Totally ordered measured trees and splitting trees with infinite variation

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Abstract

Combinatorial trees can be used to represent genealogies of asexual individuals. These individuals can be endowed with birth and death times, to obtain a so-called 'chronological tree'. In this work, we are interested in the continuum analogue of chronological trees in the setting of real trees. This leads us to consider totally ordered and measured trees, abbreviated as TOM trees. First, we define an adequate space of TOM trees and prove that under some mild conditions, every compact TOM tree can be represented in a unique way by a so-called contour function, which is right-continuous, admits limits from the left and has non-negative jumps. The appropriate notion of contour function is also studied in the case of locally compact TOM trees. Then we study the splitting property of (measures on) TOM trees which extends the notion of 'splitting tree' studied in [Lam10], where during her lifetime, each individual gives birth at constant rate to independent and identically distributed copies of herself. We prove that the contour function of a TOM tree satisfying the splitting property is associated to a spectrally positive Lévy process that is not a subordinator, both in the critical and subcritical cases of compact trees as well as in the supercritical case of locally compact trees. The genealogical trees associated to splitting trees are the celebrated Lévy trees in the subcritical case and they will be analyzed in the supercritical case in forthcoming work.

1 Introduction

1.1 Motivation

Consider the following population dynamics.

Model 1. Individuals have i.i.d. lifetimes in $(0, \infty]$ during which they give birth at times of independent Poisson point processes with the same intensity to independent copies of themselves, giving rise to exactly one offspring at each birth event.

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The global history of the population can be encoded by a so-called 'chronological tree' as depicted in Figure 1(D). In Figure 1(D), a vertical segment represents the same individual throughout its lifetime, and the vertical axis represents time flowing upwards. Hence, the lower endpoint of each vertical segment represents the birth-time while the upper endpoint represents death-time. The ancestors of segments can be found by following the dotted lines at the bottom of each segment. Following segments downwards and dotted lines to the left gives us the ancestral lines of the tree.

Conversely, starting from a non-negative piecewise linear function with positive jumps and negative slopes, we can interpret it as the contour of a tree as follows. Jumps of the function correspond to individuals and the size of the jumps are interpreted as lifetimes (which we explore from the top of the jump). The bottom of each jump is joined backwards to the first vertical line (or jump) it encounters (by joining along the dotted lines) which then becomes its parent while the height at which it is joined is the birth-time. This produces a notion of genealogy between the jumps. This interpretation of a function as giving rise to a tree was introduced and explored in [LGLJ98] and interpreted via last in first out (LIFO) queues with a single server.

[Lam10] proposes to encode the tree by its contour, obtained by using a very particular total order which is increasing when descending along ancestral lines; we then traverse the tree at unit speed respecting the total order and recording the height to obtain a piecewise linear function with jumps such as the one depicted in Figure 1(A). This gives us a notion of the individual visited at a given time t.

These chronological trees have the following self-similarity property which can be termed the **splitting property**: the ancestral line of the individual visited at time t follows the exact same dynamics, in that along it (sub)trees that have the same law as the whole tree get grafted (to its right) at constant rate. In [Lam10] a generalization of the preceding model is proposed and called a **splitting tree** following the terminology introduced in [GK97]. The generalization is such that individuals can have an infinite number of offspring which were assumed to have summable lifetimes. Now the descendant subtrees are grafted at times dictated by a Poisson point process.

The process counting the number of individuals alive as time varies in Model 1 is known as (binary, homogeneous) Crump-Mode-Jagers (CMJ) process. This process is Markovian only in the case of exponentially distributed (or a.s. infinite) lifetimes. On the other hand, the numbers of individuals in each successive generation of the genealogy evolve as a Galton-Watson process. In fact, if we forget the chronological information (the birth and death times) and retain only the genealogical information, by recording the ancestor of each individual and the order in which individuals are born, we obtain a Galton-Watson tree. This is why Model 1 can be considered as a chronological model associated to Galton-Watson trees. To visually obtain the genealogical tree from the chronological tree in the context of Figure 1d, one can contract all the solid horizontal lines (which represent the same individual throughout all its lifetime) to the same point while keeping the horizontal dashed lines to indicate descendance. One can then obtain a tree as in Figure 2. It is in this way that our binary (chronological) tree can develop hubs in its genealogy (individuals with many offspring) in particular when lifetimes are large relative to the birth rate.

Given that the process that counts the quantity of individuals alive is not typically Markovian, the real surprise of [Lam10] is that (because of the adequate choice of a total order on the chronological trees) the contours of splitting trees are Markovian: they are (finite variation) Lévy processes in the subcritical case and otherwise the splitting tree truncated at a given time has a reflected Lévy process as its contour. For descriptions of the contour of non-binary and non-homogeneous versions of Model 1, we refer to [SS15]. Note however, that the non-binary case already excludes Lévy processes as a coding
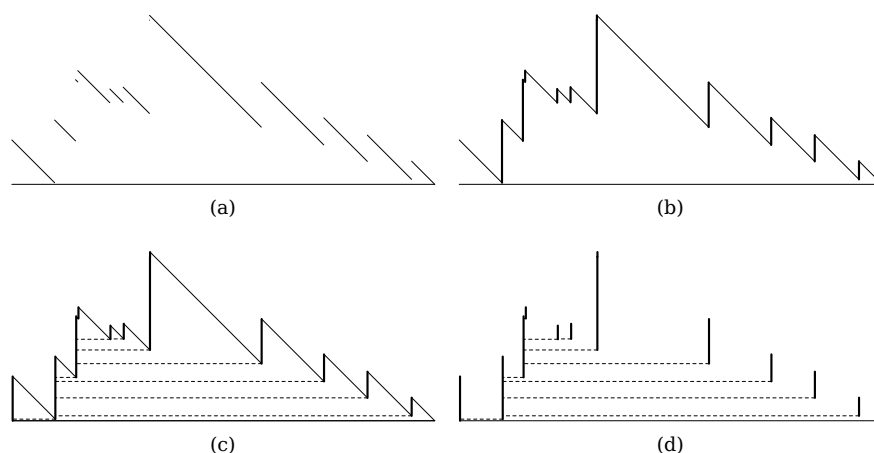
Figure 1: The coding function of a chronological tree with finite length where lifetimes are traversed at unit speed: how to recover the tree from the contour. A) Start with a càdlàg map with compact support; B) Draw vertical solid lines in the place of jumps; C) Report horizontal dashes lines from each edge bottom left to the rightmost solid point; D) erase diagonal lines.

process. Indeed, consider the following modification of Model 1, in which we declare that at each birth-time exactly two individuals arrive at the population. Then the coding process would have repeated minima over disjoint intervals, as in Figure 3 something simply does not happen for Lévy processes. Adding more than one individual at each birth event might also destroy the splitting property. In the same example with two individuals born at each birth event, we note that the ancestral line of the first sibling will differ from the ancestral line of the original individual in that, when individuals are born into the ancestral lines, sometimes they arrive in pairs and sometimes they do not, contrary to what the splitting property would enforce. To obtain non-binary trees with the splitting property, we would have to assume that at each birth event, an independent geometric quantity of individuals are born. Such a tree would have the splitting property without having contours that are coded by Lévy processes.

The chronologies of [Lam10] are basically discrete, in the sense that their associated genealogy is a discrete (or combinatorial) tree from which the chronological tree is obtained by specifying the birth and death times of individuals. In short, we wish to extend these results to real trees as follows.

1. To generalize the notion of chronological tree based on the formalism of real trees; these will be the TOM trees alluded to in the abstract.

2. To show that these TOM trees also give rise to a contour which characterizes the tree,

3. To introduce the splitting property for measures on TOM trees, using in particular Poisson random measures with σ-finite intensity to generate the random lifetimes of individuals.

4. To identify the law of contours under measures with the splitting property in terms of reflected Lévy processes, which can now be of infinite variation.

Items 1 and 2 are based on modifying the formalism of structured real trees of [Duq08] to suit our present needs. On the other hand, items 3 and 4 are motivated by [Lam10], although our arguments necessarily differ since our measures with the

Figure 2: The genealogical tree associated to the chronology displayed in Figure 1d. Individuals born first are below those born later. Notice that the generation sizes are 1,4,4 and 1.

Figure 3: A non-binary version of Model 1. Note that the second and fifth individuals are born at the same time, just as the third and fourth. The resulting coding process has repeated minima over disjoint intervals unlike compound Poisson processes with (positive jumps and negative) drift.

splitting property will be σ-finite instead of recursively defined probability measures. This is necessary in order to accomodate Lévy processes of infinite variation.

As we have remarked, the genealogical tree that is associated to the chronological tree is a Galton-Watson tree. In forthcoming work, we use the above measures with the splitting property, as a model for chronologies, and we associate a genealogical tree to them, obtaining the so-called Lévy trees, even in the supercritical case. The latter have been constructed as limits of Galton-Watson trees consistent under Bernoulli leaf percolation in [DW07] or by relating them to subcritical Lévy trees via Girsanov's theorem in [Del08] and [AD12]. In contrast, we construct the height process (based on [DLG02]) of the contours constructed in this paper. This gives us access to the chronology driving the genealogy represented by supercritical Lévy trees and lets us identify the prolific individuals (those with an infinite line of descent) introduced in [BFM08]. Finally, we believe that our construction of supercritical Lévy trees will provide a snake construction of supercritical superprocesses (with spatially independent branching mechanisms, as in [LG99] for (sub)critical cases) and give an interpretation for the backbone decomposition of [BKMS11].

1.2 Statement of the results

The central notion of this work, that of a TOM tree, is based on the metric spaces called real trees. The definition of real trees mimics the concept of a combinatorial tree defined as a combinatorial graph which is connected and has no cycles.

Definition 1.1 (From [DT96] and [EPW06])**.** *An $\mathbb{R}$-**tree** (or **real tree**) is a metric space (τ, d) satisfying the following properties:*

Completeness *(τ, d) is complete.*

Uniqueness of geodesics *For all $\sigma_1, \sigma_2 \in \tau$ there exists a unique isometric embedding*

$$\phi_{\sigma_1, \sigma_2} : [0, d(\sigma_1, \sigma_2)] \to \tau$$

such that $\phi(0) = \sigma_1$ *and* $\phi(d(\sigma_1, \sigma_2)) = \sigma_2$.

Lack of loops *For every injective continuous mapping* $\phi : [0,1] \to \tau$ *such that* $\phi(0) = \sigma_1$
and $\phi(1) = \sigma_2$, *the image of* $[0,1]$ *under* ϕ *equals the image of* $[0, d(\sigma_1, \sigma_2)]$ *under*
$\phi_{\sigma_1, \sigma_2}$.

A triple (τ, d, ρ) *consisting of a real tree* (τ, d) *and a distinguished element* $\rho \in \tau$ *is called
a* **rooted (real) tree**.

If (τ, d, ρ) is a rooted real tree and $\sigma_1, \sigma_2 \in \tau$, we define the **closed interval** $[\sigma_1, \sigma_2]$
to be the image of $[0, d(\sigma_1, \sigma_2)]$ under $\phi_{\sigma_1, \sigma_2}$; the **open interval**, obtained by removing
σ_1 and σ_2, is denoted (σ_1, σ_2). We can now define the **genealogical partial order** $\preceq$ by
stating that

$$\sigma_1 \preceq \sigma_2 \quad \text{if and only if} \quad \sigma_1 \in [\rho, \sigma_2].$$

When $\sigma_1 \preceq \sigma_2$ and $\sigma_1 \neq \sigma_2$ we write $\sigma_1 \prec \sigma_2$. Since a tree has no loops, there is a unique
element, called the **most recent common ancestor** of σ_1 and σ_2 and denoted

$$\sigma_1 \wedge \sigma_2,$$

such that

$$[\rho, \sigma_1] \cap [\rho, \sigma_2] = [\rho, \sigma_1 \wedge \sigma_2].$$

The real tree coded by a function has been introduced for continuous functions in
[LG06] and for càglàd (left-continuous with right limits) functions with negative jumps
in [Duq08]. However, when using the specific total order of [Lam10] for Model 1, we
see that the chronological trees are naturally coded instead by càdlàg functions, that is
right-continuous functions which admit limits from the left, with non-negative jumps. Let
us recall the construction. Given a càdlàg function $f : [0, m] \to [0, \infty)$ with non-negative
jumps and such that $f(m) = 0$ we can define a compact real tree (τ_f, d_f, ρ_f) as follows.
Consider the pseudo-distance d_f on $[0, m]$ given by

$$d_f(t_1, t_2) = f(t_1) + f(t_2) - 2m_f(t_1, t_2) \quad \text{where} \quad m_f(t_1, t_2) = \inf_{t \in [t_1, t_2]} f(t),$$

as well as the corresponding equivalence relationship $\sim_f$ given by

$$t_1 \sim_f t_2 \text{ if and only if } d_f(t_1, t_2) = 0.$$

Let $[s]_f$ denote the corresponding equivalence class of $s \in [0, m]$. It we equip the quotient
space

$$\tau_f = [0, m] / \sim_f = \{[t]_f : t \in [0, m]\}$$

with the induced distance (also denoted d_f) and root it at

$$\rho_f = [m]_f,$$

we obtain a compact rooted real tree. (The proof is similar to the corresponding
statement for càglàd functions in Lemma 2.1 of [Duq08]; the càdlàg property in particular
takes care of compactness and therefore completeness.) As an example, note that if
$[s_i]_f = \sigma_i$ for $i = 1, 2$ and $s \in [s_1, s_2]$ is such that $m_f(s_1, s_2) = f(s)$ then $[s]_f = [s_1]_f \wedge [s_2]_f$.
Because of this, the interval $[[s]_f, \rho_f]$ can be identified with the set of $t \in [s, m]$ such
that $f(t) = m_f(s, t)$. If in Model 1 we had instead used a total order in which lifetimes
are visited from bottom to top, we would have obtained a càglàd coding function as in

[Duq08]. However, with this different order, the coding function would not be a Lévy process in the setting of Model 1.

We now adopt (and adapt!) the insight of [Duq08] which is to notice that the function f endows the real tree τ_f with additional structure: a total order $\leq_f$ where two equivalence classes $\sigma_1, \sigma_2 \in \tau_f$ satisfy

$$\sigma_1 \leq_f \sigma_2 \text{ if and only if } \sup \sigma_1 \leq \sup \sigma_2$$

([Duq08] uses $\inf$ instead of $\sup$ and considers càglàd functions), and a measure

$$\mu_f = \text{Leb} \circ p_f^{-1},$$

where p_f is the canonical projection from $[0, m]$ to τ_f. When f is the trajectory of a normalized Brownian excursion, the resulting real tree is the celebrated Continuum Random Tree introduced in [Ald91a, Ald91b, Ald93] and the measure μ_f is called the mass measure. The triplet $((\tau_f, d_f, \rho_f), \leq_f, \mu_f)$ is called the **TOM tree coded by the function** f. The abstraction of this situation (i.e., not defining the tree from any given f) lies at the heart of the notion of TOM trees.

Definition 1.2. *A **real tree** (τ, d, ρ) is called **totally ordered** if there exists a total order $\leq$ on τ which satisfies*

Or1 $\sigma_1 \preceq \sigma_2$ *implies* $\sigma_2 \leq \sigma_1$ *and*

Or2 $\sigma_1 < \sigma_2$ *implies* $[\sigma_1, \sigma_1 \wedge \sigma_2) < \sigma_2$.

*A totally ordered real tree is called **measured** if there exists a measure μ on the Borel sets of τ satisfying:*

Mes1 μ *is locally finite and for every $\sigma_1 < \sigma_2$ we have that*

$$\mu(\{\sigma : \sigma_1 \leq \sigma \leq \sigma_2\}) > 0.$$

Mes2 μ *is diffuse.*

*A totally ordered measured tree, typically denoted $\mathbf{c} = ((\tau, d, \rho), \leq, \mu)$, will be referred to as a **TOM tree**. We will say that τ (or (τ, d, ρ)) is the tree part of $\mathbf{c}$.*

Regarding the above definition, we stress that the axiom of choice allows any infinite set to become totally ordered; however, we want the total order to be compatible with the natural partial ordering $\preceq$ on our rooted real trees. The total order will furthermore specify how we will choose to travel through the tree, visiting σ_1 before σ_2 if $\sigma_1 \leq \sigma_2$. Then, **Or1** specifies that we wish to traverse lifetimes from the top, to mimic the setting that produced a link with Lévy processes as described in Model 1. Requirement **Or2** can be interpreted as the requierement that we traverse ancestral lines until interrupted by birth events. These two requirements are of course independent since one could imagine jumping to the top of another lifetime before a birth event (taking care to come back where one left off) as in Figure 4, therefore obtaining an order satisfying **Or1** and not **Or2**. The measure μ in the definition can be intepreted as a measure of time; $\mu(A)$ can be interpreted as the time taken to traverse $A \subset \tau$. Requierement **Mes1** then just means that we wish to spend time on all parts of the tree while **Mes2** is technical but means that we do not linger at a fixed place for a positive amount of time. Note that the set involved in **Mes1** is shown to be a Borel set (actually a closed set) in Lemma A.2.

We will be exclusively interested in locally compact TOM trees. In this case, the measure μ is actually finite when the tree is compact and hence σ-finite otherwise. As defined,
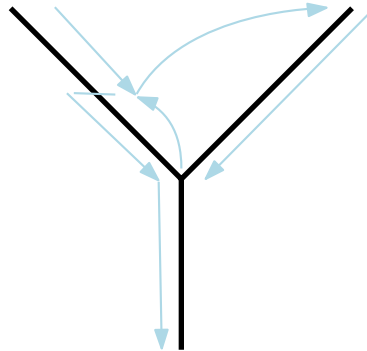
Figure 4: A tree with a total order (illustrated by the arrows) satisfying **Or1** but not **Or2**.

TOM trees are intimately linked with the structured trees of [Duq08]. **Structured trees** are also real trees which come equipped with an order and a measure. The assumptions for the measure are basically the same; we add an additional non-atomicity requirement to simplify some proofs. The main difference is in the order. In the setting of Model 1, [Duq08] would specify that lifetimes are traversed from bottom to top, interrupting the visit at each birth event. However, it is easy to go from one setting to the other: to get a structured tree from a TOM tree, it suffices to define a new total order $\leq^{\text{st}}$ so that $s \leq^{\text{st}} t$ if $t \leq s$. Then the tree $((\tau, d, \rho), \leq^{\text{st}}, \mu)$ is a structured tree. When coding a tree by a càdlàg function f, it suffices to consider the function $t \mapsto f(m - t)$ to be in the setting of [Duq08]. Recall that our definitions have been adopted to mimic the results of [Lam10] in which coding functions are Lévy processes and hence càdlàg. We do not claim any novelty in the above definition since structured trees and TOM trees are basically the same. However, for completeness, we provide (simplified and streamlined) proofs of the (deterministic results) needed in our setting, such as the fact that TOM trees can be coded by càdlàg functions.

With our definition, not every càdlàg function $f : [0, m] \to [0, \infty)$ with non-negative jumps and $f(m) = 0$ codes a compact TOM tree. We have to assume additionally that the equivalence classes f gives rise to have zero Lebesgue measure, so that the pushforward measure is non-atomic. One way to ensure this is to impose that the equivalence classes generated by f are at most countable. This assumption is satisfied with probability 1 if f is the sample path of a Lévy process with no negative jumps that is not a subordinator (cf. Proposition 2.1 in Section 2).

