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Characterizing stationary 1+1 dimensional lattice polymer models

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Abstract

Motivated by the study of directed polymer models with random weights on the square integer lattice, we define an integrability property shared by the log-gamma, strict-weak, beta, and inverse-beta models. This integrability property encapsulates a preservation in distribution of ratios of partition functions which in turn implies the so called Burke property. We show that under some regularity assumptions, up to trivial modifications, there exist no other models possessing this property.

Keywords: directed polymer; exactly solvable models; integrable models; Burke's theorem; partition function.
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1 Introduction

One method which has been used to study certain models of percolation and polymers is to introduce a version of the model with boundary conditions that possesses a stationarity property. This stationarity property allows for the exact computation of some quantities of interest, such as the free energy. In [16] O'Connell and Yor introduce a model for a directed polymer in a Brownian environment with a Burke-type stationarity property. In [20] Seppäläinen and Valkó use this stationarity to find bounds on the fluctuation exponents of the free energy and the fluctuation of the paths. In [4] Cator and Groeneboom relate a stationary version of the Hammersley process to the location of a second class particle and determine the order of the variance of the longest weakly north-east path. In [1] Balázs, Cator, and Seppäläinen use a stationary version of the last passage growth model with exponential weights to study the variance of the last passage time and transversal fluctuations of the maximal path.

We define the integrability property $T^{h,Y}$-invariance (Definition 1.1) which encapsulates this stationarity in the setting of lattice directed polymers. This property implies a preservation in distribution of ratios of partition functions. The first model discovered

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possessing this property is the log-gamma model, introduced by Seppäläinen in [19]. In his paper $T^{h,Y}$-invariance is used to prove the conjectured values for the fluctuation exponents of the free energy and the polymer path in the stationary point-to-point case and to prove upper bounds for the exponents in the point-to-point and point-to-line cases without boundary conditions. In [10] Georgiou and Seppäläinen use $T^{h,Y}$-invariance to obtain large deviation results for the log-gamma polymer. In the setting of directed polymer models, this is the first instance where precise large deviation rate functions for the free energy were derived.

Thereafter three additional models admitting $T^{h,Y}$-invariant versions were found: the strict-weak model, introduced simultaneously by Corwin, Seppäläinen, and Shen in [7] and O'Connell and Ortmann in [15], the beta model, introduced by Barraquand and Corwin in [3] as the beta RWRE, and the inverse-beta model, introduced by Thiery and Le Doussal in [23]. The stationary versions of these models were given by Balázs, Rassoul-Agha, and Seppäläinen in [2] for the beta model, Thiery in [22] for the inverse-beta model, and by Corwin, Seppäläinen, and Shen in [7] for the strict-weak model.

In this paper we present a uniqueness result for $T^{h,Y}$-invariant models. That is, under some regularity assumptions and up to the two natural modifications of reflection and scaling, the log-gamma, strict-weak, beta, and inverse-beta are the only $T^{h,Y}$-invariant models.

This work is similar in spirit to the physics works of Evans, Majumdar, and Zia ([8], [24], and [9]), who consider mass transport models on graphs and provide a characterization of the models which have a product form stationary measure. The work of Povolotsky [17] uses the framework of Evans, Majumdar, and Zia and obtains a three parameter family of zero range mass transfer models which are integrable via Bethe ansatz. In fact, both the beta and the inverse-beta models were obtained as limits of Povolotsky's family of models.

In the paper [5] we use $T^{h,Y}$-invariance along with a Mellin transform framework to simultaneously prove the conjectured values for the fluctuation exponents of the free energy and polymer path in the stationary point-to-point version of these four models.

1.1 The polymer model

The directed polymer in a random environment, first introduced by Huse and Henley [11], models a long chain of molecules in the presence of random impurities. Imbrie and Spencer [12] formulated this model as a random walk in a random environment. See the lectures by Comets [6] for a survey of results on directed polymers. We consider a class of 1+1-dimensional directed polymers on the integer lattice.

Notation: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z}_+ = \{0, 1, \ldots\}$, and $\mathbb{R}$ denotes the real numbers. On each edge e of the $\mathbb{Z}_+^2$ lattice we place a positive random weight. For $x \in \mathbb{N}^2$, let u_x and v_x denote the horizontal and vertical incoming edge weights. We assume that the collection of pairs $\{(u_x, v_x)\}_{x\in\mathbb{N}^2}$ is independent and identically distributed, but do not insist that u_x is independent of v_x (in fact we will later assume v_x is a function of u_x). Call this collection the *bulk weights*. For $x \in \mathbb{N} \times \{0\}$, let R_x^1 denote the horizontal incoming edge weight, and for $x \in \{0\} \times \mathbb{N}$, let R_x^2 denote the vertical incoming edge weight. We assume the collections $\{R_x^1\}_{x\in\mathbb{N}\times\{0\}}$ and $\{R_y^2\}_{y\in\{0\}\times\mathbb{N}}$ are independent and identically distributed, and refer to them as the *horizontal* and *vertical boundary weights*, respectively. We further assume that the horizontal boundary weights, the vertical boundary weights, and the bulk weights are independent of each other. This assignment of edge weights is illustrated in Figure 1.

For $(m, n) \in \mathbb{Z}_+^2 \backslash \{(0, 0)\}$, let $\Pi_{m,n}$ be the collection of all up-right paths from $(0, 0)$ to (m, n). See Figure 2 for an example of such a path. We identify paths $x_\bullet = (x_0, x_1, \ldots, x_{m+n})$ by their sequence of vertices, but also associate to paths their sequence

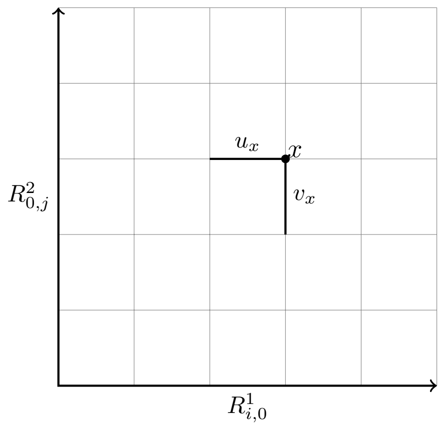

Figure 1: Assignment of edge weights.

of edges $(e_1, \dots, e_{m+n})$, where $e_i = \{x_{i-1}, x_i\}$. The point-to-point partition function for the directed polymer is defined as

$$Z_{m,n} := \sum_{x_\bullet \in \Pi_{m,n}} \prod_{i=1}^{m+n} \omega_{e_i} \qquad \text{for } (m,n) \in Z_+^2 \backslash \{(0,0)\},$$

where ω_e is the weight associated to the edge e. At the origin, define $Z_{0,0} := 1$.

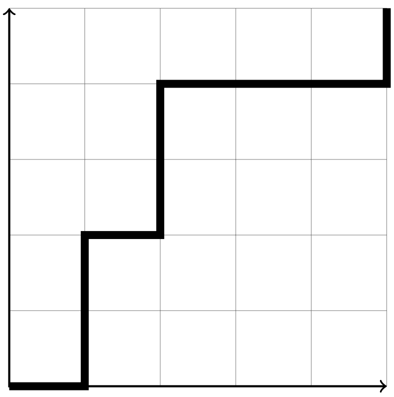

Figure 2: An up-right path from $(0,0)$ to $(5,5)$.

Write $\alpha_1 = (1,0)$, $\alpha_2 = (0,1)$. The partition functions satisfy the recurrence relation

$$Z_x = u_x Z_{x-\alpha_1} + v_x Z_{x-\alpha_2} \qquad \text{for } x \in \mathbb{N}^2. \tag{1.1}$$

For $k = 1,2$ define ratios of partition functions

$$R_x^k := \frac{Z_x}{Z_{x-\alpha_k}} \qquad \text{for all } x \text{ such that } x - \alpha_k \in \mathbb{Z}_+^2.$$

Note that these extend the definitions of $R^1_{i,0}$ and $R^2_{0,j}$, since for example $Z_{i,0} = \prod_{k=1}^{i} R^1_{k,0}$. The recurrence relation (1.1) yields the recursions

$$\begin{aligned} R_x^1 &= u_x + v_x \frac{R^1_{x-\alpha_2}}{R^2_{x-\alpha_1}} \\ R_x^2 &= u_x \frac{R^2_{x-\alpha_1}}{R^1_{x-\alpha_2}} + v_x \end{aligned} \qquad \text{for } x \in \mathbb{N}^2. \tag{1.2}$$

We look to exploit these recursions to obtain more structure of the ratios R_x^1 and R_x^2, which in turn allows us to analyze quantities of interest such as the free energy, $\log Z_{m,n}$. The notation $X \overset{d}{=} Y$ is used to specify that random vectors X and Y have the same distribution. We look for cases where $(R_x^1, R_x^2) \overset{d}{=} (R_{x-\alpha_2}^1, R_{x-\alpha_1}^2)$, under the assumption that u_x and v_x have a functional dependence of the form $(u_x, v_x) = \big(Y_x, h(Y_x)\big)$ for some positive random variable Y_x and positive function h. We further assume that there exist positive random variables R^1, R^2, Y such that the horizontal boundary weights, the vertical boundary weights, and the bulk weights are distributed as R^1, R^2, and $\big(Y, h(Y)\big)$, respectively.

