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Global fluctuations for 1D log-gas dynamics. Covariance kernel and support

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Abstract

We consider the hydrodynamic limit in the macroscopic regime of the coupled system of stochastic differential equations,

$$d\lambda_t^i = \frac{1}{\sqrt{N}} dW_t^i - V'(\lambda_t^i)dt + \frac{\beta}{2N} \sum_{j \neq i} \frac{dt}{\lambda_t^i - \lambda_t^j}, \qquad i = 1, \ldots, N, \tag{0.1}$$

with $\beta > 1$, sometimes called *generalized Dyson's Brownian motion*, describing the dissipative dynamics of a log-gas of N equal charges with equilibrium measure corresponding to a β-ensemble, with sufficiently regular convex potential V. The limit $N \to \infty$ is known to satisfy a mean-field Mc Kean-Vlasov equation. Fluctuations around this limit have been shown [39] to define a Gaussian process solving some explicit martingale problem written in terms of a generalized transport equation.

We prove a series of results concerning either the Mc Kean-Vlasov equation for the density ρ_t, notably regularity results and time-evolution of the support, or the associated hydrodynamic fluctuation process, whose space-time covariance kernel we compute explicitly.

Keywords: random matrices; Dyson's Brownian motion; log-gas; beta-ensembles; hydrodynamic limit; Stieltjes transform; fluctuations; support.
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1 Introduction and statement of main results

1.1 Introduction

Let $\beta \geq 1$ be a fixed parameter, and $N \geq 1$ an integer. We consider the following system of coupled stochastic differential equations driven by N independent standard Brownian motions $(W_t^1, \ldots, W_t^N)_{t \geq 0}$,

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$$d\lambda_t^i = \frac{1}{\sqrt{N}} dW_t^i - V'(\lambda_t^i)dt + \frac{\beta}{2N} \sum_{j \neq i} \frac{dt}{\lambda_t^i - \lambda_t^j}, \qquad i = 1, \dots, N \tag{1.1}$$

Letting

$$\mathcal{W}(\{\lambda^i\}_i) := \sum_{i=1}^N V(\lambda^i) - \frac{\beta}{4N} \sum_{i \neq j} \log(\lambda^i - \lambda^j), \tag{1.2}$$

we can rewrite (1.1) as $d\lambda_t^i = \frac{1}{\sqrt{N}} dW_t^i - \nabla_i \mathcal{W}(\lambda_t^1, \dots, \lambda_t^N)dt$. Thus the corresponding equilibrium measure,

$$d\mu_{eq}^N(\{\lambda^i\}_i) = \frac{1}{Z_V^N} e^{-2N\mathcal{W}(\{\lambda^i\}_i)} = \frac{1}{Z_V^N} \left(\prod_{j \neq i} |\lambda^j - \lambda^i| \right)^{\beta/2} \exp\left(-2N \sum_{i=1}^N V(\lambda^i) \right) d\lambda^1 \cdots d\lambda^N \tag{1.3}$$

is that of a β-log gas with confining potential V.

Let us start with a historical overview of the subject as a motivation for our study. This system of equations was originally considered in a particular case by Dyson [11] who wanted to describe the Markov evolution of a Hermitian matrix M_t with i.i.d. increments dG_t taken from the Gaussian unitary ensemble (GUE). In Dyson's idea, this matrix-valued process was to be a matrix analogue of Brownian motion. The latter time-evolution being invariant through conjugation by unitary matrices, we may project it onto a time-evolution of the set of eigenvalues $\{\lambda_t^1, \dots, \lambda_t^N\}$ of the matrix, and obtain (1.1) with $\beta = 2$ and $V \equiv 0$. Keeping $\beta = 2$, it is easy to prove that (1.1) is equivalent to a generalized matrix Markov evolution, $dM_t = dG_t - V'(M_t)dt$. The Gibbs measure

$$\mathcal{P}_V^N(M) = \frac{1}{Z_N} e^{-N\mathrm{Tr}V(M)} dM, \qquad dM = \prod_{i=1}^N dM_{ii} \prod_{1 \le i < j \le n} d\mathrm{Re}\, M_{ij}\, d\mathrm{Im}\, M_{ij}$$

can then be proved to be an equilibrium measure. Such measures, together with their projection onto the eigenvalue set, $\mu_{eq}^N(\{\lambda^1, \dots, \lambda^N\})$, are the main object of random matrix theory, see e.g. [27],[2], [31]. The *equilibrium eigenvalue distribution* can be studied by various means, in particular using orthogonal polynomials with respect to the weight $e^{-NV(\lambda)}$. The scaling in N (called *macroscopic scaling* in random matrix theory) ensures the convergence of the random point measure $X^N := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda^i}$ to a deterministic measure μ_V with *compact* support and density ρ when $N \to \infty$ (see e.g. [20], Theorem 2.1). One finds e.g. the well-known semi-circle law, $\rho(x) = \frac{1}{\pi}\sqrt{2 - x^2}$, when $V(x) = x^2/2$. Looking more closely at the limit of the point measure, one finds for arbitrary *polynomial* V (Johansson [20]) Gaussian fluctuations of order $O(1/N)$, contrasting with the $O(1/\sqrt{N})$ scaling of fluctuations for the means of N independent random variables, typical of the central limit theorem. Assuming that the support of the measure is connected (this essential "one-cut" condition holding in particular for V *convex*), Johansson proves that the *covariance* of the limiting law depends on V only through the support of the measure – it is thus *universal* up to a scaling coefficient –, while the means is equal to ρ, plus an apparently non-universal correction in $O(1/N)$.

Following Rogers and Shi [33], Li, Li and Xie [24] proved the following two facts:

(i) two arbitrary eigenvalues never collide, which implies the non-explosion of (1.1);

(ii) the random point process $X_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_t^i}$ satisfies in the limit $N \to \infty$ a deterministic hydrodynamic equation of Mc Kean Vlasov type, namely, the asymptotic density

$$\rho_t \equiv X_t := \mathrm{w\!-\!lim}_{N \to \infty} X_t^N \tag{1.4}$$

satisfies the PDE

$$\frac{\partial \rho_t(x)}{\partial t} = \frac{\partial}{\partial x}\left(\left(V'(x) - \frac{\beta}{2} p.v. \int \frac{dy}{x-y}\rho_t(y)\right)\rho_t(x)\right), \tag{1.5}$$

in a weak (i.e. distribution) sense, where $p.v. \int \frac{dy}{x-y}\rho_t(y)$ is a principal value integral. The equilibrium measure ρ_{eq}, defined as the solution of the integral equation (traditionally called: *cut equation*)

$$\frac{\beta}{2} p.v. \int \frac{dy}{x-y}\rho_{eq}(y) = V'(x), \tag{1.6}$$

cancels the right-hand side of (1.5), as is readily checked.

A complex Burgers-like PDE for the Stieltjes transform of X_t

$$U_t(z) := \int \frac{1}{x-z} X_t(dx), \qquad z \in \mathbb{C} \setminus \mathbb{R} \tag{1.7}$$

is easily derived [33, 19] from (1.5), assuming V to be polynomial,

$$\frac{\partial U_t}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\beta}{4}(U_t(z))^2 + V'(z)U_t(z) + T_t(z)\right), \tag{1.8}$$

where

$$T_t(z) := \int \frac{V'(x) - V'(z)}{x-z}\, X_t(dx). \tag{1.9}$$

In our recent article [39], in large part based on a previous paper by Israelsson [19] which dealt with the specific example of a harmonic potential, we introduced a process $Y = (Y_t)_{t\geq 0}$ interpreted as *asymptotic fluctuation process*. Let $Y_t^N := N(X_t^N - X_t)$ be the rescaled fluctuation process for finite N. Then it was proved that $Y_t^N \overset{law}{\to} Y_t$ when $N \to \infty$, where $(Y_t)_{t\geq 0}$ is the solution of a martingale problem, as can be briefly seen as follows. First, Itô's formula implies that

$$d\langle Y_t^N, f_t\rangle = \frac{1}{2}(1 - \frac{\beta}{2})\langle X_t^N, f_t''\rangle dt + \frac{1}{\sqrt{N}}\sum_{i=1}^N f_t'(\lambda_t^i) dW_t^i \tag{1.10}$$

if the test functions $(f_t)_{0\leq t\leq T}$, $f_t : \mathbb{R} \to \mathbb{R}$ solve the following linear PDE

$$\frac{\partial f_t}{\partial t}(x) = V'(x)f_t'(x) - \frac{\beta}{4}\int \frac{f_t'(x) - f_t'(y)}{x-y}(X_t^N(dy) + X_t(dy)) \tag{1.11}$$

Substituting formally to X^N its deterministic limit X in the r.-h.s. of (1.11), one gets an equation which is the asymptotic limit of (1.11) in the limit $N \to \infty$, namely,

$$\frac{\partial f_t}{\partial t}(x) = V'(x)f_t'(x) - \frac{\beta}{2}\int \frac{f_t'(x) - f_t'(y)}{x-y} X_t(dy) \tag{1.12}$$

The main task in [39] consists in proving that eq. (1.11, 1.12) is akin to a transport equation on the cut complex plane $\mathbb{C} \setminus \mathbb{R}$. In the harmonic case (i.e. when V is quadratic), then the solution of, say, (1.12) at time t with terminal condition $f_T(x) \equiv \frac{c}{x-z}$, $c \in \mathbb{C}, z \in \mathbb{C} \setminus \mathbb{R}$ is equal to $\frac{c_t}{x-z_t}$ where $z_t \equiv a_t + \mathrm{i} b_t$ and

$$\frac{da_t}{dt} = \frac{\beta}{2}\mathrm{Re}\, U_t(z_t) + V'(a_t), \qquad \frac{db_t}{dt} = \frac{\beta}{2}\mathrm{Im}\, U_t(z_t) + V''(a_t)b_t$$

$$\frac{dc_t}{dt} = \left[\frac{\beta}{2}U_t'(z_t) + 2V''(a_t)\right]c_t. \tag{1.13}$$

Thus the solution of (1.12) may be represented formally as

$$\int da\, db\, h_t(a,b) \frac{1}{x-z}, \tag{1.14}$$

where $h_t(a,b) := c_t \delta(a-a_t)\delta(b-b_t)$, interpreted as a *density* on $\mathbb{C} \setminus \mathbb{R}$, is obtained by "pushing" $h_T(a,b) := \delta(a-a_T)\delta(b-b_T)$ along the above characteristics, or equivalently, by solving the associated transport equation generated by the time-dependent operator

$$\mathcal{L}(t) := (\frac{\beta}{2}\mathrm{Re}\, U_t(z) + V'(a))\partial_a + (\frac{\beta}{2}\mathrm{Im}\, U_t(z) + V''(a)b)\partial_b + \frac{\beta}{2}U_t'(z) + 2V''(a). \tag{1.15}$$

Considering instead some arbitrary terminal condition and potential V, a similar formula holds, where the time-evolution is given up to a bounded perturbation by a transport operator whose characteristics are as (1.13) plus some extra term depending on V'''. Then (at least formally), Itô's formula (see [19], p. 29) makes it possible to find the Markov kernel in the limit $N \to \infty$. Namely, if f_t be the solution of (1.12) with terminal condition f_T, and $\phi_{f_t}(Y_t) := e^{\mathrm{i}\langle Y_t, f_t\rangle}$,

$$\mathbb{E}[\phi_{f_T}(Y_T) \big| \mathcal{F}_t] = \mathbb{E}[\phi_{f_t}(Y_t)] \exp\left(\frac{1}{2}\int_t^T \left[\mathrm{i}(1-\frac{\beta}{2})\langle X_s, f_s''\rangle - \langle X_s, (f_s')^2\rangle\right] ds\right). \tag{1.16}$$

Eq. (1.16) was proved for general potentials in our previous article [39]. Now, letting $f_T(x) := \sum_{k=1}^n \frac{c_T^k}{x - z_T^k}$, $z_T^k \in \mathbb{C} \setminus \mathbb{R}, k = 1, \dots, n$ vary in dense subspace of $L^1(\mathbb{R})$, this martingale problem is solved in Bender [3] in the case of a harmonic potential using an explicit computation of the characteristics (1.13). Such is the present state of the art.

1.2 Main results

We prove in this article two types of results. We shall generally assume that V is *polynomial* and *strictly convex*, though the reader will also find weaker sets of hypotheses, depending on the paragraph.

(A) The first series of results regards the *Mc Kean-Vlasov equation* (1.5). Little is known about it in general; the arguments in Li-Li-Xie [24] (see in particular Theorem 1.3) simply prove that it admits a unique solution in $C([0,T], \mathcal{P}(\mathbb{R}))$, which is constructed as weak limit of the sequence of stochastic processes $t \mapsto Y_N(t)$. Unicity is proved using decrease of Wasserstein distance between two arbitrary solutions. A classical large-deviation argument (reviewed here) implies under our hypotheses a bound on the support of the measure ρ_t; in particular, ρ_t is compactly supported.

Our first result is a *regularity result*: assuming that the analytic function $z \mapsto U_0(z), z \in \Pi_+ := \{\mathrm{Im}\, z > 0\}$ extends to a continuous function on the closure $\Pi_+ \cup \mathbb{R}$ of the upper half-plane, we prove that the same property holds for $U_t, t \geq 0$; see Theorem 2.1. Hence in particular (by Plemelj's formula), the density $\rho_t(\cdot) = \frac{1}{2\mathrm{i}\pi}(U_t(\cdot + \mathrm{i}0) - U_t(\cdot - \mathrm{i}0))$ is a continuous function for every $t \geq 0$.

Our second result concerns the *support*. We explain how to obtain the "external support" $[a_t, b_t]$ of ρ_t, i.e. the intersection of all intervals $[a,b]$ such that $\langle \rho_t, \phi\rangle \equiv 0$ for every test function ϕ with compact support $\subset \mathbb{R} \setminus [a,b]$. (This implies that $\mathrm{supp}(\rho_t) \subset [a_t, b_t]$ but not the reverse inclusion $[a_t, b_t] \subset \mathrm{supp}(\rho_t)$.) The external support is characterized, see eq. (2.27) and (2.28), in terms of characteristics of the generalized complex Burgers equation (1.8) – not surprisingly closely related to (1.13) – which are half-explicit in general and can be obtained in closed form in various cases, including for equilibrium dynamics or when V is harmonic. On the other hand, we do not prove any formula for the support itself. In particular, though under our hypotheses (more specifically, because V is convex) the support of the equilibrium density is a connected interval, we cannot exclude, even if $\mathrm{supp}(\rho_0)$ is connected, that e.g. $\mathrm{supp}(\rho_t) = [a,c] \cup [d,b]$ with $a < c < d < b$ for some $t > 0$.

Much stronger results have been proved by P. Biane [5] in the Hermite case, namely, for $\beta = 2$ and, say, $V(x) = \frac{x^2}{2}$ (harmonic potential), see §2.2 **A.** for more precise statements concerning regularity. Also, the number of connected components is shown to be decreasing with time, so that the above hypothetical behavior can be excluded.