There is a simpler way to define a total order satisfying **Or** when the underlying tree is binary. A real tree τ is said **binary** when $\tau \setminus \{\sigma\}$ has at most 3 connected components for every $\sigma \in \tau$. In that case, σ is called a branching point if $\tau \setminus \{\sigma\}$ has 3 connected components, the one containing ρ and two others. It can be easily seen that these three connected components are also trees. Suppose that for every branching point σ we are given an orientation, which declares, for the two components of $\tau \setminus \{\sigma\}$ which do not contain ρ, which one is the 'left' subtree rooted at σ and which one is the 'right' subtree rooted at σ. Then we can define a total order $\leq$ on τ as follows. For any $\sigma_1, \sigma_2 \in \tau$, $\sigma_1 \preceq \sigma_2$ implies $\sigma_2 \leq \sigma_1$ (so that $\leq$ satisfies **Or1**). Otherwise, note that $\sigma_1 \wedge \sigma_2$ is a branching point and we then define

$$\sigma_1 \leq \sigma_2 \quad \text{if and only if} \quad \sigma_1 \text{ belongs to the left component of } \tau \setminus \{\sigma_1 \wedge \sigma_2\}.$$

Then $\leq$ is a total order on τ which satisfies **Or**.

The importance of the notion of TOM trees is that they allow the construction of a contour which codes the tree in the aforementioned sense. To formalize this, we will

say that two compact TOM trees are **isomorphic** if there is an isometry from one to the other which respects the total order and maps the associated diffuse measures one onto the other.

Theorem 1.3. *Let* $\mathbf{c}$ *be a compact TOM tree and let* $m = \mu(\tau)$*. There exists a càdlàg function* $f_{\mathbf{c}} : [0, m] \to [0, \infty)$ *with no negative jumps such that the TOM tree coded by* $f_{\mathbf{c}}$ *is isomorphic to* $\mathbf{c}$*. In particular, if* $\mathbf{c}$ *and* $\tilde{\mathbf{c}}$ *are non-isomorphic TOM trees then* $f_{\mathbf{c}} \neq f_{\tilde{\mathbf{c}}}$*.*

The function $f_{\mathbf{c}}$ is called the **contour process** for reasons that will be clear upon its construction. Also, note that the above correspondence gives us a canonical way to explore the tree: at time t we visit the equivalence class $[t]$ of t under $f_{\mathbf{c}}$. The mapping $\phi : t \mapsto [t]$ is called the **exploration process**. We can then interpret the measure μ on the compact TOM tree as a measure of time. Indeed, if we traverse the interval $[0, m]$ from left to right at unit speed, the exploration process gives us a way to traverse the tree respecting the order $\leq$ (except at a countable number of exceptional points where one starts exploring a (sub)tree without a minimal element as in Figure 5 and one is forced to start the exploration at the root, which is the maximal element) and $\mu(V)$ is the time it takes to explore the set $V \subset \tau$. Note that whenever $f_{\mathbf{c}}$ is continuous at both s_1 and s_2, we have that $s_1 \leq s_2$ implies $[s_1]_{\mathbf{c}} \leq [s_2]_{\mathbf{c}}$. Hence, the exploration process gives us a canonical way to explore the tree in an ordered fashion. When $f_{\mathbf{c}}$ is like in Figure 1d, we explore the tree by starting at the top of the lifetimes of individuals, up to the successive branching points that we encounter, where we jump to the top of the lifetime of the individual that was born. Our order on τ_f was defined to mimic the precise order on splitting trees that was useful in [Lam10].

Theorem 1.3 follows from Theorem 1.1 in [Duq08], where the functions $f_{\mathbf{c}}$ are given a characterization (as those satisfying a certain property **Min**). For completeness, we give a proof of Theorem 1.3 in Appendix A.

In order to relate the compact and locally compact cases, we introduce the truncation of a locally compact TOM tree. Let $r > 0$ and $\mathbf{c} = ((\tau, d, \rho), \leq, \mu)$ be a locally compact TOM tree.

Definition 1.4. *The* **truncation** *of* $\mathbf{c}$ *at level* r*, is the TOM tree* $\mathbf{c}^r = ((\tau^r, d^r, \rho^r), \leq^r, \mu^r)$ *where*

$$\tau^r = \{\sigma \in \tau : d(\sigma, \rho) \leq r\},$$

d^r *is the restriction of* d *to* $\tau^r \times \tau^r$*,* $\rho^r = \rho$*,* $\leq^r$ *is the restriction of* $\leq$ *to* $\tau^r \times \tau^r$ *and* $\mu^r(A) = \mu(A \cap \tau^r)$*.*

It is simple to see that (τ^r, d^r, ρ^r) is a real tree. As noted in [DW07], the closed balls of a locally compact real tree are compact by the Hopf-Rinow theorem (cf. [Gro07] or [BBI01]); this is where the completeness assumption in our definition of real trees comes into play since, without it, $[0, 1)$ would be a locally compact tree without compact truncations at level $r \geq 1$. Hence, the truncation $\mathbf{c}^r$ at level r of a locally compact TOM tree $\mathbf{c}$ is a compact TOM tree and can therefore be coded by a function f^r. The functions f^r are related by a time-change in Proposition 3.1 of Section 3.

From Theorem 1.3, we see that the class of equivalence classes of isomorphic compact TOM trees can be put into correspondence with a subset of càdlàg functions. Therefore, this class constitute a set and we denote it by $\mathbf{C}_c$. We define $X_t : \mathbf{C}_c \to [0, \infty) \cup \{\dagger\}$ as follows, where $\dagger$ is a so-called cemetery state. Given (a representative) $\mathbf{c} = ((\tau, d, \rho), \leq, \mu) \in \mathbf{C}_c$, let f be the contour process of $\mathbf{c}$ and

$$X_t(\mathbf{c}) = \begin{cases} f(t) & t \leq \mu(\tau) \\ \dagger & \text{otherwise} \end{cases}$$

We then endow $\mathbf{C}_c$ with the σ-field $\mathscr{C} = \sigma(X_s : s \geq 0)$.

We now turn to the TOM trees coded by Lévy processes. We consider the canonical process on two functional spaces: the usual Skorohod space $\mathbf{D}$ of càdlàg trajectories $f : [0, \infty) \to [0, \infty)$ (as introduced for example in [Kal02, Ch 19, p. 380]) and the associated excursion space

$$\mathbf{E} = \{ f : [0, \infty) \to [0, \infty) \cup \{\dagger\} : $$
$$f \text{ is càdlàg}, f(t) = \dagger \text{ iff } t \geq \zeta \text{ for some } \zeta > 0 \text{ and } f(\zeta-) = 0 \}.$$

Note that the cemetery state $\dagger$ is isolated and absorbing and $\zeta = \zeta(f)$ is termed the lifetime of the excursion f.

On both functional spaces $\mathbf{D}$ and $\mathbf{E}$ we abuse our previous notation and define the canonical process X by

$$X_t(f) = f(t)$$

as well as the σ-fields $\mathscr{D}$ and $\mathscr{E}$ on $\mathbf{D}$ and $\mathbf{E}$ respectively as $\sigma(X_t : t \geq 0)$. Note that in view of the correspondence between functions and trees, if $\mathbf{c} = ((\tau, d, \rho), \leq, \mu)$ is a compact TOM tree, we can define $\zeta = \zeta(\mathbf{c}) = \mu(\tau)$. We also introduce the canonical filtration $(\mathscr{F}_t, t \geq 0)$ where

$$\mathscr{F}_t = \sigma(X_s : s \leq t).$$

We now consider spectrally positive (i.e., with no negative jumps) Lévy processes. The reader can consult [Ber96] (especially Chapter VII) for the adequate background. Let Ψ be the Laplace exponent of a spectrally positive Lévy process which is not a subordinator (equivalent to assuming $\Psi(\infty) = \infty$). This is a convex function written as

$$\Psi(\lambda) = -\kappa + \alpha\lambda + \beta\lambda^2 + \int_0^\infty \left[e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{x \leq 1} \right] \pi(dx)$$

where $\kappa, \beta \geq 0$, $\alpha \in \mathbb{R}$ and π is the Lévy measure of Ψ, satisfying $\int_0^\infty 1 \wedge x^2 \, \pi(dx) < \infty$. Let $\mathbb{P}_x$ (or $\mathbb{P}_x^\Psi$ when Ψ is not clear from the context) be the law of a spectrally positive Lévy process with Laplace exponent Ψ which starts at x. When $\kappa > 0$, the Ψ-Lévy process has the same law as the Lévy process with Laplace exponent $\Psi + \kappa$ killed after an independent exponential time with parameter κ; we will assume that a killed Lévy process jumps to the cemetery state $\dagger$ upon killing. We say that X is (sub)critical if X does not drift or jump to $+\infty$ (or equivalently $\Psi'(0) \geq 0$ and $\kappa = 0$) and otherwise it will be termed supercritical. Let

$$b = \sup \{ \lambda \geq 0 : \Psi(\lambda) = 0 \}$$

be the largest root of the convex function Ψ, so that $b > 0$ if and only if Ψ is supercritical. If $\underline{X}$ stands for the cumulative minimum process given by

$$\underline{X}_t = \inf_{s \leq t} X_s,$$

then the reflected process $X - \underline{X}$ is a strong Markov process (cf. [Ber96, Ch. VI]) and $\underline{X}$ is a version of its local time at zero. Given a supercritical exponent Ψ we can define an associated subcritical exponent $\Psi^\#(\lambda) = \Psi(\lambda + b)$ which corresponds to conditioning a Ψ-Lévy process on reaching arbitrarily low levels, in the sense that its law equals

$$\lim_{a \to -\infty} \mathbb{P}_0(\cdot \mid \underline{X}_\infty < a)$$

on every $\mathscr{F}_s$. (Cf. Lemma 7 in [Ber96, Ch. VII] and Lemme 1 in [Ber91].) We let ν stand for the excursion measure of $X - \underline{X}$ away from zero; the measure ν may charge

excursions of infinite length. Indeed, if $\mathbb{P}^{\rightarrow}$ denotes the law of the Lévy process after it approaches its overall minimum from the left for the last time then

$$\nu = \nu^{\#} + b\mathbb{P}^{\rightarrow},$$

where $\nu^{\#}$ is the excursion measure associated to $\Psi^{\#}$ which is concentrated on excursions with finite length.

Also, let us define the law $\mathbb{Q}_x$ on E as the law of the Lévy process killed (and sent to $\dagger$) upon reaching zero under $\mathbb{P}_x$. Then, under ν, the canonical process is strongly Markovian with the same semigroup as $\mathbb{Q}_x$.

We will now give examples of σ-finite measures on locally compact TOM trees satisfying the splitting property. They will turn out to contain the generic examples. We will define the measures by specifying what their truncations look like, and we do this in terms of Lévy processes reflected below a certain level. The canonical way to define a reflected Lévy process with jumps of both signs is by using the cumulative maximum (cf. [Ber96, Ch. VI§1]); this would correspond to shift all excursions of X below its cumulative maximum so that they start at the same level, r in this case. Later on we note that, in our spectrally positive case, one can also define it by time-changes to remove the excursions of X above r. Define

$$\overline{X}_t^r = \left(\max_{s \leq t} X_s - r \right)^+ \tag{1.1}$$

and let X^r be the process obtained by stopping $X - \overline{X}^r$ when it reaches zero. (In the killed case, $X - \overline{X}^r$ might be killed before $X - \overline{X}^r$ reaches zero; in this case, we simply concatenate independent copies of $X - \underline{X}^r$ started at r until a copy reaches zero). Note that large jumps of X above $\overline{X}^r$ correspond to jumps of $X - \overline{X}^r$ to r.

Let the map M associate a TOM tree to any admissible càdlàg function f.

Definition 1.5. *Let ν^r be the push-forward of ν by X^r. Then, the **splitting tree up to level** r, denoted η^r, is the pushforward of ν^r under M.*

We can think of η^r as the excursion measure above zero associated to Lévy processes below level r. Note that the measures η^r are concentrated on binary trees. Indeed, the minima of X over two disjoint intervals are almost surely different under $\mathbb{P}$. In the compound Poisson case the drift has to be negative and this follows directly. In the remaining case, the one dimensional distributions of X are non-atomic and this implies non-atomicity of $\underline{X}_t$ for $t > 0$ by Theorem 1 in [PUB12].

Our next result shows that we can define a measure η^{Ψ} whose image under truncation at level r equals η^r.

Proposition 1.6. *There exists a unique measure η^{Ψ} on locally compact TOM trees whose truncation at level r equals η^r.*

Definition 1.7. *The measure η^{Ψ} whose existence is guaranteed by the above definition will be called the **splitting tree associated to the Laplace exponent** Ψ.*

Our next result shows that the measure η^{Ψ} is characterized by a self-similarity property termed the splitting property. To define such a property, suppose that $\mathbf{c} = ((\tau, d, \rho), \leq, \mu)$ is a TOM tree. Truncate $\mathbf{c}$ at height r to obtain $\mathbf{c}^r$ and suppose that $\mathbf{c}^r$ has a total measure greater than t. Let ϕ stand for the exploration process of $\mathbf{c}^r$. Then, on the interval $[\rho, \phi(t))$ we can define left subtrees and right subtrees. The right subtrees are characterized by the existence of $\sigma \in [\rho, \phi(t))$ such that

$$\{\tilde{\sigma} : [\phi(t), \sigma) < \tilde{\sigma} \leq \sigma\} \neq \{\sigma\}.$$

The right subtree at σ is then equal to

$$R_{t,\sigma} = \{\tilde{\sigma} : [\phi(t), \sigma) < \tilde{\sigma} \leq \sigma\}$$

and we can collect all these right subtrees in the measure

$$\Xi_t^r = \sum_{R_{t,\sigma} \neq \{\sigma\}} \delta_{(d(\rho,\sigma),R_{t,\sigma})}.$$

In the following definition, recall that $X(\mathbf{c})$ is the contour associated to the tree $\mathbf{c}$.

Definition 1.8. *A measure κ on locally compact TOM trees satisfies the* **splitting property** *if for any $r > 0$, on defining κ^r as the image of κ under truncation at height r,*

1. *for any $t > 0$, $\kappa^r(\zeta > t) < \infty$ and*
2. *on the set $\{\zeta > t\}$ and conditionally on $X_t = x$, Ξ_t^r is a Poisson random measure on $[0, x] \times \mathbf{C}_c$ with intensity $Leb(ds) \otimes \kappa^{r-s}(d\mathbf{c})$: if $h : [0, \infty) \cup \{\dagger\} \to [0, \infty)$ is measurable and vanishes at $\dagger$ and $g : [0, \infty) \times \mathbf{C}_c \to [0, \infty)$ is measurable then*

$$\kappa^r\left(h(X_t)\, e^{-\Xi_t^r g}\right) = \kappa^r\left(h(X_t)\, e^{-\int_{[0,X_t] \times \mathbf{C}_c} g(s,\mathbf{c})\, \kappa^{r-s}(d\mathbf{c})\, ds}\right).$$

Note that, by definition, such a measure κ is not necessarily a probability measure but is only assumed to be σ-finite in the most interesting cases. Note also that no assumption is made concerning conditional independence of Ξ_t^r and $\sigma(X_s, s \leq t)$ given X_t. We have given an example of (probability) measures on non-binary TOM trees with the splitting property (where at each birth event a geometric quantity of individuals is added to the population) in which the above conditional independence does not hold. Hence, an important part of the proof is to show that restriction to binary trees does imply it.

Remark 1.9. As consequences of the definition, we will see that when κ is concentrated on compact TOM trees then the splitting property admits the non-truncated version

$$\kappa\left(h(X_t)\, e^{-\Xi_t^\infty g}\right) = \kappa\left(h(X_t)\, e^{-\int_{[0,X_t] \times \mathbf{C}_c} g(s,\mathbf{c})\, \kappa(d\mathbf{c})\, ds}\right)$$

and implies the integrability properties

$$\kappa(\zeta > t) < \infty \quad \text{and} \quad \int_0^\infty 1 \wedge \zeta(\mathbf{c})\, \kappa(d\mathbf{c}) < \infty.$$

See the proof of the forthcoming Theorem 1.11.

It turns out that a class of measures on binary TOM trees with the splitting property are in correspondence with Lévy processes, under an additional assumption of constant sojourn, as the next result shows.

The splitting trees of Model 1 previously introduced have the property that they treat elements of the tree in an equal manner, as in the following definition.

Definition 1.10. *A TOM tree $\mathbf{c} = ((\tau, d, \rho), \leq, \mu)$ has* **sojourn** *$a \geq 0$ if $\mu([\rho, \sigma]) = a\, d(\rho, \sigma)$ for every $\sigma \in \tau$.*

In other words, a TOM tree has sojourn a if the measure μ equals a times Lebesgue measure (on the tree). It is interesting that the sojourn a can be 0 in the above definition even if μ is a non-trivial measure. This happens in particular for the trees coded by Brownian excursions since Brownian motion spends zero time (with respect to Lebesgue measure) at its cumulative minimum process. To get a first insight into why this happens, note that for the TOM tree coded by the function f, if $\sigma = [s]_f$, then $\mu([\rho, \sigma]) = \mathrm{Leb}\left\{t \geq s : f(t) = \inf_{[s,t]} f\right\}$.

Recall the measures η^r defined after Equation (1.1).

Theorem 1.11. *The measure η^Ψ satisfies the splitting property, is concentrated on binary trees, has constant sojourn*

$$a = \lim_{\lambda \to \infty} \frac{\lambda}{\Psi(\lambda)},$$

and assigns finite measure to non-compact TOM trees. When Ψ is (sub)critical, η^Ψ is the push-forward of ν under the mapping M taking functions to trees. The measure η^Ψ charges non-compact trees if and only if Ψ is supercritical.

Conversely, if a non-zero measure κ on locally compact TOM trees has the splitting property, is concentrated on binary trees and there exists $a \geq 0$ such that under κ the tree has sojourn a almost everywhere, then there exists a spectrally positive Lévy process with Laplace exponent Ψ such that $\kappa = \eta^\Psi$.

The assumption that the measure κ has sojourn a is necessary to be able to code the tree by a Lévy process as the next example shows. Consider Model 1. Lambert shows in Theorem 4.1 and Remark 2 of [Lam10, p 373] that if b times the expected lifetime is less than or equal to 1 then the resulting real tree is compact with probability one. In this setting, one can use Lebesgue measure on each of the (finite number of) segments comprising the tree, thus obtaining a tree of sojourn 1, and the contour process of the compact TOM tree is a compound Poisson process minus a drift of slope 1 until it reaches zero. However, one might use an additional speed distribution S on $(0, \infty)$ and associate to individuals iid speeds with distribution S at which their lifetimes will be traversed (or equivalently, defining the multiple of Lebesgue measure which will be used on each interval in the tree representing the lifetime). The compact TOM tree we obtain has the splitting property but does not have sojourn a and is not coded by a Lévy process (since there is no unique drift). However, there is a (random) time-change which (by giving all individuals the same non-random speed) takes the coding process into a Lévy process and this begs the question on whether there is a characterization of binary trees with the splitting property as time-changes of Lévy processes. This question is left open. Another question that is not addressed is on characterizing splitting trees which are not binary. We do not have much insight to offer into this question, except that, as we saw in our ternary modification of Model 1, we might need a memoryless quantity of offspring at each birth-event to obtain the splitting property that we have considered.

As mentioned previously, one of our motivations was to accommodate infinite variation Lévy processes into the framework of the splitting trees defined in [Lam10]. There, the definition was explicit since one only needed the lifetime distribution of individuals and the birth rate in order to define it. Here, the definition of a measure with the splitting property is implicit but becomes explicit thanks to Theorem 1.11. However, there is a slight difference since the probability measures defined in [Lam10] are what one obtains when considering the trees that are attached to a given lifetime when it has size l. In the subcritical case, this part of the tree would be coded by a Lévy process that starts at l and is killed upon reaching zero. In contrast, we consider the excursion measure associated to such a Lévy process, something which makes sense for infinite variation Lévy processes. A posteriori, one can relate the two definitions of splitting trees thanks to Theorem 1.11.

