

## Cutoff for lamplighter chains on fractals

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### Abstract

We show that the total-variation mixing time of the lamplighter random walk on fractal graphs exhibit sharp cutoff when the underlying graph is transient (namely of spectral dimension greater than two). In contrast, we show that such cutoff can not occur for strongly recurrent underlying graphs (i.e. of spectral dimension less than two).

**Keywords:** Markov chain; total variation; mixing time; cutoff phenomenon; lamplighter group; heat kernel; fractal graphs; late points.

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## 1 Introduction

Markov chain mixing rate is an active subject of study in probability theory (see [21, 26] and the references therein). Mixing is usually measured in terms of total variation distance, which for probability measures  $\mu, \nu$  on a countable set  $H$  is

$$\|\mu - \nu\|_{\text{TV}} := \sup_{A \subset H} |\mu(A) - \nu(A)| = \frac{1}{2} \sum_{x \in H} |\mu(x) - \nu(x)| = \sum_{x \in H} [\mu(x) - \nu(x)]_+.$$

Specifically, the  $(\epsilon)$ -total variation mixing time of a Markov chain  $Y = \{Y_t\}_{t \geq 0}$  on the set of vertices of a finite graph  $G = (V, E)$ , having the invariant distribution  $\pi$ , is

$$T_{\text{mix}}(\epsilon; G) := \min \left\{ t \geq 0 \mid \max_{x \in V(G)} \|P_x(Y_t = \cdot) - \pi\|_{\text{TV}} \leq \epsilon \right\}.$$

One of the interesting topics in the study of Markov chains is the cutoff phenomena, mainly for the total variation mixing time (see e.g. [21, Chapter 18]). The study of cutoff

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phenomena for Markov chains was initiated by Aldous, Diaconis and their collaborators early in 80s, and there has been extensive work in the past several decades. Specifically, a sequence of Markov chains  $\{Y^{(N)}\}_{N \geq 1}$  on the vertices of finite graphs  $\{G^{(N)}\}_{N \geq 1}$  has cutoff with threshold  $\{a_N\}_{N \geq 1}$  iff

$$\lim_{N \rightarrow \infty} a_N^{-1} T_{\text{mix}}(\epsilon; G^{(N)}) = 1, \quad \forall \epsilon \in (0, 1).$$

In the (switch-walk-switch) lamplighter Markov chains, each vertex of a locally connected, countable (or finite) graph  $G = (V, E)$  is equipped with a lamp (from  $\mathbb{Z}_2 = \{0, 1\}$ ), and a move consists of three steps:

- (a). The walker turns on/off the lamp at the vertex where he/she is, uniformly at random.
- (b). The walker either stays at the same vertex, or moves to a randomly chosen nearest neighbor vertex.
- (c). The walker turns on/off the lamp at the vertex where he/she is, uniformly at random.

Such a lamplighter chain on the graph  $G$  is precisely the random walk on the corresponding wreath product  $G^* = \mathbb{Z}_2 \wr G$  (see Section 1.1 for the precise definitions), and the total variation mixing time of a lamplighter chain is closely related to the expected cover time of the underlying graph  $G$ , denoted hereafter by  $T_{\text{cov}}(G)$ . The study of cutoff for lamplighter chains goes back to Häggström and Jonasson [15] who showed that cutoff does not occur for the chain on one-dimensional tori, whereas for lamplighter chains on complete graphs, it occurs at the threshold  $a_N = \frac{1}{2} T_{\text{cov}}(G^{(N)})$ . Peres and Revelle [25] further explore the relation between the mixing time of lamplighter chain on  $G^{(N)}$  and  $T_{\text{cov}}(G^{(N)})$ , showing that, under suitable assumptions,

$$\left(\frac{1}{2} + o(1)\right) T_{\text{cov}}(G^{(N)}) \leq T_{\text{mix}}(\mathbb{Z}_2 \wr G^{(N)}; \epsilon) \leq (1 + o(1)) T_{\text{cov}}(G^{(N)}). \quad (1.1)$$

The bounds of (1.1) cannot be improved in general, as the lower and the upper bounds are achieved for complete graphs, and two-dimensional tori, respectively. The same bounds apply for any Markov chain on  $\mathfrak{X} \wr G^{(N)}$ , where in steps (a) and (c) the walker independently chooses the element from the finite set  $\mathfrak{X}$  according to some fixed strictly positive law. Indeed, for such chains total variation mixing time has mostly to do with the geometry of late points of  $G$ , namely those reached by the walker much later than most points. In particular, the LHS of (1.1) represents the need to visit all but  $O(\sqrt{\#V(G)})$  points before mixing of the lamps can occur and the RHS reflects having the lamps at the invariant product measure once all vertices have been visited. Miller and Peres [22] provide a large class of graphs for which the LHS of (1.1) is sharp, with cutoff at  $\frac{1}{2} T_{\text{cov}}(G^{(N)})$ . Among those are lazy simple random walkers on  $d$ -dimensional tori, any  $d \geq 3$ , for which [23] further examines the total-variation distance between the law of late points and i.i.d. Bernoulli points (c.f. [23, Section 1] and the references therein). Finally, the analysis of effective resistance on  $G^{(N)} = \mathbb{Z}_N^2 \times \mathbb{Z}_{\lfloor h \log N \rfloor}$  plays a key role in [11], where it is shown that the threshold  $a(h) T_{\text{cov}}(G^{(N)})$  for mixing time cutoff of lamplighter chain on such graphs, continuously interpolates between  $a(0) = 1$  and  $a(\infty) = \frac{1}{2}$ .

Another topic of much current interest is the long time asymptotic behavior of random walks  $\{X_t\}$  on (infinite) fractal graphs (see [1, 18, 19] and the references therein). Such random walks are typically anomalous and sub-diffusive, so generically  $E_x[d(X_0, X_t)] \asymp t^{1/d_w}$  and the walk-dimension  $d_w$  exceeds two for many fractal graphs, in contrast to the SRW on  $\mathbb{Z}^d$  for which  $d_w = 2$  (the notation  $a_t \asymp b_t$  is used hereafter whenever  $c^{-1}a_t \leq b_t \leq ca_t$  for some  $c < \infty$ ). A related important parameter is the volume growth exponent  $d_f$  such that  $\#B(x, r) \asymp r^{d_f}$ , where  $\#B(x, r)$  counts the number of vertices whose graph distance from  $x$  is at most  $r$ . The growth of the eigenvalues of

the corresponding generator is then measured by the *spectral dimension*  $d_s := 2d_f/d_w$ , with the Markov chain  $\{X_t\}$  strongly recurrent when  $d_s < 2$  and transient when  $d_s > 2$  (while  $d_f = d_s = d$  for the SRW on  $\mathbb{Z}^d$ ).

We study here the cutoff for total variation mixing time of the lamplighter chain when  $G^{(N)}$  are increasing finite subsets of a fractal graph. While gaining important insights on the geometry of late points for the corresponding walks, our main result (see Theorem 1.4), is the following dichotomy:

- When  $d_s < 2$  there is no cutoff for the corresponding lamplighter chain, whereas
- if  $d_s > 2$ , such cutoff occurs at the threshold  $a_N = \frac{1}{2}T_{\text{cov}}(G^{(N)})$ .

If the behavior of the lamplighter chain in the critical case  $d_s = 2$  is likewise universal, then it should be having a mixing cutoff at  $a_N = T_{\text{cov}}(G^{(N)})$  (as in the two-dimensional tori example from [25]).

### 1.1 Framework and main results

Given a countable, locally finite and connected graph  $G = (V(G), E(G))$ , denote by  $d(\cdot, \cdot) = d_G(\cdot, \cdot)$  the graph distance (with  $d(x, y)$  the length of the shortest path between  $x$  and  $y$ ), and by  $B(x, r) = B_G(x, r) := \{y \in V(G) \mid d(x, y) \leq r\}$  the corresponding ball of radius  $r$  centered at  $x$ . A weighted graph is a pair  $(G, \mu)$  with  $\mu : V(G) \times V(G) \rightarrow [0, \infty)$  a conductance, namely a function  $(x, y) \mapsto \mu_{xy}$  such that  $\mu_{xy} = \mu_{yx}$  and  $\mu_{xy} > 0$  if and only if  $xy \in E(G)$ . We use the notation  $V(x, r) := \mu(B(x, r))$  and more generally  $\mu(A) := \sum_{x \in A} \mu_x$  for  $A \subset V(G)$ , where

$$\mu_x := \sum_{y:xy \in E(G)} \mu_{xy}, \quad \forall x \in V.$$

The discrete time random walk  $X = \{X_t\}_{t \geq 0}$  associated with the weighted graph  $(G, \mu)$  is the Markov chain on  $V(G)$  having the transition probability

$$P(x, y) := \frac{\mu_{xy}}{\mu_x}.$$

Let  $P_t(x, y) = P_t(x, y; G) := P_x(X_t = y)$  denote the distribution of  $X_t$  with the corresponding heat kernel

$$p_t(x, y) := \frac{P_t(x, y)}{\mu_y} \quad \forall t \in \mathbb{N} \cup \{0\}$$

and Dirichlet form

$$\mathcal{E}(f, f) := \frac{1}{2} \sum_{x,y \in V(G)} (f(x) - f(y))^2 \mu_{xy} = -\langle f, (P - I)f \rangle_\mu, \quad \text{for } f : V(G) \rightarrow \mathbb{R},$$

where  $\langle f, g \rangle_\mu := \sum_x f(x)g(x)\mu(x)$ . The corresponding effective resistance  $R_{\text{eff}}(\cdot, \cdot)$  is given by

$$R_{\text{eff}}(A, B)^{-1} := \inf\{\mathcal{E}(f, f) \mid f|_A = 1, f|_B = 0\}, \quad \text{for } A, B \subset V(G).$$

We also consider the lazy random walk  $\tilde{X} = \{\tilde{X}_t\}_{t \geq 0}$  on  $(G, \mu)$ , having the transition probability

$$\tilde{P}(x, y) := \begin{cases} \frac{1}{2}P(x, y), & \text{if } x \neq y, \\ \frac{1}{2}, & \text{if } x = y. \end{cases}$$

The Dirichlet form and heat kernel of  $\tilde{X}$  are then, respectively  $\tilde{\mathcal{E}}(f, f) = \frac{1}{2}\mathcal{E}(f, f)$  and

$$\tilde{p}_t(x, y) := \frac{\tilde{P}_t(x, y)}{\mu_y} \quad \forall t \in \mathbb{N} \cup \{0\}.$$

We consider finite weighted graphs  $\{(G^{(N)}, \mu^{(N)})\}_{N \geq 1}$  with  $\#V(G^{(N)}) \rightarrow \infty$ . Using hereafter  $\cdot^{(N)}$  for objects on  $(G^{(N)}, \mu^{(N)})$  (e.g. denoting by  $R_{\text{eff}}^{(N)}(\cdot, \cdot)$  the effective resistance on  $(G^{(N)}, \mu^{(N)})$ ), we make the following assumptions, which are standard in the study of sub-Gaussian heat kernel estimates (Sub-GHKE) (c.f. [2, 19]).

**Assumption 1.1.** For some  $1 \leq d_f < \infty$ ,  $c_e, c_v < \infty$ ,  $p_0 > 0$  and all  $N \geq 1$  we have

(a) *Uniform ellipticity:*  $c_e^{-1} \leq \mu_{xy}^{(N)} \leq c_e \quad \forall xy \in E(G^{(N)})$ .

