Extremes of local times for simple random walks on symmetric trees

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Abstract

We consider local times of the simple random walk on the b-ary tree of depth n and study a point process which encodes the location of the vertex with the maximal local time and the properly centered maximum over leaves of each subtree of depth r_n rooted at the $(n - r_n)$ level, where $(r_n)_{n \geq 1}$ satisfies $\lim_{n \to \infty} r_n = \infty$ and $\limsup_{n \to \infty} r_n/n < 1$. We show that the point process weakly converges to a Cox process with intensity measure $\alpha Z_\infty(dx) \otimes e^{-2\sqrt{\log b}\, y} dy$, where $\alpha > 0$ is a constant and Z_∞ is a random measure on $[0, 1]$ which has the same law as the limit of a critical random multiplicative cascade measure up to a scale factor. As a corollary, we establish convergence in law of the maximum of local times over leaves to a randomly shifted Gumbel distribution.

1 Introduction

Much efforts have been made in the study of the so-called log-correlated random fields such as branching Brownian motion (BBM), branching random walk (BRW), and two-dimensional discrete Gaussian free field (DGFF). One of the remarkable features of these models is that laws of their maxima share common properties: each of the laws weakly converges to a randomly shifted Gumbel distribution [34, 1, 21]. It is believed that each of the limiting extremal processes of a wide class of log-correlated fields converges to a so-called randomly shifted decorated Poisson point process [39] and it is established for the BBM [2, 5], the BRW [37], and the two-dimensional DGFF [15].

It is well-known that local times of random walks on graphs have close relationships with DGFFs thanks to "the generalized second Ray-Knight theorem" [30] (this goes back to the Dynkin isomorphism [29]) which has many applications, for example, to the cover time [25, 24, 40]. Since the occupation time field of the simple random walk on the tree or on the two-dimensional lattice is closely related to the BRW or two-dimensional DGFF

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respectively, it is natural to expect that their maxima and cover times belong to the universality class mentioned above: it is known that the cover times have subleading terms similar to other log-correlated fields [27, 9] and that they are tight [20, 10, 11], but further details are still open.

In this paper, we consider local times of the simple random walk on the b-ary tree of depth n at a time much larger than the maximal hitting time of any vertex and study convergence of a point process encoding extreme local maxima of the local times as $n \to \infty$.

To state our result, we begin with some notation. We fix an arbitrary integer $b \geq 2$ throughout the paper. We will write T to denote the b-ary tree with root ρ: this is a rooted tree whose vertices have exactly b children. Let T_i be the set of vertices in the i-th generation of T. Set $T_{\leq n} := \cup_{i=0}^{n} T_i$. For $v \in T$, we will write $|v|$ to denote the depth of v, i.e. the generation to which v belongs. For $u \in T$, let T^u be the subtree of T rooted at u, and we define T_i^u and $T_{\leq n}^u$ similarly. For $v, u \in T$, let $v \wedge u$ be the most recent common ancestor of v and u.

Let $X = (X_t, \ t \geq 0, \ P_v, \ v \in T_{\leq n})$ be the continuous-time simple random walk on $T_{\leq n}$ with exponential holding times of parameter 1. We define the local time of X by

$$L_t^n(v) := \frac{1}{\deg(v)} \int_0^t 1_{\{X_s = v\}} ds, \ \ v \in T_{\leq n}, \ t \geq 0,$$

where $\deg(v)$ is the degree of v, and the inverse local time at ρ by

$$\tau(t) := \inf\{s \geq 0 : L_s^n(\rho) > t\}, \quad t \geq 0.$$

Let $E(T)$ be the set of all edges on T. Let $(Y_e)_{e \in E(T)}$ be independent and identically distributed random variables whose common law is the normal distribution with mean 0 and variance $1/2$. To each $v \in T$, we assign $h_v := \sum_{i=1}^{|v|} Y_{e_i^v}$, where $e_1^v, \ldots, e_{|v|}^v$ are the edges on the unique shortest path from ρ to v. We will call $(h_v)_{v \in T}$ a BRW on T. It is well-known that the so-called derivative martingale

$$D_n := \sum_{v \in T_n} \left(\sqrt{\log b}\, n - h_v \right) e^{-2\sqrt{\log b}\left(\sqrt{\log b}\, n - h_v \right)}$$

converges almost surely as $n \to \infty$, and the limit

$$D_\infty := \lim_{n \to \infty} D_n \tag{1.1}$$

is positive and finite almost surely (see, for example, [12, Theorem 5.1, 5.2] or [1, Proposition A.3]). To each $v \in T$, we assign a distinct label $(\overline{v}_1, \ldots, \overline{v}_{|v|})$ with $\overline{v}_i \in \{0, \ldots, b-1\}$ for all $1 \leq i \leq |v|$ so that the vertices with labels $(\overline{v}_1, \ldots, \overline{v}_{|v|}, k)$, $k \in \{0, \ldots, b-1\}$ are children of v. We define the location of $v \in T$ by

$$\sigma(v) := \sum_{i=1}^{|v|} \frac{\overline{v}_i}{b^i}. \tag{1.2}$$

For each $n \in \mathbb{N}$ and $x \in [0, 1]$, let $v(x)$ be the vertex in T_n with $x \in [\sigma(v(x)), \sigma(v(x)) + b^{-n})$. We define the random measure called a (critical) random multiplicative cascade measure by

$$Z_n(dx) := b^n \left(\sqrt{\log b}\, n - h_{v(x)} \right) e^{-2\sqrt{\log b}\left(\sqrt{\log b}\, n - h_{v(x)} \right)} dx, \tag{1.3}$$

where dx is the Lebesgue measure on $[0, 1]$. Barral, Rhodes, and Vargas [7] observed that

$$\text{the weak limit } Z_\infty := \lim_{n \to \infty} Z_n \text{ exists almost surely.} \tag{1.4}$$

For each $v \in T_n$, set $I_v := [\sigma(v), \sigma(v) + b^{-n}]$. The random measure Z_∞ satisfies that

$$(Z_\infty(I_v))_{v \in T_n} \overset{d}{=} \left(e^{-2\sqrt{\log b}\left(\sqrt{\log b}\, n - h_v\right)} D_\infty^{(v)} \right)_{v \in T_n},$$

where "$\overset{d}{=}$" means that the laws of the left and the right are the same and $D_\infty^{(v)}, v \in T_n$ are independent copies of D_∞ which are independent of $(h_v)_{v \in T_n}$. See [8, 23] for more details on Z_∞.

For each $(x, y) \in [0, 1] \times \mathbb{R}$, we write $\delta_{(x,y)}$ to denote the Dirac measure at (x, y). For each $0 \leq m \leq n$, we define the point process on $[0, 1] \times \mathbb{R}$ by

$$\Xi_{n,t}^{(m)} := \sum_{u \in T_{n-m}} \delta_{\left(\sigma\left(\operatorname{argmax}_{v \in T_m^u}\, L_{\tau(t)}^n(v) \right),\; \max_{v \in T_m^u} \sqrt{L_{\tau(t)}^n(v)} - \sqrt{t} - a_n(t) \right)}, \tag{1.5}$$

where the centering sequence $a_n(t)$ is given by

$$a_n(t) := \sqrt{\log b}\, n - \frac{3}{4\sqrt{\log b}} \log n - \frac{1}{4\sqrt{\log b}} \log \left(\frac{\sqrt{t} + n}{\sqrt{t}} \right), \tag{1.6}$$

and for each $u \in T_{n-m}$, $\operatorname{argmax}_{v \in T_m^u} L_{\tau(t)}^n(v)$ is the vertex v_* on $T_m^u \subset T_n$ with $L_{\tau(t)}^n(v_*) = \max_{v \in T_m^u} L_{\tau(t)}^n(v)$. (If two or more vertices on T_m^u attain the maximum, we take the one whose location (recall the definition from (1.2)) is the largest among such vertices.) We regard $\Xi_{n,t}^{(m)}$ as an element of all Radon measures on Borel sets of $[0, 1] \times \mathbb{R}$ topologized with the vague topology. Since this space is metrizable as a complete separable metric space, we can consider convergence in law of sequences of random measures. Given a random measure ν on $[0, 1] \times \mathbb{R}$, we will write $\mathrm{PPP}(\nu)$ to denote a point process on $[0, 1] \times \mathbb{R}$ which, conditioned on ν, is a Poisson point process with intensity measure ν (that is $\mathrm{PPP}(\nu)$ is a Cox process). We now state the main result of this paper:

Theorem 1.1. *There exists $c_1 > 0$ such that for any sequence $(t_n)_{n \geq 1}$ with $\lim_{n \to \infty} \frac{\sqrt{t_n}}{n} = \theta \in [0, \infty]$ and $t_n \geq c_1 n \log n$ for each $n \in \mathbb{N}$, and any sequence $(r_n)_{n \geq 1}$ with $\lim_{n \to \infty} r_n = \infty$ and $\limsup_{n \to \infty} r_n / n < 1$, the point process $\Xi_{n,t_n}^{(r_n)}$ converges in law to a Cox process*

$$PPP \left(\frac{4}{\sqrt{\pi}} \beta_* \gamma_* Z_\infty(dx) \otimes 2\sqrt{\log b}\, e^{-2\sqrt{\log b}\, y} dy \right) \tag{1.7}$$

as $n \to \infty$, where Z_∞ is the random measure on $[0, 1]$ in (1.4),

$$\beta_* := \begin{cases} \sqrt{\dfrac{\theta + 1}{\theta + \sqrt{\log b}}} & \text{if } \theta \in [0, \infty), \\[3mm] 1 & \text{if } \theta = \infty, \end{cases} \tag{1.8}$$

and

$$\gamma_* := \lim_{\ell \to \infty} \int_{\ell^{2/5}}^{\ell} z e^{2\sqrt{\log b}\, z} \mathbb{P} \left(\max_{v \in T_\ell} h_v > \sqrt{\log b}\, \ell + z \right) dz. \tag{1.9}$$

Remark 1.2. The existence of the limit (1.9) is non-trivial. It is proved in the proof of Proposition 5.1. Results similar to Theorem 1.1 are known for the BBM [4] and the two-dimensional DGFF [13, 14]. Our setting is inspired by [16]. The convergence of the full extremal process has been established for the BBM [2, 5], the BRW [37], and the two-dimensional DGFF [15]. Related convergence for the local times on the b-ary tree will be studied in a sequel paper.

By Theorem 1.1 and a tail estimate of the maximum of local times over leaves (Proposition 3.1(i) below), we have:

Corollary 1.3. *There exists $c_1 > 0$ such that for all $\lambda \in \mathbb{R}$ and any sequence $(t_n)_{n \geq 1}$ with $\lim_{n \to \infty} \frac{\sqrt{t_n}}{n} = \theta \in [0, \infty]$ and $t_n \geq c_1 n \log n$ for each $n \in \mathbb{N}$,*

$$\lim_{n \to \infty} P_\rho \left(\max_{v \in T_n} \sqrt{L^n_{\tau(t_n)}(v)} \leq \sqrt{t_n} + a_n(t_n) + \lambda \right) = \mathbb{E} \left[e^{-\frac{4}{\sqrt{\pi}} \beta_* \gamma_* D_\infty e^{-2\sqrt{\log b}\, \lambda}} \right], \quad (1.10)$$

where D_∞, β_* and γ_* are given by (1.1), (1.8) and (1.9), respectively.

Remark 1.4. Let $(h_v)_{v \in T}$ be a BRW on T. By Theorem 2.2 and Lemma A.7, one can show that for all $\lambda \in \mathbb{R}$ and any sequence $(t_n)_{n \geq 1}$ with $\lim_{n \to \infty} \sqrt{t_n}/n^2 = \infty$,

$$\lim_{n \to \infty} \mathbb{P} \left(\max_{v \in T_n} h_v \leq m_n + \lambda \right)$$
$$= \lim_{n \to \infty} P_\rho \left(\max_{v \in T_n} \sqrt{L^n_{\tau(t_n)}(v)} \leq \sqrt{t_n} + m_n + \lambda \right) = \mathbb{E} \left[e^{-\frac{4}{\sqrt{\pi}} \gamma_* D_\infty e^{-2\sqrt{\log b}\lambda}} \right], \quad (1.11)$$

where γ_* is given in (1.9) and the centering sequence m_n is defined by

$$m_n := \sqrt{\log b}\, n - \frac{3}{4\sqrt{\log b}} \log n.$$

(Note that the convergence of the maximum of the BRW has already been established in [6, 1, 22].) The centering sequence $a_n(t_n)$ in (1.10) is different from m_n by the term $\frac{1}{4\sqrt{\log b}} \log \left(\frac{\sqrt{t_n} + n}{\sqrt{t_n}} \right)$ which is non-negligible only when $\theta < \infty$.

The organization of the paper is as follows. Section 2 gives preliminary lemmas which we use repeatedly throughout the paper. In Section 3, we obtain tail probabilities of the maximum of local times over leaves which are essential to next sections. One can find that for each leaf v, the law of the local time process along the path from ρ to v is the same as that of a zero-dimensional squared Bessel process (see Lemma 2.3). By this and the Markov property of local time processes (see Lemma 2.1), roughly speaking, one can regard the field of local times over the set of leaves as a *branching Bessel process*. This gives hints of how to estimate the tail of the maximum of local times over leaves: we use the *constraint* first and second moment methods developed in the BBM, BRW, and two-dimensional DGFF settings. (See, for example, [18, 1, 21]. We especially use techniques in [28, 21].) Typical behavior of a vertex with extreme local time is as follows: the local time process along the path from the root to the vertex stays below the curve defined in (3.5) and finally reach the maximal value at the vertex (see the proof of Proposition 3.1). In Section 4, we show that two leaves with local times near maxima are either very close or far away. This suggests that local maximizers are distributed as a Poisson nature. More technically, this implies that $\Xi^{(n-q)}_{n,t_n}|_{[0,1] \times [z, \infty)} = \Xi^{(r_n)}_{n,t_n}|_{[0,1] \times [z, \infty)}$ with probability tending to 1 as $n \to \infty$ and then $q \to \infty$, which is one of the key steps in the proof of Theorem 1.1. In Section 5, we obtain a limiting tail of the maximum of local times over leaves which is crucial to study the Laplace functional of $\Xi^{(n-q)}_{n,t_n}$. In the estimate, *entropic repulsion* (Lemma 2.4(ii)) plays an important role: this enables us to compute the tail of the maximum by using the reflection principle of a Brownian motion. In Section 6, we give the proof of Theorem 1.1 and Corollary 1.3.

We should emphasize that it is more convenient to study "continuous" version of local times rather than the original "discrete" ones especially when we estimate tail probabilities of the maximum of the local times over leaves. To take the advantage, motivated by [35, 40], we consider the local time process of the Brownian motion on the associated metric tree as the "continuous" version.

We will write $c_1, c_2, \ldots$ to denote positive universal constants whose values are fixed within each argument. We use $c_1(M), c_2(M) \ldots$ for positive constants which depend on

M. Given sequences $(c_n)_{n \geq 1}$ and $(c'_n)_{n \geq 1}$, we write $c'_n = O(c_n)$ if there exists a universal constant C such that $|c'_n / c_n| \leq C$ for all $n \geq 1$. We write $|S|$ to denote the cardinality of a set S.

2 Preliminary lemmas

In this section, we collect some lemmas which we use repeatedly throughout the paper. We first recall the metric tree and the Brownian motions on it. In the study of local times of random walks on graphs, Lupu [35] and Zhai [40] used the corresponding metric graphs and Brownian motions. We follow the approach and find it advantageous in obtaining precise tail probabilities of the maximum of local times over leaves on the b-ary tree. Given a graph G, we will write $E(G)$ to denote the edge set of G. Let T be the b-ary tree. We regard each $e \in E(T)$ as an interval of length $1/2$ by setting $I_e := \{e\} \times \left(0, \frac{1}{2}\right)$. Set $\bar{I}_e := I_e \cup \{e^-, e^+\}$, where $e^-, e^+ \in T$ be the endpoints of the edge e. Let π_e be the map from I_e to $\left(0, \frac{1}{2}\right)$ defined by $\pi_e((e, x)) := x$. We extend π_e to the map from $\bar{I}_e$ to $\left[0, \frac{1}{2}\right]$ by setting $\pi_e(e^-) := 0$, $\pi_e(e^+) := \frac{1}{2}$. We define a metric tree of depth n by

$$\widetilde{T}_{\leq n} := T_{\leq n} \cup \bigcup_{e \in E(T_{\leq n})} I_e.$$

For each $k \in \mathbb{N}$ and $v \in T$, we will write $\widetilde{T}^v_{\leq k}$ to denote the metric tree corresponding to the subtree $T^v_{\leq k}$. We define the metric $d(\cdot, \cdot)$ on $\widetilde{T}_{\leq n}$ as follows: for $x, y \in \widetilde{T}_{\leq n}$, let e_x and e_y be the edges with $x \in I_{e_x}$ and $y \in I_{e_y}$, respectively. In the case $I_{e_x} \neq I_{e_y}$, we define $d(x, y)$ by

$$\min \left\{ |\pi_{e_x}(x) - \pi_{e_x}(v)| + \frac{1}{2} d_g(v, u) + |\pi_{e_y}(u) - \pi_{e_y}(y)| : v \in \{e_x^-, e_x^+\}, u \in \{e_y^-, e_y^+\} \right\},$$

where d_g is the graph distance on $T_{\leq n}$. In the case $I_{e_x} = I_{e_y}$, we set $d(x, y) := |\pi_{e_x}(x) - \pi_{e_y}(y)|$. We define a measure m on $\widetilde{T}_{\leq n}$ by

$$m(dx) := \sum_{e \in E(T_{\leq n})} 1_{I_e}(x) \nu_e(dx),$$

where $\nu_e := \nu \circ \pi_e$, and ν is the Lebesgue measure on $(0, 1/2)$. We have a m-symmetric Hunt process on $\widetilde{T}_{\leq n}$ with continuous sample paths such that on each I_e, it behaves like a standard Brownian motion on $(0, 1/2)$ until it hits $\{e^-, e^+\}$, and when it starts at a vertex v, it chooses one of the edges incident to v uniformly at random, and moves on it as described above. See, for example, [31, 33, 35] for the construction. We write $\widetilde{X} = (\widetilde{X}_t, t \geq 0, \widetilde{P}_x, x \in \widetilde{T}_{\leq n})$ to denote the process and call it a Brownian motion on $\widetilde{T}_{\leq n}$. It is known that $\widetilde{X}$ restricted to $T_{\leq n}$ behaves like a simple random walk on $T_{\leq n}$ in the following sense: for all $v \in T_{\leq n}$ and $1 \leq i \leq \deg(v)$,

$$\widetilde{P}_v(\widetilde{X}_{S_v} = v_i) = \frac{1}{\deg(v)}, \tag{2.1}$$

where $v_1, \ldots, v_{\deg(v)}$ are vertices on $T_{\leq n}$ adjacent to v, and S_v is the hitting time of $\{v_1, \ldots, v_{\deg(v)}\}$ by $\widetilde{X}$. See, for example, [31, Theorem 2.1] or [35, Section 2]. By [35, Section 2], $\widetilde{X}$ has a space-time continuous local time $\{\widetilde{L}^n_t(x) : (t, x) \in [0, \infty) \times \widetilde{T}_{\leq n}\}$ and the following holds for each $v \in T_{\leq n}$ under $\widetilde{P}_v$:

$$\widetilde{L}^n_{S_v}(v) \stackrel{d}{=} \mathrm{Exp}\,(\deg(v)), \tag{2.2}$$

where $\mathrm{Exp}(m)$ is an exponential random variable with mean m^{-1}. We define the inverse local time by

$$\widetilde{\tau}(t) := \inf\{s \geq 0 : \widetilde{L}^n_s(\rho) > t\}, \quad t > 0.$$

By (2.1) and (2.2), we have

$$\left(\widetilde{L}^n_{\widetilde{\tau}(t)}(v)\right)_{v \in T_{\leq n}} \text{ under } \widetilde{P}_\rho \overset{d}{=} \left(L^n_{\tau(t)}(v)\right)_{v \in T_{\leq n}} \text{ under } P_\rho. \tag{2.3}$$

The following is the Markov property of local times of the Brownian motion on $\widetilde{T}_{\leq n}$. The discrete version can be found in [24, Lemma 2.6].

Lemma 2.1. *Fix $n \in \mathbb{N}, t > 0$, and $a \in T_{\leq n} \backslash T_n$. Let $\mathcal{F}^\uparrow$ be the σ-field generated by $\widetilde{L}^n_{\widetilde{\tau}(t)}(x)$, $x \in \{a\} \cup \widetilde{T}_{\leq n} \backslash \widetilde{T}^a_{\leq n-|a|}$. Then, the law of $\left\{\widetilde{L}^n_{\widetilde{\tau}(t)}(x) : x \in \widetilde{T}^a_{\leq n-|a|} \backslash \{a\}\right\}$ under $\widetilde{P}_\rho(\cdot|\mathcal{F}^\uparrow)$ is the same as that of $\left\{\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow\left(\widetilde{L}^n_{\widetilde{\tau}(t)}(a)\right)}(x) : x \in \widetilde{T}^a_{\leq n-|a|} \backslash \{a\}\right\}$ under $\widetilde{P}_a$, where $\left\{\widetilde{L}^\downarrow_s(x) : (s,x) \in [0,\infty) \times \widetilde{T}^a_{\leq n-|a|}\right\}$ is a local time of a Brownian motion on $\widetilde{T}^a_{\leq n-|a|}$ and $\widetilde{\tau}^\downarrow(s) := \inf\{r : \widetilde{L}^\downarrow_r(a) > s\}$.*

The proof of Lemma 2.1 is given in Section A.2.

The generalized second Ray-Knight theorem connects local times and BRWs:

Theorem 2.2. *([40]) For all $t > 0$ and $n \in \mathbb{N}$, on the same probability space, one can construct a local time $(L^n_{\tau(t)}(v))_{v \in T_{\leq n}}$ and two BRWs $(h_v)_{v \in T_{\leq n}}$, $(h'_v)_{v \in T_{\leq n}}$ on $T_{\leq n}$ satisfying the following:*

$$(L^n_{\tau(t)}(v))_{v \in T_{\leq n}} \text{ and } (h_v)_{v \in T_{\leq n}} \text{ are independent,} \tag{2.4}$$

$$L^n_{\tau(t)}(v) + (h_v)^2 = \left(h'_v + \sqrt{t}\right)^2, \text{ for each } v \in T_{\leq n}, \text{ almost surely.} \tag{2.5}$$

The construction of the coupling in Theorem 2.2 can be found in the proof of Theorem 3.1 of [40]. (Note that Zhai constructed the coupling in a more general setting and that in the context of [40], the law of the DGFF on a b-ary tree is the same as that of our BRW scaled by $\sqrt{2}$.) Let $C[0, \infty)$ be the space of real-valued continuous functions on $[0, \infty)$ and $\mathcal{B}(C[0, \infty))$ be the σ-field generated by cylinder sets in $C[0, \infty)$. Given $v \in T$ and $s \in [0, |v|]$, let v_s be the point on the unique path from ρ to v with $d(\rho, v_s) = s/2$. We have a nice connection between the local time and the 0-dimensional squared Bessel process.

Lemma 2.3. *([9, Lemma 7.7]) For all $t > 0$ and $v \in T_n$,*

$$\left\{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s) : 0 \leq s \leq n\right\} \text{ under } \widetilde{P}_\rho \overset{d}{=} \left\{\frac{1}{2}X_s : 0 \leq s \leq n\right\} \text{ under } \mathbb{Q}^0_{2t},$$

where $\mathbb{Q}^d_x$ is a law on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ under which the coordinate process $\{X_s : s \geq 0\}$ is a d-dimensional squared Bessel process started at x.