The reader might have come across the so-called Lévy trees which arise when considering scaling limits of Galton-Watson trees under certain conditionings (cf. [Ald93], [DLG02], [BO18]). To explain the relationship to our measures η with the splitting property, recall that in the (sub)critical case, η is the (σ-finite!) law of the trees coded by excursions of a Lévy process above its cumulative minimum. In contrast, references [DLG02, DLG05] associate to the excursions a so-called height process, which is a continuous function with compact support, and then consider the tree that is associated to it. The height process is a functional of the excursion that features local times in its definition. However, as explained in our companion paper [LU18], the tree coded by the height process is, in a sense, the genealogical tree associated to the chronological model provided by η. Just like Galton-Watson processes, Lévy trees have the branching property, meaning that conditionally on the quantity of individuals below a certain level,

the subtrees above them have the same law as the original tree and are independent. In fact, [Wei07] has characterized Lévy trees as the ones satisfying the branching property. This is the development that motivated Theorem 1.11 since we were interested in an analogous statement for chronological rather than genealogical trees. From Model 1 and the results of [Wei07], the existence of a class of splitting trees that would be associated to Lévy processes was expected.

As a final remark, the relationship between Ψ and $\Psi^{\#}$ might be lifted to absolute continuity properties between supercritical and subcritical reflected Lévy processes. Since the latter are already consistent under truncation, one might hope to leverage the absolute continuity to prove consistency also in the subcritical case; this is the point of view explored in [Del08] and [AD12] to construct supercritical Lévy trees. We have not pursued this line of inquiry.

1.3 Organization

In Section 2, we study σ-finite measures on compact TOM trees and how they are related to Lévy processes through a proof of Theorem 1.11 in this case. In Section 3, we study the truncation operator on compact real trees and how locally compact TOM trees can be therefore constructed as direct limits of consistent families of compact TOM trees. We then give in Section 4 some constructions of the reflected Lévy processes which allow us, in Section 5 to construct σ-finite measures on locally compact TOM trees which have the splitting property and to give the proof of Theorem 1.11 in full generality. Finally, in Appendix A, we handle the deterministic aspects of the space of TOM trees culminating in a proof of Theorem 1.3.

2 Measures on compact TOM trees with the splitting property

In this section, we provide the characterization of measures on compact TOM trees with the splitting property stated in Theorem 1.11. We first show that the canonical process codes a compact real tree under the excursion measure of a (sub)critical Lévy process.

Proposition 2.1. *Let X be a (sub)critical spectrally positive Lévy process with Laplace exponent Ψ. Let ν be the intensity measure of excursions of X above its cumulative minimum. Then, ν-almost surely, the equivalence classes*

$$[s]_X = \left\{ t \in [0, \zeta) : X_t = X_s = \underline{X}_{[s,t]} \right\}$$

have at most three elements for all $s \in [0, \zeta)$.

Proof. Under ν, we have $X_t > 0$ for all $t \in (0, \zeta)$. Hence $[0]_X$ does not have more than 3 elements.

Suppose there exist $0 < t_1 < \cdots < t_4$ such that $t_i \in [t_1]_X$. Chose $u_i \in (t_i, t_{i+1}) \cap \mathbb{Q}$ for $i = 1, 2, 3$ and note that $\underline{X}_{[u_1,u_2]} = \underline{X}_{[u_2,u_3]}$. Hence

$$\{\exists t, \#[t]_X \geq 4\} \subset \bigcap_{\varepsilon \in \mathbb{Q}_+} \bigcup_{\varepsilon < u_1 < u_2 < u_3 \in \mathbb{Q}} \left\{ \underline{X}_{[u_1,u_2]} = \underline{X}_{[u_2,u_3]} \right\}.$$

We now show that the right-hand set has ν-measure zero. By countable subadditivity, it suffices to show

$$\nu\left(\underline{X}_{[u_1,u_2]} = \underline{X}_{[u_2,u_3]}, u_3 < \zeta \right) = 0.$$

for $\varepsilon < u_1 < \cdots < u_3$. However, using the Markov property at u_1 under ν, we see that

$$\nu\left(\underline{X}_{[u_1,u_2]} = \underline{X}_{[u_2,u_3]}, u_3 < \zeta \right) = \nu\left(\mathbf{1}_{\varepsilon < \zeta} \mathbb{Q}_{X_\varepsilon}\left(\underline{X}_{[u_1-\varepsilon,u_2-\varepsilon]} = \underline{X}_{[u_2-\varepsilon,u_3-\varepsilon]} \right) \right).$$

Hence, again by the Markov property, it suffices to show that $\mathbb{P}_x\left(\underline{X}_{u_1} = \underline{X}_{[u_1,u_2]}\right) = 0$. However,

$$\mathbb{P}_x\left(\underline{X}_{u_1} = \underline{X}_{[u_1,u_2]}\right) = \mathbb{E}_x\left(h\left(X_{u_1}, \underline{X}_{u_1}\right)\right) \quad \text{where} \quad h(y,z) = \mathbb{P}_y\left(\underline{X}_{u_2-u_1} = z\right).$$

However, since X is spectrally positive, then 0 is regular for $(-\infty, 0)$ (cf. Thm. 1 [Ber96, Ch. VII]) and so Lemma 1 in [PR69] (or Theorem 1 in both [Cha13] and [PUB12]) tell us that the law of $\underline{X}_{u_2-u_1}$ is non-atomic, which implies that h is actually zero. $\qquad\square$

We now do an analysis on TOM trees to reduce our proof of the splitting property of η^Ψ to a simple property of the excursion measures and to motivate the construction of the Lévy process out of a measure on trees with the splitting property.

Let $f : [0, m] \to \mathbb{R}$ be a càdlàg function which codes a TOM tree $\mathbf{c}$ and consider $s \in (0, m)$ and the class of s under $\sim_f$ denoted $\sigma = [s]_f$. We first identify the subtrees attached to the right of (σ, ρ) with the excursions of f above its cumulative minimum on $[s^*, m]$, where

$$s^* = \inf\left\{t \geq s : f(t) < f(s)\right\}.$$

We have already identified R_σ with the image of $[s^*, m]$ by the exploration process in Subsection A.3. Also, when discussing the tree coded by a function in Section 1, we identified the interval $[\sigma, \rho]$ with the image of

$$\left\{t \geq s^* : f(t) = m_f(s^*, t)\right\}.$$

Hence, $R_\sigma \setminus [\sigma, \rho]$ can be identified with the excursions of f above its cumulative minimum on the interval $[s^*, m]$. To be more precise, say that ' $\tilde{\sigma} \in [\rho, \sigma)$ is a branching point of R_σ if $R_\sigma \setminus \{\tilde{\sigma}\}$ has at least three components. One of these components contains σ and a different one contains ρ. If we join all the other components to $\tilde{\sigma}$ (of which there is only one in the case of binary trees), we obtain a compact TOM tree which can be termed the tree at $\tilde{\sigma}$ to the right of $[\sigma, \rho]$. Let us call it $\mathbf{c}_{\tilde{\sigma},\sigma}$. The set of $\tilde{\sigma} \in [\sigma, \rho]$ for which $\mathbf{c}_{\tilde{\sigma},\sigma}$ is not reduced to $\tilde{\sigma}$ is at most countable. Indeed, by compactness, there can be only finitely many of them with height exceeding a given $\varepsilon > 0$. Let $\sigma_1, \sigma_2, \ldots$ be the branching points of R_σ in $[\sigma, \rho]$. Notice that if $(\sigma_i^n, n \in \mathbb{N})$ is a sequence in $[\sigma, \sigma_i)$ converging to σ_i then the real tree of $\mathbf{c}_{\sigma_i,\sigma}$, denoted $\tau_{\sigma_i,\sigma}$ is given by

$$\tau_{\sigma_i,\sigma} = \bigcap_n R_{\sigma_i^n} \cap L_{\sigma_i}.$$

Hence, $\mathbf{c}_{\eta_i,\sigma}$ can be coded by

$$f^i = f(\cdot + s_i) - f\left(s^i\right) \text{ on } [0, s^i - s_i]$$

where

$$s_i = \lim_{n\to\infty} \mu\left(L_{\eta_i^n}\right) \quad \text{and} \quad s^i = \mu(L_{\eta_i}).$$

However, if $x_i = d(\eta_i, \rho)$, we also have the representation

$$s_i = \inf\left\{t \geq s : f(t) \leq f(s) - x_i\right\} \quad \text{and} \quad s^i = \inf\left\{t \geq s : f(t) < f(s) - x_i\right\}$$

by our previous identification of the elements of $[\sigma, \rho]$, which shows that $\mathbf{c}_{\eta_i,\sigma}$ can be coded by an excursion of f above its cumulative minimum.

Under the additional assumptions that $\mathbf{c}$ is binary with sojourn a and that $s = \mu(L_\sigma)$, we can reconstruct $f(s + \cdot)$ from $f(s)$, the sequence (x_i) and each of the functions f^i.

Indeed, for any $l \in [0, f(s)]$, let σ_l be the point of $[\sigma, \rho]$ at distance l from σ and consider S_l to be the measure of $R_\sigma \cap L_{\sigma_l}$. Then because of constant sojourn, say a,

$$S_l = \sum_{d(\eta_i, \sigma) \le l} \mu(\tau_{\eta_i, \sigma}) + al.$$

Note that $d(\eta_i, \sigma) \le l$ if and only if $f(s) - x_i \le l$ and $\mu(\tau_{\eta_i, \sigma})$ is the Lebesgue measure of the interval of definition of f^i, so that S can be constructed from the aforementioned quantities. Also,

$$S_{x_i-} = s_i \quad \text{and} \quad S_{x_i} = s^i.$$

Now let Λ be the right-continuous inverse of S. By considering the dense set of measures of left sets (cf. Lemma A.6), we see that

$$\Lambda_t = d(\sigma, \phi(t+s) \wedge \sigma).$$

for $t \in [0, \mu(R_\sigma)]$, where we have seen that $\mu(R_\sigma) = \zeta(f) - s$. Also, if $\phi(t+s)$ does not belong to $[\sigma, \rho]$ then there exists a unique i such that $\phi(t+s) \in \tau_{\eta_i, \sigma}$ and so

$$d(\phi(t+s), \phi(t+s) \wedge \phi(s)) = \sum_i \mathbf{1}_{S_{x_i-} \le t < S_{x_i}} f_i(t - S_{x_i-}).$$

We therefore obtain the representation

$$
\begin{aligned}
f(s+t) - f(s) &= d(\phi(t+s), \rho) - d(\phi(s), \rho) \qquad\qquad (2.1)\\
&= d(\phi(t+s), \phi(s) \wedge \phi(t+s)) - d(\phi(s), \phi(s) \wedge \phi(t+s))\\
&= \sum_i \mathbf{1}_{S_{x_i-} \le t < S_{x_i}} f_i(t - S_{x_i-}) - \Lambda_t.
\end{aligned}
$$

Proof of Theorem 1.11. Let us start with the proof that the tree coded by the excursion measure of a Lévy process has the splitting property. Since we have just identified right subtrees along $[\rho, \sigma)$ with excursions of the contour above its cumulative minimum, it suffices to prove that, for any $t > 0$ and on the set $\{\zeta > t\}$, the excursions of $X_{\cdot + t}$ above its cumulative minimum form a Poisson point process. Let us recall that, under ν, on the set $\{X_t \in (0, \infty)\}$ and conditionally on $X_t = x$, the path $X_{\cdot + t}$ has law $\mathbb{Q}_x$. Hence, the point process with atoms at starting levels of the excursions and the excursions themselves is a Poisson point process on $[0, x] \times \mathbf{E}$ with intensity $\text{Leb} \otimes \nu$. We conclude the splitting property of η^Ψ.

We have already remarked that η^Ψ is concentrated on binary trees. The assertion about the sojourn follows from the following arguments. First, we let $T_x = \inf\{t \ge 0 : X_t = -x\}$ and note that T is the right continuous inverse of $-\underline{X}$. Then $T = (T_x, x \ge 0)$ is a subordinator under $\mathbb{P}$ whose Laplace exponent is the right-continuous inverse Φ of Ψ (cf. Ch. 7§1 of [Ber96]). Since $\{s \le T_y : X_s = \underline{X}_s\} = \{T_x : x \ge 0\} \cap [0, T_y]$, then their Lebesgue measure equals dy thanks to Proposition 1.8 of [Ber99], where d is the drift of T. However, the drift can be computed as $d = \lim_{\lambda \to \infty} \Phi(\lambda)/\lambda$ by Proposition 2 Ch. I§1 of [Ber96] and by definition of Ψ, $d = \lim_{\lambda \to \infty} \lambda/\Psi(\lambda)$. The same result holds under $\mathbb{Q}_x$ on $[0, T_0]$ and since the law of $X_{\cdot + t}$ under η given $X_t = x$ and on the set $\zeta > t$ equals $\mathbb{Q}_x$, it follows that X has sojourn d under η.

Consider now a non-zero measure κ on compact TOM trees with the splitting property and let ν be the push-forward of κ by the contour map.

We start by proving some consequences of the splitting property of κ. First, we prove that, in the compact case, the splitting property can be recast as

$$\kappa\left(h(X_t) e^{-\Xi_t^\infty g}\right) = \kappa\left(h(X_t) e^{-\int_{[0, X_t] \times \mathbf{C}_c} g(s, \mathbf{c})\, \kappa(d\mathbf{c})\, ds}\right).$$

Indeed, note that if $\mathbf{c}$ is a compact TOM tree, then its height $\sup_{\sigma \in \tau} d(\rho, \sigma)$ is finite. Hence, for any r greater than the height, $\mathbf{c}$ coincides with its truncation $\mathbf{c}^r$ at level r. Therefore, for any measurable subset A of the set of compact TOM trees, we have that

$$A = \lim_{r \to \infty} \{\mathbf{c} : \mathbf{c^r} \in A\}.$$

We deduce that κ^r converges to κ in the following sense:

$$\kappa(A) = \lim_{r \to \infty} \kappa^r(A).$$

Hence, for any g vanishing on trees with small height:

$$
\begin{aligned}
\kappa\left(h(X_t)\, e^{-\Xi_t^\infty g}\right) &= \lim_{r \to \infty} \kappa^r\left(h(X_t)\, e^{-\Xi_t^r g}\right) \\
&= \lim_{r \to \infty} \kappa^r\left(h(X_t)\, e^{-\int_{[0,X_t] \times \mathbf{C}_c} (1 - e^{-g(s,\mathbf{c})})\, \kappa^{r-s}(d\mathbf{c})\, ds}\right) \\
&= \kappa\left(h(X_t)\, e^{-\int_{[0,X_t] \times \mathbf{C}_c} (1 - e^{-g(s,\mathbf{c})})\, \kappa(d\mathbf{c})\, ds}\right).
\end{aligned}
$$

The identity between the extremes can then be generalized to non-negative and measurable g. We now prove the integrability conditions

$$\kappa(\zeta > h) < \infty \text{ for } h > 0 \quad \text{and} \quad \int_0^\infty 1 \wedge \zeta(\mathbf{c})\, \kappa(d\mathbf{c}) < \infty.$$

Since κ is non-zero, there exists $t > 0$ such that $\kappa(\zeta > t) > 0$. Note that in Ξ_t^∞, there can only be a finite number of atoms $(s_i, \mathbf{c}_i)$ such that $\mathbf{c}_i$ has measure $> \varepsilon$ for any $\varepsilon > 0$. By the splitting property, applied with $h = \mathbf{1}_{[0,\infty)}$ and $g(s, \mathbf{c}) = \mathbf{1}_{\zeta(\mathbf{c}) > \varepsilon}$, we get:

$$
\begin{aligned}
0 < &\kappa\left(h(X_t)\, e^{-\# \text{ subtrees of measure } > \varepsilon \text{ to the right of } [\rho, \phi(t)]}\right) \\
&= \kappa\left(h(X_t)\, e^{-\int_{[0,X_t] \times \mathbf{C}_c} (1 - e^{-g(s,\mathbf{c})})\, \kappa(d\mathbf{c})\, ds}\right) \\
&= \kappa\left(h(X_t)\, e^{-(1 - e^{-1}) X_t \cdot \kappa(\zeta > \varepsilon)}\right).
\end{aligned}
$$

We conclude that $\kappa(\zeta > \varepsilon) < \infty$ for any $\varepsilon > 0$. We now choose $g(s, \mathbf{c}) = \zeta(c)$. Next, under κ, the right of $\phi(t)$ has finite measure almost everywhere, so that, using the fact that κ has constant sojourn, say equal to a:

$$
\begin{aligned}
0 < \kappa(\zeta > t) &= \kappa\left(\mathbf{1}_{\zeta > t, \text{ measure of the right of } \phi(t) < \infty}\right) \\
&= \lim_{\lambda \to 0+} \kappa\left(\mathbf{1}_{\zeta > t} e^{-\lambda \text{ measure of the right of } \phi(t)}\right) \\
&= \lim_{\lambda \to 0+} \kappa\left(h(X_t)\, e^{-a\lambda X_t - \lambda \Xi_t^\infty g}\right) \\
&= \lim_{\lambda \to 0+} \kappa\left(h(X_t)\, e^{-a X_t [\lambda + t \int (1 - e^{-\zeta(\mathbf{c})})\, \kappa(d\mathbf{c})]}\right)
\end{aligned}
$$

We conclude that $\int 1 \wedge \zeta(\mathbf{c})\, \kappa(d\mathbf{c}) < \infty$.

Let $\Xi = \sum_i \delta_{(x_i, f_i)}$ be a Poisson random measure on $[0, \infty) \times \mathbf{E}$ with intensity $\text{Leb} \otimes \nu$. Define the process Y by the following procedure inspired by Itô's synthesis theorem (cf. [Itô72]): Let

$$S_l = al + \sum_{x_i \le l} \zeta(f_i) \quad \text{and} \quad \Lambda = S^{-1}.$$

We then let

$$Y_t = \sum_{x_i \leq x} \mathbf{1}_{S_{x_i-} \leq t < S_{x_i}} f_i(t - S_{x_i-}) - \Lambda_t$$

in analogy with (2.1). Heuristically, Y is the contour process of a tree obtained by grafting trees with law ν to the right of the vertical interval $(-\infty, 0]$. We claim that Y is a (sub)critical spectrally positive Lévy process. If so, let Ψ be its Laplace exponent. Since the compact TOM tree coded by the excursion measure of Y above its cumulative minimum process has law κ by construction, and if Y is a (sub)critical Lévy process this law should equal, by definition, η^Ψ, we see that $\kappa = \eta^\Psi$.

Let us prove that Y is a (sub)critical spectrally positive Lévy process. By construction, Y has no positive jumps. Also, note that the running infimum of Y is $-\Lambda$, which goes to $-\infty$. Hence, if Y is proved to be a Lévy process, then it is spectrally positive and is critical or subcritical. It remains to see that Y has independent and stationary increments.

Let us note that the process Y^x obtained by killing Y upon reaching $-x$ has the following construction in terms of Poisson random measures:

$$Y_t^x = \begin{cases} \sum_{x_i \leq x} \mathbf{1}_{S_{x_i-} \leq t < S_{x_i}} f_i(t - S_{x_i-}) - \Lambda_t & t < S_x \\ \dagger & S_x \leq t \end{cases}.$$

Let $\tilde{\mathbb{Q}}_x$ be the law of $x + Y^x$. By translation invariance of Lebesgue measure and independence properties of Poisson random measures we see that the two processes

$$\begin{cases} x + Y_t^{x+y} & t < S_x \\ \dagger & t \geq S_x \end{cases} \quad \text{and} \quad \begin{cases} x + y + Y_{t+S_x}^{x+y} & t < S_{x+y} - S_x \\ \dagger & t \geq S_{x+y} - S_x \end{cases}$$

are independent and have respective laws $\tilde{\mathbb{Q}}_x$ and $\tilde{\mathbb{Q}}_y$. Also, note that if we concatenate a process with the same law as Y to an independent process with law $\tilde{\mathbb{Q}}_x$ when it gets killed, then we get a process with law Y.

