

Profile of a self-similar growth-fragmentation

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Abstract

A self-similar growth-fragmentation describes the evolution of particles that grow and split as time passes. Its genealogy yields a self-similar continuum tree endowed with an intrinsic measure. Extending results of Haas [13] for pure fragmentations, we relate the existence of an absolutely continuous profile to a simple condition in terms of the index of self-similarity and the so-called cumulant of the growth-fragmentation. When absolutely continuous, we approximate the profile by a function of the small fragments, and compute the Hausdorff dimension in the singular case. Applications to Boltzmann random planar maps are emphasized, exploiting recently established connections between growth-fragmentations and random maps [1, 5].

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1 Introduction

The main purpose of this work is to answer, for a specific family of continuous random trees (CRT in short), the following general question about measured metric spaces: if $m(r)$ denotes the measure assigned to the ball centered at some fixed distinguished point and with radius $r \geq 0$, is the non-decreasing function m absolutely continuous with respect to the Lebesgue measure on $[0, \infty)$? When the answer is positive, the density $m'(r)$ can then be viewed as the measure of the sphere with radius r . When furthermore the metric space is a continuum tree, the density m' is sometimes known as the *profile* of the tree.

This question has been answered by Haas [13] for the class of self-similar fragmentation trees, which notably includes Aldous' CRT. Recall that a conservative self-similar fragmentation describes the evolution of a branching particle system such that at every branching event, the sum of the masses of the children coincides with the mass of the parent, and self-similarity refers to the property that the evolution of a particle with mass $x > 0$ is a scaling transformation (depending on an index $\alpha \in \mathbb{R}$) of that of a particle

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with unit mass. Haas and Miermont [14] associated to a conservative self-similar fragmentation with index $\alpha < 0$ a self-similar continuous random tree which is furthermore naturally equipped with a root and a probability mass measure, and Haas [13] proved that under some very minor hypotheses, the non-decreasing function m is then absolutely continuous if $\alpha > -1$, and singular if $\alpha \leq -1$.

The present work should be viewed as a generalization of [13] to self-similar *growth-fragmentations*, introduced by Bertoin [4]. As the name suggests, the latter extend pure fragmentations by incorporating a growth element in the dynamics of particles, and this changes the behaviour of the system deeply. Rembart and Winkel [20] constructed recently the CRT's which describe the genealogy of self-similar growth-fragmentations with index $\alpha < 0$, whereas the so-called intrinsic area measure was introduced in [1].

The motivation of the present work is not just getting a formal extension of the results of Haas; it also stems from the connection between random surfaces and growth-fragmentations as we shall now explain informally. It was pointed out in [1] and [5] that for certain random surfaces with a boundary, the process obtained by slicing the surface at fixed distances from the boundary and measuring the lengths of the resulting cycles yields a self-similar growth-fragmentation with negative index. One might then expect that, just as for smooth surfaces, the area $A(r)$ of the components at distance at most r from the boundary can then be recovered by integrating the total cycle lengths at height $0 \leq r' \leq r$; that is, that the non-decreasing function $r \mapsto A(r)$ is absolutely continuous with density given by the total cycle lengths. It turns out that this intuition is wrong in general, and it is thus natural to wonder whether nonetheless the absolute continuity of $A(\cdot)$ holds.

The law of a growth-fragmentation is determined by the index of self-similarity and the so-called cumulant function κ , that we supposed satisfies the forthcoming condition (2.2) (more details are given in Section 2). Our main result is stated in terms of α and the smallest root ω_- of κ .

Theorem 1.1. *dA is almost surely singular with respect to the Lebesgue measure if and only if $\alpha \leq -\omega_-$, whereas $dA(x)$ is absolutely continuous almost surely whenever $\alpha > -\omega_-$.*

In particular, we shall see that for the whole family of random maps considered in [1], $x \mapsto A(x)$ is absolutely continuous. Note also that we recover Theorem 4 of Haas [13], since in the pure fragmentations case $\omega_- = 1$ (though she considers pure fragmentation processes in which a random number of particles emanates from each dislocation and not simply two).

When $dA(x) = a(x)dx$, i.e. $\alpha > -\omega_-$, the profile a can be approximated by using the small (or equivalently relatively large) fragments, under the additional assumption that $\omega_- \geq 1$. More precisely, define the processes M, N for all $\epsilon > 0, t \geq 0$ by

$$M(t, \epsilon) := \sum_{i \geq 1} X_i^{\omega_-}(t) \mathbb{1}_{\{X_i(t) \leq \epsilon\}},$$

$$N(t, \epsilon) := \sum_{i \geq 1} \mathbb{1}_{\{X_i(t) > \epsilon\}}.$$

Theorem 1.2. *Suppose that $\alpha > -\omega_-$ and $\omega_- \geq 1$. Then for almost every $t \geq 0$, we have that*

$$(i) \quad \epsilon^\alpha M(t, \epsilon) \xrightarrow[\epsilon \rightarrow 0]{\text{a.s.}} \frac{a(t)}{\alpha \kappa'(\omega_-)},$$

$$(ii) \quad \epsilon^{\omega_- + \alpha} N(t, \epsilon) \xrightarrow[\epsilon \rightarrow 0]{\text{a.s.}} \frac{a(t)}{(\omega_- + \alpha) |\kappa'(\omega_-)|}.$$

We stress that the above limits are well defined, since the forthcoming hypothesis (2.2) ensures that $\kappa'(\omega_-) \in (-\infty, 0)$. Theorem 1.2 is the analogue of Theorem 7 in [13], which deals with self-similar fragmentations (see also [11] Proposition 4.2 addressing the profile of Lévy trees associated with fragmentations that are not necessarily self-similar).

When dA is singular with respect to the Lebesgue measure, i.e. $\alpha \leq -\omega_-$, although A is strictly increasing on an interval, the Hausdorff dimension of dA can be strictly smaller than 1:

Theorem 1.3. *Suppose $-\omega_+ < \alpha \leq -\omega_-$, then almost surely, it holds that:*

$$\dim_H(dA) = \frac{\omega_-}{-\alpha}.$$

Furthermore, $\dim_H(dA) \geq -\omega_-/\alpha$ holds for any value of $\alpha \leq -\omega_-$.

Remark 1.4 (Hölder continuity). Copying the argument of Haas [13] Proposition 12(i), Theorem 1.3 directly implies that if $-\omega_+ < \alpha \leq -\omega_-$, then A is γ -Hölder continuous for every $\gamma < -\omega_-/2\alpha$.

The paper is organized as follows. In Section 2, we recall the settings of [1]. This includes the definition of a growth-fragmentation and its CRT. The construction of the intrinsic area measure from the branching random walk following in generations the collection of particles at birth is recalled. A loose description of the spinal decomposition is also given.

Section 3 is divided in three subsections. The first one is a toolbox that recalls basic properties of the major ingredients of the proof of Theorem 1.1, which is given in the second subsection. A simple corollary on the number of fragments is stated in the third subsection.

We dwell on the absolutely continuous case in Section 4 and we see that, modulo few adjustments, the proof of Haas adapts to prove Theorem 1.2.

Finally, Section 5 is devoted to the proof of Theorem 1.3. We obtain the lower bound from Frostman's Lemma, and derive the upper bound from the Hausdorff dimension of the leaves of the CRT, obtained by Rembart and Winkel [20].

In Appendix are shown two technical lemmas, including the Feller property of the growth-fragmentation (Lemma 6.2), which is needed for the arguments of Haas to apply in Section 4.

2 Preliminaries

The cell-system. We consider a positive self-similar Markov process X with index $\alpha < 0$, in the sense that its law \mathbb{P}_x started from $X_0 = x > 0$ is the same as that of $(xX_{tx^\alpha})_{t \geq 0}$ under \mathbb{P}_1 . We assume that X converges almost surely to 0. Lamperti's transformation [16] enables us to view X as a time changed of $\exp(\xi)$, where ξ is a Lévy process. As a consequence the lifetime of X , i.e. the first hitting time of the absorbing state 0, is equal to an exponential functional of ξ (we shall provide more details in the end of this section).

We follow Bertoin's construction [4] of the cell-system driven by X : let $\chi_\emptyset := (\chi_\emptyset(t))_{t \geq 0}$ have law \mathbb{P}_1 . The process χ_\emptyset is viewed as the size of the Eve cell \emptyset , evolving in time. Its birth-time b_\emptyset is taken to be 0. Let $(b_i)_{i \geq 1}$ be an exhausting sequence of its negative jump times and let $(\Delta_i)_{i \geq 1}$ be the corresponding sequence of the absolute values of the sizes of its negative jumps (the existence is ensured by the fact that χ_\emptyset converges to 0 almost surely). Each negative jump is interpreted as the birth of a new cell, that is at time b_i a cell labeled i is born and evolves independently of the other cells, with law \mathbb{P}_{Δ_i} . The other generations are defined recursively in the same manner, using the Ulam-Harris-Neveu notation, that is every cell is labeled by some $u \in \mathbb{U} := \bigcup_{n \geq 0} \mathbb{N}^n$.

We denote $|u|$ its generation and $u(k)$ its ancestor at generation $k \leq |u|$ (by convention, $|\emptyset| = 0$). For $x > 0$, we denote \mathcal{P}_x the law of the cell-system starting from a single cell of initial size x . Similarly, if $\underline{x} := (x_1, x_2, \dots)$ is a non-decreasing null sequence, $\mathcal{P}_{\underline{x}}$ is the distribution of a cell-system starting from independent cells of sizes x_1, x_2, \dots .