(B) The second series of results regards the *fluctuation process* $(Y_t)_{t\geq 0}$. While the above characteristic equations can be solved explicitly only when V is harmonic (see Bender [3]), yielding the covariance of the Stieltjes transform $(\mathcal{S}Y_t)(z) := \langle Y_t, \frac{1}{\cdot - z}\rangle$ of the fluctuation process,

$$\Lambda(t_1, z_1; t_2, z_2) := \mathrm{Cov}((\mathcal{S}Y_{t_1})(z_1), (\mathcal{S}Y_{t_2})(z_2)), \tag{1.17}$$

their "trace" on the boundary of the upper (or lower) half-planes can be solved for arbitrary V. Then the covariance kernel $\mathrm{Cov}(Y_{t_1}(x_1), Y_{t_2}(x_2))$ is found by taking boundary values $Y_{t_i}(x_i) = \frac{1}{2\mathrm{i}\pi}\Big((\mathcal{S}Y_{t_i})(x_i + \mathrm{i}0) - (\mathcal{S}Y_{t_i})(x_i - \mathrm{i}0)\Big)$, $i = 1, 2$. Our most general result in this direction is Theorem 3.1. A more explicit formula relying on Theorem 3.1 is Theorem 3.2 or Corollary 3.2 for equilibrium dynamics, see (3.70) for the specific case of a quartic (Landau-Ginzburg type) potential. The reader should compare the above results to those obtained by M. Duits [10] in a stochastic setting for fluctuations of non-colliding processes, and by N. Allegra, P. Calabrese, J. Dubail, J.-M. Stéphan and J. Viti [1],[9] in a condensed-matter context for the (real-time) propagator of the density field $\langle \rho(t_1, x_1)\rho(t_2, x_2)\rangle \equiv \langle (\psi^\dagger\psi)(t_1, x_1)(\psi^\dagger\psi)(t_2, x_2)\rangle$ of a one-dimensional Fermi gas submitted to a confining potential V. Despite the difference of language, and the fact that an analytic continuation in time is necessary to go from one situation to the other, both series of works come to a similar conclusion. Focusing on the quantum setting, and considering the low-lying spectrum of the underlying N-particle quantum Hamiltonian, the authors predict (and confirm by some numerical simulations) that (assuming the theory to be free, i.e. Gaussian at large scale) the time-evolution equation obtained for the Wigner function in the semi-classical limit is essentially correct in the large N limit. The time-evolution equation for the chiral part of the two-point function is then the same as ours (compare e.g. our equation (3.59) to eq. (6) in [9]), taking as input the equilibrium density ρ_{eq} computed by local-density approximation, see e.g. discussion in section **A.** of [6] or articles cited above. Then, in both situations, the fluctuation/density field is interpreted as a 2d Gaussian free field in a curved space with metric tensor $ds^2 = e^{2\sigma} dz\, d\bar{z}$, with coordinate transform $z = z(x, y)$ and conformal weight $\sigma = \sigma(x, y)$ chosen by requiring that $e^{\sigma(x,y)} dz = dx + \mathrm{i}\pi\rho_{eq}(x) dy$, which yields ([9], eq. (20)): $z(x, y) = \frac{1}{\pi}(G(x) + \mathrm{i}\pi y)$, where $G(x) := \int \frac{dx}{\rho_{eq}(x)}$, in exact correspondence with our Theorem 3.2. Therefore its law may be obtained from that of flat 2d Gaussian free field through a conformal transformation. The connection of our results to those is however lost at that point, since the single-time covariance kernel $\mathrm{Cov}(Y_t(x_1), Y_t(x_2))$ is (up to a simple scaling) independent of the potential, hence of ρ_{eq}. It would be interesting to obtain a deeper understanding of this difference.

2 The Mc Kean-Vlasov equation

We study in this section eq. (1.5) indirectly through the time-evolution of its Stieltjes transform

$$U_t(z) := \int dx\, \frac{\rho_t(x)}{x - z}, \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.1}$$

As shown in [33],[19], U_t satisfies following generalized complex Burgers equation,

$$\frac{\partial U_t(z)}{\partial t} = \frac{\partial}{\partial z}\big(\frac{\beta}{4}U_t^2(z) + V'(z)U_t(z) + T_t(z)\big), \tag{2.2}$$

where

$$T_t(z) := \int dx\, \rho_t(x) \frac{V'(x) - V'(z)}{x - z}. \tag{2.3}$$

When V is *harmonic*, $T_t(z)$ is a constant, whence $T_t'(z) = \frac{d}{dz}T_t(z) \equiv 0$. But in general, T_t is an unknown time-dependent quantity for which an independent equation should be provided. For V polynomial, however, say, $\deg(V) =: 2n$, $T_t(z)$ is easily seen [20] to be some explicit polynomial in z of order $\le 2n-2$, with coefficients in the linear span of the $2n-2$ first moments of the unknown density ρ_t, namely, $T_t(z) = \sum_{k=0}^{2n-2} c_k \int x^k \rho_t(x)\, dx$ for some constants $c_k = c_k[V]$. Looking at the asymptotic expansion of U_t at infinity, $-U_t(z) \sim \frac{1}{z} + \Big(\int x\rho_t(x)\, dx\Big)\frac{1}{z^2} + \Big(\int x^2\rho_t(x)\, dx\Big)\frac{1}{z^3} + \cdots$, $T_t(z)$ may also be defined (up to an additive constant) as minus the part polynomial in z of $V'(z)U_t(z)$, so that $\frac{\partial U_t(z)}{\partial t} = O(1/z^2)$ when $z \to \infty$, in coherence with the leading term of the expansion, $-U_t(z) \sim_{z\to\infty} 1/z$. Projecting (2.2) onto the linear subspace $\oplus_{k\ge 0}\mathbb{C}z^{-k-1}$ yields an infinite system of coupled ODEs for the moments $\big(\int x^k\rho_t(x)\, dx\big)_{k\ge 0}$, which in principle can be solved numerically on short time-intervals.

Notation. We let $\Pi_+ := \{z \in \mathbb{C} | \mathrm{Im}\ z > 0\}$ be the upper half-plane, and $\bar{\Pi}_+ := \Pi_+ \cup \mathbb{R}$ its closure.

We make in this section the following

Assumptions.

(i) V *is a* strictly convex *polynomial of order* $2n$, *i.e.* $\inf_{x\in\mathbb{R}} V''(x) \ge \alpha > 0$*;*

(ii) $U_0\big|_{\Pi_+}$ *extends to a* continuous function $U_0 : \bar{\Pi}_+ \to \mathbb{C}$*.*

Since (by Plemelj's formula, see §5.1), $\lim_{\varepsilon\to 0^+} \mathrm{Im}\ U_0(x + \mathrm{i}\varepsilon) = \pi\rho_0(x)$, Assumption (ii) implies in particular that ρ_0 is a continuous function.

2.1 An example: scaling solution in the Hermite case

In this paragraph, we assume that $\beta = 2$ and $V(x) = \frac{x^2}{2}$, and look for simple solution of (2.2) other than the constant solution ρ_{eq}. By reference to the underlying equilibrium unitary ensemble, we call this case the *Hermite case*.

Explicit formulas. The equilibrium density corresponds to the semi-circle law, $\rho_{eq}(x) \equiv \frac{1}{\pi}\sqrt{2-x^2}\,\mathbf{1}_{|x|<\sqrt{2}}$, with support $[-\sqrt{2}, \sqrt{2}]$ and Stieltjes transform $U_{eq}(z) \equiv -z + \sqrt{z^2-2}$ continuously extending to the real line,

$$U_{eq}(x \pm \mathrm{i}0) = -x \pm \mathrm{i}\sqrt{2-x^2} \qquad (|x| < 2), \qquad U_{eq}(x \pm \mathrm{i}0) = -x + \sqrt{x^2-2} \qquad (|x| > 2). \tag{2.4}$$

Taking the boundary value, $\frac{1}{2\mathrm{i}\pi}(U_{eq}(x + \mathrm{i}0) - U_{eq}(x - \mathrm{i}0)) \equiv \frac{1}{\pi}\mathrm{Im}\ U_{eq}(x + \mathrm{i}0)$, yields $\rho_{eq}(x)$; the functions $U_{eq}(x \pm \mathrm{i}0)$ are real-valued on $\mathbb{R} \setminus [-\sqrt{2}, \sqrt{2}]$ and $U_{eq}(\bar{z}) = \overline{U_{eq}(z)}$, hence (by Schwarz's extension lemma) U_{eq} extends to a holomorphic function (still called U_{eq}) on the cut plane $\mathbb{C} \setminus [-\sqrt{2}, \sqrt{2}]$. Note that U_{eq}' is singular in the neighbourhood of the ends of the support, $\pm\sqrt{2}$; namely, $U_{eq}'(\pm(\sqrt{2}+\varepsilon)) \sim_{\varepsilon\to 0^+} \pm c/\sqrt{\varepsilon}$ $(c > 0)$.

Scaling solution. Assume that $\rho_0(x) := \frac{1}{s}\rho_{eq}(x/s)$ $(s > 0)$, or equivalently, $U_0(z) := \frac{1}{s}U_{eq}(z/s)$. Then we use the following Ansatz,

$$U_t(z) \equiv \frac{1}{s(t)}U_{eq}(z/s(t)) \tag{2.5}$$

for some unknown scaling function $t \mapsto s(t)$, corresponding to a time-dependent support $[-s(t)\sqrt{2}, s(t)\sqrt{2}]$. From (2.2), we obtain for the stationary solution $(U_0(z)+z)U_0'+U_0 = 0$. Hence

$$\begin{aligned} 0 &= \frac{dU_t(z)}{dt} - (U_t(z)+z)U_t'(z) - U_t(z) \\ &= \frac{1}{s^2(t)}\Big\{(-U_{eq}(\frac{z}{s(t)}) + \frac{z}{s(t)}U_{eq}'(\frac{z}{s(t)}))\dot{s}(t) - (\frac{1}{s(t)}U_{eq}(\frac{z}{s(t)}) + z)U_{eq}'(\frac{z}{s(t)}) \\ &\quad - s(t)U_{eq}(\frac{z}{s(t)})\Big\} \\ &= -\frac{1}{s^2(t)}(U_{eq}(\frac{z}{s(t)}) + \frac{z}{s(t)}U_{eq}'(\frac{z}{s(t)}))\left\{\dot{s} + s - \frac{1}{s}\right\}. \end{aligned} \tag{2.6}$$

Hence our Ansatz is correct provided we choose $s(t)$ to be the solution of the ode $\dot{s} = \frac{1}{s} - s$, namely,

$$s(t) \equiv \sqrt{1 + e^{-2t}(s^2(0)-1)}. \tag{2.7}$$

Equivalently, $\frac{s^2(t)-1}{s^2(0)-1} = e^{-2t}$, which means that the "radius" $b_t := \sqrt{2}\,s(t)$ converges exponentially fast and monotonously to its equilibrium value, $\sqrt{2}$.

2.2 Regularity

As proved in our previous article [39] – extending uniform-in-time moment bounds proved in [2] in the harmonic case –, there exists $R = R(T)$ and $c, C > 0$ such that, for all $N \geq 1$, $\mathbb{P}[\sup_{0\leq t\leq T}\sup_{i=1,\ldots,N}|\lambda_t^{N,i}| > R] \leq Ce^{-cN}$ (see Proposition 3.1). Using Borel-Cantelli's lemma, one immediately deduces the following: for any test function $f : \mathbb{R} \to \mathbb{R}$ with support $\subset B(0,R)^c$, $\langle \rho_t, f\rangle = \lim_{N\to\infty}\langle X_t^n, f\rangle = 0$ a.s. Thus $\mathrm{supp}(\rho_t) \subset [-R, R]$ for every $t \leq T$. In particular, for every $n = 0, 1, \ldots$, the function $t \mapsto \int x^n \rho_t(x)\,dx$ $(0 \leq t \leq T)$ is bounded and continuous; which implies in turn that $t \mapsto T_t'(z)$ is a polynomial in z depending continuously on t.

Our main result in this subsection is

Theorem 2.1. Under the Assumptions of section 2, $U_t\Big|_{\Pi_+}$ extends to a contiuous function on $\bar{\Pi}_+$ for every $t \geq 0$. In particular, $x \mapsto \rho_t(x)$ is a continuous function for every $t \geq 0$.

A. (Case of a harmonic potential).

Then $\frac{d}{dz}T_t(z) \equiv 0$ and so (2.2) is a closed equation for U_t which can be solved on $\mathbb{C} \setminus \mathbb{R}$, where it is analytic, using the method of characteristics. We shall use this to derive the evolution of the support.

Characteristics. For definiteness we choose $V(x) = \frac{x^2}{2}$. Let $Z_t(z_0)$ be the solution at time $t \geq 0$ of the following differential equation,

$$\frac{dz}{dt} = -\frac{\beta}{2}U(t, z(t)) - z(t), \qquad z(0) = z_0 \in \Pi^+. \tag{2.8}$$

Letting $C(t) := -U(t, z(t))$ and substituting into (2.2) yields $\frac{d}{dt}C(t) = C(t)$, solved as $C(t) = e^t C(0)$. Differentiating (2.8) yields

$$\ddot{z} = \frac{\beta}{2}\dot{C} - \dot{z} = z \tag{2.9}$$

with $z(0) = z_0, \dot{z}(0) = -\frac{\beta}{2}U_0(z_0) - z_0$ hence

$$Z_t(z_0) = z_0\mathrm{ch}t - \big[\frac{\beta}{2}U_0(z_0) + z_0\big]\mathrm{sh}t = z_0 e^{-t} - \frac{\beta}{2}U_0(z_0)\mathrm{sh}t. \tag{2.10}$$

Since $\text{Im } U_0(z_0) \geq 0$ by (5.6) for $z_0 \in \Pi_+$, $t \mapsto \text{Im } Z_t(z_0)$ decreases and the characteristics may eventually cross the real axis, after which the characteristic method makes no sense because of the discontinuity. So we decide to *kill* characteristics as sooon as they cross the real axis.
Let $t_{max}(z_0) := \inf\{t > 0 \mid Z_t(z_0) \in \mathbb{R}\} \in (0, +\infty]$; for every $T < t_{max}(z_0)$, there exists a neighbourhood $\mathcal{B}(z_0)$ of z_0 in Π_+ that is mapped inside Π_+. Hence characteristics (2.8) started from $\mathcal{B}(z_0)$ are well-defined up to time T, and define for every $t \leq T$ a one-to-one mapping into a time-dependent region $Z_t(\mathcal{B}(z_0)) \subset \Pi_+$. *Denote by* $\phi_t : Z_t(\mathcal{B}(z_0)) \to \mathcal{B}(z_0)$ *the inverse mapping,* $\phi_t(z) := Z_t^{-1}(z)$. Then

$$U_t(z) = e^t U_0(\phi_t(z)), z \in Z_t(\mathcal{B}(z_0)). \tag{2.11}$$

Solving instead *backwards* in time, one gets

$$\phi_t(z) = ze^t + \frac{\beta}{2} U_t(z)\text{sh}t. \tag{2.12}$$

Since $\text{Im } U_t(z) \geq 0$, it is apparent from (2.12) that $\phi_t : \Pi_+ \to \Pi_+$, with $\text{Im } \phi_t(z) \geq \text{Im } z$; this can be deduced, even without knowing the explicit formula (2.12), from (2.8), since $-\frac{dz}{dt} \in \Pi_+$ as long as $z(t) \in \Pi_+$. Let

$$\Pi_t := \phi_t(\Pi_+) \tag{2.13}$$

and $\bar{\Pi}_t \subset \bar{\Pi}_+$ its closure in $\bar{\Pi}_+$. Since $\Pi_t = \{z \in \Pi_+ \ Z_s(z) \in \Pi_+, 0 \leq s \leq t\}$, it is clear that the family of regions $(\Pi_t)_{t \geq 0}$ is decreasing for inclusion, i.e. $\Pi_t \supset \Pi_T$ for $T \geq t$. If $w_n := \phi_t(z_n)$, $z_n \in \Pi_+$ is a sequence in Π_t converging to w, then $z_n = Z_t(w_n) \to Z_t(w)$ by (2.10) since U_0 is continuous on $\bar{\Pi}_+$. Furthermore, if $|w_n| \to \infty$, then $|z_n| \sim e^{-t}|w_n| \to \infty$. Thus (see Rudin [34], Theorem 14.19) the map ϕ_t extends to a homeomorphism $\bar{\Pi}_+ \to \bar{\Pi}_t$, while the boundary $\partial \bar{\Pi}_t$ is a Jordan curve. *Hence* $U_t : z \mapsto e^t U_0(\phi_t(z))$ *extends to a continuous function on* $\bar{\Pi}_+$.

Specifically when $\beta = 2$ *(Hermite case)*, the density at time t may be interpreted as the free convolution of the time-zero density by a semi-circular law. P. Biane [5] proves then much more. First, whatever the initial condition, the measure at time $t > 0$ has a continuous density. Then, the density ρ_t is proved to be analytic on the open subset $\{\rho_t > 0\} := \{x \in \mathbb{R} \mid \rho_t(x) > 0\}$. Furthermore, $\overline{\{\rho_t > 0\}}$ is the support of the measure at time t, and $\rho_t(x) = O((d(x, \{\rho_t > 0\}^c)^{1/3})$ for $x \in \{\rho_t > 0\}$, where $d(x, \{\rho_t > 0\}^c)$ is the distance to the complentary set. This a priori surprising $1/3$-Hölder exponent gives the correct behavior of ρ_t at a point x where two components of the support merge at time t.

