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# Point-shift foliation of a point process

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**Abstract**

A point-shift $F$ maps each point of a point process $\Phi$ to some point of $\Phi$. For all translation invariant point-shifts $F$, the $F$-foliation of $\Phi$ is a partition of the support of $\Phi$ which is the discrete analogue of the stable manifold of $F$ on $\Phi$. It is first shown that foliations lead to a classification of the behavior of point-shifts on point processes. Both qualitative and quantitative properties of foliations are then established. It is shown that for all point-shifts $F$, there exists a point-shift $F_\perp$, the orbits of which are the $F$-foils of $\Phi$, and which is measure-preserving. The foils are not always stationary point processes. Nevertheless, they admit relative intensities with respect to one another.

**Keywords:** point process; stationarity; Palm probability; point-shift; point-map; allocation rule; mass transport principle; dynamical system; stable manifold.
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## 1 Introduction

A point process is said to be stationary if its distribution is invariant by the group of translations on $\mathbb{R}^d$. A point-shift is a dynamics on the support of a stationary point process, which is itself flow-adapted with respect to the translations of $\mathbb{R}^d$.

The main new objects of the paper are the notion of foliation of a stationary point process w.r.t. a flow-adapted point-shift.

Such a foliation is a discrete version of the global stable manifold (see e.g. [9] for the general setting and below for the precise definition used here) of this dynamics, i.e., two points in the support of the point process are in the same *leaf* or *foil* of this stable manifold if they have the same “long term behavior” for this dynamics. This foliation provides a flow-adapted partition of the support of the point process in connected components and foils.

The *point foil* of a point process w.r.t. a point-shift is defined under the Palm distribution of this point process. It is the random counting measure with atoms at the points

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of the foil of the origin. The distribution of the point foil under the Palm probability of the point process is left invariant by all bijective shifts preserving the foliation. A point foil is not always *markable*, i.e., is not always a stationary point process under its Palm distribution.

The main result of the paper is the classification of point-shifts based on the cardinalities of their foils and connected components (Theorem 22). This classification states that each connected component belongs to one of the following three classes:

1. Components with finitely many points, and where the dynamics from these points exhibits a periodic behavior;
2. Components with infinitely many points and infinitely many finite foils, and where there are points with infinitely many pre-images;
3. Components with infinitely many infinite foils, and where there is no point with infinitely many pre-images; namely, if one applies the point-shift infinitely many times to the point process, there are no points left.

Another important dichotomy is whether the point foils of a component are markable or not.

This classification is complemented by quantitative results on the relative intensities of foils (Theorem 45). The existence of these relative intensities follows from the remarkable fact that one can always navigate the foil of the origin in a measure-preserving way. Two foils having a positive and finite relative intensity can be seen as *having the same dimension*.

The literature on point-shifts starts with the seminal paper by J. Mecke [12]. The fundamental result of [12] is the *point stationarity theorem*, which states that all bijective point-shifts preserve the Palm distribution of all simple and stationary point processes. The notion of *point-map* was introduced by H. Thorisson (see [13] and the references therein) and further studied by M. Heveling and G. Last [8]. These objects arise in a variety of applications ranging from drainage networks [7] to routing in wireless networks [2]. The dynamical system analysis of point-shifts which is pursued in the present paper was proposed in [4]. The last paper is focused on long term properties of iterates of point-shifts. It introduced the notion of *point-map probability*, which provides an extension of Mecke's point stationarity theorem. In contrast, the present paper is focused on the stable manifold of a point-shift, as already mentioned. It is centered on the definition of this object and on the study of both qualitative and quantitative properties of its distribution.

The paper is structured as follows. Section 2 defines the setting for point processes and point-shifts and Section 3 that for discrete foliations. Section 4 combines the two frameworks and defines the discrete foliation of a point process by a point-shift. Section 5 gives the classification. Section 6 introduces the stable group of this foliation, and shows the existence of measure-preserving dynamics on the foliation. It also defines the foil point process. Finally, Section 7 gathers the quantitative properties of foliations.

## 2 Point processes and point-shifts

Whenever $(\mathbb{R}^d, +)$ acts on a space, the action of $t \in \mathbb{R}^d$ on that space is denoted by $\theta_t$. It is assumed that $(\mathbb{R}^d, +)$ acts on the reference probability space $(\Omega, \mathcal{F})$.