When Y is a random variable taking values in the domain of h and (R^1, R^2) is a random vector taking values in $(0,\infty)^2$, define the random vector

$$T^{h,Y}(R^1, R^2) := \left(Y + h(Y)\frac{R^1}{R^2},\, Y\frac{R^2}{R^1} + h(Y)\right). \tag{1.3}$$

Note that with $(u_x, v_x) = \big(Y_x, h(Y_x)\big)$, the recursive equations (1.2) imply

$$\big(R_x^1, R_x^2\big) = T^{h,Y_x}\big(R_{x-\alpha_2}^1, R_{x-\alpha_1}^2\big) \qquad \text{for all } x \in \mathbb{N}^2. \tag{1.4}$$

Definition 1.1. *Let $O_3 \subset (0,\infty)$, $h : O_3 \to (0,\infty)$, and assume the random variable Y takes values in O_3. Let (R^1, R^2) be a random vector taking values in $(0,\infty)^2$ that is independent of Y. We say that (R^1, R^2) is $T^{h,Y}$*-invariant *if $T^{h,Y}(R^1, R^2) \overset{d}{=} (R^1, R^2)$.*

Definition 1.1, while stated in terms of the random variables (R^1, R^2) and Y, is really a property of the distributions of (R^1, R^2) and Y.

If (R^1, R^2) is $T^{h,Y}$-invariant with R^1 independent of R^2, then (1.4) and an induction argument imply that the polymer model possesses a form of stationarity:

$$\big(R_x^1, R_x^2\big) \overset{d}{=} \big(R^1, R^2\big) \qquad \text{for all } x \in \mathbb{N}^2. \tag{1.5}$$

Although our two main theorems require R^1 and R^2 to be independent, the results in Section 2 hold without this independence.

1.2 Main results

Our first main result, Theorem 1.2, consists of showing that, under some regularity assumptions, $T^{h,Y}$-invariance can only occur if h is of the form $h(y) = a + by$ for real numbers a, b satisfying $a \vee b > 0$. Our second main result, Theorem 1.4, consists of showing that if h has this form, then $T^{h,Y}$-invariance only arises as a modification of the four known invariant models (described in (1.7) through (1.10)). In the case of $h(y) = y$, which is equivalent to vertex disorder, the uniqueness of the vertex weight distributions was already shown (Lemma 3.2 of [19]).

Given a real valued function f we call $\{x : f(x) \neq 0\}$ the support of f. Note that we do not insist on taking the closure of this set. Define the non-random analogue of (1.3),

$$T^{h,y}(r_1, r_2) := \big(y + h(y)\tfrac{r_1}{r_2},\, y\tfrac{r_2}{r_1} + h(y)\big). \tag{1.6}$$

Theorem 1.2. *Let R^1, R^2, Y be positive, independent random variables with respective densities f_1, f_2, f_3. Assume that the support of f_j is $O_j \subset (0,\infty)$ for $j = 1,2,3$, where each O_j is open and O_3 is connected. Assume f_1, f_2 are twice differentiable on O_1 and O_2 respectively and that f_3 is three times differentiable on O_3. Suppose $h : O_3 \to (0,\infty)$ is four times differentiable, the mapping $O_1 \times O_2 \times O_3 \ni (r_1, r_2, y) \mapsto T^{h,y}(r_1, r_2)$ surjects onto $O_1 \times O_2$, and $\frac{r_2}{r_1} + h'(y) \neq 0$ for all $(r_1, r_2, y) \in O_1 \times O_2 \times O_3$. If (R^1, R^2) is $T^{h,Y}$-invariant, then h must be of the form $h(y) = a + by$, where a, b are real numbers satisfying $a \vee b > 0$.*

Remark 1.3. If (R^1, R^2, Y) has support $O_1 \times O_2 \times O_3$ and (R^1, R^2) is $T^{h,Y}$-invariant, then the surjectivity condition is a natural assumption. The assumption $\frac{r_2}{r_1} + h'(y) \neq 0$ is a convenience used in Lemma 2.7 which allows us to extend the preservation of distribution of the pair (R^1, R^2) to the triple (R^1, R^2, Y) (see Definition 2.2). However, this assumption can be removed by an application of Sard's theorem (see Lemma 2.8) at the expense of making a.e. statements throughout Section 2. As an example for when the assumption $\frac{r_2}{r_1} + h'(y) \neq 0$ is satisfied, we can take h to be any differentiable increasing function. Note that the assumptions do not require O_1 or O_2 to be connected.

Before giving the second main result we give the form of each of the four known invariant models.

The notation $X \sim \text{Ga}(\alpha, \beta)$ is used to denote that a random variable is gamma(α, β) distributed, i.e. has density $\Gamma(\alpha)^{-1}\beta^\alpha x^{\alpha-1}e^{-\beta x}$ supported on $(0, \infty)$, where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ is the gamma function. $X \sim \text{Be}(\alpha, \beta)$ is used to say that X is beta(α, β) distributed, i.e. has density $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ supported on $(0, 1)$. We then use $X \sim \text{Ga}^{-1}(\alpha, \beta)$ and $X \sim \text{Be}^{-1}(\alpha, \beta)$ to denote that $X^{-1} \sim \text{Ga}(\alpha, \beta)$ and $X^{-1} \sim \text{Be}(\alpha, \beta)$, respectively. We also use $X \sim \big(\text{Be}^{-1}(\alpha, \beta) - 1\big)$ to denote that $X + 1 \sim \text{Be}^{-1}(\alpha, \beta)$. The symbol $\otimes$ is used to denote (independent) product distribution.

- **Inverse-gamma**: This is also known as the log-gamma model. Assume $\mu > \lambda > 0$, $\beta > 0$ and
$$\big(R^1, R^2, Y\big) \sim \text{Ga}^{-1}(\mu - \lambda, \beta) \otimes \text{Ga}^{-1}(\lambda, \beta) \otimes \text{Ga}^{-1}(\mu, \beta). \tag{1.7}$$
Then (R^1, R^2) is $T^{h,Y}$-invariant, where $h(y) = y$. (See Lemma 3.2 of [19].)
- **Gamma**: This is also known as the strict-weak model. Assume λ, μ, $\beta > 0$ and
$$\big(R^1, R^2, Y\big) \sim \text{Ga}(\mu + \lambda, \beta) \otimes \text{Be}^{-1}(\lambda, \mu) \otimes \text{Ga}(\mu, \beta). \tag{1.8}$$
Then (R^1, R^2) is $T^{h,Y}$-invariant, where $h(y) = 1$. (See Lemma 6.3 of [7].)
- **Beta**: Assume λ, μ, $\beta > 0$ and
$$\big(R^1, R^2, Y\big) \sim \text{Be}(\mu + \lambda, \beta) \otimes \text{Be}^{-1}(\lambda, \mu) \otimes \text{Be}(\mu, \beta). \tag{1.9}$$
Then (R^1, R^2) is $T^{h,Y}$-invariant, where $h(y) = 1 - y$. (See Lemma 3.1 of [2].)
- **Inverse-beta**: Assume $\mu > \lambda > 0$, $\beta > 0$ and
$$\big(R^1, R^2, Y\big) \sim \text{Be}^{-1}(\mu - \lambda, \beta) \otimes \big(\text{Be}^{-1}(\lambda, \beta + \mu - \lambda) - 1\big) \otimes \text{Be}^{-1}(\mu, \beta). \tag{1.10}$$
Then (R^1, R^2) is $T^{h,Y}$-invariant, where $h(y) = y - 1$. (See Proposition 3.1 of [22].)

The name of each model refers to the distribution of the bulk weights. We call these models the **four basic beta-gamma models**.

Theorem 1.4. *Let $O_j \subset (0, \infty)$ for $j = 1, 2, 3$ and assume $h : O_3 \to (0, \infty)$ has the form $h(y) = a + by$, where a, b are real numbers satisfying $a \vee b > 0$. Assume the mapping $O_1 \times O_2 \times O_3 \ni (r_1, r_2, y) \mapsto T^{h,y}(r_1, r_2)$ surjects onto $O_1 \times O_2$, and R^1, R^2, Y are non-degenerate, independent random variables taking values in O_1, O_2, O_3 respectively.*

- *(a) If $a = 0$ and $b > 0$, then (R^1, R^2) is $T^{h,Y}$-invariant if and only if $\big(R^1, \frac{1}{b}R^2, Y\big)$ is distributed as in (1.7).*
- *(b) If $a > 0$ and $b = 0$, then (R^1, R^2) is $T^{h,Y}$-invariant if and only if $\big(R^1, \frac{1}{a}R^2, Y\big)$ is distributed as in (1.8).*
- *(c) If $a > 0$, $b < 0$, and $-b \notin \{\frac{y}{x} : (x, y) \in O_1 \times O_2\}$, then (R^1, R^2) is $T^{h,Y}$-invariant if and only if either $\big(-\frac{b}{a}R^1, \frac{1}{a}R^2, -\frac{b}{a}Y\big)$ or $\big(\frac{1}{a}R^2, -\frac{b}{a}R^1, 1 + \frac{b}{a}Y\big)$ is distributed as in (1.9).*
- *(d) If $a < 0$ and $b > 0$, then (R^1, R^2) is $T^{h,Y}$-invariant if and only if $\big(-\frac{b}{a}R^1, -\frac{1}{a}R^2, -\frac{b}{a}Y\big)$ is distributed as in (1.10).*

(e) If $a, b > 0$, then (R^1, R^2) is $T^{h,Y}$-invariant if and only if $\left(\frac{1}{a}R^2, \frac{b}{a}R^1, 1 + \frac{b}{a}Y\right)$ is distributed as in (1.10).

Figure 3 illustrates which one of the four basic beta-gamma models corresponds to each choice of parameters a, b.