A simple example. Assume $U_t = U_{eq}$, $t \geq 0$. Then $\phi_t(z) = ze^t + (-z + \sqrt{z^2 - 2})\text{sh}t = z\text{ch}t + \sqrt{z^2 - 2}\,\text{sh}t$, whence (for $|x| < 2$) $\phi_t(x + \text{i}0) \equiv a(x) + \text{i}b(x)$, with $a(x) = x\text{ch}t, b(x) = \sqrt{2 - x^2}\,\text{sh}t$. Thus $\partial \bar{\Pi}_t$ is the union of $(-\infty, -\text{ch}t] \cup [\text{ch}t, +\infty)$ with the semi-ellipse defined by the equation $\{\frac{a^2}{\text{ch}^2 t} + \frac{b^2}{\text{sh}^2 t} = 2,\ b \geq 0\}$. This makes it plain enough that (somewhat counter-intuitively) characteristics *do not* follow the time-evolution of the support or the singularities of U_t on the real axis (see next subsection for more).

B. General case

The general case is similar, except that the time evolution of the $(2n-2)$ first moments of the density must be determined independently. Namely, instead of (2.8), we consider the generalized characteristics $Z_t(z_0)$, solution of the o.d.e.

$$\frac{dz}{dt} = -\frac{\beta}{2} U(t, z(t)) - V'(z(t)), \qquad z(0) = z_0. \tag{2.14}$$

and the mapping $\phi_t \equiv Z_t^{-1}$. Letting $C(t) := -U(t, z(t))$ and substituting into (2.2) yields $\frac{d}{dt}C(t) = V''(z(t))C(t) - T_t'(z(t))$, solved as

$$C(t) = A_0^t C(0) - \int_0^t dt'\, A_{t'}^t T_{t'}'(z(t')), \qquad A_{t'}^t := \exp\left(\int_{t'}^t ds\, V''(z(s))\right). \tag{2.15}$$

Differentiating (2.14) yields

$$\ddot{z} = V''(z)V'(z) - \frac{\beta}{2}T_t'(z) \tag{2.16}$$

with initial condition

$$z(0) = z_0, \qquad \dot{z}(0) = -\frac{\beta}{2}U_0(z_0) - V'(z_0) \tag{2.17}$$

whence

$$\dot{z} := \pm\sqrt{(V'(z))^2 - \beta(T_t(z) - T_0(z_0)) + \beta U_0(z_0)(\frac{\beta}{4}U_0(z_0) + V'(z_0))}. \tag{2.18}$$

Solving for T_t by some independent means (e.g. numerically), (2.18) can be solved numerically for short time knowing U_0 (and even by quadrature when T_t is constant, e.g. for equilibrium dynamics). However (due to the multi-valuedness of the square-root function on $\mathbb{C}$), eq. (2.18) stops making sense in general when the function inside the square-root vanishes. On the other hand, an unambiguous definition may be given in terms of the second-order differential equation (2.16), in its matrix form

$$\frac{d}{dt}\begin{pmatrix} z \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \dot{z} \\ V''(z)V'(z) - \frac{\beta}{2}T_t'(z) \end{pmatrix}. \tag{2.19}$$

Writing $V'(z) \sim_{z\to\infty} c_n z^{2n-1} + \ldots$, we get for $0 < b < 1$: $\mathrm{Im}\, V'(a + \mathrm{i}b) \sim_{a\to\infty} (2n-1)c_n a^{2n-2} b$, whence there exists $a_{max} \geq 0$ such that:

$$\Big(0 < b < 1, |a| \geq a_{max}\Big) \Rightarrow \mathrm{Im}\, V'(a + \mathrm{i}b) > 0. \tag{2.20}$$

On the other hand, since V is strictly convex, there exists $b_{max} \in (0, 1)$ such that

$$\Big(0 < b < b_{max}, |a| \leq a_{max}\Big) \Rightarrow \mathrm{Re}\, V''(a + \mathrm{i}b) > 0; \tag{2.21}$$

for such a, b one thus gets $V'(a + \mathrm{i}b) - V'(a) = \mathrm{i}\int_0^b V''(a + \mathrm{i}y)\, dy \in \Pi_+$. Thus (see (2.14)) $-\frac{dz}{dt} \in \Pi_+$ as in the harmonic case, providing one restricts to the strip $\mathrm{Im}\, z \in (0, b_{max})$. The rest of the argument proceeds as in the previous subsection if one restricts to characteristics included either in $[-a_{max}, a_{max}] \times [0, b_{max}]$ or in $(\mathbb{R} \setminus [-a_{max}, a_{max}]) \times [0, 1]$. *Hence, letting* $z_0 \equiv \phi_t(z)$ *so that* $z(t') = Z_{t'}(\phi_t(z)) = \phi_{t-t'}(z)$*,*

$$U_t : z \mapsto A_0^t U_0(\phi_t(z)) + \int_0^t dt'\, A_{t'}^T T_{t'}^t(\phi_{t-t'}(z)), \tag{2.22}$$

see (2.14,2.15) with $z(0) = \phi_t(z)$*, extends to a continuous function on* $\bar{\Pi}_+$*,* proving Theorem 2.1 in whole generality.

2.3 Support

In this paragraph we study the time evolution of the *external support* $[a_t, b_t]$ defined as the intersection of all intervals $[a, b]$ such that $\langle \rho_t, \phi \rangle \equiv 0$ for every test function ϕ with compact support $\subset \mathbb{R} \setminus [a, b]$. Using the characteristics introduced in the previous subsection, we shall be able to give a defining formula for a_t, b_t $(t \geq 0)$.

Exactly as in the example developed in §2.1, and for the same reasons, the function U_0 has a maximal analytic extension to the cut plane $\mathbb{C} \setminus [a_0, b_0]$, which is real-valued and real-analytic on $\mathbb{R} \setminus [a_0, b_0]$. Thus the characteristics $t \mapsto Z_t(x_0)$ issued from $x_0 > b_0$, as defined by (2.14), is well-defined and real-valued for t small enough. As long as the characteristics $(z_s)_{0\le s\le t}$, $z_s := Z_s(x_0)$ remains $\gg b_0$, i.e. for x_0 large enough, the dominant term inside the square-root in (2.18) is $(V'(z))^2 \sim (c_n z^{2n-1})^2$ ($c_n > 0$), the sign is unambiguously a minus sign, $\dot z \approx -V'(z)$, and characteristics may not cross: for $t \le T$ fixed and $b_{max} > b_0$ large enough, the mapping $[b_{max}, +\infty) \to \mathbb{R}, x_0 \mapsto Z_t(x_0)$ is an increasing, real-analytic diffeomorphism on its image. On the other hand, taking the derivative of (2.19) with respect to the initial condition $\begin{pmatrix} x_0 \\ \dot x_0 \end{pmatrix}$, $\dot x_0 = -\frac{\beta}{2} U(x_0) - V'(x_0)$, one can in general only write $Z'_t(x_0)$ in terms of some time-ordered exponential of matrices,

$$Z'_t(x_0) = \Big[\overrightarrow{\exp}\Big(\int_0^t ds \begin{pmatrix} 0 & 1 \\ (V''V' - \frac{\beta}{2}T'_s)'(x_s) & 0 \end{pmatrix}\Big) \cdot \begin{pmatrix} 1 \\ -\frac{\beta}{2}U'_0(x_0) - V''(x_0) \end{pmatrix}\Big]_1, \quad (2.23)$$

$[\,\cdot\,]_1$=1st component, a complicated formula from which no general rule to guess the possible vanishing of $Z'_t(x_0)$ can be expected. Let us illustrate this on the simple Hermite case where $\beta = 2$ and $V(x) = \frac{x^2}{2}$, and characteristics are explicit (see **A.** of last subsection). When $x_0 \to \infty$, $Z_t(x_0) \sim e^{-t}x_0 + \frac{\beta}{2}\mathrm{sh}\, t\, x_0^{-1} + O(x_0^{-2})$, hence in particular $x_0 \mapsto Z_t(x_0)$ is increasing for x_0 large. On the other hand, one may expect that $\frac{dZ_t(x_0)}{dx_0} = \mathrm{ch} t - \Big[\frac{\beta}{2}U'_0(x_0) + 1\Big]\,\mathrm{sh}\, t \to_{x_0 \to b_0^+} -\infty$ for all $t > 0$, which does happen e.g. when $U_0(z) = \frac{1}{s}U_{eq}(z/s)$ is a rescaling of the equilibrium solution U_{eq}.

Define:

$$b_0^*(t) := \sup\{x_0 > b_0 \mid \min_{s\in[0,t]} Z'_s(x_0) \le 0\}. \quad (2.24)$$

By construction, $(b_0^*(t), +\infty)$ is the largest interval of the form $(x_0, +\infty)$, $x_0 \ge b_0$, such that $Z_s : (b_0^*(t), +\infty) \to (Z_s(b_0^*(t)), +\infty)$ is a diffeomorphism for all $0 \le s \le t$. Then $Z_t\Big|_{(b_0^*(t),+\infty)}$ extends analytically on some complex neighbourhood $\mathcal{B}(b_0^*(t), +\infty)$ of $(b_0^*(t), +\infty)$ to a conformal mapping with inverse ϕ_t. Thus the function U_t defined on the image $Z_t(\mathcal{B}(b_0^*(t), +\infty))$ by (2.22) is a holomorphic solution of (2.2). Hence $\mathrm{supp}(\rho_t) \subset (-\infty, b_t]$, where

$$b_t := Z_t(b_0^*(t)). \quad (2.25)$$

Also, $\min_{s\in[0,t]} Z'_s(b_0^*(t)) = 0$, so let $s_0 := \min\{s \in [0,t] \mid Z'_s(b_0^*(t)) = 0\}$. Then $Z'_s(b_0^*(t)) > 0$ for all $s < s_0$, so necessarily $\frac{d}{dx_0}\dot Z_{s_0}(b_0^*(t)) = \frac{d}{ds_0}Z'_{s_0}(b_0^*(t)) \le 0$. Since

$$\begin{aligned} \frac{d}{ds_0}Z'_{s_0}(x_0) &= \Big[\begin{pmatrix} 0 & 1 \\ (V''V' - \frac{\beta}{2}T'_{s_0})'(x_{s_0}) & 0 \end{pmatrix} \cdot \\ &\quad \cdot \overrightarrow{\exp}\Big(\int_0^{s_0} ds \begin{pmatrix} 0 & 1 \\ (V''V' - \frac{\beta}{2}T'_s)'(x_s) & 0 \end{pmatrix}\Big) \cdot \begin{pmatrix} 1 \\ -\frac{\beta}{2}U'_0(x_0) - V''(x_0) \end{pmatrix}\Big]_1 \\ &= \Big[\overrightarrow{\exp}\Big(\int_0^{s_0} ds \begin{pmatrix} 0 & 1 \\ (V''V' - \frac{\beta}{2}T'_s)'(x_s) & 0 \end{pmatrix}\Big) \cdot \begin{pmatrix} 1 \\ -\frac{\beta}{2}U'_0(x_0) - V''(x_0) \end{pmatrix}\Big]_2, \end{aligned} \quad (2.26)$$

the simultaneous vanishing of $Z'_{s_0}(b_0^*(t))$ and $\frac{d}{ds_0}Z'_{s_0}(b_0^*(t))$ implies that the two-component vector between square brackets $\Big[\,\cdot\,\Big]$ in (2.23) vanishes, which is impossible since $\overrightarrow{\exp}\Big(\,\cdot\,\Big)$ is invertible. So, actually, $\frac{d}{ds_0}Z'_{s_0}(b_0^*(t)) < 0$, which is contradictory with the definition of

$b_0^*(t)$ if $s_0 < t$. Hence $b_0^*(t)$ is also defined more simply as

$$\begin{aligned} b_0^*(t) &:= \sup\{x_0 > b_0 \mid Z_t'(x_0) \le 0\} \\ &= \inf\{x_0 > b_0 \mid Z_t(\cdot) : (x_0, +\infty) \to (Z_t(x_0), +\infty) \text{ is a diffeomorphism}\}. \end{aligned} \tag{2.27}$$

Conversely, suppose U_t were analytic at $b_0^*(t)$, then (2.14) would imply that $Z_t'(b_0^*(t)) \neq 0$, a contradiction. Hence $\mathrm{supp}(\rho_t) \not\subset (-\infty, b_t - \varepsilon)$ for any $\varepsilon > 0$.

We now claim that the function $t \mapsto b_t$ is càdlàg, i.e. right-continuous with left limits. Furthermore, it doesn't have any positive jumps, i.e. $b_t \le \lim_{t' \to t, t' < t} b_{t'}$. (On the other hand, we cannot exclude negative jumps, with $\rho_{t'}\big|_{[b_t, b_{t-}]} \to_{t' \to t, t' < t} 0$ pointwise). Namely, (i) $b_t \le \liminf_{t' \to t} b_{t'}$; otherwise (by absurd), letting $b \in (\liminf_{t' \to t} b_{t'}, b_t)$, we would have $\int_{-\infty}^b \rho_{t_n}(x)dx \to 1 > \int_{-\infty}^b \rho_t(x)dx$, where $(t_n)_{n \ge 1}$ is a sequence such that $t_n \to t$, $b_{t_n} \to \liminf_{t' \to t} b_{t'}$ and $b_{t_n} < b$, which is incompatible with the fact that the measure $\rho_s(x)dx$ depends continuously on s; (ii) $\limsup_{t' \to t, t' > t} b_{t'} \le b_t$, as follows from the characteristic method developed above; (iii) imagine (by absurd) that $b_{min}^- := \liminf_{t' \to t, t' < t} b_{t'} < b_{max}^- := \limsup_{t' \to t, t' < t} b_{t'}$. Choose $b_{min}^- < b < b' < b_{max}^-$. Let $t_n \to t$ (resp. $t_n' \to t$) a sequence such that $t_n, t_n' < t$ and $b_{t_n} \le b$ (resp. $b_{t_n'} \ge b'$). Then there exist characteristics moving by an amount $b' - b$ in arbitrary small time, which is contradictory with previous arguments.

We similarly define $a_0^*(t)$ by requiring that $(-\infty, a_0^*(t)]$ be the largest interval of the form $(-\infty, x_0)$, $x_0 \le a_0$, such that $Z_t(\cdot) : (-\infty, x_0) \to (-\infty, Z_t(x_0))$ is a diffeomorphism. Then it follows from the above that

$$[a_t, b_t] := [Z_t(a_0^*(t)), Z_t(b_0^*(t))] \tag{2.28}$$

is the external support of ρ_t.

Let us illustrate this with the example of the scaling solution of §2.1. We find from (2.10)

$$Z_t(z_0) = z_0 \mathrm{ch} t + \left[\frac{2}{b_0^2}(z_0 - \sqrt{z_0^2 - b_0^2}) - z_0\right] \mathrm{sh} t, \quad \frac{dZ_t(z_0)}{dz_0} = \mathrm{ch} t + \left[\frac{2}{b_0^2}(1 - \frac{z_0}{\sqrt{z_0^2 - b_0^2}}) - 1\right] \mathrm{sh} t \tag{2.29}$$

The Jacobian $x_0 \mapsto \frac{dZ_t(x_0)}{dx_0}$ vanishes for a single value $x_0^*(t) > b_0$, determined by

$$\frac{x_0^*(t)}{\sqrt{(x_0^*(t))^2 - b_0^2}} = 1 + \frac{b_0^2}{2}(\coth t - 1), \quad \sqrt{(x_0^*(t))^2 - b_0^2} = \left[\frac{b_0^2}{4}(\coth t - 1)^2 + (\coth t - 1)\right]^{-1/2}. \tag{2.30}$$

Easy but tedious computations yield

$$2\mathrm{sh}^2 t / ((x_0^*(t))^2 - b_0^2) = 1 + (\frac{b_0^2}{2} - 1)e^{-2t} \tag{2.31}$$

$$\begin{aligned} Z_t(x_0^*(t)) &= \sqrt{(x_0^*(t))^2 - b_0^2}\left\{(1 + \frac{b_0^2}{2}(\coth t - 1))\mathrm{ch} t \right. \\ &\qquad \left. + \left[\coth t - 1 - (1 + \frac{b_0^2}{2}(\coth t - 1))\right] \mathrm{sh} t\right\} \\ &= \sqrt{\frac{(x_0^*(t))^2 - b_0^2}{\mathrm{sh}^2 t}}\left\{(2 - r_0^2)\mathrm{sh} t \, \mathrm{ch} t + \frac{b_0^2}{2}(\mathrm{ch}^2 t + \mathrm{sh}^2 t) - 2\mathrm{sh}^2 t\right\} \\ &= \sqrt{\frac{(x_0^*(t))^2 - b_0^2}{\mathrm{sh}^2 t}}\left\{1 + (\frac{b_0^2}{2} - 1)e^{-2t}\right\} \\ &= \sqrt{2}\, s(t) \end{aligned} \tag{2.32}$$

as expected.