The branching random walk. Define the collections of logarithms of cells at birth, indexed by generations, as

$$\mathcal{Z}_n := \{\!\!\{ \ln \chi_u(0) : u \in \mathbb{N}^n \}\!\!\},$$

where $\{\!\!\{ \dots \}\!\!\}$ refers to multiset, meaning that the elements are repeated according to their multiplicities. Thanks to self-similarity, $(\mathcal{Z}_n)_{n \geq 1}$ is a branching random walk. Let (b, σ^2, Λ) be the characteristics of the Lévy process ξ and assume that there exists $p > 0$ such that $\int_1^\infty e^{py} \Lambda(dy) < \infty$ (we also assume that $\Lambda((-\infty, 0)) > 0$ as there are no children otherwise). We thus have that the Laplace exponent of ξ , given by $\psi(q) := \log \mathbb{E}(\exp(q\xi(1)))$ is finite at least on $[0, p]$ (see e.g. [15] Theorem 3.6); we set $\psi(q) = \infty$ whenever the expectation is infinite. The so-called *cumulant function* is defined as

$$\kappa : q \mapsto \psi(q) + \int_{(-\infty, 0)} (1 - e^y)^q \Lambda(dy), \quad q > 0. \tag{2.1}$$

The mean Laplace transform of \mathcal{Z}_1 is then given by $q \mapsto 1 - \kappa(q)/\psi(q)$, when this makes sense (see [4] Lemma 3). Hence, as soon as $\kappa(q) = 0$, the process $(\sum_{|u|=n} \chi_u(0)^q)_{n \geq 0}$ is a martingale. We thus naturally assume that there exists $\omega_- > 0$ such that¹

$$\kappa(\omega_-) = 0, \quad -\infty < \kappa'(\omega_-) < 0. \tag{2.2}$$

The so-called *intrinsic martingale* introduced in [1] is then defined as

$$\mathcal{M}(n) := \sum_{|u|=n} \chi_u(0)^{\omega_-}, \quad n \geq 0.$$

This martingale is moreover uniformly integrable with mean 1 under \mathcal{P}_1 (see [1] Lemma 2.3). We shall also denote $\omega_+ := \sup \{q \geq 0 : \kappa(q) < 0\}$, which is strictly greater than ω_- thanks to (2.2). (In [1], ω_+ is a second root of κ , which if it exists, is consistent with our definition.) Finally, we rule out the case where X is the negative of a subordinator, as this induces fragmentation processes, which are fully addressed by [13].

The Ulam tree, the CRT and the intrinsic area measure. In [1], the authors define a random measure on the boundary of the Ulam tree $\partial\mathbb{U}$, which is the set of infinite integer sequences, endowed with the distance $d(\ell, \ell') := \exp(-\sup\{n \geq 0 : \ell(n) = \ell'(n)\})$ which makes it a complete metric space (recall that $\ell(n) \in \mathbb{N}^n$ denotes the ancestor of ℓ at generation n , that is the sequence ℓ up to n). Specifically, for every $u \in \mathbb{U}$ with $|u| = n$, let $B(u) := \{\ell \in \partial\mathbb{U} : \ell(n) = u\}$ be the ball in $\partial\mathbb{U}$ generated by u . The *intrinsic area measure* on $\partial\mathbb{U}$ is then the unique measure that satisfies

$$\mathcal{A}(B(u)) = \lim_{k \rightarrow \infty} \sum_{|v|=k, v(n)=u} \chi_v(0)^{\omega_-},$$

see [18] for background and references. (This is well-defined thanks to the uniform integrability of $(\mathcal{M}(n))_{n \geq 0}$ and the branching property.) The total mass is denoted

¹In the context of branching random walks, this assumption is known as the Cramér hypothesis and ω_- is called the Malthusian parameter.

$\mathcal{M} := \lim_{n \rightarrow \infty} \mathcal{M}(n) = \mathcal{A}(\partial\mathbb{U})$. Note that almost surely, a cell begets infinitely many children before dying, hence the branching property ensures that $\mathcal{M} > 0$ almost surely.

Rembart and Winkel [20] built a CRT from the cell-system as follows: construct a first segment of length equal to the lifetime ζ_\emptyset of \emptyset , endowed with a metric corresponding to the age of the cell. It means that each point of this branch corresponds to the Eve cell at a particular time of its life; the root ρ is thus naturally taken to be the point corresponding to 0. On this branch, at every jump location b_i , glue a new branch of length equal to the lifetime ζ_i of the cell i , with the corresponding metric. This yields a CRT (\mathcal{T}_1, d_1) . For all $n \geq 1$, to obtain $(\mathcal{T}_{n+1}, d_{n+1})$, repeat this procedure on every branch $u \in \mathbb{N}^n$ at locations $\{b_{uj} : j \geq 1\}$.

Theorem 1.7 in [20] shows that, whenever $\psi(-\alpha) < 0$, $(\mathcal{T}_n, d_n)_{n \geq 1}$ converges almost surely in the Gromov-Hausdorff topology to some compact CRT (\mathcal{T}, d) . Even though it is not explicitly given in their construction, there is a very natural way to define simultaneously the analogue of the intrinsic area measure on $\mathcal{L}(\mathcal{T})$, the set of leaves of \mathcal{T} , that we now introduce. Fix $n \geq 1$ and consider \mathcal{T}_n . For every $u \in \mathbb{N}^n$ and $j \geq 1$, put a mass $\chi_{uj}(0)^{\omega_-}$ at location b_{uj} on the branch of u . This defines a measure \mathcal{A}_n on \mathcal{T}_n , with total mass given by $\mathcal{M}(n+1)$. As for the Ulam tree, it is clear that $(\mathcal{A}_n)_{n \geq 1}$ converges weakly toward a measure $\mathcal{A}_{\mathcal{T}}$ with total mass \mathcal{M} and supported on $\mathcal{L}(\mathcal{T})$, the set of leaves of \mathcal{T} . The correspondance between \mathcal{T} and $\bar{\mathbb{U}} := \mathbb{U} \cup \partial\mathbb{U}$ is straightforward from the two constructions, that is every $x \in \mathcal{T}$ corresponds to either a unique $\chi_u(t)$ for some $u \in \mathbb{U}, t \in [0, \zeta_u]$, or a unique $\ell \in \partial\mathbb{U}$. In particular, $\mathcal{A}_{\mathcal{T}}$ and \mathcal{A} are essentially the same, this is even clearer when looking at the masses at heights.

Recall that the height function on \mathcal{T} is defined as the distance to the root

$$\text{ht}(x) := d(\rho, x).$$

We then define $A_{\mathcal{T}} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$A_{\mathcal{T}} : t \mapsto \mathcal{A}_{\mathcal{T}}(\{\ell \in \mathcal{L}(\mathcal{T}) : \text{ht}(\ell) \leq t\}).$$

This coincides exactly with

$$A : t \mapsto \mathcal{A}(\{\ell \in \partial\mathbb{U} : \zeta_\ell \leq t\}), \tag{2.3}$$

where $\zeta_\ell := \lim_{n \rightarrow \infty} b_{\ell(n)}$. (Actually, $\mathcal{L}(\mathcal{T})$ also contains cells at death-times, but they do not generate area since there are only countably many.) Since the cell system carries more informations, we shall rather work with A than $A_{\mathcal{T}}$.

The elements of $\mathcal{T} \setminus \mathcal{L}(\mathcal{T})$ at a fixed height $t \geq 0$ correspond to the collection of cells alive at time t :

$$\mathbf{X}(t) := \{\{\chi_u(t - b_u) : u \in \mathbb{U}, b_u \leq t < b_u + \zeta_u\}\}.$$

This is the definition of Bertoin [4] of the growth-fragmentation process induced by the cell-system. Shi [21] showed that the distribution of \mathbf{X} is characterized by the pair (κ, α) .² The lifetime of \mathbf{X} is defined as $\zeta := \inf \{t > 0 : \mathbf{X}(t) = \emptyset\}$. In [4], it is shown that the Cramér hypothesis (2.2) ensures that the following properties hold:

- Almost surely, for any fixed $\epsilon > 0$, there are finitely many fragments larger than ϵ in $\mathbf{X}(t)$ for all $t \geq 0$.
- $\zeta < \infty$ almost surely. ([4] Corollary 3)
- \mathbf{X} enjoys the self-similarity and branching properties, as stated in [4] Theorem 2.

²However, this is not the case of the distribution of the cell-system.

As a consequence, \mathcal{T} satisfies a Markov-branching type property, that we express in terms of A as follows: let $t \geq 0$ and let $(A_i)_{i \geq 1}$ be a sequence of i.i.d. copies of A , independent of $(\mathbf{X}(u); u \leq t)$, then for all $s \geq 0$:

$$A(t+s) - A(t) \stackrel{d}{=} \sum_{i \geq 1} X_i(t)^{\omega_-} A_i(s X_i(t)^\alpha), \tag{2.4}$$

where for $i \geq 1$, $X_i(t)$ denotes the size of the i th largest fragment in $\mathbf{X}(t)$ (being possibly 0).

Spinal decomposition We now give an informal description of the spinal decomposition induced by \mathcal{A} , introduced in [1] Section 4. The statements are provided without proof, the reader is referred to this paper for a rigorous treatment.

We introduce a probability measure $\widehat{\mathcal{P}}_1$ describing the joint distribution of a cell-system and a random leaf $\sigma \in \partial\mathbb{U}$. Under $\widehat{\mathcal{P}}_1$, the law of the cell-system is absolutely continuous with respect to \mathcal{P}_1 , with density \mathcal{M} . The random leaf σ is then tagged according to the intrinsic area. In particular we have

Lemma 2.1. *Under $\widehat{\mathcal{P}}_1$ and conditionally on the cell-system, the probability measure dA/\mathcal{M} satisfies*

$$\frac{dA(t)}{\mathcal{M}} = \widehat{\mathcal{P}}_1(\zeta_\sigma \in dt | (\chi_u)_{u \in \mathbb{U}}).$$

Let $\phi : q \mapsto \kappa(\omega_- + q)$, $q \geq 0$. It is known that ϕ can be viewed as the Laplace exponent of a Lévy process³ that we denote η . We then define the positive self-similar Markov process $(Y_t)_{t \geq 0}$ with index α , associated with η by Lamperti’s transformation, that is

$$(Y_t)_{t \geq 0} := (\exp(\eta(\tau_t)))_{t \geq 0},$$

where the time-change τ_t is defined for all $t \geq 0$ by

$$\tau_t := \inf \left\{ s \geq 0 : \int_0^s e^{-\alpha\eta(u)} du \geq t \right\}. \tag{2.5}$$

The absorption time of Y is thus given by the following exponential functional

$$I = \int_0^\infty e^{-\alpha\eta(t)} dt. \tag{2.6}$$

Since $\kappa'(\omega_-) < 0$ by (2.2), we know that η drifts to $-\infty$ and $I < \infty$ almost surely. We shall denote $\widehat{\mathbb{P}}_x$ the law of Y starting from $x > 0$.

The *spine* $(\sigma(t))_{t \geq 0}$ is the process following the size of the ancestors of σ in time. Remark that we can write $\zeta_\sigma = \inf \{t > 0 : \sigma(t) = 0\}$. We thus call ζ_σ the *lifetime* of σ to emphasize that we will look at σ as a random process rather than a random element of a random metric space. In this direction, we have

Lemma 2.2. *Under $\widehat{\mathcal{P}}_1$, the spine σ is distributed as Y under $\widehat{\mathbb{P}}_1$. In particular, it holds that $\zeta_\sigma \stackrel{d}{=} I$.*

Lemma 2.1 relates A to the lifetime of the spine, which in turns is distributed as the variable I by Lemma 2.2. Let $\mathcal{C}_0^\infty(\mathbb{R}_+^*)$ be the set of infinitely differentiable functions on \mathbb{R}_+^* vanishing together with their derivatives at infinity. Equation (2.6) plays a crucial role to obtain distributional properties of I . The next lemma collects some that we shall extensively use throughout the rest of this work.