### 2.1 Counting measures and point processes

Let $\mathbf{N}$ be the space of all locally finite and simple counting measures on $\mathbb{R}^d$. It contains all measures $\phi$ on $\mathbb{R}^d$ such that for all bounded (relatively compact) Borel subsets $B$ of $\mathbb{R}^d$, $\phi(B) \in \mathbb{N}$ (counting measure condition) and for all $x \in \mathbb{R}^d$, $\phi(\{x\}) \leq 1$

(simplicity condition). Let $\mathcal{N}$ be the cylindrical $\sigma$-field on $\mathbf{N}$ generated by the functionals $\phi \mapsto \phi(B)$, where $B$ ranges over the elements of $\mathcal{B}$, the Borel $\sigma$-field of $\mathbb{R}^d$. The flow $\theta_t$ acts on counting measures as

$$(\theta_t \phi)(B) = \phi(B + t),$$

and therefore on $\mathbb{R}^d$ as $\theta_t x = x - t$.

Let $\mathbf{N}^0$ be the subspace of $\mathbf{N}$ of counting measures with an atom at the origin.

A *(random) point process* is a couple $(\Phi, \mathbb{P})$ where $\mathbb{P}$ is a probability measure on a measurable space $(\Omega, \mathcal{F})$ and $\Phi$ is a measurable mapping from $(\Omega, \mathcal{F})$ to $(\mathbf{N}, \mathcal{N})$.

If, for all $t \in \mathbb{R}^d$, $\Phi(\theta_t \omega) = \theta_t \Phi(\omega)$ (this is the *flow-adapted* assumption on $\Phi$) and $\theta_t \mathbb{P} = \mathbb{P}$, the point process $(\Phi, \mathbb{P})$ is called *stationary*. This implies

$$\begin{aligned}
&\forall B_1, \ldots, B_k \in \mathcal{B}, n_1, \ldots, n_k \in \mathbb{N}, \\
&\mathbb{P}[\Phi(B_1) = n_1, \ldots, \Phi(B_k) = n_k] = \mathbb{P}[\Phi(\theta_t B_1) = n_1, \ldots, \Phi(\theta_t B_k) = n_k].
\end{aligned}$$

Note that being flow-adapted is a property of $\Phi$, while being stationary is a property of $(\Phi, \mathbb{P})$.

When the point process $(\Phi, \mathbb{P})$ has a finite and positive intensity, its Palm probability [5] is denoted by $\mathbb{P}_\Phi$. Expectation w.r.t. $\mathbb{P}_\Phi$ is denoted by $\mathbb{E}_\Phi$.

### 2.2 Flow-adapted point-shifts

A point-shift on $\mathbf{N}$ is a measurable function $F$ which is defined for all pairs $(\phi, x)$, where $\phi \in \mathbf{N}$ and $x \in \phi$, and satisfies the relation $F_\phi(x) \in \phi$.

In order to define flow-adapted point-shifts, it is convenient to use the notion of point-map. A measurable function $f$ from the set $\mathbf{N}^0$ to $\mathbb{R}^d$ is called a *point-map* if for all $\phi$ in $\mathbf{N}^0$, $f(\phi)$ belongs to $\phi$.

If $f$ is a point-map, the associated *flow-adapted point-shift*, $F = F_f$, is a function which is defined for all pairs $(\phi, x)$, where $\phi \in \mathbf{N}$ and $x \in \phi$, and

$$F_\phi : \phi \to \phi, \quad F_\phi(x) := f(\theta_x \phi) + x.$$

Note that with abuse of notation, $F_f$ and $F_\phi$ were used with different meanings. The two can be distinguished from the context depending on whether $f$ is a point-map or $\phi$ a counting measure. It is easy to verify that the point-shift $F$ is flow-adapted. In the rest of this article, point-shift always means flow-adapted point-shift. Point-shifts will be denoted by capital letters and the point-map of a given point-shift will be denoted by the associated small letter ($F$'s point-map is hence denoted by $f$).

**Remark 1.** The term *flow-adapted* is borrowed from [11]. The term *compatible* is also used for the same notion in the literature ([3] and [4]). Since the former is more common in dynamical systems, *flow-adapted* will be used in the present paper.