(b)  *$p_0$ -condition:*  $\frac{\mu_{xy}^{(N)}}{\mu_x^{(N)}} \geq p_0 \quad \forall xy \in E(G^{(N)})$ .

(c)  *$d_f$ -set condition:*  $c_v^{-1}r^{d_f} \leq V^{(N)}(x, r) \leq c_v r^{d_f} \quad \forall x \in V(G^{(N)}), 1 \leq r \leq \text{diam}\{G^{(N)}\} \rightarrow \infty$ .

**Assumption 1.2 (Uniform Parabolic Harnack Inequality).** For some  $2 \leq d_w < \infty$ ,  $C_{\text{PHI}} < \infty$ ,  $c_{\text{PHI}} \in (0, 1]$  and all  $N \geq 1$ , whenever  $u : [0, \infty) \times V(G^{(N)}) \rightarrow [0, \infty)$  satisfies

$$u(t+1, x) - u(t, x) = (P^{(N)} - I)u(t, x), \quad \forall t \in [0, 4T], x \in B^{(N)}(x_0, 2R), \quad (1.2)$$

for some  $x_0 \in V(G^{(N)})$ ,  $R \leq c_{\text{PHI}} \text{diam}\{G^{(N)}\}$  and  $T \geq 2R$ ,  $T \asymp R^{d_w}$ , one also has that

$$\max_{\substack{z \in B^{(N)}(x_0, R) \\ s \in [T, 2T]}} \{u(s, z)\} \leq C_{\text{PHI}} \min_{\substack{z \in B^{(N)}(x_0, R) \\ s \in [3T, 4T]}} \{u(s, z) + u(s+1, z)\}.$$

**Remark 1.3.** Thanks to the  $p_0$ -condition we have that  $1 \geq \deg_{G^{(N)}}(x)p_0$ , so the graphs  $\{G^{(N)}\}$  are of uniformly bounded degrees

$$\sup_N \sup_{x \in V(G^{(N)})} \{\deg_{G^{(N)}}(x)\} < \infty.$$

Together with the uniform ellipticity, this implies that for some  $\tilde{c} < \infty$

$$\tilde{c}^{-1} \leq \mu_x^{(N)} \leq \tilde{c}, \quad \forall N \geq 1, x \in V(G^{(N)}),$$

and thereby

$$\tilde{c}^{-1} \#A \leq \mu^{(N)}(A) \leq \tilde{c} \#A, \quad \forall N \geq 1, A \subset V(G^{(N)}). \quad (1.3)$$

To any finite underlying graph  $G = (V, E)$  corresponds the wreath product  $G^* = \mathbb{Z}_2 \wr G$  such that

$$V(G^*) = \mathbb{Z}_2^V \times V,$$

$$E(G^*) = \{(f, x), (g, y) \mid f = g \text{ and } xy \in E, \text{ or } x = y \text{ and } f(v) = g(v) \text{ for } v \neq x\}$$

and we adopt throughout the convention of using  $\mathbf{y} = (f, y)$  for the vertices of  $\mathbb{Z}_2 \wr G$ . The lazy random walk  $\tilde{X}$  on  $(G, \mu)$  induces the switch-walk-switch lamplighter chain, namely the random walk  $Y = \{Y_t = (f_t, \tilde{X}_t)\}_{t \geq 0}$  on  $\mathbb{Z}_2 \wr G$  whose transition probability is

$$P^*((f, x), (g, y)) = \begin{cases} \frac{1}{2}\tilde{P}(x, y) = \frac{1}{4}, & \text{if } x = y \text{ and } f(v) = g(v) \text{ for any } v \neq x, \\ \frac{1}{4}\tilde{P}(x, y) = \frac{1}{8}P(x, y), & \text{if } x \neq y \text{ and } f(v) = g(v) \text{ for any } v \neq x, y, \\ 0, & \text{otherwise.} \end{cases}$$

One way to describe the moves of the Markov chain  $Y$  is as done before: first  $Y$  switches the lamp of the current position, then moves on  $G$  according to  $\tilde{P}$ , and finally switches the lamp on vertex on which it landed. We denote by  $Y^{(N)} = \{Y_t^{(N)} = (f_t, \tilde{X}_t^{(N)})\}_{t \geq 0}$  the lamplighter chain on weighted graphs  $(G^{(N)}, \mu^{(N)})$ , using  $P^*(\cdot, \cdot; G)$  whenever we wish to emphasize its underlying graph. The invariant (reversible) distribution of each  $X^{(N)}$ , and its lazy version  $\tilde{X}^{(N)}$ , is clearly

$$\pi^{(N)}(x) = \frac{\mu_x^{(N)}}{\mu^{(N)}(G^{(N)})}, \quad \forall x \in V(G^{(N)}) \tag{1.4}$$

with the corresponding invariant distribution of  $Y^{(N)}$  being

$$\pi^*(\mathbf{y}; G^{(N)}) = 2^{-\#V(G^{(N)})} \pi^{(N)}(y), \quad \forall \mathbf{y} = (f, y) \in V(\mathbb{Z}_2 \wr G^{(N)}).$$

We next state our main result.

**Theorem 1.4.** *Consider lamplighter chains  $Y^{(N)}$  whose underlying weighted graphs  $(G^{(N)}, \mu^{(N)})$  satisfy Assumptions 1.1, 1.2.*

- (a) *If  $d_f < d_w$ , then there is no cutoff for the total variation mixing time of  $Y^{(N)}$ .*
- (b) *If  $d_f > d_w$ , then the total variation mixing time for  $Y^{(N)}$  admits cutoff at  $a_N = \frac{1}{2}T_{\text{cov}}(G^{(N)})$ .*

Note that for countable, infinite weighted graph  $(G, \mu)$ , having  $d_f < d_w$  (resp.  $d_f > d_w$ ), corresponds to a strongly recurrent (resp. transient), random walk  $\tilde{X}$  in the sense of [6, Definition 1.2] (see [6, Theorem 1.3, Proposition 3.5 and Lemma 3.6]). In Section 2 we provide a host of fractal graphs satisfying Assumptions 1.1 and 1.2, with the Sierpinski gaskets and the two-dimensional Sierpinski carpets as typical examples of Theorem 1.4(a), while high-dimensional Sierpinski carpets with small holes serve as typical examples of Theorem 1.4(b).

In Section 3, we adapt to the setting of large finite weighted graphs, certain consequences of Assumptions 1.1 and 1.2 which are standard for infinite graphs. In case  $d_f < d_w$ , the relevant time scale for the cover time  $\tau_{\text{cov}}(G^{(N)})$  is shown there to be

$$T_N := (R_N)^{d_w} \quad \text{where} \quad R_N := \text{diam}\{G^{(N)}\}. \tag{1.5}$$

Applying in Section 4 results from Section 3 that apply for  $d_f < d_w$ , we derive the following uniform exponential tail decay for  $\tau_{\text{cov}}(G^{(N)})/T_N$ , which is of independent interest.

**Proposition 1.5.** *If Assumptions 1.1, 1.2 hold with  $d_f < d_w$ , then for some  $c_0$  finite and all  $t, N$ ,*

$$\sup_{z \in V(G^{(N)})} \{P_z(\tau_{\text{cov}}(G^{(N)}) > t)\} \leq c_0 e^{-t/(c_0 T_N)}. \tag{1.6}$$

Starting with all lamps off, namely at  $Y_0 = \mathbf{x} := (\mathbf{0}, x)$ , on the event  $\{\sup_{0 \leq s \leq t} d(\tilde{X}_0, \tilde{X}_s) \leq \frac{1}{4}R_N\}$ , all lamps outside  $B^{(N)}(x, \frac{1}{4}R_N)$  are off at time  $t$ . Hence, then  $\|P_t^*(\mathbf{x}, \cdot; G^{(N)}) - \pi^*(\cdot; G^{(N)})\|_{\text{TV}}$  is still far from 0. Using this observation, we prove in Section 5 the following uniform lower bound on the lamplighter chain distance from equilibrium at time  $t \asymp T_N$ .

**Proposition 1.6.** *If Assumptions 1.1, 1.2 hold, then for some finite  $c_1, N_1$ , any  $t$  and  $N \geq N_1$ ,*

$$\max_{\mathbf{x} \in V(\mathbb{Z}_2 \wr G^{(N)})} \|P_t^*(\mathbf{x}, \cdot; G^{(N)}) - \pi^*(\cdot; G^{(N)})\|_{\text{TV}} \geq c_1^{-1} e^{-c_1 t/T_N} - \tilde{c} c_v R_N^{-d_f}. \tag{1.7}$$

In Proposition 5.1 we bound the LHS of (1.7) by  $\max_x P_x(\tau_{\text{cov}}(G^{(N)}) > t)$  provided  $t/S_N$  is large (for  $S_N$  of (3.10)). Since  $S_N \asymp T_N$  when  $d_f < d_w$ , contrasting Propositions 1.5 and 1.6 yields Theorem 1.4(a) (c.f. Remark 5.2 for information about  $T_{\text{mix}}(\epsilon; \mathbb{Z}_2 \wr G^{(N)})/T_{\text{cov}}(G^{(N)})$  and lack of concentration of  $\tau_{\text{cov}}(G^{(N)})/T_N$ ).

Propositions 1.6 and 5.1 apply also when  $d_w < d_f$ , but in that case  $\tau_{\text{cov}}(G^{(N)}) \geq \sharp V(G^{(N)}) \gg T_N$ , and the proof of Theorem 1.4(b), provided in Section 5.2, amounts to verifying the sufficient conditions of [22, Theorem 1.5] for cutoff at  $\frac{1}{2}T_{\text{cov}}(G^{(N)})$ . Indeed, the required uniform Harnack inequality follows from the UPHI of Assumption 1.2, which as we see in Section 2 is more amenable to analytic manipulations than the Harnack inequality. (Yet, we note that quite recently stability of the (elliptic) Harnack inequalities is proved in [8].)

## 2 Cutoff in fractal graphs

We provide here a few examples for which Theorem 1.4 applies, starting with the following.

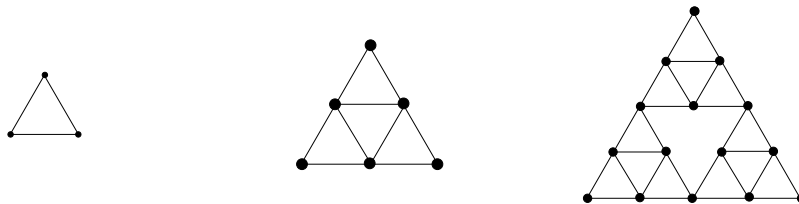


Figure 1: A sequence of the Sierpinski gasket graphs  $(G^{(0)}, G^{(1)}, G^{(2)})$  respectively.

**Example 2.1** (Sierpinski gasket graph in two dimension). Let  $G^{(0)}$  denote the equilateral triangle of side length 1. That is,

$$V(G^{(0)}) = \left\{ x_0 = (0, 0), x_1 = (1, 0), x_2 = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \right\}, \quad E(G^{(0)}) = \{x_0x_1, x_0x_2, x_1x_2\}.$$

Setting  $\psi_i(x) := (x + x_i)/2$  for  $i = 0, 1, 2$ , we define the graphs  $\{G^{(N)}\}_{N \geq 1}$  via

$$V(G^{(N+1)}) = 2 \cdot \left( \bigcup_{i=1}^3 \psi_i(V(G^{(N)})) \right) \quad \text{and} \quad E(G^{(N+1)}) = 2 \cdot \left( \bigcup_{i=1}^3 \psi_i(E(G^{(N)})) \right).$$

The limit graph  $G = (V(G), E(G))$ , where  $V(G) = \cup_{N \geq 0} V(G^{(N)})$  and  $E(G) = \cup_{N \geq 0} E(G^{(N)})$ , is called the Sierpinski gasket graph. It is easy to confirm that if Assumption 1.1(a) holds for weight  $\mu^{(N)}$  on  $G^{(N)}$  then such  $\mu^{(N)}$  satisfies also Assumption 1.1(b) and Assumption 1.1(c) for  $d_f = \log 3 / \log 2$ .