³This fact is stated in [1] Lemma 2.1 for $q \mapsto \kappa(\omega_+ + q)$, however it is also true for ϕ by the same arguments.

Lemma 2.3. *The law of I is absolutely continuous with a bounded density in $\mathcal{C}_0^\infty(\mathbb{R}_+^*)$, which we denote by k . Furthermore, $\lim_{x \rightarrow 0^+} k(x) = 0$ and*

$$\widehat{\mathbb{E}}_1(I^{-1}) = \alpha \kappa'(\omega_-) < \infty.$$

These results are already known. Hence in the following proof, we only provide references and check that the hypotheses of the cited theorems are fulfilled.

Proof. Theorem 3.9 in [6] ensures the existence of k . Recently, Patie and Savov [19] have shown that k is infinitely differentiable. The Lévy measure Π of η is given (see [1] Section 4.3) by

$$\Pi(dy) := e^{\omega y}(\Lambda + \widetilde{\Lambda})(dy). \tag{2.7}$$

where $\widetilde{\Lambda}$ is the push-forward of Λ by $y \mapsto 1_{\{y < 0\}} \log(1 - e^y)$. We see that if $\Lambda(\mathbb{R}_-) = \infty$, then Π also has infinite total mass. Notice that either $\Lambda(\mathbb{R}_-) = \infty$, or $\sigma^2 > 0$, or η is a compound Poisson process with a non-negative drift. Theorem 2.4.(3)⁴ in [19] thus shows that $k \in \mathcal{C}_0^\infty(\mathbb{R}_+^*)$ (see in particular Remark 2.5 in the same paper). Finally, the limit at 0 of k is given in [19] by Theorem 2.15. The statement on the moment of order -1 can be found in [9] Proposition 3.1(iv). \square

We conclude this section by recalling the following essential fact on the spinal decomposition: conditionally on $(\sigma(t))_{t \geq 0}$, a child depends only on the spine through its own initial value, given by the size x of the negative jump who generated it, and then evolves with law \mathbb{P}_x , independently from $(\sigma(t))_{t \geq 0}$ and the other children.

Notation: In the sequel, the expectations under $\mathbb{P}_x, \widehat{\mathbb{P}}_x, \mathcal{P}_x, \widehat{\mathcal{P}}_x$ are denoted respectively by $\mathbb{E}_x, \widehat{\mathbb{E}}_x, \mathcal{E}_x, \widehat{\mathcal{E}}_x$.

3 Existence of the profile

In this section we provide the proof of Theorem 1.1. Recall that this theorem can be read as a statement on A stemming from either ∂U or \mathcal{T} , as explained in Section 2.

3.1 Toolbox

We introduce in this subsection the main tools of the proof of Theorem 1.1. Let μ be a measure on \mathbb{R} . We denote its Fourier-Stieltjes transform

$$\mathcal{F}_\mu(\theta) := \int_{\mathbb{R}} e^{i\theta x} \mu(dx), \quad \theta \in \mathbb{R}.$$

Recall from Plancherel's Theorem that

$$\mu(dx) \ll dx \quad \text{with} \quad \mu(dx)/dx \in \mathbb{L}^2(dx) \quad \Leftrightarrow \quad \mathcal{F}_\mu \in \mathbb{L}^2(dx). \tag{3.1}$$

We shall use (3.1) to prove the next lemma, which is the main ingredient in the proof of the absolute continuity of dA . It was used in [13] but somewhat implicitly. We state it in a general setting.

Let \mathbf{P} be a probability measure on a generic random space. Let (E, d, μ, ρ) be a random measured metric space, where μ is a measure on E with finite total mass \mathbf{P} -almost surely and $\rho \in E$ is a distinguished element. Let $B(\rho, r)$ be the open ball centered in ρ with radius $r > 0$. Let γ, γ' be two random variables in E such that γ and γ' are conditionally independent given (E, d, μ, ρ) , with conditional law $\mu(\cdot)/\mu(E)$.

⁴In [19], the authors use the equivalent convention that the process drifts to $+\infty$ and they take the exponential of the negative of the process to define I .

Lemma 3.1. *If the law of $\nabla := d(\rho, \gamma) - d(\rho, \gamma')$ is absolutely continuous with a density h bounded in a neighbourhood of 0, then $m : r \mapsto \mu(B(\rho, r))$ is absolutely continuous with respect to the Lebesgue measure, with a density in $\mathbb{L}^2(dx)$ \mathbf{P} -almost surely.*

Proof. Recall (3.1), we thus look at

$$\begin{aligned} \frac{|\mathcal{F}_{dm}(\theta)|^2}{\mu(E)^2} &= \frac{\mathcal{F}_{dm}(\theta)}{\mu(E)} \cdot \frac{\mathcal{F}_{dm}(-\theta)}{\mu(E)} = \int_0^\infty \int_0^\infty \frac{dm(x)}{\mu(E)} \cdot \frac{dm(y)}{\mu(E)} e^{i\theta(x-y)} \\ &= \mathbf{E} (e^{i\theta\nabla} | (E, d, \mu, \rho)), \end{aligned}$$

where \mathbf{E} is the expectation operator induced by \mathbf{P} . We see in particular that $\theta \mapsto \mathbf{E} (e^{i\theta\nabla}) \geq 0$. Theorem 9 in [8] ensures that if ∇ has a density bounded in a neighbourhood of 0, then its Fourier transform is integrable, that is

$$\int_{\mathbb{R}} \mathbf{E} (e^{i\theta\nabla}) d\theta = \mathbf{E} \left(\int_{\mathbb{R}} \frac{|\mathcal{F}_{dm}(\theta)|^2}{\mu(E)^2} \right) < \infty.$$

We conclude by Plancherel’s Theorem (3.1). □

We shall use this Lemma for two suitable choices of \mathbf{P} , taking (E, d, μ, ρ) as $(\mathcal{T}, d, \mathcal{A}, 0)$, the distance d being the age, see Section 2. This means in particular that $m = A$.

We now state an easy but important consequence of Lemma 2.1. Recall that ζ denotes the lifetime of \mathbf{X} .

Lemma 3.2. *The function $t \mapsto A(t)$ is strictly increasing on $(0, \zeta)$ and it holds that*

$$\mathcal{E}_1 (A(\epsilon)) = o(\epsilon), \quad \epsilon \rightarrow 0.$$

Proof. Fix $\epsilon > 0$. Recall that $(\sigma(t))_{t \geq 0}$ denotes the size process of the spine. We write

$$\mathcal{E}_1(A(\epsilon)) = \widehat{\mathcal{E}}_1 \left(\frac{A(\epsilon)}{\mathcal{M}} \right) = \widehat{\mathcal{P}}_1 (\zeta_\sigma \leq \epsilon),$$

where the last identity is seen from Lemma 2.1. Lemma 2.2 combined with the fact that $k(x) \rightarrow 0$ as $x \rightarrow 0+$ from Lemma 2.3 entail that

$$\mathcal{E}_1 (A(\epsilon)) = o(\epsilon), \quad \epsilon \rightarrow 0.$$

Very similar arguments to those used in [13] Proposition 10(*iv*) can be applied to show that A is strictly increasing on $(0, \zeta)$. □

3.2 Proof of Theorem 1.1

In the case of self-similar pure fragmentations, Haas [13] exploited the unit interval representation to tag two fragments by sampling two independent uniform random variables on $[0, 1]$. In the context of Lemma 3.1, it means that the measure μ is uniform over the leaves, given the tree. Recall that we work on $\overline{\mathbb{U}}$. In our case, \mathcal{A} is not uniform on $\partial\mathbb{U}$. However it is not required to apply Lemma 3.1.

There are two main difficulties stemming from the presence of growth. The first one is to deal with the dependence of the two leaves selected according to the random measure \mathcal{A} . It is done in the proof of the next lemma, thanks to the use of optional projections. The second obstacle is that the total area \mathcal{M} is not necessarily square integrable.

We divide the proof into two subsections to separately answer these two issues. Even though the second subsection would be enough to prove Theorem 1.1 in great generality, it is very similar to the first one but involves considerations that can be avoided in some cases, and we do so for the sake of clarity.

3.2.1 The case $\omega_+/\omega_- > 2$ and $\alpha > -\omega_-$

We assume throughout this subsection that $\omega_+/\omega_- > 2$ and $\alpha > -\omega_-$. The reason is that thanks to Lemma 2.3 in [1], $\mathcal{E}_1(\mathcal{M}^2) < \infty$. We can thus define a probability measure $\tilde{\mathcal{P}}_x$, that is absolutely continuous with respect to $\hat{\mathcal{P}}_x$ with density $\mathcal{M}/\hat{\mathcal{E}}_x(\mathcal{M}) = x^{\omega_-} \mathcal{M}/\mathcal{E}_x(\mathcal{M}^2)$. In particular, $\tilde{\mathcal{P}}_x$ has density $\mathcal{M}^2/\mathcal{E}_x(\mathcal{M}^2)$ with respect to \mathcal{P}_x . In particular, we can choose $\mathbf{P} = \tilde{\mathcal{P}}_1$ and try to apply Lemma 3.1. (This argument does not apply when $\omega_+/\omega_- \leq 2$ since we then know, still from [1] Lemma 2.3, that $\mathcal{E}_1(\mathcal{M}^2) = \infty$.)

In this subsection, we write $C = 1/\mathcal{E}_1(\mathcal{M})$. We extend k on \mathbb{R} to be such that $k(x) = 0$ if $x \leq 0$.

Lemma 3.3. Consider ∇ as in Lemma 3.1. Under $\tilde{\mathcal{P}}_1$, ∇ has a density $h : \mathbb{R} \rightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$h(x) = C\hat{\mathcal{E}}_1 \left(\sum_{s>0} |\Delta_- Y(s)|^{\omega_- + \alpha} Y(s)^\alpha \int_0^\infty duk(u|\Delta_- Y(s)|^\alpha)k((u-x)Y(s)^\alpha) \right),$$

where k denotes the density of the law of I from Lemma 2.3.