Note that our setting is different from that of [9, Lemma 7.7]. Notwithstanding, given Lemma A.1, the proof of Lemma 2.3 is almost the same as that of [9, Lemma 7.7], so we omit the proof of Lemma 2.3. It is known that the laws of 0-dimensional and 1-dimensional squared Bessel processes are related to each other by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^0_x}{d\mathbb{Q}^1_x}\big|_{\mathcal{F}_t \cap \{H_0 > t\}} = \left(\frac{x}{X_t}\right)^{1/4} \exp\left(-\frac{3}{8}\int_0^t \frac{1}{X_s}ds\right), \tag{2.6}$$

for all $t > 0$ and $x > 0$, where $H_0 := \inf\{t \geq 0 : X_t = 0\}$ and $\mathcal{F}_t$ is the σ-field generated by $\{X_s : s \leq t\}$. See, for example, [9, (7.31)]. The transition semigroup $\{Q^0_t : t \geq 0\}$ of a 0-dimensional squared Bessel process is given by

$$Q^0_t(x, \cdot) = \exp\left(-\frac{x}{2t}\right)\delta_0 + \widetilde{Q}_t(x, \cdot), \quad x > 0, \tag{2.7}$$

where δ_0 is the Dirac measure at 0. $\widetilde{Q}_t(x, \cdot)$ in (2.7) has the density

$$q_t^0(x, y) = \frac{1}{2t} \sqrt{\frac{x}{y}} \exp\left(-\frac{x+y}{2t}\right) I_1\left(\frac{\sqrt{xy}}{t}\right), \quad x, y \in (0, \infty), \tag{2.8}$$

where $I_1(\cdot)$ is the modified Bessel function of the first kind

$$I_1(z) = \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^{2k+1} \frac{1}{k!(k+1)!}. \tag{2.9}$$

We will use the following asymptotic behavior of $I_1(\cdot)$:

$$I_1(z) = \frac{e^z}{\sqrt{2\pi z}}(1 + O(1/z)), \quad \text{as } z \to \infty. \tag{2.10}$$

See, for example, [38, Chapter XI, §1] or [17, Section 2] for the details on squared Bessel processes.

Let $(B_s, s \geq 0, P_x^B, x \in \mathbb{R})$ be a Brownian motion on $\mathbb{R}$ with variance $1/2$. To estimate tail probabilities of the maximum of local times over leaves, we frequently use the following.

Lemma 2.4. *([21, Lemma 3.6]) Fix a constant $C > 0$. For $z > 0$ and $s > 0$, set*

$$\mu_{s,z}(x)dx := P_0^B\left(B_s \in dx, B_r \leq z, \ 0 \leq \forall r \leq s\right)$$
$$= \frac{1}{\sqrt{\pi s}}\left(e^{-\frac{x^2}{s}} - e^{-\frac{(2z-x)^2}{s}}\right) dx, \quad x \leq z, \tag{2.11}$$

$$\mu_{s,z}^*(x)dx := P_0^B\left(B_s \in dx, B_r \leq z + z^{\frac{1}{20}} + C(r \wedge (s-r))^{\frac{1}{20}}, \ 0 \leq \forall r \leq s\right).$$

(i) There exists $c_1 > 0$ such that for any $z > 1$, $s > 0$, and $x \leq z + z^{\frac{1}{20}}$,

$$\mu_{s,z}^*(x) \leq c_1 z(z + z^{\frac{1}{20}} - x)s^{-\frac{3}{2}}e^{-\frac{x^2}{s}}.$$

(ii) There exists δ_z with $\lim_{z\to\infty} \delta_z = 0$ such that for all $z > 1$, $x \leq 0$, and $s \geq x^2 + z^2$,

$$\mu_{s,z}^*(x) \leq (1 + \delta_z)\mu_{s,z}(x).$$

3 Tail of maximum of local time over leaves

The aim of this section is to obtain the following tail estimates of the maximum of local times of the simple random walk on the b-ary tree over leaves. Recall the definition of $a_n(t)$ from (1.6).

Proposition 3.1. *(i) There exist $c_1, c_2 \in (0, \infty)$ such that for all $t > 0, y \geq 0$, and $n \in \mathbb{N}$,*

$$P_\rho\left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y\right) \leq c_1(1 + y)e^{-2\sqrt{\log b}\, y}e^{-c_2\frac{y^2}{n}}. \tag{3.1}$$

(ii) There exist $c_3 > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $y \in [0, 2\sqrt{n}]$, and $t \geq n$,

$$P_\rho\left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y\right) \geq c_3(1 + y)e^{-2\sqrt{\log b}\, y}. \tag{3.2}$$

An outline of the proof of Proposition 3.1(i) is as follows: Recall the definition of v_s defined before Lemma 2.3. We will prove in Lemma 3.2 that for each $v \in T_n$, the local time process $(\sqrt{\widetilde{L}_{\tilde\tau(t)}^n(v_s)})_{s \in [0,n]}$ does not cross the curve $g_{y,t,n}(\cdot)$ (see (3.5) for definition)

with probability at least $1 - c(1+y)e^{-2\sqrt{\log b}\, y}e^{-c'\frac{y^2}{n}}$. Thus, we can focus our attention on a truncated version of the event in Proposition 3.1 which says that there exists a leaf $v \in T_n$ satisfying the following: the local time process $(\sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s)})_{s \in [0,n]}$ stays below the barrier $g_{y,t,n}(\cdot)$ and reaches the maximal value $\sqrt{t} + a_n(t) + y$ at $s = n$. Using Lemma 2.3 and (2.6), we reduce the barrier estimate for the local time process to an analogue for a Brownian motion and then apply Lemma 2.4.

In the proof of Proposition 3.1(ii), we will apply the second moment method to $Z = \sum_{v \in T_n} 1_{A^n_v(t)}$, where $A^n_v(t), v \in T_n$ are events which will be defined in (3.25). In order to get the lower bound on the first moment $\widetilde{E}_\rho[Z]$, we apply Lemma 2.3 and (2.6) and reduce the estimate of $\widetilde{E}_\rho[Z]$ to a barrier estimate for a Brownian motion. We need more work to obtain the upper bound on the second moment $\widetilde{E}_\rho[Z^2]$. Fix leaves $u, v \in T_n$ with $u \neq v$ and set $\ell = |u \wedge v|$. The event $A^n_u(t) \cap A^n_v(t)$ is described in terms of local time processes $L^\uparrow = (\sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s)})_{s \in [0,\ell]}$ and $L^{\downarrow,w} = (\sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(w_s)})_{s \in [\ell,n]}$, $w = u, v$. By the Markov property of the local time (Lemma 2.1) and independence of Brownian motions on different subtrees, conditioned on $L^\uparrow$, $L^{\downarrow,u}$ and $L^{\downarrow,v}$ are independent. Thanks to the independence, the estimate of $\widetilde{P}_\rho[A^n_u(t) \cap A^n_v(t)]$ is reduced to barrier estimates for $L^\uparrow$, $L^{\downarrow,u}$, and $L^{\downarrow,v}$.

We first prove Proposition 3.1(i). Fix $\kappa \in \left(\frac{1}{2\sqrt{\log b}}, \infty\right)$. For $y > 0$ and $n \in \mathbb{N}$, we define the event $G^n_y(t)$ by

$$\left\{ \exists v \in T_n, \exists s \in [0,n] : \sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s)} \geq \sqrt{t} + \frac{a_n(t)}{n}s + \kappa(\log(s \wedge (n-s)))_+ + y + 1 \right\}, \quad (3.3)$$

where $c_+ := \max\{c, 0\}$. We prove that $G^n_y(t)$ is a rare event, that is, every local time process along the path from the root to a leaf stays below the curve $s \mapsto \sqrt{t} + \frac{a_n(t)}{n}s + \kappa(\log(s \wedge (n-s)))_+ + y + 1$ with high probability:

Lemma 3.2. *There exist $c_1, c_2 \in (0, \infty)$ such that for all $t > 0$, $n \in \mathbb{N}$, and $y \geq 0$,*

$$\widetilde{P}_\rho(G^n_y(t)) \leq c_1(1+y)e^{-2\sqrt{\log b}\, y}e^{-c_2\frac{y^2}{n}}. \quad (3.4)$$

Proof. We first consider the case $y > M$, where M is sufficiently large constant. Set

$$g_{y,t,n}(s) := \sqrt{t} + \frac{a_n(t)}{n}s + \kappa(\log(s \wedge (n-s)))_+ + y + 1, \quad 0 \leq s \leq n, \quad (3.5)$$

$$m_{y,t,n}(j) := \sqrt{t} + \frac{a_n(t)}{n}j + \kappa \min_{j \leq r \leq j+1}(\log(r \wedge (n-r)))_+ + y + 1, \quad 0 \leq j \leq n-1.$$

Recall the probability measure $\mathbb{Q}^d_x$ defined in Lemma 2.3 and set

$$\tau := \inf\left\{ s \geq 0 : \sqrt{X_s/2} \geq g_{y,t,n}(s) \right\},$$

where X is a coordinate process. Fix $\delta \in (0,1)$. By Lemma 2.3, $\widetilde{P}_\rho(G^n_y(t))$ is bounded from above by

$$\widetilde{P}_\rho\left(\begin{array}{c} 0 \leq \exists j \leq n-1, \exists v \in T_{j+1}, \exists s \in (j, j+1] : \\ \sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_r)} < g_{y,t,n}(r), \ 0 \leq \forall r \leq j, \ \sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s)} \geq g_{y,t,n}(s) \end{array} \right)$$

$$\leq \sum_{j=0}^{n-1} b^{j+1} \mathbb{Q}^0_{2t}\left(\tau \in (j, j+1], \ \sqrt{X_{j+1}/2} \geq \delta m_{y,t,n}(j) \right)$$

$$+ \sum_{j=0}^{n-1} b^{j+1} \mathbb{Q}_{2t}^0 \left(\tau \in (j, j+1], \ \sqrt{X_{j+1}/2} < \delta m_{y,t,n}(j) \right)$$

$$=: \sum_{j=0}^{n-1} b^{j+1} I_j^{(1)} + \sum_{j=0}^{n-1} b^{j+1} I_j^{(2)}. \tag{3.6}$$

Fix $0 \le j \le n-1$. We first estimate $I_j^{(2)}$. By the strong Markov property of a 0-dimensional squared Bessel process and (2.7), we have

$$I_j^{(2)} = \mathbb{Q}_{2t}^0 \left[1_{\{\tau \in (j,j+1)\}} \mathbb{Q}_{X_\tau}^0 \left(\sqrt{X_{j+1-\tau}/2} < \delta m_{y,t,n}(j) \right) \right]$$

$$= \mathbb{Q}_{2t}^0 \left[1_{\{\tau \in (j,j+1)\}} \exp \left(-\frac{X_\tau}{2(j+1-\tau)} \right) \right]$$

$$+ \mathbb{Q}_{2t}^0 \left[1_{\{\tau \in (j,j+1)\}} \int_0^{2\delta^2(m_{y,t,n}(j))^2} \frac{\sqrt{X_\tau/z}}{2(j+1-\tau)} e^{-\frac{X_\tau + z}{2(j+1-\tau)}} I_1 \left(\frac{\sqrt{X_\tau z}}{j+1-\tau} \right) dz \right]$$

$$=: J_1 + J_2. \tag{3.7}$$

By the definition of τ, we have

$$J_1 \le \exp\{-(m_{y,t,n}(j))^2\}. \tag{3.8}$$

Assume that $\tau \in (j, j+1)$. Recall the definition of I_1 from (2.9). If $z \le \frac{M(j+1-\tau)^2}{X_\tau}$, then we have

$$I_1 \left(\frac{\sqrt{X_\tau z}}{j+1-\tau} \right) \le \sum_{k=0}^\infty (\sqrt{M}/2)^{2k+1} \frac{1}{k!(k+1)!} \le c_1(M). \tag{3.9}$$

If $z > \frac{M(j+1-\tau)^2}{X_\tau}$, then by (2.10) and the assumption that M is sufficiently large, we have

$$I_1 \left(\frac{\sqrt{X_\tau z}}{j+1-\tau} \right) \le c_2 \frac{e^{\frac{\sqrt{X_\tau z}}{j+1-\tau}}}{\sqrt{\frac{\sqrt{X_\tau z}}{j+1-\tau}}}. \tag{3.10}$$

By (3.9) and (3.10), we have

$$J_2 \le c_3(M) \left(\max_{j \le r \le j+1} g_{y,t,n}(r) \right) \exp\{-c_4(m_{y,t,n}(j))^2\}, \tag{3.11}$$

where we have used the inequality $(j+1-\tau)^{-1/2} e^{-\frac{(\sqrt{X_\tau} - \sqrt{z})^2}{2(j+1-\tau)}} \le e^{-c_4(m_{y,t,n}(j))^2}$ for all $z \in [0, 2\delta^2(m_{y,t,n}(j))^2]$. By (3.8) and (3.11), we have

$$\sum_{j=0}^{n-1} b^{j+1} I_j^{(2)} \le c_5(M) \sum_{j=0}^{n-1} b^{j+1} e^{-c_6(\sqrt{t}+y+1)^2} e^{-c_7(\sqrt{t}+y+1)j} \le c_8(M) e^{-c_6(\sqrt{t}+y+1)^2}, \tag{3.12}$$

where in the last inequality, we have used the assumption that $y > M$ and M is sufficiently large.

Next, we will estimate $I_j^{(1)}$. Fix $1 \le j \le n-1$. By (2.6), $I_j^{(1)}$ is equal to

$$\mathbb{Q}_{2t}^1 \left[\left(\frac{2t}{X_{j+1}} \right)^{1/4} \exp \left(-\frac{3}{8} \int_0^{j+1} \frac{ds}{X_s} \right) 1_{\left\{ \tau \in (j,j+1], \sqrt{X_{j+1}/2} \ge \delta m_{y,t,n}(j) \right\} \cap \{H_0 > j+1\}} \right]. \tag{3.13}$$

Recall that $(B_s, s \geq 0, P_x^B, x \in \mathbb{R})$ is a Brownian motion on $\mathbb{R}$ with variance $1/2$. Since the law of a 1-dimensional squared Bessel process is the same as that of a square of a standard Brownian motion on $\mathbb{R}$, (3.13) is bounded from above by $\sqrt{\frac{\sqrt{t}}{\delta m_{y,t,n}(j)}}$ times

$$P_{\sqrt{t}}^B \left(B_r < g_{y,t,n}(r), \ \forall r \in [0,j], \ B_s \geq g_{y,t,n}(s), \ \exists s \in [j, j+1] \right)$$

$$\leq E_0^B \left[1_{\left\{ B_r < g_{y,t,n}(r) - \sqrt{t}, \ \forall r \in [0,j] \right\}} P_{B_j}^B \left(B_s \geq g_{y,t,n}(j+s) - \sqrt{t}, \ \exists s \in [0,1] \right) \right], \quad (3.14)$$

where we have used the translation invariance and Markov property of B in the last inequality.

Let $\widetilde{P}_j^B$ be the probability measure defined by

$$\widetilde{P}_j^B(A) = E_0^B \left[1_A \exp \left\{ \frac{2 a_n(t)}{n} B_j - \frac{(a_n(t))^2}{n^2} j \right\} \right], \quad A \in \sigma(B_s : s \leq j). \quad (3.15)$$

By the Girsanov theorem, under $\widetilde{P}_j^B$, the process

$$\left\{ \widetilde{B}_s := B_s - \frac{a_n(t)}{n} s \ : \ 0 \leq s \leq j \right\} \quad (3.16)$$

is a Brownian motion on $\mathbb{R}$ with variance $1/2$ started at 0. By the change of measure (3.15), the right of (3.14) is bounded from above by

$$\widetilde{E}_j^B \left[e^{-\frac{2 a_n(t)}{n} \widetilde{B}_j - \frac{(a_n(t))^2}{n^2} j} 1_{\left\{ \widetilde{B}_r < y + c_9 + \kappa \log(j \wedge (n-j)) + c_{10}(r \wedge (j-r))^{1/20}, \ \forall r \in [0,j] \right\}} \right. $$
$$\left. \times P_{\widetilde{B}_j}^B \left[\max_{0 \leq s \leq 1} B_s \geq y - c_{11} + \kappa \log(j \wedge (n-j)) \right] \right]$$

$$\leq \widetilde{E}_j^B \left[\begin{array}{l} e^{-\frac{2 a_n(t)}{n} \widetilde{B}_j - \frac{(a_n(t))^2}{n^2} j} 1_{\left\{ \widetilde{B}_r < y + c_9 + \kappa \log(j \wedge (n-j)) + c_{10}(r \wedge (j-r))^{1/20}, \ \forall r \in [0,j] \right\}} \\ \times 1_{\left\{ \widetilde{B}_j < y - c_{11} + \kappa \log(j \wedge (n-j)) \right\}} \\ \times P_{\widetilde{B}_j}^B \left[\max_{0 \leq s \leq 1} B_s \geq y - c_{11} + \kappa \log(j \wedge (n-j)) \right] \end{array} \right]$$

$$+ \widetilde{E}_j^B \left[\begin{array}{l} e^{-\frac{2 a_n(t)}{n} \widetilde{B}_j - \frac{(a_n(t))^2}{n^2} j} 1_{\left\{ \widetilde{B}_r < y + c_9 + \kappa \log(j \wedge (n-j)) + c_{10}(r \wedge (j-r))^{1/20}, \ \forall r \in [0,j] \right\}} \\ \times 1_{\left\{ \widetilde{B}_j \in [y - c_{11} + \kappa \log(j \wedge (n-j)), \ y + c_9 + \kappa \log(j \wedge (n-j))] \right\}} \end{array} \right]. \quad (3.17)$$

To estimate the tail of $\max_{0 \leq s \leq 1} B_s$ in the first term of the right of (3.17), we use the following:

$$P_0^B \left(\max_{0 \leq s \leq 1} B_s \geq \lambda \right) \leq e^{-\lambda^2}, \quad \text{for each } \lambda > 0 \quad (3.18)$$

(see, for example, [32, Chapter 2, (8.4)]). By Lemma 2.4(i) and (3.18), the right of (3.17) is bounded from above by

$$c_{12} e^{-\frac{2 a_n(t)}{n} y - \frac{(a_n(t))^2}{n^2} j} \left(1 + \frac{y}{j+1} \right) (y + c_9 + \kappa \log(j \wedge (n-j)))$$

$$\times j^{-\frac{3}{2}} (j \wedge (n-j))^{-2\sqrt{\log b} \, \kappa} e^{-c_{13} \frac{y^2}{j+1}}. \quad (3.19)$$

Similarly, in the case $j = 0$, by (2.6) and (3.18), we have

$$I_0^{(1)} \leq c_{14} P_0^B \left(\max_{0 \leq s \leq 1} B_s \geq y + 1 \right) \leq c_{14} e^{-(y+1)^2}. \quad (3.20)$$

Thus, by (3.19), (3.20), and the condition $\kappa > 1/(2\sqrt{\log b})$, we have

$$\sum_{j=0}^{n-1} b^{j+1} I_j^{(1)} \le c_{15}(1+y)e^{-2\sqrt{\log b}\, y} e^{-c_{16} \frac{y^2}{n}}. \tag{3.21}$$

Thus, by (3.6), (3.12) and (3.21), we have (3.4) for $y > M$. For $y \le M$, (3.4) holds if we take c_1 in (3.4) sufficiently large depending on M. $\qquad\square$

We now prove Proposition 3.1(i).

Proof of Proposition 3.1(i). Recall the definitions of the event $G_y^n(t)$ and the function $g_{y,t,n}(\cdot)$ from (3.3) and (3.5). In view of Lemma 3.2, it is natural to impose the restriction that local time processes stay below the curve $s \mapsto g_{y,t,n}(s)$: we have

$$P_\rho\left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \ge \sqrt{t} + a_n(t) + y\right) \le \widetilde{P}_\rho\left(\cup_{v \in T_n} E_v^n(t)\right) + \widetilde{P}_\rho(G_y^n(t)), \tag{3.22}$$

where for each $v \in T_n$, we define the event $E_v^n(t)$ by

$$\left\{\sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v_s)} \le g_{y,t,n}(s),\ \forall s \in [0,n],\ \sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v)} \in \left[\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1\right]\right\}.$$

Fix any $v \in T_n$. Recall the process $\widetilde{B}$ from (3.16). By Lemma 2.3, (2.6), and the change of measure (3.15) with $j = n$, $\widetilde{P}_\rho(E_v^n(t))$ is bounded from above by

$$\sqrt{\frac{\sqrt{t}}{\sqrt{t} + a_n(t) + y}} \widetilde{E}_n^B\left[e^{-\frac{2a_n(t)}{n}\widetilde{B}_n - \frac{(a_n(t))^2}{n}} 1_{\left\{\widetilde{B}_s \le y+1+c_1\kappa(s\wedge(n-s))^{\frac{1}{20}},\ \forall s\in[0,n],\ \widetilde{B}_n\in[y,y+1]\right\}}\right]. \tag{3.23}$$

By Lemma 2.4(i), the right of (3.23) is bounded from above by

$$c_2 b^{-n}(1+y)e^{-2\sqrt{\log b}\, y} e^{-c_3 \frac{y^2}{n}}. \tag{3.24}$$

Thus, by (3.22), (3.24), and Lemma 3.2, we have (3.1). $\qquad\square$

Next, we prove Proposition 3.1(ii). Fix $\delta \in (0,1)$. For $v \in T_n$, set the event

$$A_v^n(t) := \left\{\begin{array}{l} \delta\sqrt{t} \le \sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v_s)} < \sqrt{t} + \frac{a_n(t)}{n}s + y + 1,\ \forall s \in [0,n], \\ \sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v)} \in [\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1) \end{array}\right\}. \tag{3.25}$$

To obtain Proposition 3.1(ii), we will apply the second moment method to $\sum_{v \in T_n} 1_{A_v^n(t)}$. We first need the following:

Lemma 3.3. *There exist $c_1 > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $y \in [0, 2\sqrt{n}]$, $t \ge n$, and $v \in T_n$,*

$$\widetilde{P}_\rho(A_v^n(t)) \ge c_1 b^{-n}(1+y)e^{-2\sqrt{\log b}\, y}. \tag{3.26}$$

Proof. Fix any $t \ge n$. By Lemma 2.3 and (2.6), $\widetilde{P}_\rho(A_v^n(t))$ is bounded from below by

$$c_1 \sqrt{\frac{\sqrt{t}}{\sqrt{t} + a_n(t) + y + 1}} P_0^B\left(\begin{array}{l} -(1-\delta)\sqrt{t} \le B_s < \frac{a_n(t)}{n}s + y + 1,\ \forall s \in [0,n], \\ B_n \in [a_n(t) + y, a_n(t) + y + 1) \end{array}\right)$$

$$\ge c_1 \sqrt{\frac{\sqrt{t}}{\sqrt{t} + a_n(t) + y + 1}}\, (J_1 - J_2), \tag{3.27}$$

where we set

$$J_1 := P_0^B \left(B_s < \frac{a_n(t)}{n} s + y + 1, \ \forall s \in [0, n], \ B_n \in [a_n(t) + y, a_n(t) + y + 1) \right),$$

$$J_2 := P_0^B \left(B_s < -(1 - \delta)\sqrt{t}, \ \exists s \in [0, n], \ B_n \in [a_n(t) + y, a_n(t) + y + 1) \right).$$

We first obtain an upper bound of J_2. By using the density

$$P_0^B \left(B_s \in dx, \ \max_{r \in [0,s]} B_r \in dz \right) = \frac{4(2z - x)}{\sqrt{\pi s^3}} e^{-\frac{(2z-x)^2}{s}} dx dz, \ \ s > 0, \ x \le z, \ z \ge 0, \quad (3.28)$$

(see, for example, [32, Chapter 2, Proposition 8.1]), for all $n \in \mathbb{N}$ and $y \in [0, 2\sqrt{n}]$, we have

$$J_2 = P_0^B \left(\max_{0 \le s \le n} B_s > (1 - \delta)\sqrt{t}, \ B_n \in (-a_n(t) - y - 1, -a_n(t) - y] \right)$$

$$\le c_2 b^{-n} n e^{-c_3 \sqrt{t}} \sqrt{\frac{\sqrt{t} + n}{\sqrt{t}}} e^{-2\sqrt{\log b} \, y}, \quad (3.29)$$

where we have used the symmetry of B in the first equality.