For all $n \geq 0$, all $\phi \in \mathbf{N}$ and $x \in \phi$, the *$n$-th order iterate* of the point-shift $F$ is defined inductively by $F^0_\phi(x) = x$ and

$$F^{k+1}_\phi(x) = F_\phi(F^k_\phi(x)), \quad k \geq 0.$$

For all $n$, $F^n$ is a flow-adapted point-shift.

Let $m^n_f(\phi, y) = \mathrm{card}(F^{-n}_\phi(y))$. The *$n$-th image counting measure* (of $\phi$ by $F$) is then defined as the counting measure $F^n_\phi(\phi)$ with support $\{y \in \phi; F^{-n}_\phi(y) \neq \emptyset\}$, and such that the multiplicity of $y$ in the support of $F^n_\phi(\phi)$ is $m^n_f(\phi, y)$. Notice that $F^n_\phi(\phi)$ is not simple in general.

### 2.3 Examples

This subsection introduces a few basic examples which will be used to illustrate the results below. These examples will be based on two types of point processes: Poisson point processes and Bernoulli grids. The latter are defined as follows: it is well known that the $d$ dimensional lattice $\mathbb{Z}^d$ can be transformed into a stationary point process in $\mathbb{R}^d$ by a uniform random shift of the origin in the $d$ unit cube. The *Bernoulli grid* of $\mathbb{R}^d$ is obtained in the same way when keeping or discarding each of the lattice points independently with probability $p$. The result is again a stationary point process whose distribution will be denoted by $\mathcal{P}_p$.

#### 2.3.1 Strip point-shift

The Strip Point-Shift was introduced by Ferrari, Landim and Thorisson [6]. For all points $x = (x_1, x_2)$ in the plane, let $St(x)$ denote the half strip $(x_1, \infty) \times [x_2 - 1/2, x_2 + 1/2]$. Then $S(x)$ is the left most point of $\phi$ in $St(x)$. It is easy to verify that $S$ is flow-adapted. Denote its point-map by $s$.

The strip point-shift is not well-defined when there are more than one left most point in $St(x)$, nor when the point process has no point in $St(x)$. Note that such ambiguities can always be removed, and some refined version of the strip point-shift can always be defined by fixing, in some flow-adapted manner, the choice of the image and by choosing $f_\phi(x) = x$ in the case of non-existence. By doing so one gets a refined point-shift defined for all $(\phi, x)$.

#### 2.3.2 Mutual nearest neighbor point-shift

The Mutual Nearest Neighbor point-shift was defined by Olle Häggström. Two points $x$ and $y$ in $\phi$ are mutual nearest neighbors if $x$ (resp. $y$) is the closest point of $\phi$ to $y$ (resp. $x$). The Mutual Nearest Neighbor Point-Shift $N$ is the involution which maps $x$ to $y$ when these two points are mutual nearest neighbors and maps $z$ to itself if $z$ has no mutual nearest neighbor. This point-shift is bijective.

#### 2.3.3 Drainage point-shift on the Bernoulli grid

The *Drainage* point-shift, which is variant of drainage network [7], is denoted by $D$ and is defined on the 2-dimensional Bernoulli grid as follows:

$$D_\phi(x_1, x_2) = (x_1', x_2 - 1),$$

where

$$x_1' = \min\{y \geq x_1; (y, x_2 - 1) \in \phi\}.$$

It is easy to verify that $0 < p < 1$, $R$ is a.s. well-defined.

#### 2.3.4 Condenser point-shifts

Assume each point $x \in \phi$ is marked with

$$m_c(x) = \#(\phi \cap B(x, 1)).$$

Note that $m_c(x)$ is always positive. The *condenser point-shift* acts on marked point processes as follows: it goes from each point $x \in \phi$ to the closest point $y$ with a larger first coordinate such that $m_c(y) = m_c(x) + 1$. It is easy to verify that the condenser point-shift is flow-adapted and almost surely well-defined on the homogeneous Poisson point process.

### 2.4 On finite subsets of point process supports

This subsection contains some of the key technical results to be used in the forthcoming proofs.

A *disjoint subset collection of counting measures*, is a map $\mathcal{T}$ defined on a subset $A$ of $\mathbf{N}(\mathbb{R}^d)$, which associates to each $\phi \in A$, a collection $\mathcal{T}(\phi) = \{T_i(\phi); i \in I\}$, where $I$ is a countable index set, of pairwise disjoint non-empty subsets of $\phi$.