For any $t \geq 0$, let

$$\underline{Y}_t = \inf_{s \leq t} Y_s = -\Lambda_t,$$
$$g_t = \sup\{s \leq t : Y_s = \underline{Y}_s\} = S_{\Lambda_t -}$$

and

$$d_t = \inf\{s \geq t : Y_s = \underline{Y}_s\} = S_{\Lambda_t}.$$

Notice that the interval $[g_t, d_t]$ can be reduced to a point but that when it isn't, it is an interval of constancy for Λ. Also, note that $g_t = d_t$ if and only if S is continuous at Λ_t. Let Ξ_t be the restriction of Ξ to $[0, \Lambda_t] \times \mathbf{E}$ and define the random measure Ξ^t characterized by

$$\Xi^t(A \times B) = \Xi((A + \Lambda_t) \times B),$$

where A and B are Borel subsets of $[0, \infty)$ and of the excursion space $\mathbf{E}$ respectively. Let $\mathscr{G}_l$ and $\mathscr{G}^l$ be the σ-fields generated by the restriction of Ξ to $[0, l] \times \mathbf{E}$ and $[l, \infty) \times \mathbf{E}$. Then $\mathscr{G}_l$ is independent of $\mathscr{G}^l$. It is simple to see that Λ_t is a stopping time for $(\mathscr{G}_l)$ for any $t \geq 0$ and using the independence of $\mathscr{G}_l$ and $\mathscr{G}^l$, the translation invariance of Lebesgue measure and discretizing Λ_t by $\lfloor \Lambda_t 2^n \rfloor / 2^n$, one shows the independence of Ξ_t and Ξ^t and that Ξ^t has the same law as Ξ by mimicking the proof of the strong Markov property for Feller processes. Define

$$\nu_l(A) = \frac{\nu(A, \zeta > l)}{\nu(\zeta > l)}.$$

Using the analysis of the excursions straddling a given time of [RY99, Ch. XIII§3, p. 488], we see that defining the σ-field $\mathscr{F}_{g_t}^Y$ as $\sigma(Y_{s \wedge g_t} : s \geq 0)$, then the law of the excursion of Y straddling t given $\mathscr{F}_{g_t}^Y$ equals

$$\mathbf{1}_{s<t}\nu_{t-s} + \mathbf{1}_{s=t}\delta_{\dagger}$$

on the set $g_t = s$. The above discussion gives us access to 3 different parts of the path of Y: before g_t, between g_t and d_t (conditionally on $\mathscr{F}_{g_t}^Y$) and after d_t (which, if we shift it to start it at zero, is independent of the pre-d_t part and has the same law as Y since it is constructed from Ξ^t in the same manner as Y is constructed from Ξ). We now study the measure ν to make a further description of the parts of the path between g_t and t and between t and d_t.

Consider the shift operators θ_t on canonical space which take f to the function $s \mapsto f(t+s)$. Thanks to the splitting property of κ, we see that conditionally on X_s and under ν, the post-s process has law $\tilde{\mathbb{Q}}_{X_s}$. Formally the splitting property translates into

$$\nu(h(X_s)\, F \circ \theta_s) = \nu\Big(h(X_s)\, \tilde{\mathbb{Q}}_{X_s}(F)\Big) \tag{2.2}$$

for nonnegative measurable h and F defined on $[0,\infty)$ and $\mathbf{E}$ respectively (with g vanishing at $\dagger$). We now prove an analogous result under $\tilde{\mathbb{Q}}_x$:

$$\tilde{\mathbb{Q}}_x(h(X_s)\, F \circ \theta_s) = \tilde{\mathbb{Q}}_x\Big(h(X_s)\, \tilde{\mathbb{Q}}_{X_s}(F)\Big). \tag{2.3}$$

It is obtained from (2.2) as follows. Fix $s > 0$. Using the Poissonian description of $\tilde{\mathbb{Q}}_x$, we see that under this measure, the trajectory splits into 3 independent subpaths: 1) the subpath $X_{\cdot \wedge g_s}$ before g_s, 2) the subpath $X_{g_s + \cdot \wedge d_s} - X_{g_s-}$ between g_s and d_s, and 3) the subpath $X_{d_s+\cdot}$ after d_s. We now work conditionally on $\underline{X}_s = y, X_s = z, g_s = s'$; note that $X_{s-} = z$ also. After d_s, the process has law $\tilde{\mathbb{Q}}_y$. Between g_s and d_s, the process has law $\nu_{s-s'}$. Using (2.2), we see that under $\nu_{s-s'}$, the post $s - g_s$ part of the trajectory has law $\tilde{\mathbb{Q}}_{z-y}$. By concatenation, we see that under $\tilde{\mathbb{Q}}_x$ and on the set $\underline{X}_s = y, X_s = z, g_s = s'$, the post-$s$ part of the trajectory has law $\tilde{\mathbb{Q}}_{X_s}$, which implies (2.3). We deduce that $\tilde{\mathbb{Q}}_x$ is Markovian and that, therefore ν also is.

Finally, we finish the proof that Y is a Lévy process. Let $t > 0$ and note that the process Y has 3 independent subpaths by our analysis of its Poissonian construction: before g_t, between g_t and d_t (shifted by $-\Lambda_t$ to end at zero) and after d_t (shifted by $-\Lambda_t$ to start at zero). Also, the law of the (shifted) process between g_t and d_t is ν_{t-g_t} and by the Markov property (under ν) we see that the conditional law of the process between t and d_t (minus Λ_t, to start at zero) given the pre-t process is $\tilde{\mathbb{Q}}_{X_t - L_t}$. By concatenating the post d_t-part, which has law the law of Y (when shifted to start at zero) we see that the law of $Y_{t+\cdot} - Y_t$ given $\sigma(Y_s : s \leq t)$ is the law of Y. Hence, Y is a Lévy process. $\qquad\square$

3 Locally compact TOM trees

We have already noted in Section 1 that truncations of locally compact real trees are compact and defined the truncation of a locally compact TOM tree. In this short section, we explore the relationship between the contour processes of two truncations of a same locally compact TOM tree. We then use this information to see how to build locally compact TOM trees out of sequences of consistent compact TOM trees.

The truncation at level r of a locally compact TOM tree $\mathbf{c}$ is the compact TOM tree

$$\mathbf{c}^r = \left((\tau^r, d|_{\tau^r}, \rho), \leq|_{\tau_r}, \mu|_{\tau^r}\right).$$

We now study the relationship of the contour processes of two different truncations of the same tree.

Proposition 3.1. *Let* **c** *be a compact TOM tree. The contour processes* f *and* f^r *of* **c** *and* $\mathbf{c}^r$ *are related by a time-change as follows: let* $C^r(f)$ *be the right-continuous inverse of the continuous and non-decreasing function* $A^r = A^r(f)$ *given by*

$$A_t^r = \int_0^t \mathbf{1}_{f(s)\leq r}\, ds.$$

Then

$$f^r = f \circ C^r.$$

Proof. As verified at the end of the proof of Proposition A.9, we have the equality $\mu = \text{Leb} \circ \phi^{-1}$, where ϕ is the exploration process of **c**. Hence

$$\mu^r(L_\sigma) = \mu(L_\sigma \cap \tau^r) = \text{Leb}([0, \mu(L_\sigma)] \cap \{t : d(\phi(t), \rho) \leq r\})$$
$$= \text{Leb}(\{t \leq \mu(L_\sigma) : f(t) \leq r\}).$$

Let $\sigma \in \tau^r$ and define $s = \mu^r(L_\sigma)$ and $t = \mu(L_\sigma)$ so that

$$f^r(s) = d(\sigma, \rho) = f(t).$$

As we just computed,

$$s = \mu^r(L_\sigma) = A_t^r.$$

However, by taking $\sigma_n \in [\rho, \sigma)$ satisfying $\sigma \leq \sigma_{n+1} \leq \sigma_n$ and setting $t_n = \mu(L_{\sigma_n})$, we see that t_n decreases to t and $f(t_n) < r$, so that A^r increases in any right neighborhood of t and so

$$t = C_s^r.$$

Hence $f^r = f \circ C^r$ on the set $\{\mu^r(L_\sigma) : \sigma \in \tau^r\}$, which by Lemma A.6 is dense. We conclude that $f^r = f \circ C^r$ on $[0, \mu(\tau^r)]$. $\qquad\square$

We now use the preceding proposition to identify the space of locally compact TOM trees with the direct limit of compact TOM trees. Indeed, note that if **c** is a locally compact TOM tree and $\mathbf{c}^n$ is its truncation at height n, then the truncation of $\mathbf{c}^{n+1}$ at height n equals $\mathbf{c}^n$. Conversely, let $(r_n, n \in \mathbb{N})$ be a sequence of non-negative real numbers increasing to ∞ and $(\mathbf{c}_n, n \in \mathbb{N})$ be a sequence of compact TOM trees.

Definition 3.2. *The sequence* $(\mathbf{c}_n, n \in \mathbb{N})$ *is said to be* **consistent** *(at levels* $(r_n, n \in \mathbb{N})$*) if the truncation of* $\mathbf{c}_{n+1}$ *at height* r_n *is isomorphic to* $\mathbf{c}_n$.

Proposition 3.3. *If* $(\mathbf{c}_n, n \in \mathbb{N})$ *is a consistent sequence of compact TOM trees at levels* $r_1 < r_2 < \cdots$ *then there exists a unique locally compact TOM tree* **c** *such that the truncation of* **c** *at level* r_n *is isomorphic to* $\mathbf{c}_n$.

Proof. Let us construct a locally compact TOM tree from a consistent sequence $(\mathbf{c}_n, n \in \mathbb{N})$ at levels $r_1 < r_2 < \cdots$; suppose that $\mathbf{c}_n = ((\tau_n, d_n, \rho_n), \leq_n, \mu_n)$, let ψ_n be an isomorphism between $\mathbf{c}_n$ and the truncation of $\mathbf{c}_{n+1}$ at level r_n (which is actually a mapping between τ_n and τ_{n+1} preserving distances, root, total order and which maps the measure μ_n into the restriction of μ_{n+1} to $\tau_{n+1}^{r_n}$). We then define ψ_m^n for $m \leq n$ as the composition $\psi_{n-1} \circ \cdots \circ \psi_m$.

We begin by specifying the tree part τ of **c**. Let τ be the direct limit of $(\tau_n, n \in \mathbb{N})$ with respect to the mappings (ψ_n). In other words, τ has elements of the type (l, σ) where $\sigma \in \tau_l$ and we identify (l, σ) and $(m, \tilde{\sigma})$ if there exists $n \geq l, m$ such that $\psi_l^n(\sigma) = \psi_m^n(\tilde{\sigma})$. We then deduce that the preceeding equality holds for any $n \geq l, m$.

Let us specify the distance d to be placed on τ: we define it by

$$d((l, \sigma), (m, \tilde{\sigma})) = d_n(\psi_l^n(\sigma), \psi_m^n(\tilde{\sigma})).$$

for any $n \geq l, m$.

The root of τ is taken as the equivalence class of $(1, \rho_1)$.

The triple (τ, d, ρ) is locally compact rooted real tree. Indeed, note that if (l, σ) and $(m, \tilde{\sigma})$ are two representatives of elements of τ, we can embed them in τ_n for any $n \geq m, l$ and since τ_n is a tree, we can obtain an isometry of an interval which starts at $\psi_l^n(\sigma)$ and ends at $\psi_m^n(\tilde{\sigma})$. Lack of loops can be proved by noting that an isometry of an interval into τ must remain bounded, so that it can be transformed into an isometry of an interval into τ_n for some n. Local compactness and completeness can also be proved by a similar argument.

A total order $\leq$ on τ can be defined by stating that

$$(l, \sigma) \leq (m, \tilde{\sigma}) \text{ if and only if } \psi_l^n(\sigma) \leq_n \psi_m^n(\tilde{\sigma})$$

for some $n \geq l, m$. It is easy to see that $\leq$ satisfies **Or1** and **Or2**.

Finally, we place the measure μ on τ. We abuse our notation to say that μ_n is the push-forward of the measure μ_n by the mapping that sends σ to the equivalence class of (n, σ). Note that (μ_n) increases and so we can define the measure μ by

$$\mu = \lim_{n \to \infty} \mu_n.$$

It is simple to see that μ satisfies **Mes1** and **Mes2**.

We end the proof by noting that $\mathbf{c} = ((\tau, d, \rho), \leq, \mu)$ is a locally compact TOM tree whose truncation at height r_n is isomorphic to $\mathbf{c}_n$. $\qquad \square$

Let (f_n) be a sequence of càdlàg functions where f_n is defined on $[0, m_n]$, $f_n(0) = 0$ and such that f_n codes a tree $\mathbf{c}_n$.

Definition 3.4. *We say that the sequence (f_n) is **consistent under time change** if there exists a sequence of heights $r_1, r_2, \ldots$ increasing to infinity and such that*

$$f_n = f_{n+1} \circ C^{r_n}(f_{n+1}).$$

From Propositions 3.1 and 3.3 we see that (the equivalence class under isomorphism of) a locally compact TOM tree can be identified with a sequence of functions consistent under time change (and such that their induced equivalence classes have zero Lebesgue measure). Let us then define the function X_t^r defined on TOM trees and with values in $\mathbb{R} \cup \{\dagger\}$ by stating that $X_t^r(\mathbf{c})$ is the value at time t of the contour process of the truncation $\mathbf{c}^r$ of $\mathbf{c}$ at level r (if the measure of $\mathbf{c}^r$ is greater than t; otherwise we define it as $\dagger$). We can then use the σ-field $\mathscr{F} = \sigma(X_t^r, t, r \geq 0)$ when discussing measurability issues for functions defined on (or with values on) locally compact TOM trees.

Corollary 3.5. *Let $\mathbb{P}^r$ be a sequence of measures on Skorohod space such that the push-forward of $\mathbb{P}^r$ under the time change at level $r' \leq r$ gives $\mathbb{P}^{r'}$. Suppose that under $\mathbb{P}^r$, the canonical process is non-negative, approaches zero at death time and that the equivalence classes $[s]_X$ induced by the canonical process have zero Lebesgue measure. Then there exists a unique measure $\mathbb{P}$ on locally compact TOM trees such that, under $\mathbb{P}$ the contour process of the truncation at level r has law $\mathbb{P}^r$.*

4 Consistent families of reflected Lévy processes

The objective of this section is to give a further study of a family of stochastic processes consistent under time-change: the reflected Lévy processes introduced in Section 1. This will allow us, through Corollary 3.5, to construct random locally compact TOM trees. In this section, we give two additional constructions of reflected Lévy processes. Both constructions make sense even in the case of positive killing coefficient. The first

construction is based on time-changes and has a pathwise time-change consistency which is useful in constructing locally compact TOM trees. This enables us to construct the measure η^Ψ in the statement of Theorem 1.11 in the non-compact case. The second construction is through Poisson random measures. It is a consequence of the time-change construction and is fundamental to our proof of Theorem 1.11 in the locally compact case.

4.1 Reflected Lévy processes through time-change

We place ourselves on the canonical space of càdlàg trajectories from $[0, \infty)$ to $\mathbb{R}$ together with the canonical process $X = (X_s, s \geq 0)$ and the canonical filtration $(\mathscr{F}_t, t \geq 0)$. Let $\mathbb{P}_x$ be the law of a spectrally positive Lévy process (spLp) with Laplace exponent Ψ started at x. We will suppose throughout that X is not a subordinator under $\mathbb{P}_0$.

Let us consider a reflecting threshold r and define the cumulative maximum process from r, $\overline{X}^r$ given by

$$\overline{X}_t^r = \left(\max_{s \leq t} X_s - r \right)^+ .$$

We will write $\overline{X}$ when r is clear from the context, as in the next definition. Recall that a spectrally positive Lévy process with Laplace exponent ψ started at $x \leq r$ and reflected below r is a stochastic process having the law $\mathbb{P}_x^r$ of $X - \overline{X}$ under $\mathbb{P}_x$. Under $\mathbb{P}_x^r$, the canonical process is a strong Markov process as proved in Proposition 1 of [Ber96, Ch. VI]. We will also make use of the Lévy process reflected below r and killed upon reaching zero and denote its law by $\mathbb{Q}_x^r$; recall that killing preserves the strong Markov property. The laws $\mathbb{Q}_x^r$ were introduced in Definition 4.5.4.1 of [Lam08] since they coincide with the law of the (jump-chronological) contour process J^r of a splitting tree truncated at height r associated to a finite-variation spLp with Laplace exponent Ψ whose progenitor has lifetime x. The relationship between the contour processes at different heights (cf. Proposition 3.1) suggests the following corresponding property of the laws $\mathbb{Q}_x^r$: let

$$A_t^r = \int_0^t \mathbf{1}_{X_s \leq r} \, ds.$$

Let C^r be the right-continuous inverse of A^r.

Proposition 4.1. *If $x \leq r_1 < r_2$ then the law of $X \circ C^{r_1}$ under $\mathbb{P}_x^{r_2}$ is $\mathbb{P}_x^{r_1}$.*

By stopping at the first hitting time of zero, we obtain:

Corollary 4.2. *If $0 \leq x \leq r_1 < r_2$ then the law of $X \circ C^{r_1}$ under $\mathbb{Q}_x^{r_2}$ is $\mathbb{Q}_x^{r_1}$.*

Indeed, [Lam10] proves Corollary 4.2 when the underlying Lévy process is of finite variation. We are interested in extending the consistency under time-change in order to construct random locally compact TOM trees out of the trajectories of more general Lévy processes.

In the Brownian case (when $\Psi(\lambda) = \lambda^2/2$), the equality in law between $X \circ C^0$ under $\mathbb{P}_x^{r_2}$ and $\mathbb{P}_x^{r_1}$ is reduced to the equality in law between Brownian motion reflected below its supremum and Brownian motion time changed by C^0 to remain negative. This can be proved using Tanaka's formula, Knight's Theorem and Skorokhod's reflection lemma. These tools are not available for general spectrally positive Lévy processes. However, this already tells us that some local time argument should be involved in the proof. Indeed, our argument is based on excursion theory for Markov processes, in which local time plays a fundamental role.

Proposition 4.1 is a generalization of the results for Brownian motion or finite variation Lévy processes. The only caveat is to see that the equality between time-change and

reflection holds as a consequence of the assumed lack of negative jumps. For example, if $\mathbb{P}_0$ is the law of the difference of two independent stable subordinators of index $\alpha \in (0,1)$, then under $\mathbb{P}_x^{r_1}$, the barrier r_1 is touched. However, since points are polar (see for example [Ber96, p. 63]) then $X \circ C^{r_1}$ does not touch the barrier r_1 almost surely under $\mathbb{P}_x^{r_2}$ so that no equality in law can be valid in this case.

Proof of Proposition 4.1. Recall that the time-changed processes $X \circ C^{r_1}$ under $\mathbb{P}_x^{r_2}$ are also strongly Markovian, as shown for example in [Dyn65] or [Sha88].

The argument is based on excursion theory, as introduced in [Ber96, Ch. 4]. Indeed, we prove that the (regenerative) sets of times at which the two (strong Markov) processes visit r_1, being the images of subordinators as in [Ber99, Ch. 2], have the same drift (zero in this case), and that the excursions of both processes below r_1 admit the same stochastic descriptions. To see that the drift is equal to zero, by Proposition 1.9 of [Ber99], it suffices to see that the sets of times at which the two processes visit r_1 have zero Lebesgue measure almost surely.