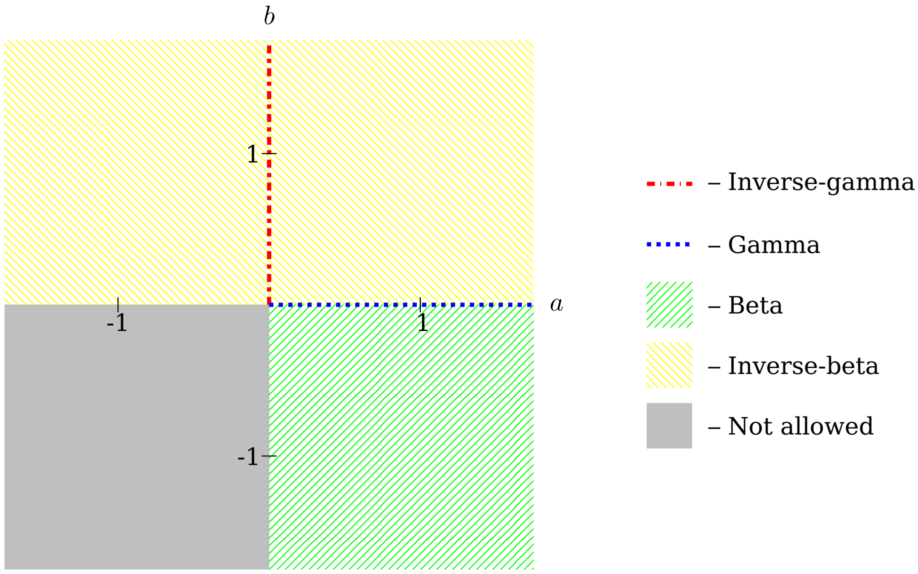

Figure 3: Modifications of the four beta-gamma models.

In the related physics paper [23], Thiery and Le Doussal study the implications of Bethe ansatz solvability in the context of $1+1$-dimensional lattice directed polymers. In their work, they consider the model without boundary and do not impose the additional assumption that the weights on incoming horizontal and vertical edges, u_x and v_x, have a functional dependence. Making the assumption of coordinate Bethe ansatz solvability (for a precise definition see II.B in [23]), they arrive at a formula for the joint moments of u_x and v_x. This is carried out under the assumption that all joint moments of u_x and v_x are finite. Ignoring the finiteness of these moments, they consider the implications of the joint moment conditions in an attempt to classify all weights u_x and v_x leading to coordinate Bethe ansatz solvability. From this classification they are able to retrieve the four basic beta-gamma models. This suggests a direct connection between the integrability properties of Bethe ansatz solvability and stationarity. In the current paper we do not further explore this connection, but consider it an interesting direction for future research.

Structure of the paper: In Section 2 we define the stronger property T^h-invariance, and give conditions for when $T^{h,Y}$-invariance is equivalent to T^h-invariance. T^h-invariance will be used as a tool in proving our main theorems. The proof of Theorem 1.2 is then given in Section 3. In Section 4 we describe the natural modifications of reflection and scaling. The proof of Theorem 1.4 is given in Section 5.

2 Equivalences between $T^{h,Y}$-invariance and T^h-invariance

First define

$$T_1^h(r_1, r_2, y) := y + h(y)\frac{r_1}{r_2} \qquad\qquad T_2^h(r_1, r_2, y) := y\frac{r_2}{r_1} + h(y). \tag{2.1}$$

Notice that (R^1, R^2) is $T^{h,Y}$-invariant if and only if

$$\big(T_1^h, T_2^h\big)\big(R^1, R^2, Y\big) := \big(T_1^h(R^1, R^2, Y), T_2^h(R^1, R^2, Y)\big) \stackrel{d}{=} \big(R^1, R^2\big).$$

In this section we determine conditions which allow us to construct a function T_3^h such that $\big(T_1^h, T_2^h, T_3^h\big)(R^1, R^2, Y) \stackrel{d}{=} (R^1, R^2, Y)$. Moreover, T_3^h will be such that $T := \big(T_1^h, T_2^h, T_3^h\big)$ is an involution. Recall that a function T is an involution if $T \circ T$ is the identity function. This augmentation of our mapping $T^{h,Y}$ to an involution T encapsulates a form of reversibility of the polymer model.

Definition 2.1. *Let $O \subset (0,\infty)^2$, $O_3 \subset (0,\infty)$, and $h : O_3 \to (0,\infty)$. We say that an involution $T : O \times O_3 \to O \times O_3$ is a* polymer involution adapted to h *if its first two coordinates are as in* (2.1).

Existence and uniqueness of polymer involutions is addressed in Lemma 2.4. When the polymer involution adapted to h is unique we write T^h. In our two main theorems we assume that R^1 and R^2 are independent and therefore take $O = O_1 \times O_2$. We allow for arbitrary $O \subset (0,\infty)^2$ since the results in this section allow for dependence between R^1 and R^2.

Definition 2.2. *Suppose (R^1, R^2, Y) is a random vector taking values in $O \times O_3$, where $O \subset (0,\infty)^2$, $O_3 \subset (0,\infty)$, and Y is independent of (R^1, R^2). Let $h : O_3 \to (0,\infty)$. If there exists a polymer involution T on $O \times O_3$ adapted to h such that $T(R^1, R^2, Y) \stackrel{d}{=} (R^1, R^2, Y)$, then we say (R^1, R^2, Y) is T-*invariant (with respect to h).

If (R^1, R^2, Y) is T-invariant, the polymer model with weight distributions (R^1, R^2, Y) not only has property (1.5), but possesses a stronger form of stationarity called the Burke property (see Theorem 3.3 of [19]), named after Burke's theorem on the output distribution of M/M/1 queues (see the reference [13]). In Definition 2.1 we restrict our attention to involutions, as T-invariance not only implies stationarity, but also a form of reversibility: the construction of a dual measure (see Section 3.2 of [19] and Proposition III.3 of [22] for more details).

The four basic beta-gamma models are not only $T^{h,Y}$-invariant, but are in fact T^h-invariant as well. The rest of this section is dedicated to relating the properties of $T^{h,Y}$-invariance and T^h-invariance, as given in the following proposition.

Proposition 2.3. *Let $O \subset (0,\infty)^2$, $O_3 \subset (0,\infty)$, and $h : O_3 \to (0,\infty)$. Assume (R^1, R^2, Y) is a random vector taking values in $O \times O_3$ and that Y is independent of (R^1, R^2). Then the following two conditions are equivalent.*

(a) The mapping $O \times O_3 \ni (r_1, r_2, y) \mapsto T^{h,y}(r_1, r_2)$ surjects onto O, for every $(r_1, r_2) \in O$ the function $O_3 \ni y \mapsto y\frac{r_2}{r_1} + h(y)$ is injective, and (R^1, R^2) is $T^{h,Y}$-invariant.

(b) There exists a unique polymer involution T^h adapted to h on $O \times O_3$ and (R^1, R^2, Y) is T^h-invariant.

The proof of Proposition 2.3 follows from combining Lemmas 2.4, 2.6, and Remark 2.5 below.

We use the notation $\pi_j : (0,\infty)^2 \to (0,\infty)$ to denote the projection onto the j-th coordinate for $j = 1, 2$. Given $O \subset (0,\infty)^2$, $Q(O)$ will denote the set $\big\{\frac{y}{x} : (x,y) \in O\big\}$. When $O = O_1 \times O_2$ we will write $\frac{O_2}{O_1}$ for $Q(O)$.

When T is a polymer involution adapted to h we will often use the following notation

$$(\widetilde{r}_1, \widetilde{r}_2, \widetilde{y}) := T(r_1, r_2, y). \tag{2.2}$$

More precisely, by equations (2.1)

$$\widetilde{r}_1 := y + h(y)\frac{r_1}{r_2}, \qquad \widetilde{r}_2 := y\frac{r_2}{r_1} + h(y), \qquad \widetilde{y} := T_3^h(r_1, r_2, y).$$

Note that these definitions imply that

$$\frac{\widetilde{r}_2}{\widetilde{r}_1} = \frac{r_2}{r_1}. \tag{2.3}$$

This equality of ratios will turn out to be quite useful.

The following lemma gives an equivalence to the existence of a unique polymer involution.

Lemma 2.4. *Let $O \subset (0,\infty)^2$, $O_3 \subset (0,\infty)$, $h : O_3 \to (0,\infty)$, and T_1^h, T_2^h be as in (2.1). Then the following are equivalent:*

(a) $\big(T_1^h, T_2^h\big)(O \times O_3) = O$ and for every $(r_1,r_2) \in O$ the function $O_3 \ni y \mapsto T_2^h(r_1,r_2,y) = y\frac{r_2}{r_1} + h(y)$ is injective.

(b) $G(s,y) := \Big(y + \frac{h(y)}{s},\ ys + h(y)\Big)$ is a bijection between $Q(O) \times O_3$ and O.

(c) There exists a unique polymer involution T^h on $O \times O_3$ adapted to h. Moreover,

$$T^h = (G \otimes \mathit{id}) \circ \psi_{2,3} \circ (G \otimes \mathit{id})^{-1}, \tag{2.4}$$

where $\psi_{2,3}(a,b,c) = (a,c,b)$ and $(G \otimes \mathit{id})(a,b,c) := (G(a,b),c)$.

(d) There exists a polymer involution on $O \times O_3$ adapted to h such that T_3^h has no y-dependence.

Proof. $(a) \Rightarrow (b)$: Note that

$$G\big(\tfrac{r_2}{r_1}, y\big) = \big(T_1^h, T_2^h\big)(r_1,r_2,y) \tag{2.5}$$

implies $G\big(Q(O) \times O_3\big) = O$. Injectivity of G follows from $\frac{\pi_2 \circ G(s,y)}{\pi_1 \circ G(s,y)} = s$ and the injectivity condition on T_2^h.