A disjoint subset collection $\mathcal{R}$ is called an *inclusion* of $\mathcal{T}$ if for each $T_i \in \mathcal{T}$ there exists a unique $R_i \in \mathcal{R}$ such that $R_i \cap T_i \neq \emptyset$ and in addition $R_i \subset T_i$. With abuse of notation, $R_i$ is referred as the *inclusion* of $T_i$.

Note that a disjoint collection can be seen as a *partition of counting measures* by adding $\phi \setminus \cup_{i \in I} T_i(\phi)$ as another element to the collection. Therefore with abuse of notation the term partition will also be used to refer to a disjoint subset collection.

If $A \subset \mathbf{N}(\mathbb{R}^d)$ is closed under the action of $\mathbb{R}^d$, the partition $\mathcal{T}$ on the elements of $A$ is called *flow-adapted* if for all $\phi \in A$, and all $t \in \mathbb{R}^d$,

$$\mathcal{T}(\phi) = \{T_i(\phi),\ i \in I\} \Rightarrow \mathcal{T}(\theta_t \phi) = \{\theta_t T_i(\phi); i \in I\}.$$

One of the simplest cases of flow-adapted partitions is the *singleton partition*; i.e.,

$$\mathcal{T}(\phi) = \{\{t\}; t \in \phi\}.$$

**Theorem 2.** Let $(\Phi, \mathbb{P})$ be a stationary point process and $\mathcal{T}$ be a flow-adapted partition of the point process. If $\mathcal{R}$ is a flow-adapted inclusion of $\mathcal{T}$, then almost surely $T_i \in \mathcal{T}$ is finite if and only if its inclusion, $R_i$, is finite.

*Proof.* For $x \in \phi$, let $T_x := T_x(\phi)$ denote the element of $\mathcal{T}$ which includes $x$ and $R_x := R_x(\phi)$ denote the inclusion of $T_x$. Let $\mathbb{P}_\Phi$ be the Palm probability of the point process. It is sufficient to prove almost surely $T_0$ is finite if and only if $R(0)$ is finite.

Clearly if $T_0$ is finite so is $R_0$. To prove the converse, define the mass transport $w$ (see Lemma 51)

$$w_\phi(x, y) = \begin{cases} \mathbf{1}_{x=y} & T_x \text{ is finite or } R_x \text{ is infinite,} \\ \mathbf{1}_{y \in R_x}/|R_x| & T_x \text{ is infinite and } R_x \text{ is finite.} \end{cases} \tag{2.1}$$

The mass transport principle implies

$$\mathbb{E}_\Phi[w^-(0)] = \mathbb{E}_\Phi[w^+(0)] = \mathbb{E}_\Phi[1] = 1.$$

Therefore $w^-(0)$ is $\mathbb{P}_\Phi$-a.s. finite and hence $\mathbb{P}_\Phi$-a.s. there is no point satisfying the second case of (2.1), which completes the proof. □

**Remark 3.** As mentioned earlier, a flow-adapted disjoint collection can be completed to a flow-adapted partition and hence the statement of Theorem 2 is also valid when $\mathcal{T}$ is a flow-adapted disjoint collection and $\mathcal{R}$ is a flow-adapted inclusion of $\mathcal{T}$.

**Corollary 4.** Let $\mathcal{T}$ be the partition with a single element including all points of the point process. Theorem 2 gives that one cannot choose a finite non-empty subset of a stationary point process in a flow-adapted manner.

Following the notation of the proof of Theorem 2, for a flow-adapted partition $\mathcal{T}$, one can consider $T_0 : \mathbf{N}^0 \to \mathbf{N}^0$ which maps $\phi$ to $T_0(\phi)$. It is easy to see that each flow-adapted partition $\mathcal{T}$ of counting measures is fully characterized by the measurable map $T_0 : \mathbf{N}^0 \to \mathbf{N}^0$. Indeed,

$$T_t(\phi) = \theta_{-t} T_0(\theta_t \phi) = t + T_0(\theta_t \phi). \tag{2.2}$$

An enumeration of the elements of a set is an injective function $\nu$ from from this set to $\mathbb{N}$ (or equivalently to $\mathbb{Z}$). There are several enumerations of the elements of the partition $\mathcal{T}$; e.g., based on the distance to the origin. Any element $T$ of the partition is a countable collection of points of $\phi$. Since $\phi$ has no accumulation points, one can define the distance of $T$ to the origin as the minimum of the distances from the points of $T$ to the origin. If the set of distances of the sets of the partition to the origin are all different, one defines $T_0$ as the element of the partition with the smallest distance to the origin, $T_1$ as the one with the second smallest distance to the origin, and so on. Ties are treated in the usual way, e.g. by using lexicographic ordering. Note that this enumeration is not flow-adapted.