We further prove in Section 2.2 the following.

**Proposition 2.2.** *The weighted graphs  $\{(G^{(N)}, \mu^{(N)})\}_{N \geq 0}$  of Example 2.1 further satisfy Assumption 1.2 with  $d_w = \log 5 / \log 2$ .*

In view of Proposition 2.2 and having  $d_f < d_w$ , we deduce from Theorem 1.4(a) that the total variation mixing time of the lamplighter chains of Example 2.1, admits no cutoff.

**Remark 2.3.** For  $d \geq 3$ , the  $d$ -dimensional Sierpinski gasket graph is similarly defined, and by the same reasoning the corresponding lamplighter chains admit no mixing cutoff. In fact, one can deduce for a more general family of nested fractal graphs (see for instance [16, Section 2] for definition), that no cutoff applies.

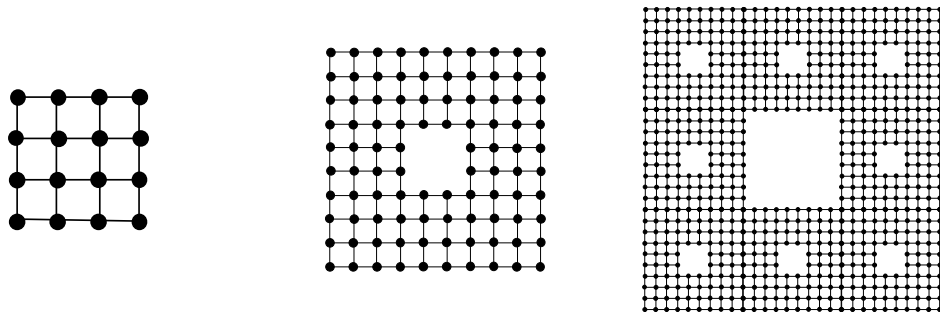


Figure 2: A sequence of the Sierpinski carpet graphs.

**Example 2.4** (Sierpinski carpet graph). Fixing integers  $L \geq 2$  and  $K \in [L, L^d]$ , partition the  $d$ -dimensional unit cube  $H_0 = [0, 1]^d$  into the collection  $Q := \{\prod_{i=1}^d [\frac{(k_i-1)}{L}, \frac{k_i}{L}] \mid 1 \leq k_i \leq L \text{ for all } i \in \{1, 2, \dots, d\}\}$  of  $L^d$  sub-cubes. Then fixing  $L$ -similitudes  $\{\psi_i, 1 \leq i \leq K\}$  of  $H_0$  onto mutually distinct elements of  $Q$ , such that  $\psi_1(x) := L^{-1}x$ , there exists a unique non-empty compact  $F \subset H_0$  such that  $F = \bigcup_{i=1}^K \psi_i(F)$ . We call  $F$  the generalized Sierpinski carpet if the following four conditions hold:

- (a) (Symmetry)  $H_1 := \bigcup_{i=1}^K \psi_i(H_0)$  is preserved by all isometries of  $H_0$ .
- (b) (Connectedness)  $\text{Int}(H_1)$  is connected, and contains a path connecting the hyperplanes  $\{x_1 = 0\}$  and  $\{x_1 = 1\}$ .
- (c) (Non-diagonality) If  $\text{Int}(H_1 \cap B)$  is nonempty for some  $d$ -dimensional cube  $B \subset H_0$  which is the union of  $2^d$  distinct elements of  $Q$ , then  $\text{Int}(H_1 \cap B)$  is a connected set.
- (d) (Borders included)  $H_1$  contains the line segment  $\{(x_1, 0, \dots, 0) \mid 0 \leq x_1 \leq 1\}$ .

For a generalized Sierpinski carpet, let  $V^{(0)}$  and  $E^{(0)}$  denote the the  $2^d$  corners of  $H_0$  and  $d2^{d-1}$  edges on the boundary of  $H_0$  respectively, with

$$\begin{aligned}
 V(G^{(N)}) &:= \bigcup_{i_1, i_2, \dots, i_N=1}^K L^N \psi_{i_1, i_2, \dots, i_N}(V^{(0)}), & V(G) &:= \bigcup_{N \geq 1} V(G^{(N)}). \\
 E(G^{(N)}) &:= \bigcup_{i_1, i_2, \dots, i_N=1}^K L^N \psi_{i_1, i_2, \dots, i_N}(E^{(0)}), & E(G) &:= \bigcup_{N \geq 1} E(G^{(N)}).
 \end{aligned}$$

Once again, it is easy to check that if Assumption 1.1(a) holds for weight  $\mu^{(N)}$  on  $G^{(N)}$ , then such  $\mu^{(N)}$  satisfies also Assumptions 1.1(b) and 1.1(c) for  $d_f = \log K / \log L$ .

We prove in Section 2.2 the following.

**Proposition 2.5.** *For any generalized Sierpinski carpet, the weighted graphs  $\{(G^{(N)}, \mu^{(N)})\}_{N \geq 0}$  of Example 2.4 further satisfy Assumption 1.2 for some finite  $d_w = \log(\rho K) / \log L$ .*

Whereas directly verifying Assumption 1.2 is often difficult, as shown in Section 2.1, certain conditions from the research on Sub-GHKE are equivalent to PHI and more robust. Indeed those equivalent conditions are key to our proof of Propositions 2.2 and 2.5.

In the context of Example 2.4, for carpets with central block of size  $b^d$  removed (so  $K = L^d - b^d$ ), for some  $1 \leq b \leq L - 1$ , one always have  $\rho > 1$  when  $d = 2$  (see [4, LHS of (5.9)]), hence by Theorem 1.4(a) no cutoff for the corresponding lamplighter chain. In contrast, from [4, RHS of (5.9)] we know that  $\rho < 1$  for high-dimensional carpets of small central hole (specifically, whenever  $b^{d-1} < L^{d-1} - L$ ), so by Theorem 1.4(b) the corresponding lamplighter chains then admit cutoff at  $a_N = \frac{1}{2} T_{\text{cov}}(G^{(N)})$ .

**2.1 Stability of heat kernel estimates and parabolic Harnack inequality**

We recall here various stability results for Heat Kernel Estimates (HKE) and Parabolic Harnack Inequalities (PHI), in case of a countably infinite weighted graph  $(G, \mu)$ . To this end, we assume

- *Uniform ellipticity:*  $c_e^{-1} \leq \mu_{xy} \leq c_e$  for some  $c_e < \infty$  and all  $xy \in E(G)$ ,
- *$p_0$ -condition:*  $\frac{\mu_{xy}}{\mu_x} \geq p_0$  for some  $p_0 > 0$  and all  $xy \in E(G)$ ,

and recall few relevant properties of such  $(G, \mu)$ .

**Definition 2.6.** Consider the following properties for  $d_w \geq 2$  and  $d_f \geq 1$ :

- (VD) There exists  $C_D < \infty$  such that

$$V(x, 2r) \leq C_D V(x, r) \quad \text{for all } x \in V(G) \text{ and } r \geq 1.$$

- (V( $d_f$ )) There exists  $C_V < \infty$  such that

$$C_V^{-1} r^{d_f} \leq V(x, r) \leq C_V r^{d_f} \quad \text{for all } x \in V(G) \text{ and } r \geq 1.$$

- (CS( $d_w$ )) There exist  $\theta > 0$ ,  $C_{CS} < \infty$  and for each  $z_0 \in V(G)$ ,  $R \geq 1$  there exists a cut-off function  $\psi = \psi_{z_0, R} : V(G) \rightarrow \mathbb{R}$  such that:

(a)  $\psi(x) \geq 1$  when  $d(x, z_0) \leq R/2$ , while  $\psi(x) \equiv 0$  when  $d(x, z_0) > R$ ,

(b)  $|\psi(x) - \psi(y)| \leq C_{CS} (d(x, y)/R)^\theta$ ,

(c) for any  $z \in V(G)$ ,  $f : B(z, 2s) \rightarrow \mathbb{R}$  and  $1 \leq s \leq R$

$$\begin{aligned} & \sum_{x \in B(z, s)} f(x)^2 \sum_{y \in V(G)} |\psi(x) - \psi(y)|^2 \mu_{xy} \\ & \leq C_{CS}^2 \left(\frac{s}{R}\right)^{2\theta} \left( \sum_{x, y \in B(z, 2s)} |f(x) - f(y)|^2 \mu_{xy} + s^{-d_w} \sum_{y \in B(z, 2s)} f(y)^2 \mu_y \right). \end{aligned}$$

- (PI( $d_w$ )) There exists  $C_{PI} < \infty$  such that

$$\sum_{x \in B(z, R)} (f(x) - \bar{f}_{B(z, R)})^2 \mu_x \leq C_{PI} R^{d_w} \sum_{x, y \in B(z, 2R)} (f(x) - f(y))^2 \mu_{xy}$$

for all  $R \geq 1$ ,  $x \in V(G)$  and  $f : V(G) \rightarrow \mathbb{R}$ , where  $\bar{f}_{B(z, R)} = \frac{1}{V(z, R)} \sum_{x \in B(z, R)} f(x) \mu_x$ .

- (HKE( $d_w$ )) There exists  $C_{HK} < \infty$  such that

$$p_t(x, y) \leq \frac{C_{HK}}{V(x, t^{1/d_w})} \exp \left[ - \frac{1}{C_{HK}} \left( \frac{d(x, y)^{d_w}}{t} \right)^{1/(d_w-1)} \right]$$

for all  $x, y \in V(G)$  and  $t \geq 0$ , whereas

$$p_t(x, y) + p_{t+1}(x, y) \geq \frac{1}{C_{HK} V(x, t^{1/d_w})} \exp \left[ - C_{HK} \left( \frac{d(x, y)^{d_w}}{t} \right)^{1/(d_w-1)} \right]$$

for all  $x, y \in V(G)$  and  $t \geq d(x, y)$ .

- (PHI( $d_w$ )) There exists  $C_{PHI} < \infty$  such that if  $u : [0, \infty) \times V(G) \rightarrow [0, \infty)$  satisfies

$$u(t + 1, x) - u(t, x) = (P - I)u(t, x), \quad \forall (t, x) \in [0, 4T] \times B(x_0, 2R)$$

for some  $x_0 \in V(G)$ ,  $T \geq 2R$  with  $T \asymp R^{d_w}$ , then, for such  $x_0, T, R$ ,

$$\max_{\substack{z \in B(x_0, R) \\ s \in [T, 2T]}} u(s, z) \leq C_{PHI} \min_{\substack{z \in B(x_0, R) \\ s \in [3T, 4T]}} \{u(s, z) + u(s + 1, z)\}.$$



**Theorem 2.7** ([5, Theorems 1.2, 1.5]). *The following are equivalent for any uniformly elliptic, countably infinite  $(G, \mu)$  satisfying the  $p_0$ -condition:*

- (a) (VD), (PI( $d_w$ )) and (CS( $d_w$ )).
- (b) (HKE( $d_w$ )).
- (c) (PHI( $d_w$ )).