Next, we obtain a lower bound of J_1. Recall the process $\widetilde{B}$ from (3.16). By the change of measure (3.15) with $j = n$, we have

$$J_1 = \widetilde{E}_n^B \left[e^{-\frac{2a_n(t)}{n} \widetilde{B}_n - \frac{(a_n(t))^2}{n}} 1_{\{\widetilde{B}_s < y+1, \ \forall s \in [0,n], \ \widetilde{B}_n \in [y, \ y+1)\}} \right]. \quad (3.30)$$

By the reflection principle (2.11), for all $n \ge n_0$ (n_0 is sufficiently large) and $y \in [0, 2\sqrt{n}]$, (3.30) is bounded from below by

$$c_4 b^{-n} \sqrt{\frac{\sqrt{t} + n}{\sqrt{t}}} (y + 1) e^{-2\sqrt{\log b} \, y}, \quad (3.31)$$

where we used $e^{-\frac{(y+1-z)^2}{n}} - e^{-\frac{(y+1+z)^2}{n}} \ge c_5 \frac{y+1}{n}$ for all $z \in [1/2, 1]$. Thus, by (3.27), (3.29), and (3.31), we have (3.26). $\qquad \square$

To obtain upper bounds of $\widetilde{P}_\rho \left(A_u^n(t) \cap A_v^n(t) \right), u, v \in T_n$, we need the following:

Lemma 3.4. *(1) There exists $c_1 > 0$ such that for all $n \in \mathbb{N}$, $t > 0$, $v \in T_n$, $0 \le \ell \le n - 1$, $s < (\sqrt{t} + \frac{a_n(t)}{n} \ell + y + 1)^2$, and $y \ge 0$,*

$$\widetilde{P}_{v_\ell} \left(\begin{array}{l} \sqrt{\widetilde{L}_{\widetilde{\tau}^\downarrow(s)}^\downarrow(v_r)} < \sqrt{t} + \frac{a_n(t)}{n} r + y + 1, \ \forall r \in [\ell, \ n], \\ \sqrt{\widetilde{L}_{\widetilde{\tau}^\downarrow(s)}^\downarrow(v)} \in [\sqrt{t} + a_n(t) + y, \ \sqrt{t} + a_n(t) + y + 1) \end{array} \right)$$

$$\le c_1 (n - \ell)^{-3/2} \sqrt{\frac{\sqrt{s}}{\sqrt{t} + a_n(t) + y}} \left(\sqrt{t} - \sqrt{s} + \frac{a_n(t)}{n} \ell + y + 1 \right) e^{-\frac{(\sqrt{t} - \sqrt{s} + a_n(t) + y)^2}{n - \ell}}, \quad (3.32)$$

where $\left\{ \widetilde{L}_r^\downarrow(x) : (r, x) \in [0, \infty) \times \widetilde{T}_{\le n - \ell}^{v_\ell} \right\}$ is a local time of a Brownian motion on $\widetilde{T}_{\le n - \ell}^{v_\ell}$ and $\widetilde{\tau}^\downarrow(s) := \inf\{r \ge 0 : \widetilde{L}_r^\downarrow(v_\ell) > s\}$.
(2) There exists $c_3 > 0$ such that for all $n \in \mathbb{N}$, $t > 0$, $v \in T_n$, and $y \ge 0$,

$$\widetilde{P}_\rho(A_v^n(t)) \le c_3 b^{-n} (y + 1) e^{-2\sqrt{\log b} \, y}. \quad (3.33)$$

Proof. We first prove (1). Recall the process $\widetilde{B}$ from (3.16). By Lemma 2.3, (2.6), and the change of measure (3.15), the left of (3.32) is bounded from above by

$$\sqrt{\frac{\sqrt{s}}{\sqrt{t}+a_n(t)+y}}\,\widetilde{E}^B_{n-\ell}\left[\begin{array}{l} e^{-\frac{2a_n(t)}{n}\widetilde{B}_{n-\ell}-\frac{(a_n(t))^2}{n^2}(n-\ell)}1_{\left\{\widetilde{B}_{n-\ell}\in\sqrt{t}-\sqrt{s}+\frac{a_n(t)}{n}\ell+y+[0,1)\right\}} \\ \times 1_{\left\{\widetilde{B}_r<\sqrt{t}-\sqrt{s}+\frac{a_n(t)}{n}\ell+y+1,\ \forall r\in[0,n-\ell]\right\}} \end{array}\right]. \quad (3.34)$$

By the reflection principle (2.11), (3.34) is bounded from above by

$$c_1\sqrt{\frac{\sqrt{s}}{\sqrt{t}+a_n(t)+y}}(n-\ell)^{-3/2}\left(\sqrt{t}-\sqrt{s}+\frac{a_n(t)}{n}\ell+y+1\right)e^{-\frac{(\sqrt{t}-\sqrt{s}+a_n(t)+y)^2}{n-\ell}},$$

where we have used the inequality $1-e^{-x}\le x$ for each $x\ge 0$. Thus, we have obtained (3.32). The inequality (3.33) immediately follows from (1) with $s=t$ and $\ell=0$. $\qquad\square$

Proof of Proposition 3.1(ii). Fix any $n\ge n_0$, $t\ge n$ and $y\in[0,2\sqrt{n}]$, where we take $n_0\in\mathbb{N}$ large enough. Set

$$Z:=\sum_{v\in T_n}1_{A^n_v(t)}.$$

We have

$$P_\rho\left(\max_{v\in T_n}\sqrt{L^n_{\tau(t)}(v)}\ge\sqrt{t}+a_n(t)+y\right)\ge\widetilde{P}_\rho(Z\ge 1)\ge\frac{\left(\widetilde{E}_\rho[Z]\right)^2}{\widetilde{E}_\rho[Z^2]}. \quad (3.35)$$

By Lemma 3.3, we have

$$\widetilde{E}_\rho[Z]\ge c_1(1+y)e^{-2\sqrt{\log b}\,y}. \quad (3.36)$$

The rest of the proof focuses on obtaining an upper bound of $\widetilde{E}_\rho[Z^2]$. We have

$$\widetilde{E}_\rho[Z^2]=\widetilde{E}_\rho[Z]+\sum_{\ell=0}^{n-1}\sum_{\substack{v,u\in T_n,\\|v\wedge u|=\ell}}\widetilde{P}_\rho(A^n_v(t)\cap A^n_u(t)). \quad (3.37)$$

By Lemma 3.4 (2), we have

$$\widetilde{E}_\rho[Z]\le c_2(y+1)e^{-2\sqrt{\log b}\,y}. \quad (3.38)$$

Fix $1\le\ell\le n-1$ and $v,u\in T_n$ with $|v\wedge u|=\ell$. Let $\left\{\widetilde{L}^\downarrow_s(x):(s,x)\in[0,\infty)\times\widetilde{T}^{v_\ell}_{\le n-\ell}\right\}$ be a local time of a Brownian motion on $\widetilde{T}^{v_\ell}_{\le n-\ell}$. Set $\widetilde{\tau}^\downarrow(s):=\inf\{r\ge 0:\widetilde{L}^\downarrow_r(v_\ell)>s\}$. For $w\in\{v,u\}$ and $s\ge 0$, we define the event $C^\downarrow_w(s)$ by

$$\left\{\begin{array}{l} \delta\sqrt{t}\le\sqrt{\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow(s)}(w_r)}<\sqrt{t}+\frac{a_n(t)}{n}r+y+1,\ \forall r\in[\ell,\,n], \\ \sqrt{\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow(s)}(w)}\in[\sqrt{t}+a_n(t)+y,\ \sqrt{t}+a_n(t)+y+1) \end{array}\right\}.$$

By Lemma 2.1, we have

$$\widetilde{P}_\rho(A^n_v(t)\cap A^n_u(t))$$

$$=\widetilde{E}_\rho\left[1_{\left\{\delta\sqrt{t}\le\sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s)}<\sqrt{t}+\frac{a_n(t)}{n}s+y+1,\ \forall s\in[0,\ell]\right\}}\widetilde{P}_{v_\ell}\left(\bigcap_{w\in\{v,u\}}C^\downarrow_w\left(\widetilde{L}^n_{\widetilde{\tau}(t)}(v_\ell)\right)\right)\right]$$

$$=\widetilde{E}_\rho\left[1_{\left\{\delta\sqrt{t}\le\sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s)}<\sqrt{t}+\frac{a_n(t)}{n}s+y+1,\ \forall s\in[0,\ell]\right\}}\prod_{w\in\{v,u\}}\widetilde{P}_{v_\ell}\left(C^\downarrow_w\left(\widetilde{L}^n_{\widetilde{\tau}(t)}(v_\ell)\right)\right)\right]$$

$$\le\sum_{i=0}^{\lceil(1-\delta)\sqrt{t}+\frac{a_n(t)}{n}\ell+y\rceil}\widetilde{E}_\rho\left[\begin{array}{l} 1_{\left\{\delta\sqrt{t}\le\sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s)}<\sqrt{t}+\frac{a_n(t)}{n}s+y+1,\ \forall s\in[0,\ell]\right\}} \\ \times 1_{\left\{\sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_\ell)}\in\sqrt{t}+\frac{a_n(t)}{n}\ell+y+1+[-i-1,\ -i)\right\}} \\ \times\prod_{w\in\{v,u\}}\widetilde{P}_{v_\ell}\left(C^\downarrow_w\left(\widetilde{L}^n_{\widetilde{\tau}(t)}(v_\ell)\right)\right) \end{array}\right], \quad (3.39)$$

where we have used the independence of $C_v^{\downarrow}(s)$ and $C_u^{\downarrow}(s)$ for each $s \geq 0$. (The independence follows from that of two types of excursions of a Brownian motion around v_ℓ on $\widetilde{T}_{\leq n-\ell-1}^{v_{\ell+1}} \cup I_{\{v_\ell, v_{\ell+1}\}}$ or on $\widetilde{T}_{\leq n-\ell-1}^{u_{\ell+1}} \cup I_{\{v_\ell, u_{\ell+1}\}}$.) Fix $i \leq \left\lceil (1-\delta)\sqrt{t} + \frac{a_n(t)}{n}\ell + y \right\rceil$. Lemma 3.4 implies that under $\left\{ \sqrt{\widetilde{L}_{\tau(t)}^n(v_\ell)} \in \sqrt{t} + \frac{a_n(t)}{n}\ell + y + 1 + [-i-1, \; -i) \right\}$, for all $w \in \{v, u\}$, $\widetilde{P}_{v_\ell}[C_w^{\downarrow}(\widetilde{L}_{\tau(t)}^n(v_\ell)]$ is bounded from above by

$$c_3(n-\ell)^{-\frac{3}{2}}(i+1)\sqrt{\frac{\sqrt{t}+\frac{a_n(t)}{n}\ell+y+1-i}{\sqrt{t}+a_n(t)+y}}\; e^{-\frac{\left(\frac{a_n(t)}{n}(n-\ell)+i-1\right)^2}{n-\ell}}. \tag{3.40}$$

By almost the same argument as the proof of Lemma 3.4 (1), we have

$$\widetilde{P}_\rho \left(\begin{array}{c} \sqrt{\widetilde{L}_{\tau(t)}^n(v_s)} < \sqrt{t} + \frac{a_n(t)}{n}s + y + 1, \; \forall s \in [0, \ell], \\ \sqrt{\widetilde{L}_{\tau(t)}^n(v_\ell)} \in \sqrt{t} + \frac{a_n(t)}{n}\ell + y + 1 + [-i-1, \; -i) \end{array} \right)$$

$$\leq c_4 \sqrt{\frac{\sqrt{t}}{\sqrt{t}+\frac{a_n(t)}{n}\ell+y-i}} b^{-\ell}\ell^{-\frac{3}{2}}(i+1)(y+1)e^{-2\sqrt{\log b}\, y}e^{\frac{2a_n(t)}{n}i}e^{\frac{3\log n}{2n}\ell}e^{\frac{\log\left(\frac{\sqrt{t}+n}{\sqrt{t}}\right)}{2n}\ell}. \tag{3.41}$$

By (3.40) and (3.41), the right of (3.39) is bounded from above by

$$c_5 b^{-2n+\ell}\ell^{-\frac{3}{2}}(n-\ell)^{-3}n^3(y+1)e^{-2\sqrt{\log b}\, y}e^{-\frac{3\log n}{2n}\ell}e^{-\frac{\log\left(\frac{\sqrt{t}+n}{\sqrt{t}}\right)}{2n}\ell}\sqrt{\frac{\sqrt{t}+\frac{a_n(t)}{n}\ell+y}{\sqrt{t}}}. \tag{3.42}$$

By (3.42), we have

$$\sum_{\ell=1}^{n-1} \sum_{\substack{v,u\in T_n, \\ |v\wedge u|=\ell}} \widetilde{P}_\rho(A_v^n(t) \cap A_u^n(t)) \leq c_6(y+1)e^{-2\sqrt{\log b}\, y}. \tag{3.43}$$

In the case $\ell = 0$, by Lemma 3.4 (2), we have

$$\sum_{\substack{v,u\in T_n, \\ |v\wedge u|=0}} \widetilde{P}_\rho(A_v^n(t) \cap A_u^n(t)) = \sum_{\substack{v,u\in T_n, \\ |v\wedge u|=0}} \widetilde{P}_\rho(A_v^n(t))\widetilde{P}_\rho(A_u^n(t)) \leq c_7(y+1)e^{-2\sqrt{\log b}\, y}, \tag{3.44}$$

where we have used the independence of $A_v^n(t)$ and $A_u^n(t)$ for each $v, u \in T_n$ with $|v \wedge u| = 0$ in the first equality. (The independence follows from that of two types of excursions of a Brownian motion on $\widetilde{T}_{\leq n}$ around ρ restricted to $\widetilde{T}_{\leq n-1}^{v_1} \cup I_{\{\rho, v_1\}}$ or to $\widetilde{T}_{\leq n-1}^{u_1} \cup I_{\{\rho, u_1\}}$). Thus, by (3.35)–(3.38) and (3.43)–(3.44), we have (3.2). $\qquad \square$

4 Geometry of near maxima

In this section, we will prove that two leaves with local times near maxima are either very close or far away. More specifically, the following is the aim of this section.

Proposition 4.1. *There exist* $c_1, c_2 \in (0, \infty)$, $n_0, r_0 \in \mathbb{N}$, *and* $t_0 > 0$ *such that for all* $n \geq n_0$, $t \geq t_0$, *and* $r_0 \leq r \leq n/4$,

$$P_\rho \left(\begin{array}{c} \exists v, u \in T_n \text{ with } r \leq |v \wedge u| \leq n - r : \\ \sqrt{L_{\tau(t)}^n(v)}, \; \sqrt{L_{\tau(t)}^n(u)} \geq \sqrt{t} + a_n(t) - c_1 \log r \end{array} \right) \leq c_2 r^{-1/8}. \tag{4.1}$$

Remark 4.2. Results similar to Proposition 4.1 are known for the BBM [3] and the two-dimensional DGFF [28].

For $n \in \mathbb{N}$, $t > 0$, and $k \in \mathbb{Z}$, set

$$\Gamma_k^n(t) := \left\{ v \in T_n : \sqrt{L_{\tau(t)}^n(v)} \in [\sqrt{t} + a_n(t) - k - 1, \sqrt{t} + a_n(t) - k] \right\}.$$

Remark 4.3. Fix $n' > n \geq 1$, $t > 0$, and $k \in \mathbb{Z}$. Set

$$\Gamma_k^{n',n}(t) := \left\{ v \in T_n : \sqrt{L_{\tau(t)}^{n'}(v)} \in [\sqrt{t} + a_n(t) - k - 1, \sqrt{t} + a_n(t) - k] \right\}.$$

Since the law of the simple random walk on $T_{\leq n'}$ watched only on $T_{\leq n}$ is the same as that of the simple random walk on $T_{\leq n}$, we have $|\Gamma_k^{n',n}(t)| \stackrel{d}{=} |\Gamma_k^n(t)|$.

An outline of the proof of Proposition 4.1 is as follows: In Lemma 4.4, we will estimate a tail probability for the size of a level set of the local time and reduce the estimate to a tail estimate for the maximum of the local time. Note that the event in Proposition 4.1 implies that there exist $k \in \{r, \ldots, n - r\}$ and $w \in T_k$ satisfying the following: there exist children w_1, w_2 of w such that the maxima of local times over $T_{n-k-1}^{w_i}$, $i = 1, 2$ are larger than $\sqrt{t} + a_n(t) - c_1 \log r$. We will estimate the probability of this event by conditioning on the level set in T_k and applying the Markov property (Lemma 2.1) and independence of Brownian motions on different subtrees. The estimate will be reduced to tail estimates for the size of the level set and for the maximum of the local time.

In the proof of Proposition 4.1, we will use the following repeatedly.

Lemma 4.4. *(i) There exist $c_1 > 0$ and $t_0 > 0$ such that for all $n \in \mathbb{N}$, $t \geq t_0$, $k \leq -1$, and $\lambda \in \mathbb{R}$ with $k + \lambda \geq 0$,*

$$P_\rho\left(|\Gamma_k^n(t)| \geq e^{2\sqrt{\log b}\,(k+\lambda)}\right)$$

$$\leq c_1(\lambda + 1)e^{-2\sqrt{\log b}\left(\lambda - \frac{3}{4\sqrt{\log b}}\log(\lceil (k+\lambda)^2 \rceil \vee 1) - \frac{1}{4\sqrt{\log b}}\log\left(\frac{\sqrt{t}+a_n(t)-k-1+\lceil (k+\lambda)^2\rceil \vee 1}{\sqrt{t}+a_n(t)-k-1}\right)\right)}. \quad (4.2)$$

(ii) There exist $c_2 > 0$ and $t_0 > 0$ such that for all $n \in \mathbb{N}$, $t \geq t_0$, $\lambda > 0$, and $k \geq 0$ with $\sqrt{t} + a_n(t) - k - 1 \geq c_2$, (4.2) holds.

Proof. (i) Fix $n \in \mathbb{N}$, $t > 0$, $k \leq -1$, and $\lambda \in \mathbb{R}$ with $k + \lambda \geq 0$. Set $r := \lceil (k + \lambda)^2 \rceil \vee 1$ and $y := \lambda - \frac{3}{4\sqrt{\log b}}\log r - \frac{1}{4\sqrt{\log b}}\log\left(\frac{\sqrt{t}+a_n(t)-k-1+r}{\sqrt{t}+a_n(t)-k-1}\right)$. If $y < 0$, then it is clear that (4.2) holds because $P_\rho\left(|\Gamma_k^n(t)| \geq e^{2\sqrt{\log b}\,(k+\lambda)}\right) \leq 1 \leq (\lambda + 1)e^{-2\sqrt{\log b}\,y}$. So, we may assume that $y \geq 0$. Fix any $K > 0$. We have

$$P_\rho\left(\max_{v \in T_{n+r}} \sqrt{L_{\tau(t)}^{n+r}(v)} \geq \sqrt{t} + a_{n+r}(t) + y\right)$$

$$\geq P_\rho\left(|\Gamma_k^{n+r,n}(t)| \geq K, \max_{v \in T_{n+r}} \sqrt{L_{\tau(t)}^{n+r}(v)} \geq \sqrt{t} + a_{n+r}(t) + y\right)$$

$$= P_\rho\left(|\Gamma_k^{n+r,n}(t)| \geq K\right) - P_\rho\left(|\Gamma_k^{n+r,n}(t)| \geq K, \max_{v \in T_{n+r}} \sqrt{L_{\tau(t)}^{n+r}(v)} < \sqrt{t} + a_{n+r}(t) + y\right). \quad (4.3)$$

We estimate the second term on the right-hand side of (4.3). By Lemma 2.1, we have

$$P_\rho\left(|\Gamma_k^{n+r,n}(t)| \geq K, \max_{v \in T_{n+r}} \sqrt{L_{\tau(t)}^{n+r}(v)} < \sqrt{t} + a_{n+r}(t) + y\right)$$

$$\leq \sum_{\substack{S \subset T_n, \\ |S| \geq K}} P_\rho\left(\Gamma_k^{n+r,n}(t) = S, \max_{v \in T_r^u} \sqrt{L_{\tau(t)}^{n+r}(v)} < \sqrt{t} + a_{n+r}(t) + y, \forall u \in S\right)$$

$$= \sum_{\substack{S \subset T_n, \\ |S| \geq K}} E_\rho\left[\mathbb{1}_{\{\Gamma_k^{n+r,n}(t)=S\}} \prod_{u \in S} \widetilde{P}_u\left(\max_{v \in T_r^u} \sqrt{\widetilde{L}_{\widetilde{\tau}^\downarrow\left(L_{\tau(t)}^{n+r}(u)\right)}^\downarrow(v)} < \sqrt{t} + a_{n+r}(t) + y\right)\right], \quad (4.4)$$

where for each $u \in T_n$, $\left\{ \widetilde{L}^{\downarrow}_s(x) : (s,x) \in [0, \infty) \times \widetilde{T}^u_{\leq r} \right\}$ is a local time of a Brownian motion on $\widetilde{T}^u_{\leq r}$ and $\widetilde{\tau}^{\downarrow}(q) := \inf\{s \geq 0 : \widetilde{L}^{\downarrow}_s(u) > q\}$. We omit the subscript u in $\widetilde{L}^{\downarrow}_s(x)$ and $\widetilde{\tau}^{\downarrow}$ to simplify the notation.

We estimate each probability on the right-hand side of (4.4). Fix $S \subset T_n$ with $|S| \geq K$ and $u \in S$. Note that under the event that $\Gamma^{n+r,n}_k(t) = S$, we have

$$\sqrt{t} + a_{n+r}(t) + y \leq \sqrt{L^{n+r}_{\tau(t)}(u)} + a_r\left(L^{n+r}_{\tau(t)}(u)\right) + k + \lambda + 1.$$

By this and Proposition A.3 for $t \geq t_0$, where t_0 is sufficiently large, we have

$$\widetilde{P}_u \left(\max_{v \in T^u_r} \sqrt{\widetilde{L}^{\downarrow}_{\widetilde{\tau}^{\downarrow}\left(L^{n+r}_{\tau(t)}(u)\right)}(v)} < \sqrt{t} + a_{n+r}(t) + y \right)$$

$$\leq 1 - \widetilde{P}_u \left(\max_{v \in T^u_r} \sqrt{\widetilde{L}^{\downarrow}_{\widetilde{\tau}^{\downarrow}\left(L^{n+r}_{\tau(t)}(u)\right)}(v)} \geq \sqrt{L^{n+r}_{\tau(t)}(u)} + a_r\left(L^{n+r}_{\tau(t)}(u)\right) + k + \lambda + 1 \right)$$

$$\leq 1 - c_1 e^{-2\sqrt{\log b}\ (k+\lambda+1)}. \tag{4.5}$$

By (4.3)–(4.5), we have

$$P_\rho \left(\max_{v \in T_{n+r}} \sqrt{L^{n+r}_{\tau(t)}(v)} \geq \sqrt{t} + a_{n+r}(t) + y \right)$$

$$\geq \left(1 - \exp\left\{ -c_1 K e^{-2\sqrt{\log b}(k+\lambda+1)} \right\} \right) P_\rho \left(|\Gamma^{n+r,n}_k(t)| \geq K \right). \tag{4.6}$$

By Remark 4.3, (4.6) with $K := e^{2\sqrt{\log b}(k+\lambda)}$, and Proposition 3.1(i), we have (4.2).

(ii) The proof of (ii) is almost the same as that of (i), so we omit the detail. $\qquad\square$

For the rest of this section, we focus on proving the following.

Lemma 4.5. *Fix* $0 < \underline{c} < \overline{c} < \frac{3}{4\sqrt{\log b}}$. *There exist* $c_1 > 0$, $n_0, s_0 \in \mathbb{N}$, *and* $t_0 > 0$ *such that for all* $n \geq n_0$, $t \geq t_0$, *and* $s_0 \leq s \leq n - s_0$,

$$P_\rho \left[\begin{array}{l} \exists v, u \in T_n \text{ with } |v \wedge u| = s : \\ \sqrt{L^n_{\tau(t)}(v)}, \sqrt{L^n_{\tau(t)}(u)} \geq \sqrt{t} + a_n(t) - (\overline{c} - \underline{c}) \log(s \wedge (n-s)) \end{array} \right]$$

$$\leq c_1 (\log(s \wedge (n-s)))^8 (s \wedge (n-s))^{-3+4\overline{c}\sqrt{\log b} - 2\underline{c}\sqrt{\log b}}$$

$$+ c_1 (\log(s \wedge (n-s)))^6 (s \wedge (n-s))^{-2\underline{c}\sqrt{\log b}}. \tag{4.7}$$

Before we prove this, let us show that Lemma 4.5 implies Proposition 4.1.