$(b) \Rightarrow (c)$: We first show uniqueness. Suppose $T = \big(T_1^h, T_2^h, T_3^h\big)$ is a polymer involution on $O \times O_3$ adapted to h. For fixed $(r_1,r_2,y) \in O \times O_3$, with notation as in (2.2), we have $T(\widetilde{r}_1, \widetilde{r}_2, \widetilde{y}) = (r_1,r_2,y)$ since T is an involution. Using (2.3) we have

$$(r_1,r_2) = \big(T_1^h, T_2^h\big)(\widetilde{r}_1,\widetilde{r}_2,\widetilde{y}) = G\big(\tfrac{r_2}{r_1}, \widetilde{y}\big).$$

Therefore

$$G^{-1}(r_1,r_2) = \big(\tfrac{r_2}{r_1},\ T_3^h(r_1,r_2,y)\big). \tag{2.6}$$

Since G^{-1} has no y-dependence, neither does T_3^h. One can now check that

$$T = (G \otimes \mathrm{id}) \circ \psi_{2,3} \circ (G \otimes \mathrm{id})^{-1} \tag{2.7}$$

proving uniqueness. Existence follows by simply setting $T_3^h(r_1,r_2,y) = \pi_2 \circ G^{-1}(r_1,r_2)$. This forces equality (2.7), the right side of which is indeed a polymer involution adapted to h.

$(c) \Rightarrow (d)$ is clear.

$(d) \Rightarrow (a)$: Let T be a polymer involution on $O \times O_3$ adapted to h for which T_3^h has no y-dependence. Clearly the first two components of T, $\big(T_1^h, T_2^h\big)$, surject onto O. Now fix $(r_1,r_2) \in O$. Since $T_1^h(r_1,r_2,y) = \frac{r_1}{r_2} T_2^h(r_1,r_2,y)$ and T is itself injective, we have injectivity of $y \mapsto T_2^h(r_1,r_2,y)$. $\square$

Remark 2.5. Note that the conditions in part (a) of Lemma 2.4 depend only on the sets O, O_3, and the function h. The condition $\big(T_1^h, T_2^h\big)(O \times O_3) = O$ in part (a) is equivalent to the condition that the mapping $O \times O_3 \ni (r_1,r_2,y) \mapsto T^{h,y}(r_1,r_2)$ surjects onto O (recall definition (1.6)).

When the polymer involution T is such that T_3^h has no y-dependence, we will simply write $T_3^h(r_1, r_2)$. The following lemma gives conditions for when $T^{h,Y}$-invariance is equivalent to T^h-invariance.

Lemma 2.6. *Suppose O, O_3, and h satisfy one of the equivalent conditions in Lemma 2.4. Let (R^1, R^2, Y) be a random vector taking values in $O \times O_3$ and assume that Y is independent of the pair (R^1, R^2). Let T^h be the unique polymer involution adapted to h, defined by* (2.4)*, and write $\widetilde{Y} = T_3^h(R^1, R^2)$. Then the following are equivalent:*

(a) (R^1, R^2) *is $T^{h,Y}$-invariant.*
(b) R^2/R^1 *is independent of $\widetilde{Y}$ and $\widetilde{Y} \overset{d}{=} Y$.*
(c) (R^1, R^2, Y) *is T^h-invariant.*

Proof. $(a) \Leftrightarrow (b)$: Put $\big(\widetilde{R}^1, \widetilde{R}^2\big) = \big(T_1^h, T_2^h\big)(R^1, R^2, Y)$. Using equations (2.5) and (2.6),

$$\begin{aligned} &G\big(R^2/R^1, Y\big) = \big(\widetilde{R}^1, \widetilde{R}^2\big) \overset{d}{=} \big(R^1, R^2\big) \\ \Leftrightarrow\ &\big(R^2/R^1, Y\big) \overset{d}{=} G^{-1}\big(R^1, R^2\big) = \big(R^2/R^1, \widetilde{Y}\big) \\ \Leftrightarrow\ &R^2/R^1 \text{ is independent of } \widetilde{Y} \text{ and } Y \overset{d}{=} \widetilde{Y}. \end{aligned}$$

$(c) \Rightarrow (a)$ is clear. We now show that (a) and (b) imply (c). Since T_3^h has no y-dependence, Y is independent of the pair $(R^2/R^1, \widetilde{Y})$. Therefore the triple $(R^2/R^1, Y, \widetilde{Y})$ is independent. Thus $\big(\widetilde{R}^1, \widetilde{R}^2\big) = G(R^2/R^1, Y)$ is independent of $\widetilde{Y}$. Now combining (a) and $\widetilde{Y} \overset{d}{=} Y$ we get $\big(\widetilde{R}^1, \widetilde{R}^2, \widetilde{Y}\big) \overset{d}{=} (R^1, R^2, Y)$. □

We now give an analogue of Lemma 2.4 in which h and T^h are continuously differentiable. We compute the Jacobian matrix and determinant of T^h in order to later give an explicit form for the density of $T^h(R^1, R^2, Y)$ in terms of the density of (R^1, R^2, Y) (see proof of Proposition 3.1).

Given a differentiable transformation $F : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is open, use the notations $DF(u)$ and $D[F](u)$ to denote the Jacobian matrix of F evaluated at the point $u \in U$. When $m = n$ we say F is a C^1-diffeomorphism if F is injective, continuously differentiable, and its Jacobian matrix is invertible throughout U.

Lemma 2.7. *Let $O \subset (0, \infty)^2$, $O_3 \subset (0, \infty)$, $h : O_3 \to (0, \infty)$, and T_1^h, T_2^h be as in* (2.1)*. Further assume O and O_3 are open, O_3 is connected, and h is continuously differentiable. Then the following are equivalent:*

(a) $\big(T_1^h, T_2^h\big)(O \times O_3) = O$ *and the following function does not vanish on $Q(O) \times O_3$*

$$L(s, y) := s + h'(y). \tag{2.8}$$

(b) $G(s, y) := \Big(y + \frac{h(y)}{s},\ ys + h(y)\Big)$ *is a C^1-diffeomorphism between $Q(O) \times O_3$ and O. Moreover its Jacobian matrix and determinant are given by*

$$DG(s, y) = \begin{bmatrix} -h(y)/s^2 & L(s, y)/s \\ y & L(s, y) \end{bmatrix}, \quad \det DG(s, y) = -\frac{L(s, y)}{s}\left(y + \frac{h(y)}{s}\right). \tag{2.9}$$

(c) There exists a unique C^1-diffeomorphic polymer involution T^h on $O \times O_3$ adapted to h. Moreover T_3^h has no y dependence and the Jacobian matrix and determinant of T^h are given by

$$DT^h(r_1, r_2, y) = \frac{1}{r_1}\begin{bmatrix} h(y)/s & -h(y)/s^2 & L(s, y) r_1/s \\ -ys & y & L(s, y) r_1 \\ \widetilde{y}s/L(s, \widetilde{y}) & h(\widetilde{y})/\big(sL(s, \widetilde{y})\big) & 0 \end{bmatrix}, \tag{2.10}$$

$$\det DT^h(r_1, r_2, y) = -\left(\frac{y}{r_1} + \frac{h(y)}{r_2}\right)\frac{L(s,y)}{L(s,\widetilde{y})},$$

where $s = \frac{r_2}{r_1}$ *and* $\widetilde{y} = T_3^h(r_1, r_2)$.

(d) *There exists a differentiable polymer involution on* $O \times O_3$ *adapted to* h.

Proof. $(a) \Rightarrow (b)$: For fixed $(r_1, r_2) \in O$, since $y \mapsto \frac{\partial T_2^h}{\partial y}(r_1, r_2, y) = L\big(\frac{r_2}{r_1}, y\big)$ does not vanish on the connected set O_3, the conditions of Lemma 2.4-(a) are satisfied. Therefore G is a bijection. The continuous differentiability of h now implies that G is continuously differentiable. The Jacobian matrix and determinant of G can now be calculated. Notice that for all $(s, y) \in Q(O) \times O_3$, $y + h(y)/s = \pi_1 \circ G(s, y) \in \pi_1(O) \subset (0, \infty)$. Thus the Jacobian determinant of G does not vanish on $Q(O) \times O_3$, which shows it is a C^1-diffeomorphism.

$(b) \Rightarrow (c)$: Since G is a bijection, Lemma 2.4 gives existence and uniqueness of the polymer involution $T^h = (G \otimes \mathrm{id}) \circ \psi_{2,3} \circ (G \otimes \mathrm{id})^{-1}$. Since G is a C^1-diffeomorphism, the inverse function theorem tells us T^h is a C^1-diffeomorphism as well. Now fix $(r_1, r_2, y) \in O \times O_3$ and put $(s, \widetilde{y}) = \big(\frac{r_2}{r_1},\ T_3^h(r_1, r_2)\big)$. By (2.6)

$$(s, \widetilde{y}) = G^{-1}(r_1, r_2). \tag{2.11}$$

$DG^{-1}(r_1, r_2)$ is now the inverse of the matrix $DG\big(G^{-1}(r_1, r_2)\big) = DG(s, \widetilde{y})$. (2.11) implies $(r_1, r_2) = G(s, \widetilde{y}) = \big(\widetilde{y} + h(\widetilde{y})/s,\ \widetilde{y}s + h(\widetilde{y})\big)$. Using this one can show that

$$DG^{-1}(r_1, r_2) = \frac{1}{r_1}\begin{bmatrix} -s & 1 \\ \frac{s\widetilde{y}}{L(s,\widetilde{y})} & \frac{h(\widetilde{y})}{sL(s,\widetilde{y})} \end{bmatrix} \quad \text{and} \quad \det DG^{-1}(r_1, r_2) = -\frac{s}{r_1 L(s, \widetilde{y})}. \tag{2.12}$$