3 Kernel of the fluctuation process

We give in this section formulas for the distribution-valued covariance kernel

$$g_{1,2}(t_1,x_1;t_2,x_2) := \mathrm{Cov}\Big(Y_{t_1}(x_1), Y_{t_2}(x_2)\Big) \tag{3.1}$$

of the asymptotic fluctuation process $(Y_t)_{t\geq 0}$. The proof is indirect. First we obtain an evolution equation for the Stieltjes transformed covariance kernels

$$g_{1,2}^{\varepsilon_1,\varepsilon_2}(t_1,x_1;t_2,x_2) := \lim_{y\to 0^+} \Lambda(t_1, x_1+\mathrm{i}\varepsilon_1 y; t_2, x_2+\mathrm{i}\varepsilon_2 y), \qquad \varepsilon_1,\varepsilon_2=\pm \tag{3.2}$$

which are the boundary values of the kernel $\Lambda : (\mathbb{C}\setminus\mathbb{R})\times(\mathbb{C}\setminus\mathbb{R})\to\mathbb{C}$ defined by

$$\Lambda(t_1,z_1;t_2,z_2) := \mathrm{Cov}\Big((\mathcal{S}Y_{t_1})(z_1), (\mathcal{S}Y_{t_2})(z_2)\Big). \tag{3.3}$$

where $\mathrm{Cov}(Z_1,Z_2)$ for two complex-valued random variables $Z_1 = X_1+\mathrm{i}Y_1, Z_2 = X_2+\mathrm{i}Y_2$ means $\mathrm{Cov}(X_1,X_2)-\mathrm{Cov}(Y_1,Y_2)+\mathrm{i}(\mathrm{Cov}(X_1,Y_2)+\mathrm{Cov}(X_2,Y_1))$. In (3.3), $\mathcal{S}Y_t : \mathbb{C}\setminus\mathbb{R}\to\mathbb{C}$ is the Stieltjes transform of Y_t, $(\mathcal{S}Y_t)(z) := \langle Y_t, \frac{1}{\cdot - z}\rangle$. Then we use Plemelj's formula, $\frac{1}{x-(x_j+\mathrm{i}0)} - \frac{1}{x-(x_j-\mathrm{i}0)} = 2\mathrm{i}\pi\delta(x-x_j)$, $j=1,2$, and obtain

$$\begin{aligned} g_{1,2}(t_1,x_1;t_2,x_2) &= -\frac{1}{4\pi^2}\Big[g_{1,2}^{+,+} - g_{1,2}^{-,+} - g_{1,2}^{+,-} + g_{1,2}^{-,-}\Big](t_1,x_1;t_2,x_2) \\ &= -\frac{1}{2\pi^2}\,\mathrm{Re}\,\Big[g_{1,2}^{+,+} - g_{1,2}^{+,-}\Big](t_1,x_1;t_2,x_2) \end{aligned} \tag{3.4}$$

All these formulas are to be understood in a distribution sense.

Though we are not able to solve (1.16) for an arbitrary test function f_T, it turns out that the limiting evolution equation for $f_T(x) := \sum_{k=1}^n \frac{c_T^k}{x-z_T^k}$ when $\mathrm{Im}\, z_T^k \to 0^+$ (see Introduction) is an explicit transport equation, which is the key to the PDE we obtain for the kernel $g^{\pm,\pm}$; see Theorem 3.1. This PDE can be solved in terms of the characteristics (see (3.52)). In the stationary case one gets a more explicit formula (see Theorem 3.2 and Corollary 3.8).

We end this section with the interesting case of a quartic potential, $V(x) = \frac{1}{4}t^4 + \frac{c}{2}t^2 + d$ ($c>0$), for which computations can be made totally explicit (see eq. (3.70)).

3.1 General framework

We collect here those notations and results proved in our previous article [39] which are necessary for the present study.

3.1.1 Assumptions

Our Assumptions in this section are of three different types.

Assumptions on the potential.

We assume that V is convex and C^{11}.

The convexity assumption on V is essential for the convergence of the finite N-density to the solution ρ_t of the Mc Kean-Vlasov equation, see [24], and for Johansson's universal formula for equilibrium fluctuations to apply [20], see §3.4 below. The extra regularity assumptions on V have been used in [39] for semi-group estimates and in some perturbation arguments. Later on (see end of §3.3, and §3.4), we shall further assume that

V extends analytically to an entire function $V : \mathbb{C} \to \mathbb{C}$ in order to get more explicit formulas.

Assumptions on the initial measure.

Let $\mu_0^N = \mu_0(\{\lambda_0^i\}_i)$ be the initial measure of the stochastic process $\{\lambda_t^i\}_{t\geq 0, i=1,\ldots,N}$, and $X_0^N := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_0^i}$ be the initial empirical measure. Since N varies, we find it useful here to add an extra upper index $(\lambda_0^{N,i})_{i=1,\ldots,N}$ to denote the initial condition of the process for a given value of N. *We assume that:*

(i) (large deviation estimate for the initial support) there exist some constants $C_0, c_0, R_0 > 0$ such that, for every $N \geq 1$,

$$\mathbb{P}[\max_{i=1,\ldots,N} |\lambda_0^{N,i}| > R_0] \leq C_0 e^{-c_0 N}. \tag{3.5}$$

(ii) $X_0^N \overset{law}{\to} \rho_0(x)\,dx$ when $N \to \infty$, where $\rho_0(x)$ is a deterministic measure;

(iii) (rate of convergence)

$$\left(\mathbb{E}[|U_0^N(z) - U_0(z)|^2]\right)^{1/2} = O(\frac{1}{Nb}) \tag{3.6}$$

for $z = a + \mathrm{i}b \in \mathbb{C} \setminus \mathbb{R}$, where $U_0(z) := \int dx\, \frac{\rho_0(x)}{x-z}$ is the Stieltjes transform of ρ_0.

As proved in [39], the initial large deviation estimate (i) implies a uniform-in-time large deviation estimate for the support of the random point measure:

Proposition 3.1. (see [39], Lemma 5.1) Assume (i) holds for some constants $R_0, c_0, C_0 > 0$. Let $T > 0$. There exists some radius $R = R(T)$ and constant c, depending on V and R_0, c_0 but uniform in N, such that

$$\mathbb{P}\left[\sup_{0\leq t\leq T}\sup_{i=1,\ldots,N} |\lambda_t^{N,i}| > R\right] \leq Ce^{-cN}. \tag{3.7}$$

Finally, as in section 2, we add a

Regularity assumption on the initial density.

We assume that the Stieltjes transform $U_0\Big|_{\Pi_+}$ of the initial density ρ_0 on the upper-half plane extends to a continuous function $\bar{\Pi}_+ \to \mathbb{C}$.

Though this Assumption is probably unnecessary, it is natural, holds true in all examples treated below, and allows stating convergence results in a stronger sense.

3.1.2 Summary of results

All results presented here come from our previous article [39].

Notation. Generally speaking and without further mention, if $z, z_1, z_2, \ldots$ are complex numbers, then we write $z = a + \mathrm{i}b, z_1 = a_1 + \mathrm{i}b_1, z_2 = a_2 + \mathrm{i}b_2, \ldots$ their decomposition into real/imaginary part.

Definition 3.2 (Sobolev spaces)**.** Let $H_n := \{f \in L^2(\mathbb{R}) \mid ||f||_{H_n} < \infty\}$ $(n \geq 0)$, where $||f||_{H_n} := \left(\int d\xi\,(1+|\xi|^2)^n |\mathcal{F}f(\xi)|^2\right)^{1/2}$, and $H_{-n} := (H_n)'$ its dual.

The measure-valued process

$$Y^N := N(X_t^N - X_t) \tag{3.8}$$

has been shown in [39] to converge in $C([0,T], H_{-14})$:

Proposition 3.3 (Gaussianity of limit fluctuation process)**.** (see [39], Main Theorem) Let Y_t^N be the finite N fluctuation process (3.8). Then:

1. $Y^N \stackrel{law}{\to} Y$ when $N \to \infty$, where Y is a Gaussian process. More precisely, Y^N converges to Y weakly in $C([0,T], H_{-14})$;
2. let $\phi_f(Y_t) := e^{\mathrm{i}\langle Y_t, f\rangle}$. Then

$$\mathbb{E}[\phi_{f_T}(Y_T)\big|\mathcal{F}_t] = \phi_{f_t}(Y_t)\ \exp\left(\frac{1}{2}\int_t^T \left[\mathrm{i}(1-\frac{\beta}{2})\langle X_s, (f_s)''\rangle - \langle X_s, ((f_s)')^2\rangle\right] ds\right) \tag{3.9}$$

   where $(f_s)_{0\le s\le T}$ is the solution of the asymptotic equation (1.12).

The main point of the proof has been to rewrite the evolution equation for $(f_t)_{0\le t\le T}$ in terms of a "quasi"-transport operator on functions on the upper half-plane. Let us briefly recapitulate how this is done.

Definition 3.4 (Stieltjes transform)**.** (i) Let $\mathfrak{f}_z(x) := \frac{1}{x-z}$ ($x \in \mathbb{R}$, $z \in \mathbb{C}\setminus\mathbb{R}$).
(ii) Let, for $z \in \mathbb{C}\setminus\mathbb{R}$,

$$U_t^N(z) := \langle X_t^N, \mathfrak{f}_z\rangle \equiv (\mathcal{S}X_t^N)(z) = \sum_{i=1}^N \frac{1}{\lambda_t^i - z} \tag{3.10}$$

and

$$U_t(z) := \langle X_t, \mathfrak{f}_z\rangle \equiv (\mathcal{S}X_t)(z) = \int \frac{\rho_t(x)}{x-z}\, dx \tag{3.11}$$

be the Stieltjes transform of X_t^N, resp. X_t.

Definition 3.5 (upper half-plane)**.** 1. Let $\Pi^+ := \{z \in \mathbb{C} \mid \mathrm{Im}\ (z) > 0\}$.
2. For $b_{max} > 0$, let $\Pi^+_{b_{max}} := \{z \in \mathbb{C} \mid 0 < \mathrm{Im}\ (z) < b_{max}\}$.
3. Let $\Pi^- := -\Pi^+$, $\Pi^-_{b_{max}} := -\Pi^+_{b_{max}}$ and $\Pi := \Pi^+ \uplus \Pi^-$, $\Pi_{b_{max}} := \Pi^+_{b_{max}} \uplus \Pi^-_{b_{max}}$.

Definition 3.6. Let, for $p \in [1,+\infty]$ and $b_{max} > 0$,

$$L^p(\Pi_{b_{max}}) := \{h : \Pi_{b_{max}} \to \mathbb{C} \mid h(\bar{z}) = \overline{h(z)}\ (z \in \Pi^+_{b_{max}}) \text{ and } ||h||_{L^p(\Pi_{b_{max}})} < \infty\}, \tag{3.12}$$

where

$$||h||_{L^p(\Pi_{b_{max}})} := \left(\int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} db\, |h(a,b)|^p\right)^{1/p} (p<\infty), \quad ||h||_{L^\infty(\Pi_{b_{max}})} := \sup_{z\in\Pi_{b_{max}}} |h(z)|. \tag{3.13}$$

The value of b_{max} is unessential, so we fix some constant $b_{max} > 0$ (e.g. $b_{max} = 1$) and omit the b_{max}-dependence in the estimates.

Definition 3.7 (**Stieltjes decomposition**)**.** Let $\kappa = 0, 1, 2, \dots$ and $h \in L^1(\Pi_{b_{max}})$. Let $R = R(T)$ be as in Proposition 3.1. We say that $f : \mathbb{R} \to \mathbb{R}$ *has Stieltjes decomposition* h *of order* κ *and cut-off* b_{max} *on* $[-R, R]$ if, for all $|x| \le R$,

$$f(x) = (\mathcal{C}^\kappa h)(x) := \int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} db\, (-\mathrm{i}b)\, \frac{|b|^\kappa}{(1+\kappa)!}\ \mathfrak{f}_z(x) h(a,b). \tag{3.14}$$

Thanks to the symmetry condition, $h(\bar{z}) = \overline{h(z)}$, (3.14) may be rewritten in the form

$$(\mathcal{C}^\kappa h)(x) = 2\int_{-\infty}^{+\infty} da \int_0^{b_{max}} db\, \frac{b^{1+\kappa}}{(1+\kappa)!}\ \mathrm{Im}\ [\mathfrak{f}_z(x) h(a,b)]\,, \tag{3.15}$$

from which it is apparent that f is indeed real-valued.

Various Stieltjes decompositions, following Israelsson [19], have been constructed in [39]. The simplest one consists in defining $h:(a,b)\mapsto \mathcal{K}^\kappa_{b_{max}}(f)(a)$ where $\mathcal{K}^\kappa_{b_{max}}:f\mapsto \mathcal{F}^{-1}(K^\kappa_{b_{max}})*f$ is the Fourier multiplication operator by $K^\kappa_{b_{max}}(s):=\Big(2\int_0^{b_{max}}db\,|b|^{1+\kappa}\cdot \mathcal{F}(\mathrm{Im}\,(\mathfrak{f}_{\mathrm{i}b}))(s)\Big)^{-1}$. When κ is *even*, it is proved (see [39], (2.13)) that

$$\mathcal{K}^\kappa_{b_{max}}=(1+\underline{\mathcal{K}}^\kappa_{b_{max}})\Big(-(\kappa+1)!\,\partial_x^{2+\kappa}+(-1)^{\kappa/2}(2+\kappa)b_{max}^{-2-\kappa}\Big), \tag{3.16}$$

where $|||\underline{\mathcal{K}}^\kappa_{b_{max}}|||_{L^1(\mathbb{R})\to L^1(\mathbb{R})},|||\underline{\mathcal{K}}^\kappa_{b_{max}}|||_{L^\infty(\mathbb{R})\to L^\infty(\mathbb{R})}=O(1)$. (We shall only need to consider $\kappa=0$ in the present article). Since in the sequel we want to focus on narrow strips around the real axis, one might think of taking the limit $b_{max}\to 0$. However, this introduces awkward boundary terms. Instead we fix $b_{max}>0$ and define $h:(a,b)\mapsto e^{-b/\varepsilon}\mathcal{K}^\kappa_{b_{max},\varepsilon}$ ($\varepsilon>0$), where $\mathcal{K}^\kappa_{b_{max},\varepsilon}$ is the Fourier multiplication operator by $K^\kappa_{b_{max},\varepsilon}(s):=\Big(\int_0^{b_{max}}db\,b^{1+\kappa}e^{-b/\varepsilon}\cdot\mathcal{F}(\mathrm{Im}\,(\mathfrak{f}_{\mathrm{i}b}))(s)\Big)^{-1}$. Similarly to (3.16), we get (specifically for $\kappa=0$)

$$\mathcal{K}^0_{b_{max},\varepsilon}=(1+\underline{\mathcal{K}}^0_{b_{max},\varepsilon})\Big(\mathcal{F}^{-1}((|s|+\varepsilon^{-1})^2)*\Big) \tag{3.17}$$

where $|||\underline{\mathcal{K}}^0_{b_{max},\varepsilon}|||_{L^1(\mathbb{R})\to L^1(\mathbb{R})},|||\underline{\mathcal{K}}^0_{b_{max},\varepsilon}|||_{L^\infty(\mathbb{R})\to L^\infty(\mathbb{R})}=O(1)$. The Fourier multiplication operator in the r.h.s. of (3.17) is not a differential operator any more:

$$\Big(\mathcal{F}^{-1}((|s|+\varepsilon^{-1})^2)*f\Big)(x)=\varepsilon^{-2}\Big(\int_{-\infty}^{+\infty}f(y)\,dy\Big)+\Big(2\varepsilon^{-1}H\partial_x-\partial_x^2\Big)f(x) \tag{3.18}$$

where H is the Hilbert transform (see Appendix). Note that the most singular term in $O(\varepsilon^{-2})$ is simply a constant.