A natural question is about the existence of translation invariant enumerations. This is not always granted. For example, it is well known, and can be seen from Corollary 4, that the singleton partition of a stationary point process $(\Phi, \mathbb{P})$ cannot be enumerated in a measurable and flow-adapted manner.

**Definition 5.** A flow-adapted partition of a stationary point process will be said to be *markable* if there exists an enumeration of the elements of the partition which is invariant by translations.

The reason for this terminology is that if the partition is defined by a selection of the points of $\Phi$ based on marks (see e.g. [5] for the definition of marks of a point process), then such an enumeration exists[1]. For instance, the singleton partition of a stationary point process is flow-adapted but is not a markable partition.

**Definition 6.** Let $\mathcal{T}$ be a flow-adapted partition of the support of $\Phi$. Let $H$ be a point-shift. One says $H$ *preserves* $\mathcal{T}$ if for all $T \in \mathcal{T}$, $H^{-1}(T) = T$. If $H$ is bijective, this is equivalent to the property that for all $T \in \mathcal{T}$, $H(T) = T$.

**Definition 7.** Let $\Gamma_{\mathcal{T}} := \Gamma_{\mathcal{T}}(\Phi)$ be the set of all bijective and $\mathcal{T}$-preserving point-shifts. The set $\Gamma_{\mathcal{T}}$ can be equipped with a group structure by composition of point-shifts. This group, which is as subgroup of the symmetric group on the support of $\Phi$, is called the *$\mathcal{T}$-stable group*. An element $H$ of this $\mathcal{T}$-stable group is said *$\mathcal{T}$-dense* if $\mathbb{P}$-almost surely, for all $x \in \phi$, the orbit of $x$ under $H$ spans the whole set of the partition that contains $x$; i.e.,

$$\{H^n(x); n \in \mathbb{Z}\} = \mathcal{T}_x(\phi).$$

# 3 Discrete foliations

## 3.1 Foils and connected components

The notion of discrete foliation can be defined for any function on any set. Since the present paper is focused on stochastic objects, only measurable functions on measurable spaces will be considered.

Assume $(X, \mathcal{F})$ is a measurable space where all singletons are measurable; i.e., for all $x \in X$ one has $\{x\} \in \mathcal{F}$ and let $g$ be a measurable map (or dynamics) on $X$. Note that in general $X$ is not a topological space[2].

[1] In fact the following result holds: there exists an enumeration invariant by translation if and only if there exists a decomposition of the stationary point process into a collection of stationary sub-point processes with disjoint supports and with positive intensities. The proof of this result is skipped as it will not be used below.

[2] When $X$ is a topological space and $g$ is continuous, $g$ defines a topological dynamical system; when $X$ is equipped with a probability measure which is preserved by $g$, the latter defines a measure preserving dynamical system.

**Definition 8.** The graph $G^g = G^g(X) = (V, E)$ has for set of vertices $V = X$ and $E = \{(x, g(x)), x \in X\}$ for set of edges[3]. Note that this graph can be considered either as undirected or as directed, with each edge from $x$ to $g(x)$.

For $x \in X$, denote by $C^g(x)$ the undirected connected component of $G^g$ which contains $x$; i.e., the set of all points $y \in X$ for which there exist non-negative integers $m$ and $n$ such that $g^m(x) = g^n(y)$. The set of connected components of $G^g$ will be denoted by $\mathcal{C}^g(X)$.

$C^g(x)$ will be said to be *$g$-acyclic*, if the restriction of $G^g$ to $C^g(x)$ is a tree.

**Lemma 9.** The connected component $C = C^g(x)$ of $G^g$ is either an infinite tree or it has exactly one (directed) cycle $K(C)$; in the latter case, for all $y \in C$, there exists $n \in \mathbb{N}$ such that $g^n(y) \in K(C)$.