Note that in each implication of Theorem 2.7 the resulting values of  $(C_D, C_{PI}, \theta, C_{CS})$ ,  $C_{HK}$  and  $C_{PHI}$  depend only on  $p_0, c_e, d_w$  and the assumed constants. For example, in  $(PHI(d_w)) \Rightarrow (HKE(d_w))$ , the value of  $C_{HK}$  depends only on  $C_{PHI}, p_0, c_e$  and  $d_w$ .

The stability of such equivalence involves the following notion of rough isometry (see [16, Definition. 5.9]).

**Definition 2.8.** *Weighted graphs  $(G^{(1)}, \mu^{(1)})$  and  $(G^{(2)}, \mu^{(2)})$  are rough isometric if there exist  $C_{QI} < \infty$  and a map  $T : V^{(1)} \rightarrow V^{(2)}$  such that*

$$\begin{aligned} C_{QI}^{-1} d^{(1)}(x, y) - C_{QI} &\leq d^{(2)}(T(x), T(y)) \leq C_{QI} d^{(1)}(x, y) + C_{QI}, & \forall x, y \in V^{(1)}, \\ d^{(2)}(x', T(V^{(1)})) &\leq C_{QI}, & \forall x' \in V^{(2)}, \\ C_{QI}^{-1} \mu_x^{(1)} &\leq \mu_{T(x)}^{(2)} \leq C_{QI} \mu_x^{(1)}, & \forall x \in V^{(1)}, \end{aligned}$$

where  $d^{(i)}(\cdot, \cdot)$  and  $V^{(i)}$  denote the graph distance and vertex set of  $G^{(i)}$ ,  $i = 1, 2$ , respectively. Similarly, weighted graphs  $\{(G^{(N)}, \mu^{(N)})\}_N$  are uniformly rough isometric to a fixed, weighted graph  $(G, \mu)$  if each  $(G^{(N)}, \mu^{(N)})$  is rough isometric to  $(G, \mu)$  for some  $C_{QI} < \infty$  which does not depend on  $N$ .

Recall [16, Lemma 5.10], that rough isometry is an equivalence relation. Further, (VD), (PI( $d_w$ )) and (CS( $d_w$ )) are stable under rough isometry. That is,

**Theorem 2.9** ([16, Proposition 5.15]). *Suppose  $(G^{(1)}, \mu^{(1)})$  and  $(G^{(2)}, \mu^{(2)})$  have the  $p_0$ -condition and are rough isometric with constant  $C_{QI}$ . If  $(G^{(1)}, \mu^{(1)})$  satisfies (VD), (PI( $d_w$ )), (CS( $d_w$ )) with constants  $(C_D, C_{PI}, \theta, C_{CS})$ , then so does  $(G^{(2)}, \mu^{(2)})$  with constants which depend only on  $(C_D, C_{PI}, \theta, C_{CS})$ ,  $d_w, p_0$  and  $C_{QI}$ .*

Combining Theorems 2.7 and 2.9 we have the following useful corollary.

**Corollary 2.10.** *Suppose uniformly elliptic weighted graphs  $\{(G^{(N)}, \mu^{(N)})\}_N$  satisfy the  $p_0$ -condition and are uniformly rough isometric to some countably infinite uniformly elliptic  $(G, \mu)$  that also has the  $p_0$ -condition. If  $(G, \mu)$  further satisfies (PHI( $d_w$ )), then so do  $\{(G^{(N)}, \mu^{(N)})\}_N$  with finite constant  $C'_{PHI}$  which is independent of  $N$ .*

## 2.2 Proof of Propositions 2.2 and 2.5

*Proof of Proposition 2.2.* Recall that for random walks on the Sierpinski gasket, namely  $\mu_{xy} \equiv 1$  and the limit graph  $G$  of Example 2.1 (or its  $d$ -dimensional analog,  $d \geq 3$ ), Jones [17, Theorems 17,18] established (HKE( $d_w$ )), which by Theorem 2.7 implies that such  $(G, \mu)$  must also satisfy (PHI( $d_w$ )).

Proceeding to construct for each  $N \geq 1$  a new weighted graph  $(G, \mu^{(N)})$ , recall that  $G^{(N+1)}$  consists of three copies  $G^{(N,i)}$  of  $G^{(N)}$ , with  $2^N x_i \in G^{(N,i)}$  for  $i = 0, 1, 2$ . Note that  $G^{(N,0)} = G^{(N)}$  whereas each  $G^{(N,i)}$ ,  $i = 1, 2$  is the reflection of  $G^{(N,0)}$  across a certain line  $\ell^{N,i}$ . Reflecting the weight  $\mu^{(N,0)} := \mu^{(N)}$  on  $G^{(N,0)}$ , across  $\ell^{N,i}$  yields weights  $\mu^{(N,i)}$  on  $G^{(N,i)}$ ,  $i = 1, 2$  (see Figure 3). With  $\{\mu^{(N,i)}, i = 0, 1, 2\}$  forming a new weight on  $G^{(N+1)} \subset G$ , we thus set

$$\mu_{xy}^{(N)} := \begin{cases} \mu_{xy}^{(N,i)}, & \text{if } xy \in E(G^{(N+1)}), \\ 1, & \text{otherwise.} \end{cases}$$

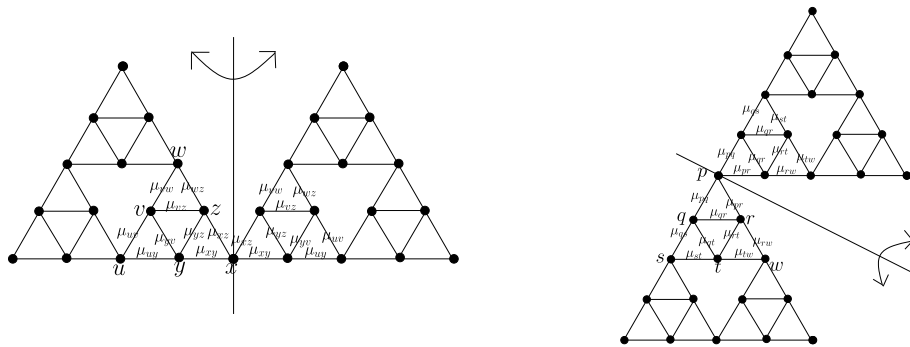


Figure 3: The construction of a weighted graph  $(G^{(N+1)}, \mu^{(N+1)})$  for a given  $(G^{(N)}, \mu^{(N)})$ .

Fixing a solution  $u^{(N)} : [0, \infty) \times V(G^{(N)}) \rightarrow [0, \infty)$  of the heat equation (1.2) on the time-space cylinder of center  $y_0 \in V(G^{(N)})$  and size  $2R \leq T \asymp R^{d_w}$ ,  $R \leq \frac{1}{4}R_N$ , we extend  $u^{(N)}(t, \cdot)$  to the non-negative function on  $V(G)$

$$\tilde{u}^{(N)}(t, x) := \begin{cases} u^{(N)}(t, x), & \text{if } x \in V(G^{(N)}), \\ u^{(N)}(t, x'), & \text{if } x \notin V(G^{(N)}) \text{ and } x' \text{ are symmetric w.r.t. } \ell^{N,1} \text{ or } \ell^{N,2}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Having  $R \leq \frac{1}{4}R_N$  guarantees that  $B_G(y_0, 2R) \subseteq G^{(N+1)}$ , hence from our construction of  $\mu'^{(N)}$  it follows that  $\tilde{u}^{(N)}(t, x)$  satisfy the heat equation corresponding to  $(G, \mu'^{(N)})$  on the time-space cylinder defined by  $(y_0, R, T)$ . Since  $G$  has uniformly bounded degrees, the weighted graphs  $\{(G, \mu'^{(N)})\}_N$  satisfy a  $p'_0$ -condition (for some  $p'_0 > 0$  independent of  $N$ ). Further,  $\{(G, \mu'^{(N)})\}_N$  are uniformly rough isometric to  $(G, \mu)$  (thanks to the uniform ellipticity of  $\mu^{(N)}$ ). Hence, by Corollary 2.10, for some  $C'_{\text{PHI}} < \infty$ , which does not depend on  $N$ , nor on the specific choice of  $y_0, R$  and  $T$ ,

$$\max_{\substack{t \in [T, 2T] \\ y \in B_G(y_0, R)}} \tilde{u}^{(N)}(t, y) \leq C'_{\text{PHI}} \min_{\substack{t \in [3T, 4T] \\ y \in B_G(y_0, R)}} \{\tilde{u}^{(N)}(t, y) + \tilde{u}^{(N)}(t+1, y)\}. \tag{2.2}$$

Since  $\tilde{u}^{(N)}$  of (2.1) coincides with  $u^{(N)}$  on  $B^{(N)}(y_0, R) \subseteq B_G(y_0, R)$ , replacing  $\tilde{u}^{(N)}$  and  $B_G(y_0, R)$  in (2.2) by  $u^{(N)}$  and  $B^{(N)}(y_0, R)$ , respectively, may only decrease its LHS and increase its RHS. That is, (2.2) applies also for  $u^{(N)}(\cdot, \cdot)$  and  $B^{(N)}(y_0, R)$ . This holds for all  $N$  and any of the preceding choices of  $y_0, R, T$ , yielding Assumption 1.2, as stated.  $\square$

*Proof of Proposition 2.5.* Consider the random walk, namely  $\mu_{xy} \equiv 1$ , on a limiting graph  $G$  that corresponds to a generalized Sierpinski carpet, as in Example 2.4. Clearly,  $(G, \mu)$  is uniformly elliptic and of uniformly bounded degrees (so  $p_0$ -condition holds as well). Further, such random walk has properties  $(V(d_f))$  and  $(\text{HKE}(d_w))$ , with  $d_f = \log K / \log L \geq 1$  and  $d_w = \log(\rho K) / \log L$  (see [3]). In particular, by Theorem 2.7  $(G, \mu)$  satisfies  $(\text{PHI}(d_w))$ . With  $G^{(N+1)}$  consisting of  $K$  copies of  $G^{(N)}$ , we extend the given weight  $\mu^{(N)}$  on  $G^{(N)}$  to a weight  $\mu'^{(N)}$  on  $G$ . Specifically, the weight on the edges of the reflected part of  $G^{(N)}$ , as in Figure 4, is  $\mu'_e{}^{(N)} = K_e \mu_{e'}^{(N)}$ , where  $K_e \in [1, K]$  is the number of overlaps of  $e$ , and  $e'$  is the edge which moves to  $e$  by the reflection (so in Figure 4, we set  $\mu'_e{}^{(N)} = 2\mu_e^{(N)}$  for each edge  $e$  lying on a reflection axis). Taking  $\mu'_e{}^{(N)} \equiv 1$  for all other  $e \in E(G)$ , the graphs  $\{(G, \mu'^{(N)})\}_N$  are uniformly elliptic, satisfy a  $p'_0$ -condition (for some  $p'_0 > 0$  independent of  $N$ ), and are uniformly rough isometric to  $(G, \mu)$ . Thus, by Corollary 2.10 the  $(\text{PHI}(d_w))$  holds for  $\{(G, \mu'^{(N)})\}_N$  with a constant  $C'_{\text{PHI}}$  which does not depend