In [39], we wrote down an explicit time-dependent operator $\mathcal{H}(t)$ such that the right-hand side of (1.12) for f_t decomposed as

$$f_t(x)\equiv(\mathcal{C}^\kappa h_t)(x)=\int_{-\infty}^{+\infty}da\int_{-b_{max}}^{b_{max}}db\,(-\mathrm{i}b)\,\frac{|b|^\kappa}{(1+\kappa)!}\;\mathfrak{f}_z(x)\,h_t(a,b) \tag{3.19}$$

(see Definition 3.7) is equal to

$$\mathcal{C}^\kappa(\mathcal{H}(t)h_t)(x)=\int_{-\infty}^{+\infty}da\int_{-b_{max}}^{b_{max}}db\,(-\mathrm{i}b)\,\frac{|b|^\kappa}{(1+\kappa)!}\,\mathfrak{f}_z(x)\,\mathcal{H}(t)(h_t)(a,b) \tag{3.20}$$

Note that, since Stieltjes decompositions are not unique, the operator $\mathcal{H}(t)$ is very under-determined. The essential features of the operator $\mathcal{H}(t)$ chosen in [39] are recapitulated in Appendix, see section 4; in particular, for $\kappa\ge 0$, $\mathcal{H}(t)$ is the generator of a time-inhomogeneous semi-group of L^p, $p\ge 1$, which is a bounded perturbation of a transport operator. Moving around the operator $\mathcal{H}(t)$ to the function $\mathfrak{f}_z(x)$, one obtains an operator $\mathcal{L}(t)$ which is a "twisted adjoint" of $\mathcal{H}(t)$,

$$\mathcal{L}(t):=w^{-1}(a,b)\mathcal{H}^\dagger(t)w(a,b) \tag{3.21}$$

with $w(a,b):=(-\mathrm{i}b)\frac{|b|^\kappa}{(1+\kappa)!}$ (see [39], eq. (3.18) for details). For $\kappa=-1$ (at least formally), $\mathcal{L}(t)=\mathcal{H}^\dagger(t)$, and h_t may be directly interpreted as a density in $\mathbb{C}\setminus\mathbb{R}$ exactly as in (1.14), so that $\mathcal{L}(t)$ is the direct generalization of (1.15) to an arbitrary potential.

3.2 The stationary covariance kernel in the Hermite case

We rewrite in appropriate coordinates the formulas for the covariance found by Israelsson in the stationary regime, in the Hermite case, i.e. when V is harmonic ($V(x) = x^2/2$) and $\beta = 2$.

The covariance kernel Λ of the Stieltjes transform of the fluctuation field satisfies the following obvious properties due to stationarity, where one assumes $t_1 \geq t_2$,

$$\Lambda(t_1, z_1; t_2, z_2) =: \Lambda(\Delta t; z_1, z_2) = \lim_{t\to+\infty} \Lambda(t + \Delta t, z_1; t, z_2) \tag{3.22}$$

with $\Delta t := t_1 - t_2 \geq 0$.

The covariance function Λ can be found by taking the $t \to \infty$ limit in (Bender [3], Corollary 2.4 p. 7) with $\beta = 2$, (take $\sigma = 1/2$ and $U(t, z) = \frac{1}{\sqrt{2}} U_{Bender}(t, \frac{z}{\sqrt{2}})$)

$$\Lambda(\Delta t; z_1, z_2) = e^{-\Delta t} \frac{\frac{1}{2} f'_\mu(\frac{z_1}{\sqrt{2}}) \cdot \frac{1}{2} f'_\mu(\frac{z_2}{\sqrt{2}})}{2 \left(\frac{1}{2} \left(\frac{1}{\sqrt{2}} f_\mu(\frac{z_1}{\sqrt{2}}) \right) \cdot \left(\frac{1}{\sqrt{2}} f_\mu(\frac{z_2}{\sqrt{2}}) \right) e^{-\Delta t} - 1 \right)^2} \tag{3.23}$$

where:

(i) $\frac{1}{\sqrt{2}} f_\mu(\frac{z}{\sqrt{2}}) = U_{eq}(z) = -z + \sqrt{z^2 - 2}$ is the Stieltjes transform of Wigner's semi-circle law; the boundary values on the support $[-\sqrt{2}, \sqrt{2}]$ are $U_{eq}(x \pm \mathrm{i}0) = -x \pm \mathrm{i}\sqrt{2 - x^2}$.

(ii) $\frac{1}{2} f'_\mu(\frac{z}{\sqrt{2}}) = U'_{eq}(z) = \frac{z}{\sqrt{z^2-2}} - 1$ is its derivative, with boundary values on the support $U'_{eq}(x \pm \mathrm{i}0) = \frac{x \mp \mathrm{i}\sqrt{2-x_1^2}}{\pm \mathrm{i}\sqrt{2-x_1^2}}$.

Letting $x_j \pm \mathrm{i}\sqrt{2 - x_j^2} =: \sqrt{2}\, e^{\pm i\theta_j}$ $(0 \leq \theta_j \leq \pi)$ for convenience, i.e. $x_j = \sqrt{2}\cos(\theta_j)$, $\pi\rho_{eq}(x) = \sqrt{2 - x_j^2} = \sqrt{2}\sin\theta_j$, we get

$$\Lambda(\Delta t; x_1 \pm \mathrm{i}0, x_2 \pm \mathrm{i}0) = \frac{-1}{2\sin(\pm\theta_1)\, \sin(\pm\theta_2)} \frac{e^{-\mathrm{i}(\pm\theta_1 \pm\theta_2) - \Delta t}}{(e^{-\mathrm{i}(\pm\theta_1 \pm\theta_2) - \Delta t} - 1)^2}, \tag{3.24}$$

from which

$$\begin{aligned} 2\sin(\theta_1)\sin(\theta_2) g_{1,2}(t_1, x_1; t_2, x_2) &= \frac{1}{4\pi^2} \left[\frac{e^{\mathrm{i}(\theta_1+\theta_2)-\Delta t}}{(e^{\mathrm{i}(\theta_1+\theta_2)-\Delta t} - 1)^2} + \frac{e^{\mathrm{i}(-\theta_1+\theta_2)-\Delta t}}{(e^{\mathrm{i}(-\theta_1+\theta_2)-\Delta t} - 1)^2} + \right. \\ &\quad \left. + \frac{e^{\mathrm{i}(\theta_1-\theta_2)-\Delta t}}{(e^{\mathrm{i}(\theta_1-\theta_2)-\Delta t} - 1)^2} + \frac{e^{-\mathrm{i}(\theta_1+\theta_2)-\Delta t}}{(e^{-\mathrm{i}(\theta_1+\theta_2)-\Delta t} - 1)^2} \right] \\ &= -\frac{1}{8\pi^2} \mathrm{Re} \left[\frac{1}{\sin^2 \frac{\theta_1 - \theta_2 + \mathrm{i}\Delta t}{2}} + \frac{1}{\sin^2 \frac{\theta_1 + \theta_2 + \mathrm{i}\Delta t}{2}} \right]. \end{aligned} \tag{3.25}$$

When $\Delta t = 0$, we find

$$\begin{aligned} 2\sin(\theta_1)\sin(\theta_2) g_{1,2}(t, \theta_1; t, \theta_2) &= -\frac{1}{8\pi^2} \left(\frac{1}{\sin^2(\frac{\theta_1 - \theta_2}{2})} + \frac{1}{\sin^2(\frac{\theta_1 + \theta_2}{2})} \right) \\ &= -\frac{1}{2\pi^2} \frac{1 - \cos\theta_1 \cos\theta_2}{(\cos\theta_1 - \cos\theta_2)^2} \end{aligned} \tag{3.26}$$

Eq. (3.26) is in agreement with Johansson's formula for equilibrium fluctuations (compare with the kernel of the operator $h \mapsto \delta^h$ of eq. (2.10) in [20], Theorem 2.4), in our case

$$g_{1,2}(t, x_1; t, x_2) = \frac{1}{2\pi^2} \frac{1}{\sqrt{2 - x_1^2}} \partial_{x_2} \left(\frac{\sqrt{2 - x_2^2}}{x_2 - x_1} \right) = -\frac{1}{2\pi^2} \frac{1}{\sqrt{(2 - x_1^2)(2 - x_2^2)}} \frac{2 - x_1 x_2}{(x_2 - x_1)^2}. \tag{3.27}$$

We shall now be able to formulate a *hydrodynamic fluctuation equation*, compare with H. Spohn's formulas (3.30) or (4.14) in [35].

From (3.24), one sees trivially that the boundary values $\Lambda(\Delta t; x_1 \pm \mathrm{i}0, x_2 + \pm \mathrm{i}0)$ satisfy the following evolution equation,

$$(\partial_t \mp \mathrm{i}\partial_{\theta_1})\,(\sin(\theta_1)\,\Lambda(\Delta t; x_1 \pm \mathrm{i}0, \cdot)) = 0 \tag{3.28}$$

or equivalently, considering the coordinate-transformed

$$\tilde{g}_{1,2}^{\varepsilon_1,\varepsilon_2}(\Delta t; \theta_1, \theta_2) := 2\sin(\theta_1)\sin(\theta_2) g_{1,2}^{-\varepsilon_1,-\varepsilon_2}(\Delta t; x_1, x_2) \tag{3.29}$$

so that $\tilde{g}_{1,2}^{\varepsilon_1,\varepsilon_2}(\cdot; \theta_1, \theta_2) d\theta_1\, d\theta_2 = g_{1,2}^{-\varepsilon_1,-\varepsilon_2}(\cdot; x_1, x_2) dx_1\, dx_2$,

$$(\partial_{t_1} \pm \mathrm{i}\partial_{\theta_1})\tilde{g}_{1,2}^{\pm,\cdot}(t_1, \theta_1; \cdot) = 0. \tag{3.30}$$

Since $\cos : (0,\pi) - \mathrm{i}\mathbb{R}_+^* \to \Pi_+$ and $\cos : (0,\pi) + \mathrm{i}\mathbb{R}_+^* \to \Pi_-$ are biholomorphisms, the boundary value identity $\frac{1}{2\mathrm{i}\pi}(\tilde{g}^+ - \tilde{g}^-) \equiv g$ on the real line may equally be seen as a boundary value identity on the unit circle in the variable $\tilde{z}_1 := e^{\mathrm{i}\theta_1}$, with $\tilde{g}^+$, resp. $\tilde{g}^-$ extending on $\{|z_1| < 1\}$, resp. $\{|z_1| > 1\}$.

Summing over $\varepsilon_1, \varepsilon_2$, we obtain the *hydrodynamic fluctuation equation*

$$\partial_{t_1}\tilde{g}_{1,2}(t_1, \theta_1; t_2, \theta_2) = \partial_{\theta_1}(H_1\tilde{g}_{1,2})(t_1, \theta_1; t_2, \theta_2). \tag{3.31}$$

where H_1 is the *periodic Hilbert transform* acting on the first variable (see Appendix, section 5). Since $\tilde{g}_{1,2}(t_1, \theta_1; t_2, \theta_2)$ is invariant under the parity symmetries $\theta_i \to -\theta_i$, $i = 1, 2$, we bother only about the first term $\tilde{g}_-(t_1, \theta_1; t_2, \theta_2) := -\frac{1}{8\pi^2}\mathrm{Re}\, \frac{1}{\sin^2 \frac{\theta_1 - \theta_2 + \mathrm{i}\Delta t}{2}}$ in (3.25), and add the other term, $\tilde{g}_+(t_1, \theta_1; t_2, \theta_2) = \tilde{g}_-(t_1, \theta_1; t_2, -\theta_2)$ by hand in the end. Since $\frac{\partial}{\partial\theta_1}\left(\frac{1}{2}\cot(\frac{\theta_1-\theta_2}{2})\right) = -\frac{1}{4}\frac{1}{\sin^2(\frac{\theta_1-\theta_2}{2})}$, compare with (5.11), we get in Fourier modes (see Appendix, section 6)

$$\hat{K}_\infty(n) = \frac{1}{2\pi^2}|n|, \qquad \frac{d}{d\Delta t}\Big|_{\Delta t=0}\hat{K}_\infty(t + \Delta t, t; n) = |n|\hat{K}_\infty(n) \qquad (n \in \mathbb{Z}) \tag{3.32}$$

where $K_\infty(\cdot,\cdot) = \tilde{g}_-(t, \cdot; t, \cdot)$ is the stationary covariance kernel. Thus *the asymptotic fluctuation process Y is the stationary solution of the Ornstein-Uhlenbeck (linear Langevin) equation*

$$\partial_t Y(t,\theta) = -\sqrt{-\partial^2/\partial\theta^2}\, Y(t,\theta) + \frac{1}{\pi}\,\partial_\theta\eta(t,\theta), \tag{3.33}$$

where $\sqrt{-\partial^2/\partial\theta^2} = \partial_\theta H$ is the convolution operator acting by multiplication on Fourier modes, viz. $\widehat{\sqrt{-\partial^2/\partial\theta^2}\phi}_n = |n|\hat{\phi}_n$, and $\eta = \eta(t,\theta)$ is a 2π-periodic space-time white noise admitting the parity symmetry, $\eta(t,-\theta) = \eta(t,\theta)$.

It is very instructive to compute the short-distance asymptotics in a scaled limit, $\Delta t = \varepsilon\delta t_{12} \to 0$, $x_1 - x_2 = \varepsilon\delta x_{12} \to 0$. Formula (3.25) implies in the angular coordinates

$$\tilde{g}_{1,2}(t + \varepsilon\delta t_{12}, \theta + \varepsilon\delta\theta_{12}; t, \theta) \sim_{\varepsilon\to 0} -\frac{1}{2\pi^2}\varepsilon^{-2}\mathrm{Re}\left[\frac{1}{(\delta\theta_{12} - \mathrm{i}\delta t_{12})^2}\right] \tag{3.34}$$

independently of θ, from which

$$\begin{aligned} g_{1,2}(t + \varepsilon\delta t_{12}, x + \varepsilon\delta x_{12}; t, x) \quad &\sim_{\varepsilon\to 0} \quad -\frac{1}{2\pi^2}\varepsilon^{-2}\mathrm{Re}\left[\frac{1}{(\delta x_{12} + \mathrm{i}\sqrt{2 - x^2}\,\delta t_{12})^2}\right] \\ &= \quad -\frac{1}{2\pi^2}\varepsilon^{-2}\mathrm{Re}\left[\frac{1}{(\delta x_{12} + \mathrm{i}\pi\rho_{eq}(x)\delta t_{12})^2}\right]. \end{aligned} \tag{3.35}$$

Note that only the first term in the r.-h.s. of (3.25) contributes to (3.35).

3.3 PDE for the covariance kernel: the general case

We shall now derive a PDE for $g_{1,2}^{+;\pm}$ in whole generality. (A PDE for $g_{1,2}^{-;\pm}$ is then obtained by complex-conjugating the first space coordinate.)

Theorem 3.1 (hydrodynamic fluctuation equation for general V)**.** The kernel $g_{1,2}^{+;\pm}(t_1,x_1;t_2,x_2)$ satisfies the following PDE in a weak sense,

$$\partial_{t_1} g_{1,2}^{+;\pm}(t_1,x_1;t_2,x_2) = -\partial_{x_1}\Big(\Big(\frac{\beta}{2}U_{t_1}(x_1+\mathrm{i}0)+V'(x_1)\Big)g_{1,2}^{+;\pm}(t_1,x_1;t_2,x_2)\Big), \tag{3.36}$$

that is, for any smooth, compactly supported test function $\psi=\psi(x_1)$,

$$\partial_{t_1}\int dx_1\,\psi(x_1)\,g_{1,2}^{+;\pm}(t_1,x_1;\cdot) = \int dx_1\,\psi'(x_1)\,\Big(\frac{\beta}{2}U_{t_1}(x_1+\mathrm{i}0)+V'(x_1)\Big)g_{1,2}^{+;\pm}(t_1,x_1;\cdot). \tag{3.37}$$

Remark. The product $U_{t_1}(x_1+\mathrm{i}0)g_{1,2}^{\pm;\pm}(t_1,x_1;\cdot)$ makes sense as a distribution because both $x_1\mapsto U_{t_1}(x_1+\mathrm{i}0)$ and $x_1\mapsto g_{1,2}^{\pm;\pm}(t_1,x_1;\cdot)$ are obtained by convolution with respect to the function $x\mapsto\frac{1}{x+\mathrm{i}0}$, hence have Fourier support $\subset \mathrm{supp}\Big(\mathcal{F}(x\mapsto\frac{1}{x+\mathrm{i}0})\Big)=\mathbb{R}_+$.