Proof of Proposition 4.1 via Lemma 4.5. Fix any $n \geq n_0$, $t \geq t_0$, and $r_0 \leq r \leq n/4$, where we take $n_0, r_0 \in \mathbb{N}$ and $t_0 > 0$ sufficiently large. By Lemma 4.5 with $\underline{c} = \frac{5}{8\sqrt{\log b}}$ and $\overline{c} = \frac{11}{16\sqrt{\log b}}$, the left of (4.1) is bounded from above by

$$c_1 \sum_{s=r}^{n-r} \left\{ (\log(s \wedge (n-s)))^8 (s \wedge (n-s))^{-3/2} + (\log(s \wedge (n-s)))^6 (s \wedge (n-s))^{-5/4} \right\}.$$

This is bounded from above by $c_2 r^{-1/8}$. $\qquad\square$

Proof of Lemma 4.5. Fix any $n \geq n_0$, $t \geq t_0$, $s_0 \leq s \leq n - s_0$, where we take $n_0, s_0 \in \mathbb{N}$ and $t_0 > 0$ sufficiently large. Set $z := \underline{c} \log(s \wedge (n-s))$. The left of (4.7) is bounded from

above by

$$P_\rho\left[\begin{array}{l}\exists k\in\mathbb{Z}\text{ with }\sqrt{t}+a_s(t)-k>0,\ \exists j\geq -k\text{ s.t.}\\ |\Gamma_k^{n,s}(t)|\in\left[e^{2\sqrt{\log b}(k+j)},\ e^{2\sqrt{\log b}(k+j+1)}\right),\\ \exists w\in\Gamma_k^{n,s}(t),\ \exists w_1,w_2\in T_1^w\text{ with }w_1\neq w_2\text{ s.t. }\forall i\in\{1,2\},\\ \max_{v\in T_{n-s-1}^{w_i}}\sqrt{L_{\tau(t)}^n(v)}\geq\sqrt{t}+a_n(t)-\bar{c}\log(s\wedge(n-s))+z\end{array}\right].\tag{4.8}$$

For $k\in\mathbb{Z}$, we set $j^*(k,z):=\lceil\max\{|k|,\ z\}\rceil$. Fix sufficiently large constant $t_*>0$. We will decompose (4.8) into three terms with respect to k and j: (i) $k\leq\sqrt{t}+a_s(t)-t_*-1$ and $-k\leq j\leq j^*(k,z)$; (ii) $k\leq\sqrt{t}+a_s(t)-t_*-1$ and $j>j^*(k,z)$; (iii) $k\geq\sqrt{t}+a_s(t)-t_*-1$. Then, (4.8) is bounded from above by

$$\sum_{\substack{k\in\mathbb{Z},\\ \sqrt{t}+a_s(t)-k-1\geq t_*}}\sum_{j=-k}^{j^*(k,z)}P_\rho\left[\begin{array}{l}|\Gamma_k^{n,s}(t)|\in\left[e^{2\sqrt{\log b}(k+j)},\ e^{2\sqrt{\log b}(k+j+1)}\right),\\ \exists w\in\Gamma_k^{n,s}(t),\exists w_1,w_2\in T_1^w\text{ with }w_1\neq w_2\text{ s.t.}\forall i\in\{1,2\},\\ \max_{v\in T_{n-s-1}^{w_i}}\sqrt{L_{\tau(t)}^n(v)}\geq\sqrt{t}+a_n(t)-\bar{c}\log(s\wedge(n-s))+z\end{array}\right]$$

$$+\sum_{\substack{k\in\mathbb{Z},\\ \sqrt{t}+a_s(t)-k-1\geq t_*}}P_\rho\left(|\Gamma_k^{n,s}(t)|\geq e^{2\sqrt{\log b}(k+j^*(k,z))}\right)$$

$$+\sum_{\substack{k\in\mathbb{Z},\\ 0<\sqrt{t}+a_s(t)-k\leq t_*+1}}P_\rho\left[\begin{array}{l}\exists w\in T_s\text{ with }\sqrt{L_{\tau(t)}^n(w)}\leq t_*+1,\\ \exists w_1,w_2\in T_1^w\text{ with }w_1\neq w_2\text{ s.t. }\forall i\in\{1,2\},\\ \max_{v\in T_{n-s-1}^{w_i}}\sqrt{L_{\tau(t)}^n(v)}\geq\sqrt{t}+a_n(t)-\bar{c}\log(s\wedge(n-s))+z\end{array}\right]$$

$$=:\sum_{\substack{k\in\mathbb{Z},\\ \sqrt{t}+a_s(t)-k-1\geq t_*}}\sum_{j=-k}^{j^*(k,z)}J_1(k,j)+J_2+\sum_{\substack{k\in\mathbb{Z},\\ 0<\sqrt{t}+a_s(t)-k\leq t_*+1}}J_3(k).\tag{4.9}$$

For each $w\in T_s$, let $\widetilde{L}^\downarrow$ be a local time of a Brownian motion on $\widetilde{T}_{\leq n-s}^w$ and set $\widetilde{\tau}^\downarrow(q):=\inf\{p\geq 0:\widetilde{L}_p^\downarrow(w)>q\}$. We omit the subscript w in $\widetilde{L}^\downarrow$ and $\widetilde{\tau}^\downarrow(q)$. Fix $k\leq\sqrt{t}+a_s(t)-t_*-1$ and $-k\leq j\leq j^*(k,z)$. Set $I_{k,j}:=[e^{2\sqrt{\log b}(k+j)},\ e^{2\sqrt{\log b}(k+j+1)})$. $J_1(k,j)$ is bounded from above by

$$\sum_{\substack{S\subset T_s,\\ |S|\in I_{k,j}}}\sum_{w\in S}\sum_{\substack{w_1,w_2\in T_1^w,\\ w_1\neq w_2}}J_1^{k,j}(S,w,w_1,w_2),\tag{4.10}$$

where $J_1^{k,j}(S,w,w_1,w_2)$ is given by

$$P_\rho\left[\Gamma_k^{n,s}(t)=S,\ \max_{v\in T_{n-s-1}^{w_i}}\sqrt{L_{\tau(t)}^n(v)}\geq\sqrt{t}+a_n(t)-\bar{c}\log(s\wedge(n-s))+z,\ \forall i\in\{1,2\}\right].\tag{4.11}$$

Fix $S\subset T_s$ with $|S|\in I_{k,j}$, $w\in S$, and $w_1,w_2\in T_1^w$ with $w_1\neq w_2$. By Lemma 2.1 and the independence of two types of excursions restricted to $\widetilde{T}_{\leq n-s-1}^{w_1}\cup I_{\{w,w_1\}}$ or to $\widetilde{T}_{\leq n-s-1}^{w_2}\cup I_{\{w,w_2\}}$, (4.11) is equal to

$$E_\rho\left[\begin{array}{l}\mathbf{1}_{\{\Gamma_k^{n,s}(t)=S\}}\\ \times\prod_{i\in\{1,\ 2\}}\widetilde{P}_w\left(\max_{v\in T_{n-s-1}^{w_i}}\sqrt{\widetilde{L}_{\widetilde{\tau}^\downarrow\left(L_{\tau(t)}^n(w)\right)}^\downarrow(v)}\geq\sqrt{t}+a_n(t)-\bar{c}\log(s\wedge(n-s))+z\right)\end{array}\right].\tag{4.12}$$

By the symmetry of the b-ary tree and (2.3), (4.12) is bounded from above by

$$P_\rho \left[\Gamma_k^{n,s}(t) = S \right]$$
$$\times P_\rho \left(\max_{v \in T_{n-s}} \sqrt{L^{n-s}_{\tau\left((\sqrt{t}+a_s(t)-k)^2\right)}(v)} \geq \sqrt{t} + a_n(t) - \bar{c}\log(s \wedge (n-s)) + z \right)^2. \quad (4.13)$$

By (4.13), (4.10) is bounded from above by

$$c_1 e^{2\sqrt{\log b}(k+j)} P_\rho \left(|\Gamma_k^{n,s}(t)| \geq e^{2\sqrt{\log b}(k+j)} \right)$$
$$\times P_\rho \left(\max_{v \in T_{n-s}} \sqrt{L^{n-s}_{\tau\left((\sqrt{t}+a_s(t)-k)^2\right)}(v)} \geq \sqrt{t} + a_n(t) - \bar{c}\log(s \wedge (n-s)) + z \right)^2. \quad (4.14)$$

We estimate (4.14) in different ways according to three cases: (a) $k \geq -z$ and $j \geq 0$; (b) $k < -z$; (c) $j < 0$. In the case (a), we use Proposition 3.1(i), Lemma 4.4, and Remark 4.3. In the case (b), we only use Lemma 4.4 and Remark 4.3 and estimate the square of the probability in (4.14) just by 1. In the case (c), we only use Proposition 3.1(i) and estimate the probability in the first display of (4.14) just by 1. Note that for $k \in \mathbb{Z}$ with $k \geq -z$, we have

$$\sqrt{t} + a_n(t) - \bar{c}\log(s \wedge (n-s)) + z \geq (\sqrt{t} + a_s(t) - k) + a_{n-s}((\sqrt{t} + a_s(t) - k)^2)$$
$$+ \left(\frac{3}{4\sqrt{\log b}} - \bar{c} \right) \log(s \wedge (n-s)) + k + z - c_2.$$

Recall that $z = \underline{c}\log(s \wedge (n-s))$. By these observations, for sufficiently large t_*, the first term of (4.9) is bounded from above by

$$c_3 (\log(s \wedge (n-s)))^8 (s \wedge (n-s))^{-3+4\bar{c}\sqrt{\log b}-2\underline{c}\sqrt{\log b}}$$
$$+ c_3 (\log(s \wedge (n-s)))(s \wedge (n-s))^{-2\underline{c}\sqrt{\log b}}. \quad (4.15)$$

Next, we estimate the second term of (4.9). By Lemma 4.4 and Remark 4.3, we have

$$J_2 \leq c_4 (\log(s \wedge (n-s)))^6 (s \wedge (n-s))^{-2\underline{c}\sqrt{\log b}}. \quad (4.16)$$

Finally, we estimate the third term of (4.9). Fix $k \geq \sqrt{t} + a_s(t) - t_* - 1$. By Lemma 2.1, $J_3(k)$ is bounded from above by

$$\sum_{w \in T_s} \sum_{\substack{w_1, w_2 \in T_1^w, \\ w_1 \neq w_2}} J_3^k(w, w_1, w_2), \quad (4.17)$$

where $J_3^k(w, w_1, w_2)$ is given by

$$E_\rho \left[\begin{array}{l} \mathbb{1}\left\{ \sqrt{L^n_{\tau(t)}(w)} \leq t_*+1 \right\} \\ \times \prod_{i \in \{1,2\}} \widetilde{P}_w \left[\max_{v \in T_{n-s-1}^{w_i}} \sqrt{\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow\left(L^n_{\tau(t)}(w)\right)}(v)} \geq \sqrt{t} + a_n(t) - \bar{c}\log(s \wedge (n-s)) + z \right] \end{array} \right]. \quad (4.18)$$

By (4.18) together with the symmetry of the b-ary tree, (4.17) is bounded from above by

$$c_5 b^s P_\rho \left(\max_{v \in T_{n-s}} \sqrt{L^{n-s}_{\tau((t_*+1)^2)}(v)} \geq \sqrt{t} + a_n(t) - \bar{c}\log(s \wedge (n-s)) + z \right)^2. \quad (4.19)$$

Since t_0 and n_0 are sufficiently large, we have

$$\sqrt{t} + a_n(t) - \overline{c}\log(s \wedge (n-s)) + z \geq (t_* + 1) + a_{n-s}((t_* + 1)^2)$$
$$+ \frac{1}{2}\sqrt{\log b}\, s - \overline{c}\log(s \wedge (n-s)) + z.$$

By Proposition 3.1(i) and (4.19), the third term of (4.9) is bounded from above by

$$c_6 s^2 b^{-s}(s \wedge (n-s))^{4(\overline{c}-\underline{c})\sqrt{\log b}}. \qquad (4.20)$$

Thus, by (4.9), (4.15), (4.16), and (4.20), we have (4.7). $\qquad \square$

5 Limiting tail of the maximum of local times

The aim of this section is to prove the exact asymptotics of the tail of the maximum of local times. Recall the constants β_* and γ_* from (1.8) and (1.9).

Proposition 5.1. *Fix positive sequences* $(y_j^+)_{j\geq 1}, (y_j^-)_{j\geq 1}$ *with* $y_j^- \leq y_j^+$ *for each* $j \geq 1$ *and* $\lim_{j\to\infty} y_j^- = \infty$. *For each* $j \geq 1$, *fix sequences* $(t_n^+(j))_{n\geq 1}, (t_n^-(j))_{n\geq 1}$ *with* $\lim_{n\to\infty} \frac{\sqrt{t_n^+(j)}}{n} = \lim_{n\to\infty} \frac{\sqrt{t_n^-(j)}}{n} = \theta \in [0,\infty]$ *and* $t_n^-(j) \leq t_n^+(j)$ *for each* $n \geq 1$. *For all* $\varepsilon > 0$, *there exists* $j_0 \in \mathbb{N}$ *such that the following holds for each* $j \geq j_0$: *there exists* $n_0(j) \in \mathbb{N}$ *such that for all* $n \geq n_0(j)$,

$$\frac{P_\rho\left(\max_{v\in T_n}\sqrt{L_{\tau(t)}^n(v)} > \sqrt{t} + a_n(t) + y_j\right)}{y_j e^{-2\sqrt{\log b}\, y_j}} \in \left[\frac{4}{\sqrt{\pi}}\beta_*\gamma_* - \varepsilon,\ \frac{4}{\sqrt{\pi}}\beta_*\gamma_* + \varepsilon\right], \qquad (5.1)$$

uniformly in y_j *and* t *with*

$$y_j^- \leq y_j \leq y_j^+, \quad t \geq c_* n\log n \text{ and } t_n^-(j) \leq t \leq t_n^+(j), \qquad (5.2)$$

for some constant $c_* > 0$ *not depending on* ε, j, n.

Fix $\delta \in (0,1)$ and $\kappa > \frac{1}{2\sqrt{\log b}}$. Fix $t > 0$, $y > 0$, $\ell(y) \in \mathbb{N}$, $n > \ell(y)$. We will approximate the tail probability of the maximum of local times by $\widetilde{E}_\rho[\Lambda_{y,\ell(y)}^n(t)]$, where

$$\Lambda_{y,\ell(y)}^n(t) := \sum_{v\in T_{n-\ell(y)}} 1_{F_{v,y,\ell(y)}^n(t)},$$

and for each $v \in T_{n-\ell(y)}$,

$$F_{v,y,\ell(y)}^n(t) := \left\{ \begin{array}{l} \delta\sqrt{t} \leq \sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n}s + y,\ \forall s \in [0,\ n-\ell(y)], \\ \max_{u\in T_{\ell(y)}^v}\sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(u)} > \sqrt{t} + a_n(t) + y \end{array} \right\}.$$

To do so, we need an intermediate approximation $\widetilde{E}_\rho[\widetilde{\Lambda}_{y,\ell(y)}^n(t)]$, where

$$\widetilde{\Lambda}_{y,\ell(y)}^n(t) := \sum_{v\in T_{n-\ell(y)}} 1_{\widetilde{F}_{v,y,\ell(y)}^n(t)},$$

and for each $v \in T_{n-\ell(y)}$,

$$\widetilde{F}_{v,y,\ell(y)}^n(t) := \left\{ \begin{array}{l} \delta\sqrt{t} \leq \sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n}s + y + y^{\frac{1}{20}} + \kappa(\log(s\wedge(n-\ell(y)-s)))_+, \\ \forall s \in [0,\ n-\ell(y)],\ \max_{u\in T_{\ell(y)}^v}\sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(u)} > \sqrt{t} + a_n(t) + y \end{array} \right\}.$$

An outline of the proof of Proposition 5.1 is as follows: The difference between the upper barriers in $F^n_{v,y,\ell(y)}(t)$ and $\widetilde{F}^n_{v,y,\ell(y)}(t)$ is so tiny that thanks to the entropic repulsion (Lemma 2.4 (ii)), we can approximate $\widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)]$ by $\widetilde{E}_\rho[\widetilde{\Lambda}^n_{y,\ell(y)}(t)]$ (Lemma 5.2). In Lemma 5.3, applying an argument similar to the proof of Proposition 3.1(ii), we will show that the second moment $\widetilde{E}_\rho[(\Lambda^n_{y,\ell(y)}(t))^2]$ is close to $\widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)]$. In Lemma 5.4, we will prove that the tail probability of the maximum of the local time is approximated by $\widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)]$: the upper bound follows from Lemmas 3.2, 5.2 and the lower bound follows from Lemma 5.3. Applying Lemmas 2.1, 2.3, (2.6), and the reflection principle (2.11), we can represent $\widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)]$ in terms of a tail probability of the local time at a relatively large time. The generalized second Ray-Knight theorem (Theorem 2.2) implies that the tail probability is approximated by an analogue for the maximum of BRW. Proposition 5.1 follows from these approximations.

In Lemma 5.4, we show that $P_\rho(\max_{v\in T_n}\sqrt{L^n_{\tau(t)}(v)} \geq \sqrt{t}+a_n(t)+y)/\widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)]$ is close to 1. To obtain the upper bound in this statement, we need the following:

Lemma 5.2. *There exist $c_1, c_2 \in (0,\infty)$, $y_0 > 0$, and $\{\delta_{y'} : y' > 0\}$ with $\lim_{y'\to\infty}\delta_{y'} = 0$ such that the following holds: for all $y \geq y_0$ and $\ell(y) > e^{\frac{8\sqrt{\log b}}{3}y^{1/20}}$, there exists $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ such that for all $n \geq n_0$ and $t \geq c_1 n\log n$,*

$$\frac{\widetilde{E}_\rho\left[\Lambda^n_{y,\ell(y)}(t)\right]}{\widetilde{E}_\rho\left[\widetilde{\Lambda}^n_{y,\ell(y)}(t)\right]} \geq (1-\delta_y)\left(1 - c_2(\ell(y))^{-1/2} - \delta_y\right). \tag{5.3}$$

Proof. Fix any $y \geq y_0$ and $\ell(y) > e^{\frac{8\sqrt{\log b}}{3}y^{1/20}}$, where we take $y_0 > 0$ large enough. Throughout the proof, given $n \in \mathbb{N}$, we assume that $t \geq c_* n\log n$, where c_* is a sufficiently large positive constant. Fix $v \in T_{n-\ell(y)}$. Let $\widetilde{L}^\downarrow$ be a local time of a Brownian motion on $\widetilde{T}^v_{\leq\ell(y)}$ and set $\widetilde{\tau}^\downarrow(s) := \inf\{r \geq 0 : \widetilde{L}^\downarrow_r(v) > s\}$. Recall the definitions of $\mu_{n-\ell(y),y}$, $\mu^*_{n-\ell(y),y}$, and δ_y from Lemma 2.4. By Lemma 2.1, we have

$$\widetilde{P}_\rho\left(\widetilde{F}^n_{v,y,\ell(y)}(t)\right) - \widetilde{P}_\rho\left(F^n_{v,y,\ell(y)}(t)\right)$$

$$= \widetilde{E}_\rho\left[\begin{matrix}1_{\left\{\delta\sqrt{t}\leq\sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s)}\leq\sqrt{t}+\frac{a_n(t)}{n}s+y+y^{\frac{1}{20}}+\kappa(\log(s\wedge(n-\ell(y)-s)))_+,\ \forall s\in[0,\ n-\ell(y)]\right\}} \\ \times \widetilde{P}_v\left(\max_{u\in T^v_{\ell(y)}}\sqrt{\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow}\left(L^n_{\tau(t)}(v)\right)}(u) > \sqrt{t}+a_n(t)+y\right)\end{matrix}\right]$$

$$- \widetilde{E}_\rho\left[\begin{matrix}1_{\left\{\delta\sqrt{t}\leq\sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s)}\leq\sqrt{t}+\frac{a_n(t)}{n}s+y,\ \forall s\in[0,\ n-\ell(y)]\right\}} \\ \times \widetilde{P}_v\left(\max_{u\in T^v_{\ell(y)}}\sqrt{\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow}\left(L^n_{\tau(t)}(v)\right)}(u) > \sqrt{t}+a_n(t)+y\right)\end{matrix}\right]. \tag{5.4}$$

Recall the process $\widetilde{B}$ from (3.16). By Lemma 2.3, (2.6), and the change of measure (3.15), (5.4) is bounded from above by

$$\widetilde{E}^B_{n-\ell(y)}\left[1_{\left\{-(1-\delta)\sqrt{t}-\frac{a_n(t)}{n}s\leq\widetilde{B}_s\leq y+y^{\frac{1}{20}}+\kappa(\log(s\wedge(n-\ell(y)-s)))_+,\ \forall s\in[0,\ n-\ell(y)]\right\}}\psi(\widetilde{B}_{n-\ell(y)})\right]$$

$$- \widetilde{E}^B_{n-\ell(y)}\left[1_{\left\{-(1-\delta)\sqrt{t}-\frac{a_n(t)}{n}s\leq\widetilde{B}_s\leq y,\ \forall s\in[0,\ n-\ell(y)]\right\}}\psi(\widetilde{B}_{n-\ell(y)})\right], \tag{5.5}$$

where

$$\psi(x) := \sqrt{\frac{\sqrt{t}}{\sqrt{t}+\frac{a_n(t)}{n}(n-\ell(y))+x}} \cdot e^{-\frac{2a_n(t)}{n}x-\frac{(a_n(t))^2}{n^2}(n-\ell(y))}$$

$$\times \widetilde{P}_v\left(\max_{u\in T^v_{\ell(y)}}\sqrt{\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow\left((\sqrt{t}+\frac{a_n(t)}{n}(n-\ell(y))+x)^2\right)}}(u) > \sqrt{t}+a_n(t)+y\right). \tag{5.6}$$

By Lemma 2.4, (5.5) is bounded from above by

$$\int_{[0,\ y+y^{\frac{1}{20}}]\cup[-(1-\delta)\sqrt{t}-\frac{a_n(t)}{n}(n-\ell(y)),\ -\ell(y)]} \mu^*_{n-\ell(y),\ y}(x)\psi(x)dx$$

$$+ \int_{-\ell(y)}^0 \delta_y \mu_{n-\ell(y),\ y}(x)\psi(x)dx$$

$$=: J_1 + J_2. \tag{5.7}$$

By Proposition 3.1(i) and the assumption $\ell(y) > e^{\frac{8\sqrt{\log b}}{3}y^{1/20}}$, taking $n_0(y,\ell(y)) \in \mathbb{N}$ large enough, we have for all $n \geq n_0(y,\ell(y))$

$$J_1 \leq c_1 b^{-(n-\ell(y))}ye^{-2\sqrt{\log b}\,y}(\ell(y))^{-1/2}. \tag{5.8}$$

By the change of measure (3.15), J_2 is bounded from above by

$$\delta_y E_0^B \left[\begin{array}{l} 1_{\{\frac{a_n(t)}{n}(n-\ell(y))-\ell(y)\leq B_{n-\ell(y)}\leq \frac{a_n(t)}{n}(n-\ell(y))\}} \\ \times 1_{\{-(1-\delta)\sqrt{t}\leq B_s\leq \frac{a_n(t)}{n}s+y,\ \forall s\in[0,\ n-\ell(y)]\}}\sqrt{\frac{\sqrt{t}}{\sqrt{t}+B_{n-\ell(y)}}} \\ \times \widetilde{P}_v\left(\max_{u\in T^v_{\ell(y)}}\sqrt{\widetilde{L}^{\downarrow}_{\widetilde{\tau}^{\downarrow}}((\sqrt{t}+B_{n-\ell(y)})^2)}(u) > \sqrt{t}+a_n(t)+y\right) \end{array} \right]$$

$$+ \delta_y E_0^B \left[\begin{array}{l} 1_{\left\{\frac{a_n(t)}{n}(n-\ell(y))-\ell(y)\leq B_{n-\ell(y)}\leq \frac{a_n(t)}{n}(n-\ell(y)),\ \min_{s\in[0,\ n-\ell(y)]}B_s<-(1-\delta)\sqrt{t}\right\}} \\ \times\sqrt{\frac{\sqrt{t}}{\sqrt{t}+B_{n-\ell(y)}}}\widetilde{P}_v\left(\max_{u\in T^v_{\ell(y)}}\sqrt{\widetilde{L}^{\downarrow}_{\widetilde{\tau}^{\downarrow}}((\sqrt{t}+B_{n-\ell(y)})^2)}(u) > \sqrt{t}+a_n(t)+y\right) \end{array} \right]$$

$$=: J_{2,1} + J_{2,2}. \tag{5.9}$$

By Lemma 2.3 and (2.6), we have for each $n \geq 1$

$$J_{2,1} \leq c_2\delta_y\widetilde{P}_\rho(F^n_{v,y,\ell(y)}(t)), \tag{5.10}$$

where we have used the fact that under the event that $\sqrt{X_s/2} \geq \delta\sqrt{t}$ for all $0 \leq s \leq n-\ell(y)$, we have $\exp\left(\frac{3}{8}\int_0^{n-\ell(y)}\frac{ds}{X_s}\right) \leq c_2$ under the assumption $t \geq c_*n\log n$.