The law of X under $\mathbb{P}_x^{r_1}$ until hitting r_1 is equal to the law of X under $\mathbb{P}_x$ until it surpasses r_1. After X hits r_1 under $\mathbb{P}_x^{r_1}$, the process has law $\mathbb{P}_{r_1}^{r_1}$, which is the push-forward of $\mathbb{P}_0^0$ under $f \mapsto r_1 + f$. Similarly, X equals $X \circ C^{r_1}$ until X surpasses r_1, and under $\mathbb{P}_x^{r_2}$, these processes have the law of X under $\mathbb{P}_x$ until it surpasses r_1. After $X \circ C^{r_1}$ hits r_1, it has the law under $\mathbb{P}_x^{r_2}$ of $X \circ C^{r_1}$ under $\mathbb{P}_{r_1}^{r_2}$, which by spatial homogeneity is the image of the law of $X \circ C^0$ under $\mathbb{P}_0^{r_2}$. Hence, it suffices to focus on the case $0 = x = r_1 < r_2$; set $r_2 = r$ to simplify notation.

Formula 9.2.9 of [Don07, p. 98] implies that the Laplace exponent of the upward ladder time process of X under $\mathbb{P}_0$ is equal to $\alpha/\Phi(\alpha)$ where Φ is the right-continuous inverse of Ψ and since $\mathbb{P}_0$ is not the law of a subordinator, then $\Phi(\alpha) \to \infty$ as $\alpha \to \infty$ implying that the drift of the upward ladder time process, which is equal to $\lim_{\alpha \to \infty} 1/\Phi(\alpha)$, is zero. Hence, the set of times X is at its supremum under $\mathbb{P}_0$ has zero Lebesgue measure almost surely and so $\int_0^\infty \mathbf{1}_{X_s = 0} \, ds = 0$ almost surely under $\mathbb{P}_0^0$.

Let us obtain a corresponding statement for $X \circ C^0$ under $\mathbb{P}_0^r$. Since the time-change just removes parts of the trajectory, we are reduced to proving that X spends zero time at 0 under $\mathbb{P}_0^r$ which is equivalent to proving that $X - \overline{X}^r$ spends zero time at zero under $\mathbb{P}_0$. Note however that X spends zero time at 0 under $\mathbb{P}_0$, even if X is compound Poisson minus a drift since the drift is forced to be nonzero having excluded subordinators. Hence, until the trajectory of $X - \overline{X}^r$ reaches r, it spends zero time at zero under $\mathbb{P}_0$. After this process reaches r, its law is that of $r + X - \overline{X}$ under $\mathbb{P}_0$. Note that

$$0 = \mathbb{E}_{\mathbb{P}_0}\left(\int_0^\infty \mathbf{1}_{X_t - \overline{X}_t = -r} \, dt\right) \text{ if and only if } 0 = \mathbb{E}_{\mathbb{P}_0}\left(\int_0^\infty \lambda e^{-\lambda t} \mathbf{1}_{X_t - \overline{X}_t = -r} \, dt\right).$$

Hence, we now show that if T is a standard exponential time independent of X then $X_T - \overline{X}_T$ is almost surely not $-r$. By the Pečerskiĭ-Rogozin formula (cf. Th. 5 in [Ber96, Ch. VI] or Equation (5) in [PUB12]), we know that $X_T - \overline{X}_T$ is a negative infinitely divisible random variable with zero drift and Lévy measure equal to

$$\nu(A) = \int_0^\infty \frac{e^{-t}}{t} \mathbb{P}_0(X_t \in A \cap (-\infty, 0)) \, dt.$$

Recall that since X is not a subordinator under $\mathbb{P}_0$ then 0 is regular for $(-\infty, 0)$, as verified in Theorem 1 of [Ber96, Ch. VII]. By Rogozin's criterion for regularity (cf. Proposition 11 of [Ber96, Ch. VI] or equation (8) in [PUB12]), we then see that

$$\nu((-\infty, 0)) = \int_0^\infty \frac{e^{-t}}{t} \mathbb{P}_0(X_t < 0) \, dt = \infty.$$

Since ν is an infinite measure the law of $X_T - \overline{X}_T$ has no atoms under $\mathbb{P}_0$ (cf. Theorem 27.4 in [Sat99]).

Since the laws $\mathbb{P}_0^r$ and $\mathbb{P}_0^0$ are strongly Markovian, the sets

$$\left\{ t \geq 0 : X \circ C_t^0 = 0 \right\} \quad \text{and} \quad \left\{ t \geq 0 : X_t = 0 \right\}$$

are regenerative under $\mathbb{P}_0^r$ and $\mathbb{P}_0^0$; we have just proved that in both cases their corresponding inverse local time at zero has zero drift.

We now describe the excursions below 0. Let $\mathbf{E}^-$ be the set of negative excursions. That is, the set of functions $f : [0, \infty) \to (-\infty, 0] \cup \{\dagger\}$ for which there exists $\zeta = \zeta(f) > 0$, termed the lifetime, such that f is strictly negative and càdlàg with no negative jumps on $(0, \zeta)$ and equal to $\dagger$ after ζ; hence, $\dagger$ is again interpreted as a cemetery state. The set $\mathbf{E}^-$ is equipped with the natural filtration of the canonical process. Since we have proved that the canonical process X under $\mathbb{P}_x^{r_1}$ (resp. $X \circ C^{r_1}$) under $\mathbb{P}_x^{r_2}$) spends zero time at r_1, excursion theory tells us that X under $\mathbb{P}_x^{r_1}$ has the same law as the stochastic process Z given by

$$Z_t = \sum_i \mathbf{1}_{\tau_{\Lambda_i -} \leq t < \tau_{\Lambda_i}} E_i(t - \tau_{\Lambda_i -}),$$

where the random measure $\sum_i \delta_{(\Lambda_i, E_i)}$ is a Poisson random measure on $[0, \infty) \times \mathbf{E}^-$ with intensity $\mathrm{Leb} \otimes \nu$ for some σ-finite measure $\nu = \nu^0$ (resp. $\nu = \nu^r$) on $\mathbf{E}^-$ and

$$\tau_l = \sum_{\Lambda_i \leq l} \zeta(E_i).$$

The measure ν^0 (resp. ν^r) is characterized, up to a multiplicative constant which does not affect the law of Z, as follows. Let T_ε denote the hitting time of $-\varepsilon$ by the canonical process. Since functions in $\mathbf{E}^-$ start at zero and leave 0 immediately, any measure on $\mathbf{E}^-$ is determined by its values on $\cup_{\varepsilon > 0} \sigma(X_{t + T_\varepsilon} : t \geq 0)$. Let $\tilde{\mathbb{P}}_0^r$ be the law of $X \circ C^0$ under $\mathbb{P}_0^r$. Let

$$\mathscr{O} = \{t \geq 0 : X_t < 0\}.$$

Since X has non-negative jumps under $\mathbb{P}_0^0$ and $\tilde{\mathbb{P}}_0^r$ then $\mathscr{O}$ is almost surely open under both measures. Its connected components are termed the excursion intervals of X below 0. If (a, b) is a connected component of $\mathscr{O}$ then the non-negativity of jumps forces $X_a = 0$, and so the path e given by $e_s = X_{a+s}$ if $0 \leq s < b - a$ and $e_s = \dagger$ if $s \geq b - a$ is an element of $\mathbf{E}^-$. The depth of the excursion e is defined as $\min_{s < b-a} e_s$. For any $\varepsilon > 0$, there is a well-defined sequence of successive excursions of depths falling below $-\varepsilon$ which are iid with common laws ν_ε^0 (respectively ν_ε^r) under $\mathbb{P}_0^0$ (respectively under $\tilde{\mathbb{P}}_0^r$); this follows from the strong Markov property under $\mathbb{P}_0^0$ and $\tilde{\mathbb{P}}_0^r$. Then, the intensity measure ν^0 is the unique σ-finite measure (up to a multiplicative constant that does not affect the law of Z) such that ν^0 is finite on the set of excursions falling below $-\varepsilon$ and ν_ε^0 equals ν^0 conditioned on falling below $-\varepsilon$. Indeed, the measure ν^0 can be defined by

$$\nu^0 = \lim_{\varepsilon \to 0} \frac{\nu^\varepsilon}{\nu^\varepsilon(\text{depth} < 1)}.$$

We construct ν^r in the same manner, based on the probability measures ν_ε^r giving us the law of successive excursions of $X \circ C^0$ below zero of depth below $-\varepsilon$ under $\mathbb{P}_0^r$. Hence, to prove that $\nu^0 = \nu^r$, it suffices to see that $\nu_\varepsilon^0 = \nu_\varepsilon^r$. Let us describe the left-hand side: under $\mathbb{P}_0$, consider the process $X \circ \theta_{T_\varepsilon}$ until it becomes positive, after which we send it to ∞; this law equals the image $\mathbb{P}_{-\varepsilon}$ by killing upon becoming positive by lack of positive

jumps and this is exactly the measure $\nu_\varepsilon^0 \circ \theta_{T_\varepsilon}^{-1}$. However, the law $\nu_\varepsilon^r \circ \theta_{T_\varepsilon}$ admits the same description. Indeed, it suffices to note that if we define the stopping time

$$T_\varepsilon^0 = \inf \left\{ t \geq 0 : X \circ C_t^0 = -\varepsilon \right\}$$

for the process $X \circ C^0$ and note that the process $X(C_{T_\varepsilon^0+.}^0)$ is equal to $X_{T_\varepsilon+t}$ until the latter becomes positive. Hence $\nu^0 = \nu^r$ and so $\mathbb{P}_0^0$ equals the law of $X \circ C^0$ under $\mathbb{P}_0^r$. Note that the proof remains valid even in the subcritical case where there are excursions of infinite length. $\qquad\square$

As mentioned in [Lam08], Corollary 4.2 and Kolmogorov's consistency theorem imply the existence of a doubly indexed process X_t^r such that X^r has law $\mathbb{P}_x^r$ for every $r \geq x$ and which is consistent under time-change in the following pathwise sense:

$$X^r = X^{r'} \circ C^{r',r}$$

for $r < r'$ where $C^{r',r}$ is the inverse of $A^{r',r}$ and

$$A_t^{r',r} = \int_0^t \mathbf{1}_{X_s^{r'} \leq r} \, ds.$$

We now give a representation of this doubly indexed process by time-changing independent copies of a Lévy process.

Let $X^1, X^2, \dots$ be independent copies of X and define:

$$T_1 = \inf \left\{ t \geq 0 : x + X_t^1 = 0 \right\}, \quad T_n = \inf \left\{ t \geq 0 : r + X_t^n = 0 \right\} \text{ for } n \geq 2$$

as well as

$$A_t^1 = \int_0^t \mathbf{1}_{x + X_{s \wedge T_1}^1 \leq r} \, ds$$

and, for $n \geq 2$,

$$A_t^n = \int_0^t \mathbf{1}_{r + X_{s \wedge T_n}^n \leq r} \, ds,$$

Let C^n be the right continuous inverse of A^n for any $n \geq 1$. Consider the process

$$Y_t^r = \begin{cases} x + X^1 \circ \alpha_t^1 & \text{if } t < C_\infty^1 \\ r + X^n \circ \alpha_{t - C_\infty^1 - \cdots - C_\infty^{n-1}}^n & \text{if } C_\infty^1 + \cdots + C_\infty^{n-1} \leq t < C_\infty^1 + \cdots + C_\infty^n \text{ for } n \geq 2 \end{cases},$$
(4.1)

which, by construction, is consistent under time-change.

Proposition 4.3. *The process Y^r has law $\mathbb{Q}_x^r$.*

Proof. Note that if X is (sub)critical, then $C_\infty^1 = \infty$ since $T < \infty$; hence, we only need X^1 in the definition of Y^r. But then, Y^r coincides with $X \circ C^r$ under $\mathbb{Q}_x^r$, so that the result is valid in this case.

It remains to consider the supercritical case. Also, before Y^r hits r it has the same law as the Lévy process until it exceeds r, so that it suffices to consider the case $x = r$.

If $\tilde{Y}^r$ admits the same construction as Y^r but without killing upon reaching zero (that is, we use $\tilde{A}_t^n = \int_0^t \mathbf{1}_{r + X_s^n \leq r} \, ds$ instead of A^n), we now prove that $\tilde{Y}^r$ has law $\mathbb{P}_r^r$, which proves the stated result.

The process $\tilde{Y}$ is strongly Markovian and, using the proof of Proposition 4.1, one sees that

$$\int_0^t \mathbf{1}_{Y_s^r = r} \, ds = 0$$

almost surely.

Also, for each $\varepsilon > 0$, the successive excursions of Y^r reaching $r - \varepsilon$ are also the excursions of some X^i (below r and reaching $r - \varepsilon$). After they reach $r - \varepsilon$ their law is the image of $\mathbb{P}_{r-\varepsilon}$ by killing upon reaching (r, ∞). Therefore, the intensity of excursions of Y^r below r is ν^0. By the arguments of the proof of Proposition 4.1, we see that Y^r has law $\mathbb{P}_r^r$. $\square$

4.2 Poissonian construction of reflected Lévy processes

We now present a Poissonian construction of reflected Lévy processes which will be useful in establishing Theorem 1.11 in the locally compact case.

Let Ψ be the Laplace exponent of a spectrally positive Lévy process which is not a subordinator and let Φ be its right-continuous inverse. Let d be the drift of Φ, so that

$$d = \lim_{\lambda \to \infty} \frac{\Phi(\lambda)}{\lambda}$$

As before, we let $\mathbb{P}_x^r$ be the law of a Ψ-Lévy process reflected under $r > 0$ and started at $x \leq r$. Suppose for the moment that Ψ is subcritical or critical (that is, $\Psi'(0+) \geq 0$). Then we can perform the following Poissonian construction of $\mathbb{P}_x^r$: let ν be the excursion measure of $X - \underline{X}$ under $\mathbb{P}_0$ and define ν^r as the push-forward of ν by the truncation operator above r (which sends f to $f \circ C^r(f)$).

Let

$$\Xi^r = \sum_n \delta_{(s_n, f_n)}$$

be a Poisson random measure on $[0, \infty) \times \mathbf{E}$ with intensity $\mathrm{Leb}(ds) \otimes \nu^{r-x+s}(df)$ and define

$$S_t^r = dt + \sum_{s_n \leq t} \zeta(f_n) \,, \quad M^r = -(S^r)^{-1} \quad \text{and} \quad W_t = x + M_t^r + \sum_n \mathbf{1}_{S_{s_n-}^r \leq t < S_{s_n}^r} f_n\big(t - S_{s_n-}^r\big)$$

$$(4.2)$$

Proposition 4.4. *For a (sub)critical Laplace exponent Ψ, the law of W is $\mathbb{P}_x^r$.*

Proof. It suffices to consider the case $x = 0$ because the push-forward of $\mathbb{P}_0^{r-x}$ by the operator which adds x to a given function is $\mathbb{P}_x^r$.

Recall that under $\mathbb{P}_0$, $-\underline{X}$ is the local time of $X - \underline{X}$ at zero and its right-continuous inverse, say S, is a subordinator with Laplace exponent Φ, as shown in [Ber96, Ch. VII]. Also, the point process of excursions of $X - \underline{X}$ above zero under $\mathbb{P}$, say

$$\Xi' = \sum_n \delta_{(s_n', f_n')},$$

is a Poisson point process on $[0, \infty) \times \mathbf{E}$ with intensity $\mathrm{Leb}(ds) \otimes \nu(df)$. Finally, since excursion intervals become jumps of S and S is a subordinator with drift d, X and S can be recovered from Ξ' and d by means of the formulae

$$S_t = dt + \sum_{s_n' \leq t} \zeta(f_n') \,, \quad -\underline{X} = S^{-1}$$

and

$$X_t - \underline{X}_t = \sum_{n>0} \mathbf{1}_{S_{s_n'-} \leq t < S_{s_n'}} f_n'\big(t - S_{s_n'-}\big) \,.$$

Since the law of $X \circ C^r$ under $\mathbb{P}_0$ is $\mathbb{P}_0^r$, we only need to see that the excursion process of $X \circ C^r$ has the same law as Ξ^r and that $X \circ C^r$ can be written in terms of its point process of excursions as in (4.2).

An excursion of $X \circ C^r$ above its cumulative minimum, when the latter equals s is an excursion of $X - \underline{X}$ reflected below level $r + s$ by time-change. Hence the point process of excursions of $X \circ C^r$ above its cumulative minimum is

$$\tilde{\Xi} = \sum_n \delta_{(\zeta(f_n' \circ C^r), f_n' \circ C^r)},$$

which is a Poisson point process with intensity $\text{Leb}(ds) \otimes \nu^{r+s}(df)$.

Also, the cumulative minimum process of $X \circ C^r$ is obtained from $\underline{X}$ by erasing the time projection of excursions above the reflection threshold, which means erasing parts of jumps of S. Hence, the process

$$S^r = (-\underline{X} \circ C^r)^{-1}$$

is given by

$$S_t^r = dt + \sum_{s_n \leq t} \zeta\left(f_n' \circ C^{r - s_n'}(f_n')\right).$$

Finally, $X \circ C^r$ minus its cumulative minimum can be obtained by concatenating excursions,

$$X \circ C_t^r - \min_{s \leq t} X \circ C_s^r = \sum_{s_n'} \mathbf{1}_{S_{s_n'-}^r \leq t < S_{s_n'}^r} f_n' \circ C^{r - S_{s_n'-}^r}\left(t - S_{s_n'-}^r\right)$$

so that indeed $X \circ C^r$ can be reconstructed from its excursion process as in (4.2). $\square$

When Ψ is supercritical, let $b > 0$ be the largest root of the convex function Ψ and $\Psi^\#(\lambda) = \Psi(\lambda + b)$. Recall that $\Psi^\#$ is the Laplace exponent of the (subcritical) spectrally positive Lévy process with Laplace exponent Ψ conditioned to reach arbitrarily low levels in the sense that its law equals

$$\lim_{z \to -\infty} \mathbb{P}_0(\cdot \mid \underline{X}_\infty < z)$$

on every $\mathscr{F}_s$. (Cf. Lemma 7 in [Ber96, Ch. VII] and Lemme 1 in [Ber91].) Let us use the notation $\mathbb{P}_x$ and $\mathbb{P}_x^\#$ to distinguish both Lévy processes. Let H_a denote the hitting time of $-a$, defined by $H_a = \inf\{t \geq 0 : X_t \leq -a\}$. Recall that conditionally on $H_a < \infty$, the laws of X stopped at H_a coincide under $\mathbb{P}$ and $\mathbb{P}^\#$. We deduce that, conditionally on $H_a < \infty$, the point process of excursions of X above its cumulative minimum that start at levels deeper than $-a$ under $\mathbb{P}$ is a Poisson point process with the same intensity as under $\mathbb{P}^\#$ (let us call it $\nu^\#$). Furthermore, $-\underline{X}_\infty$ under $\mathbb{P}$ has an exponential law with rate b and the post minimum process is independent of the pre-minimum process (cf. Theorem 25 in [Don07, Ch. 8, p.85]). The law of the former had been denoted $\mathbb{P}^\rightarrow$. With these preliminaries, we can give a construction of the excursion process of X above its cumulative minimum $\underline{X}$ under $\mathbb{P}$ and $\mathbb{P}^r$. Let

$$\Xi = \sum_n \delta_{(s_n, f_n)}$$

be a Poisson point process with intensity $\nu^\# + b\mathbb{P}^\rightarrow$, interpreting the law $\mathbb{P}^\rightarrow$ as that of an excursion of infinite length. Construct the processes

$$S_t = dt + \sum_{s_n \leq t} \zeta(f_n), \quad M = -\left(S^{-1}\right) \quad \text{and} \quad W_t = x + M_t + \sum_{S_{s_n-} \leq t < S_{s_n}} f_n(t - S_{s_n-}).$$

Proposition 4.5. *If Ψ is supercritical, the law of W is $\mathbb{P}_x$.*

An analogous Poissonian construction is valid in the reflected case. To construct the excursion measure, let V^r stand for the operator that removes (by time change) the trajectory above level r and let ν^r be the law of the concatenation of $\mathbb{P}^{\rightarrow} \circ (V^r)^{-1}$ followed by $\mathbb{Q}_r^r$ (which is the law of the Lévy process started at r, reflected below r, and killed (or sent to †) upon reaching zero. Consider also the push-forward $\nu^{\#,r}$ of $\nu^{\#}$ by truncation at level r. Finally, let

$$\Xi^r = \sum_n \delta_{(s_n, f_n)}$$

be a Poisson point process with intensity $\mathrm{Leb}(ds) \otimes \left(\nu^{\#, r-x+s} + b\nu^{r-x+s} \right)$. As before, consider

$$S_t^r = dt + \sum_{s_n \leq t} \zeta(f_n), \quad M^r = -(S^r)^{-1} \quad \text{and} \quad W_t^r = x + M_t^r + \sum_{S_{s_n-}^r \leq t < S_{s_n}^r} f_n\left(t - S_{s_n-}^r\right).$$

Proposition 4.6. *If Ψ is supercritical, the law of W^r is $\mathbb{P}_x^r$.*

Proof sketch. Again, it suffices to consider the case $x = 0$. The arguments are very close to those of Proposition 4.4. The differences stem from the fact that in the supercritical case, under $\mathbb{P}_0$, there are two types of excursions: those corresponding to the associated subcritical Lévy process with exponent $\Psi^{\#}$, and the unique excursion of infinite length (which appears at rate b for the cumulative minimum process). Appealing to the time-change construction of $\mathbb{P}_0^r$ in the proof of Proposition 4.3 we also see the two types of excursions. The first type of excursions come from the associated subcritical process, which are then affected by a time-change to remain below level $r - x + s$ if the cumulative minimum takes the value $-s$. On the other hand, the excursion of infinite length under $\mathbb{P}_0$ will also be affected by a time-change to remain below $r - x + s$, except that after death time, we must concatenate a process with law $\mathbb{P}_r^r$ which is killed upon reaching $-s$. These second types of excursions arrive at rate b. $\qquad\square$

5 Measures on non-compact TOM trees satisfying the splitting property

In this section, we will establish the relationship between Lévy processes and measures on locally compact TOM trees stated in Proposition 1.6 and Theorem 1.11. Part of the proof of the latter theorem is similar to the proof in the compact case of Theorem 1.11 which we will follow closely, highlighting the differences.

