_{\xi_n}}}}{s^{\frac{1}{\alpha_{\xi_1}}\cdots\frac{1}{\alpha_{\xi_n}}}} = 0,$$

η-a.e. $\bar{\xi}$, that means for any $0<s<\infty$, $d(\bar{\xi},s)=0$. Combined with the fact that $f_{\xi_0}(0)=0$, this example satisfies Assumption ($\mathbf{A1}$) and ($\mathbf{A2}$).

(2) Since $Q_{\xi_i}(s) = \frac{f'_{\xi_i}(s)(1-s)}{1-f_{\xi_i}(s)} = \alpha_{\xi_i}$, $\mathbb{P}\left(\{\xi_0 : \sup_{0<s<1} Q_{\xi_0}(s)\leqslant 1-\epsilon\}\right) = 1$. From Corollary 4.2 we know that $\{Z_n\}$ is a regular branching process.

(3) If we choose

$$U(\bar{\xi},x) = \log x, \quad c_n(\bar{\xi}) = \frac{1}{\alpha_{\xi_0}}\cdots\frac{1}{\alpha_{\xi_{n-1}}},$$

by calculation,

$$\begin{aligned}\lim_{n\to\infty} U\left(\bar{\xi},\frac{1}{h_n(\bar{\xi},s)}\right)/c_n(\bar{\xi}) &= \lim_{n\to\infty} U\left(\bar{\xi},\frac{1}{r_n(\bar{\xi},1-e^{-s})}\right)/c_n(\bar{\xi}) \\ &= \frac{\log(1-e^{-s})^{-\frac{1}{\alpha_{\xi_0}}\cdots\frac{1}{\alpha_{\xi_{n-1}}}}}{\frac{1}{\alpha_{\xi_0}}\cdots\frac{1}{\alpha_{\xi_{n-1}}}} \\ &= -\log(1-e^{-s}) \in (0,\infty).\end{aligned}$$

Then from Theorem 2.11 we know that in this case

$$\lim_{n\to\infty}\frac{\log\left(Z_n(\bar{\xi})\right)}{c_n\left(\bar{\xi}\right)} = -\log\left(1-e^{-T(\bar{\xi})}\right) = -\log\left(1-Y(\bar{\xi})\right) \in (0,\infty).$$

Thus this is a suitable choice for the normalization of $\{Z_n\}$.

(4) Note that in this example $\lim_{n\to\infty} \frac{c_{n-1}(\theta\bar{\xi})}{c_n(\bar{\xi})} = \alpha_{\xi_0}$. If we use $F_{\bar{\xi}}$ to denote the distribution function of the limit of $\frac{\log(Z_n(\bar{\xi}))}{c_n(\bar{\xi})}$, we have

$$F_{\bar{\xi}}(x) = P_{\bar{\xi}}\left(-\log(1-Y(\bar{\xi})) \leqslant x\right).$$

Since $\{Z_n\}$ is a regular branching process, Theorem 2.10 tells us that for η-a.s. $\bar{\xi}$, $Y(\bar{\xi})$ is uniformly distributed on $(0,1)$. Thus $F_{\bar{\xi}}(x) = 1 - e^{-x}$. Then

$$f_{\xi_0}\left(F_{\theta\bar{\xi}}(u)\right) = f_{\xi_0}\left(1-e^{-u}\right) = 1 - e^{-\alpha_{\xi_0} u} = F_{\bar{\xi}}(\alpha_0 u),$$

which coincides with (2.7). □

Remark 4.4. All results still hold if the environmental sequence $\bar{\xi}$ is supposed to be stationary and ergodic instead of $i.i.d.$. We thank the referee for pointing this out.

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