By the symmetry of B, (3.28), and Proposition 3.1(i), taking $n_0 = n_0(y,\ell(y)) \in \mathbb{N}$ large enough, we have for all $n \geq n_0$

$$J_{2,2} \leq c_3\delta_y b^{-(n-\ell(y))}e^{-2\sqrt{\log b}\,y}, \tag{5.11}$$

where we have used the assumption $t \geq c_*n\log n$, c_* is large enough. Thus, by (5.4)–(5.11), we have

$$(1+c_2\delta_y)\widetilde{E}_\rho(\Lambda^n_{y,\ell(y)}(t)) \geq \widetilde{E}_\rho(\widetilde{\Lambda}^n_{y,\ell(y)}(t)) - c_1ye^{-2\sqrt{\log b}\,y}(\ell(y))^{-1/2} - c_3\delta_ye^{-2\sqrt{\log b}\,y}. \tag{5.12}$$

In the remainder of the proof, we obtain a lower bound of $\widetilde{E}_\rho(\widetilde{\Lambda}^n_{y,\ell(y)}(t))$. Recall the definition of the event $G^{n-\ell(y)}_{y+y^{1/20}-2}(t)$ from (3.3). Let $\widetilde{G}$ be the slightly modified version of this event given by

$$\left\{ \begin{array}{l} \exists v \in T_{n-\ell(y)}, \exists s \in [0, n-\ell(y)]: \\ \sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s)} > \sqrt{t} + \frac{a_n(t)}{n}s + \kappa(\log(s\wedge(n-\ell(y)-s)))_+ + y + y^{1/20} \end{array} \right\}.$$

$\widetilde{E}_\rho(\widetilde{\Lambda}^n_{y,\ell(y)}(t))$ is bounded from below by

$$\widetilde{P}_\rho\left[\left\{ \begin{array}{l} \exists v \in T_{n-\ell(y)}: \begin{array}{l} \min_{s\in[0,\ n-\ell(y)]}\sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s)} \geq \delta\sqrt{t}, \\ \max_{u\in T^v_{\ell(y)}}\sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(u)} > \sqrt{t}+a_n(t)+y \end{array} \end{array} \right\} \cap \left(\widetilde{G}\right)^c \right]$$

$$\geq P_\rho \left[\max_{u \in T_n} \sqrt{L^n_{\tau(t)}(u)} > \sqrt{t} + a_n(t) + y \right] - \widetilde{P}_\rho \left[G^{n-\ell(y)}_{y+y^{1/20}-2}(t) \right]$$

$$- \widetilde{P}_\rho \left[\exists v \in T_{n-\ell(y)} : \begin{array}{l} \min_{s \in [0, \, n-\ell(y)]} \sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s)} < \delta\sqrt{t}, \\ \max_{u \in T^v_{\ell(y)}} \sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(u)} > \sqrt{t} + a_n(t) + y \end{array} \right], \tag{5.13}$$

where we have used the inequality $\frac{a_n(t)}{n} s \geq \frac{a_{n-\ell(y)}(t)}{n-\ell(y)} s - 1$, $s \in [0, n - \ell(y)]$, which implies that

$$\widetilde{P}_\rho[\widetilde{G}] \leq \widetilde{P}_\rho \left[G^{n-\ell(y)}_{y+y^{1/20}-2}(t) \right]. \tag{5.14}$$

(Note that we have also used the fact that the law of $\{\widetilde{L}^n_{\widetilde{\tau}(t)}(x) : x \in \widetilde{T}_{\leq n-\ell(y)}\}$ is the same as that of $\{\widetilde{L}^{n-\ell(y)}_{\widetilde{\tau}(t)}(x) : x \in \widetilde{T}_{\leq n-\ell(y)}\}$.)

Fix $v \in T_{n-\ell(y)}$. Recall the definitions of $\widetilde{L}^\downarrow$ and $\widetilde{\tau}^\downarrow(\cdot)$ from the beginning of the proof. By Lemma 2.1, the third term on the right-hand side of (5.13) is bounded from above by

$$b^{n-\ell(y)} \widetilde{E}_\rho \left[\begin{array}{l} \mathbf{1}_{\left\{ \min_{s \in [0, \, n-\ell(y)]} \sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s)} < \delta\sqrt{t}, \, \sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v)} > 0 \right\}} \\ \times \widetilde{P}_v \left(\max_{u \in T^v_{\ell(y)}} \sqrt{\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow \left(\widetilde{L}^n_{\widetilde{\tau}(t)}(v) \right)}(u)} > \sqrt{t} + a_n(t) + y \right) \end{array} \right]. \tag{5.15}$$

By Lemma 2.3, (2.6), and the symmetry of the Brownian motion, (5.15) is bounded from above by

$$b^{n-\ell(y)} E^B_0 \left[\begin{array}{l} \mathbf{1}_{\left\{ \max_{s \in [0, \, n-\ell(y)]} B_s > (1-\delta)\sqrt{t}, \, B_{n-\ell(y)} < \sqrt{t} \right\}} \sqrt{\frac{\sqrt{t}}{\sqrt{t} - B_{n-\ell(y)}}} \\ \times \widetilde{P}_v \left(\max_{u \in T^v_{\ell(y)}} \sqrt{\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow \left((\sqrt{t} - B_{n-\ell(y)})^2 \right)}(u)} > \sqrt{t} + a_n(t) + y \right) \end{array} \right]. \tag{5.16}$$

We estimate the indicator function in the expectation in (5.16) from above by $\sum_{i=1}^{3} \mathbf{1}_{E_i}$, where

$$E_1 := \{(1 - \delta)\sqrt{t} \leq B_{n-\ell(y)} \leq \sqrt{t}\},$$
$$E_2 := \{-a_n(t) + \sqrt{\log b}\, \ell(y) \leq B_{n-\ell(y)} < (1-\delta)\sqrt{t}, \max_{s \in [0, n-\ell(y)]} B_s > (1-\delta)\sqrt{t}\},$$
$$E_3 := \{B_{n-\ell(y)} < -a_n(t) + \sqrt{\log b}\, \ell(y), \max_{s \in [0, n-\ell(y)]} B_s > (1-\delta)\sqrt{t}\}.$$

Let $H_i, i \in \{1, 2, 3\}$ be the expectation obtained from the one in (5.16) by replacing the indicator function in it with $\mathbf{1}_{E_i}$. In particular, the right of (5.16) is bounded from above by $b^{n-\ell(y)}(H_1 + H_2 + H_3)$.

To estimate H_1, we use Proposition 3.1(i) and the density of $B_{n-\ell(y)}$. To estimate H_2, we use Proposition 3.1(i) and (3.28). To estimate H_3, we use (3.28) and bound the probability in H_3 from above just by 1. Taking $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ large enough, for all $n \geq n_0$, (5.16) is bounded from above by

$$c_4 \varepsilon_n y e^{-2\sqrt{\log b}\, y}, \tag{5.17}$$

where $\varepsilon_n, n \geq 0$ is a sequence with $\varepsilon_n \to 0$ as $n \to \infty$. By (5.13) and (5.17) together with Proposition 3.1(ii) and Lemma 3.2, taking $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ large enough, we have for all $n \geq n_0$

$$\widetilde{E}_\rho(\widetilde{\Lambda}^n_{y,\ell(y)}(t)) \geq c_5 y e^{-2\sqrt{\log b}\, y}. \tag{5.18}$$

By (5.12) and (5.18), we have (5.3). $\qquad\square$

To obtain a lower bound for $P_\rho(\max_{v \in T_n} \sqrt{L^n_{\tau(t)}(v)} \geq \sqrt{t} + a_n(t) + y)/\widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)]$, we need the following:

Lemma 5.3. *There exist $c_1, c_2 \in (0, \infty)$, $y_0 > 0$ such that the following holds: for all $y \geq y_0$ and $\ell(y) > e^{\frac{8\sqrt{\log b}}{3} y^{1/20}}$, there exists $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ such that for all $n \geq n_0$ and $t \geq c_1 n \log n$,*

$$\frac{\widetilde{E}_\rho\left[\left(\Lambda^n_{y,\ell(y)}(t)\right)^2\right]}{\widetilde{E}_\rho\left[\Lambda^n_{y,\ell(y)}(t)\right]} \leq 1 + c_2 y^{-1/2}. \tag{5.19}$$

Proof. Fix any $y \geq y_0$ and $\ell(y) > e^{\frac{8\sqrt{\log b}}{3} y^{1/20}}$, where we take $y_0 > 0$ large enough. Throughout the proof, given $n \in \mathbb{N}$, we assume $t \geq c_* n \log n$ for some sufficiently large $c_* > 0$. We have

$$\widetilde{E}_\rho\left[\left(\Lambda^n_{y,\ell(y)}(t)\right)^2\right] = \widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)] + \sum_{k=1}^{n-\ell(y)} \sum_{\substack{v,w \in T_{n-\ell(y)}, \\ |v \wedge w| = n-\ell(y)-k}} \widetilde{P}_\rho\left(F^n_{v,y,\ell(y)}(t) \cap F^n_{w,y,\ell(y)}(t)\right). \tag{5.20}$$

Fix $1 \leq k \leq n - \ell(y) - 1$ and $v, w \in T_{n-\ell(y)}$ with $|v \wedge w| = n - \ell(y) - k$. Let $\widetilde{L}^\downarrow$ be a local time of a Brownian motion on $\widetilde{T}^{v \wedge w}_{\leq \ell(y)+k}$ and set $\widetilde{\tau}^\downarrow(s) := \inf\{r \geq 0 : \widetilde{L}^\downarrow_r(v \wedge w) > s\}$. By Lemma 2.1, $\widetilde{P}_\rho(F^n_{v,y,\ell(y)}(t) \cap F^n_{w,y,\ell(y)}(t))$ is equal to

$$\widetilde{E}_\rho\left[1_{\left\{\delta\sqrt{t} \leq \sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n}s+y, \ \forall s \in [0, \ n-\ell(y)-k]\right\}} \prod_{x \in \{v, \ w\}} P^x\right], \tag{5.21}$$

where for each $x \in \{v, w\}$,

$$P^x := \widetilde{P}_{v \wedge w}\begin{pmatrix} \forall s \in [n-\ell(y)-k, \ n-\ell(y)], \\ \delta\sqrt{t} \leq \sqrt{\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow\left(\widetilde{L}^n_{\widetilde{\tau}(t)}(v \wedge w)\right)}(x_s)} \leq \sqrt{t} + \frac{a_n(t)}{n}s + y, \\ \max_{u \in T^x_{\ell(y)}} \sqrt{\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow\left(\widetilde{L}^n_{\widetilde{\tau}(t)}(v \wedge w)\right)}(u)} > \sqrt{t} + a_n(t) + y \end{pmatrix}.$$

Fix $0 \leq i \leq \lfloor (1-\delta)\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)-k) + y \rfloor$. Assume that

$$\sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v \wedge w)} \in \sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)-k) + y - i + (-1, \ 0]. \tag{5.22}$$

Under the assumption (5.22), we estimate the probabilities in (5.21). Fix $x \in \{v, w\}$. Let $\widetilde{L}^{\downarrow\downarrow}$ be a local time of a Brownian motion on $\widetilde{T}^x_{\leq \ell(y)}$. We define the inverse local time by $\widetilde{\tau}^{\downarrow\downarrow}(s) := \inf\{r \geq 0 : \widetilde{L}^{\downarrow\downarrow}_r(x) > s\}$. By Lemma 2.1, P^x is equal to

$$\widetilde{E}_{v \wedge w}\left[1_{\left\{\delta\sqrt{t} \leq \sqrt{\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow\left(\widetilde{L}^n_{\widetilde{\tau}(t)}(v \wedge w)\right)}(x_s)} \leq \sqrt{t} + \frac{a_n(t)}{n}s+y, \ \forall s \in [n-\ell(y)-k, \ n-\ell(y)]\right\}} \times \widetilde{P}_x\left(\max_{u \in T^x_{\ell(y)}} \sqrt{\widetilde{L}^{\downarrow\downarrow}_{\widetilde{\tau}^{\downarrow\downarrow}\left(\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow\left(\widetilde{L}^n_{\widetilde{\tau}(t)}(v \wedge w)\right)}(x)\right)}(u)} > \sqrt{t} + a_n(t) + y\right)\right]. \tag{5.23}$$

By Lemma 2.3, (2.6), and the change of measure (3.15)–(3.16), (5.23) is bounded from

above by

$$
\widetilde{E}_k^B \left[\begin{array}{l} e^{-\frac{2a_n(t)}{n}\widetilde{B}_k - \frac{(a_n(t))^2}{n^2}k} \sqrt{\dfrac{\sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v \wedge w)}}{\sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v \wedge w)} + \frac{a_n(t)}{n}k + \widetilde{B}_k}} \\ \times \mathbf{1}_{\left\{\delta\sqrt{t} - \frac{a_n(t)}{n}s \le \sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v \wedge w)} + \widetilde{B}_s \le \sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)-k)+y, \ \forall s\in[0,k]\right\}} \\ \times \widetilde{P}_x\left(\max_{u\in T_{\ell(y)}^x} \sqrt{\widetilde{L}_{\widetilde{\tau}^{\downarrow\downarrow}}^{\downarrow\downarrow}\left(\left(\sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v \wedge w)} + \frac{a_n(t)}{n}k + \widetilde{B}_k\right)^2\right)}(u) > \sqrt{t} + a_n(t) + y\right) \end{array} \right]. \quad (5.24)
$$

By (5.22) and (2.11), (5.24) is bounded from above by

$$
\int_0^{(1-\delta)\sqrt{t}+\frac{a_n(t)}{n}(n-\ell(y))+y+1} \frac{1}{\sqrt{\pi k}} \left(e^{-\frac{(i+1-z)^2}{k}} - e^{-\frac{(i+1+z)^2}{k}}\right)
$$
$$
\times e^{\frac{2a_n(t)}{n}z - \frac{2a_n(t)}{n}(i+1) - \frac{(a_n(t))^2}{n^2}k} \sqrt{\frac{\sqrt{t}+\frac{a_n(t)}{n}(n-\ell(y)-k)+y-i}{\sqrt{t}+\frac{a_n(t)}{n}(n-\ell(y))+y-z}}
$$
$$
\times \widetilde{P}_x\left(\max_{u\in T_{\ell(y)}^x} \sqrt{\widetilde{L}_{\widetilde{\tau}^{\downarrow\downarrow}}^{\downarrow\downarrow}\left(\left(\sqrt{t}+\frac{a_n(t)}{n}(n-\ell(y))+y+1-z\right)^2\right)}(u) > \sqrt{t} + a_n(t) + y\right) dz. \quad (5.25)
$$

We use the following estimate in the integrand of (5.25): $e^{-\frac{(i+1-z)^2}{k}} - e^{-\frac{(i+1+z)^2}{k}} \le 1$ for each $1 \le k \le \lfloor y \rfloor$, and $e^{-\frac{(i+1-z)^2}{k}} - e^{-\frac{(i+1+z)^2}{k}} \le \frac{4(i+1)z}{k}$ for each $\lfloor y \rfloor \le k \le n - \ell(y) - 1$. By this and Proposition 3.1(i), the right-hand side of (5.25) is bounded from above by

$$
c_1 A b^{-k} e^{-\frac{2a_n(t)}{n}i} e^{\frac{3k\log n}{2n}} e^{\frac{k\log\left(\frac{\sqrt{t}+n}{\sqrt{t}}\right)}{2n}} \sqrt{\frac{\sqrt{t}+\frac{a_n(t)}{n}(n-\ell(y)-k)+y-i}{\sqrt{t}+n}}, \quad (5.26)
$$

where $A := k^{-1/2}(\ell(y))^{-1/2}$ if $1 \le k \le \lfloor y \rfloor$, and $A := k^{-3/2}(i+1)$ if $\lfloor y \rfloor \le k \le n - \ell(y) - 1$. Recall the events in the indicator function in (5.21) and in (5.22). We estimate the probability of the intersection of these events. Using Lemma 2.3, (2.6), and and the change of measure (3.15)–(3.16) for $n \ge n_0$ ($n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ large enough), we have

$$
\widetilde{P}_\rho \left(\begin{array}{l} \delta\sqrt{t} \le \sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v_s)} \le \sqrt{t} + \frac{a_n(t)}{n}s + y, \ \forall s \in [0, \ n-\ell(y)-k], \\ \sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v \wedge w)} \in \sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)-k)+y-i+(-1, \ 0] \end{array} \right)
$$
$$
\le \widetilde{E}_{n-\ell(y)-k}^B \left[\begin{array}{l} \mathbf{1}_{\left\{\widetilde{B}_s \le y, \ \forall s\in[0, \ n-\ell(y)-k], \ \widetilde{B}_{n-\ell(y)-k}-y\in(-i-1, \ -i]\right\}} \\ \times e^{-\frac{2a_n(t)}{n}\widetilde{B}_{n-\ell(y)-k} - \frac{(a_n(t))^2}{n^2}(n-\ell(y)-k)} \sqrt{\dfrac{\sqrt{t}}{\sqrt{t}+\frac{a_n(t)}{n}(n-\ell(y)-k)+\widetilde{B}_{n-\ell(y)-k}}} \end{array} \right]. \quad (5.27)
$$

By (2.11), the right of (5.27) is bounded from above by

$$
\int_i^{i+1} \frac{1}{\sqrt{\pi(n-\ell(y)-k)}} \left(e^{-\frac{(y-z)^2}{n-\ell(y)-k}} - e^{-\frac{(y+z)^2}{n-\ell(y)-k}}\right)
$$
$$
\times e^{\frac{2a_n(t)}{n}z - \frac{2a_n(t)}{n}y - \frac{(a_n(t))^2}{n^2}(n-\ell(y)-k)} \sqrt{\frac{\sqrt{t}}{\sqrt{t}+\frac{a_n(t)}{n}(n-\ell(y)-k)+y-z}} \ dz. \quad (5.28)
$$

We will use the following in the integrand of (5.28): $e^{-\frac{(y-z)^2}{n-\ell(y)-k}} - e^{-\frac{(y+z)^2}{n-\ell(y)-k}} \le \frac{4yz}{n-\ell(y)-k}$ for each $1 \le k \le n - \ell(y) - \lfloor y \rfloor$, and $e^{-\frac{(y-z)^2}{n-\ell(y)-k}} - e^{-\frac{(y+z)^2}{n-\ell(y)-k}} \le 1$ for each $n - \ell(y) - \lfloor y \rfloor \le$

$k \le n - \ell(y) - 1$. By this, the right-hand side of (5.28) is bounded from above by

$$
c_2 A' b^{-(n-\ell(y)-k)} e^{\frac{2a_n(t)}{n} i} e^{-2\sqrt{\log b}\, y}
$$
$$
\times e^{\frac{3(n-\ell(y)-k)}{2n}} \log n \, e^{\frac{(n-\ell(y)-k)\log\left(\frac{\sqrt{t}+n}{\sqrt{t}}\right)}{2n}} \sqrt{\frac{\sqrt{t}}{\sqrt{t}+\frac{a_n(t)}{n}(n-\ell(y)-k)+y-i-1}}, \tag{5.29}
$$

where $A' := (n-\ell(y)-k)^{-3/2}(i+1)y$ if $1 \le k \le n-\ell(y)-\lfloor y \rfloor$, and $A' := (n-\ell(y)-k)^{-1/2}$ if $n-\ell(y)-\lfloor y \rfloor \le k \le n-\ell(y)-1$.

We divide the sum over $1 \le k < n-\ell(y)$ in (5.20) into sums over the following: (a) $1 \le k \le \lfloor y \rfloor$; (b) $\lfloor y \rfloor < k \le \lfloor (n-\ell(y))/2 \rfloor$; (c) $\lfloor (n-\ell(y))/2 \rfloor < k \le n-\ell(y)-\lfloor y \rfloor$; (d) $n-\ell(y)-\lfloor y \rfloor < k \le n-\ell(y)-1$. We make remarks on how to estimate the sums over difficult regimes, (b) and (c): In the regime (b), we use the fact that $k^{-\frac{3}{2}} e^{\frac{3k\log n}{2n}} e^{\frac{k\log\left(\frac{\sqrt{t}+n}{\sqrt{t}}\right)}{2n}}$ is bounded from above by a universal constant. In the regime of (c), we use the estimate

$$
\frac{1}{(n-\ell(y)-k)^{\frac{3}{2}}} \sqrt{\frac{\sqrt{t}+\frac{a_n(t)}{n}(n-\ell(y)-k)+y}{\sqrt{t}}} \le \frac{c}{(n-\ell(y)-k)^{\frac{3}{2}}} + \frac{c}{t^{\frac{1}{4}}(n-\ell(y)-k)},
$$

for some universal constant c.

By (5.21), (5.26), and (5.29), we have

$$
\sum_{k=1}^{n-\ell(y)-1} \sum_{\substack{v,w \in T_{n-\ell(y)}, \\ |v \wedge w| = n-\ell(y)-k}} \widetilde{P}_\rho\left(F^n_{v,y,\ell(y)}(t) \cap F^n_{w,y,\ell(y)}(t)\right) \le c_3 y^{1/2} e^{-2\sqrt{\log b}\, y}. \tag{5.30}
$$

In the case $k = n - \ell(y)$, by the independence of excursions of a Brownian motion around ρ, we have

$$
\sum_{\substack{v,w \in T_{n-\ell(y)}, \\ |v \wedge w| = 0}} \widetilde{P}_\rho\left(F^n_{v,y,\ell(y)}(t) \cap F^n_{w,y,\ell(y)}(t)\right) = \sum_{\substack{v,w \in T_{n-\ell(y)}, \\ |v \wedge w| = 0}} \widetilde{P}_\rho\left(F^n_{v,y,\ell(y)}(t)\right) \widetilde{P}_\rho\left(F^n_{w,y,\ell(y)}(t)\right)
$$
$$
\le \left(\widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)]\right)^2. \tag{5.31}
$$

Thus, by (5.20), (5.30), and (5.31), we have

$$
\widetilde{E}_\rho\left[\left(\Lambda^n_{y,\ell(y)}(t)\right)^2\right] \le \widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)] + c_3 y^{1/2} e^{-2\sqrt{\log b}\, y} + \left(\widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)]\right)^2. \tag{5.32}
$$

We will obtain upper and lower bounds of $\widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)]$. By (5.12) and (5.18), taking $n_0 = n_0(y, \ell(y))$ large enough, we have for all $n \ge n_0$

$$
\widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)] \ge c_4 y e^{-2\sqrt{\log b}\, y}. \tag{5.33}
$$

By the arguments in (5.4) and (5.5), $\widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)]$ is bounded from above by $b^{n-\ell(y)}$ times the second term of (5.5). By this together with (2.11) and Proposition 3.1(i), we have

$$
\widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)] \le c_5 y e^{-2\sqrt{\log b}\, y}. \tag{5.34}
$$

Thus, by (5.32)–(5.34), we have (5.19). $\qquad\square$

Using Lemma 5.2 and 5.3, we prove the following:

Lemma 5.4. *There exist $c_1, c_2, c_3 \in (0, \infty)$, $y_0 > 0$, and $\{\delta_{y'} : y' > 0\}$ with $\lim_{y' \to \infty} \delta_{y'} = 0$ such that the following holds: for all $y \geq y_0$ and $\ell(y) > e^{\frac{8\sqrt{\log b}}{3} y^{1/20}}$, there exists $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ such that for all $n \geq n_0$ and $t \geq c_1 n \log n$,*

$$\frac{P_\rho \left(\max_{v \in T_n} \sqrt{L^n_{\tau(t)}(v)} > \sqrt{t} + a_n(t) + y \right)}{\widetilde{E}_\rho \left[\Lambda^n_{y,\ell(y)}(t) \right]} \leq (1 + \delta_y) \left(1 + e^{-c_2 \, y^{1/20}} + \delta_y \right), \qquad (5.35)$$

$$\frac{P_\rho \left(\max_{v \in T_n} \sqrt{L^n_{\tau(t)}(v)} > \sqrt{t} + a_n(t) + y \right)}{\widetilde{E}_\rho \left[\Lambda^n_{y,\ell(y)}(t) \right]} \geq 1 - c_3 y^{-1/2}. \qquad (5.36)$$