Proof. A short but non rigorous proof goes as follows. Fix $\kappa=-1$. Since $b_1(\mathcal{H}^{0,-1}_{nonlocal}(t)h)(a_1,b_1)=O(b_1)\to_{b_1\to 0}0$ (see (4.2)), we consider the limit when $N\to\infty$ and $b_1\to 0^+$ of the characteristic equations associated to $\mathcal{L}_{transport}:=\mathcal{H}^\dagger_{transport}$, see (3.21) and below, thus obtaining directly the solution of the evolution equation with terminal condition $f_{t_1}=\mathfrak{f}_{z_1}$, where $z_1\equiv a_{t_1}+\mathrm{i}b_{t_1}$. One finds

$$\begin{aligned}\mathcal{L}_{transport} &= \Big(v_{hor}(t,z)\partial_a+v_{vert}(t,z)\partial_b+\tau^{-1}(t,z)\Big)^\dagger\\ &= -v_{hor}(t,z)\partial_a-v_{vert}(t,z)\partial_b+\Big(\tau^{-1}(t,z)-\frac{\partial v_{hor}}{\partial a}(t,z)-\frac{\partial v_{vert}}{\partial b}(t,z)\Big).\end{aligned} \tag{3.38}$$

Explicit formulas (4.4,4.5,4.6) for $v_{hor},v_{vert},\tau^{-1}$ yield (as follows from easy explicit computation, of from [39], eq. (3.15),(3.41), (3.45) and (3.48), where one has set $b\equiv 0^+$)

$$\frac{da_t}{dt} = -\frac{\beta}{2}\mathrm{Re}\,U_t(a_t+\mathrm{i}0)-V'(a_t) \tag{3.39}$$

$$\frac{db_t}{dt} = -\frac{\beta}{2}\mathrm{Im}\,U_t(a_t+\mathrm{i}0) \tag{3.40}$$

$$\frac{dc_t}{dt} = \left[-\frac{\beta}{2}U'_t(a_t+\mathrm{i}0)-V''(a_t)\right]c_t \tag{3.41}$$

Consider these to be the characteristics of a generalized transport operator $\mathcal{L}_{hol}$ acting on a function $\mathfrak{f}_{z_1}$ *analytic* on Π_+, so that $\partial_{z_1}\equiv\partial_{z_1}+\partial_{\bar z_1}\equiv\partial_{x_1}$: then

$$\mathcal{L}_{hol}(t) = -\left(\frac{\beta}{2}U_t(x_1+\mathrm{i}0)+V'(x_1)\right)\partial_{x_1}-\left(\frac{\beta}{2}U'_t(x_1+\mathrm{i}0)+V''(x_1)\right), \tag{3.42}$$

which is exactly the operator featuring in the r.-h.s. of (3.36), acting on the x_1-variable. This makes it possible to keep $b_t\equiv 0^+$ during the time evolution.

Consider now the time-evolution of

$$\phi(t_1,t_2;\lambda_1,\lambda_2;z_1,z_2) =: \mathbb{E}\left[e^{\mathrm{i}\lambda_1\langle Y_{t_1},\mathfrak{f}_{z_1}\rangle}\,e^{\mathrm{i}\lambda_2\langle Y_{t_2},\mathfrak{f}_{z_2}\rangle}\right]-\mathbb{E}\left[e^{\mathrm{i}\lambda_1\langle Y_{t_1},\mathfrak{f}_{z_1}\rangle}\right]\mathbb{E}\left[e^{\mathrm{i}\lambda_2\langle Y_{t_2},\mathfrak{f}_{z_2}\rangle}\right], \tag{3.43}$$

with $z_1=x_1+\mathrm{i}0$ and $t_1\ge t_2$. Taylor expanding around $(\lambda_1,\lambda_2)=(0,0)$ yields

$$\phi(t_1,t_2;\lambda_1,\lambda_2;z_1,z_2) = 1-\lambda_1\lambda_2\Lambda(t_1,z_1;t_2,z_2)+O((|\lambda_1|+|\lambda_2|)^3). \tag{3.44}$$

On the other hand, by (3.9),

$$\phi(t_1,t_2;\lambda_1,\lambda_2;z_1,z_2)=\exp\left(\frac{1}{2}\int_{t_2}^{t_1}ds\left(\mathrm{i}(1-\frac{\beta}{2})\langle X_s,\lambda_1(f_s)''\rangle-\langle X_s,(\lambda_1(f_s)')^2\rangle\right)\right)\cdot$$
$$\cdot\left(\mathbb{E}\left[e^{\mathrm{i}\lambda_1\langle Y_{t_2},f_{t_2}\rangle}e^{\mathrm{i}\lambda_2\langle Y_{t_2},\mathfrak{f}_{z_2}\rangle}\right]-\mathbb{E}\left[e^{\mathrm{i}\lambda_1\langle Y_{t_2},f_{t_2}\rangle}\right]\mathbb{E}\left[e^{\mathrm{i}\lambda_2\langle Y_{t_2},\mathfrak{f}_{z_2}\rangle}\right]\right) \tag{3.45}$$

where f_s is the solution at time $s\le t_1$ of (1.12) with terminal condition $f_{t_1}\equiv\mathfrak{f}_{z_1}$. The second line of (3.45) is of the form $-C(t_1,z_1;t_2,z_2)\lambda_1\lambda_2+O((|\lambda_1|+|\lambda_2|)^3)$, hence (by identification) $C(t_1,z_1;t_2,z_2)\equiv\Lambda(t_1,z_1;t_2,z_2)$. Thus the time-evolution of Λ may be computed by considering the sole contribution of (3.42) acting on the x_1 variable.

For a rigorous proof we proceed instead as follows. Consider an arbitrary terminal condition $f=f_{t_1}\in L^1(\mathbb{R})\cap C^\infty(\mathbb{R})$ such that $\int_{-\infty}^{+\infty}f(y)\,dy=0$ (this can always be achieved by modifying f outside of the support of the density), and rewrite it as $f\equiv f_++f_-\equiv\mathcal{C}^0(h)+\mathcal{C}^0(\bar h)$, where $h(z)\equiv h_{t_1}(z)=\mathbf{1}_{b>0}e^{-b/\varepsilon}(\mathcal{K}^0_{1,\varepsilon}f)(z)$ (see (3.17), (3.18)) and $\bar h(\bar z):=\overline{h(z)}$, for $\varepsilon>0$. Later on we let $\varepsilon\to0$ to obtain the time-evolution of Λ near the real axis. Define

$$\Phi_f(t_1,t_2;\lambda_1,\lambda_2;z_2)=:\mathbb{E}\left[e^{\mathrm{i}\lambda_1\langle Y_{t_1},f_+\rangle}e^{\mathrm{i}\lambda_2\langle Y_{t_2},\mathfrak{f}_{z_2}\rangle}\right]-\mathbb{E}\left[e^{\mathrm{i}\lambda_1\langle Y_{t_1},f_+\rangle}\right]\mathbb{E}\left[e^{\mathrm{i}\lambda_2\langle Y_{t_2},\mathfrak{f}_{z_2}\rangle}\right]. \tag{3.46}$$

Taylor expanding around $(\lambda_1,\lambda_2)=(0,0)$ yields

$$\Phi_f(t_1,t_2;\lambda_1,\lambda_2;z_2)=-\lambda_1\lambda_2\Lambda_f(t_1;t_2,z_2)+O((|\lambda_1|+|\lambda_2|)^3) \tag{3.47}$$

where

$$\Lambda_f(t_1;t_2,z_2):=\int da_1\int_0^{b_{max}}(-\mathrm{i}b_1)e^{-|b_1|/\varepsilon}(\mathcal{K}^0_{1,\varepsilon}f)(a_1)\Lambda(t_1,z_1;t_2,z_2). \tag{3.48}$$

Eq. (3.45) for $\phi(t_1,t_2;\lambda_1,\lambda_2;z_1,z_2)$ holds equally well for $\Phi_f(t_1,t_2;\lambda_1,\lambda_2;z_2)$, with only the terminal condition f_{t_1} changing. Thus we get two expressions for the time-derivative of Λ_f,

$$\begin{aligned}\frac{\partial}{\partial t_1}\Lambda_f(t_1;t_2,z_2) &= \int da_1\int_0^{b_{max}}db_1\,(-\mathrm{i}b_1)\,e^{-b_1/\varepsilon}(\mathcal{K}^0_{1,\varepsilon}f)(a_1)\partial_{t_1}\Lambda(t_1,z_1;t_2,z_2)\\ &= \int da_1\int_0^{b_{max}}db_1\,(-\mathrm{i}b_1)\,\mathcal{H}^0(t_1)\Big(e^{-b_1/\varepsilon}(\mathcal{K}^0_{1,\varepsilon}f)(a_1)\Big)\Lambda(t_1,z_1;t_2,z_2).\end{aligned} \tag{3.49}$$

The condition $\int_{-\infty}^{+\infty}f=0$ ensures that $(\mathcal{K}^0_{1,\varepsilon}f)(a_1)\sim_{\varepsilon\to0}2\varepsilon^{-1}(Hf)'(a_1)$.

Main terms in (3.49) are due to $\mathcal{H}^0_{transport}(t_1)\equiv v_{hor}(t,z_1)\partial_{a_1}+v_{vert}(t,z_1)\partial_{b_1}+\tau^0(t,z_1)$. Consider first the velocity terms; since $U_t\Big|_{\Pi_+}$ extends continuously to the real axis by hypothesis, $e^{-|b_1|/\varepsilon}v_{hor}(t,z_1)=e^{-|b_1|/\varepsilon}v^{asympt}_{hor}(t,a_1)+o(1)$ when $\varepsilon\to0$, where $v^{asympt}_{hor}(t,a_1):=\frac{\beta}{2}\mathrm{Re}\,U_t(a_1+\mathrm{i}0)+V'(a_1)$. Then (integrating by parts) the adjoint operator $(v_{vert}(t,z_1)\partial_{b_1})^\dagger=-v_{vert}(t,z_1)\partial_{b_1}-\frac{\partial v_{vert}(t,z_1)}{\partial b_1}$, $(\cdots)$, acts on the product $b_1\Lambda(t_1,z_1;t_2,z_2)$. Since $z_1\mapsto\Lambda(t_1,z_1;\cdot)$ is holomorphic, $\partial_{b_1}\Lambda(t_1,z_1;\cdot)\equiv\mathrm{i}\partial_{a_1}\Lambda(t_1,z_1;\cdot)$. Thus the action of $v_{vert}(t,z_1)\partial_{b_1}=\Big((v_{vert}(t,z_1)\partial_{b_1})^\dagger\Big)^\dagger$ is equivalent to that of the transport operator $\mathrm{i}v_{vert}(t,z_1)\partial_{a_1}$. Now $e^{-|b_1|/\varepsilon}(v_{hor}(t,z_1)+\mathrm{i}v_{vert}(t,z_1))=e^{-|b_1|/\varepsilon}v^{asympt}(t,a_1)+o(1)$, with

$v^{asympt}(t,a_1) = \frac{\beta}{2}U_t(a_1+\mathrm{i}0) + V'(a_1)$, a function whose product with $\Lambda(t_1,z_1;t_2,z_2)$ is well-defined (see Remark before the proof). Then:

$$\begin{aligned}
&\int_0^{b_{max}} db_1\,(-\mathrm{i}b_1)\,e^{-b_1/\varepsilon}\int da_1\, v^{asympt}(t,a_1)\Lambda(t_1,z_1;t_2,z_2)\partial_{a_1}(\mathcal{K}^0_{1,\varepsilon}f)(a_1)\\
&= \int_0^{b_{max}} db_1\,(-\mathrm{i}b_1)\,e^{-b_1/\varepsilon}\mathbb{E}\Big[(\mathcal{S}Y)(t_2,z_2)\,\langle Y_{t_1}, x\mapsto \int da_1\, v^{asympt}(t,a_1)\frac{\partial_{a_1}(\mathcal{K}^0_{1,\varepsilon}f)(a_1)}{x-a_1-\mathrm{i}b_1}\rangle\Big]\\
&\sim_{\varepsilon\to 0} -\mathrm{i}\varepsilon^{-1}I(\varepsilon)\int da_1\,\psi'(a_1)\Big(\frac{\beta}{2}U_{t_1}(x_1+\mathrm{i}0)+V'(x_1)\Big)\Lambda(t_1,x_1+\mathrm{i}0;t_2,z_2)
\end{aligned} \tag{3.50}$$

where $\psi(a_1) := 2(Hf)'(a_1)$ and $I(\varepsilon) := \int_0^{b_{max}} db_1\, b_1 e^{-b_1/\varepsilon} \sim_{\varepsilon\to 0} \varepsilon^2$ is simply a coefficient. Missing multiplicative terms are easily checked (as we did in the above "short" proof) to compensate all terms in (4.6) with $\kappa = 0$, except the last one, $\frac{dc_t^0}{dt} = \cdots - \mathrm{i}V'''(a_t)b_t c_t^0$; however, the corresponding multiplication operator $-\mathrm{i}V'''(a_1)b_1$, as well as $b_1\mathcal{H}^{1,0}_{nonloca}(t_1)$, vanish in the limit $\varepsilon \to 0$ when multiplied by $e^{-|b_1|/\varepsilon}$.

On the other hand, working directly on the first line of eq. (3.49) yields $\frac{\partial}{\partial t_1}\Lambda_f(t_1;t_2,z_2) \sim_{\varepsilon\to 0} -\mathrm{i}\varepsilon^{-1}I(\varepsilon)\int da_1\,\psi(a_1)\partial_{t_1}\Lambda(t_1,a_1+\mathrm{i}0;t_2,z_2)$. Comparing the latter expression with (3.50) yields (3.37). □

Assume from now on that V *extends analytically to an entire function* $V:\mathbb{C}\to\mathbb{C}$ (e.g. V is a polynomial). Then the above equation (3.36) may be solved formally in terms of its initial condition, namely, the one-time covariance kernel $g^{+;\pm}_{1,2}(t_2,\cdot;t_2,\cdot)$, as follows. First one uses the adjoint equation in order to obtain a transport operator without multiplicative term. Namely, letting ψ_t solve the PDE $\partial_t\psi_t(x) = -(\frac{\beta}{2}U_t(x+\mathrm{i}0)+V'(x))\psi_t(x)$, we have $\partial_{t_1}\langle\psi_{t_1}, g^{+,\pm}(t_1,\cdot;t_2,x_2)\rangle = 0$. *Assume* $\psi_{t_1}(x) \equiv \psi_{t_1}(x+\mathrm{i}0)$ is the boundary value of a function $\psi_{t_1} = \psi_{t_1}(z)$ holomorphic on Π_+; then, for $t \le t_1$, $\psi_t(x) \equiv \psi_t(x+\mathrm{i}0)$ is the boundary value of $\psi_t = \psi_t(z)$ solution of

$$\partial_t\psi_t(z) = -(\frac{\beta}{2}U_t(z)+V'(z))\psi_t(z). \tag{3.51}$$

Denote by $\Phi_t^{t_1}(z_1)$ $(0 < t < t_1)$ the solution of the characteristic equation on Π_+, $\dot z = -(\frac{\beta}{2}U_t(z)+V'(z))$ with terminal condition $z(t_1) \equiv z_1 \in \Pi_+$. Note that we considered exactly the same characteristics in §2.2 **B**, where the notation Z_{t_1} for $\Phi_t^{t_1}$ was used. Then $\psi_{t_2}(z_1) = \psi_{t_1}(\Phi_0^{t_1}(z_1))$. Hence (considering the scalar product $\langle\cdot,\cdot\rangle$ in $L^2(\mathbb{R})$)

$$\langle\psi_{t_1}, g^{+;\pm}_{1,2}(t_1,\cdot;t_2,x_2)\rangle = \langle\psi_{t_2}, g^{+;\pm}_{1,2}(t_2,\cdot;t_2,x_2)\rangle \tag{3.52}$$

Eq. (3.52) does not suffice to characterize the law of the fluctuation process in general. However, Theorem 3.1 can easily be extended to provide a full answer. Namely, letting $\varepsilon_i = \pm$, $i = 1,2$

(i) $\partial_t g^{\varepsilon_1,\varepsilon_2}_{1,2}(t,x_1;t,x_2) = \Big(\mathcal{L}^{(1)}_{\varepsilon_1}(t)+\mathcal{L}^{(2)}_{\varepsilon_2}(t)\Big)g^{\varepsilon_1,\varepsilon_2}_{1,2}(t,x_1;t,x_2)$, where

$$\mathcal{L}^{(i)}_+(t) = -\left(\frac{\beta}{2}U_t(x_i+\mathrm{i}0)+V'(x_i)\right)\partial_{x_i} - \left(\frac{\beta}{2}U_t'(x_i+\mathrm{i}0)+V''(x_i)\right). \tag{3.53}$$

is $\mathcal{L}_{hol}$, see (3.42), acting on the x_i-variable, while $\mathcal{L}_- := \overline{\mathcal{L}_+}$. Solving as in (3.52) by following characteristics on both space variables simultaneously, one obtains $g^{\varepsilon_1,\varepsilon_2}_{1,2}(t,x_1;t,x_2)$ in terms of the initial covariance $g^{\varepsilon_1,\varepsilon_2}_{1,2}(0,\cdot;0,\cdot)$.