*Proof.* All statements follow from the fact that all vertices of $C$, seen as a directed graph, have out-degree equal to one and from the fact that $C$ is connected (as an undirected graph). □

**Remark 10.** If for all $x \in X$, $\mathrm{Card}(g^{-1}(x))$ is finite, then $C^g(x)$ is countable.

Whenever it is clear from the context, the superscript $g$ is dropped.

Connected components can be partitioned into finer sets. Let $\sim_g$ be the binary relation on the elements of $X$ defined by

$$x \sim_g y \Leftrightarrow \exists n \in \mathbb{N}; g^n(x) = g^n(y).$$

It is immediate that $\sim_g$ is an equivalence relation.

**Definition 11.** The partition of $X$ generated by the equivalence classes of $\sim_g$ will be called the *$g$-foliation of $X$*. Denote it by $\mathcal{L}^g(X)$ or $\mathcal{L}^g_X$. Each equivalence class is called a *foil*. The equivalence class of $x \in X$ is denoted by $L^g(x)$.

**Remark 12.** In the terminology of geometry, foils are called leaves. But since the paper uses graphs which are mostly trees, to avoid confusion with tree leaves, the word foil will be used here.

**Remark 13.** If $x \sim_g y$ then $x$ and $y$ are in the same connected component of $C^g(x)$. In other words, the foliation is a subdivision of $\mathcal{C}^g(X)$.

One can also see $L^g(x)$ as the limit of the increasing sets $L^g_n(x)$, where

$$L^g_n(x) := \{y \in X; g^n(y) = g^n(x)\}.$$

The cardinality of $L^g(x)$ (resp. $L^g_n(x)$) will be denoted by $l^g(x)$ (resp. $l^g_n(x)$).

For reasons that will be explained below, the class of $g(x)$, namely $L^g(g(x))$ will be denoted by $L^g_+(x)$. If there exists a point $y \in X$ such that $g(y) \in L^g(x)$, $L^g(y)$ is denoted by $L^g_-(x)$. One can verify that $L^g_+(x)$ is well-defined and that both $L^g_-(x)$ and $L^g_+(x)$ are *class* objects; i.e., they do not depend on the choice of the element of the equivalence class.

**Remark 14.** For a homeomorphism $g$ on a metric space, the *stable manifold* of a point $x \in X$ with respect to $g$ is

$$W^s(g, x) = \{y \in X; \lim_{n\to\infty} d(g^n(x), g^n(y)) = 0\}.$$

Hence, in the case where the space $X$ is equipped with a discrete metric, the stable manifold foliation is the $g$-foliation of $X$ as defined above. This explains the chosen terminology.

[3] In all cases to be considered, the connected components of $G^g$ will always have a countable collection of nodes and a finite degree, even when $X$ is not countable; see the next remark.

The measurability of $g$ implies all foils are measurable subsets of $X$.

A partition $\mathcal{L}$ of $X$ into measurable sets is called $g$-invariant if for all $L \in \mathcal{L}$

$$g^{-1}(L) = \{x \in X; g(x) \in L\} \in \mathcal{L},$$

provided that $g^{-1}(L) \neq \emptyset$.

### 3.2 Foil order

The $g$-foliation of each connected component of $X$ can be equipped with some form of order. Consider $g(x)$ as the *father* of $x$. Then $L^g(x)$ denotes the $g$-generation of $x$ i.e., the set of its $g$-cousins of all orders; $L^g_n(x)$ denotes the set of its $g$-cousins with common $n$-th $g$-ancestor. In addition, $L^g_+(x)$ is the $g$-generation *senior* to $x$'s, i.e., that of its father, whereas $L^g_-(x)$ (if it exists) is the $g$-generation *junior* to $x$'s, i.e., that of its sons (if any) or that of the sons of its cousins (again if any).

**Definition 15.** Note that if $C(x)$ is acyclic, this definition of generations gives a linear order on the foils of $C(x)$ which is that of seniority: by definition $L^g(y) < L^g_+(y)$ for all $y \in C(x)$. This order is then similar to the order of either $\mathbb{Z}$ or $\mathbb{N}$ (total order with either no minimal element or with a minimal element).