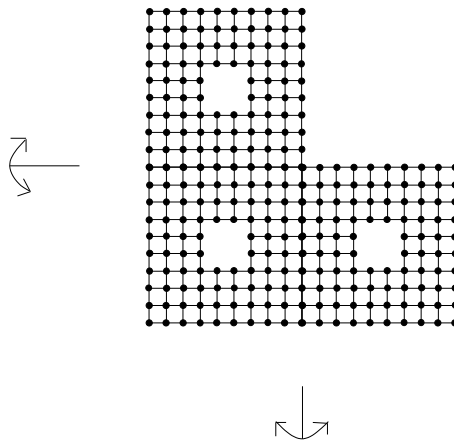


Figure 4: An example of the reflection

on  $N$ . Fixing center  $y_0 \in V(G^{(N)})$  and size parameters  $2R \leq T \asymp R^{d_w}$ ,  $R \leq \frac{1}{4}R_N$ , we extend any given solution  $u^{(N)} : [0, \infty) \times V(G^{(N)}) \rightarrow [0, \infty)$  of the heat equation (1.2) on the corresponding time-space cylinder, to the non-negative  $\tilde{u}^{(N)} : [0, \infty) \times V(G) \rightarrow [0, \infty)$ , symmetrically along reflections, analogously to (2.1). Since  $R \leq \frac{1}{4} \text{diam}\{G^{(N)}\}$  all edges of  $B_G(y_0, 2R)$  not in  $G^{(N)}$  are among those reflected to  $G^{(N)}$ , with our construction of  $\mu^{(N)}$  guaranteeing that  $\tilde{u}^{(N)}(\cdot, \cdot)$  satisfy the heat equation on the corresponding time-space cylinder of  $(G, \mu^{(N)})$ . Thanks to the (PHI( $d_w$ )) for  $\{(G, \mu^{(N)})\}_N$ , we have (2.2), and since  $\tilde{u}^{(N)}$  coincides with  $u^{(N)}$  on  $B^{(N)}(y_0, R) \subseteq B_G(y_0, R)$ , the same applies when replacing  $\tilde{u}^{(N)}$  and  $B_G(y_0, R)$  by  $u^{(N)}$  and  $B^{(N)}(y_0, R)$ , respectively. As in our proof of Proposition 2.2, this holds for all relevant values of  $N$ ,  $y_0$ ,  $R$  and  $T$ , thereby establishing Assumption 1.2.  $\square$

### 3 Random walk consequences of Assumptions 1.1 and 1.2

We summarize here those consequences of Assumptions 1.1 and 1.2 we need for Theorem 1.4, starting with Sub-GHKE, an upper bound on the uniform mixing times and a covering statement which are applicable for all values of  $(d_f, d_w)$ . Then, focusing in Section 3.1 on the case  $d_f < d_w$ , we control  $R_{\text{eff}}^{(N)}(\cdot, \cdot)$  and relate it to  $T_N$  of (1.5), complemented in Section 3.2 by upper bounds on the Green functions, in case  $d_f > d_w$ . We provide only proof outlines since most of these results, and their proofs, are pretty standard.

Our first result is the uniform Sub-GHKE one has on  $\{(G^{(N)}, \mu^{(N)})\}_{N \geq 1}$ , up to time of order  $T_N$ .

**Proposition 3.1.** *Under Assumptions 1.1 and 1.2, for any  $\eta < \infty$ , there exist  $c_{\text{HK}} = c_{\text{HK}}(\eta) < \infty$ , such that for all  $N$ , any  $x, y \in V(G^{(N)})$  and  $t \leq \eta T_N$ ,*

$$p_t^{(N)}(x, y) \leq \frac{c_{\text{HK}}}{t^{d_f/d_w}} \exp \left[ - \frac{1}{c_{\text{HK}}} \left( \frac{d^{(N)}(x, y)^{d_w}}{t} \right)^{1/(d_w-1)} \right]. \quad (3.1)$$

Further, for all  $N$ , any  $x, y \in V(G^{(N)})$  and  $d^{(N)}(x, y) \leq t \leq \eta T_N$ ,

$$p_t^{(N)}(x, y) + p_{t+1}^{(N)}(x, y) \geq \frac{1}{c_{\text{HK}} t^{d_f/d_w}} \exp \left[ - c_{\text{HK}} \left( \frac{d^{(N)}(x, y)^{d_w}}{t} \right)^{1/(d_w-1)} \right]. \quad (3.2)$$

*Proof.* (Sketch:) This is a finite graph analogue of (PHI( $d_w$ )) $\Rightarrow$  (HKE( $d_w$ )) of Theorem 2.7, which is standard for a countably infinite weighted graph (see [14, Theorem 3.1, (ii)]  $\Rightarrow$

(i)). Such implication holds also for metric measure space with a local regular Dirichlet form, as [7, Theorem 3.2 ( $c' \Rightarrow a''$ )], and we sketch below how to adapt the latter proof, specifically [7, Sections 4.3 and 5], to the finite graph setting. First note that for  $t \leq \eta T_N$  the derivation of the (near-)diagonal upper-bound (3.1) (without the exponential term), follows as in the proof of [27, Proposition 7.1]. Setting  $p_t^{(N,x,R)}$  for the heat kernel of the process killed upon exiting  $B^{(N)}(x, R)$ , upon adapting the arguments in [7, Section 4.3.4], one thereby establishes the corresponding (near-)diagonal lower bound, analogous to [7, (4.63)]. Namely, showing that for some  $c'_{\text{PHI}} \in (0, 1)$  and  $c'_{\text{HK}} = c'_{\text{HK}}(\eta')$  finite, any  $\eta' < \infty$ , all  $N \geq 1$ ,  $x \in V(G^{(N)})$  and  $R \leq c'_{\text{PHI}} R_N$ , if  $c'_{\text{HK}} d^{(N)}(x, y) \leq t^{1/d_w} \leq \eta' R$ , then

$$p_t^{(N,x,R)}(x, y) + p_{t+1}^{(N,x,R)}(x, y) \geq \frac{1}{c'_{\text{HK}} t^{d_f/d_w}}. \tag{3.3}$$

Combining (3.3) and the (near-)diagonal upper bound, one then deduces (3.1) as done in [7, Sections 4.3.5-4.3.6]. Similarly, by adapting the proof of [7, Proposition 5.2(i) and (iii)], the near-diagonal lower bound (3.3) yields the full lower-bound of (3.2). Since all these arguments involve only  $\eta$  and the constants from Assumptions 1.1-1.2, we can indeed choose the constant  $c_{\text{HK}}(\eta)$  in (3.1)-(3.2) independently of  $N$ .  $\square$

Proposition 3.1 has the following immediate consequence.

**Corollary 3.2.** *Under Assumptions 1.1 and 1.2 there exist  $R_0$  and  $c_2$  finite, such that for any  $N \geq 1$ ,  $x \in V(G^{(N)})$  and  $R_0 \leq r \leq R_N$*

$$P_x\left(\max_{0 \leq j \leq t} d^{(N)}(x, X_j^{(N)}) \leq r\right) \geq c_2^{-1} \exp(-c_2 t/r^{d_w}).$$

*Proof.* Using the same arguments as in the proof of [20, Proposition 3.3], from (3.1) and (3.2) we get the finite graph analogs of [20, Lemma 3.1] and [20, Lemma 3.4], respectively. Combining these bounds and the Markov property, as done in [20, Lemma 3.5], results with the stated bound for  $k[r^{d_w}] \leq t < (k + 1)[r^{d_w}]$ . All steps of the proof involve only our universal constants  $c_e, c_v, p_0, C_{\text{PHI}}, c_{\text{PHI}}, c_{\text{HK}}$  and with  $X_j^{(N)}$  confined to certain balls, having our Sub-GHKE restricted to  $t \leq \eta T_N$  is immaterial here.  $\square$

Another consequence of (3.1) is the following upper bound on uniform mixing times.

**Proposition 3.3.** *Suppose Assumptions 1.1 and 1.2 hold. Then, for the invariant measures  $\pi^{(N)}(\cdot)$  of (1.4), some finite  $c(\cdot)$ , all  $N \geq 1$  and  $\epsilon > 0$ ,*

$$T_{\text{mix}}^U(\epsilon; G^{(N)}) := \min \left\{ t \geq 0 \mid \max_{x,y \in V(G^{(N)})} \left| \frac{P_x(\tilde{X}_t = y; G^{(N)})}{\pi^{(N)}(y)} - 1 \right| \leq \epsilon \right\} \leq c(\epsilon) T_N. \tag{3.4}$$

For the proof of Proposition 3.3, consider the normalized Dirichlet forms of  $\tilde{X}^{(N)}$  and  $X^{(N)}$ ,

$$\begin{aligned} \mathcal{E}_{\text{norm}}^{(N)}(f, f) &:= -\langle f, (P^{(N)} - I)f \rangle_{\pi^{(N)}}, \\ \tilde{\mathcal{E}}_{\text{norm}}^{(N)}(f, f) &:= -\langle f, (\tilde{P}^{(N)} - I)f \rangle_{\pi^{(N)}} = \frac{1}{2} \mathcal{E}_{\text{norm}}^{(N)}(f, f). \end{aligned}$$

Let  $\mathcal{H}_0^+(S) := \{f : V(G^{(N)}) \rightarrow [0, \infty) \mid f \text{ not a constant function, } \text{Supp}\{f\} \subseteq S\}$  for  $S \subseteq V(G^{(N)})$  and define the spectral quantities

$$\begin{aligned} \lambda^{(N)}(S) &:= \inf \left\{ \frac{\mathcal{E}_{\text{norm}}^{(N)}(f, f)}{\text{Var}^{\pi^{(N)}}(f)} \mid f \in \mathcal{H}_0^+(S) \right\}, \\ \tilde{\lambda}^{(N)}(S) &:= \inf \left\{ \frac{\tilde{\mathcal{E}}_{\text{norm}}^{(N)}(f, f)}{\text{Var}^{\pi^{(N)}}(f)} \mid f \in \mathcal{H}_0^+(S) \right\} = \frac{1}{2} \lambda^{(N)}(S). \end{aligned}$$

Recall the following upper bound on uniform mixing times in terms of the corresponding spectral profile.

**Lemma 3.4** ([12, Corollary 2.1]). *For  $r \geq \pi_*^{(N)} := \inf_{x \in V(G^{(N)})} \{\pi^{(N)}(x)\}$ , let*

$$\tilde{\Lambda}^{(N)}(r) := \inf \left\{ \tilde{\lambda}^{(N)}(S) \mid \pi^{(N)}(S) \leq r \right\}.$$

Then, for any  $\epsilon > 0$  and all  $N$ ,

$$T_{\text{mix}}^U(\epsilon; G^{(N)}) \leq \int_{4\pi_*^{(N)}}^{4/\epsilon} \frac{4 dr}{r \tilde{\Lambda}^{(N)}(r)}. \tag{3.5}$$

Our next lemma controls the spectral profiles on the RHS of (3.5) en-route to Proposition 3.3.