Proof. Fix any $y \geq y_0$, $\ell(y) > e^{\frac{8\sqrt{\log b}}{3} y^{1/20}}$, $n \geq n_0$, and $t \geq c_* n \log n$, where we take $y_0 > 0$, $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$, $c_* > 0$ large enough. We first obtain the upper bound. Recall the event $G^{n-\ell(y)}_{y+y^{1/20}-2}(t)$ from (3.3) and ε_n from (5.17). Recall the inequality in (5.14). We have

$$P_\rho \left(\max_{v \in T_n} \sqrt{L^n_{\tau(t)}(v)} > \sqrt{t} + a_n(t) + y \right)$$

$$= P_\rho \left(\exists v \in T_{n-\ell(y)} : \max_{u \in T^v_{\ell(y)}} \sqrt{L^n_{\tau(t)}(u)} > \sqrt{t} + a_n(t) + y \right)$$

$$\leq \widetilde{E}_\rho[\widetilde{\Lambda}^n_{y,\ell(y)}(t)] + \widetilde{P}_\rho \left(G^{n-\ell(y)}_{y+y^{1/20}-2}(t) \right)$$

$$+ \widetilde{P}_\rho \left(\exists v \in T_{n-\ell(y)} : \begin{array}{l} \max_{u \in T^v_{\ell(y)}} \sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(u)} > \sqrt{t} + a_n(t) + y, \\ \min_{s \in [0, \, n-\ell(y)]} \sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v_s)} < \delta\sqrt{t} \end{array} \right)$$

$$\leq \left(1 + c_1 e^{-2\sqrt{\log b} \, y^{1/20}} + \varepsilon_n \right) \widetilde{E}_\rho[\widetilde{\Lambda}^n_{y,\ell(y)}(t)], \qquad (5.37)$$

where we have used Lemma 3.2, (5.17), and (5.18) in the last inequality. By (5.37) and Lemma 5.2, we have (5.35). By Lemma 5.3, we have

$$P_\rho \left(\max_{v \in T_n} \sqrt{L^n_{\tau(t)}(v)} > \sqrt{t} + a_n(t) + y \right) \geq \widetilde{P}_\rho \left(\Lambda^n_{y,\ell(y)}(t) \geq 1 \right) \geq \frac{\left(\widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)] \right)^2}{\widetilde{E}_\rho \left[\left(\Lambda^n_{y,\ell(y)}(t) \right)^2 \right]}$$

$$\geq \left(1 - c_2 y^{-1/2} \right) \widetilde{E}_\rho[\Lambda^n_{y,\ell(y)}(t)],$$

which proves (5.36). $\qquad \square$

For each interval $I \subset \mathbb{R}$, set

$$\Lambda^n_{y,\ell(y),I}(t) := \sum_{v \in T_{n-\ell(y)}} 1_{F^n_{v,y,\ell(y)}(t) \cap \left\{ \sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v)} \in \sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)) + I \right\}}. \qquad (5.38)$$

Set the interval

$$J_{\ell(y)} := \left(-\ell(y) + y, \, -(\ell(y))^{2/5} + y \right]. \qquad (5.39)$$

Then the following holds:

Lemma 5.5. *There exist $c_1, c_2 \in (0, \infty)$ and $y_0 > 0$ such that the following holds: for all $y \geq y_0$ and $\ell(y) > e^{\frac{8\sqrt{\log b}}{3} y^{1/20}}$, there exists $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ such that for all $n \geq n_0$ and $t \geq c_1 n \log n$,*

$$\frac{\widetilde{E}_\rho \left[\Lambda^n_{y,\ell(y),J_{\ell(y)}}(t) \right]}{\widetilde{E}_\rho \left[\Lambda^n_{y,\ell(y)}(t) \right]} \geq 1 - c_2 (\ell(y))^{-1/5}. \qquad (5.40)$$

Proof. Fix any $y \geq y_0$ and $\ell(y) > e^{\frac{8\sqrt{\log b}}{3} y^{1/20}}$, where we take $y_0 > 0$ large enough. Throughout the proof, given $n \in \mathbb{N}$, we assume that $t \geq c_* n \log n$ for some sufficiently large $c_* > 0$. Fix $v \in T_{n-\ell(y)}$ and any interval $I \subset \mathbb{R}$. Recall the definition of ψ from (5.6). By similar arguments to those in (5.4) and (5.5), taking $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ large enough, we have for all $n \geq n_0$,

$$\widetilde{P}_\rho \left(F^n_{v,y,\ell(y)}(t) \cap \left\{ \sqrt{\widetilde{L}^n_{\widetilde{\tau}(t)}(v)} \in \sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + I \right\} \right)$$
$$\leq \widetilde{E}^B_{n-\ell(y)} \left[\mathbb{1}_{\left\{ -(1-\delta)\sqrt{t} - \frac{a_n(t)}{n} s \leq \widetilde{B}_s \leq y, \, \forall s \in [0, \, n-\ell(y)], \, \widetilde{B}_{n-\ell(y)} \in I \right\}} \psi(\widetilde{B}_{n-\ell(y)}) \right]. \quad (5.41)$$

By Proposition 3.1(i) and (2.11), (5.41) is bounded from above by

$$c_1 b^{-(n-\ell(y))} (\ell(y))^{-3/2} y e^{-2\sqrt{\log b}\, y}$$

$$\times \int_{(y-I) \cap I_{n,t,y}} z(\log \ell(y) + z) e^{-c_2 \frac{z^2}{\ell(y)}} \sqrt{\frac{\sqrt{t} + n}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + y - z + \ell(y)}} \; dz, \quad (5.42)$$

where $I_{n,t,y} := [0, \, (1-\delta)\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + y]$ and we have used the inequality $e^{-\frac{(y-z)^2}{n-\ell(y)}} - e^{-\frac{(y+z)^2}{n-\ell(y)}} \leq \frac{4yz}{n-\ell(y)}$ for $z \geq 0$. In the cases $I = (-\infty, \, y - \ell(y)]$ and $I = \left(y - (\ell(y))^{\frac{2}{5}}, \, \infty \right)$, the right-hand side of (5.42) is bounded from above by

$$c_3 b^{-(n-\ell(y))} (\ell(y))^{-1/5} y e^{-2\sqrt{\log b}\, y}$$

for all $n \geq n_0$, where $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ large enough. By this and (5.33), we have

$$\widetilde{E}_\rho \left[\Lambda^n_{y,\ell(y)}(t) \right]$$
$$= \widetilde{E}_\rho \left[\Lambda^n_{y,\ell(y),J_{\ell(y)}}(t) \right] + \widetilde{E}_\rho \left[\Lambda^n_{y,\ell(y),(-\infty, \, y-\ell(y)]}(t) \right] + \widetilde{E}_\rho \left[\Lambda^n_{y,\ell(y),\left(y-(\ell(y))^{2/5}, \, \infty\right)}(t) \right]$$
$$\leq \widetilde{E}_\rho \left[\Lambda^n_{y,\ell(y),J_{\ell(y)}}(t) \right] + c_4 (\ell(y))^{-1/5} \widetilde{E}_\rho \left[\Lambda^n_{y,\ell(y)}(t) \right],$$

which implies (5.40). $\qquad\square$

Proof of Proposition 5.1. Let $(h_v)_{v \in T}$ be a BRW on T defined in Section 1. By Lemma A.7, one can show that the sequence

$$\left(\int_{\ell^{2/5}}^{\ell} z e^{2\sqrt{\log b}\, z} \mathbb{P} \left[\max_{u \in T_\ell} h_u > \sqrt{\log b}\, \ell + z \right] dz \right)_{\ell \geq 1} \quad (5.43)$$

is bounded from above and away from 0. Fix a nondecreasing sequence $(\ell_0(y_k^+))_{k \geq 1}$ with $\ell_0(y_k^+) > e^{\frac{8\sqrt{\log b}}{3}(y_k^+)^{1/20}}$ for each $k \geq 1$. By the boundedness of the sequence (5.43), there exists a subsequence $(\ell_0(y_{k_j}^+))_{j \geq 1}$ of $(\ell_0(y_k^+))_{k \geq 1}$ such that the limit

$$\widetilde{\gamma}_* := \lim_{j \to \infty} \int_{(\ell_0(y_{k_j}^+))^{2/5}}^{\ell_0(y_{k_j}^+)} z e^{2\sqrt{\log b}\, z} \mathbb{P} \left[\max_{u \in T_{\ell_0(y_{k_j}^+)}} h_u > \sqrt{\log b}\, \ell_0(y_{k_j}^+) + z \right] dz \in (0, \infty) \quad (5.44)$$

exists. We set

$$\ell_j := \ell_0(y_{k_j}^+), \;\; j \geq 1.$$

Note that by the definition of $\ell_0(y_j^+)$, for any y_j with $y_j \leq y_j^+$, we have

$$\ell_j \geq \ell_0(y_j^+) > e^{\frac{8\sqrt{\log b}}{3}(y_j^+)^{1/20}} \geq e^{\frac{8\sqrt{\log b}}{3}(y_j)^{1/20}}.$$

Fix $j \geq 1$ and y_j with $y_j^- \leq y_j \leq y_j^+$. Recall definitions J_{ℓ_j} and $\Lambda_{y_j, \ell_j, J_{\ell_j}}^n(t)$ from (5.39) and (5.38). Fix $v^* \in T_{n-\ell_j}$. Let $\widetilde{L}^{\downarrow}$ be a local time of a Brownian motion on $\widetilde{T}_{\leq \ell_j}^{v^*}$. We set $\widetilde{\tau}^{\downarrow}(s) := \inf\{r \geq 0 : \widetilde{L}_r^{\downarrow}(v^*) > s\}$. Recall $\widetilde{P}_{n-\ell_j}^B$ and $\widetilde{B}$ from (3.15) and (3.16). We define $\phi(x)$ by replacing y, $\ell(y)$, and v in the definition of $\psi(x)$ in (5.6) by y_j, ℓ_j, and v^*, respectively. By similar arguments to those in (5.4) and (5.5), we have for $t \geq n \log n$,

$$
\widetilde{P}_{\rho} \left[F_{v^*, y_j, \ell_j}^n(t) \cap \left\{ \sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v^*)} \in \sqrt{t} + \frac{a_n(t)}{n}(n - \ell_j) + J_{\ell_j} \right\} \right]
$$

$$
= \left(1 + O\left(\frac{1}{\log n} \right) \right) \widetilde{E}_{n-\ell_j}^B \left[1_{\left\{ \widetilde{B}_s \leq y_j,\, \forall s \in [0,\, n-\ell_j],\, \widetilde{B}_{n-\ell_j} \in J_{\ell_j} \right\}} \phi\left(\widetilde{B}_{n-\ell_j} \right) \right]
$$

$$
- \left(1 + O\left(\frac{1}{\log n} \right) \right) \widetilde{E}_{n-\ell_j}^B \left[\begin{array}{l} 1_{\left\{ \widetilde{B}_s \leq y_j,\, \forall s \in [0,\, n-\ell_j],\, \widetilde{B}_{n-\ell_j} \in J_{\ell_j} \right\}} \\ \times 1_{\left\{ \widetilde{B}_s < -(1-\delta)\sqrt{t} - \frac{a_n(t)}{n}s,\, \exists s \in [0,\, n-\ell_j] \right\}} \phi\left(\widetilde{B}_{n-\ell_j} \right) \end{array} \right]
$$

$$
=: K_1 - K_2, \tag{5.45}
$$

where we have used the fact that for $t \geq n \log n$, $\exp\left(-\frac{3}{8} \int_0^{n-\ell_j} \frac{ds}{X_s} \right) = 1 + O((\log n)^{-1})$ under the event that $\sqrt{X_s/2} \geq \delta\sqrt{t}$ for all $0 \leq s \leq n - \ell_j$.

We first estimate K_2. By (3.28) and Proposition 3.1(i), for all $\varepsilon > 0$ and $j \geq 1$, there exists $n_0(j) = n_0(y_j^-, y_j^+, \ell_j) \in \mathbb{N}$ such that for all $n \geq n_0(j)$, we have

$$
K_2 \leq c_1 b^{-(n-\ell_j)} \sqrt{\ell_j}\, n^{1-(1-\delta)^2 c_*} y_j e^{-2\sqrt{\log b}\, y_j} \leq \varepsilon b^{-(n-\ell_j)} y_j e^{-2\sqrt{\log b}\, y_j}, \tag{5.46}
$$

uniformly in y_j and t satisfying (5.2) (we take c_* large enough). Next, we estimate K_1. By the density (2.11), K_1 is equal to

$$
\int_{(\ell_j)^{2/5}}^{\ell_j} \frac{1 + O(1/\log n)}{\sqrt{\pi(n - \ell_j)}} \left(e^{-\frac{(z - y_j)^2}{n - \ell_j}} - e^{-\frac{(z + y_j)^2}{n - \ell_j}} \right)
$$

$$
\times e^{\frac{2a_n(t)}{n} z - \frac{2a_n(t)}{n} y_j - \frac{(a_n(t))^2}{n^2}(n - \ell_j)} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell_j) + y_j - z}}\, \widetilde{P}_{v^*}[A_{y_j, \ell_j}^{n,t}(z)]\, dz
$$

$$
= b^{-(n-\ell_j)}(1 + O(1/\log n)) \frac{4}{\sqrt{\pi}} y_j e^{-2\sqrt{\log b}\, y_j}
$$

$$
\times \int_{(\ell_j)^{2/5}}^{\ell_j} z e^{2\sqrt{\log b}\, z} \sqrt{\frac{\sqrt{t} + n}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell_j) + y_j - z}}\, P_{\rho}[B_{y_j, \ell_j}^{n,t}(z)]\, dz, \tag{5.47}
$$

where we have set

$$
A_{y_j, \ell_j}^{n,t}(z) := \left\{ \max_{u \in T_{\ell_j}^{v^*}} \sqrt{\widetilde{L}_{\widetilde{\tau}^{\downarrow}(s_{y_j, \ell_j}^{n,t}(z))}^{\downarrow}(u)} > \sqrt{t} + a_n(t) + y_j \right\},
$$

$$
B_{y_j, \ell_j}^{n,t}(z) := \left\{ \frac{\max_{u \in T_{\ell_j}} L_{\tau(s_{y_j, \ell_j}^{n,t}(z))}^{\ell_j}(u) - s_{y_j, \ell_j}^{n,t}(z)}{2\sqrt{s_{y_j, \ell_j}^{n,t}(z)}} > \sqrt{\log b}\, \ell_j + z + \Delta_{y_j, \ell_j}^{n,t}(z) \right\}.
$$

Here,

$$
s_{y_j, \ell_j}^{n,t}(z) := \left(\sqrt{t} + \frac{a_n(t)}{n}(n - \ell_j) + y_j - z \right)^2
$$

and $\Delta_{y_j, \ell_j}^{n,t}(z)$ is the remainder term so that $\widetilde{P}_{v^*}[A_{y_j, \ell_j}^{n,t}(z)] = P_{\rho}[B_{y_j, \ell_j}^{n,t}(z)]$. (Note that by the symmetry of the b-ary tree, the law of the Brownian motion on $\widetilde{T}_{\leq \ell_j}^{v^*}$ starting at v^* is

the same as that of the Brownian motion on $\widetilde{T}_{\leq \ell_j}$ starting at ρ.) One can show that for each $j \geq 1$,

$$\lim_{n \to \infty} \Delta^{n,t}_{y_j, \ell_j}(z) = 0, \quad \text{uniformly in } z \in [(\ell_j)^{2/5}, \ell_j] \text{ and } y_j, t \text{ satisfying (5.2).} \qquad (5.48)$$

By Theorem 2.2, we have for any fixed $m \in \mathbb{N}$

$$\left(\frac{L^m_{\tau(s)}(v) - s}{2\sqrt{s}} \right)_{v \in T_m} \longrightarrow (h_v)_{v \in T_m} \quad \text{in law as } s \to \infty,$$

where $(h_v)_{v \in T}$ is a BRW. By this together with the definition of $\widetilde{\gamma}_*$ in (5.44) and (5.48), for all $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that the following holds: for each $j \geq j_0$, there exists $n_0(j) = n_0(y_j^-, y_j^+, \ell_j) \in \mathbb{N}$ such that for all $n \geq n_0(j)$,

$$\left| \int_{(\ell_j)^{2/5}}^{\ell_j} P_\rho \left[B^{n,t}_{y_j, \ell_j}(z) \right] \cdot z e^{2\sqrt{\log b}\, z} dz \; - \; \widetilde{\gamma}_* \right| < \varepsilon, \qquad (5.49)$$

uniformly in y_j and t satisfying (5.2). Thus, by (5.45)–(5.47), (5.49), and the definition (1.8) of β_*, for all $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that the following holds: for each $j \geq j_0$, there exists $n_0(j) = n_0(y_j^-, y_j^+, \ell_j) \in \mathbb{N}$ such that for all $n \geq n_0(j)$,

$$\left| \widetilde{E}_\rho \left[\Lambda^n_{y_j, \ell_j, J_{\ell_j}}(t) \right] (y_j)^{-1} e^{2\sqrt{\log b}\, y_j} - \frac{4}{\sqrt{\pi}} \beta_* \widetilde{\gamma}_* \right| < \varepsilon, \qquad (5.50)$$

uniformly in y_j and t satisfying (5.2). By Lemma 5.4, 5.5, and (5.50), for all $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that the following holds: for each $j \geq j_0$, there exists $n_0(j) = n_0(y_j^-, y_j^+, \ell_j) \in \mathbb{N}$ such that for all $n \geq n_0(j)$, (5.1) holds by replacing γ_* with $\widetilde{\gamma}_*$ uniformly in y_j and t satisfying (5.2) (we take $c_* > 0$ large enough). Let $\widehat{\gamma}_*$ be the limit of any convergent subsequence of (5.43). By taking a sub-subsequence, if necessary, and repeating the above argument, we have (5.1) if we replace γ_* with $\widehat{\gamma}_*$. Thus, the full sequence (5.43) converges to a finite positive constant and we write γ_* to denote the limit. Therefore, we have (5.1). $\qquad \square$

6 Proof of Theorem 1.1 and Corollary 1.3

In this section, we prove Theorem 1.1 and Corollary 1.3. We begin with preliminary lemmas. Let $(h_v)_{v \in T}$ be a BRW on T defined in Section 1. For each $n \in \mathbb{N}$, we set

$$D_n^{(2)} := \sum_{v \in T_n} \left(\sqrt{\log b}\, n - h_v \right)^2 e^{-4\sqrt{\log b} \left(\sqrt{\log b}\, n - h_v \right)}.$$

Then the following holds:

Lemma 6.1. *For all $\varepsilon > 0$,*

$$\lim_{n \to \infty} \mathbb{P} \left(D_n^{(2)} \geq \varepsilon \right) = 0. \qquad (6.1)$$

Proof. Set $m_n := \sqrt{\log b}\, n - \frac{3}{4\sqrt{\log b}} \log n$. Fix any $y > 0$. By the simple equality $1_{\{\max_{v \in T_n} h_v \leq m_n + y\}} + 1_{\{\max_{v \in T_n} h_v > m_n + y\}} = 1$, we bound the probability $\mathbb{P}(D_n^{(2)} \geq \varepsilon)$ from above by

$$\mathbb{P} \left(\sum_{v \in T_n} \left(\sqrt{\log b}\, n - h_v \right)^2 e^{-4\sqrt{\log b} \left(\sqrt{\log b}\, n - h_v \right)} 1_{\{h_v \leq m_n + y\}} \geq \varepsilon \right)$$

$$+ \mathbb{P} \left(\max_{v \in T_n} h_v > m_n + y \right). \qquad (6.2)$$

Since h_v is a Gaussian random variable with mean 0 and variance $n/2$ for each $v \in T_n$, a simple calculation implies that

$$\lim_{n\to\infty} \mathbb{E}\left[\sum_{v\in T_n} \left(\sqrt{\log b}\, n - h_v\right)^2 e^{-4\sqrt{\log b}\,\left(\sqrt{\log b}\, n - h_v\right)} 1_{\{h_v \le m_n + y\}}\right] = 0.$$

By this, Lemma A.7(i), and (6.2), we have

$$\limsup_{n\to\infty} \mathbb{P}\left(D_n^{(2)} \ge \varepsilon\right) \le c_1(1+y)e^{-2\sqrt{\log b}\, y}.$$

Since we can take arbitrary $y \in (0, \infty)$, this implies (6.1). □

Recall the definition of $\sigma(\cdot)$ from (1.2). For $a \in \mathbb{R}$, an interval $I \subset \mathbb{R}$, $t > 0$, and $r, n \in \mathbb{N}$ with $r < n$, we set

$$\mathcal{B}_{r,t,n}^{\sigma}(I; a) := \left\{\sqrt{L_{\tau(t)}^n(v)} - \sqrt{t} - a_n(t) \notin I,\ \forall v \in T_n \text{ with } \sigma(v_r) \in \left[a - b^{-r},\, a\right]\right\}. \quad (6.3)$$

Then we have the following:

Lemma 6.2. *There exists $c_1 > 0$ such that the following holds: for any finite interval $I \subset \mathbb{R}$, there exist $c_2(I) > 0$ and $r_0(I) \in \mathbb{N}$ such that for all $t > 0$, $a \in \mathbb{R}$, $r \ge r_0(I)$, and $n > r$,*

$$P_\rho\left[\left(\mathcal{B}_{r,t,n}^{\sigma}(I; a)\right)^c\right] \le c_2(I)r^3 e^{-c_1 r}. \quad (6.4)$$

Proof. Fix any finite interval $I \subset \mathbb{R}$. Let $\{\overline{I}, \underline{I}\}$ be the boundary of I with $\underline{I} < \overline{I}$. Fix any $t > 0$, $a \in \mathbb{R}$, $r \ge r_0(I)$, and $n > r$, where we take $r_0(I) > |\overline{I}|$ large enough. Recall the event $G_r^n(t)$ from (3.3). $P_\rho[(\mathcal{B}_{r,t,n}^{\sigma}(I; a))^c]$ is bounded from above by

$$\widetilde{P}_\rho\left[\begin{array}{l} \exists v \in T_n \text{ with } \sigma(v_r) \in [a - b^{-r},\, a] \text{ s.t. } \forall s \in [0,\, n], \\ \sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v_s)} \le \sqrt{t} + \frac{a_n(t)}{n}s + \kappa(\log(s \wedge (n-s)))_+ + r + 1, \\ \sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v)} - \sqrt{t} - a_n(t) \in I \end{array}\right] + \widetilde{P}_\rho\left(G_r^n(t)\right). \quad (6.5)$$

Next, we estimate the number of $v \in T_n$ satisfying $\sigma(v_r) \in [a - b^{-r},\, a]$. We may assume $a \in [0, 1]$. We have a sequence $(x_i)_{i \ge 1}$ with $x_i \in \{0, \dots, b-1\}$ such that $a = \sum_{i=1}^{\infty} \frac{x_i}{b^i}$. In particular, we have $\sum_{i=1}^{r} \frac{\overline{v}_i - x_i}{b^i} \in [-b^{-r},\, b^{-r}]$ for all $v \in T_n$ with $\sigma(v_r) \in [a - b^{-r},\, a]$ and the label $(\overline{v}_1, \dots, \overline{v}_n)$. By this and a simple observation, one can see that $\{v \in T_n : \sigma(v_r) \in [a - b^{-r},\, a]\}$ is a subset of

$$\{v \in T_n : \overline{v}_i = x_i,\ 1 \le \forall i \le r\}$$

$$\cup \bigcup_{i=1}^{r} \{v \in T_n : \overline{v}_j = x_j,\ 1 \le \forall j \le i-1,\ \overline{v}_i - x_i > 0,\ \overline{v}_{j'} = x_{j'} - (b-1),\ i < \forall j' \le r\}$$

$$\cup \bigcup_{i=1}^{r} \{v \in T_n : \overline{v}_j = x_j,\ 1 \le \forall j \le i-1,\ \overline{v}_i - x_i < 0,\ \overline{v}_{j'} = x_{j'} + (b-1),\ i < \forall j' \le r\}.$$

This implies

$$\left|\{v \in T_n : \sigma(v_r) \in [a - b^{-r},\, a]\}\right| \le c_1 r b^{n-r}. \quad (6.6)$$

By (6.6) and similar arguments to those in (3.23) and (3.24), the first term on the right-hand side of (6.5) is bounded from above by $c_2(I)b^{-r}r^3$. By this and Lemma 3.2, we have (6.4). □