Note that $g^n(X)$ is a sequence of decreasing sets in $n$. Its limit (which may be the empty set) is denoted by $g^\infty(X)$.

**Definition 16.** Let $n$ be a positive integer. For all $x \in X$, let $D_n(x) = D_n(g, x)$ be the set of all descendants of $x$ which belong to the $n$-th generation w.r.t. $x$; i.e.,

$$D_n(x) := \{y \in X; g^n(y) = x\}.$$

The cardinality of $D_n(x)$ (which may be zero, finite or infinite) is denoted by $d_n(x)$. Also, let $D(x) = D(g, x)$ denote the set of all descendants of $x$; i.e.,

$$D(x) := \{y \in X; \exists\, n \geq 0 : g^n(y) = x\} = \bigcup_{n=1}^{\infty} D_n(x).$$

Finally the cardinality of $D(x)$ is denoted by $d(x)$.

## 4 Point-map foliations

This section introduces two dynamics associated with a flow-adapted point-shift $F = F_\phi$ (or equivalently to its associated point-map $f$) and discuss the associated foliations. Dynamics 1 will be used for the classification result and Dynamics 2 will be used for the results of Section 7.

1. For all fixed $\phi \in \mathbf{N}$, consider the map $g = F_\phi$, from the discrete space $\mathrm{support}(\phi)$ to itself. The $F_\phi$-foliation of $\phi$ is a partition of the set $\mathrm{support}(\phi)$. It will be denoted by $\mathcal{L}^{F_\phi}_\phi$. The set of connected components will be denoted by $\mathcal{C}^{F_\phi}_\phi$. Whenever the context allows it, the subscript $\phi$ is dropped, so that the latter set is denoted by $\mathcal{C}^F$ and the former by $\mathcal{L}^F$.
2. $(\mathbf{N}^0, \theta_f)$: for all $\phi \in \mathbf{N}^0$, let $g(\phi) = \theta_f \phi := \theta_{f(\phi)}\phi$. The map $\theta_f$ is a measurable dynamics on $\mathbf{N}^0$, a non-discrete measure space. The definition of the $\theta_f$-foliation is nevertheless that of Definition 11[4]. The reason for this choice of definition is given in Corollary 17 below. The associated foliation (resp. set of connected

[4] Rather than that of the stable manifold alluded to in Remark 14

components) is denoted by $\mathcal{L}^{\theta_f}_{\mathbf{N}^0}$ or simply by $\mathcal{L}^{\theta_f}$ (resp. $\mathcal{C}^{\theta_f}_{\mathbf{N}^0}$ or $\mathcal{C}^{\theta_f}$). Note that this partition of $\mathbf{N}^0$ is very different in nature from that discussed for dynamics 1 above: each connected component of $\mathcal{L}^{\theta_f}$ (and hence each foil or each component of the graph $G^{\theta_f}$) is still discrete, whereas $\mathbf{N}^0$ is a non-countable set. So the number of connected components of this foliation must be non-countable.

Although $\mathcal{L}^F$ and $\mathcal{L}^{\theta_f}$ are defined on different spaces, they are closely related because of the following statement, which follows from the compatibility of the point-shift $F$.

**Corollary 17.**

$$x \sim_{F_\phi} y \Leftrightarrow \theta_x\phi \sim_{\theta_f} \theta_y\phi. \tag{4.1}$$

**Example 18.** Consider the Drainage point-shift $D$ on the 2-dimensional Bernoulli grid defined in Subsection 2.3.3. If $p \in (0,1)$, one can show that

$$L^D(x_1,x_2) = \{(y,x_2) \in \Phi\},$$

and

$$C^D(x_1,x_2) = \{(y_1,y_2) \in \Phi\}.$$

Thus each foil looks like a 1-dimensional Bernoulli grid and each connected component looks like a 2-dimensional Bernoulli grid (Figure 1). If $d = 2$, the graph $G^R$ has a single connected component.