Proof. In what follows, we shall implicitly use several times Tonelli’s Theorem, since every term involved in the proof is non-negative. Let $(\sigma'(t))_{t \geq 0}$ be the size process of the ancestor of a leaf in $\partial\mathbb{U}$, chosen according to \mathcal{A} , independently from σ given $(\chi_u)_{u \in \mathbb{U}}$. Note that the independence between σ and σ' does not hold without the conditioning. Let $f : \mathbb{R} \rightarrow \mathbb{R}_+$ be any non-negative measurable function. We have

$$\begin{aligned} \check{\mathcal{E}}_1(f(\nabla)) &= C\hat{\mathcal{E}}_1(\mathcal{M}f(\nabla)) = C\hat{\mathcal{E}}_1(\mathcal{M}f(\zeta_\sigma - \zeta_{\sigma'})) \\ &= C\hat{\mathcal{E}}_1\left(\mathcal{M}\hat{\mathcal{E}}_1(f(\zeta_\sigma - \zeta_{\sigma'}) | (\chi_u)_{u \in \mathbb{U}}, \sigma)\right) \\ &= C\hat{\mathcal{E}}_1\left(\int_{\partial\mathbb{U}} \mathcal{A}(d\ell)f(\zeta_\sigma - \zeta_\ell)\right), \end{aligned}$$

where we used the conditional independence of σ and σ' given $(\chi_u)_{u \in \mathbb{U}}$. We now choose a subtree among those generated by the children of the spine $(\sigma(s))_{s \geq 0}$ (the ancestors of σ), according to the intrinsic area. Denoting $\partial\mathbb{U}_s$ the leaves of the tree descending from the negative jump (if any) of the spine at time s and using that the jump times are measurable with respect to the natural filtration of $(\sigma(t))_{t \geq 0}$, it reads as

$$\begin{aligned} \check{\mathcal{E}}_1(f(\nabla)) &= C\hat{\mathcal{E}}_1\left(\sum_{s>0} \hat{\mathcal{E}}_1\left(\mathcal{M}_s \int_{\partial\mathbb{U}_s} \frac{\mathcal{A}(d\ell)}{\mathcal{M}_s} f(\zeta_\sigma - \zeta_\ell) \Big| (\sigma(t))_{t \geq 0}\right)\right) \\ &= C\hat{\mathcal{E}}_1\left(\sum_{s>0} \hat{\mathcal{E}}_1(f(\zeta_\sigma - s - \zeta_{\hat{\sigma}}) | (\sigma(t))_{t \geq 0})\right), \end{aligned}$$

where $\hat{\sigma}$ is a random leaf of $\partial\mathbb{U}_s$ tagged according to the restriction of the intrinsic area on this subtree and $\zeta_{\hat{\sigma}}$ denotes the height of $\hat{\sigma}$ seen in $\bar{\mathbb{U}}_s$; note that the root of the latter is at height s , therefore the height of $\hat{\sigma}$ in the full tree $\bar{\mathbb{U}}$ is $s + \zeta_{\hat{\sigma}}$. Hence, $\zeta_{\hat{\sigma}}$ is distributed as the lifetime of a spine under $\hat{\mathcal{P}}_{|\Delta_- \sigma(s)|^{\omega_-}}$, conditionally on $(\sigma(t))_{t \geq 0}$. More precisely, Theorem 4.7 in [1] ensures that $\zeta_{\hat{\sigma}}$ depends on σ only through $\Delta_- \sigma(s)$, and is independent of what happens to the spine at other times. The scaling property yields that the above is equal to

$$C\hat{\mathcal{E}}_1\left(\sum_{s>0} |\Delta_- \sigma(s)|^{\omega_-} \hat{\mathcal{E}}_1(f(\zeta_\sigma - s - |\Delta_- \sigma(s)|^{-\alpha} \zeta_{\hat{\sigma}}) | (\sigma(t))_{t \geq 0})\right), \tag{3.2}$$

where now the law of ζ_σ is independent of σ and is that of I under $\widehat{\mathbb{P}}_1$ by Lemma 2.2. We thus get

$$\begin{aligned} \check{\mathcal{E}}_1(f(\nabla)) &= C\widehat{\mathcal{E}}_1\left(\sum_{s>0}|\Delta_{-}\sigma(s)|^{\omega_{-}}\int_0^\infty dxk(x)f(\zeta_\sigma-s-|\Delta_{-}\sigma(s)|^{-\alpha}x)\right) \\ &= C\widehat{\mathbb{E}}_1\left(\sum_{s>0}|\Delta_{-}Y(s)|^{\omega_{-}}\int_0^\infty dxk(x)f(I-s-|\Delta_{-}Y(s)|^{-\alpha}x)\right), \end{aligned}$$

by Lemma 2.2. Note that $I-s$ can be seen as the lifetime of a process distributed as Y started from $Y(s)$, that is by self-similarity $I-s \stackrel{d}{=} Y(s)^{-\alpha}I'$ where I' is independent of $(Y(t))_{t\leq s}$ and is distributed as I under $\widehat{\mathbb{P}}_1$. We then view the above sum as the Stieltjes integral of

$$S : s \mapsto \int_0^\infty dxk(x)f(I-s-|\Delta_{-}Y(s)|^{-\alpha}x)$$

with respect to the non-decreasing process $\sum_{t\leq s}|\Delta_{-}Y(s)|^{\omega_{-}}$ (which is adapted with respect to the natural filtration of Y). Since both processes are non-negative, we can use the optional projection Theorem ([10] Theorem 57), which consists of replacing $S(s)$ by its conditional expectation with respect to the natural filtration of Y without changing the expectation of the sum (see also Theorem 43 in the same book for a definition of optional projection). Informally, this eliminates the randomness of the remaining lifetime after the jumps, and the right-hand side above becomes

$$\begin{aligned} C\widehat{\mathbb{E}}_1\left(\sum_{s>0}|\Delta_{-}Y(s)|^{\omega_{-}}\int_0^\infty dxk(x)\int_0^\infty dyk(y)f(Y(s)^{-\alpha}y-|\Delta_{-}Y(s)|^{-\alpha}x)\right) \\ = C\widehat{\mathbb{E}}_1\left(\sum_{s>0}|\Delta_{-}Y(s)|^{\omega_{-}+\alpha}Y(s)^\alpha\int_0^\infty dxk(x|\Delta_{-}Y(s)|^\alpha)\int_0^\infty dyk(yY(s)^\alpha)f(y-x)\right), \end{aligned}$$

which gives the claim. □

We now provide the proof of the absolute continuity of dA .

Proof of Theorem 1.1, case $\alpha > \omega_{-}$ and $\omega_{+}/\omega_{-} > 2$. By the lemmas 3.1 and 3.3, it is sufficient to show that the supremum of h is finite, that is

$$\sup_{x\in\mathbb{R}}\widehat{\mathbb{E}}_1\left(\sum_{s>0}|\Delta_{-}Y(s)|^{\omega_{-}+\alpha}Y(s)^\alpha\int_0^\infty duk(u|\Delta_{-}Y(s)|^\alpha)k((u-x)Y(s)^\alpha)\right) < \infty.$$

For all $x \in \mathbb{R}$, we can bound $h(x)$ after a suitable change of variable by

$$h(x) \leq C\widehat{\mathbb{E}}_1\left(\sum_{s>0}|\Delta_{-}Y(s)|^{\omega_{-}}(Y(s) \vee |\Delta_{-}Y(s)|)^\alpha\right)\|k\|_\infty\int_0^\infty k(u)du, \tag{3.3}$$

the last integral being equal to 1 since k is a density. It remains to show that the expectation is finite. Since the terms in the sum do not depend on α , we can assume without loss of generality that Y is homogeneous, that is $Y(s) = \exp(\eta(s))$ for all $s \geq 0$. The compensation formula for Poisson point processes then yields that the expectation in the right-hand side above is equal to

$$\begin{aligned} \int_0^\infty ds\widehat{\mathbb{E}}_1\left(e^{(\omega_{-}+\alpha)\eta(s)}\right)\int_{(-\infty,0)}\Pi(dy)(1-e^y)^{\omega_{-}}(e^y \vee (1-e^y))^\alpha \\ \leq \int_0^\infty ds e^{\kappa(2\omega_{-}+\alpha)s}\int_{(-\infty,0)}\Lambda(dy)e^{\omega_{-}y}(1-e^y)^{\omega_{-}-2-\alpha}. \end{aligned}$$

The last integral is finite by definition of ω_- , as well as the first one when $\kappa(2\omega_- + \alpha) < 0$, that is $-\omega_- < \alpha < \omega_+ - 2\omega_-$. Since $\alpha > -\omega_-$ and $\omega_+/\omega_- > 2$ by assumption, we always have $-\omega_- < \alpha < 0 < \omega_+ - 2\omega_-$, which ends the proof. \square

3.2.2 The case $\omega_+/\omega_- \leq 2$ and $\alpha > -\omega_-$

The main issue when $\omega_+/\omega_- \leq 2$ is that the measure $\check{\mathcal{P}}$, defined earlier in order to select two spines, has now infinite total mass. In particular, this prevents us from using Lemma 3.1. We overcome this by defining a new measure $\check{\mathcal{P}}^{(K)}$ in such a way that the two random leaves are tagged among those whose ancestors' sizes have never been too large, which, we will see, entails that $\check{\mathcal{P}}^{(K)}$ has finite total mass. In this direction, let $K > 0$ intended to tend to ∞ . Define B_K as

$$B_K := \{\ell \in \partial\mathbb{U} : \ell^* \leq K\},$$

where $\ell^* := \sup_{t \geq 0} \ell(t)$. Let \mathcal{A}_K be the restriction of the measure \mathcal{A} to B_K , and $\mathcal{M}_K := \mathcal{A}_K(B_K)$. We now define for all $x > 0$ the probability measure $\widehat{\mathcal{P}}_x^{(K)}$ such that it is absolutely continuous with respect to \mathcal{P}_x , with density $\mathcal{M}_K/\mathcal{E}_x(\mathcal{M}^{(K)})$. In the same vein, we would like to define $\check{\mathcal{P}}_x^{(K)}$ to be absolutely continuous with respect to $\widehat{\mathcal{P}}_x^{(K)}$ with density $\mathcal{M}_K/\widehat{\mathcal{E}}_x^{(K)}(\mathcal{M}_K)$. To ease the expressions, we shall write C for the finite and strictly positive constants that appear when changing of measures. Even if C may vary from line to line, it is always known and only depends on the starting point and K , which will play no role in the proofs. We need the following