**Lemma 3.5** (Faber-Krahn inequality). *For any  $N$  and  $S \subseteq V(G^{(N)})$  let*

$$\lambda_1^{(N)}(S) := \inf \left\{ \frac{\mathcal{E}^{(N)}(f, f)}{\|f\|_{L^2(\mu^{(N)})}} \mid f \in \mathcal{H}_0(S) \right\}, \tag{3.6}$$

where  $\mathcal{H}_0(S) := \{f : V(G^{(N)}) \rightarrow \mathbb{R} \mid \text{Supp}\{f\} \subseteq S\}$ . If Assumptions 1.1 and 1.2 hold, then for some  $c_{\text{FK}} > 0$  and all  $N$ ,

$$\lambda_1^{(N)}(S) \geq c_{\text{FK}} \mu^{(N)}(S)^{-d_w/d_f} \quad \forall S \subseteq V(G^{(N)}). \tag{3.7}$$

*Proof.* (Sketch:) For countably infinite  $(G, \mu)$  satisfying the  $p_0$ -condition, such Faber-Krahn inequality is a standard consequence of  $(V(d_f))$  and the on-diagonal  $(\text{HKE}(d_w))$  upper bound. Indeed, its proof in [9, Theorem 5.4], while written for  $d_w = 2$ , is easily adapted to any  $d_w > 0$ , upon suitably adjusting various exponents (e.g. taking  $\nu = d_w/d_f$  and  $r = t^{1/d_w}$ , c.f. the discussion in [13, Proposition 5.1]). To get (3.7) one instead relies on (3.1) at  $y = x$ , and on Assumption 1.1, noting that all steps of the proof involve only the universal  $d_f, d_w, p_0, c_e, c_v$  and  $c_{\text{HK}}$ . Further, following the proof of [9, Theorem 5.4] it now suffices to take only  $r \leq R_N$ , hence  $t \leq \eta T_N$  for some fixed  $\eta < \infty$ .  $\square$

*Proof of Proposition 3.3.* Recall that  $\mu^{(N)}(S) = \pi^{(N)}(S)\mu^{(N)}(V(G^{(N)}))$ . By (3.6) we further have that  $\lambda^{(N)}(S) \geq \lambda_1^{(N)}(S)$  for any choice of  $S$  and  $N$ , hence Lemma 3.5 results with

$$\tilde{\Lambda}^{(N)}(r) \geq \frac{c_{\text{FK}}}{2} \left[ r \mu^{(N)}(V(G^{(N)})) \right]^{-d_w/d_f}. \tag{3.8}$$

By the assumed  $d_f$ -set condition,  $\mu^{(N)}(V(G^{(N)})) \leq c_v \text{diam}\{G^{(N)}\}^{d_f}$ . Thus, combining (3.5) and (3.8) yields the bound

$$T_{\text{mix}}^U(\epsilon; G^{(N)}) \leq \frac{8}{c_{\text{FK}}} \frac{d_f}{d_w} \left( \frac{4c_v}{\epsilon} \right)^{d_w/d_f} T_N,$$

as claimed.  $\square$

We conclude with a very useful covering property.

**Proposition 3.6.** *Assumption 1.1 implies that for any  $\eta \in (0, 1]$ , there exist  $L = L(\eta, d_f, \tilde{c}c_v) < \infty$  such that each  $G^{(N)}$  can be covered by  $L$  balls  $\{B^{(N)}(x_i, \eta R_N)\}_{i=1}^L$  of  $V(G^{(N)})$ .*

*Proof.* Covering  $V(G^{(N)})$  by a single ball of radius  $R_N$ , thanks to (1.3) and the assumed  $d_f$ -set condition  $\sharp V(G^{(N)}) \leq \tilde{c}c_v (R_N)^{d_f}$ . Further,  $G^{(N)}$  can be covered by  $L$  balls  $B^{(N)}(x_i, \eta R_N)$  such that  $\{B^{(N)}(x_i, \eta R_N/2)\}$  are disjoint (e.g. [2, Lemma 6.2(a)]). Consequently,  $L(\tilde{c}c_v)^{-1}(\eta R_N/2)^{d_f} \leq \sharp V(G^{(N)})$  and we conclude that  $L \leq (\tilde{c}c_v)^2(2/\eta)^{d_f}$  for all  $N$ , as claimed.  $\square$

**3.1 Strongly recurrent case:**  $d_f < d_w$

A consequence of Assumptions 1.1, 1.2 for  $d_f < d_w$  is the following relation between the resistance metric and the graph distance.

**Proposition 3.7.** *Suppose Assumptions 1.1, 1.2 and  $d_f < d_w$ . Then, for some  $c_R$  finite, all  $N \geq 1$  and any  $x, y \in V(G^{(N)})$ ,*

$$c_R^{-1} d^{(N)}(x, y)^{d_w - d_f} \leq R_{\text{eff}}^{(N)}(x, y) \leq c_R d^{(N)}(x, y)^{d_w - d_f}. \tag{3.9}$$

*Proof.* (Sketch:) For a single infinite weighted graph this is a well known consequence of (HKE( $d_w$ )), see for example [6, Theorem 1.3]. In our setting, the upper bound on  $R_{\text{eff}}^{(N)}$  is derived from Proposition 3.1 by going via (PI( $d_w$ )), as done in the proof of [6, Lemma 2.3(ii), Proposition 4.2(1)]. The corresponding lower bound in (3.9) is proved as in [6, Proposition 4.2(2)], by showing instead the property (SRL( $d_w$ )) (see remark at [6, bottom of Pg. 1650]). As in Proposition 3.1, all steps use only constants from Assumptions 1.1–1.2 and require our Sub-GHKE only at  $t \leq \eta_0 T_N$ . Hence, we end with finite  $c_R$  which is independent of  $N$ .  $\square$

The following corollary of Proposition 3.7 is immediate.

**Corollary 3.8.** *Suppose Assumptions 1.1 and 1.2 hold for some  $d_f < d_w$  and let*

$$r(G^{(N)}) := \max_{x, y \in V(G^{(N)})} \{R_{\text{eff}}^{(N)}(x, y)\}, \quad S_N := \mu^{(N)}(V(G^{(N)}))r(G^{(N)}). \tag{3.10}$$

*Then, for some finite  $c_\star$*

$$c_\star^{-1} T_N \leq S_N \leq c_\star T_N, \quad \forall N \geq 1.$$

*Proof.* By our  $d_f$ -set condition  $\mu^{(N)}(G^{(N)}) \asymp (R_N)^{d_f}$ , whereas  $r(G^{(N)}) \asymp (R_N)^{d_w - d_f}$ , thanks to Proposition 3.7. With  $T_N := (R_N)^{d_w}$  we are thus done.  $\square$

**3.2 Transient case:**  $d_f > d_w$

When  $d_f > d_w$ , Proposition 3.3 and (3.1) yield the following decay rate of the Green functions.

**Proposition 3.9.** *Suppose Assumptions 1.1, 1.2 and  $d_f > d_w$ . Then, for some  $c_g(\cdot)$  finite, any  $\epsilon > 0$  and finite  $N$ ,*

$$\tilde{g}^{(N)}(x, y) := \sum_{t=0}^{T_{\text{mix}}^U(\epsilon; G^{(N)})} \tilde{p}_t^{(N)}(x, y) \leq c_g(\epsilon) d^{(N)}(x, y)^{d_w - d_f}, \quad \forall y \neq x \in V(G^{(N)}). \tag{3.11}$$

*Proof.* Clearly  $\tilde{p}_t^{(N)}(x, y) = \sum_s q_t(s) p_s^{(N)}(x, y)$  with  $q_t(s)$  the probability that a Binomial( $t, 1/2$ ) equals  $s$ . Consequently,  $\tilde{g}^{(N)}(x, y) \leq 2g^{(N)}(x, y)$  (since  $\sum_t q_t(s) = 2$ ). We further replace  $T_{\text{mix}}^U(\epsilon; G^{(N)})$  in (3.11) by  $\eta T_N$ , for  $\eta := c(\epsilon)$  of Proposition 3.3. Hence, from (3.1) for some  $c_{\text{HK}} = c_{\text{HK}}(\eta)$ , all  $N$  and  $x \neq y$ ,

$$\tilde{g}^{(N)}(x, y) \leq 2c_{\text{HK}} \sum_{t=1}^{\infty} t^{-d_f/d_w} \exp \left[ -c_{\text{HK}}^{-1} \left( \frac{d^{(N)}(x, y)^{d_w}}{t} \right)^{1/(d_w - 1)} \right].$$

Since  $d_f/d_w > 1$ , the series on the RHS converges (even when  $d^{(N)}(x, y) = 0$ ), and it is easy to further bound it by  $c'_g d^{(N)}(x, y)^{d_w - d_f}$  for some  $c'_g = c'_g(c_{\text{HK}})$  finite, as we claim in (3.11).  $\square$

### 4 Cover time: Proof of Proposition 1.5

We recall  $S_N, r(G^{(N)})$  of (3.10) and use the following notations for  $x, y \in V(G^{(N)})$ ,  $r \in [0, 1]$ ,

$$\widehat{R}_{\text{eff}}^{(N)}(x, y) := \frac{R_{\text{eff}}^{(N)}(x, y)}{r(G^{(N)})} \in [0, 1], \quad B_R^{(N)}(x, r) := \{y \in V(G^{(N)}) \mid \widehat{R}_{\text{eff}}^{(N)}(x, y) \leq r\}.$$

We show in Lemma 4.1 that for some  $\epsilon' > 0$ , with positive probability, during its first  $S_N$  steps, a random walk on  $G^{(N)}$  makes at least  $\epsilon' r(G^{(N)})$  visits to the starting point. Combining this with the modulus of continuity of the relevant local times (of Lemma 4.2), we show in Proposition 4.3 and Corollary 4.4 that for some  $\kappa > 0$ , with positive probability, by time  $4S_N$  a (small) ball  $B_R^{(N)}(x, \kappa)$  is covered by the random walk trajectory. In view of Propositions 3.6 and 3.7, if in addition  $d_f < d_w$ , then for some  $L = L(\kappa, c_R)$  finite and all  $N$ , the set  $V(G^{(N)})$  is covered by some  $\{B_R^{(N)}(z_i, \kappa)\}_{i=1}^L$ . Proposition 1.5 then follows by using this fact, the Markov property and having  $S_N \asymp T_N$  (see Corollary 3.8).

We now implement the details of the preceding proof strategy.

**Lemma 4.1.** *Under Assumptions 1.1 and 1.2, there exists  $\epsilon > 0$  such that*

$$\max_{N \geq 1} \max_{x \in V(G^{(N)})} P_x \left( \widehat{L}_{S_N}^{(N)}(x) \leq 2\epsilon \right) \leq \frac{1}{8}, \quad \widehat{L}_t^{(N)}(x) := \frac{1}{r(G^{(N)})\mu_x^{(N)}} \sum_{s=0}^{t-1} 1_x(X_s^{(N)}). \tag{4.1}$$

*Proof.* Recall that the successive times in which the walk  $X_t^{(N)}$  re-visits  $x = X_0^{(N)}$ , form a partial sum, whose i.i.d. N-valued increments  $\{\eta_x^{(N)}(i)\}_{i \geq 1}$  have mean

$$E_x[\eta_x^{(N)}] = \frac{1}{\pi^{(N)}(x)} = \frac{\mu^{(N)}(G^{(N)})}{\mu_x^{(N)}}.$$

Setting  $m_x^{(N)} := \lceil 2\epsilon \mu_x^{(N)} r(G^{(N)}) \rceil$  we thus have by Markov's inequality that

$$P_x \left( \widehat{L}_{S_N}^{(N)}(x) \leq 2\epsilon \right) = P_x \left( \sum_{i=1}^{m_x^{(N)}} \eta_x^{(N)}(i) \geq S_N \right) \leq \frac{m_x^{(N)}}{S_N} E_x[\eta_x^{(N)}] \leq 2\epsilon,$$

yielding (4.1) when  $\epsilon \leq 2^{-4}$ . □

With our graphs having uniform volume growth, [10, Theorem 1.4] applies here, giving the following modulus of continuity result.