We first use excursion theory for Lévy processes as in the compact case to construct the measure η^{Ψ} of Theorem 1.11 in the non-compact case and to prove that it has the splitting property.

Proof of Proposition 1.6. Let Ψ be a supercritical Laplace exponent. Recall the definition of the truncated excursion measure ν^r associated to Ψ given in Section 1.

From Proposition 2.1, the local absolute continuity of $\mathbb{P}^{\Psi}$ and $\mathbb{P}^{\#}$, and the fact that Y^r is a concatenation of time-changed trajectories of $\mathbb{P}_0^{\Psi}$ we see that the canonical process codes a compact TOM tree ν^r-almost surely. Let η^r be the image measure of ν^r under the mapping sending functions to TOM trees. Thanks to Corollary 4.2, we see that the measures $(\nu^r, r \geq 0)$ are consistent under time-change. Hence, Corollary 3.5 implies the existence of a measure η^{Ψ} on locally compact TOM trees such that the image of η^{Ψ} by truncation at level r equals η^r. $\qquad\square$

Proof of Theorem 1.11 in the non-compact case: Splitting property of η_Ψ . By the Markov property under ν^r, we see that on the set $\zeta > t$ and conditionally on $X_t = x$, the post-t process has law $\mathbb{Q}_x^r$. By the Poissonian construction of $\mathbb{P}_x^r$ of Proposition 4.6, we see that the point process of excursions above the cumulative minimum of the post-t process is a Poisson random measure with intensity $\text{Leb}(ds) \otimes \nu^{r-x+s}(df)$. We conclude that η^Ψ has the splitting property. $\qquad\square$

We now consider a measure η on locally compact TOM trees which is not concentrated on compact ones. Our aim is to construct the Laplace exponent of a supercritical spectrally positive Lévy process such that $\eta = \eta^\Psi$. In the compact case, we could construct the corresponding Lévy process directly, while in the locally compact case, we will construct first the Lévy process reflected at a given level r and obtain the Lévy process in the limit as $r \to \infty$.

Proof of Theorem 1.11 in the non-compact case. (Construction of a Laplace exponent such that $\eta = \eta^\Psi$) Let η^r be the push-forward of η by truncation at level r and let n^r the law of the contour process under η^r. Let

$$\Xi = \sum_i \delta_{(x_i, f_i)}$$

be a Poisson random measure on $[0, \infty) \times \mathbf{E}$ with intensity $\text{Leb}(ds) \otimes \eta^{r+s}$ (the definition of the intensity was simpler in the compact case since truncation was not necessary). If a is the sojourn of η, define

$$S_l = al + \sum_{x_i \leq l} \zeta(f_i) \quad \text{and} \quad \Lambda = S^{-1}$$

as well as

$$Y_t^r = \sum_{x_i \leq x} \mathbf{1}_{S_{x_i-} \leq t < S_{x_i}} f_i(t - S_{x_i-}) - \Lambda_t.$$

We claim that Y^r is a reflected Lévy process. To prove it, however, we need to let $r \to \infty$ to see that we get a limit which is a Lévy process. However, truncation allows us to use the splitting property in a way that is parallel to the compact case. Note that the measure η^r satisfies an r-dependent version of the splitting property: under η^r and conditionally on $X_t = x$, Ξ_t is a Poisson random measure on $[0, x] \times \mathbf{C}_c$ with intensity $\text{Leb}(ds) \otimes \eta^{r-s}(df)$. This is an important difference with the compact case. For example, if we denote by $Y^{r,x}$ the process obtained by killing Y^r when it reaches level $-x$, then (with the same proof as in the compact case) we see that the two processes

$$\begin{cases} Y_t^{r,x+y} & t \leq S_x \\ \dagger & t \geq S_x \end{cases} \quad \text{and} \quad \begin{cases} Y_{t+S_x}^{r,x+y} + x & t \leq S_{x+y} - S_x \\ \dagger & t \geq S_{x+y} - S_x \end{cases}$$

are independent and have the same laws as $Y^{r,x}$ and $Y^{r+x,y}$ respectively.

Denote the law of $x + Y^{r-x,x}$ by $\mathbb{Q}_x^r$. Just as in the compact case, use of the splitting property implies that both n^r and $\mathbb{Q}_x^r$ are Markovian:

$$n^r(g_1(X_{s_1}) \cdots g_n(X_{s_n}) F \circ \theta_{s_n}) = n^r \Big(g_1(X_{s_1}) \cdots g_n(X_{s_n}) \mathbb{Q}_{X_{s_n}}^r(F) \Big)$$

and

$$\mathbb{Q}_x^r(g_1(X_{s_1}) \cdots g_n(X_{s_n}) F \circ \theta_{s_n}) = \mathbb{Q}_x^r \Big(g_1(X_{s_1}) \cdots g_n(X_{s_n}) \mathbb{Q}_{X_{s_n}}^r(F) \Big).$$

Let $\mathbb{P}_x^r$ be the law of $x + Y^{r-x}$ (defined for $x \leq r$). The above Markov property for $\mathbb{Q}_x^r$ implies that $\mathbb{P}_x^r$ is also Markovian. Using Proposition 3.1 we see that if $r < r'$ then

the image of $n^{r'}$ under truncation at level r equals n^r. Hence, the image of $\mathbb{P}_x^{r'}$ under truncation at level r equals $\mathbb{P}_x^r$. If H_r^+ equals the hitting time of $[r, \infty)$ by the canonical process we then see that for any $A \in \mathscr{F}_t$

$$\mathbb{P}_x^r\big(A, H_r^+ > t\big) = \mathbb{P}_x^{r'}\big(A, H_r^+ > t\big)$$

which implies, through a projective limit theorem such as Theorem 3.2 in [Par05, Ch. V, p. 139], the existence of a probability measure $\mathbb{P}_x$, on càdlàg trajectories $f : [0, \infty) \to \mathbb{R} \cup \{\infty\}$ for which there exists $\zeta(f) \in [0, \infty]$ such that $f(t) = \infty$ if and only if $t \geq \zeta(f)$ and such that $\lim_{s \uparrow t} f(s) = \infty$ implies $\zeta(f) \leq t$, such that

$$\mathbb{P}_x\big(A, H_r^+ > t\big) = \lim_{r' \to \infty} \mathbb{P}_x^{r'}\big(A, H_r^+ > t\big) .$$

This law is then easily seen to be subMarkovian. To see that $\mathbb{P}_0$ is the law of a Lévy process, note that

$$\mathbb{P}_x^r(F(y + X)) = \mathbb{P}_{x+y}^{r-y}(F(X)) .$$

If F is $\mathscr{F}_t$-measurable and bounded and vanishes if $H_r^+ \leq t$, we can take limits as $r \to \infty$ to obtain

$$\mathbb{E}_x(F(y + X)) = \mathbb{E}_{x+y}(F(X))$$

which implies that the family of laws $(\mathbb{P}_x, x \in \mathbb{R})$ is spatially homogeneous and so $\mathbb{P}_0$ is the law of a (possibly killed) Lévy process. The Lévy process has to be spectrally positive since contour processes have only positive jumps. Let Ψ be its Laplace exponent. We now assert that $\mathbb{P}_x^r$ has the law of a Lévy process with Laplace exponent Ψ reflected below r, following the proof of Proposition 4.1. Indeed, recall that under $\mathbb{P}_x^r$ the canonical process is Markov. Also, if we stop the canonical process when it surpasses r, this stopped process has the same law under $\mathbb{P}_x^r$ and under $\mathbb{P}_x$ by construction. Finally, note that under $\mathbb{P}_x$, the level set at r $\{t \geq 0 : X_t = r\}$ has Lebesgue measure zero and by truncating at $r' > r$ and time changing, we see that the same holds under $\mathbb{P}_x^r$. Hence, $\mathbb{P}_x^r$ coincides with the reflection of $\mathbb{P}_x$ at level r. We conclude that $\eta = \eta^{\Psi}$. Note that Ψ is then supercritical since otherwise η^{Ψ} would be concentrated on compact TOM trees and η was assumed to charge non-compact trees. $\qquad\square$

A Preliminaries on TOM trees and a Proof of Theorem 1.3

The objective of this section is to collect some technical results on TOM trees in Subsection A.1 which will enable us to construct a coding function for them in Subsection A.2 and to prove Theorem 1.3.

A.1 Preliminary results

We start with an analysis of the greatest common ancestor operator denoted $\wedge$ in Section 1 (page 5). Recall that $\sigma_1 \wedge \sigma_2$ is characterized by

$$[\rho, \sigma_1] \cap [\rho, \sigma_2] = [\rho, \sigma_1 \wedge \sigma_2].$$

Since on the interval $[\sigma_1, \sigma_2]$ the distance d coincides with the usual distance on an interval, for every $\sigma \in [\sigma_1, \sigma_2]$ we have:

$$d(\sigma_1, \sigma_2) = d(\sigma_1, \sigma) + d(\sigma, \sigma_2). \tag{A.1}$$

As usual, the open ball of radius ε centered at $\sigma \in \tau$ is denoted $B_\varepsilon(\sigma)$.

Lemma A.1. *The $\wedge$ operator is bicontinuous.*

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Proof. Let $\sigma \in \tau$; for every $\varepsilon \in (0, d(\rho, \sigma))$, let σ^ε be the unique point on $[\rho, \sigma]$ at distance ε from σ, so that

$$(\sigma^\varepsilon, \sigma] \in B_\varepsilon(\sigma) \quad \text{and} \quad [\rho, \sigma^\varepsilon] \notin B_\varepsilon(\sigma).$$

Note that if $\tilde\sigma \in B_\varepsilon(\sigma)$, then, by considering the unique isometry to an interval, $[\sigma, \tilde\sigma] \subset B_\varepsilon(\sigma)$ and $\sigma^\varepsilon \prec \sigma \wedge \tilde\sigma$. Hence

$$[\rho, \sigma^\varepsilon] \subset [\rho, \tilde\sigma] \subset [\rho, \sigma^\varepsilon] \cup B_\varepsilon(\sigma). \tag{A.2}$$

There are 3 different geometries to consider to prove that $\wedge$ is continuous at σ_1, σ_2, using the fact that $\wedge$ is symmetric:

1. $\sigma_1 \wedge \sigma_2 \neq \sigma_1, \sigma_2$: In this case, we choose

$$0 < \varepsilon < d(\sigma_1, \sigma_1 \wedge \sigma_2) \wedge d(\sigma_1 \wedge \sigma_2, \sigma_2) \leq d(\sigma_1, \sigma_2)/2$$

where the last inequality holds by equation (A.1). Note then that

$$\sigma_1^\varepsilon \wedge \sigma_2^\varepsilon = \sigma_1 \wedge \sigma_2 \quad \text{and} \quad B_\varepsilon(\sigma_1) \cap B_\varepsilon(\sigma_2) = \emptyset$$

by choice of ε. Equation (A.2) implies that if $\tilde\sigma_i \in B_\varepsilon(\sigma_i)$ then

$$[\rho, \sigma_1 \wedge \sigma_2] \subset [\rho, \tilde\sigma_1] \cap [\rho, \tilde\sigma_2] \subset [\rho, \sigma_1 \wedge \sigma_2]$$

so that $\tilde\sigma_1 \wedge \tilde\sigma_2 = \sigma_1 \wedge \sigma_2$.

2. $\sigma_1 \wedge \sigma_2 = \sigma_1 = \sigma_2$: We write σ instead of σ_i and consider

$$0 < \varepsilon < d(\rho, \sigma)).$$

We then note that if $\tilde\sigma_1, \tilde\sigma_2 \in B_\varepsilon(\sigma)$ then, by Equation (A.2),

$$[\rho, \sigma^\varepsilon] \subset [\rho, \tilde\sigma_1] \cap [\rho, \tilde\sigma_2] \subset [\rho, \sigma^\varepsilon] \cup B_\varepsilon(\sigma),$$

so that

$$\tilde\sigma_1 \wedge \tilde\sigma_2 \in \{\sigma^\varepsilon\} \cup B_\varepsilon(\sigma) \subset B_{2\varepsilon}(\sigma).$$

3. $\sigma_1 \wedge \sigma_2 = \sigma_1 \prec \sigma_2$: We consider

$$0 < \varepsilon < \frac{1}{2} d(\rho, \sigma_1) \wedge d(\sigma_1, \sigma_2),$$

so that

$$B_\varepsilon(\sigma_1) \cap B_\varepsilon(\sigma_2) = \emptyset \quad \text{and} \quad \sigma_1^\varepsilon \prec \sigma_2^\varepsilon.$$

If $\tilde\sigma_i \in B_\varepsilon(\sigma_i)$ then Equation (A.2) gives

$$[\rho, \sigma_1^\varepsilon] \subset [\rho, \tilde\sigma_1] \cap [\rho, \tilde\sigma_2] \subset [\rho, \sigma_1^\varepsilon] \cup B_\varepsilon(\sigma_1)$$

so that $\tilde\sigma_1 \wedge \tilde\sigma_2 \in B_{2\varepsilon}(\sigma_1)$. $\qquad\square$

We tacitly assumed the sets involved in **Mes1** of the definition of a TOM tree are measurable. They are actually closed as we now show. For any $\sigma \in \tau$, define the **left, strict left, right, and strict right** of σ as follows:

$$L_\sigma = \{\tilde\sigma \in \tau : \tilde\sigma \leq \sigma\}, \qquad\qquad L_{\sigma-} = \{\tilde\sigma \in \tau : \tilde\sigma < \sigma\},$$
$$R_\sigma = \{\tilde\sigma \in \tau : \tilde\sigma \geq \sigma\}, \qquad\qquad R_{\sigma+} = \{\tilde\sigma \in \tau : \tilde\sigma > \sigma\}.$$

Lemma A.2. *For any $\sigma \in \tau$, the strict left of σ is open.*

Proof. Let $\sigma_1 < \sigma_2$ and consider $\delta = d(\sigma_1, \sigma_1 \wedge \sigma_2)$. Since $\sigma_1 < \sigma_2$, $\delta > 0$ since if $\sigma_1 \wedge \sigma_2$ were equal to one of the σ_i, it would have to be σ_2 thanks to **Or1**. We now show that $B_\delta(\sigma_1) \subset L_{\sigma_2-}$. Let $\sigma \in B_\delta(\sigma_1)$.

Note that $\sigma \wedge \sigma_1 \in B_\delta(\sigma_1)$ since

$$d(\sigma, \sigma \wedge \sigma_1) + d(\sigma \wedge \sigma_1, \sigma_1) = d(\sigma, \sigma_1) < \delta$$

Using the isometry ϕ_{ρ,σ_1}, we see that

$$\begin{aligned}
d(\rho, \sigma \wedge \sigma_1) &= d(\rho, \sigma_1) - d(\sigma_1, \sigma \wedge \sigma_1) \\
&= d(\rho, \sigma_1 \wedge \sigma_2) + d(\sigma_1 \wedge \sigma_2, \sigma_1) - d(\sigma_1, \sigma \wedge \sigma_1) \\
&= d(\rho, \sigma_1 \wedge \sigma_2) + \delta - d(\sigma_1, \sigma \wedge \sigma_1) \\
&> d(\rho, \sigma_1 \wedge \sigma_2)
\end{aligned}$$

we obtain

$$\sigma_1 \wedge \sigma \in (\sigma_1 \wedge \sigma_2, \sigma_1];$$

property **Or2** then implies $\sigma_1 \wedge \sigma < \sigma_2$ and then, using **Or1**, we get

$$\sigma \le \sigma \wedge \sigma_1 < \sigma_2. \qquad \square$$

Note however that the strict right is in general not open. Indeed, if $\sigma_1 < \sigma_2$ and $\sigma_1, \sigma_2 \neq \sigma_1 \wedge \sigma_2$, then $\sigma_1 \wedge \sigma_2 \in R_{\sigma_2+}$ but every neighborhood of $\sigma_1 \wedge \sigma_2$ intersects $[\sigma_1, \sigma_1 \wedge \sigma_2) \subset L_{\sigma_2}$. (In the context of the above proof, δ could be 0.)

The previous lemma ensures that right sets are closed, as complementary sets of strict left sets. To prove that left (and so strict right) sets are measurable, we use Lemma A.2 in conjunction with Lemma A.3 below.

Consider the set $\tau_\sigma = \{\sigma' \in \tau : \sigma \preceq \sigma'\}$.

Lemma A.3. *If $\sigma_1 \le \sigma_2 \le \sigma_3$ then $\sigma_1 \wedge \sigma_2 \in [\sigma_1, \sigma_1 \wedge \sigma_3]$ and $\sigma_2 \in \tau_{\sigma_1 \wedge \sigma_3}$.*

Proof. It suffices to prove the result when inequalities are strict. Note that the assumption $\sigma_1 \wedge \sigma_2 \in [\rho, \sigma_1 \wedge \sigma_3)$ leaves us with the impossible cases

$\sigma_2 > \sigma_1 \wedge \sigma_3$ since $\sigma_1 \wedge \sigma_3 \ge \sigma_3$ by **Or1** and $\sigma_3 > \sigma_2$ and

$\sigma_2 \le \sigma_1 \wedge \sigma_3$ since **Or2** and $\sigma_1 < \sigma_2$ imply $\sigma_1 \wedge \sigma_3 \in [\sigma_1, \sigma_1 \wedge \sigma_2) < \sigma_2$ which implies $\sigma_1 \wedge \sigma_3 < \sigma_2$.