Using equations (2.9), (2.11), and (2.12) we can compute

$$\begin{aligned}
DT^h(r_1, r_2, y) &= \big[D(G \otimes \mathrm{id})\left(\psi_{2,3} \circ (G^{-1} \otimes \mathrm{id})(r_1, r_2, y)\right)\big] \cdot \big[D\psi_{2,3}\left((G^{-1} \otimes \mathrm{id})(r_1, r_2, y)\right)\big] \\
&\quad \cdot \big[D\big(G^{-1} \otimes \mathrm{id}\big)(r_1, r_2, y)\big] \\
&= [DG(s, y) \otimes 1] \cdot [D\psi_{2,3}] \cdot \big[DG^{-1}(r_1, r_2) \otimes 1\big] \\
&= \begin{bmatrix} \frac{-h(y)}{s^2} & \frac{L(s,y)}{s} & 0 \\ y & L(s,y) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \frac{1}{r_1}\begin{bmatrix} -s & 1 & 0 \\ \frac{s\widetilde{y}}{L(s,\widetilde{y})} & \frac{h(\widetilde{y})}{sL(s,\widetilde{y})} & 0 \\ 0 & 0 & r_1 \end{bmatrix} \\
&= \frac{1}{r_1}\begin{bmatrix} h(y)/s & -h(y)/s^2 & L(s,y)r_1/s \\ -ys & y & L(s,y)r_1 \\ \widetilde{y}s/L(s,\widetilde{y}) & h(\widetilde{y})/\big(sL(s,\widetilde{y})\big) & 0 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\det\left(DT^h(r_1, r_2, y)\right) &= \det\left(DG(s, y)\right)\det\left(D\psi_{2,3}\right)\det\left(DG^{-1}(r_1, r_2)\right) \\
&= -\frac{L(s,y)}{s}\left(y + \frac{h(y)}{s}\right)(-1)\left(-\frac{s}{r_1 L(s,\widetilde{y})}\right) \\
&= -\left(\frac{y}{r_1} + \frac{h(y)}{r_2}\right)\frac{L(s,y)}{L(s,\widetilde{y})}.
\end{aligned}$$

$(c) \Rightarrow (d)$ is clear.

$(d) \Rightarrow (a)$: If T is a differentiable polymer involution adapted to h, then its Jacobian matrix has the same entries as the 2×3 upper portion of (2.10), as $\big(T_1^h, T_2^h\big)$ are completely determined. Therefore the determinant of the top-left 2×2 minor of the Jacobian matrix of T is zero. Thus L vanishing at a point $(s, y) \in Q(O) \times O_3$ would imply the Jacobian determinant of T vanishes at any point $(r_1, r_2, y) \in O \times O_3$ such that $\frac{r_2}{r_1} = s$. Since $T \circ T$ is the identity function, the Jacobian determinant of T cannot vanish on $O \times O_3$. Thus L cannot vanish on $Q(O) \times O_3$. □

Let $B := \{(r_1, r_2, y) \in O \times O_3 : \frac{r_2}{r_1} + h'(y) = 0\}$. The following Lemma shows that when h is twice continuously differentiable, the image of B under the mapping (T_1^h, T_2^h) has Lebesgue measure zero.

Lemma 2.8. *Take all assumptions from Lemma 2.7 with the addition that h is twice continuously differentiable. Then $(T_1^h, T_2^h)(B)$ has Lebesgue measure zero.*

Proof. For convenience define $H(r_1, r_2, y) := (T_1^h, T_2^h)(r_1, r_2, y)$. The Jacobian matrix of H is given by the top 2×3 portion of the matrix (2.10). Therefore (r_1, r_2, y) is a critical point of H, meaning the rank of $DH(r_1, r_2, y) < 2$, if and only if $L(\frac{r_2}{r_1}, y) = 0$ (since $y\frac{r_2}{r_1} + h(y) = \widetilde{r}_2 > 0$), which occurs if and only if $(r_1, r_2, y) \in B$. Sard's theorem [18] yields the desired result. □

3 Proof of Theorem 1.2

We begin by using Lemma 2.7 to give another useful equivalence to T-invariance under some regularity assumptions. In the appendix of [22], Thiery uses a specific case of the following proposition to prove the invariance of the inverse-beta model. It can also be used to prove invariance of the other three basic beta-gamma models.

Proposition 3.1. *Let (R^1, R^2, Y) be a random vector with density ρ and assume Y is independent of (R^1, R^2). Suppose the support of ρ equals $O \times O_3$ where $O \subset (0, \infty)^2$ is open and $O_3 \subset (0, \infty)$ is open and connected. Let $h : O_3 \to (0, \infty)$ be continuously differentiable and T be a differentiable polymer involution adapted to h on $O \times O_3$. Then (R^1, R^2, Y) is T-invariant if and only if*

$$q \circ T(x) = q(x) \qquad \text{for a.e. } x \in O \times O_3$$

where $q(r_1, r_2, y) := \frac{r_2}{|L(r_2/r_1, y)|}\rho(r_1, r_2, y)$ and $L(s, y) = s + h'(y)$, as given in (2.8).

Proof of Proposition 3.1. Recall the notation (2.2). By Lemma 2.7, L does not vanish on $Q(O) \times O_3$ and T is in fact a C^1-diffeomorphism with

$$\det DT(r_1, r_2, y) = -\left(\frac{y}{r_1} + \frac{h(y)}{r_2}\right)\frac{L(r_2/r_1, y)}{L(r_2/r_1, \widetilde{y})} = -\frac{\widetilde{r}_2 L(r_2/r_1, y)}{r_2 L(r_2/r_1, \widetilde{y})}.$$

Therefore $T(R^1, R^2, Y)$ has density

$$\widehat{\rho}(x) := \rho(T^{-1}(x))\,|\det DT^{-1}(x)| = \rho(T(x))\,|\det DT(x)|$$

supported on $x \in O \times O_3$. Thus T-invariance of (R^1, R^2, Y) is equivalent to $\rho(x) = \widehat{\rho}(x)$ a.e. on $O \times O_3$.

Using (2.3) we can explicitly write $\widehat{\rho}(r_1, r_2, y) = \rho(\widetilde{r}_1, \widetilde{r}_2, \widetilde{y})\left|\frac{\widetilde{r}_2 L(r_2/r_1, y)}{r_2 L(\widetilde{r}_2/\widetilde{r}_1, \widetilde{y})}\right|$. After rearranging terms, the condition $\rho(x) = \widehat{\rho}(x)$ for a.e. $x \in O \times O_3$ yields the desired result. □

We now prove the first main result.

Proof of Theorem 1.2. (R^1, R^2, Y) has density $\rho(r_1, r_2, y) = f_1(r_1)f_2(r_2)f_3(y)$. By Lemma 2.7, there exists a unique differentiable polymer involution T^h on $O_1 \times O_2 \times O_3$ adapted to h and the function $L(s, y) = s + h'(y)$ does not vanish on the set $\frac{O_2}{O_1} \times O_3$. By Lemma 2.6, (R^1, R^2, Y) is T^h-invariant. Applying Proposition 3.1 gives $q \circ T^h = q$ a.e. on $O_1 \times O_2 \times O_3$. Since f_1, f_2, f_3, and T^h are continuous, this equality holds everywhere on $O_1 \times O_2 \times O_3$. Since the support of f_j equals O_j, we can further assume $f_j(x) = \exp(\eta_j(x))$ for $x \in O_j$, $j = 1, 2, 3$. Note that η_j has the same differentiability properties as f_j. Set $s = \frac{r_2}{r_1}$ and recall the notation (2.2). Taking logarithms of the equality $q \circ T^h = q$ then computing the total derivative we obtain

$$D[\log q](\widetilde{r}_1, \widetilde{r}_2, \widetilde{y}) \cdot DT^h(r_1, r_2, y) = D[\log q](r_1, r_2, y), \tag{3.1}$$

where DT^h is given in (2.10) and

$$D[\log q](r_1, r_2, y) = \left[\frac{r_2}{r_1^2 L(s,y)} + \eta_1'(r_1), \frac{h'(y)}{r_2 L(s,y)} + \eta_2'(r_2), -\frac{h''(y)}{L(s,y)} + \eta_3'(y)\right].$$

Using the fact that T^h is an involution and (2.3), $r_2 = T_2^h(\widetilde{r}_1, \widetilde{r}_2, \widetilde{y}) = \widetilde{y}s + h(\widetilde{y})$. One can then check that

$$DT^h(r_1, r_2, y) \cdot [r_1,\, r_2,\, 0]^T = [0,\, 0,\, r_2/L(s, \widetilde{y})]^T.$$

Thus multiplying both sides of equation (3.1) on the right by $[r_1,\, r_2,\, 0]^T$ gives

$$1 + r_1\eta_1'(r_1) + r_2\eta_2'(r_2) = r_2 g(s, \widetilde{y}), \tag{3.2}$$

where

$$g(s, y) := \frac{\eta_3'(y)}{L(s,y)} - \frac{h''(y)}{L(s,y)^2}.$$

Applying the operator $\frac{\partial^2}{\partial r_1 \partial r_2}$ to the left-hand side of (3.2) gives zero. We now exploit the fact that $\frac{\partial^2}{\partial r_1 \partial r_2}$ applied to the right hand side must equal zero to ultimately arrive at the conclusion that $h''(y) = 0$.