Lemma 3.4. *For all $x > 0$, it holds that $\widehat{\mathcal{E}}_x^{(K)}(\mathcal{M}_K) = \mathcal{E}_x(\mathcal{M}_K^2) < \infty$. In particular, $\{\check{\mathcal{P}}_x^{(K)}; x > 0\}$ is a family of probability measures.*

Proof. By self-similarity, it is enough to show that $\mathcal{E}_1(\mathcal{M}_K^2) < \infty$. Let σ_K denote a random leaf sampled on B_K with conditional law $\mathcal{A}_K(\cdot)/\mathcal{M}_K$ given $(\chi_u)_{u \in \mathbb{U}}$. Similarly as in the previous subsection, we write

$$\mathcal{E}_1(\mathcal{M}_K^2) = C\widehat{\mathcal{E}}_1^{(K)}(\mathcal{M}_K) = C\widehat{\mathcal{E}}_1^{(K)}\left(\sum_{s>0} |\Delta_- \sigma_K(s)|^{\omega_-} \widehat{\mathcal{E}}_1^{(K)}(\mathcal{M}_{K,s} | (\sigma_K(t))_{t \geq 0})\right)$$

where $\mathcal{M}_{K,s}$ is the rescaled truncated mass generated by the cell born at time s (truncated when cells reach a size $K/|\Delta_- \sigma_K(s)|^{\omega_-}$ by the scaling property). Since $\mathcal{M}_{K,s}$ cannot be greater than the non-truncated area, that has expectation 1, we get

$$\begin{aligned} \mathcal{E}_1(\mathcal{M}_K^2) &\leq C\widehat{\mathcal{E}}_1^{(K)}\left(\sum_{s>0} |\Delta_- \sigma_K(s)|^{\omega_-}\right) = C\widehat{\mathcal{E}}_1\left(\sum_{s>0} \mathbb{1}_{\{\sigma^* \leq K\}} |\Delta_- \sigma_K(s)|^{\omega_-}\right) \\ &= C\widehat{\mathbb{E}}_1\left(\sum_{s>0} \mathbb{1}_{\{Y^* \leq K\}} |\Delta_- Y(s)|^{\omega_-}\right). \end{aligned}$$

As previously we assume without loss of generality that Y is homogeneous. Now we fix $p \in (0, \omega_+ - \omega_-)$ and we bound the latter from above by

$$CK^{\omega_- - p} \widehat{\mathbb{E}}_1\left(\sum_{s>0} e^{p\eta(s^-)} (1 - e^{\Delta_- \eta(s)})^{\omega_-}\right).$$

The compensation formula yields that the expectation is equal to

$$\int_0^\infty ds e^{s\kappa(\omega_- + p)} \int_{(-\infty, 0)} \Pi(dy) (1 - e^y)^{\omega_-},$$

which is finite since $\kappa(\omega_- + p) < 0$ and by definition of ω_- and Π . This shows that $\mathcal{E}_x(\mathcal{M}_K^2) < \infty$ for any $x > 0$ by self-similarity. \square

Thanks to Lemma 3.4, Lemma 3.1 applies with $\mathbf{P} = \check{\mathcal{P}}_1^{(K)}$ and $\nabla_K := \zeta_{\sigma_K} - \zeta_{\sigma'_K}$, where σ_K and σ'_K are conditionally independent random leaves in B_K with conditional law $\mathcal{A}_K(\cdot)/\mathcal{M}_K$ given $(\chi_u)_{u \in \mathbb{U}}$. We claim that when K is large enough, $B_K = \partial\mathbb{U}$, that is

$$\mathcal{P}_1(\exists K > 0 : \mathcal{A} = \mathcal{A}_K) = 1.$$

Indeed, as a direct consequence of Corollary 4 in [4], we have that $\sup_{u \in \mathbb{U}} \chi_u^* < \infty$ \mathcal{P} -a.s., where $\chi_u^* := \sup_{t \geq 0} \chi_u(t)$. By Lemma 3.1, it is hence enough to show that ∇_K has a bounded density in a neighbourhood of 0 to prove the Theorem. Similarly to Lemma 3.3, we have

Lemma 3.5. Under $\check{\mathcal{P}}_1^{(K)}$, ∇_K has a density $h : \mathbb{R} \rightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$h : x \mapsto C \widehat{\mathbb{E}}_1 \left(\sum_{s>0} \mathbb{1}_{\{\sup_{t \leq s} Y(t) \leq K\}} |\Delta_- Y(s)|^{\omega_- + \alpha} Y(s)^\alpha \times \int_0^\infty du g_{2,s}(u |\Delta_- Y(s)|^\alpha) g_{1,s}((u-x)Y(s)^\alpha) \right),$$

where C is known and comes from the change of measure, and $g_{1,s}, g_{2,s}$ are non-negative random functions, measurable with respect to the natural filtration of Y at time s for all $s \geq 0$, that are all pointwise bounded by k .

The proof being very similar to that of Lemma 6, we do not provide all the steps, but only those where new arguments are needed.

Proof. We have the following analogue of (3.2):

$$\begin{aligned} \check{\mathcal{E}}_1^{(K)}(f(\nabla_K)) &= C \widehat{\mathcal{E}}_1 \left(\sum_{s>0} \mathbb{1}_{\{\sigma^* \leq K\}} |\Delta_- \sigma(s)|^{\omega_-} \right. \\ &\quad \left. \times \widehat{\mathcal{E}}_1 \left(\mathbb{1}_{\{\hat{\sigma}^* \leq K | \Delta_- \sigma(s)|^{-\omega_-}\}} f(\zeta_\sigma - s - |\Delta_- \sigma(s)|^{-\alpha} \zeta_{\hat{\sigma}}) \Big| (\sigma(t))_{t \geq 0} \right) \right) \\ &= C \widehat{\mathbb{E}}_1 \left(\sum_{s>0} \mathbb{1}_{\{Y_1^* \leq K\}} |\Delta_- Y_1(s)|^{\omega_-} \right. \\ &\quad \left. \times \widehat{\mathbb{E}}_1 \left(\mathbb{1}_{\{Y_2^* \leq K | \Delta_- Y_1(s)|^{-\omega_-}\}} f(I_1 - s - |\Delta_- Y_1(s)|^{-\alpha} I_2) \Big| Y_1 \right) \right), \end{aligned}$$

by Lemma 2.2, where I_1 and I_2 are the respective absorption times at 0 of two independent positive self-similar Markov processes Y_1, Y_2 with same distribution $\widehat{\mathbb{P}}_1$. The expectations in the right-hand side then becomes

$$\widehat{\mathbb{E}}_1 \left(\sum_{s>0} \mathbb{1}_{\{Y_1^* \leq K\}} |\Delta_- Y_1(s)|^{\omega_-} \int_0^\infty dx g_{2,s}(x) f(I_1 - s - |\Delta_- Y_1(s)|^{-\alpha} x) \right),$$

where for any $s > 0$ such that $\Delta_- Y_1(s) < 0$, the random function

$$g_{2,s} : x \mapsto k(x) \widehat{\mathbb{P}}(Y_2^* \leq K | \Delta_- Y_1(s)|^\alpha | I_2 = x)$$

is measurable with respect to the natural filtration of Y_1 at time s . Clearly, $g_{2,s}(x) \leq k(x)$ for all $x > 0$. As before, applying [10] Theorem 57, we obtain that

$$\begin{aligned} \check{\mathcal{E}}_1(f(\nabla_{|B})) &= C \widehat{\mathbb{E}}_1 \left(\sum_{s>0} \mathbb{1}_{\{\sup_{t \leq s} Y_1(t) \leq K\}} |\Delta_- Y_1(s)|^{\omega_-} \int_0^\infty dx g_{2,s}(x) \int_0^\infty dy g_{1,s}(y) \right. \\ &\quad \left. \times f(Y_1(s)^{-\alpha} y - |\Delta_- Y_1(s)|^{-\alpha} x) \right), \end{aligned}$$

where $g_{1,s}$ is defined in the same way as $g_{2,s}$. □

Proof of Theorem 1.1, the case $\alpha > -\omega_-$ and $\omega_+/\omega_- \leq 2$. As previously, we only need to show that h given in Lemma 3.5 is bounded to conclude by Lemma 3.1. Recall that the (random) functions $g_{1,s}, g_{2,s}, s \geq 0$ in the definition of h are bounded by k . For any $x \in \mathbb{R}$, similarly to (3.3), we thus have that

$$h(x) \leq C \widehat{\mathbb{E}}_1 \left(\sum_{s>0} \mathbb{1}_{\{\sup_{t \leq s} Y(t) \leq K\}} |\Delta_- Y(s)|^{\omega_-} (Y(s) \vee |\Delta_- Y(s)|)^\alpha \right) \|k\|_\infty \|k\|_1.$$

As before, we can consider the simpler homogeneous case to show that h is bounded, without loss of generality. Let $\eta(s) = \log(Y(s)), s > 0$. We rewrite the expectation above as

$$\begin{aligned} & \widehat{\mathbb{E}}_1 \left(\sum_{s>0} \mathbb{1}_{\{\sup_{t \leq s} \eta(t) \leq \log K\}} e^{(\omega_- + \alpha)\eta(s^-)} (1 - e^{\Delta_- \eta(s)})^{\omega_-} (e^{\Delta_- \eta(s)} \vee (1 - e^{\Delta_- \eta(s)}))^\alpha \right) \\ & \leq 2^{-\alpha} \widehat{\mathbb{E}}_1 \left(\sum_{s>0} \mathbb{1}_{\{\sup_{t \leq s} \eta(t) \leq \log K\}} e^{(\omega_- + \alpha)\eta(s^-)} (1 - e^{\Delta_- \eta(s)})^{\omega_-} \right) \\ & = 2^{-\alpha} K^{\omega_- + \alpha - p} \widehat{\mathbb{E}}_1 \left(\sum_{s>0} e^{p\eta(s^-)} (1 - e^{\Delta_- \eta(s)})^{\omega_-} \right), \end{aligned}$$

for any arbitrary fixed $p \in (0, \omega_+ - \omega_-)$. The compensation formula then shows that the last expectation is equal to

$$\int_0^\infty ds e^{\kappa(\omega_- + q)s} \int_{(-\infty, 0)} \Lambda(dy) e^{\omega_- y} (1 - e^y)^{\omega_-},$$

which is clearly finite from the choice of q and the definition of ω_- . We have thus proved that h is bounded. By Lemma 3.1, this shows that $t \mapsto \mathcal{A}(\{\ell \in B_K : \zeta_\ell \leq t\})$ is absolutely continuous with respect to the Lebesgue measure $\check{\mathcal{P}}$ -a.s., and therefore \mathcal{P} -a.s., and we conclude using that $\mathcal{P}(\exists K > 0 : \mathcal{A}_K = \mathcal{A}) = 1$ as stated earlier. □

3.2.3 The singular case, $\alpha \leq -\omega_-$

We finish the proof of Theorem 1.1, that is we show that when $\alpha \leq -\omega_-$, dA is almost surely singular with respect to the Lebesgue measure.