Finally, since $\sigma_1 \wedge \sigma_3 \preceq \sigma_1 \wedge \sigma_2 \preceq \sigma_2$, we see that $\sigma_2 \in \tau_{\sigma_1 \wedge \sigma_3}$. $\qquad \square$

To prove that L_σ is measurable, note first that this is true when $\sigma = \rho$. When $\sigma \neq \rho$, choose $\sigma_n \in [\rho, \sigma)$ converging to σ. We assert that

$$L_\sigma = \cap_n L_{\sigma_n-},$$

where the right-hand side is measurable by Lemma A.2. Indeed, $\tilde\sigma \le \sigma$ implies $\tilde\sigma < \sigma_n$ for all n. On the other hand, if $\tilde\sigma < \sigma_n$ for all n and $\sigma < \tilde\sigma$, then Lemma A.3 gives $\tilde\sigma \in \tau_{\sigma \wedge \sigma_n}$. However, Lemma A.1 tells us that $\sigma \wedge \sigma_n \to \sigma$, so that $\tilde\sigma \in \tau_\sigma$ which contradicts $\sigma < \tilde\sigma$.

We now give a simple sufficient condition for a sequence to converge.

Lemma A.4. *If τ is compact, any $\le$-monotonic sequence on τ converges.*

Proof. Let $\sigma_1, \sigma_2, \ldots$ a $\le$-monotonic sequence. Since τ is compact, we must only show that all convergent subsequences have the same limit.

Suppose there exist two subsequences $\sigma_{n_k^1}$ and $\sigma_{n_k^2}$ converging to σ^1 and σ^2 where $\sigma^1 < \sigma^2$. Since $\sigma^1 < \sigma^2$, it follows that $\sigma^1 \neq \sigma^1 \wedge \sigma^2$, so that we can choose $0 < \delta < d(\sigma^1, \sigma^1 \wedge \sigma^2)$.

Let $\sigma^{1,\delta}$ be the element of $[\sigma^1, \rho]$ at distance δ from σ^1; in terms of isometries between intervals and paths on the tree, $\sigma^{1,\delta} = \phi_{\sigma^1, \rho}(\delta)$. Note that for $\sigma, \tilde{\sigma} \in B_\delta(\sigma^1)$, since $\sigma \wedge \sigma^1 \in B_\delta(\sigma^1)$ and similarly for $\tilde{\sigma}$, then

$$[\rho, \sigma^{1,\delta}] \subsetneq [\rho, \sigma \wedge \sigma_1] \cap [\rho, \tilde{\sigma} \wedge \sigma_1] \subset [\rho, \sigma] \cap [\rho, \tilde{\sigma}] = [\rho, \sigma \wedge \tilde{\sigma}]$$

and so $\sigma^{1,\delta} \prec \sigma \wedge \tilde{\sigma}$.

Let $K \geq 1$ be such that for $k \geq K$, $d\left(\sigma_{n_k^1}, \sigma^1\right) < \delta$. For $n \geq n_K^1$ and k large enough (depending on n)

$$\sigma_{n_K^1} \leq \sigma_n \leq \sigma_{n_k^1} \quad \text{or} \quad \sigma_{n_K^1} \geq \sigma_n \geq \sigma_{n_k^1}$$

Lemma A.3 implies

$$\sigma_n \in \tau_{\sigma_{n_K^1} \wedge \sigma_{n_k^1}} \subset \tau_{\sigma^{1,\delta}},$$

the last inclusion holding because of the preceding paragraph.

Hence $\sigma_1 \wedge \sigma_2 \prec \phi_{\sigma^1, \rho}(\delta) \prec \sigma_n$ so that

$$d\left(\sigma_n, \sigma^2\right) \geq d\left(\sigma^{1,\delta}, \sigma_1 \wedge \sigma_2\right) > 0$$

and so $\sigma_{n_k^2}$ cannot converge to σ^2. $\qquad \square$

In order to characterize measures on TOM trees, note that the set

$$\mathscr{L} = \{L_\sigma : \sigma \in \tau\} \cup \{\emptyset\}$$

is a π-system (since $\leq$ is a total order). Let us define $\mathscr{A} = \sigma(\mathscr{L})$.

Lemma A.5. *The set $\mathscr{L}$ generates the Borel subsets of τ.*

Proof. Note that $\overline{B_\delta(\sigma)} = \{\tilde{\sigma} : d(\tilde{\sigma}, \sigma) \leq \delta\}$ and denote the latter set by C. If suffices to show that $C \in \mathscr{A}$ (for every $\sigma \in \tau$ and $\delta > 0$). Since $\tau \setminus C$ is open, it is the union of a countable set of open components, say $C_1, C_2, \ldots$. Note that real trees are locally path connected (one way to see this is because closed balls of a real tree are real trees themselves and another one is to prove directly that if $\sigma_1, \sigma_2 \in B_\delta(\sigma)$ then $[\sigma_1, \sigma_2] \subset B_\delta(\sigma)$). Hence the notions of path-wise connectedness and connectedness coincide for real trees.

We assert that if $\sigma_1, \sigma_2 \in C_i$ and $\sigma_1 \leq \tilde{\sigma} \leq \sigma_2$ then $\tilde{\sigma} \in C_i$. Indeed, note first that $[\sigma_1, \sigma_2] \subset C_i$: this follows since $[\sigma_1, \sigma_2] \subset \tau \setminus C$ (or σ_1 and σ_2 would belong to different path-connected components of $\tau \setminus C$) and then since $[\sigma_1, \sigma_2]$ is connected and has non-empty intersection with C_i, it follows that $C_i \cup [\sigma_1, \sigma_2]$ is connected and by definition of connected component, $[\sigma_1, \sigma_2] \subset C_i$. If $\sigma_1 < \tilde{\sigma} < \sigma_2$, Lemma A.3 implies $\tilde{\sigma} \wedge \sigma_1 \in [\sigma_1, \sigma_1 \wedge \sigma_2]$. Hence $C_i \cup [\sigma_1 \wedge \tilde{\sigma}, \tilde{\sigma}]$ is connected so that $[\sigma_1 \wedge \tilde{\sigma}, \tilde{\sigma}] \subset C_i$ and $\tilde{\sigma} \in C_i$.

Let

$$L_i = \{L_{\tilde{\sigma}} : \tilde{\sigma} \in C_i\}, \quad I_i = \inf L_i \quad \text{and} \quad S_i = \sup L_i.$$

If $I_i \in L_i$, there exists a $\leq$-smallest element $\underline{\sigma}$ of C_i, and likewise if $S_i \in L_i$ there exists a $\leq$-largest element $\overline{\sigma}$ of C_i which would imply

$$C_i = L_{\overline{\sigma}} \setminus L_{\underline{\sigma}-} \in \mathscr{A}.$$

If $I_i \notin L_i$ but $S_i \in L_i$, consider a decreasing sequence $\underline{\sigma}_n$ such that $\mu\left(L_{\underline{\sigma}_n}\right) \to I_i$ and note that in this case

$$C_i = \bigcup_n L_{\overline{\sigma}} \setminus L_{\sigma_n} \in \mathscr{A}.$$

Remaining cases are handled similarly, which shows that

$$\tau \setminus C = \bigcup C_i \in \mathscr{A},$$
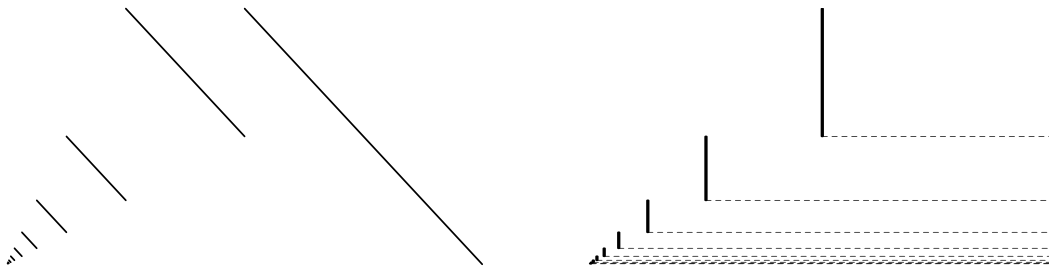
which terminates the proof. $\qquad \square$

Figure 5: A càdlàg function coding a TOM tree without a first element (visualized on the right through vertical segments joined via dashed lines).

Hence, any measure on the Borel sets of τ is determined by its values on $\mathscr{L}$.

A.2 The contour process of a compact TOM tree

The reason for introducing compact TOM trees is that they provide us with a simple way to code their elements by real numbers. This is formally done through exploration process and the height function now defined. First, define $\psi : \tau \to [0, \mu(\tau)]$ by

$$\psi(\sigma) = \mu(L_\sigma),$$

where we recall that L_σ is the left of σ. Note that ψ is strictly increasing thanks to **Mes**.

Lemma A.6. *The set $D = \psi(\tau)$ is dense in $[0, \mu(\tau)]$.*

Remark A.7. If a tree does not have a first element, then $0 \notin D$; an actual example can be seen in Figure 5.

Proof. Let $t \in (0, \mu(\tau))$. Set

$$G_t = \{\sigma \in \tau : \psi(\sigma) < t\} \text{ and } D_t = \{\sigma \in \tau : \psi(\sigma) \geq t\}.$$

Also set

$$s_t = \sup \{\psi(\sigma) : \sigma \in G_t\} \quad \text{and} \quad i_t = \inf \{\psi(\sigma) : \sigma \in D_t\},$$

so that in particular $s_t \leq t \leq i_t$.

First, notice that for any $\sigma \in G_t$ we have the inclusion $L_\sigma \subset G_t$, so that G_t is necessarily of the form L_σ or $L_\sigma \setminus \{\sigma\}$, which yields $\mu(G_t) = s_t$.

Now, by definition of i_t, there is some $\leq$-decreasing sequence (σ_n) of elements of D_t such that $\mu(\sigma_n) \downarrow i_t$. Since (σ_n) is decreasing, the sequence L_{σ_n} is also decreasing; let L denote its limit. If there were two elements in $L \setminus G_t$, say $\sigma_1 < \sigma_2$, we would have $t \leq \psi(\sigma_1) < \psi(\sigma_2) \leq i_t$ by **Mes1**. Now this contradicts the definition of i_t since $L \setminus G_t \subset D_t$, so $L \setminus G_t$ contains at most one element. By **Mes2**, this shows that $\mu(L) = \mu(G_t)$. Now recall that $\mu(G_t) = s_t$, so that

$$i_t = \lim_n \psi(\sigma_n) = \lim_n \mu(L_{\sigma_n}) = \mu(L) = \mu(G_t) = s_t,$$

which shows that $i_t = s_t = t$. $\qquad\square$

We can now define the **exploration process** $\phi : [0, \mu(\tau)] \to \tau$ by means of

$$\phi(t) = \lim_{\substack{s \to t+ \\ s \in D}} \psi^{-1}(s);$$

notice that $\psi^{-1}(s)$ is $\leq$-decreasing if s decreases. Therefore, if $s_n \downarrow t$ along D, then $\psi^{-1}(s_n)$ converges thanks to Lemma A.4. To see that the limit does not depend on the subsequence, notice that if s_n and $\tilde{s}_n$ both decrease to t along D, they can be intertwined into a decreasing sequence which contains them both as subsequences.

Lemma A.8. *The exploration process is the unique càdlàg extension of ψ^{-1} to $[0, \mu(\tau)]$. Furthermore, it satisfies the following property: for any $t_1 \in [0, \mu(\tau)]$,*

$$\phi([t_1, \psi(\phi(t_1))]) \subset \tau_{\phi(t_1)}.$$

Proof. Note first that $\phi = \psi^{-1}$ on D. Indeed, if $t \in D$, say $t = \mu(L_\sigma)$, and $t_n \in D$ decreases to t, we let $\sigma_n = \psi^{-1}(t_n) \geq \sigma$. Since σ_n is decreasing it converges thanks to Lemma A.4. Call the limit $\tilde{\sigma}$. The inequality $\tilde{\sigma} < \sigma$ is impossible since $\tilde{\sigma} < \sigma \leq \sigma_n$ and Lemma A.2 imply that $\sigma_n \not\to \tilde{\sigma}$. Bu the same argument, $\tilde{\sigma} \leq \sigma_n$ for all n. Then, as in A.6, we note that $\bigcap_n L_{\sigma_n} \setminus L_{\tilde{\sigma}}$ has at most one element, since if it contains an element $\tilde{\tilde{\sigma}} > \tilde{\sigma}$, then $\sigma_n \not\to \sigma$. We conclude that $\mu(L_{\sigma_n}) \to \mu(L_\sigma)$. Then, the inequality $\sigma < \tilde{\sigma}$ implies the contradiction

$$t = \lim_n t_n = \lim_n \mu(L_{\sigma_n}) = \mu(L_{\tilde{\sigma}}) > \mu(L_\sigma) = t$$

by use of **Mes1**, so that σ_n decreases to σ and so $\phi(t) = \psi^{-1}(t)$.

We now prove that ϕ is right-continuous. Let $t < \mu(\tau)$ and $t_n \downarrow t$ with $t_n > t$. Then there exists $\tilde{t}_n \in D$ such that

$$t < t_n < \tilde{t}_n, \quad \tilde{t}_n - t_n \to 0 \quad \text{and} \quad d\big(\phi(t_n), \psi^{-1}(\tilde{t}_n)\big) \to 0.$$

By monotonicity of ψ^{-1} on D, we see that

$$\phi(t) = \lim_{n \to \infty} \psi^{-1}(\tilde{t}_n) = \lim_{n \to \infty} \phi(t_n).$$

A similar argument shows that ϕ has left limits: note first that by monotonicity the limit

$$\lim_{\substack{s \to t- \\ s \in D}} \psi^{-1}(s)$$

exists. If $t_n \uparrow t$ with $t_n < t$, there exist $\tilde{t}_n \in D$ such that

$$t_n < \tilde{t}_n < t, \quad \tilde{t}_n - t_n \to 0 \quad \text{and} \quad d\big(\phi(t_n), \psi^{-1}(\tilde{t}_n)\big) \to 0.$$

By monotonicity we see that

$$\lim_{\substack{s \to t- \\ s \in D}} \psi^{-1}(s) = \lim_n \psi^{-1}(t_n),$$

so that

$$\phi(t-) = \lim_{\substack{s \to t- \\ s \in D}} \psi^{-1}(s).$$

Uniqueness of the extension follows from density of D.

We finally prove that

$$\phi\big([t_1, \mu\big(L_{\phi(t_1)}\big)]\big) \subset \tau_{\phi(t_1)}.$$

Let us then consider

$$t_1 < t < \mu\big(L_{\phi(t_1)}\big).$$

Take $t_1^n \in D$ decreasing to t_1 and $t^n \in D$ decreasing to t and let $\sigma_1^n = \phi(t_1^n)$ and $\sigma^n = \phi(t^n)$. Then, by definition, $\sigma_1^n \to \phi(t_1)$ and $\sigma^n \to \phi(t)$. Since ψ^{-1} is order preserving and coincides with ϕ on D, we see that

$$\sigma_1^n \leq \sigma^n \leq \phi(t_1).$$

Lemma A.3 implies

$$\sigma^n \wedge \sigma_1^n \in [\sigma_1^n, \sigma_1^n \wedge \phi(t_1)].$$

By passing to the limit in the expression

$$\sigma_1^n \wedge \phi(t_1) \preceq \sigma_1^n \wedge \sigma^n,$$

using Lemma A.1, we get

$$\phi(t_1) \preceq \phi(t_1) \wedge \phi(t),$$

which ends the proof. $\qquad \square$

We now define the **contour process** $f : [0, \mu(\tau)] \to \tau$ of $\mathbf{c} = ((\tau, d, \rho), \le, \mu)$ by

$$f_t = d(\rho, \phi(t)).$$

Proposition A.9. *The contour process f is càdlàg with no negative jumps and the compact TOM tree coded by f is isomorphic to $\mathbf{c}$.*

Proof. While it is clear that f is càdlàg, it is less evident that its jumps are non-negative. However, if $f(t) < f(t-)$ then $\phi(t-) \notin [\rho, \phi(t)]$. We obtain a contradiction by analyzing the cases $\phi(t-) < \phi(t)$ or $\phi(t) < \phi(t-)$. In the first case, any element of $(\phi(t-), \phi(t-) \wedge \phi(t))$ has a measure exceeding t and is $< \phi(t)$. Indeed let $\sigma \in (\phi(t-), \phi(t-) \wedge \phi(t))$ and suppose that $t_n \in D$ increase to t so that $\phi(t_n) \to \phi(t-)$. By Lemma A.1

$$\phi(t_n) \wedge \phi(t-) \to \phi(t-)$$

and so for large enough n, $\phi(t_n) \le \phi(t_n) \wedge \phi(t-) < \sigma$ so that

$$t_n = \mu\big(L_{\phi(t_n)}\big) < \mu(L_\sigma).$$

In the second case, for any $\sigma \in L_{\phi(t)}$, we have the bound

$$d(\phi(t-), \sigma) \ge d(\phi(t-), \phi(t) \wedge \phi(t-))$$

(since σ cannot belong to $[\phi(t) \wedge \phi(t-), \phi(t-)]$). However, $\phi(t_n) \in L_{\phi(t)}$, which contradicts the fact that $\phi(t_n) \to \phi(t-)$.

Let us now analyze the tree coded by f. Recall that if we define the pseudo-distance

$$d_f(t_1, t_2) = f(t_1) + f(t_2) - 2 \inf_{t \in [t_1, t_2]} f(t),$$

then the tree coded by f is the quotient space of $[0, \mu(\tau)]$ under the equivalence relation $\sim_f$ given by $t_1 \sim_f t_2$ if $d_f(t_1, t_2) = 0$.

We first prove that ϕ is constant on the equivalence classes of $\sim_f$. Indeed, consider $t_1 < t_2$ such that $\phi(t_1) \ne \phi(t_2)$ and let us prove that $t_1 \not\sim_f t_2$. We have 3 cases:

$\phi(t_1) \prec \phi(t_2)$ Then $f(t_1) < f(t_2)$ and so $t_1 \not\sim_f t_2$.

$\phi(t_2) \prec \phi(t_1)$ Then $f(t_2) < f(t_1)$ and so $t_1 \not\sim_f t_2$.

$\phi(t_1) \wedge \phi(t_2) \ne \phi(t_1), \phi(t_2)$ Note that $t_1 < t_2 \le \mu\big(L_{\phi(t_2)}\big)$. We now use Lemma A.8 and the fact that $\phi(t_2) \notin \tau_{\phi(t_1)}$ to deduce

$$t_1 \le \mu\big(L_{\phi(t_1)}\big) < t_2 \le \mu\big(L_{\phi(t_2)}\big).$$

Hence $\phi(t_1) < \phi(t_2)$. Let $\sigma \in (\phi(t_1), \phi(t_1) \wedge \phi(t_2))$. Then

$$t_1 \le \mu\big(L_{\phi(t_1)}\big) < \mu(L_\sigma) < \mu\big(L_{\phi(t_2)}\big).$$

Also, since $\sigma \notin \tau_{\phi(t_2)}$, then, actually, $\mu(L_\sigma) < t_2$. We then have:

$$\mu(L_\sigma) \in (t_1, t_2) \quad \text{and} \quad f(\mu(L_\sigma)) = d(\rho, \sigma) < d(\rho, \phi(t_1)) = f(t_1),$$

so that $t_1 \not\sim_f t_2$.

Abusing notation, we can then define ϕ on the equivalence class $[t]_f$ of t under $\sim_f$ as $\phi(t)$. ϕ is a bijection from the set τ_f of equivalence classes under $\sim_f$ to τ: it is surjective because ϕ is surjective on $[0, \mu(\tau)]$, since

$$\phi(\psi(\sigma)) = \sigma,$$

and it is injective thanks to the following argument. If $\phi(t_1) = \phi(t_2)$ and $t_1 < t_2$, then

$$f(t_1) = f(t_2) \quad \text{and} \quad t_2 \leq \mu\big(L_{\phi(t_2)}\big) = \mu\big(L_{\phi(t_1)}\big).$$

By Lemma A.8, we see that $\phi([t_1, t_2]) \subset \tau_{\phi(t_1)}$, so that for every $t \in [t_1, t_2]$, $f(t) \geq f(t_1)$. Hence, $t_1 \sim_f t_2$.