**Lemma 4.2.** *Suppose Assumptions 1.1 and 1.2. Then, for  $\varphi(\kappa) := \sqrt{\kappa(1 + |\log \kappa|)}$  we have that*

$$\Delta(\lambda) := \sup_{\kappa \in (0,1], N \geq 1} \sup_{z \in V(G^{(N)})} P_z \left( \max_{t \leq S_N} \max_{\substack{x, y \in V(G^{(N)}) \\ \widehat{R}_{\text{eff}}^{(N)}(x, y) \leq \kappa}} |\widehat{L}_t^{(N)}(x) - \widehat{L}_t^{(N)}(y)| \geq \lambda \varphi(\kappa) \right) \rightarrow 0,$$

as  $\lambda \rightarrow \infty$ .

Combining Lemmas 4.1 and 4.2 yields the following uniform lower bound on the minimum over  $y \in B_R^{(N)}(x, \kappa)$ , of the normalized local time at  $y$  during the first  $4S_N$  moves of the random walker.

**Proposition 4.3.** *Under Assumptions 1.1 and 1.2, for some positive  $\epsilon, \kappa$*

$$\inf_{N \geq 1} \inf_{x, z \in V(G^{(N)})} P_z \left( \min_{y \in B_R^{(N)}(x, \kappa)} \{ \widehat{L}_{4S_N}^{(N)}(y) \} \geq \epsilon \right) \geq \frac{1}{2}. \tag{4.2}$$

*Proof.* Step 1. Taking  $\epsilon > 0$  as in Lemma 4.1, we first show that for some  $\kappa > 0$ ,

$$\inf_{N \geq 1} \inf_{x \in V(G^{(N)})} P_x \left( \min_{y \in B_R^{(N)}(x, \kappa)} \{ \widehat{L}_{S_N}^{(N)}(y) \} \geq \epsilon \right) \geq \frac{3}{4}.$$

To this end considering Lemma 4.2 for  $\lambda < \infty$  such that  $\Delta(\lambda) < 2^{-3}$  and  $\kappa > 0$  such that  $\lambda\varphi(\kappa) \leq \epsilon$ , we obtain that, for all  $N$  and any  $z \in V(G^{(N)})$ ,

$$P_z \left( \max_{x, y \in B_R^{(N)}(x, \kappa)} \{ | \widehat{L}_{S_N}^{(N)}(x) - \widehat{L}_{S_N}^{(N)}(y) | \} \geq \epsilon \right) \leq \frac{1}{8}.$$

Consequently, by Lemma 4.1,

$$\begin{aligned} \frac{7}{8} &\leq P_x \left( \widehat{L}_{S_N}^{(N)}(x) \geq 2\epsilon \right) \leq \frac{1}{8} + P_x \left( \widehat{L}_{S_N}^{(N)}(x) \geq 2\epsilon, \max_{y \in B_R^{(N)}(x, \kappa)} \{ | \widehat{L}_{S_N}^{(N)}(x) - \widehat{L}_{S_N}^{(N)}(y) | \} \leq \epsilon \right) \\ &\leq \frac{1}{8} + P_x \left( \min_{y \in B_R^{(N)}(x, \kappa)} \{ \widehat{L}_{S_N}^{(N)}(y) \} \geq \epsilon \right), \end{aligned}$$

thereby completing Step 1.

Step 2. Turning to prove (4.2) when  $z \neq x$ , let  $\tau_x^{(N)} := \inf\{t \geq 0 \mid X_t^{(N)} = x\}$  denote the first hitting time of  $x \in V(G^{(N)})$  by the random walk. Recall the commute time identity (see [21, Proposition 10.6]), that for any  $N$  and  $x \neq z$  in  $V(G^{(N)})$ ,

$$E_x[\tau_z^{(N)}] + E_z[\tau_x^{(N)}] = R_{\text{eff}}^{(N)}(z, x)\mu^{(N)}(G^{(N)}). \tag{4.3}$$

Hence,

$$P_z \left( \tau_x^{(N)} \geq 3S_N \right) \leq \frac{1}{3S_N} E_z[\tau_x^{(N)}] \leq \frac{1}{3} \tag{4.4}$$

so by the strong Markov property at  $\tau_x^{(N)}$ , we see that for any  $z \in V(G^{(N)})$ ,

$$\begin{aligned} P_z \left( \min_{y \in B_R^{(N)}(x, \kappa)} \{ \widehat{L}_{4S_N}^{(N)}(y) \} \geq \epsilon \right) &\geq \sum_{t=0}^{3S_N} P_z \left( \min_{y \in B_R^{(N)}(x, \kappa)} \{ \widehat{L}_{4S_N}^{(N)}(y) - \widehat{L}_t^{(N)}(y) \} \geq \epsilon, \tau_x^{(N)} = t \right) \\ &= \sum_{t=0}^{3S_N} P_z(\tau_x^{(N)} = t) P_x \left( \min_{y \in B_R^{(N)}(x, \kappa)} \{ \widehat{L}_{4S_N-t}^{(N)}(y) \} \geq \epsilon \right) \\ &\geq P_z \left( \tau_x^{(N)} \leq 3S_N \right) P_x \left( \min_{y \in B_R^{(N)}(x, \kappa)} \{ \widehat{L}_{S_N}^{(N)}(y) \} \geq \epsilon \right) \geq \frac{1}{2}, \end{aligned}$$

by combining Step 1 and (4.4). □

Denoting the range of the random walk by  $\text{Range}_t^{(N)} := \{X_0^{(N)}, X_1^{(N)}, \dots, X_{t-1}^{(N)}\}$ , we have the following consequence of Proposition 4.3.

**Corollary 4.4.** *If Assumptions 1.1 and 1.2 hold, then for some  $\kappa > 0$  and any  $t$ ,*

$$\sup_{N \geq 1} \sup_{x, z \in V(G^{(N)})} P_z \left( \text{Range}_t^{(N)} \not\supseteq B_R^{(N)}(x, \kappa) \right) \leq 2^{1-t/(4S_N)}.$$



*Proof.* Taking  $\kappa > 0$  as in Proposition 4.3, we have that for all  $N$  and  $x, z \in V(G^{(N)})$ ,

$$P_z \left( \text{Range}_{4S_N}^{(N)} \supseteq B_R^{(N)}(x, \kappa) \right) \geq \frac{1}{2}.$$

Applying the Markov property at times  $\{4iS_N\}$  for  $i = 1, \dots, k - 1$ , it follows that

$$P_z \left( \text{Range}_{4kS_N}^{(N)} \not\supseteq B_R^{(N)}(x, \kappa) \right) \leq 2^{-k}$$

and we are done, since  $t \mapsto \text{Range}_t^{(N)}$  is non-decreasing. □

*Proof of Proposition 1.5.* From Proposition 3.7, if  $c_R^2 \eta^{d_w - d_f} \leq \kappa$ , then for any  $N$  and  $x \in V(G^{(N)})$ ,

$$B^{(N)}(x, \eta R_N) \subseteq B_R^{(N)}(x, \kappa).$$

Setting such  $\eta = \eta(c_R, \kappa) > 0$  we deduce from Proposition 3.6 that for any  $\kappa > 0$  there exist  $L = L(\kappa)$  finite and  $x_1, \dots, x_L \in V(G^{(N)})$ , such that for all  $N$ ,

$$V(G^{(N)}) = \bigcup_{i=1}^L B_R^{(N)}(x_i, \kappa).$$

We embed the walk  $X_s^{(N)}$  within the sample path  $s \mapsto \tilde{X}_s^{(N)}$  of its lazy counterpart, such that the number of steps  $M_t$  made by the lazy walk during the first  $t$  steps of  $\{X_s^{(N)}\}$  is the sum of  $t$  i.i.d. Geometric(1/2) variables, which are further independent of  $\{X_s^{(N)}\}$ . Since the range of the lazy random walk at time  $M_t$  is then  $\text{Range}_t^{(N)}$ , we have for any  $t$ ,  $N$  and  $z \in V(G^{(N)})$

$$P_z \left( \tau_{\text{cov}}(G^{(N)}) > 3t \right) \leq P(M_t > 3t) + \sum_{i=1}^L P_z \left( \text{Range}_t^{(N)} \not\supseteq B_R^{(N)}(x_i, \kappa) \right).$$

By Cramer-Chernoff bound, the first term on the RHS is at most  $\theta^t$  for some  $\theta < 1$ . With  $L = L(\kappa)$  independent of  $N$ ,  $z$ , and  $S_N \leq c_* T_N$  (see Corollary 3.8), we thus reach (1.6) upon choosing  $\kappa > 0$  as in Corollary 4.4 and  $c_0 \geq 2L(\kappa) + 1$  such that  $e^{-3/c_0} \geq \max(\theta, 2^{-1/(4c_*)})$ . □

## 5 Lamplighter mixing: Theorem 1.4 and Proposition 1.6

*Proof of Proposition 1.6.* W.l.o.g. we may and do assume that  $x_0 = (0, x_0)$  for some  $x_0 \in V(G^{(N)})$ . Let

$$A_N^* := \left\{ (f, x) \in V(\mathbb{Z}_2 \wr G^{(N)}) \mid \exists y \in V(G^{(N)}) \text{ such that } f(b) \equiv 0, \quad \forall b \in B^{(N)}(y, r_N) \right\},$$

where taking  $r_N := \lceil (2d_f \tilde{c}_v \log_2 R_N)^{1/d_f} \rceil$  we have thanks to (1.3) and the  $d_f$ -set condition, that

$$\#B^{(N)}(y, r_N) \geq \tilde{c}^{-1} V^{(N)}(y, r_N) \geq (\tilde{c}_v)^{-1} (r_N)^{d_f} \geq 2d_f \log_2 R_N.$$

By the same reasoning  $\#V(G^{(N)}) \leq \tilde{c}_v (R_N)^{d_f}$ , so for the invariant distribution  $\pi^*(\cdot; G^{(N)})$  of the lamplighter chain  $Y^{(N)}$  on  $\mathbb{Z}_2 \wr G^{(N)}$

$$\pi^*(A_N^*; G^{(N)}) \leq \sum_{y \in V(G^{(N)})} 2^{-\#B^{(N)}(y, r_N)} \leq \tilde{c}_v (R_N)^{-d_f}. \tag{5.1}$$

Part of our  $d_f$ -set condition is having  $R_N \rightarrow \infty$ , so there exists  $N_1$  finite such that  $R_0 \leq r_N \leq \frac{1}{4}R_N$  for  $R_0$  of Corollary 3.2 and any  $N \geq N_1$ . Since  $\max_y \{d^{(N)}(x_0, y)\} \geq \frac{1}{2}R_N$  for any  $x_0 \in V(G^{(N)})$ , whenever  $N \geq N_1$  the event

$$\tilde{\Gamma}_t^{(N)} := \left\{ \max_{0 \leq s \leq t} d(\tilde{X}_0^{(N)}, \tilde{X}_s^{(N)}) \leq \frac{1}{4}R_N \right\}$$

implies that  $\{Y_t^{(N)} \in A_N^*\}$ . Consequently, for any such  $N$  we have by (5.1) that

$$\begin{aligned} \max_{\mathbf{x} \in V(\mathbb{Z}_2 \wr G^{(N)})} \|P_t^*(\mathbf{x}, \cdot; G^{(N)}) - \pi^*(\cdot; G^{(N)})\|_{\text{TV}} &\geq P_{\mathbf{x}_0}^*(Y_t^{(N)} \in A_N^*; G^{(N)}) - \pi^*(A_N^*; G^{(N)}) \\ &\geq P_{x_0}(\tilde{\Gamma}_t^{(N)}; G^{(N)}) - \tilde{c} c_v(R_N)^{-d_f}. \end{aligned} \quad (5.2)$$

Let  $c_1 := 4^{d_w} c_2$  for  $c_2 < \infty$  of Corollary 3.2. Then, by Corollary 3.2 at  $r = \frac{1}{4}R_N$ , we have for all  $N \geq N_1$

$$P_{x_0}(\tilde{\Gamma}_t^{(N)}; G^{(N)}) \geq P_x \left( \max_{0 \leq s \leq t} d^{(N)}(X_0^{(N)}, X_s^{(N)}) \leq \frac{1}{4}R_N \right) \geq c_1^{-1} e^{-c_1 t / T_N},$$

which together with (5.2) completes the proof. □

As shown next, at  $t \gg S_N$  the lazy walk is near equilibrium (in total variation), and the total variation distance of  $P_t^*(\mathbf{x}, \cdot; G^{(N)})$  from its equilibrium law is then controlled by the tail probabilities of  $\tau_{\text{cov}}(G^{(N)})$ .