Since $t \mapsto A(t)$ is non-decreasing, almost surely, for almost every $t \geq 0$, the derivative of A at t exists. Therefore by Fubini's Theorem, for almost every $t \geq 0$ there exists a finite random variable $A'(t)$ such that $A'(t) := \lim_{\epsilon \rightarrow 0} (A(t + \epsilon) - A(t))/\epsilon$. For such t , applying (2.4) and using the same notation we obtain:

$$A(t + \epsilon) - A(t) \stackrel{d}{=} \sum_{i \geq 1} X_i^{\omega_-}(t) A_i(\epsilon X_i^\alpha(t)).$$

Suppose that there are infinitely many fragments at time t . Then for all $n \geq 1$, set $\epsilon_n := X_n^{-\alpha}(t)$ and divide the last expression by ϵ_n to get

$$\begin{aligned} \epsilon_n^{-1} (A(t + \epsilon_n) - A(t)) &= \sum_{i \geq 1} X_n^\alpha(t) X_i^{\omega_-}(t) A_i(X_n^{-\alpha}(t) X_i^\alpha(t)) \\ &\geq X_n^{\alpha + \omega_-}(t) A_n(1). \end{aligned}$$

Since the A_n 's are i.i.d. copies of A , there are almost surely infinitely many $A_n(1)$'s which are greater than any given constant. But if $\alpha \leq -\omega_-$, we see that $\limsup_{n \rightarrow \infty} \epsilon_n^{-1} (A(t +$

$\epsilon_n) - A(t)) = \infty$ which is in contradiction with the fact that A admits a derivative at t . This implies that there is only a finite number of fragments at time t , say $N \in \mathbb{N}$, therefore we can switch the sum and the limit and we obtain :

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-1}(A(t + \epsilon) - A(t)) = \sum_{i \leq N} X_i^{\alpha + \omega_-}(t) \lim_{\epsilon \rightarrow 0} A_i(\epsilon X_i^\alpha(t))/\epsilon,$$

where the limits are well defined and equal to 0 by Lemma 3.2. Hence $A'(t) = 0$ for almost every t and dA is singular.

3.3 Number of fragments

As we just saw in the proof of the singular case of Theorem 1.1, there is a link between the number of fragments in the growth-fragmentation and the regularity of A . Even though we shall not use this relation later on, we state it in a corollary as it might be of independent interest.

Corollary 3.6. *Suppose that $\alpha > -\omega_-$, then almost surely the number of fragments with positive mass is infinite for every t such that $a(t) > 0$.*

Conversely if $\alpha \leq -\omega_-$, then almost surely the number of fragments with positive mass is finite for almost every $t \geq 0$.

Proof. The second statement has been established in the subsection 3.2.3. If $\alpha > -\omega_-$, then by Theorem 1.1 we have $dA(t) = a(t)dt$. Fix $s, t > 0$ such that $t < \zeta$. By (2.4) we can write

$$A(t + s) - A(t) \stackrel{d}{=} \sum_{i=1}^{N_t} X_i^{\omega_-}(t) A_i(s X_i^\alpha(t)), \tag{3.4}$$

where N_t is the number of fragments (possibly infinite) with positive mass at time t , $\{A_i; i = 1..N_t\}$ are i.i.d. copies of A . Then using the above identity we have

$$\frac{1}{\epsilon} \sum_{i=1}^{N_t} X_i^{\omega_-}(t) A_i(\epsilon X_i^\alpha(t)) \xrightarrow[\epsilon \rightarrow 0]{a.s.} a(t).$$

Suppose moreover that $N_t < \infty$, then Lemma 3.2 implies that $a(t) = 0$. □

4 Approximation of the profile

Throughout this section, we assume that $dA(t) = a(t)dt$, or equivalently $\alpha > -\omega_-$, by Theorem 1.1. We moreover suppose that $\omega_- \geq 1$, which ensures the Feller property of \mathbf{X} as stated in Lemma 6.2 in Appendix. Our main goal in what follows is to adapt the arguments of Haas [13] Section 5 to the growth-fragmentations case to prove 1.2. Recall the definitions of the processes M, N for all $\epsilon > 0, t \geq 0$:

$$M(t, \epsilon) := \sum_{i \geq 1} X_i^{\omega_-}(t) \mathbb{1}_{\{X_i(t) \leq \epsilon\}},$$

$$N(t, \epsilon) := \sum_{i \geq 1} \mathbb{1}_{\{X_i(t) > \epsilon\}}.$$

As Haas we shall focus on the small fragments, since the behaviour of $N(t, \epsilon)$ as $\epsilon \rightarrow 0$ can be deduced from that of $M(t, \epsilon)$ applying Tauberian theorems (as discussed in the end of this section).

Lemma 4.1 (Analogue of Lemma 8 in [13]). *Let I be a random variable with density k , independent of \mathbf{X} . If $\alpha > -\omega_-$ and $\omega_- \geq 1$, then for almost every $t > 0$,*

$$\lim_{\epsilon \rightarrow 0} \epsilon^\alpha \mathcal{E} \left(M(t, \epsilon I^{\frac{1}{\alpha}}) \middle| \mathbf{X} \right) \stackrel{a.s.}{=} a(t).$$

Provided that \mathbf{X} is a Feller process, the proof is almost identical to that of Haas, with the difference that one has to work on an event having a probability arbitrarily close to 1 and that ensures $\int_0^\infty a^2(t)dt$ to have finite expectation (this event can be $H_K := \{\sup_{u \in \mathbb{U}} \chi_u^* \leq K\}$ where $\chi_u^* := \sup_{t \geq 0} \chi_u(t)$). We skip the details of the proof of Lemma 4.1, the Feller property and its proof are given in Appendix.

We can now make the proof of Theorem 1.2.

Proof of Theorem 1.2(i). This proof is again very similar to that of Haas, we thus just focus on verifying that the hypotheses of the Wiener-Pitt theorem (Theorem 4.8.0 of [7]) are satisfied, and refer to the proof of [13] Theorem 7 to see how it applies to show the convergence of small fragments. What has to be shown is that the Mellin transform of I , defined as $\mathcal{M}_I(ix) := \widehat{\mathbb{E}}_1(I^{ix-1})$, exists and is non zero for all $x \in \mathbb{R}$. We already know by Lemma 2.3 that $\widehat{\mathbb{E}}_1(I^{-1}) = \alpha\kappa'(\omega_-) \in (0, \infty)$. Let Ψ be the characteristic exponent of η defined as $\Psi : \theta \mapsto -\log \widehat{\mathbb{E}}(e^{i\theta\eta(1)})$. Theorem 2.7(1) in [19] shows that

$$\mathcal{M}_I(ix) = \frac{\Psi(-\alpha x)}{ix} \mathcal{M}_I(1 + ix).$$

It is not hard to check that $\Psi(-\alpha x) \neq 0$. Moreover, it is also stated in the same theorem that

$$\mathcal{M}_I(1 + ix) = \Phi_+(0) \frac{\Gamma(1 + ix)}{W_{\Phi_-}(1 + ix)} W_{\Phi_+}(-ix),$$

where Φ_+ (respectively Φ_-) is the characteristic exponent of the ascending (respectively descending) ladder height process of η and W_{Φ_+} (respectively W_{Φ_-}) is the generalized Weierstrass product of Φ_+ (respectively Φ_-) as in [19] (see Kyprianou [15] or Bertoin [2] for definitions and details on ladder height processes). In particular, it is well-known that since η drifts to $-\infty$, we have $\Phi_+(0) > 0$. Therefore $W_{\Phi_+}(z) \neq 0$ for all $z \in \mathbb{C}$ with $\text{Re}(z) \geq 0$ by Theorem 3.2 of [19]. Furthermore, it also ensures that W_{Φ_-} is holomorphic on $\{z \in \mathbb{C} : \text{Re}(z) > 0\}$ therefore $|W_{\Phi_-}(1 + ix)| < \infty$. Since $\Gamma(1 + ix)$ is non-zero, we see that $\mathcal{M}_I(1 + ix) \neq 0$, we can then apply the Wiener-Pitt theorem, as planned, giving the claim. \square

To conclude the proof of Theorem 1.2, it remains to show that for almost every $t \geq 0$,

$$\frac{-\alpha}{\omega_-} M(t, \epsilon) \underset{\epsilon \rightarrow 0}{\sim} \frac{\omega_- + \alpha}{\omega_-} \epsilon^{\omega_-} N(t, \epsilon). \tag{4.1}$$

Let $\mu := \sum_{i \geq 1} \delta_{X_i(t)^{\omega_-}}$ and let $\bar{\mu}(x) := \mu((x, \infty))$. Define $df(y) = y\mu(dy)$. Equation (4.1) can be shown using Tauberian theorems, we refer to the proof of equation (4) in [3] to see that f is regularly varying at 0 with index $1 - \beta \in (0, 1)$ if and only if $\bar{\mu}$ is regularly varying at 0 with index $-\beta$. In that case, it holds that

$$\beta \epsilon \bar{\mu}(\epsilon) \underset{\epsilon \rightarrow 0}{\sim} (1 - \beta) f(\epsilon).$$

Now notice that $\bar{\mu}(\epsilon) = N(t, \epsilon^{1/\omega_-})$ and $f(\epsilon) = M(t, \epsilon^{1/\omega_-})$ which implies with Theorem 1.2(i) that $1 - \beta = -\alpha/\omega_- \in (0, 1)$. This proves that (4.1) holds.

5 Hausdorff dimension

We now study the case $\alpha \leq -\omega_-$ so that dA is singular with respect to the Lebesgue measure almost surely, by Theorem 1.1. Theorem 1.3 describes the set on which dA is concentrated through its Hausdorff dimension (note that its support is exactly $[0, \zeta]$ by Lemma 3.2), see [12] for background.

5.1 The lower bound

Frostman’s Lemma (see e.g. [12] Corollary 6.6(a)), that we now recall, is the key to the lower bound.

Lemma 5.1 (Frostman’s Lemma). *Let $b \in (0, 1]$ and let μ be a finite measure on \mathbb{R} . If*

$$\mathcal{I}_b(\mu) := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{d\mu(u)d\mu(v)}{|u - v|^b} < \infty,$$

then $\dim_H(\mu) \geq b$.