It remains to see that ϕ is an isometry between τ_f and τ which preserves root, order and measure.

Since $\tau = L_\rho$, we see that $\rho = \phi(\mu(\tau))$ and that f reaches its minimum (which is zero) at $\mu(\tau)$. Hence, $\rho_f = [\mu(\tau)]_f$ and $\phi(\rho_f) = \rho$. This implies that for all $t \in \tau_f$:

$$d(\phi(t), \rho) = f(t) = d_f(t, \mu(\tau)). \tag{A.3}$$

The bijection $\phi : \tau_f \to \tau$ is an isometry by the following observations: firstly, on every compact rooted real tree distances between every pair of elements is determined by distances to the root, thanks to the formula

$$d(\sigma_1, \sigma_2) = d(\sigma_1, \rho) + d(\sigma_2, \rho) - 2d(\sigma_1 \wedge \sigma_2, \rho).$$

Hence, if $\tilde{d}$ is a metric on τ and $(\tau, \tilde{d}, \rho)$ is a real tree, then $\tilde{d}(\sigma, \rho) = d(\sigma, \rho)$ implies that $d = \tilde{d}$. Equation (A.3) then proves that ϕ is an isometry.

To see that ϕ preserves order, note first that $\sup[t]_f = \mu(L_{\phi(t)})$ for every t. Indeed, if $\mu(L_{\phi(t)}) < \tilde{t}$, since $\tilde{t} \leq \mu(L_{\phi(\tilde{t})})$, we see that

$$\mu\big(L_{\phi(t)}\big) < \mu\big(L_{\phi(\tilde{t})}\big)$$

so that $\phi(t) < \phi(\tilde{t})$ and $\tilde{t} \notin [t]_f$. Hence, if t_1, t_2 are such that $\sup[t_1]_f < \sup[t_2]_f$ then $\mu(L_{\phi(t_1)}) < \mu(L_{\phi(t_2)})$ by **Mes1**, so that $\phi(t_1) < \phi(t_2)$.

Finally, to see that ϕ is measure preserving, recall that p_f stands for the projection of $[0, \mu(\tau)]$ into τ_f (sending t to its equivalence class) defined on page 6 and that $\mu_f = \text{Leb} \circ p_f^{-1}$ by definition. We need to prove the equality $\mu = \text{Leb} \circ p_f^{-1} \circ \phi^{-1}$ (where, due to our abuse of notation, ϕ is defined on τ_f). Note first that $\phi \circ p_f$ is the exploration process (also denoted ϕ). Then, by Lemma A.5, it suffices to prove that $\mu = \text{Leb} \circ \phi^{-1}$ on $\mathscr{L}$. However, $\{t : \phi(t) \leq \sigma\} = [0, \mu(L_\sigma)]$ since $\phi(t) = \phi\big(\mu(L_{\phi(t)})\big)$. Hence $\mu = \text{Leb} \circ \phi^{-1}$ on $\mathscr{L}$. □

We can finally turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. Existence of a coding function for the tree follows from Proposition A.9.

However, if two trees $\mathbf{c}_1$ and $\mathbf{c}_2$ have the same coding function f, then Proposition A.9 also implies that $\mathbf{c}_1$ and $\mathbf{c}_2$ are isomorphic to $\mathbf{c^f}$, so that they are isomorphic to each other. □

A.3 Elementary operations on compact TOM trees

Consider a compact TOM tree $\mathbf{c} = ((\tau, d, \rho), \leq, \mu)$. The objective of this section is to consider two operations on $\mathbf{c}$ which allow one to construct another TOM tree and we explore the relationships between the contour processes. Consider any element σ of τ.

Definition A.10. *We define the **subtree to the right of** σ, denoted $\mathbf{c}_{\geq\sigma}$, equal to R_σ with the same root as $\mathbf{c}$, and with the distance, order and measure obtained by restricting those of $\mathbf{c}$ to R_σ.*

*We define the **subtree rooted at** σ, denoted $\mathbf{c}_{\succeq\sigma}$, equal to $\tau_\sigma = \{\sigma' \in \tau : \sigma \preceq \sigma'\}$ with root σ, with distance, total order and measure restricted to τ_σ.*

Proposition A.11. *For any compact TOM tree $\mathbf{c}$ and any $\sigma \in \tau$, $\mathbf{c}_{\geq\sigma}$ and $\mathbf{c}_{\succeq\sigma}$ are compact TOM trees.*

The contour process of $\mathbf{c}_{\geq\sigma}$ *If $f : [0, m] \to [0, \infty)$ is the contour process of $\mathbf{c}$ and $s^* = \mu(L_\sigma)$ then the contour process $f_{\geq\sigma}$ of $\mathbf{c}_{\geq\sigma}$ is constructed as follows:*

$$f_{\geq\sigma}(t) = f(s^* + t) \quad \text{defined on } [0, m - s^*].$$

Furthermore, if $\phi(s) = \sigma$ then

$$s^* = \inf\{t \geq s : f(t) < f(s)\} = \sup[s]_f.$$

The contour process of $\mathbf{c}_{\succeq\sigma}$ *The contour process $f_{\succeq\sigma}$ of $\mathbf{c}_{\succeq\sigma}$ is defined on $[0, s^* - s_*]$ where*

$$s^* = \mu(L_\sigma) \text{ and } s_* = \inf\{\mu(L_{\tilde\sigma}) : \tilde\sigma \in \tau_\sigma\}.$$

We have the explicit formula

$$f_{\succeq\sigma}(t) = f(t + s_*) - f(s^*) \quad \text{for } t \in [0, s^* - s_*].$$

Furthermore, if $\phi(s) = \sigma$ then

$$s^* = \inf\{t \geq s : f(t) < f(s)\} = \sup[s]_f$$

and

$$s_* = \sup\{t \leq s : f(t) < f(s)\} = \inf[s]_f.$$

Proof. Note that $\rho \in R_\sigma$. We have seen in Lemma A.2 that R_σ is closed. If $\sigma \leq \sigma_1 \leq \sigma_2$ then $[\sigma_1, \sigma_2] \subset R_\sigma$, so that R_σ is a closed and pathwise connected subset of a compact real tree. Hence R_σ, with the induced distance is a compact real tree. The total order, when restricted to R_σ, satisfies properties **Or1** and **Or2**. Restriction to R_σ does not alter **Mes1** or **Mes2**. This implies that $\mathbf{c}_{\geq\sigma}$ is a compact TOM tree. To find its contour process $f_{\geq\sigma}$, note that if $\sigma' \in R_\sigma$ then

$$\mu(L_{\sigma'} \cap R_\sigma) = \mu(L_{\sigma'}) - \mu(L_\sigma) = \mu(L_{\sigma'}) - s^*.$$

Hence, the exploration process $\phi_{\geq\sigma}$ of $\mathbf{c}_{\geq\sigma}$ is given by

$$\phi_{\geq\sigma}(s) = \phi(s^* + s)$$

and we deduce the equality $f_{\geq\sigma}(s) = f(s^* + s)$.

It remains to prove that if $\phi(s) = \sigma$ (or in other words, s is a representative of σ when considering $\mathbf{c}$ as the tree coded by f) then

$$\mu(L_\sigma) = \inf\{t \geq s : f(t) < f(s)\}.$$

By Lemma A.8, between s and $\mu(L_\sigma)$, ϕ explores τ_σ, so that $f \geq f(s)$ on $[s, \mu(L_\sigma)]$ which implies the inequality $\mu(L_\sigma) \leq s^*$. On the other hand, if $s' \in D$ belongs to a right

neighborhood of $\mu(L_\sigma)$ then $\sigma < \phi(s')$ which implies that $\sigma \wedge \phi(s') \in (\sigma, \rho]$ which in turns tells us that $m_f(\mu(L_\sigma), s') < f(s) = f(s^*)$. Hence $s^* \leq s'$ and since s' is an arbitrary element of D and any right neighborhood of $\mu(L_\sigma)$, we obtain $s^* \leq \mu(L_\sigma)$ and so in fact $s^* = \mu(L_\sigma)$.

The last argument is also valid in the setting of $\mathbf{c}_{\succeq\sigma}$ so that $s^* = \inf\{t \geq s : f(t) < f(s)\}$. Actually, we have the equality

$$\psi(\tau_\sigma) = [s_*, s^*] \cap D :$$

since if $\sigma \preceq \sigma'$ and $\phi(s') = \sigma'$ then $m_f(s', s^*) = f(s^*)$, so that $\psi(\tau_\sigma) \subset [s_*, s^*] \subset D$ and on the other hand, if $s' \in [s_*, s^*] \cap D$ with $s' = \mu(L_{\sigma'})$ then $m_f(s', s^*) \geq f(s^*)$ by definition of s_* and so $\sigma' \in \tau_\sigma$. We now take s_n decreasing to s_* with $s_n = \psi(\sigma_n)$ through D and note that

$$\tau_\sigma = \bigcup_n L_\sigma \setminus L_{\sigma_n}.$$

This implies

$$\mu(L_{\sigma'} \cap \tau_\sigma) = \lim_{n\to\infty} \mu(L_{\sigma'}) - \mu(L_{\sigma_n}) = \psi(\sigma') - s_*.$$

for any $\sigma' \in \tau_\sigma$.

We therefore obtain the following relationship between the exloration process $\phi_{\succeq\sigma}$ of τ_σ and ϕ:

$$\phi_{\succeq\sigma}(s) = \phi(s + s_*).$$

Finally, we note that for any $\sigma' \in \tau_\sigma$

$$d(\sigma', \sigma) = d(\sigma', \rho) - d(\sigma', \sigma)$$

to conclude that

$$f_{\succeq\sigma}(s) = f(s) - f(s_*)$$

for $s \leq s^* - s_*$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A.4 Topological remarks about the space of compact TOM trees

Given that compact TOM trees can be identified with a subset of càdlàg functions, thanks to Theorem 1.3, we see that TOM trees constitute a set (without the need of passing to isometry classes). Also, we can define the distance between compact TOM trees $\mathbf{c}_1$ and $\mathbf{c}_2$ in terms of the distance of their contours f_1 and f_2 as follows. Suppose that the supports of f_i is $[0, m_i]$ and that $m_1 < m_2$, say. First extend f_1 to $[0, m_2]$ by declaring it constant on $[m_1, m_2]$. We then define

$$d(\mathbf{c_1}, \mathbf{c}_2) = d_{m_2}(f_1, f_2) + |m_2 - m_1|,$$

where d_{m_2} is the Skorohod J_1 distance on $[0, m_2]$ defined as

$$d_{m_2}(f_1, f_2) = \sup_\lambda \sup_{s \leq m_2} |f_1(s) - f_2 \circ \lambda(s)|$$

and λ runs over all strictly increasing continuous functions of $[0, m_2]$ into itself. By uniqueness of the contour in Theorem 1.3, we see that d is a metric on compact TOM trees. If $\mathbf{c_n} = ((\tau_n, d_n, \rho_n), \leq_n, \mu_n)$ is a sequence of compact TOM trees converging to $\mathbf{c}$ under the metric d, then, a slight generalization of Proposition 2.10 of [ADH14] (to cover càdlàg functions as done in [Lam17]) shows us that the rooted measured metric spaces $((\tau_n, d_n, \rho_n), \mu_n)$ converge in the Gromov-Hausdorff-Prokhorov topology associated to the metric d_{GHP} defined as follows:

$$d_{GHP}(((\tau_1, d_1, \rho_1), \mu_1), ((\tau_n, d_n, \rho_n), \mu_n))$$
$$= \inf_{\phi_1, \phi_2, \tau} \left[d_H^\tau(\phi_1(\tau_1), \phi_2(\tau_2)) + d^\tau(\phi_1(\rho_1), \phi_2(\rho_2)) + d_P^\tau(\mu_1 \circ \phi_1^{-1}, \mu_2 \circ \phi_2^{-1}) \right]$$

where the infimum runs over all isometric embeddings ϕ_i from τ_i into a common metric space τ, d_H^τ denotes the Hausdorff distance between subsets of τ, and d_P^τ stands for the Prokhorov distance between finite measures on τ.

References

[AD12]   Romain Abraham and Jean-François Delmas, *A continuum-tree-valued Markov process*, Ann. Probab. **40** (2012), no. 3, 1167–1211. MR-2962090

[ADH14]   Romain Abraham, Jean-François Delmas, and Patrick Hoscheit, *Exit times for an increasing Lévy tree-valued process*, Probab. Theory Related Fields **159** (2014), no. 1-2, 357–403. MR-3201925

[Ald91a] David Aldous, *The continuum random tree. I*, Ann. Probab. **19** (1991), no. 1, 1–28. MR-1085326

[Ald91b] David Aldous, *The continuum random tree. II. An overview*, Stochastic analysis (Durham, 1990), London Math. Soc. Lecture Note Ser., vol. 167, Cambridge Univ. Press, Cambridge, 1991, pp. 23–70. MR-1166406

[Ald93]   David Aldous, *The continuum random tree. III*, Ann. Probab. **21** (1993), no. 1, 248–289. MR-1207226

[BBI01]   Dmitri Burago, Yuri Burago, and Sergei Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001. MR-1835418

[Ber91]   Jean Bertoin, *Sur la décomposition de la trajectoire d'un processus de Lévy spectralement positif en son infimum*, Ann. Inst. H. Poincaré Probab. Statist. **27** (1991), no. 4, 537–547. MR-1141246

[Ber96]   Jean Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996. MR-1406564

[Ber99]   Jean Bertoin, *Subordinators: examples and applications*, Lecture Notes in Math., vol. 1717, pp. 1–91, Springer, Berlin, 1999. MR-1746300

[BFM08]   Jean Bertoin, Joaquin Fontbona, and Servet Martínez, *On prolific individuals in a supercritical continuous-state branching process*, J. Appl. Probab. **45** (2008), no. 3, 714–726. MR-2455180

[BKMS11] J. Berestycki, A. E. Kyprianou, and A. Murillo-Salas, *The prolific backbone for supercritical superprocesses*, Stochastic Process. Appl. **121** (2011), no. 6, 1315–1331. MR-2794978

[BO18]   Gabriel Hernán Berzunza Ojeda, *On scaling limits of multitype Galton-Watson trees with possibly infinite variance*, ALEA Lat. Am. J. Probab. Math. Stat. **15** (2018), no. 1, 21–48. MR-3748121

[Cha13]   L. Chaumont, *On the law of the supremum of Lévy processes*, Ann. Probab. **41** (2013), no. 3A, 1191–1217. MR-3098676

[Del08]   J.-F. Delmas, *Height process for super-critical continuous state branching process*, Markov Process. Related Fields **14** (2008), no. 2, 309–326. MR-2437534

[DLG02]   Thomas Duquesne and Jean-François Le Gall, *Random trees, Lévy processes and spatial branching processes*, Astérisque (2002), no. 281, vi+147. MR-1954248

[DLG05]   Thomas Duquesne and Jean-François Le Gall, *Probabilistic and fractal aspects of Lévy trees*, Probab. Theory Related Fields **131** (2005), no. 4, 553–603. MR-2147221

[Don07]   Ronald A. Doney, *Fluctuation theory for Lévy processes*, Lecture Notes in Mathematics, vol. 1897, Springer, Berlin, 2007. MR-2320889

[DT96]   A. W. M. Dress and W. F. Terhalle, *The real tree*, Adv. Math. **120** (1996), no. 2, 283–301. MR-1397084

[Duq08]   Thomas Duquesne, *The coding of compact real trees by real valued functions*, http://arxiv.org/abs/math/0604106, 2008.

[DW07]   Thomas Duquesne and Matthias Winkel, *Growth of Lévy trees*, Probab. Theory Related Fields **139** (2007), no. 3-4, 313–371. MR-2322700

Splitting trees

[Dyn65]   E. B. Dynkin, *Markov processes. Vols. I, II*, Die Grundlehren der Mathematischen Wissenschaften, vol. 122, Academic Press Inc.; Springer-Verlag, 1965. MR-0193671

[EPW06]   Steven N. Evans, Jim Pitman, and Anita Winter, *Rayleigh processes, real trees, and root growth with re-grafting*, Probab. Theory Related Fields **134** (2006), no. 1, 81–126. MR-2221786

[GK97]   J. Geiger and G. Kersting, *Depth-first search of random trees, and Poisson point processes*, Classical and modern branching processes (Minneapolis, MN, 1994), IMA Vol. Math. Appl., vol. 84, Springer, New York, 1997, pp. 111–126. MR-1601713

[Gro07]   Misha Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Birkhäuser, 2007. MR-2307192

[Itô72]   Kiyosi Itô, *Poisson point processes attached to Markov processes*, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. III: Probability theory (Berkeley, Calif.), Univ. California Press, 1972, pp. 225–239. MR-0402949

[Kal02]   Olav Kallenberg, *Foundations of modern probability*, 2nd ed., Springer-Verlag, 2002. MR-1876169

[Lam08]   Amaury Lambert, *Population dynamics and random genealogies*, Stoch. Models **24** (2008), no. suppl. 1, 45–163. MR-2466449

[Lam10]   Amaury Lambert, *The contour of splitting trees is a Lévy process*, Ann. Probab. **38** (2010), no. 1, 348–395. MR-2599603

[Lam17]   Amaury Lambert, *Probabilistic models for the (sub)tree(s) of life*, Braz. J. Probab. Stat. **31** (2017), no. 3, 415–475. MR-3693976

[LG99]   Jean-François Le Gall, *Spatial branching processes, random snakes and partial differential equations*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1999. MR-1714707

[LG06]   Jean-François Le Gall, *Random real trees*, Ann. Fac. Sci. Toulouse Math. (6) **15** (2006), no. 1, 35–62. MR-2225746

[LGLJ98]   Jean-Francois Le Gall and Yves Le Jan, *Branching processes in Lévy processes: the exploration process*, Ann. Probab. **26** (1998), no. 1, 213–252. MR-1617047

[LU18]   A. Lambert and G. Uribe Bravo, *Totally Ordered Measured Trees and Splitting Trees with Infinite Variation II: Prolific Skeleton Decomposition*, ArXiv e-prints (2018), 1803.05421.

[Par05]   K. R. Parthasarathy, *Probability measures on metric spaces*, AMS Chelsea Publishing, Providence, RI, 2005, Reprint of the 1967 original. MR-2169627

[PR69]   E. A. Pecherskii and B. A. Rogozin, *On joint distributions of random variables associated with fluctuations of a process with independent increments*, Theory of Probability and its Applications **14** (1969), no. 3, 410–14. MR-0260005

[PUB12]   Jim Pitman and Gerónimo Uribe Bravo, *The convex minorant of a Lévy process*, Ann. Probab. **40** (2012), no. 4, 1636–1674. MR-2978134

[RY99]   Daniel Revuz and Marc Yor, *Continuous martingales and Brownian motion*, 3rd ed., Grundlehren der Mathematischen Wissenschaften, vol. 293, Springer-Verlag, Berlin, 1999. MR-1725357

[Sat99]   Ken-iti Sato, *Lévy processes and infinitely divisible distributions*, Cambridge Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 1999. MR-1739520

[Sha88]   Michael Sharpe, *General theory of Markov processes*, Pure and Applied Mathematics, vol. 133, Academic Press, Inc., Boston, MA, 1988. MR-0958914

[SS15]   Emmanuel Schertzer and Florian Simatos, *Height and contour processes of Crump-Mode-Jagers forests (i): general distribution and scaling limits in the case of short edges*, 2015, arXiv:1506.03192. MR-3835473

[Wei07]   Mathilde Weill, *Regenerative real trees*, Ann. Probab. **35** (2007), no. 6, 2091–2121. MR-2353384

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