(ii) The one-point function $\mathbb{E}[Y(x)]$ does not follow from the above computations; actually, Johansson gave a general but not very explicit formula for $\mathbb{E}[Y(x)]$ at equilibrium, showing that it vanishes for $\beta = 2$ but not for $\beta \neq 2$ in general (see [20], Theorem 2.4, Remark 2.5 and eq. (3.54)). Following the method used in the proof of Theorem 3.1, one can consider the time-evolution of the generating function

$$\phi(t;\lambda;z) := \mathbb{E}\left[e^{\mathrm{i}\lambda\langle Y_t,\mathfrak{f}_z\rangle}\right] \tag{3.54}$$

with $z = x + \mathrm{i}0$. Taylor expanding around $\lambda = 0$ yields $\phi(t;\lambda;z) = 1 + \mathrm{i}\lambda\mathbb{E}[(\mathcal{S}Y_t)(z)] + O(\lambda^2)$. On the other hand,

$$\phi(t;\lambda;z) = \exp\left(\frac{1}{2}\int_0^t ds\left(\mathrm{i}(1-\frac{\beta}{2})\langle X_s, \lambda(f_s)''\rangle - \langle X_s, (\lambda(f_s)')^2\rangle\right)\right)\cdot\ \mathbb{E}\left[e^{\mathrm{i}\lambda\langle Y_0,f_0\rangle}\right] \tag{3.55}$$

where f_s is the solution at time $s \le t$ of (1.12) with terminal condition $f_t \equiv \mathfrak{f}_z$. Differentiating w.r.t. to t and Taylor expanding to order 1 in λ yields $\frac{d}{dt}\mathbb{E}[(\mathcal{S}Y_t)(z)] = \left(\frac{\mathrm{i}}{2}(1-\frac{\beta}{2})U_t''(z) + \mathcal{L}_\pm(t,z)\right)\mathbb{E}[(\mathcal{S}Y_t)(z)]$, where $\mathcal{L}_\pm = \mathcal{L}_+$ if $z \in \Pi_+$, resp. $\mathcal{L}_-$ if $z \in \Pi_-$, a generalized transport equation on $\mathbb{C}\setminus\mathbb{R}$ with the same characteristics as above.

How explicit can these formulas be made ? One may of course try to answer this question through case-by-case inspection. Let us point out at two specific but sufficiently general cases. The first one is the *harmonic case*, i.e. $V(x) = \frac{x^2}{2}$, treated in an exhaustive way by Bender [3] (see in particular Theorem 2.3) for an arbitrary parameter $\beta > 1$ and an arbitrary initial condition. Though the mapping $\Phi_t^{t_1}$ is explicit (see (2.10)), the *inverse mapping*, $(\Phi_t^{t_1})^{-1}$, of course, is not in general. It requires some skill to provide explicit formulas not relying on the use of $(\Phi_t^{t_1})^{-1}$, see e.g. the beautiful result using Schwartzian derivatives ([3], Theorem 2.7) for $\mathrm{Cov}(\langle Y_{t_1}, F_1\rangle, \langle Y_{t_2}, F_2\rangle)$ when F_1, F_2 are bounded analytic functions on a neighbourhood of the real axis. The second one is the *stationary case*, where β and V are general but $\rho_t = \rho_{eq}$ is assumed to be the equilibrium measure. This is the subject of the next subsection.

3.4 Solution of the PDE in the stationary case

We restrict to the *stationary case* in this subsection, and assume as stated before that *V extends analytically to an entire function $V : \mathbb{C} \to \mathbb{C}$*. Let us first state two essential facts. First, the *universality* (up to simple scaling and translation) *of Johansson's formula for equilibrium fluctuations* implies, assuming that $\mathrm{supp}(\rho_{eq}) = [-A, A]$ $(A > 0)$:

$$\begin{aligned} g_{1,2}^{+;\pm}(t_1,x_1;t_1,x_2) &= (\frac{\sqrt{2}}{A})^2\Lambda_{(3.24)}(0;\frac{\sqrt{2}}{A}(x_1+\mathrm{i}0),\frac{\sqrt{2}}{A}(x_2\pm\mathrm{i}0)) \\ &= (\frac{\sqrt{2}}{A})^2\frac{1}{8\sin(\theta_1)\sin(\pm\theta_2)\sin^2\frac{\theta_1\pm\theta_2}{2}} \end{aligned} \tag{3.56}$$

where $\Lambda_{(3.24)}$ is as in (3.24), and $A\cos(\theta_j) = x_j, A\sin(\theta_j) = \sqrt{A^2 - x_j^2}$, $j = 1, 2$ is up to scaling the change of variables used in the Hermite case. Second, using (1.6) and (5.3), (5.4),

$$U_{eq}(x \pm \mathrm{i}0) = -\frac{2}{\beta}V'(x) \pm \mathrm{i}\pi\rho_{eq}(x), \tag{3.57}$$

which may also be interpreted as saying that ρ_{eq} extends analytically to $\Pi_\pm$ as $\frac{1}{\pm\mathrm{i}\pi}(U_{eq}(z) + \frac{2}{\beta}V'(z))$. Therefore, *Theorem 3.1 may be restated in this simple form,*

$$\partial_{t_1} g_{1,2}^{+;\pm}(t_1,x_1;t_2,x_2) = -\mathrm{i}\pi\frac{\beta}{2}\partial_{x_1}\left(\rho_{eq}(x_1)g_{1,2}^{+;\pm}(t_1,x_1;t_2,x_2)\right). \tag{3.58}$$

which generalizes (3.31). Solving (3.58) for short time and $x_1 \to x_2$, with initial condition $(t_1 = t_2)$ given by Johansson's equilibrium formula, one finds the same short-distance asymptotics as in (3.35), namely,

$$g_{1,2}^{+;+}(t+\varepsilon\delta t_{12}, x+\varepsilon\delta x_{12}; t, x) \sim_{\varepsilon\to 0} -\frac{1}{4\pi^2}\varepsilon^{-2}\left[\frac{1}{(\delta x_{12} + \mathrm{i}\pi\rho_{eq}(x)\delta t_{12})^2}\right]. \tag{3.59}$$

See discussion in the Introduction.

The hydrodynamic fluctuation equation (3.58) may be solved as follows:

Theorem 3.2 (equilibrium fluctuation covariance kernel)**.** Let $A\cos(\theta_1) = x_1$, $A\sin(\theta_1) = \sqrt{2-x_1^2}$ as above, and:

(i) $G = G(z)$ be the analytic continuation to Π_+ of the function G defined on the support of ρ_{eq} by $G(x) := \frac{2}{\beta}\int_0^x \frac{dy}{\rho_{eq}(y)}$;

(ii) $F_0^{\pm}(\cdot, x_2) := F_0^{\pm}(z, x_2)$ be the analytic continuation to Π_+ of the function $F_0^{\pm}(x_1, x_2) := \rho_{eq}(x_1) g_{1,2}^{+;\pm}(0, x_1; 0, x_2) = (\frac{\sqrt{2}}{A})^2 \rho_{eq}(A\cos\theta_1) \frac{1}{8\sin(\theta_1)\sin(\pm\theta_2)\sin^2\frac{\theta_1\pm\theta_2}{2}}$.

Then

$$g_{1,2}^{+;\pm}(t, x_1; 0, x_2) = \frac{1}{\rho_{eq}(x_1)} F_0^{\pm}(G^{-1}(G(x_1) + \mathrm{i}\pi t), x_2). \tag{3.60}$$

Remark. Extend continuously G, resp. $F_0^{\pm}$ to $[-A, A] \uplus \Pi^+ \uplus \Pi^-$ by letting $G(\bar{z}) := \overline{G(z)}$, resp. $F_0^{\pm}(\bar{z}) := \overline{F_0^{\pm}(z)}$. Then

$$g_{1,2}^{-;\pm}(t, x_1; 0, x_2) = \overline{g_{1,2}^{+;\pm}(t, x_1; 0, x_2)} = \frac{1}{\rho_{eq}(x_1)} F_0^{\pm}(G^{-1}(G(x_1) - \mathrm{i}\pi t), x_2)$$

satisfies the time-reversed evolution equation $\overline{(3.58)}$,

$$\partial_{t_1} g_{1,2}^{-;\pm}(t_1, x_1; t_2, x_2) = +\mathrm{i}\pi\frac{\beta}{2}\partial_{x_1}\Big(\rho_{eq}(x_1) g_{1,2}^{-;\pm}(t_1, x_1; t_2, x_2)\Big).$$

Proof. The product $\psi(t, x_1; 0, x_2) := \rho_{eq}(x_1) g_{1,2}^{+;\pm}(t, x_1; 0, x_2)$ solves the transport equation $\partial_t \psi = -\mathrm{i}\pi\frac{\beta}{2}\rho_{eq}(x_1)\partial_{x_1}\psi$ with initial condition $\psi(0, x_1; 0, x_2) = \rho_{eq}(x_1) g_{1,2}^{+;\pm}(0, x_1; 0, x_2) = F_0^{\pm}(x_1, x_2)$. Since $-\mathrm{i}\pi\frac{\beta}{2}\rho(x_1) = \lim_{z\to x_1, z\in\Pi_+} -(\frac{\beta}{2}U_{eq}(z) + V'(z))$, $\psi(t, x_1; 0, x_2)$ may be obtained exactly as in (3.51) by considering the time-homogeneous characteristic equation in Π_+, $\dot{z}_1 = -(\frac{\beta}{2}U_{eq}(z) + V'(z))$ with terminal condition $z_1(t) \equiv x_1$. Solving by quadrature, one gets

$$t = -\int_{z_1(0)}^{x_1} \frac{dw}{\frac{\beta}{2}U_{eq}(w) + V'(w)} \equiv \frac{1}{\mathrm{i}\pi}(G(z_1(0)) - G(x_1)). \tag{3.61}$$

Whence $\psi(t, x_1; 0, x_2) = F_0^{\pm}(z_1(0), x_2)$, equivalent to (3.58). □

Using (3.4), one obtains:

$$g_{1,2}(t_1, x_1; t_2, x_2) = -\frac{1}{2\pi^2}\frac{1}{\rho_{eq}(x_1)}\,\mathrm{Re}\,\Big[F_0^{+}(G^{-1}(G(x_1) + \mathrm{i}\pi(t_1 - t_2)), x_2)$$
$$-F_0^{-}(G^{-1}(G(x_1) + \mathrm{i}\pi(t_1 - t_2)), x_2)\Big] \tag{3.62}$$

Using time-translation invariance for $g, g^{\pm,\pm}$ and reversibility for g, a more symmetric-looking formula can be obtained. Namely, assume $t := t_1 - t_2 > 0$ and let $t' \in (\frac{t}{2}, t)$. Then

$$\begin{aligned} g_{1,2}(t_2 + t', x_1; t_2, x_2) &= g_{1,2}(t' - \frac{t}{2}, x_1; -\frac{t}{2}, x_2) = g_{1,2}(-\frac{t}{2}, x_1; t' - \frac{t}{2}, x_2) \\ &= -\frac{1}{2\pi^2}\mathrm{Re}\,[g_{1,2}^{+,+} - g_{1,2}^{+,-}](-\frac{t}{2}, x_1; t' - \frac{t}{2}, x_2). \end{aligned} \tag{3.63}$$

Then $\psi'(-\frac{t}{2},x_1;t'-\frac{t}{2},x_2) := \rho_{eq}(x_2)g^{+,\pm}(-\frac{t}{2},x_1;t'-\frac{t}{2},x_2)$ solves the transport equation $\partial_{t'}\psi' = -\mathrm{i}\pi\frac{\beta}{2}\rho_{eq}(x_2)\partial_{x_2}\psi'$ with initial condition $\psi'(-\frac{t}{2},x_1;0,x_2) = \rho_{eq}(x_2)\overline{g^{+,\pm}(\frac{t}{2},x_1;0,x_2)}$ by the previous Remark, solved as in the proof of Theorem 3.2 by solving a characteristic equation in the *second* space variable x_2. The conjugation plays no rôle in the end upon taking the real part. The sign in the denominator $\sin(\pm\theta_2) = \pm\sin\theta_2$ of $F_0^{\pm}$ can also be removed since (3.4) is an alternate sum. Hence the result of Theorem 3.2 can be reformulated as follows:

Corollary 3.8. Let

$$\tilde{g}^{\pm}(x_1,x_2) := (\frac{\sqrt{2}}{A})^2\rho_{eq}(A\cos\theta_1)\rho_{eq}(A\cos\theta_2)\frac{1}{8\sin(\theta_1)\sin(\theta_2)\sin^2\frac{\theta_1\pm\theta_2}{2}}. \tag{3.64}$$

Then

$$\begin{aligned} g_{1,2}(t_1,x_1;t_2,x_2) = -\frac{1}{2\pi^2}\frac{1}{\rho_{eq}(x_1)\rho_{eq}(x_2)}\ \mathrm{Re}\ \Big[\tilde{g}^{+}\Big(G^{-1}(G(x_1)+\mathrm{i}\frac{\pi}{2}(t_1-t_2)),\\ G^{-1}(G(x_2)+\mathrm{i}\frac{\pi}{2}(t_1-t_2))\Big)\ +\ \tilde{g}^{-}\Big(G^{-1}(G(x_1)+\mathrm{i}\frac{\pi}{2}(t_1-t_2)),\\ G^{-1}(G(x_2)-\mathrm{i}\frac{\pi}{2}(t_1-t_2))\Big)\Big]. \end{aligned} \tag{3.65}$$

The notation $\tilde{g}^{\pm}_{1,2}$ for the covariance multiplied by the density is reminiscent of (3.29). In the Hermite case (see §3.2), the ρ_{eq}-factors in the denominator of $\tilde{g}^{\pm}$ cancel the sines in the denominator, $G(x) = \pi\theta$ and $G^{-1}(G(x)+\mathrm{i}\pi\frac{t}{2}) = G^{-1}(\pi(\theta+\mathrm{i}\frac{t}{2})) = \sqrt{2}\cos(\theta+\mathrm{i}\frac{t}{2})$, thus time-evolution amounts to an imaginary translation of the angle coordinates θ_1,θ_2, and one easily retrieves (3.25).