Proof of Theorem 1.3: the lower bound. In the light of Lemma 5.1, it is sufficient to show that

$$\mathcal{E}_1(\mathcal{I}_b(dA)) < \infty, \tag{5.1}$$

for all $b < \frac{\omega_-}{-\alpha}$. We write

$$\mathcal{E}_1(\mathcal{I}_b(dA)) = \mathcal{E}_1 \left(\int_0^\infty \int_0^\infty \frac{dA(u)dA(v)}{|u - v|^b} \right).$$

As previously, we sample a first spine applying Lemma 2.1 and we get

$$\mathcal{E}_1(\mathcal{I}_b(dA)) = \widehat{\mathcal{E}}_1 \left(\mathcal{E}_1 \left(\int_0^\infty \frac{dA(v)}{|\zeta_\sigma - v|^b} \middle| (\sigma(t))_{t \leq \zeta_\sigma} \right) \right).$$

We then decompose A as in the proof of Theorem 1.1, we write

$$\mathcal{E}_1(\mathcal{I}_b(dA)) = \widehat{\mathcal{E}}_1 \left(\sum_{s < \zeta_\sigma} \mathcal{E}_{|\Delta_- \sigma(s)|} \left(\int_s^\infty \frac{dA_s(v)}{|\zeta_\sigma - v|^b} \middle| (\sigma(t))_{t \leq \zeta_\sigma} \right) \right),$$

where A_s is the intrinsic area function associated with the restriction of \mathcal{A} to ∂U_s , the leaves of the subtree generated by the cell born at time s . In particular, A_s has same conditional distribution as A under $\mathcal{P}_{|\Delta_- \sigma(s)|}$ shifted by s (see [1] Theorem 4.7). We rewrite the right hand-side above denoting $A_s^* := A_s(s + \cdot)$ and get

$$\begin{aligned} &\mathcal{E}_1(\mathcal{I}_b(dA)) \\ &= \widehat{\mathcal{E}}_1 \left(\sum_{s < \zeta_\sigma} \mathcal{E}_{|\Delta_- \sigma(s)|} \left(\int_0^\infty \frac{dA_s^*(v)}{|\zeta_\sigma - s - v|^b} \middle| (\sigma(t))_{t \leq \zeta_\sigma} \right) \right) \\ &= \widehat{\mathcal{E}}_1 \left(\sum_{s < \zeta_\sigma} |\Delta_- \sigma(s)|^{\omega_-} \mathcal{E}_1 \left(\int_0^\infty \frac{dA_s^*(v)}{|\zeta_\sigma - s - |\Delta_- \sigma(s)|^{-\alpha} v|^b} \middle| (\sigma(t))_{t \leq \zeta_\sigma} \right) \right), \end{aligned}$$

where we applied the self-similarity of A_s^* for the last equality. Using Lemmas 2.1 and 2.2, let I_2 be a random variable with density k independent of σ and write

$$\begin{aligned} \mathcal{E}_1(\mathcal{I}_b(dA)) &= \widehat{\mathcal{E}}_1 \left(\sum_{s < \zeta_\sigma} |\Delta_- \sigma(s)|^{\omega_-} \widehat{\mathcal{E}}_1 \left(|\zeta_\sigma - s - |\Delta_- \sigma(s)|^{-\alpha} I_2|^{-b} \middle| (\sigma(t))_{t \leq \zeta_\sigma} \right) \right) \\ &= \widehat{\mathbb{E}}_1 \left(\sum_{s < I_1} |\Delta_- Y(s)|^{\omega_-} |I_1 - s - |\Delta_- Y(s)|^{-\alpha} I_2|^{-b} \right). \end{aligned}$$

As in the proof of Lemma 3.3, we use Theorem 57 of [10], justified again by positivity: the optional projection of $s \mapsto |I_1 - s - |\Delta_- Y(s)|^{-\alpha} I_2|^{-b}$ being given by

$$s \mapsto \int_0^\infty du \int_0^\infty dv \frac{k(u)k(v)}{|Y(s)^{-\alpha} u - |\Delta_- Y(s)|^{-\alpha} v|^b}.$$

We hence obtain

$$\mathcal{E}_1(\mathcal{I}_b(A)) = \widehat{\mathbb{E}}_1 \left(\sum_{s < I_1} |\Delta_- Y(s)|^{\omega_-} \int_0^\infty du \int_0^\infty dv \frac{k(u)k(v)}{|Y(s)^{-\alpha}u - |\Delta_- Y(s)|^{-\alpha}v|^b} \right).$$

Assuming without loss of generality that Y is homogeneous, we see that the latter is equal to

$$\widehat{\mathbb{E}}_1 \left(\sum_{s > 0} e^{(\omega_- + \alpha b)\eta(s^-)} (1 - e^{\Delta_- \eta(s)})^{\omega_-} \int_0^\infty du \int_0^\infty dv \frac{k(u)k(v)}{|e^{-\alpha \Delta_- \eta(s)}u - (1 - e^{\Delta_- \eta(s)})^{-\alpha}v|^b} \right).$$

Lemma 6.1 in Appendix finally yields that

$$\mathcal{E}_1(\mathcal{I}_b(A)) \leq C \widehat{\mathbb{E}}_1 \left(\sum_{s > 0} e^{(\omega_- + \alpha b)\eta(s^-)} (1 - e^{\Delta_- \eta(s)})^{\omega_-} \right),$$

where C is a deterministic finite constant. The compensation formula then shows that the right-hand side above is equal to

$$C \int_0^\infty e^{s\kappa(2\omega_- + \alpha b)} ds \int_{\mathbb{R}_-} (1 - e^y)^{\omega_-} e^{\omega_- y} (\Lambda + \tilde{\Lambda})(dy).$$

The last integral being finite by definition of ω_- , we see that this expression is finite whenever $\kappa(2\omega_- + \alpha b) < 0$, which is the case if and only if $b \in \left(\frac{2\omega_- - \omega_+}{-\alpha}, \frac{\omega_-}{-\alpha} \right)$ (this interval is never empty since we assume that $\omega_- < \omega_+$). We thus have shown that

$$b \in \left(\frac{2\omega_- - \omega_+}{-\alpha}, \frac{\omega_-}{-\alpha} \right) \Rightarrow (5.1) \Rightarrow \mathcal{I}_b(A) < \infty \text{ a.s.},$$

which by Lemma 5.1 gives the lower bound for any $\alpha \leq \omega_-$. □

5.2 The upper bound

In the pure fragmentation setting, the analogue of A is the function M of the loss of mass. The upper bound of $\dim_H(dM)$ has been obtained by Haas and Miermont in [14] by constructing the CRT induced by the fragmentation. They first investigated the Hausdorff dimension of the leaves of the tree, then they deduced the upper bound for $\dim_H(dM)$ using the fact that the image of a set by any surjective Lipschitz mapping (in their case the cumulative height profile) has Hausdorff dimension at most equal to that of the original set (this is a direct consequence of Lemma 1.8 in [12]). Since Rembart and Winkel [20] already provided the Hausdorff dimension of the leaves $\mathcal{L}(\mathcal{T})$ of the CRT, we can use the same argument as Haas and Miermont to obtain the upper bound. For this reason we now work on $(\mathcal{T}, \mathcal{A}_\mathcal{T})$ instead of $(\overline{\mathbb{U}}, \mathcal{A})$.

It is not hard to see from its definition in Section 2 that the height function ht is Lipschitz with respect to the metric d .

Proof of the upper bound. Recall that $\mathcal{A}_\mathcal{T}$ is supported on $\mathcal{L}(\mathcal{T})$ (more precisely the subset of $\mathcal{L}(\mathcal{T})$ corresponding to leaves in $\partial\mathbb{U}$). By definition of A (2.3), $dA(\text{ht}(\mathcal{L}(\mathcal{T})))$ is equal to its total mass \mathcal{M} . Therefore,

$$\dim_H(dA) \leq \dim_H(\text{ht}(\mathcal{L}(\mathcal{T}))) \leq \dim_H(\mathcal{L}(\mathcal{T}))$$

since ht is Lipschitz. By Theorem 4.5 in [20], $\dim_H(\mathcal{L}(\mathcal{T})) = -\omega_-/\alpha$, which gives the claim. □

6 Application to Boltzmann random planar maps

In [1], the authors showed that by cutting particular Boltzmann random maps at heights, one obtains a collection of cycles whose lengths are described in scaling limit by a specific family of growth-fragmentations with cumulant function of the form

$$\kappa_\theta(q) := \frac{\cos(\pi(q - \theta))}{\sin(\pi(q - 2\theta))} \cdot \frac{\Gamma(q - \theta)}{\Gamma(q - 2\theta)}, \quad q \in (\theta, 2\theta + 1),$$

with self-similarity index $\alpha = 1 - \theta$, for some parameter $\theta \in (1, 3/2]$. (The case $\theta = 3/2$ corresponds to the Brownian map.)

The Cramér hypothesis (2.2) holds with $\omega_- = \theta + 1/2$ and $\omega_+ = \theta + 3/2$, so that the intrinsic area of the ball of radius r is an absolutely continuous function of r , by Theorem 1.1. The small cycle lengths in the random maps are related to a by Theorem 1.2 in this paper and Theorem 6.8 in [1].

Appendix

We state and prove here a technical results on the density k which is used in the proof of the lower bound in Theorem 1.3.

Lemma 6.1. *Let $b, c \in (0, 1)$. We have that*

$$\int_0^\infty du \int_0^\infty dv \frac{k(u)k(v)}{|uc^{-\alpha} - v(1-c)^{-\alpha}|^b} \leq C,$$

where C is a finite constant not depending on c .