Proof of Theorem 1.1. Recall definitions of Z_∞, $\Xi_{n,t}^{(m)}$, β_*, and γ_* from (1.4), (1.5), (1.8), and (1.9). Fix any sequence of positive integers $(r_n)_{n\geq 1}$ with $\lim_{n\to\infty} r_n = \infty$ and $\limsup_{n\to\infty} r_n/n < 1$. Fix any $(t_n)_{n\geq 1}$ with $\lim_{n\to\infty} \sqrt{t_n}/n = \theta \in [0,\infty]$ and $t_n \geq c_* n\log n$, where c_* is a sufficiently large positive constant. By, for example, [26, Proposition 11.1.VIII], in order to show the convergence of the point process $\Xi_{n,t_n}^{(r_n)}$ to the Cox process (1.7) as $n \to \infty$, it is enough to prove the following: for all positive values $a_1,\ldots,a_m$ and finite intervals $A_i := (\underline{A}_i,\ \overline{A}_i] \subset [0,1]$, $B_i := (\underline{B}_i,\ \overline{B}_i]$, $1 \leq i \leq m$ with $(A_i \times B_i) \cap (A_j \times B_j) = \emptyset$ for any $i \neq j$,

$$\lim_{n\to\infty} E_\rho\left[\exp\left\{-\sum_{i=1}^m a_i \Xi_{n,t_n}^{(r_n)}(A_i \times B_i)\right\}\right]$$
$$= \mathbb{E}\left[\exp\left\{-\frac{4}{\sqrt{\pi}}\beta_*\gamma_*\sum_{i=1}^m\left(1-e^{-a_i}\right)Z_\infty(A_i)\left(e^{-2\sqrt{\log b}\ \underline{B}_i} - e^{-2\sqrt{\log b}\ \overline{B}_i}\right)\right\}\right]. \qquad (6.7)$$

Let $q < n$ be a positive integer. To show (6.7), we first prove convergence of $\Xi_{n,t_n}^{(n-q)}$ as $n \to \infty$, $q \to \infty$. Recall the events (6.3). We have

$$E_\rho\left[\exp\left\{-\sum_{i=1}^m a_i \Xi_{n,t_n}^{(n-q)}(A_i \times B_i)\right\}\right]$$
$$= E_\rho\left[\exp\left\{-\sum_{i=1}^m a_i \Xi_{n,t_n}^{(n-q)}(A_i \times B_i)\right\}\mathbf{1}_{\bigcap_{i=1}^m \mathcal{B}_{q,t_n,n}^\sigma(B_i;\underline{A}_i)\cap\mathcal{B}_{q,t_n,n}^\sigma(B_i;\overline{A}_i)}\right]$$
$$+ E_\rho\left[\exp\left\{-\sum_{i=1}^m a_i \Xi_{n,t_n}^{(n-q)}(A_i \times B_i)\right\}\mathbf{1}_{\bigcup_{i=1}^m\left(\mathcal{B}_{q,t_n,n}^\sigma(B_i;\underline{A}_i)\right)^c\cup\left(\mathcal{B}_{q,t_n,n}^\sigma(B_i;\overline{A}_i)\right)^c}\right]$$
$$=: J_1 + J_2. \qquad (6.8)$$

We estimate J_1 in (6.8). Under the event $\bigcap_{i=1}^m \mathcal{B}_{q,t_n,n}^\sigma\left(B_i;\overline{A}_i\right) \cap \mathcal{B}_{q,t_n,n}^\sigma\left(B_i;\underline{A}_i\right)$, we have for all $v \in T_q$ and $1 \leq i \leq m$,

$$\mathbf{1}_{\left\{\sigma\left(\arg\max_{u\in T_{n-q}^v} L_{\tau(t_n)}^n(u)\right)\in A_i,\ \ \max_{u\in T_{n-q}^v}\sqrt{L_{\tau(t_n)}^n(u)}-\sqrt{t_n}-a_n(t_n)\in B_i\right\}}$$
$$= \mathbf{1}_{\left\{\sigma(v)\in A_i,\ \ \max_{u\in T_{n-q}^v}\sqrt{L_{\tau(t_n)}^n(u)}-\sqrt{t_n}-a_n(t_n)\in B_i\right\}}. \qquad (6.9)$$

Thus, we have

$$J_1 = E_\rho\left[\exp\left\{-\sum_{v\in T_q}\sum_{i=1}^m a_i\mathbf{1}_{\left\{\sigma(v)\in A_i,\ \ \max_{u\in T_{n-q}^v}\sqrt{L_{\tau(t_n)}^n(u)}-\sqrt{t_n}-a_n(t_n)\in B_i\right\}}\right\}\right]$$
$$- E_\rho\left[\exp\left\{-\sum_{v\in T_q}\sum_{i=1}^m a_i\mathbf{1}_{\left\{\sigma(v)\in A_i,\ \ \max_{u\in T_{n-q}^v}\sqrt{L_{\tau(t_n)}^n(u)}-\sqrt{t_n}-a_n(t_n)\in B_i\right\}}\right\}\right.$$
$$\left. \mathbf{1}_{\bigcup_{i=1}^m\left(\mathcal{B}_{q,t_n,n}^\sigma(B_i;\underline{A}_i)\right)^c\cup\left(\mathcal{B}_{q,t_n,n}^\sigma(B_i;\overline{A}_i)\right)^c}\right]$$
$$=: J_{1,1} - J_{1,2}. \qquad (6.10)$$

By Lemma 6.2, we have

$$\max\{J_2, J_{1,2}\} \leq \sum_{i=1}^m c_1(B_i)q^3 e^{-c_2 q}. \qquad (6.11)$$

We estimate $J_{1,1}$. By Theorem 2.2, on the same probability space (we will write P to denote the probability measure), we can construct a local time $(L_{\tau(t_n)}^n(v))_{v\in T_{\leq n}}$ and two

BRWs $(h_v)_{v \in T_{\leq n}}$, $(h'_v)_{v \in T_{\leq n}}$ satisfying (2.4) and (2.5). Fix $\delta \in (0, 1/3)$. We set

$$C_q := \left\{ |h_v|, |h'_v| \leq \sqrt{\log b} \, q - \frac{3}{4\sqrt{\log b}} (1 - \delta) \log q, \; \forall v \in T_q \right\}. \tag{6.12}$$

For each $v \in T_q$, let $\widetilde{L}^{\downarrow}$ be a local time of a Brownian motion on $\widetilde{T}^v_{\leq n-q}$ and set $\widetilde{\tau}^{\downarrow}(p) :=$ $\inf\{s \geq 0 : \widetilde{L}^{\downarrow}_s(v) > p\}$. We omit the subscript v in $\widetilde{L}^{\downarrow}$ and $\widetilde{\tau}^{\downarrow}$. By Lemma 2.1, we have

$$J_{1,1} = E\left[1_{C_q} \prod_{v \in T_q} K_v \right] + E\left[1_{(C_q)^c} \prod_{v \in T_q} K_v \right] =: J_{1,1,1} + J_{1,1,2}, \tag{6.13}$$

where for each $v \in T_q$, we have set

$$K_v := \widetilde{E}_v\left[\exp\left\{ -\sum_{i=1}^m a_i 1_{\left\{ \sigma(v) \in A_i, \; \max_{u \in T^v_{n-q}} \sqrt{\widetilde{L}^{\downarrow}_{\widetilde{\tau}^{\downarrow}\left(L^n_{\tau(t_n)}(v)\right)}(u)} - \sqrt{t_n} - a_n(t_n) \in B_i \right\}} \right\} \right]. \tag{6.14}$$

By Lemma A.7, we have

$$J_{1,1,2} \leq c_3(\log q) \cdot (q)^{-\frac{3}{2}\delta}. \tag{6.15}$$

We estimate $J_{1,1,1}$. Fix $v \in T_q$. For each $1 \leq i \leq m$, we set

$$E^{(i)}_v(n, t_n, q) := \left\{ \max_{u \in T^v_{n-q}} \sqrt{\widetilde{L}^{\downarrow}_{\widetilde{\tau}^{\downarrow}\left(L^n_{\tau(t_n)}(v)\right)}(u)} - \sqrt{t_n} - a_n(t_n) \in B_i \right\}. $$

Since $A_1 \times B_1, \ldots, A_m \times B_m$ are disjoint, K_v is equal to

$$\widetilde{E}_v\left[\prod_{i=1}^m \left\{ 1 - (1 - e^{-a_i}) 1_{\{\sigma(v) \in A_i\}} 1_{E^{(i)}_v(n, t_n, q)} \right\} \right]$$

$$= \widetilde{E}_v\left[1 - \sum_{i=1}^m (1 - e^{-a_i}) 1_{\{\sigma(v) \in A_i\}} 1_{E^{(i)}_v(n, t_n, q)} \right]$$

$$= \exp\left\{ \log\left(1 - \sum_{i=1}^m \left(1 - e^{-a_i} \right) 1_{\{\sigma(v) \in A_i\}} \widetilde{P}_v[E^{(i)}_v(n, t_n, q)] \right) \right\}. \tag{6.16}$$

On the event C_q, by (2.5), we have for all $v \in T_q$

$$\sqrt{t_n} + a_n(t_n) = \sqrt{L^n_{\tau(t_n)}(v)} + a_{n-q}\left(L^n_{\tau(t_n)}(v)\right) + \sqrt{\log b} \, q - h'_v + \delta^n_{v,q}, \tag{6.17}$$

where

$$\delta^n_{v,q} := \frac{3 \log(1 - q/n)}{4\sqrt{\log b}} + \frac{\log\left(1 + O\left(q/\sqrt{t_n} \right) \right)}{4\sqrt{\log b}} + O\left(q^2/\sqrt{t_n} \right). \tag{6.18}$$

We take any sufficiently small $\varepsilon > 0$ and sufficiently large $q_0 \in \mathbb{N}$ which depends on $B_1, \ldots, B_m$ and ε. We assume that $q \geq q_0$ and $n \geq n_0$, where we take sufficiently large $n_0 = n_0(q, \varepsilon) \in \mathbb{N}$. By Proposition 5.1 and (6.17)–(6.18), under the event C_q, we have for all $1 \leq i \leq m$ and $v \in T_q$, $\widetilde{P}_v[E^{(i)}_v(n, t_n, q)]$ is bounded from below by

$$\widetilde{P}_v\left[\begin{array}{l} \max_{u \in T^v_{n-q}} \sqrt{\widetilde{L}^{\downarrow}_{\widetilde{\tau}^{\downarrow}\left(L^n_{\tau(t_n)}(v)\right)}(u)} - \sqrt{L^n_{\tau(t_n)}(v)} - a_{n-q}\left(L^n_{\tau(t_n)}(v)\right) \\ \geq \sqrt{\log b} \, q - h'_v + \underline{B}_i + \delta^n_{v,q} \end{array} \right]$$

$$- \widetilde{P}_v \left[\begin{array}{l} \max_{u \in T_{n-q}^v} \sqrt{\widetilde{L}_{\widetilde{\tau}^\downarrow}^\downarrow \left(L_{\tau(t_n)}^n(v) \right)}(u) - \sqrt{L_{\tau(t_n)}^n(v)} - a_{n-q} \left(L_{\tau(t_n)}^n(v) \right) \\ \geq \sqrt{\log b}\, q - h_v' + \overline{B}_i + \delta_{v,q}^n \end{array} \right]$$

$$\geq (1 - c_4 \varepsilon) \frac{4}{\sqrt{\pi}} \beta_* \gamma_* \left(\sqrt{\log b}\, q - h_v' + \underline{B}_i \right) e^{-2\sqrt{\log b}\, \left(\sqrt{\log b}\, q - h_v' + \underline{B}_i \right)}$$

$$- (1 + c_4 \varepsilon) \frac{4}{\sqrt{\pi}} \beta_* \gamma_* \left(\sqrt{\log b}\, q - h_v' + \overline{B}_i \right) e^{-2\sqrt{\log b}\, \left(\sqrt{\log b}\, q - h_v' + \overline{B}_i \right)}. \tag{6.19}$$

We set

$$D_q^{(2)} := \sum_{v \in T_q} \left(\sqrt{\log b}\, q - h_v' \right)^2 e^{-4\sqrt{\log b}\, \left(\sqrt{\log b}\, q - h_v' \right)},$$

$$W_q := \sum_{v \in T_q} e^{-2\sqrt{\log b}\, \left(\sqrt{\log b}\, q - h_v' \right)}.$$

Recall the random measure Z_q from (1.3). By (6.16), (6.19), and Taylor's expansion of the function $x \mapsto \log(1 - x)$, under the event $C_q \cap \left\{ D_q^{(2)} < \varepsilon \right\} \cap \{ W_q < \varepsilon \}$, $\prod_{v \in T_q} K_v$ is bounded from above by

$$e^{c_5(B_1,\ldots,B_m)\varepsilon} \cdot \exp \left\{ -\frac{4}{\sqrt{\pi}} \beta_* \gamma_* \sum_{i=1}^m \left(1 - e^{-a_i} \right) Z_q(A_i) \cdot \left(e^{-2\sqrt{\log b}\, \underline{B}_i} - e^{-2\sqrt{\log b}\, \overline{B}_i} \right) \right\}$$

$$\times \exp \left\{ c_4 \varepsilon \frac{4}{\sqrt{\pi}} \beta_* \gamma_* \sum_{i=1}^m \left(1 - e^{-a_i} \right) Z_q(A_i) \left(e^{-2\sqrt{\log b}\, \underline{B}_i} + e^{-2\sqrt{\log b}\, \overline{B}_i} \right) \right\}. \tag{6.20}$$

We can obtain a similar lower bound of $\prod_{v \in T_q} K_v$. By Lemma A.7, Lemma 6.1, and the fact that $\lim_{q \to \infty} W_q = 0$ almost surely (see [36]), we have

$$\lim_{q \to \infty} P \left((C_q)^c \cup \left\{ D_q^{(2)} \geq \varepsilon \right\} \cup \{ W_q \geq \varepsilon \} \right) = 0. \tag{6.21}$$

Thus, by the above estimates, taking $n \to \infty$, then $q \to \infty$, and finally $\varepsilon \to 0$, we have

$$\left| E_\rho \left[e^{-\sum_{i=1}^m a_i \Xi_{n,t_n}^{(n-q)}(A_i \times B_i)} \right] - \mathbb{E} \left[e^{-\frac{4}{\sqrt{\pi}} \beta_* \gamma_* \sum_{i=1}^m \left(1 - e^{-a_i} \right) Z_\infty(A_i) \left(e^{-2\sqrt{\log b}\, \underline{B}_i} - e^{-2\sqrt{\log b}\, \overline{B}_i} \right)} \right] \right|$$

$$\to 0. \tag{6.22}$$

Next, by using (6.22), we will prove (6.7). Let z_* be a real number with $z_* < \min_{1 \leq i \leq m} \underline{B}_i$. Take $q_0 = q_0(z_*) \in \mathbb{N}$ large enough and fix any $q \geq q_0$. Take $n \in \mathbb{N}$ large enough so that $q < n - r_n < n - q$ and $q < n/4$. We set

$$U_{z_*,q}^n(t_n) := \left\{ \begin{array}{l} \exists v, u \in T_n \text{ with } q \leq |v \wedge u| \leq n - q \text{ s.t.} \\ \sqrt{L_{\tau(t_n)}^n(v)}, \ \sqrt{L_{\tau(t_n)}^n(u)} \geq \sqrt{t_n} + a_n(t_n) + z_* \end{array} \right\}.$$

Under the event $\left(U_{z_*,q}^n(t_n) \right)^c$, we have

$$\left\{ \arg \max_{u \in T_{r_n}^v} L_{\tau(t_n)}^n(u) : v \in T_{n-r_n}, \ \max_{u \in T_{r_n}^v} \sqrt{L_{\tau(t_n)}^n(u)} \geq \sqrt{t_n} + a_n(t_n) + z_* \right\}$$

$$= \left\{ \arg \max_{u \in T_{n-q}^v} L_{\tau(t_n)}^n(u) : v \in T_q, \ \max_{u \in T_{n-q}^v} \sqrt{L_{\tau(t_n)}^n(u)} \geq \sqrt{t_n} + a_n(t_n) + z_* \right\}. \tag{6.23}$$

By (6.22), (6.23), and Proposition 4.1, we have (6.7). □

Proof of Corollary 1.3. Corollary 1.3 immediately follows form Theorem 1.1 and Proposition 3.1(i). We omit the details. □

A Appendix

In this section, we give proof of some technical estimates.

A.1 Gaussian process associated with Brownian motion on metric tree

We adopt the notation in Section 2. Let $\widetilde{X} = (\widetilde{X}_t, t \geq 0, \widetilde{P}_x, x \in \widetilde{T}_{\leq n})$ be a Brownian motion on $\widetilde{T}_{\leq n}$. Let $\{\widetilde{h}_x : x \in \widetilde{T}_{\leq n}\}$ be a centered Gaussian process with $\mathbb{E}(\widetilde{h}_x \widetilde{h}_y) = \widetilde{E}_x(\widetilde{L}^n_{H_\rho}(y))$ for all $x, y \in \widetilde{T}_{\leq n}$, where $H_x := \inf\{t \geq 0 : \widetilde{X}_t = x\}$. We will call $\widetilde{h}$ a Gaussian process associated with the Brownian motion $\widetilde{X}$. We have an explicit representation of the covariance of $\widetilde{h}$.

Lemma A.1. *For all $x, y \in \widetilde{T}_{\leq n}$,*

$$\mathbb{E}(\widetilde{h}_x \widetilde{h}_y) = d(\rho, x) + d(\rho, y) - d(x, y).$$

In particular, the law of $\{\widetilde{h}_{v_s} : 0 \leq s \leq n\}$ is the same as that of a standard Brownian motion $\{W_s : 0 \leq s \leq n\}$ on $\mathbb{R}$, where v_s is the point on the unique path in $\widetilde{T}_{\leq n}$ from ρ to v with $d(\rho, v_s) = s/2$.

Proof. Let $\{z_1, z_2\}, \{w_1, w_2\}$ be edges of T with $x \in I_{\{z_1, z_2\}}$ and $y \in I_{\{w_1, w_2\}}$. Let $\theta_s, s \geq 0$ be shift operators. We have

$$\mathbb{E}(\widetilde{h}_x \widetilde{h}_y) = \widetilde{E}_x(\widetilde{L}^n_{H_\rho}(y))$$
$$= \widetilde{E}_x\left(\widetilde{L}^n_{H_{z_1} \wedge H_{z_2}}(y) + \widetilde{L}^n_{H_\rho}(y) \circ \theta_{H_{z_1} \wedge H_{z_2}}\right)$$
$$= \widetilde{E}_x(\widetilde{L}^n_{H_{z_1} \wedge H_{z_2}}(y)) + \widetilde{P}_x(H_{z_1} < H_{z_2})\widetilde{E}_{z_1}(\widetilde{L}^n_{H_\rho}(y)) + \widetilde{P}_x(H_{z_2} < H_{z_1})\widetilde{E}_{z_2}(\widetilde{L}^n_{H_\rho}(y)). \quad \text{(A.1)}$$

Note that we have $\widetilde{E}_w(\widetilde{L}^n_{H_\rho}(w')) = \widetilde{E}_{w'}(\widetilde{L}^n_{H_\rho}(w))$, $w, w' \in \widetilde{T}_{\leq n}$; see, for example, [30, Lemma 5.1]. Since $\widetilde{X}$ behaves like a standard Brownian motion on each $I_{\{w, w'\}}$ ($\{w, w'\}$ is an edge of T), we have $\widetilde{P}_x(H_w < H_{w'}) = d(x, w')/d(w, w')$, $x \in I_{\{w, w'\}}$; see, for example, [32, Chapter 2, Exercise 8.13]. By these, the right hand side of (A.1) is equal to

$$\widetilde{E}_x(\widetilde{L}^n_{H_{z_1} \wedge H_{z_2}}(y)) + \frac{d(x, z_2)}{d(z_1, z_2)}\widetilde{E}_y(\widetilde{L}^n_{H_\rho}(z_1)) + \frac{d(x, z_1)}{d(z_1, z_2)}\widetilde{E}_y(\widetilde{L}^n_{H_\rho}(z_2)). \quad \text{(A.2)}$$

Similar arguments imply that for each $i \in \{1, 2\}$,

$$\widetilde{E}_y(\widetilde{L}^n_{H_\rho}(z_i)) = \frac{d(y, w_2)}{d(w_1, w_2)}\widetilde{E}_{w_1}(\widetilde{L}^n_{H_\rho}(z_i)) + \frac{d(y, w_1)}{d(w_1, w_2)}\widetilde{E}_{w_2}(\widetilde{L}^n_{H_\rho}(z_i)). \quad \text{(A.3)}$$

(In (A.3), we have used the equality $\widetilde{E}_y(\widetilde{L}^n_{H_{w_1} \wedge H_{w_2}}(z_i)) = 0$ for each $i \in \{1, 2\}$.) Note that by (2.3) and [25, Lemma 2.1], for all $i, j \in \{1, 2\}$,

$$\widetilde{E}_{w_i}(\widetilde{L}^n_{H_\rho}(z_j)) = E_{w_i}(L^n_{H_\rho}(z_j)) = d(\rho, w_i) + d(\rho, z_j) - d(w_i, z_j). \quad \text{(A.4)}$$

We have

$$\widetilde{E}_x(\widetilde{L}^n_{H_{z_1} \wedge H_{z_2}}(y)) = 0, \quad \text{if } I_{\{z_1, z_2\}} \neq I_{\{w_1, w_2\}}. \quad \text{(A.5)}$$

Assume that $I_{\{z_1, z_2\}} = I_{\{w_1, w_2\}}$. Note that a Brownian motion on $\widetilde{T}_{\leq n}$ starting at x killed upon $H_{z_1} \wedge H_{z_2}$ has the same law as a standard Brownian motion on $(0, \frac{1}{2})$ stating at $d(x, z_1)$ killed upon $H_0 \wedge H_{1/2}$. Let $\{L^B_t(z) : t \geq 0\}$ be a local time at z of a standard Brownian motion $\{B_t, t \geq 0, P^B_w, w \in \mathbb{R}\}$ on $\mathbb{R}$. By this and [38, Chapter VI, Exercise 2.8],

we have

$$
\begin{aligned}
\widetilde{E}_x(\widetilde{L}^n_{H_{z_1} \wedge H_{z_2}}(y)) &= E^B_{d(x,z_1)}(L^B_{H_0 \wedge H_{1/2}}(d(y,z_1))) \\
&= \frac{d(x,z_2)}{d(z_1,z_2)} d(y,z_1) + \frac{d(x,z_1)}{d(z_1,z_2)} d(y,z_2) - d(x,y), \ \ \text{if } I_{\{z_1,z_2\}} = I_{\{w_1,w_2\}}.
\end{aligned}
$$

(A.6)

Thus, by (A.1)–(A.6), we have

$$
\mathbb{E}(\widetilde{h}_x \widetilde{h}_y) = d(\rho,x) + d(\rho,y) - d(x,y). \qquad \square
$$

A.2 Proof of Lemma 2.1

In this section, we use notation in Section 2. Let $\widetilde{X} = (\widetilde{X}_t, t \geq 0, \widetilde{P}_x, x \in \widetilde{T}_{\leq n})$ be a Brownian motion on $\widetilde{T}_{\leq n}$. Let $\{h_x : x \in \widetilde{T}_{\leq n}\}$ be a centered Gaussian process associated with $\widetilde{X}$ (see Section A.1 for the definition). In the proof of Lemma 2.1, we will use a variation of Theorem 2.2:

Theorem A.2. *([30, Theorem 1.1]) Fix $n \in \mathbb{N}$. For any $c \in \mathbb{R}$ and $t > 0$,*

$$
\text{the law of } \left\{ \widetilde{L}^n_{\widetilde{\tau}(t)}(x) + \frac{1}{2}(\widetilde{h}_x + c)^2 : x \in \widetilde{T}_{\leq n} \right\} \text{ under } \widetilde{P}_\rho \times \mathbb{P} \text{ is the same as}
$$

$$
\text{that of } \left\{ \frac{1}{2}(\widetilde{h}_x + \sqrt{2t + c^2})^2 : x \in \widetilde{T}_{\leq n} \right\} \text{ under } \mathbb{P}.
$$

Proof of Lemma 2.1. Fix $n \in \mathbb{N}$, $t > 0$, and $a \in T_{\leq n} \backslash T_n$. Let $\widetilde{X}^\downarrow = (\widetilde{X}^\downarrow_s, s \geq 0, \widetilde{P}^\downarrow_a)$ be a Brownian motion on $\widetilde{T}^a_{\leq n-|a|}$ starting at a. Let $\{\widetilde{L}^\downarrow_s(x) : (s,x) \in [0,\infty) \times \widetilde{T}^a_{\leq n-|a|}\}$ be a local time of $\widetilde{X}^\downarrow$. Set $\widetilde{\tau}^\downarrow(s) := \inf\{r \geq 0 : \widetilde{L}^\downarrow_r(a) > s\}$. By Lemma A.1, we have

$$
\left\{ \widetilde{h}_x : x \in \widetilde{T}_{\leq n} \right\} \text{ under } \mathbb{P}
$$

$$
\overset{d}{=} \left\{ \widetilde{h}^\downarrow_x + \widetilde{h}_a : x \in \widetilde{T}^a_{\leq n-|a|} \right\} \cup \left\{ \widetilde{h}_x : x \in \widetilde{T}_{\leq n} \backslash \widetilde{T}^a_{\leq n-|a|} \right\} \text{ under } \mathbb{P}^\downarrow \times \mathbb{P}, \quad (A.7)
$$

where $\{\widetilde{h}^\downarrow_x : x \in \widetilde{T}^a_{\leq n-|a|}\}$ is a Gaussian process associated with $\widetilde{X}^\downarrow$ under $\mathbb{P}^\downarrow$.