Note that if f is differentiable then for all non-negative integers k and n,

$$D\left[\frac{s^k f(y)}{L(s,y)^n}\right](s,y) = s^{k-1}\left[\frac{1}{L(s,y)^n}, \frac{-nf(y)}{L(s,y)^{n+1}}\right] \cdot \begin{bmatrix} kf(y) & sf'(y) \\ s & sh''(y) \end{bmatrix}. \tag{3.3}$$

First calculate, using (2.10) and (3.3),

$$\begin{aligned}
\frac{\partial}{\partial r_1}\big(r_2 g(s, \widetilde{y})\big) &= r_2 Dg(s, \widetilde{y}) \cdot \left[\frac{\partial s}{\partial r_1}, \frac{\partial \widetilde{y}}{\partial r_1}\right]^T \\
&= s^2 Dg(s, \widetilde{y}) \cdot \left[-1, \frac{\widetilde{y}}{L(s, \widetilde{y})}\right]^T \\
&= s\left[\frac{1}{L(s,\widetilde{y})}, -\frac{\eta_3'(\widetilde{y})}{L(s,\widetilde{y})^2}\right] \cdot \begin{bmatrix} 0 & s\eta_3''(\widetilde{y}) \\ s & sh''(\widetilde{y}) \end{bmatrix} \cdot \left[-1, \frac{\widetilde{y}}{L(s,\widetilde{y})}\right]^T \\
&\quad - s\left[\frac{1}{L(s,\widetilde{y})^2}, -\frac{2h'(\widetilde{y})}{L(s,\widetilde{y})^3}\right] \cdot \begin{bmatrix} 0 & sh''(\widetilde{y}) \\ s & sh''(\widetilde{y}) \end{bmatrix} \cdot \left[-1, \frac{\widetilde{y}}{L(s,\widetilde{y})}\right]^T \\
&= t(s, \widetilde{y}) := \sum_{j=2}^{4} \frac{s^2 \kappa_j(\widetilde{y})}{L(s,\widetilde{y})^j},
\end{aligned}$$

where

$$\begin{aligned}
\kappa_2(y) &= y\eta_3''(y) + \eta_3'(y) \\
\kappa_3(y) &= -yh''(y)\eta_3'(y) - yh'''(y) - 2h''(y) \\
\kappa_4(y) &= 2yh''(y)^2.
\end{aligned}$$

Taking an r_2 partial derivative and multiplying by r_1, by (2.10)

$$\begin{aligned}
0 = r_1 \frac{\partial^2}{\partial r_2 \partial r_1}\big(r_2 g(s, \widetilde{y})\big) &= r_1 \frac{\partial}{\partial r_2} t(s, \widetilde{y}) \\
&= r_1 Dt(s, \widetilde{y}) \cdot \left[\frac{\partial s}{\partial r_2}, \frac{\partial \widetilde{y}}{\partial r_2}\right]^T \\
&= Dt(s, \widetilde{y}) \cdot \left[1, \frac{h(\widetilde{y})}{sL(s,\widetilde{y})}\right]^T.
\end{aligned}$$

This equality holds for all $(r_1, r_2, y) \in O_1 \times O_2 \times O_3$. Since T^h is an involution on $O_1 \times O_2 \times O_3$, it also holds after interchanging $(r_1, r_2, y) \leftrightarrow (\widetilde{r}_1, \widetilde{r}_2, \widetilde{y})$. Notice that, by (2.3), $s = \frac{r_2}{r_1}$ is unaffected by this interchange. Therefore, applying this interchange and using (3.3)

$$\begin{aligned} 0 &= Dt(s,y) \cdot \left[1,\ \frac{h(y)}{sL(s,y)}\right]^T \\ &= \sum_{j=2}^{4} s \left[\frac{1}{L(s,y)^j}, \frac{-j\kappa_j(y)}{L(s,y)^{j+1}}\right] \cdot \begin{bmatrix} 2\kappa_j(y) & s\kappa_j'(y) \\ s & sh''(y) \end{bmatrix} \cdot \left[1,\ \frac{h(y)}{sL(s,y)}\right]^T \end{aligned}$$

for all $(s,y) \in \frac{O_2}{O_1} \times O_3$. Multiplying by $L(s,y)^6/s$ gives

$$0 = \sum_{j=2}^{4} \left[L(s,y)^{5-j}, -j\kappa_j(y)L(s,y)^{4-j}\right] \cdot \begin{bmatrix} 2\kappa_j(y) & \kappa_j'(y) \\ s & h''(y) \end{bmatrix} \cdot \left[L(s,y),\ h(y)\right]^T. \tag{3.4}$$

Now fix $y \in O_3$. The right hand side is now a fourth degree polynomial in s which vanishes on the open set $\frac{O_2}{O_1}$. It must therefore vanish at all values $s \in \mathbb{R}$. Taking $s = -h'(y)$ so that $L(s,y) = 0$, (3.4) gives

$$0 = -4\kappa_4(y)h(y)h''(y) = -8yh(y)h''(y)^3.$$

The fact that y and $h(y)$ are positive implies $h''(y) = 0$. Since this holds for all $y \in O_3$, which we assumed to be connected, h has the form $h(y) = a + by$ where a, b are real numbers. The condition $a \vee b > 0$ follows from the fact that h maps a subset of $(0,\infty)$ into $(0,\infty)$. □

4 Reflection and scaling

We describe two procedures which preserve T-invariance. By applying these procedures to the four basic beta-gamma models, we can obtain a T-invariant model corresponding to $h(y) = a + by$ for each choice of a, b such that $a \vee b > 0$.

We first define the reflection procedure. Let T be a polymer involution adapted to h on $O_1 \times O_2 \times O_3$ and assume that h is injective so that $h : O_3 \to h(O_3)$ is a bijection. Define the mapping $\rho(r_1, r_2, y) := (r_2, r_1, h(y))$. Define the mapping and the random vector

$$\widehat{T} := \rho \circ T \circ \rho^{-1} \quad \text{and} \quad \big(\widehat{R}^1, \widehat{R}^2, \widehat{Y}\big) := \big(R^2, R^1, h(Y)\big). \tag{4.1}$$

One can then check that $\widehat{T}$ is a polymer involution adapted to h^{-1} on $O_2 \times O_1 \times h(O_3)$. Furthermore, (R^1, R^2, Y) is T-invariant with respect to h if and only if $\big(\widehat{R}^1, \widehat{R}^2, \widehat{Y}\big)$ is $\widehat{T}$-invariant with respect to h^{-1}.

In the directed polymer setting, this procedure of mapping

$$h \mapsto h^{-1} \qquad \big(R^1, R^2, Y\big) \mapsto \big(\widehat{R}^1, \widehat{R}^2, \widehat{Y}\big) \qquad T \mapsto \widehat{T}$$

corresponds to interchanging the horizontal and vertical coordinates while remaining in the same framework. This is illustrated in Figure 4.

We now define the scaling procedure. If $O \subset (0,\infty)$ and c is a positive constant, define $cO := \{cx : x \in O\}$. Note that $cO \subset (0,\infty)$. Let c_1, c_2 be positive constants. Let T be a polymer involution adapted to h on $O_1 \times O_2 \times O_3$. Define the mapping $\sigma(r_1, r_2, y) := (c_1 r_1, c_2 r_2, c_1 y)$. Define the two mappings and the random vector

$$\breve{T} := \sigma \circ T \circ \sigma^{-1} \qquad \breve{h}(y) := c_2 h\big(\tfrac{y}{c_1}\big) \qquad \big(\breve{R}^1, \breve{R}^2, \breve{Y}\big) := \big(c_1 R^1, c_2 R^2, c_1 Y\big). \tag{4.2}$$

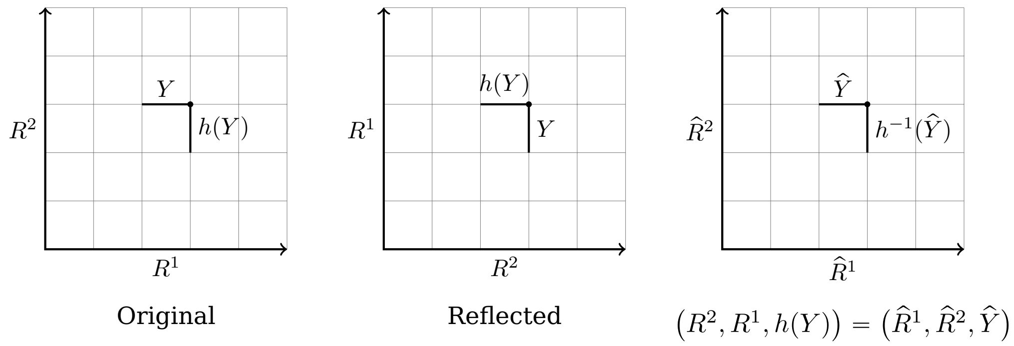

Figure 4: Reflection.

One can check that $\breve{T}$ is a polymer involution adapted to $\breve{h}$ on $c_1O_1 \times c_2O_2 \times c_1O_3$. Furthermore, (R^1, R^2, Y) is T-invariant with respect to h if and only if $\left(\breve{R}^1, \breve{R}^2, \breve{Y}\right)$ is $\breve{T}$-invariant with respect to $\breve{h}$.

In the directed polymer setting, this procedure of mapping

$$h \mapsto \breve{h} \qquad\qquad \left(R^1, R^2, Y\right) \mapsto \left(\breve{R}^1, \breve{R}^2, \breve{Y}\right) \qquad\qquad T \mapsto \breve{T}$$

corresponds to scaling the horizontal axis weights by c_1 and the vertical axis weights by c_2 while remaining in the same framework. This procedure is illustrated in Figure 5.

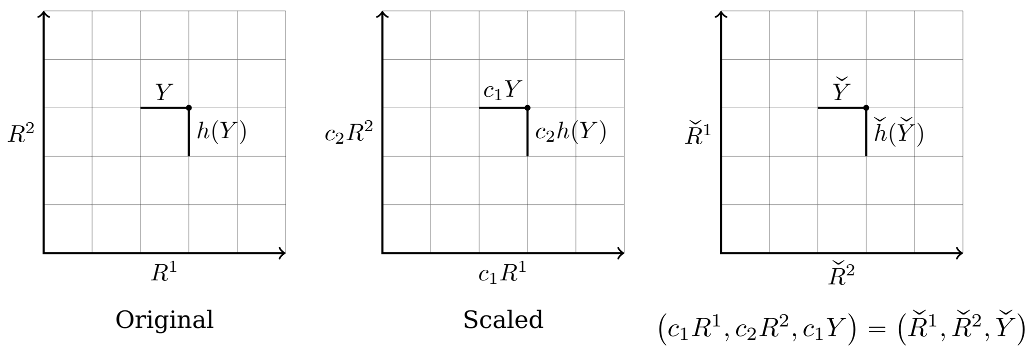

Figure 5: Scaling.