Proof. We have

$$\int_0^\infty du \int_0^\infty dv \frac{k(u)k(v)}{|uc^{-\alpha} - v(1-c)^{-\alpha}|^b} = c^\alpha(1-c)^\alpha \int_0^\infty k(uc^\alpha)du \int_0^\infty dv \frac{k(v(1-c)^\alpha)}{|u-v|^b}.$$

Consider the last integral, we have that

$$\begin{aligned} \int_0^\infty \frac{k(v(1-c)^\alpha)}{|u-v|^b} dv &\leq \int_{u-1}^{u+1} \frac{\|k\|_\infty}{|u-v|^b} dv + \int_0^\infty k(v(1-c)^\alpha)dv \\ &\leq \int_{u-1}^{u+1} \frac{\|k\|_\infty}{|u-v|^b} dv + \int_0^\infty k(v)dv \\ &= C, \end{aligned}$$

where C denotes a constant not depending on c . This yields that

$$\begin{aligned} \int_0^\infty du \int_0^\infty dv \frac{k(u)k(v)}{|uc^{-\alpha} - v(1-c)^{-\alpha}|^b} &\leq c^\alpha(1-c)^\alpha \int_0^\infty Ck(uc^\alpha)du \\ &= (1-c)^\alpha C \end{aligned}$$

Notice that the same arguments apply when the roles of c and $(1-c)$ are exchanged, which entails that the upper bound that we just obtained holds with $(c \vee (1-c))^\alpha$ instead of $(1-c)^\alpha$. One remarks that $c \vee (1-c) \geq 1/2$ and the claim follows. \square

For $q \geq 1$, we define $\ell^{q\downarrow}$ the subset of ℓ^q of non-increasing null sequences with finite q -norm, denoted by $\|\cdot\|_q$.

Lemma 6.2 (Feller's Property). *Suppose that $\omega_- \geq 1$. Then, the law of the growth-fragmentation \mathbf{X} satisfies the following Feller's property: let \underline{x}_n , $n \in \mathbb{N}$ and \underline{x} be elements of $\ell^{\omega_- \downarrow}$ such that $(\underline{x}_n)_{n \geq 1}$ converges in $\ell^{\omega_- \downarrow}$ to \underline{x} . Then it holds that*

$$\mathcal{P}_{\underline{x}_n} \Rightarrow \mathcal{P}_{\underline{x}}, \quad \text{as } n \rightarrow \infty,$$

where \Rightarrow means weak convergence in the sense of finite-dimensional distributions in $\ell^{\omega-\downarrow}$.

Proof of Lemma 6.2. We denote $\underline{x}_n := (x_{n,1}, x_{n,2}, \dots)$ and $\underline{x} := (x_1, x_2, \dots)$. Let $\mathbf{X}^{(n)}$ (respectively \mathbf{Y}) be a self-similar growth-fragmentation process with distribution $\mathcal{P}_{\underline{x}_n}$ (respectively $\mathcal{P}_{\underline{x}}$). We shall show that the Wasserstein distance between $\mathcal{P}_{\underline{x}_n}$ and $\mathcal{P}_{\underline{x}}$ converges to zero, which will entail the claim. Thanks to the assumption $\omega_- \geq 1$, the distance of two sequences in $\ell^{\omega-}$ of non-negative numbers decreases after rearranging them in the non-increasing order (see [17] Theorem 3.5). Let $t > 0$, we thus have that

$$\mathcal{E} \left(\|\mathbf{X}^{(n)}(t) - \mathbf{Y}(t)\|_{\omega_-}^{\omega_-} \right) \leq \mathcal{E} \left(\sum_{k \geq 1} \|\mathbf{X}_k^{(n)}(t) - \mathbf{Y}_k(t)\|_{\omega_-}^{\omega_-} \right), \tag{6.1}$$

where $\mathbf{X}_k^{(n)}$ and \mathbf{Y}_k are growth-fragmentations with respective distributions $\mathcal{P}_{x_{n,k}}$, \mathcal{P}_{x_k} . In the same vein as in the proof of Proposition 2 in [4] (viz the branching property for growth-fragmentations), we fix $\epsilon > 0$ and define $\mathbf{X}_{k,\epsilon}^{(n)}$, (respectively $\mathbf{Y}_{k,\epsilon}$) the growth-fragmentation obtained from $\mathbf{X}_k^{(n)}$ (respectively \mathbf{Y}_k) by killing every fragment - and those it generates in the future - as soon as it reaches a size smaller than ϵ . Using that $(a + b + c)^d \leq 3^d(a^d + b^d + c^d)$ for all $a, b, c, d > 0$, the right-hand side of (6.1) entails that

$$\mathcal{E} \left(\|\mathbf{X}^{(n)}(t) - \mathbf{Y}(t)\|_{\omega_-}^{\omega_-} \right) \leq 3^{\omega_-} (A_n + B_n + C), \tag{6.2}$$

where

$$\begin{aligned} A_n &:= \mathcal{E} \left(\sum_{k \geq 1} \|\mathbf{X}_k^{(n)}(t) - \mathbf{X}_{k,\epsilon}^{(n)}(t)\|_{\omega_-}^{\omega_-} \right) \\ B_n &:= \mathcal{E} \left(\sum_{k \geq 1} \|\mathbf{X}_{k,\epsilon}^{(n)}(t) - \mathbf{Y}_{k,\epsilon}(t)\|_{\omega_-}^{\omega_-} \right) \\ C &:= \mathcal{E} \left(\sum_{k \geq 1} \|\mathbf{Y}_{k,\epsilon}(t) - \mathbf{Y}_k(t)\|_{\omega_-}^{\omega_-} \right). \end{aligned}$$

Recall that $X_j(t)$ is the size of the j th largest fragment in a growth-fragmentation at time t . We define $X_j^*(t)$ an infimum of the sizes of the ancestors of $X_j(t)$ before time t . In particular, if $X_j(t) > 0$, then $X_j^*(t) > 0$. Fix $a > \epsilon$ and write

$$\begin{aligned} A_n &= \sum_{k \geq 1} \mathbb{1}_{\{x_{n,k} \leq \epsilon\}} \mathcal{E}_{x_{n,k}} \left(\|\mathbf{X}_k^{(n)}(t)\|_{\omega_-}^{\omega_-} \right) \\ &\quad + \mathbb{1}_{\{x_{n,k} > \epsilon\}} \mathcal{E}_{x_{n,k}} \left(\sum_{j \geq 1} X_j(t)^{\omega_-} \mathbb{1}_{\{0 < X_j^*(t) \leq \epsilon, X_j(t) \leq a\}} \right) \\ &\quad + \mathbb{1}_{\{x_{n,k} > \epsilon\}} \mathcal{E}_{x_{n,k}} \left(\sum_{j \geq 1} X_j(t)^{\omega_-} \mathbb{1}_{\{0 < X_j^*(t) \leq \epsilon, X_j(t) > a\}} \right). \end{aligned}$$

The first part is smaller than $\sum_{k \geq 1} x_{n,k}^{\omega_-} \mathbb{1}_{\{x_{n,k} \leq \epsilon\}}$ by Theorem 2 in [4]. Applying Proposition 4.6 and Theorem 4.7 in [1], we bound the second part by

$$\sum_{k \geq 1} \mathbb{1}_{\{x_{n,k} > \epsilon\}} \mathcal{E}_{x_{n,k}} \left(\sum_{j \geq 1} X_j(t)^{\omega_-} \mathbb{1}_{\{X_j(t) \leq a\}} \right) \leq \sum_{k \geq 1} \mathbb{1}_{\{x_{n,k} > \epsilon\}} x_{n,k}^{\omega_-} \widehat{\mathbb{P}}_{x_{n,k}} (0 < Y(t) \leq a).$$

Hence, since Y is a Feller process, we have that

$$\limsup_{n \rightarrow \infty} A_n \leq \sum_{k \geq 1} \mathbb{1}_{\{x_k \leq \epsilon\}} x_k^{\omega_-} + \mathbb{1}_{\{x_k \geq \epsilon\}} x_k^{\omega_-} \widehat{\mathbb{P}}_{x_k} (0 < Y(t) \leq a) + \mathbb{1}_{\{x_k \geq \epsilon\}} \mathcal{E}_{x_k} \left(\sum_{j \geq 1} X_j(t)^{\omega_-} \mathbb{1}_{\{0 < X_j^*(t) \leq \epsilon, X_j(t) \geq a\}} \right) \quad (6.3)$$

(the latter expectations follow from self-similarity, Fatou’s Lemma and stochastic continuity). The sum of the terms smaller than ϵ can be taken arbitrarily close to 0, which is also the case of the second part of the sum with a , by dominated convergence. To see that the last terms of (6.3) can also be taken as small as one wishes with ϵ and a , we refer to the proof of Proposition 2 in [4]. Therefore, it holds that $\lim_{n \rightarrow \infty} A_n = 0$. Similar arguments can be used to deal with C .

It remains to show that $\lim_{n \rightarrow \infty} B_n = 0$ and the claim will follow from (6.2). Using the self-similarity, we have that

$$B_n = \mathcal{E} \left(\sum_{k \geq 1} \|\mathbb{1}_{\{x_{n,k} > \epsilon\}} x_{n,k} \mathbf{X}_{k,\epsilon/x_{n,k}}^{(n)}(tx_{n,k}^\alpha) - \mathbb{1}_{\{x_k > \epsilon\}} x_k \mathbf{Y}_{k,\epsilon/x_k}(tx_k^\alpha)\|_{\omega_-}^{\omega_-} \right).$$

Now each growth-fragmentation has distribution \mathcal{P}_1 , and since the Wasserstein metric is given by the infimum over the set of joint distributions, we can assume that $\mathbf{Y}_k = \mathbf{X}_k^{(n)}$. We drop the indicator functions, letting $\mathbf{Y}_{k,\delta}$ be the null sequence whenever $\delta \geq 1$. Since $x_n \xrightarrow{\ell^{\omega_-}} x$, we just need to show the following convergence:

$$\sum_{k \geq 1} x_k^{\omega_-} \mathcal{E} \left(\|\mathbf{Y}_{k,\epsilon/x_{n,k}}(tx_{n,k}^\alpha) - \mathbf{Y}_{k,\epsilon/x_k}(tx_k^\alpha)\|_{\omega_-}^{\omega_-} \right) \xrightarrow{n \rightarrow \infty} 0. \quad (6.4)$$

The left-hand side is bounded by

$$\sum_{k \geq 1} x_k^{\omega_-} \mathcal{E} \left(\|\mathbf{Y}_{k,\epsilon/x_{n,k}}(tx_{n,k}^\alpha) - \mathbf{Y}_{k,\epsilon/x_k}(tx_{n,k}^\alpha)\|_{\omega_-}^{\omega_-} \|\mathbf{Y}_{k,\epsilon/x_k}(tx_{n,k}^\alpha) - \mathbf{Y}_{k,\epsilon/x_k}(tx_k^\alpha)\|_{\omega_-}^{\omega_-} \right).$$

The second part converges to 0 as $n \rightarrow \infty$ by stochastic continuity. The first part contains only fragments whose ancestors have minimum size between $\epsilon/(x_{n,k} \vee x_k)$ and $\epsilon/(x_{n,k} \wedge x_k)$. The dominated convergence theorem yields the claim, that is (6.4) holds, which concludes the proof. \square

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