Fix $m, r \in \mathbb{N}$, $\lambda_1, \cdots, \lambda_m \in [0,\infty)$, $\mu_1, \cdots, \mu_r \in [0,\infty)$, $x_1, \cdots, x_m \in \widetilde{T}^a_{\leq n-|a|} \backslash \{a\}$, and $y_1, \cdots, y_r \in (\widetilde{T}_{\leq n} \backslash \widetilde{T}^a_{\leq n-|a|}) \cup \{a\}$. Let E, $\widetilde{E}$, $E^\downarrow$ be expectations of the product measures $\widetilde{P}_\rho \times \widetilde{P}^\downarrow_a \times \mathbb{P}$, $\widetilde{P}_\rho \times \mathbb{P}$, $\widetilde{P}^\downarrow_a \times \mathbb{P}^\downarrow$, respectively. By (A.7), we have

$$
E \left[e^{-\sum_{i=1}^m \lambda_i \left(\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow \left(\widetilde{L}^n_{\widetilde{\tau}(t)}(a) \right)}(x_i) + \frac{1}{2}(\widetilde{h}_{x_i})^2 \right)} \cdot e^{-\sum_{i=1}^r \mu_i \left(\widetilde{L}^n_{\widetilde{\tau}(t)}(y_i) + \frac{1}{2}(\widetilde{h}_{y_i})^2 \right)} \right]
$$

$$
= \widetilde{E} \left[e^{-\sum_{i=1}^r \mu_i \left(\widetilde{L}^n_{\widetilde{\tau}(t)}(y_i) + \frac{1}{2}(\widetilde{h}_{y_i})^2 \right)} E^\downarrow \left[e^{-\sum_{i=1}^m \lambda_i \left(\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow \left(\widetilde{L}^n_{\widetilde{\tau}(t)}(a) \right)}(x_i) + \frac{1}{2}(\widetilde{h}^\downarrow_{x_i} + \widetilde{h}_a)^2 \right)} \right] \right]. \quad (A.8)
$$

Applying Theorem A.2 to the local time and the associated Gaussian process on $\widetilde{T}^a_{\leq n-|a|}$, the right of (A.8) is equal to

$$
\widetilde{E} \left[e^{-\sum_{i=1}^r \mu_i \left(\widetilde{L}^n_{\widetilde{\tau}(t)}(y_i) + \frac{1}{2}(\widetilde{h}_{y_i})^2 \right)} \mathbb{E}^\downarrow \left[e^{-\sum_{i=1}^m \lambda_i \left(\frac{1}{2} \left(\widetilde{h}^\downarrow_{x_i} + \sqrt{2\widetilde{L}^n_{\widetilde{\tau}(t)}(a) + (\widetilde{h}_a)^2} \right)^2 \right)} \right] \right]. \quad (A.9)
$$

Again, by Theorem A.2, the expectation of (A.9) is equal to

$$
\mathbb{E} \left[e^{-\sum_{i=1}^r \mu_i \left(\frac{1}{2}(\widetilde{h}_{y_i} + \sqrt{2t})^2 \right)} \mathbb{E}^\downarrow \left[e^{-\sum_{i=1}^m \lambda_i \left(\frac{1}{2} \left(\widetilde{h}^\downarrow_{x_i} + \sqrt{(\widetilde{h}_a + \sqrt{2t})^2} \right)^2 \right)} \right] \right]. \quad (A.10)
$$

By the symmetry of $\widetilde{h}^{\downarrow}$, conditioned on $\widetilde{h}_a$, the law of $(\widetilde{h}^{\downarrow} + |\widetilde{h}_a + \sqrt{2t}|)^2$ is the same as that of $(\widetilde{h}^{\downarrow} + \widetilde{h}_a + \sqrt{2t})^2$. By this and (A.7), the expectation in (A.10) is equal to

$$
\mathbb{E}\left[e^{-\sum_{i=1}^{r} \mu_i \left(\frac{1}{2}(\widetilde{h}_{y_i} + \sqrt{2t})^2 \right)} \cdot e^{-\sum_{i=1}^{m} \lambda_i \left(\frac{1}{2}(\widetilde{h}_{x_i} + \sqrt{2t})^2 \right)} \right]. \tag{A.11}
$$

By Theorem A.2, the expectation in (A.11) is equal to

$$
\widetilde{E}\left[e^{-\sum_{i=1}^{r} \mu_i \left(\widetilde{L}_{\widetilde{\tau}(t)}^n(y_i) + \frac{1}{2}(\widetilde{h}_{y_i})^2 \right)} \cdot e^{-\sum_{i=1}^{m} \lambda_i \left(\widetilde{L}_{\widetilde{\tau}(t)}^n(x_i) + \frac{1}{2}(\widetilde{h}_{x_i})^2 \right)} \right]. \tag{A.12}
$$

The above argument implies that

$$
\left\{ \widetilde{L}_{\widetilde{\tau}^{\downarrow}\left(\widetilde{L}_{\widetilde{\tau}(t)}^n(a)\right)}^{\downarrow}(x) : x \in \widetilde{T}_{\leq n-|a|}^a \right\} \cup \left\{ \widetilde{L}_{\widetilde{\tau}(t)}^n(x) : x \in \widetilde{T}_{\leq n} \setminus \widetilde{T}_{\leq n-|a|}^a \right\} \text{ under } \widetilde{P}_\rho \times \widetilde{P}_a^{\downarrow}
$$

$$
\stackrel{d}{=} \left\{ \widetilde{L}_{\widetilde{\tau}(t)}^n(x) : x \in \widetilde{T}_{\leq n} \right\} \text{ under } \widetilde{P}_\rho.
$$

This yields the statement of Lemma 2.1. $\qquad\square$

A.3 Tail of maximum of local time revisited

In the proof of Proposition 4.1, we need a version of Proposition 3.1(ii):

Proposition A.3. *There exist $c_1 > 0$ and $t_0 > 0$ such that for all $n \in \mathbb{N}$, $t \geq t_0$, and $y \in [0, 2\sqrt{n}]$,*

$$
P_\rho\left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y \right) \geq c_1 e^{-2\sqrt{\log b}\, y}. \tag{A.13}
$$

Remark A.4. The assumption of t in Proposition A.3 is weaker than that of Proposition 3.1(ii). This is the main requirement in the proof of Proposition 4.1.

Fix $\varepsilon \in (0, 1/4)$. For $n \in \mathbb{N}$, $t > 0$, $y > 0$, $0 < r < n$, and $v \in T_n$, set

$$
A_{v,r}^n(t) := \left\{
\begin{array}{l}
\sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v_s)} \in \sqrt{t} + \ell_{y,t,n}(s) + [-g_n(s),\ -f_n(s)),\ \forall s \in [r,\ n-r], \\[4pt]
\sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v)} \in [\sqrt{t} + a_n(t) + y,\ \sqrt{t} + a_n(t) + y + 1), \\[4pt]
\sqrt{\widetilde{L}_{\widetilde{\tau}(t)}^n(v_{s'})} \geq \sqrt{t} + \ell_{y,t,n}(s') - r^{1/2+2\varepsilon},\ \forall s' \in [0,\ r] \cup [n-r,\ n]
\end{array}
\right\},
$$

where

$$
\ell_{y,t,n}(s) := \left(\frac{a_n(t)}{n} + \frac{y}{n} \right) s, \qquad 0 \leq s \leq n,
$$

$$
f_n(s) := \min\left\{ s^{1/2-\varepsilon},\ (n-s)^{1/2-\varepsilon} \right\}, \qquad g_n(s) := \min\left\{ s^{1/2+\varepsilon},\ (n-s)^{1/2+\varepsilon} \right\}.
$$

To prove Proposition A.3, we will apply the second moment method to $\sum_{v \in T_n} 1_{A_{v,r}^n(t)}$. We first need the following:

Lemma A.5. *There exist $r_0 \in \mathbb{N}$ and $c_1 > 0$ such that for all $r_0 \leq r \leq n/4$, $t > 4r^{1+4\varepsilon}$, $y \in [0, 2\sqrt{n}]$, and $v \in T_n$,*

$$
\widetilde{P}_\rho(A_{v,r}^n(t)) \geq c_1 b^{-n} e^{-2\sqrt{\log b}\, y}. \tag{A.14}
$$

Proof. Fix $r_0 \leq r \leq n/4$, $t > 4r^{1+4\varepsilon}$, and $y \in [0, 2\sqrt{n}]$, where we take $r_0 \in \mathbb{N}$ large enough. By Lemma 2.3 and (2.6), $\widetilde{P}_\rho(A_{v,r}^n(t))$ is bounded from below by

$$
c_1 \sqrt{\frac{\sqrt{t}}{\sqrt{t} + a_n(t) + y + 1}}\, P_0^B\left[
\begin{array}{l}
-g_n(s) \leq B_s - \ell_{y,t,n}(s) < -f_n(s),\ \forall s \in [r,\ n-r], \\[4pt]
B_n \in [a_n(t) + y,\ a_n(t) + y + 1), \\[4pt]
B_{s'} \geq \ell_{y,t,n}(s') - r^{1/2+2\varepsilon},\ \forall s' \in [0,\ r] \cup [n-r,\ n]
\end{array}
\right], \tag{A.15}
$$

where we have used the following: under the event that $\sqrt{X_s/2} \geq \sqrt{t} + \ell_{y,t,n}(s) - g_n(s)$ for all $s \in [r,\, n-r]$ and $\sqrt{X_{s'}/2} \geq \sqrt{t} + \ell_{y,t,n}(s') - r^{1/2+2\varepsilon}$ for all $s' \in [0,\, r] \cup [n-r,\, n]$, we have $\exp\left(-\frac{3}{8}\int_0^n \frac{ds}{X_s}\right) \geq c_1$. Since $\{B_s - B_n \frac{s}{n} : 0 \leq s \leq n\}$ is independent of B_n and has the same law as that of a Brownian bridge from 0 to 0 on $[0,\, n]$, the right-hand side of (A.15) is bounded from below by

$$c_1 \sqrt{\frac{\sqrt{t}}{\sqrt{t} + a_n(t) + y + 1}} P^n_{0\to 0} \begin{bmatrix} -g_n(s) \leq X_s < -f_n(s) - 1,\ \forall s \in [r,\, n-r], \\ X_{s'} \geq -r^{1/2+2\varepsilon},\ \forall s' \in [0,\, r] \cup [n-r,\, n] \end{bmatrix}$$
$$\times P^B_0 \left(B_n \in [a_n(t) + y,\, a_n(t) + y + 1]\right), \tag{A.16}$$

where for $T > 0$ and $p, q \in \mathbb{R}$, $P^T_{p\to q}$ is a probability law on $(C[0,T], \mathcal{B}(C[0,T]))$ ($C[0,T]$ is the space of all continuous functions on $[0,T]$ and $\mathcal{B}(C[0,T])$ is the σ-field generated by cylinder sets in $C[0,T]$) under which the coordinate process $\{X_s : 0 \leq s \leq T\}$ is a Brownian bridge with variance $1/2$ from p to q on $[0,\, T]$. We have

$$P^n_{0\to 0} \begin{bmatrix} -g_n(s) \leq X_s < -f_n(s) - 1,\ \forall s \in [r,\, n-r], \\ X_{s'} \geq -r^{1/2+2\varepsilon},\ \forall s' \in [0,\, r] \cup [n-r,\, n] \end{bmatrix}$$
$$= E^n_{0\to 0} \begin{bmatrix} 1_{\{-g_n(r) \leq X_r < -f_n(r)-1,\ -g_n(n-r) \leq X_{n-r} < -f_n(n-r)-1\}} \\ \times P^{n-2r}_{X_r \to X_{n-r}} \left(-g_n(s+r) \leq X_s < -f_n(s+r) - 1, \forall s \in [0,\, n-2r]\right) \\ \times P^r_{0 \to X_r} \left(X_s \geq -r^{1/2+2\varepsilon},\ \forall s \in [0,\, r]\right) \\ \times P^r_{X_{n-r} \to 0} \left(X_s \geq -r^{1/2+2\varepsilon},\ \forall s \in [0,\, r]\right) \end{bmatrix}$$
$$\geq c_2 P^n_{0\to 0} \left(-g_n(s) \leq X_s < -f_n(s) - 1,\ \forall s \in [r,\, n-r]\right), \tag{A.17}$$

where we have used [19, Lemma 2.2(a)] in the second inequality which implies that the last two probabilities in the expectation in the second display of (A.17) are bounded from below by some constants. By the symmetry of a Brownian bridge, [19, Lemma 2.7 and Proposition 6.1], we have for $r_0 \leq r < n/4$ with r_0 large enough

$$P^n_{0\to 0} \left(-g_n(s) \leq X_s < -f_n(s) - 1,\ \forall s \in [r,\, n-r]\right) \geq c_3 P^n_{0\to 0} \left(X_s \geq 0,\ \forall s \in [r,\, n-r]\right). \tag{A.18}$$

Using [19, Lemma 2.2(a)], one can show that the right of (A.18) is bounded from below by c_4/n for some $c_4 > 0$. Thus, by (A.15)–(A.18), we have (A.14). $\square$

Next, we need the following:

Lemma A.6. *(i) There exist $r_0 \in \mathbb{N}$ and $c_1 > 0$ such that for all $t > 0$, $r_0 \leq r \leq n/4$, $y \in [0,\, 2\sqrt{n}]$, $k \in \{r, r+1, \ldots, n-r-1\}$, $v \in T_n$, and $\left(\sqrt{t} + \ell_{y,t,n}(k) - g_n(k)\right)^2 \leq q \leq \left(\sqrt{t} + \ell_{y,t,n}(k) - f_n(k)\right)^2$,*

$$\widetilde{P}_{v_k} \begin{bmatrix} \sqrt{\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow(q)}(v_s)} < \sqrt{t} + \ell_{y,t,n}(s) - f_n(s),\ \forall s \in [k,\, n-r], \\ \sqrt{\widetilde{L}^\downarrow_{\widetilde{\tau}^\downarrow(q)}(v)} \in [\sqrt{t} + a_n(t) + y,\ \sqrt{t} + a_n(t) + y + 1) \end{bmatrix}$$
$$\leq c_1 r^{1/2+\varepsilon} \sqrt{\frac{\sqrt{t} + \ell_{y,t,n}(k) - f_n(k)}{\sqrt{t} + a_n(t) + y}} \frac{g_n(k)}{(n-k-r)\sqrt{n-k}} b^{-(n-k)}$$
$$\times e^{\frac{3\log n}{2n}(n-k)} e^{\frac{\log\left(\frac{\sqrt{t}+n}{\sqrt{t}}\right)}{2n}(n-k)} e^{-\frac{2a_n(t)}{n}f_n(k)}, \tag{A.19}$$

where $\{\widetilde{L}^\downarrow_s(x) : (s,x) \in [0,\, \infty) \times \widetilde{T}^{v_k}_{\leq n-k}\}$ is a local time of a Brownian motion on $\widetilde{T}^{v_k}_{\leq n-k}$ and $\widetilde{\tau}^\downarrow(p) := \inf\{s \geq 0 : \widetilde{L}^\downarrow_s(v_k) > p\}$.

(ii) There exist $r_0 \in \mathbb{N}$ and $c_2 > 0$ such that for all $t > 0$, $r_0 \leq r \leq n/4$, $y \in [0, 2\sqrt{n}]$, and $v \in T_n$,

$$\widetilde{P}_\rho\left(A^n_{v,r}(t)\right) \leq c_2 r b^{-n} e^{-2\sqrt{\log b}\, y}.$$

Proof. Recall the probability measure $P_{0 \to 0}^T$ defined in the proof of Lemma A.5. By Lemma 2.3 and (2.6), the left of (A.19) is bounded from above by

$$\sqrt{\frac{\sqrt{q}}{\sqrt{t} + a_n(t) + y}} P_0^B \begin{bmatrix} B_s < \sqrt{t} - \sqrt{q} + \ell_{y,t,n}(k+s) - f_n(k+s), \ \forall s \in [0, \ n-k-r], \\ B_{n-k} \in [\sqrt{t} - \sqrt{q} + a_n(t) + y, \ \sqrt{t} - \sqrt{q} + a_n(t) + y + 1) \end{bmatrix}. \quad \text{(A.20)}$$

Since the process $\left\{ B_s - \frac{s}{n-k} \cdot B_{n-k} : 0 \le s \le n-k \right\}$ is independent of B_{n-k} and has the same law as that of a Brownian bridge with variance $1/2$ from 0 to 0 on $[0, \ n-k]$, the probability in (A.20) is bounded from above by

$$P_{0 \to 0}^{n-k} \left[X_s < \sqrt{t} - \sqrt{q} + \ell_{y,t,n}(k+s) - \frac{s}{n-k}(\sqrt{t} - \sqrt{q} + a_n(t) + y), \ \forall s \le n-k-r \right]$$
$$\times P_0^B \left[B_{n-k} \in [\sqrt{t} - \sqrt{q} + a_n(t) + y, \ \sqrt{t} - \sqrt{q} + a_n(t) + y + 1) \right]. \quad \text{(A.21)}$$

We estimate two probabilities in (A.21). By the assumption of q, we have

$$P_0^B \left(B_{n-k} \in [\sqrt{t} - \sqrt{q} + a_n(t) + y, \ \sqrt{t} - \sqrt{q} + a_n(t) + y + 1) \right)$$
$$\le \frac{c_1}{\sqrt{n-k}} b^{-(n-k)} e^{\frac{3 \log n}{2n}(n-k)} e^{\frac{\log\left(\frac{\sqrt{t}+n}{\sqrt{t}}\right)}{2n}(n-k)} e^{-\frac{2a_n(t)}{n} f_n(k)}. \quad \text{(A.22)}$$

To estimate the other probability, we use [3, Lemma 3.4]: for any $x_1, x_2 \in [0, \infty)$, $r_1, r_2 \in [0, \infty)$, and $T > r_1 + r_2$,

$$P_{0 \to 0}^T \left(X_s \le \left(1 - \frac{s}{T}\right) x_1 + \frac{s}{T} x_2, \ r_1 \le s \le T - r_2 \right)$$
$$\le \frac{2}{T - r_1 - r_2} \left\{ \left(1 - \frac{r_1}{T}\right) x_1 + \frac{r_1}{T} x_2 + \sqrt{r_1} \right\} \left\{ \frac{r_2}{T} x_1 + \left(1 - \frac{r_2}{T}\right) x_2 + \sqrt{r_2} \right\}. \quad \text{(A.23)}$$

By (A.23), we have

$$P_{0 \to 0}^{n-k} \left(X_s < \sqrt{t} - \sqrt{q} + \ell_{y,t,n}(k+s) - \frac{s}{n-k}(\sqrt{t} - \sqrt{q} + a_n(t) + y), \ \forall s \le n-k-r \right)$$
$$\le c_2 \frac{r^{1/2+\varepsilon}}{n-k-r} g_n(k). \quad \text{(A.24)}$$

Thus, by (A.20)–(A.24), we have (A.19). By repeating a similar argument, we can also prove (ii). We omit the detail. $\square$

Proof of Proposition A.3. Fix a sufficiently large positive constant r. We set $Z := \sum_{v \in T_n} 1_{A_{v,r}^n(t)}$. For each $k \in \{0, \cdots, n-1\}$, set $S_k := \sum_{\substack{v,u \in T_n \\ |v \wedge u| = k}} \widetilde{P}_\rho \left[A_{v,r}^n(t) \cap A_{u,r}^n(t) \right]$.
We have

$$\widetilde{E}_\rho[Z^2] = \widetilde{E}_\rho[Z] + \sum_{k=0}^{r-1} S_k + \sum_{k=r}^{n-r-1} S_k + \sum_{k=n-r}^{n-1} S_k. \quad \text{(A.25)}$$

For each $0 \le k \le n-1$, set

$$I_{y,t,n}(k) := [(\sqrt{t} + \ell_{y,t,n}(k) - g_n(k))^2, (\sqrt{t} + \ell_{y,t,n}(k) - f_n(k))^2].$$

For each $q \in I_{y,t,n}(k)$ and $v \in T_n$, let $P_{v_k}(q)$ be the probability in (A.19). In the estimate of $\sum_{k=r}^{n-r-1} S_k$, we use the following: by Lemma 2.1, for any $r \le k \le n-r-1$ and $v, u \in T_n$ with $|v \wedge u| = k$, $\widetilde{P}_\rho \left[A_{v,r}^n(t) \cap A_{u,r}^n(t) \right]$ is bounded from above by

$$\widetilde{P}_\rho[A_{u,r}^n(t)] \cdot \sup_{q \in I_{y,t,n}(k)} P_{v_k}(q). \quad \text{(A.26)}$$

By (A.26) and Lemma A.6, dividing the sum $\sum_{k=r}^{n-r-1} S_k$ into $\sum_{k=r}^{\lfloor n/2 \rfloor} S_k + \sum_{\lfloor n/2 \rfloor+1}^{\lfloor n-n/\log n \rfloor} S_k + \sum_{k=\lfloor n-n/\log n \rfloor+1}^{n-r-1} S_k$, we have $\sum_{k=r}^{n-r-1} S_k \leq c_1 e^{-2\sqrt{\log b}\, y}$. Similarly, by Lemma 2.1, for any $0 \leq k \leq r-1$ and $v, u \in T_n$ with $|v \wedge u| = k$, $\widetilde{P}_\rho \left[A_{v,r}^n(t) \cap A_{u,r}^n(t) \right]$ is bounded from above by $\widetilde{P}_\rho[A_{u,r}^n(t)] \cdot \sup_{q \in I_{y,t,n}(r)} P_{v_r}(q)$. By this and Lemma A.6, $\sum_{k=0}^{r-1} S_k \leq c_2 e^{-2\sqrt{\log b}\, y}$. For any $n-r \leq k \leq n-1$ and $v, u \in T_n$ with $|v \wedge u| = k$, we bound $\widetilde{P}_\rho \left[A_{v,r}^n(t) \cap A_{u,r}^n(t) \right]$ just by $\widetilde{P}_\rho \left[A_{u,r}^n(t) \right]$. Then, by Lemma A.6(ii), we have $\sum_{k=n-r}^{n-1} S_k \leq c_3 e^{-2\sqrt{\log b}\, y}$.

By these estimates, Lemma A.5, and an argument similar to the proof of Proposition 3.1(ii), we obtain the desired result (A.13). $\qquad\square$

A.4 Tail of maximum of BRW

In the proof of Proposition 5.1, we use tail estimates of the maximum of the BRW on T. Let $(h_v)_{v \in T}$ be a BRW on T defined in Section 1.

Lemma A.7. *(i) There exist $c_1, c_2 \in (0, \infty)$ such that for all $y > 0$ and $n \in \mathbb{N}$,*

$$\mathbb{P}\left(\max_{v \in T_n} h_v > \sqrt{\log b}\, n - \frac{3}{4\sqrt{\log b}} \log n + y \right) \leq c_1(1+y)e^{-2\sqrt{\log b}\, y} e^{-c_2 \frac{y^2}{n}}.$$

(ii) There exist $c_3 > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $y \in [1, \sqrt{n}]$,

$$\mathbb{P}\left(\max_{v \in T_n} h_v > \sqrt{\log b}\, n - \frac{3}{4\sqrt{\log b}} \log n + y \right) \geq c_3 y e^{-2\sqrt{\log b}\, y}.$$

Lemma A.7(ii) is a special version of Lemma 2.7 of [22]. One can easily modify the proof of Lemma 3.8 in [21] (this is basically a tail estimate of the maximum of a BRW on a 4-ary tree) to prove Lemma A.7(i). We omit the details.

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