One can also check that the reflection and scaling procedures commute. By using the reflection and scaling procedures, the following lemma reduces the existence and uniqueness of T-invariant models corresponding to $h(y) = a + by$ where $a \vee b > 0$ to the existence and uniqueness for values $(a, b) = (0, 1), (1, 0), (1, -1)$, and $(-1, 1)$.

For real numbers a, b such that $a \vee b > 0$, define

$$T^{(a,b)}(r_1, r_2, y) := \left(y + (a + by)\tfrac{r_1}{r_2},\ y\tfrac{r_2}{r_1} + (a + by),\ \tfrac{r_1(r_2-a)}{r_2+br_1}\right). \tag{4.3}$$

One can check that when $h(y) = a + by$, (2.4) implies that $T^h = T^{(a,b)}$. The domain of $T^{(a,b)}$ is discussed prior to Lemma 5.4.

Lemma 4.1. *Let a, b be real numbers satisfying $a \vee b > 0$, $h(y) = a + by$, and $T = T^{(a,b)}$ as defined in* (4.3)*. Let R^1, R^2, and Y be random variables.*

(a) If $a = 0$ and $b > 0$, then (R^1, R^2, Y) is T-invariant with respect to h if and only if $\left(R^1, \frac{1}{b}R^2, Y\right)$ is $T^{(0,1)}$-invariant with respect to $\breve{h}(y) = y$.

(b) *If $a > 0$ and $b = 0$, then (R^1, R^2, Y) is T-invariant with respect to h if and only if $\left(R^1, \frac{1}{a}R^2, Y\right)$ is $T^{(1,0)}$-invariant with respect to $\breve{h}(y) = 1$*
(c) *If $a > 0$ and $b < 0$, then (R^1, R^2, Y) is T-invariant with respect to h if and only if $\left(-\frac{b}{a}R^1, \frac{1}{a}R^2, -\frac{b}{a}Y\right)$ is $T^{(1,-1)}$-invariant with respect to $\breve{h}(y) = 1 - y$.*
(d) *If $a < 0$ and $b > 0$, then (R^1, R^2, Y) is T-invariant with respect to h if and only if $\left(-\frac{b}{a}R^1, -\frac{1}{a}R^2, -\frac{b}{a}Y\right)$ is $T^{(-1,1)}$-invariant with respect to $\breve{h}(y) = y - 1$.*
(e) *If $a > 0$ and $b > 0$, then (R^1, R^2, Y) is T-invariant with respect to h if and only if $\left(\frac{b}{a}R^1, \frac{1}{a}R^2, \frac{b}{a}Y\right)$ is $T^{(1,1)}$-invariant with respect to $\breve{h}(y) = y + 1$.*
(f) *If $a = 1$ and $b = 1$, then (R^1, R^2, Y) is T-invariant with respect to h if and only if $(R^2, R^1, 1 + Y)$ is $T^{(-1,1)}$-invariant with respect to $h^{-1}(y) = y - 1$.*

Proof. Let c_1, c_2 be positive constants. After applying the scaling procedure with (c_1, c_2), with notation as in (4.2), one can check that

$$\breve{h}(y) = ac_2 + \tfrac{bc_2}{c_1}y \quad \text{and} \quad \breve{T} = T^{(ac_2, bc_2/c_1)}.$$

Recall that $\left(\breve{R}^1, \breve{R}^2, \breve{Y}\right) = \left(c_1R^1, c_2R^2, c_1Y\right)$ is $\breve{T}$-invariant with respect to $\breve{h}$ if and only if $\left(R^1, R^2, Y\right)$ is T-invariant with respect to h. Now (a) through (e) follow by taking

$$(c_1, c_2) = \left(1, \tfrac{1}{b}\right), \left(1, \tfrac{1}{a}\right), \left(-\tfrac{b}{a}, \tfrac{1}{a}\right), \left(-\tfrac{b}{a}, -\tfrac{1}{a}\right), \left(\tfrac{b}{a}, \tfrac{1}{a}\right)$$

respectively.

For part (f), after applying the reflection procedure, with notation as in (4.1), one can check that $\widehat{T} = T^{(-1,1)}$. Since $\left(\widehat{R}^1, \widehat{R}^2, \widehat{Y}\right) = (R^2, R^1, 1 + Y)$ is $\widehat{T}$-invariant with respect to $h^{-1}(y) = y - 1$ if and only if (R^1, R^2, Y) is T-invariant with respect to $h(y) = y + 1$, the result follows. □

5 Proof of Theorem 1.4

The following two theorems, due to Seshadri and Wesołowski (2003) and Lukacs (1955) give characterizations of gamma and beta random variables, which will be used in the sequel.

Theorem 5.1 ([21])**.** *Let A and B be non-degenerate independent random variables taking values in $(0, 1)$. Then the pair $(C, D) := \left(\frac{1-B}{1-AB}, 1 - AB\right)$ is independent if and only if there exist positive constants p, q, r such that $(A, B) \sim$ Be$(p, q) \otimes$ Be$(p + q, r)$, in which case $(C, D) \sim$ Be$(r, q) \otimes$ Be$(r + q, p)$.*

Theorem 5.2 ([14])**.** *Let A and B be non-degenerate independent positive random variables. Then the pair $(C, D) := \left(A + B, \frac{A}{A+B}\right)$ is independent if and only if there exist positive constants $\lambda_A, \lambda_B, \beta$ such that $(A, B) \sim$ Ga$(\lambda_A, \beta) \otimes$ Ga(λ_B, β), in which case $(C, D) \sim$ Ga$(\lambda_A + \lambda_B, \beta) \otimes$ Be(λ_A, λ_B).*

Notice that the mapping $(A, B) \mapsto \left(A + B, A/(A + B)\right)$ has the inverse $(A, B) \mapsto \left(AB, A(1 - B)\right)$. The following statement is a corollary of Theorem 5.2.

Corollary 5.3. *Let A and B be non-degenerate independent random variables. Further assume that A is positive and B takes values in $(0, 1)$. Then the pair $(C, D) := \left(AB, A(1-B)\right)$ is independent if and only if there exist positive constants $\lambda_A, \lambda_B, \beta$ such that $(A, B) \sim$ Ga$(\lambda_A + \lambda_B, \beta) \otimes$ Be(λ_A, λ_B) in which case $(C, D) \sim$ Ga$(\lambda_A, \beta) \otimes$ Ga(λ_B, β).*

The next lemma constrains the sets on which $T^{(a,b)}$ (as defined by (4.3)) can be a polymer involution. To specify this constraint, we define the following sets. For real numbers (a, b) such that $a \vee b > 0$,

$$V_a^{\pm} := \{x > 0 : \pm(x - a) > 0\}, \qquad W_{a,b}^{\pm} := \{x > 0 : \pm(a + bx) > 0\}$$
$$D_{a,b}^{\pm} := W_{a,b}^{\pm} \times V_a^{\pm} \times W_{a,b}^{+}.$$

Lemma 5.4. *Let a, b be real numbers satisfying $a \vee b > 0$. Let $O_j \subset (0, \infty)$ for $j = 1, 2, 3$ such that O_3 is not a singleton. If $T^{(a,b)}$, as defined in* (4.3)*, is a polymer involution on $O_1 \times O_2 \times O_3$ with respect to h of the form $h(y) = a + by$ then $O_1 \times O_2 \times O_3 \subset D^+_{a,b}$ or $O_1 \times O_2 \times O_3 \subset D^-_{a,b}$ assuming $D^{\pm}_{a,b}$ is non-empty.*

Proof. We first show the following holds:

(i) For all $(r_1, r_2) \in O_1 \times O_2$, the three numbers $a + br_1, \frac{r_2}{r_1} + b, r_2 - a$ are all either strictly positive, strictly negative, or equal to zero.

Fix $(r_1, r_2, y) \in O_1 \times O_2 \times O_3$ and put $\widetilde{y} = T_3^{(a,b)}(r_1, r_2) = \frac{r_1(r_2-a)}{r_2+br_1}$. Then the following two equalities hold

$$r_2 - a = \widetilde{y}\big(\tfrac{r_2}{r_1} + b\big), \qquad a + br_1 = \tfrac{r_1}{r_2}\big(a + b\widetilde{y}\big)\big(\tfrac{r_2}{r_1} + b\big). \tag{5.1}$$

Since $T^{(a,b)}$ is an involution on $O_1 \times O_2 \times O_3$, $\widetilde{y} \in O_3$. Recall that, by Definition 2.1, h maps $O_3 \to (0, \infty)$. Therefore $O_3 \subset W^+_{a,b}$ and the four numbers r_1, r_2, $\widetilde{y}$, and $h(\widetilde{y}) = a + b\widetilde{y}$ are all positive. (5.1) now gives (i).

By Lemma 2.4, for all $(r_1, r_2) \in O_1 \times O_2$ the mapping $O_3 \ni y \mapsto T_2^{(a,b)}(r_1, r_2, y) = y(\frac{r_2}{r_1} + b) + a$ is injective. Therefore $\frac{r_2}{r_1} + b$ does not vanish for any $(r_1, r_2) \in O_1 \times O_2$. Thus, by (i)

$$O_1 \times O_2 \subset \big(W^+_{a,b} \times V^+_a\big) \cup \big(W^-_{a,b} \times V^-_a\big). \tag{5.2}$$

If $O_1 \cap W^+_{a,b} = \varnothing$, then by (5.2) $O_1 \times O_2 \subset W^-_{a,b} \times V^-_a$. In this case $O_1 \times O_2 \times O_3 \subset D^-_{a,b}$. On the other hand, if $O_1 \cap W^+_{a,b} \neq \varnothing$ then there exists $r_1 \in O_1$ such that $a + br_1 > 0$. By (i), $r_2 - a > 0$ for all $r_2 \in O_2$. Thus $O_2 \subset V^+_a$. Now (5.2) implies that $O_1 \times O_2 \subset W^+_{a,b} \times V^+_a$ which gives $O_1 \times O_2 \times O_3 \subset D^+_{a,b}$, completing the proof. □

Using (5.1) one can in fact check that $T^{(a,b)}$ is an involution on both $D^+_{a,b}$ and $D^-_{a,b}$ assuming they are non-empty.