Let us illustrate this explicit formula in the case when $V(x) = \frac{1}{4}t^4 + \frac{c}{2}t^2 + d$ is quartic. The additive constant d does not play any rôle, so we may forget it. The equilibrium density ρ_{eq} and its Stieltjes transform U_{eq}, when $\mathrm{supp}(\rho_{eq})$ is *connected*, are given by a general explicit integral formula, see e.g. [20], eq. (3.9), which can be solved when V is polynomial. The result in the particular case when V is quartic can be found e.g. in Johansson [20], Example 3.2; note the slight discrepancy of notations in that article with respect to ours, namely, $U_{Johansson} = -U$, $V_{Johansson} = \frac{4}{\beta}V$, $T_{Johansson} = \frac{4}{\beta}T$. Thus, letting $\mathrm{supp}(\rho_{eq}) =: [-A,A]$, $A = A(c)$, we find:

$$U_{eq}(z) = \frac{2}{\beta}\Big(-V'(z)+(z^2+\frac{1}{2}A^2+c)\sqrt{z^2-A^2}\Big),\ \rho_{eq}(x) = \frac{2}{\beta\pi}(x^2+\frac{1}{2}A^2+c)\sqrt{A^2-x^2}\ \mathbf{1}_{|x|<A}. \tag{3.66}$$

This is the same density as Johansson's up to a rescaling,
$\rho_{eq}(x) = (\frac{\beta}{2})^{-1/4}\rho_{eq,Johansson}((\frac{\beta}{2})^{-1/4}x;(\frac{\beta}{2})^{-1/4}A,(\frac{\beta}{2})^{-1/2}c)$, which means that $A(c) = (\frac{\beta}{2})^{1/4}A_{Johansson}((\frac{\beta}{2})^{1/2}c) = (\frac{\beta}{2})^{1/2}\sqrt{-\frac{2}{3}c+2\sqrt{\frac{1}{9}c^2+\frac{8}{3\beta}}}$. Then (reversing the arrow of time for commodity of notation) we solve forward characteristics with fixed initial condition,

$$\dot{z} = +(\frac{\beta}{2}U(z)+V'(z)) = (z^2+\frac{1}{2}A^2+c)\sqrt{z^2-A^2}, \qquad z(0)\equiv x_1\in\Pi_+. \tag{3.67}$$

Changing variables, $\theta\equiv\arcsin(x/A)$, then $h := \mathrm{cotan}\theta = \sqrt{\frac{A^2}{z^2}-1}$, one finds by quadrature

$$t = \mathrm{i}\tau\Big\{\arctan\Big(C\sqrt{(A^2/z_t^2)-1}\Big) - \arctan\Big(C\sqrt{(A^2/x_1^2)-1}\Big)\Big\} \tag{3.68}$$

where $\tau := \frac{1}{\sqrt{(\frac{3}{2}A^2+c)(\frac{1}{2}A^2+c)}}$, $C := \sqrt{\frac{\frac{1}{2}A^2+c}{\frac{3}{2}A^2+c}}$. In the notations of Theorem 3.2 or Corollary 3.8, one has found:

$$G(z) = \pi\tau \arctan\Big(C\sqrt{(A^2/z^2)-1}\Big). \tag{3.69}$$

Inverting this formula, we get

$$\begin{aligned} z_t &\equiv G^{-1}(G(x_1+\mathrm{i}\pi t)) \\ &= A\Big(1+C^{-2}\tan^2(-\mathrm{i}\frac{t}{\tau}+\arctan(C\sqrt{(A^2/x_1^2)-1})\Big)^{-1/2} \\ &= A\Big(1+C^{-2}\Big[\frac{-\mathrm{i}\tanh(\frac{t}{\tau})+C\sqrt{(A^2/x_1^2)-1}}{1+\mathrm{i}C\tanh(\frac{t}{\tau})\sqrt{(A^2/x_1^2)-1}}\Big]^2\Big)^{-1/2} \\ &= A\Big(1+\mathrm{i}C\tanh(\frac{t}{\tau})\sqrt{(A^2/x_1^2)-1}\Big) \\ &\qquad \cdot\Big[\ (1-C^2\tanh^2(t/\tau))\frac{A^2}{x_1^2}+\tanh^2(t/\tau)(C^2-C^{-2}) \\ &\qquad +2\mathrm{i}(C-C^{-1})\tanh(t/\tau)\sqrt{(A^2/x_1^2)-1}\ \Big]^{-1/2}. \end{aligned} \tag{3.70}$$

4 Appendix. Generator and semi-group estimates

A large part of the work in our previous article [39] has been to write down explicitly a time-dependent operator $\mathcal{H}^\kappa(t)$ (called: *generator*) such that, assuming $f_T = \mathcal{C}^\kappa(h_T)$, the function $f_t = \mathcal{C}^\kappa(h_t)$ with h_t solution for $t \le T$ of the backwards evolution equation

$$\frac{dh}{dt}(t;a,b) = \mathcal{H}^\kappa(t)(h(t))(a,b), \qquad h(T)\equiv h_T \tag{4.1}$$

is solution of (1.12).

The operator $\mathcal{H}^\kappa(t)$ exhibited in [39] is of the following form:

$$\mathcal{H}^\kappa(t)(h(t))(a,b) \equiv \mathcal{H}^\kappa_{transport}(t)(h(t))(a,b) + b\mathcal{H}^{\kappa+1,\kappa}_{nonlocal}(h(t))(a,b), \tag{4.2}$$

where (assuming $\mathrm{supp}(\rho_t)\subset[-R,R]$, $0\le t\le T$):

1. $\mathcal{H}^\kappa_{transport}$ is a (generalized) transport operator (see [39], section 6 for a brief exposition of the characteristic method, and §3.6 for the formulas below, where we have taken the limit $N\to\infty$),

$$\mathcal{H}^\kappa_{transport}(t) = v_{hor}(t,z)\frac{\partial}{\partial a} + v_{vert}(t,z)\frac{\partial}{\partial b} + \tau^\kappa(t,z) \tag{4.3}$$

with associated characteristics on $[-3R,3R]\times[-b_{max},b_{max}]$,

$$\frac{da_t}{dt} = v_{hor}(t,z_t) \equiv \frac{\beta}{2}\mathrm{Re}\ (U_t(a_t+\mathrm{i}b_t)) + V'(a_t) - \frac{1}{2}V'''(a_t)b_t^2 \tag{4.4}$$

$$\frac{db_t}{dt} = v_{vert}(t,z_t) \equiv \frac{\beta}{2}\mathrm{Im}\ (U_t(a_t+\mathrm{i}b_t)) + V''(a_t)b_t \tag{4.5}$$

$$\begin{aligned}\frac{dc^\kappa_t}{dt} = \tau^\kappa(t,z_t) \equiv \Big[\frac{\beta}{2}\Big(\frac{1+\kappa}{b_t}\mathrm{Im}\ (U_t(a_t+\mathrm{i}b_t)) + (\bar{U}_t)'(a_t+\mathrm{i}b_t)\Big) \\ +(2+\kappa)V''(a_t) - \mathrm{i}V'''(a_t)b_t\Big]c^\kappa_t.\end{aligned} \tag{4.6}$$

Let (a_t, b_t) be the solution of (4.4,4.5) with terminal condition $(a_T, b_T) \equiv (a, b)$, and $c_t^\kappa \equiv \exp\Big(- \int_t^T \tau^\kappa(a_s, b_s)\, ds\Big)$ the solution of (4.6) with terminal condition $c_T^\kappa \equiv 1$. Then the solution of the evolution equation $\frac{\partial f_t}{\partial t}(x) = \mathcal{H}^\kappa_{transport}(t) f_t(x)$ with terminal condition $f_T \equiv f$ is: $f_t(a,b) = c_t^\kappa f(a_t, b_t)$.

2. (non-local term)

$$|||\mathcal{H}^{\kappa+1;\kappa}_{nonlocal}|||_{(L^1\cap L^\infty)(\Pi_{b_{max}})\to(L^1\cap L^\infty)(\Pi_{b_{max}})} = O(||V'||_{8+\kappa,[-3R,3R]}), \tag{4.7}$$

(see [39], eq. (4.26),(4.27)).

The transport operator $\mathcal{H}^\kappa_{transport}(t)$ can be exponentiated backward in time for $\kappa \geq 0$ as results from the sign of $\mathrm{Re}\ \tau^\kappa$: namely,

Proposition 4.1. (see [39], Lemma 3.5) Let $u_T \in (L^1 \cap L^\infty)([-3R, 3R] \times (0, b_{max}])$ and $\kappa = 0, 1, 2\ldots$ Then the backward evolution equation $\frac{du}{dt} = \mathcal{H}^\kappa_{transport}(t)u(t), u\big|_{t=T} = u_T$ $(0 \leq t \leq T)$ has a unique solution $u(t) := U^\kappa_{transport}(t, T)u_T$, such that

$$||u_t||_{L^1([-3R,3R]\times(0,b_{max}])} \leq ||u_T||_{L^1([-3R,3R]\times(0,b_{max}])} \tag{4.8}$$

$$||u_t||_{L^\infty([-3R,3R]\times(0,b_{max}])} \leq ||u_T||_{L^\infty([-3R,3R]\times(0,b_{max}])} \tag{4.9}$$

5 Appendix. Stieltjes and Hilbert transforms

We collect in this section some definitions and elementary properties concerning Stieltjes and Hilbert transforms, in the periodic and in the non-periodic cases.

5.1 Non-periodic case

We make use of the Fourier transform normalized as follows, $\mathcal{F}(f)(s) = \int_{-\infty}^{+\infty} f(x) e^{-\mathrm{i}xs} dx$.

Let, for $z = a + \mathrm{i}b \in \mathbb{C}\setminus\mathbb{R}$, $\mathfrak{f}_z(x) = \frac{1}{x-z} (x \in \mathbb{R})$, so that the *Stieltjes transform* of $\phi : \mathbb{R} \to \mathbb{R}$ is $(\mathcal{S}\phi)(z) := \langle \phi, \mathfrak{f}_z \rangle \equiv \int dx\, \phi(x) \mathfrak{f}_z(x)$. Note that $\mathrm{Im}\ (\mathfrak{f}_z)(x) = \frac{b}{(x-a)^2+b^2}$ is of the same sign as b. The Plemelj formula, $\frac{1}{x-\mathrm{i}0} = p.v.\left(\frac{1}{x}\right) + \mathrm{i}\pi\delta_0$ implies the following boundary values,

$$\frac{1}{2\mathrm{i}\pi}(\phi_+(x) - \phi_-(x)) = \frac{1}{\pi}\mathrm{Im}\ \phi_+(x) = \phi(x) \tag{5.1}$$

with $\phi_\pm(x) := \lim_{b\to 0^+}(\mathcal{S}\phi)(x \pm \mathrm{i}b)$, and

$$\frac{1}{2\pi}(\phi_+(x) + \phi_-(x)) = -(H\phi)(x), \tag{5.2}$$

where $H\phi$ is the *Hilbert transform* of ϕ,

$$(H\phi)(x) := \frac{1}{\pi} p.v. \int \frac{dy}{x-y}\phi(y). \tag{5.3}$$

Conversely,

$$\frac{1}{\pi}\phi_\pm(x) = -(H\phi)(x) \pm \mathrm{i}\phi(x). \tag{5.4}$$

Since

$$\mathcal{F}\mathfrak{f}_{\mathrm{i}b}(s) = 2\mathrm{i}\pi \mathrm{sgn}(b) e^{-|b|\,|s|}\mathbf{1}_{\mathrm{sgn}(s)=-\mathrm{sgn}(b)},$$

the Fourier kernel of the Hilbert transform is $\mathcal{F}H(s) = -\mathrm{i}\mathrm{sgn}(s)$, implying in particular $H^2 = -I$. Applying this identity to a function ϕ yields

$$(\mathcal{F}\phi^+)(s) = 2\mathrm{i}\pi \mathbf{1}_{s<0}(\mathcal{F}\phi)(s), \qquad (\mathcal{F}\phi^-)(s) = -2\mathrm{i}\pi \mathbf{1}_{s>0}(\mathcal{F}\phi)(s). \tag{5.5}$$

An essential property of the Stieltjes transform $(\mathcal{S}\rho)(z) := \langle \rho, \mathfrak{f}_z \rangle$ *of a density is the following,*

$$\mathrm{Im}\,(\mathcal{S}\rho)(z) = \langle \rho, x \mapsto \frac{b}{(x-a)^2+b^2} \rangle > 0, \qquad z \in \Pi_+ \tag{5.6}$$

from which it follows that the functions $U_t = \mathcal{S}\rho_t$ map Π_+ into Π_+ and Π_- into Π_-.

5.2 Periodic case

The Fourier integral is replaced by Fourier series, $\phi(\theta) = \sum_{n\in\mathbb{Z}} c_n(\phi) e^{in\theta}$ with

$$c_n(\phi) \equiv \hat{\phi}(n) := \frac{1}{2\pi} \int_0^{2\pi} d\theta\, \phi(\theta) e^{-\mathrm{i}n\theta}. \tag{5.7}$$

We consider only functions ϕ *with vanishing means, i.e. such that* $c_0(\phi) = \frac{1}{2\pi}\int_0^{2\pi} \phi \equiv 0$. The Stieltjes transform (still denoted $\mathcal{S}$) is now a Cauchy integral on the unit circle, $(\mathcal{S}\phi)(z) := \oint_{|\zeta|=1} d\zeta\, \frac{\phi(\zeta)}{\zeta - z}$ ($|z| \neq 1$). In terms of the Fourier coefficients,

$$(\mathcal{S}\phi)(z) = \sum_{n\ge 1} c_n(\phi) z^n \qquad (|z|<1), \qquad -\sum_{n\le -1} c_n(\phi) z^n \qquad (|z|>1). \tag{5.8}$$

Letting $\phi_\pm(\theta) := \lim_{r\to 1^\mp} (\mathcal{S}\phi)(re^{\mathrm{i}\theta})$, one has:

$$\phi_+(\theta) = 2\mathrm{i}\pi \sum_{n\ge 1} c_n(\phi) e^{\mathrm{i}n\theta}, \qquad \phi_-(\theta) = -2\mathrm{i}\pi \sum_{n\le -1} c_n(\phi) e^{\mathrm{i}n\theta}.$$

By analogy with the real line case, we let

$$(H\phi)(\theta) := -\frac{1}{2\pi}(\phi_+(\theta) + \phi_-(\theta)) = \sum_{n\ge 1} (-\mathrm{i}) c_n(\phi) e^{\mathrm{i}n\theta} + \sum_{n\le -1} (+\mathrm{i}) c_n(\phi) e^{\mathrm{i}n\theta}, \tag{5.9}$$

also given, using $\cot\frac{\theta}{2} = 2\mathrm{Re}\,\left\{(-\mathrm{i})\sum_{n\ge 1} e^{\mathrm{i}n\theta}\right\}$, by the following convolution kernel,

$$(H\phi)(\theta) = \frac{1}{2\pi} p.v. \int_0^{2\pi} dt\, f(t) \cot(\frac{\theta - t}{2}), \tag{5.10}$$

and get $\phi = \frac{1}{2\mathrm{i}\pi}(\phi_+ - \phi_-), H^2 = -I$. Letting $\widetilde{\cot}(\theta) := \frac{1}{2}\cot\frac{\theta}{2}$, we obtain an equivalent formula,

$$(H\phi)(\theta) = \frac{1}{\pi} p.v. \int_0^{2\pi} dt\, f(t) \widetilde{\cot}(\theta - t), \tag{5.11}$$

making it plain that the periodic Hilbert transformation is a rational generalization of the Hilbert transform (5.3) on the real line.

6 On Ornstein-Uhlenbeck processes

An Ornstein-Uhlenbeck process is a (Hilbert-space valued) stochastic process $Y(t)$ satisfying a linear stochastic differential equation of the form

$$\dot{Y}(t) = -AY(t) + \Sigma\, \eta(t) \tag{6.1}$$

where η is delta-correlated white noise, time-derivative of a Wiener process, and A, Σ are some operators; see [18], §5 for details. If $Y : \mathbb{R}_+ \to \mathbb{R}$ is one-dimensional, and $\Sigma = \sqrt{T} > 0$, $Y(t)$ modelizes either the velocity of a massive Brownian particle under the influence of friction, or the position of an infinitely massive Brownian particle submitted to friction and to a harmonic potential $V(Y) = \frac{1}{2}AY^2$; in the first interpretation, A is the friction coefficient. In both cases T plays the rôle of a temperature, as appears in the Maxwell-like equilibrium distribution $e^{-AY^2/2T} = e^{-V(Y)/T}$. In our context $Y = Y(t,x)$ is the random fluctuation process, $\eta = \eta(t,y)$ is space-time white noise, and (6.1) is a Langevin equation for Y. Under adequate assumptions, notably on the analytic properties and long-time behavior of the semi-group e^{-tA}, $t \geq 0$ generated by A, this equation has a unique stationary measure μ_∞, and the law μ_t of Y_t converges to μ_∞ for any reasonable initial measure μ_0. Furthermore, μ_∞ is Gaussian, with covariance kernel $K_\infty = K_\infty(x,y)$ defined uniquely by

$$\mathrm{Sym}(K_\infty A^\dagger) = \frac{1}{2}\Sigma\Sigma^\dagger \tag{6.2}$$

with $\mathrm{Sym}(B) := \frac{1}{2}(B+B^\dagger)$ (see [18], Theorem 5.22). If Σ, A are self-adjoint and commute, and $A \geq 0$, then (starting from any initial measure) $Y(t) = e^{-tA}Y(0) + e^{-tA}\int_0^t e^{sA}\Sigma\eta(s)$, so $K_\infty = \lim_{t\to\infty}\int_0^t ds\, e^{-(t-s)A}\Sigma\Sigma^\dagger e^{-(t-s)A} = \frac{1}{2}\Sigma^2/A$, confirming (6.2).

Assume conversely that some stationary Gaussian process $Y(t)$ is given, with known two-time covariance kernel $K_\infty(t_1,x_1;t_2,x_2) = K_\infty(t_1-t_2;x_1,x_2)$. Then Y is the solution of (6.1) with

$$AK_\infty = \frac{d}{dt}\Big|_{t=0} K_\infty(t,0), \qquad \frac{1}{2}\Sigma\Sigma^\dagger = \mathrm{Sym}(K_\infty A^\dagger). \tag{6.3}$$

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