**Proposition 5.1.** *For any  $t$ , weighted graphs  $(G^{(N)}, \mu^{(N)})$  and  $\mathbf{x} \in V(\mathbb{Z}_2 \wr G^{(N)})$ ,*

$$\begin{aligned} \|P_t^*(\mathbf{x}, \cdot; G^{(N)}) - \pi^*(\cdot; G^{(N)})\|_{\text{TV}} &\leq P_x(\tau_{\text{cov}}(G^{(N)}) > t) + \|\tilde{P}_t(x, \cdot; G^{(N)}) - \pi(\cdot; G^{(N)})\|_{\text{TV}} \\ &\leq P_x(\tau_{\text{cov}}(G^{(N)}) > t) + \frac{\sqrt{S_N}}{2\sqrt{t}}. \end{aligned} \quad (5.3)$$

*Proof.* Using the uniform (invariant) distribution of lamp configurations at  $t \geq \tau_{\text{cov}}(G^{(N)})$ , yields

$$\begin{aligned} &\|P_t^*(\mathbf{x}, \cdot; G^{(N)}) - \pi^*(\cdot; G^{(N)})\|_{\text{TV}} \\ &\leq \sum_{\mathbf{y} \in V(\mathbb{Z}_2 \wr G^{(N)})} P_{\mathbf{x}}^*(Y_t^{(N)} = \mathbf{y}, \tau_{\text{cov}}(G^{(N)}) > t) \\ &\quad + \sum_{\mathbf{y} \in V(\mathbb{Z}_2 \wr G^{(N)})} [P_{\mathbf{x}}^*(Y_t^{(N)} = \mathbf{y}, t \geq \tau_{\text{cov}}(G^{(N)})) - \pi^*(\mathbf{y}; G^{(N)})]_+ \\ &\leq P_x(\tau_{\text{cov}}(G^{(N)}) > t) + \sum_{y \in V(G^{(N)})} [P_x(\tilde{X}_t^{(N)} = y) - \pi^{(N)}(y)]_+. \end{aligned}$$

Applying the definition of total variation distance for  $\tilde{X} = \{\tilde{X}_t\}_{t \geq 0}$  yields the first inequality in (5.3). Next, let  $\tilde{\tau}_x^{(N)} := \min\{t \geq 0 \mid \tilde{X}_t^{(N)} = x\}$ . By the embedding of  $X^{(N)}$  within  $\tilde{X}^{(N)}$  (as in the proof of Proposition 1.5), and the commute time identity (see (4.3)), we have that for all  $N$  and  $x, z \in V(G^{(N)})$

$$E_z[\tilde{\tau}_x^{(N)}] = 2E_z[\tau_x^{(N)}] \leq 2S_N. \quad (5.4)$$

While proving [24, Lemma 4.1], it shown that for all  $N, t$  and  $x \in V(G^{(N)})$ ,

$$\left( \|\tilde{P}_t(x, \cdot; G^{(N)}) - \pi(\cdot; G^{(N)})\|_{\text{TV}} \right)^2 \leq \frac{1}{8t} \max_{z \in V(G^{(N)})} \{E_z[\tilde{\tau}_x^{(N)}]\} \quad (5.5)$$

and we get the second inequality in (5.3) by combining (5.4) and (5.5). □

**5.1 The strongly recurrent case:**  $d_f < d_w$

For  $d_f < d_w$  we get Theorem 1.4(a) by combining the lower bounds of Proposition 1.6 with the upper bounds of Propositions 1.5 and 5.1.

*Proof of Theorem 1.4(a).* Since  $R_N \rightarrow \infty$ , we deduce from Proposition 1.6 that for any  $\epsilon \in (0, 1)$ ,

$$\liminf_{N \rightarrow \infty} \left\{ \frac{T_{\text{mix}}(\epsilon; \mathbb{Z}_2 \wr G^{(N)})}{T_N} \right\} \geq -c_1^{-1} \log(c_1 \epsilon). \tag{5.6}$$

In contrast, with  $S_N \leq c_* T_N$  and  $\gamma = \gamma(\epsilon)$  denoting the unique solution of

$$\epsilon = c_0 e^{-\gamma/c_0} + \frac{\sqrt{c_*}}{2\sqrt{\gamma}},$$

we get from Propositions 1.5 and 5.1 that

$$\limsup_{N \rightarrow \infty} \left\{ \frac{T_{\text{mix}}(\epsilon; \mathbb{Z}_2 \wr G^{(N)})}{T_N} \right\} \leq \gamma(\epsilon). \tag{5.7}$$

The RHS of (5.6) blows up as  $\epsilon \rightarrow 0$ , while the RHS of (5.7) is uniformly bounded above for  $\epsilon \in [\frac{1}{2}, 1]$ . Hence, there can be no cutoff for these lamplighter chains.  $\square$

**Remark 5.2.** In view of Proposition 1.5, here  $T_{\text{mix}}(\epsilon; \mathbb{Z}_2 \wr G^{(N)})/T_{\text{cov}}(G^{(N)}) \gg 1$  for small  $\epsilon$ . From Section 4 we also learn that, when  $d_f < d_w$ , the lamplighter chains have no mixing cutoff mainly because the laws of  $\tau_{\text{cov}}(G^{(N)})/T_{\text{cov}}(G^{(N)})$  do not concentrate as  $N \rightarrow \infty$  (unlike the transient case of  $d_f > d_w$ ).

**5.2 The transient case:**  $d_f > d_w$

As mentioned before, in case  $d_f > d_w$ , we establish the cutoff for total-variation mixing time of the lamplighter chains by verifying that our weighted graphs  $\{(G^{(N)}, \mu^{(N)})\}_{N \geq 1}$  satisfy the sufficient conditions from [22, Theorem 1.5]. To this end, recall the uniform mixing times  $T_{\text{mix}}^U(G^{(N)})$  and Green functions  $\tilde{g}^{(N)}(\cdot, \cdot)$  that correspond to  $\epsilon = \frac{1}{4}$  in (3.4) and (3.11), respectively. In [22], uniformly elliptic, finite weighted graphs  $\{(G^{(N)}, \mu^{(N)})\}_{N \geq 1}$  are called *uniformly locally transient* if for all  $N$ ,

$$g(x, A; G^{(N)}) := \sum_{y \in A} \tilde{g}^{(N)}(x, y) \leq \rho(d^{(N)}(x, A), \text{diam}\{A\}), \quad \forall x \in V(G^{(N)}), A \subseteq V(G^{(N)}),$$

where  $\rho : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is such that  $\rho(r, s) \downarrow 0$  as  $r \rightarrow \infty$ , for each fixed  $s$ . Further setting

$$\bar{\Delta}(G) := \max_{x \in V} \{\mu_x\}, \quad \underline{\Delta}(G) := \min_{x \in V} \{\mu_x\}, \quad \Delta(G) := \frac{\bar{\Delta}(G)}{\underline{\Delta}(G)},$$

the following two assumptions are made in [22].

**Assumption 5.3** (Transience). *The finite weighted graphs  $\{(G^{(N)}, \mu^{(N)})\}_{N \geq 1}$  are such that for any fixed  $r < \infty$ , as  $N \rightarrow \infty$ ,*

- (a)  $\mu^{(N)}(G^{(N)}) \rightarrow \infty$ .
- (b)  $\sup_N \{\Delta(G^{(N)})\} < \infty$ .
- (c)  $\sup_x \{\log V^{(N)}(x, r)\} = o(\log \mu^{(N)}(G^{(N)}))$ .
- (d)  $T_{\text{mix}}^U(G^{(N)}) (\bar{\Delta}(G^{(N)}))^r = o(\mu^{(N)}(G^{(N)}))$ .

**Assumption 5.4** (Uniform Harnack inequalities). *For some  $C(\alpha) < \infty$  and all  $N, r \geq 1$ ,  $\alpha > 1$ ,  $x \in V(G^{(N)})$ , if  $h(\cdot)$  is a positive  $\mu^{(N)}$ -harmonic on  $B^{(N)}(x, \alpha r)$ , then*

$$\max_{y \in B^{(N)}(x, r)} \{h(y)\} \leq C(\alpha) \min_{y \in B^{(N)}(x, r)} \{h(y)\}.$$

We next prove Theorem 1.4(b) by relying on the following restatement of [22, Theorem 1.5].

**Theorem 5.5.** *If uniformly locally transient  $\{(G^{(N)}, \mu^{(N)})\}_{N \geq 1}$  satisfy Assumptions 5.3 and 5.4, then the lamplighter chains  $\{Y^{(N)}\}_{N \geq 1}$  have cutoff at the threshold  $\frac{1}{2}T_{\text{cov}}(G^{(N)})$ .*

**Remark 5.6.** The derivation of [22, Theorem 1.5] is limited to lazy SRW on graphs  $G^{(N)}$ , namely with  $\mu_{xy} \equiv 1$  for all  $xy \in E(G)$ . However, up to the obvious modifications we made in Assumptions 5.3 and 5.4, the same argument applies for uniformly elliptic weighted graphs, as re-stated in Theorem 5.5.

*Proof of Theorem 1.4(b).* Thanks to Proposition 3.9 and (1.3) we confirm that  $(G^{(N)}, \mu^{(N)})$  are uniformly locally transient for  $\rho(r, s) = c_g \tilde{c} c_v r^{d_w - d_f} s^{d_f}$ . Having  $\mu^{(N)}(G^{(N)}) \geq c_v^{-1}(R_N)^{d_f} \rightarrow \infty$  and  $G^{(N)}$  of uniformly bounded degrees (see Remark 1.3), conditions (a)-(c) of Assumption 5.3 also hold here. Further, with  $d_w < d_f$ , the bound  $T_{\text{mix}}^U(G^{(N)}) \leq c(R_N)^{d_w}$  of Proposition 3.3 yields Assumption 5.3(d). Considering Assumption 1.2 for  $u(t, \cdot) = h(\cdot)$  results with the lazy version  $\tilde{P}^{(N)}$  satisfying the uniform Harnack inequality of Assumption 5.4 for any  $\alpha > \max(2, 1/c_{\text{PHI}})$ . By our  $p_0$ -condition this is equivalent to the full Assumption 5.4 (see [28, Proposition 3.5]), and we complete the proof by applying Theorem 5.5.  $\square$

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