The following proposition characterizes T^h-invariant models corresponding to $h(y) = a + by$ when $(a, b) = (0, 1)$, $(1, 0)$, $(1, -1)$, and $(-1, 1)$.

Proposition 5.5. *For a, b real numbers, let $h(y) = a + by$ and assume $T^{(a,b)}$, as defined in* (4.3)*, is a polymer involution adapted to h on $O_1 \times O_2 \times O_3 \subset (0, \infty)^3$. Assume that (R^1, R^2, Y) are non-degenerate independent random variables taking values in $O_1 \times O_2 \times O_3$.*

(a) If $(a, b) = (0, 1)$, then (R^1, R^2, Y) is $T^{(0,1)}$-invariant if and only if $\big(R^1, R^2, Y\big)$ is distributed as in (1.7)

(b) If $(a, b) = (1, 0)$, then (R^1, R^2, Y) is $T^{(1,0)}$-invariant if and only if $\big(R^1, R^2, Y\big)$ is distributed as in (1.8)

(c) If $(a, b) = (1, -1)$, then (R^1, R^2, Y) is $T^{(1,-1)}$-invariant if and only if either $\big(R^1, R^2, Y\big)$ or $(R^2, R^1, 1 - Y)$ is distributed as in (1.9)

(d) If $(a, b) = (-1, 1)$, then (R^1, R^2, Y) is $T^{(-1,1)}$-invariant if and only if $\big(R^1, R^2, Y\big)$ is distributed as in (1.10)*.*

Proof. Observe that $T_3^{(a,b)}$ has no y-dependence. Thus, by Lemma 2.4, $T^{(a,b)}$ is the unique polymer involution adapted to h on $O_1 \times O_2 \times O_3$. By Lemma 2.6, (R^1, R^2, Y) is $T^{(a,b)}$-invariant if and only if the following two properties hold:

(i) $\frac{R^2}{R^1}$ is independent of $T_3^{(a,b)}(R^1, R^2)$.

(ii) $Y \overset{d}{=} T_3^{(a,b)}(R^1, R^2)$.

Recall that
$$T_3^{(a,b)}(R^1,R^2)=\frac{R^1(R^2-a)}{R^2+bR^1}.$$

We now prove (a). Put $(A,B):=\big((R^1)^{-1},(R^2)^{-1}\big)$. Then (A,B) are non-degenerate independent positive random variables. Now
$$\frac{R^2}{R^1}=\frac{A}{B}\quad\text{and}\quad T_3^{(0,1)}(R^1,R^2)=(A+B)^{-1}.$$
So (i) holds if and only if $A/(A+B)=(1+B/A)^{-1}$ is independent of $A+B$. By Theorem 5.2 this occurs if and only if there exist positive constants $\lambda_A,\lambda_B,\beta$ such that $(A,B)\sim\text{Ga}(\lambda_A,\beta)\otimes\text{Ga}(\lambda_B,\beta)$. In such a case, $A+B=C\sim\text{Ga}(\lambda_A+\lambda_B,\beta)$. Thus $T_3^{(0,1)}(R^1,R^2)=(A+B)^{-1}\sim\text{Ga}^{-1}(\lambda_A+\lambda_B,\beta)$. Now put $(\mu,\lambda)=(\lambda_A+\lambda_B,\lambda_B)$ and use (ii) to get $(R^1,R^2,Y)\sim\text{Ga}^{-1}(\mu-\lambda,\beta)\otimes\text{Ga}^{-1}(\lambda,\beta)\otimes\text{Ga}^{-1}(\mu,\beta)$. This completes the proof of (a).

We now prove (b). Notice that $D^-_{1,0}=\varnothing$. Therefore by Lemma 5.4 we have that (R^1,R^2,Y) takes values in $D^+_{1,0}=(0,\infty)\times(1,\infty)\times(0,\infty)$. Put $(A,B):=\big(R^1,(R^2)^{-1}\big)$. Then (A,B) are non-degenerate independent random variables taking values in $(0,\infty)\times(0,1)$. Now
$$\frac{R^2}{R^1}=\frac{1}{AB}\quad\text{and}\quad T_3^{(1,0)}(R^1,R^2)=A(1-B).$$
So (i) holds if and only if AB is independent of $A(1-B)$. By Corollary 5.3, this occurs if and only if there exist positive constants $\lambda_A,\lambda_B,\beta$ such that $(A,B)\sim\text{Ga}(\lambda_A+\lambda_B,\beta)\otimes\text{Be}(\lambda_A,\lambda_B)$. In such a case, $T_3^{(1,0)}(R^1,R^2)=A(1-B)=D\sim\text{Ga}(\lambda_B,\beta)$. Now put $(\mu,\lambda)=(\lambda_B,\lambda_A)$ and use (ii) to get $(R^1,R^2,Y)\sim\text{Ga}(\mu+\lambda,\beta)\otimes\text{Be}^{-1}(\lambda,\mu)\otimes\text{Ga}(\mu,\beta)$. This completes the proof of (b).

We now prove (c). By Lemma 5.4, (R^1,R^2,Y) either takes values in
$$D^+_{1,-1}=(0,1)\times(1,\infty)\times(0,1)\quad\text{or}\quad D^-_{1,-1}=(1,\infty)\times(0,1)\times(0,1).$$
First consider the case when (R^1,R^2,Y) takes values in $D^+_{1,-1}$. Put $(A,B):=\big((R^2)^{-1},R^1\big)$. Then (A,B) are non-degenerate independent random variables, both taking values in $(0,1)$. Now
$$\frac{R^2}{R^1}=\frac{1}{AB}\quad\text{and}\quad T_3^{(1,-1)}(R^1,R^2)=1-\frac{1-B}{1-AB}.$$
So (i) holds if and only if $1-AB$ is independent of $(1-B)/(1-AB)$. By Theorem 5.1, this occurs if and only if there exist positive constants p,q,r such that $(A,B)\sim\text{Be}(p,q)\otimes\text{Be}(p+q,r)$. In such a case, $1-T_3^{(1,-1)}(R^1,R^2)=(1-B)/(1-AB)=C\sim\text{Be}(r,q)$. Thus $T_3^{(1,-1)}(R^1,R^2)\sim 1-\text{Be}(r,q)=\text{Be}(q,r)$. Now put $(\mu,\lambda,\beta)=(q,p,r)$ and use (ii) to get $(R^1,R^2,Y)\sim\text{Be}(\mu+\lambda,\beta)\otimes\text{Be}^{-1}(\lambda,\mu)\otimes\text{Be}(\mu,\beta)$.

In the case where (R^1,R^2,Y) takes values in $D^-_{1,-1}$, applying the reflection procedure as in (4.1), one can check that $\widehat{T}=T^{(1,-1)}$ and the resulting random variables $\big(\widehat{R}^1,\widehat{R}^2,\widehat{Y}\big)=(R^2,R^1,1-Y)$ take values in $D^+_{1,-1}$. By the first case, we are done. This completes the proof of (c).

We now prove (d). Notice that $D^-_{-1,1}=\varnothing$. Therefore by Lemma 5.4 (R^1,R^2,Y) must take values in $D^+_{-1,1}=(1,\infty)\times(0,\infty)\times(1,\infty)$. Put $(A,B):=(1-(R^1)^{-1},\,1-(R^2+1)^{-1})$. Then (A,B) are non-degenerate independent random variables, both taking values in $(0,1)$. Therefore
$$\left(1+\frac{R^2}{R^1}\right)^{-1}=\frac{1-B}{1-AB}\quad\text{and}\quad T_3^{(-1,1)}(R^1,R^2)=\frac{1}{1-AB}.$$
So (i) holds if and only if $(1-B)/(1-AB)$ is independent of $1-AB$. By Theorem

5.1, this occurs if and only if there exist positive constants p, q, r such that $(A, B) \sim \text{Be}(p, q) \otimes \text{Be}(p+q, r)$. In such a case, $T_3^{(-1,1)}(R^1, R^2) = (1-AB)^{-1} = D^{-1} \sim \text{Be}^{-1}(r+q, p)$. Now put $(\mu, \lambda, \beta) = (r+q, r, p)$ and use (ii) to get $(R^1, R^2, Y) \sim \text{Be}^{-1}(\mu-\lambda, \beta) \otimes (\text{Be}^{-1}(\lambda, \beta+\mu-\lambda) - 1) \otimes \text{Be}^{-1}(\mu, \beta)$. This completes the proof of (d). □

We now prove the second main result.

Proof of Theorem 1.4. When $h(y) = a + by$, for all fixed $(r_1, r_2) \in O_1 \times O_2$ the mapping $y \mapsto T_2^h(r_1, r_2, y) = y\big(\frac{r_2}{r_1} + b\big) + a$ is injective whenever $b \geqslant 0$. In the case $b < 0$ and $a > 0$ this injectivity follows from the assumption $-b \notin \big\{\frac{y}{x} : (x, y) \in O_1 \times O_2\big\}$. Therefore the conditions of Proposition 2.3-(a) are satisfied in all cases, which gives the existence of a unique polymer involution T^h adapted to $h(y) = a + by$ such that (R^1, R^2, Y) is T^h-invariant. By (2.4), $T^h = T^{(a,b)}$ as defined in (4.3). Now applying Lemma 4.1 then Proposition 5.5 completes the proof. □

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