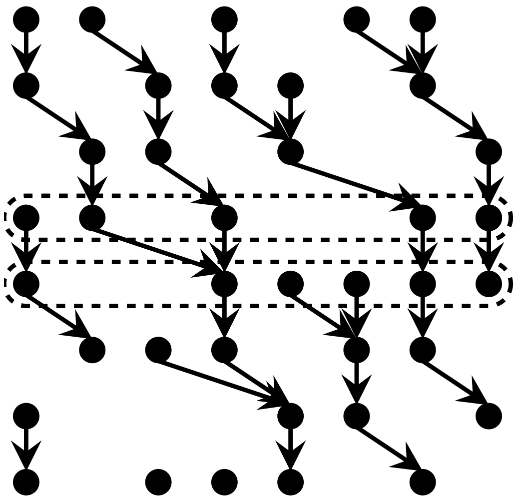

Figure 1: The Drainage point-shift on 2-dimensional Bernoulli grid. The dashed lines indicate two foils of this point-shift.

**Example 19.** Consider the Strip Point-Shift of Subsection 2.3.1 on the 2-dimensional Poisson point process. There is a single connected component. The foil of the origin is depicted in Figure 2.

Consider now a stationary point process $(\Phi,\mathbb{P})$, with Palm version denoted by $(\Phi,\mathbb{P}_\Phi)$. The expectation with respect to $\mathbb{P}$ (resp. $\mathbb{P}_\Phi$) is denoted by $\mathbb{E}$ (resp. $\mathbb{E}_\Phi$). The above dynamics lead to the following stochastic objects:

1. $F_\Phi$ is a random map from the discrete random set $\mathrm{support}(\Phi)$ to itself. Both the component partition $\mathcal{C}^F = \mathcal{C}^{F_\Phi}_\Phi$ and the foil partition $\mathcal{L}^F = \mathcal{L}^{F_\Phi}_\Phi$ are flow-adapted partitions of this random set, with the latter being a refinement of the former.

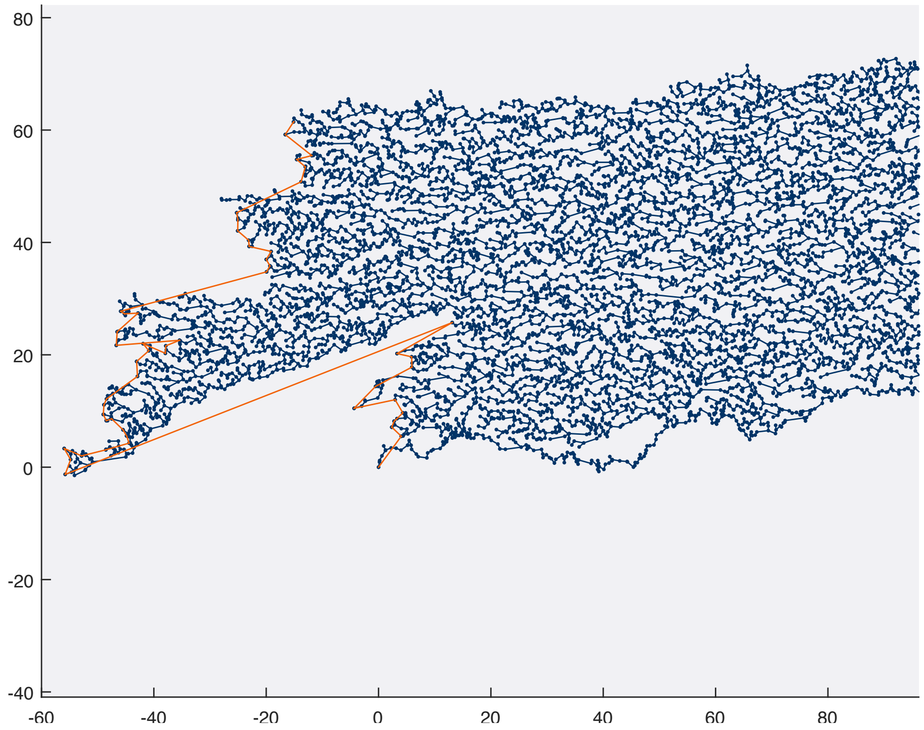


Figure 2: The strip point-shift on the 2-dimensional Poisson point process. The graph in blue is the restriction of the graph of the point-shift to the points of older generations (see Remark 21) than the foil of the origin and larger than the origin in the total order of the foil (see Subsection 6.2). The half-foil of the origin is in red.

2. $\mathcal{L}^{\theta_f} = \mathcal{L}^{\theta_f}_{\mathbf{N}^0}$ is a deterministic partition of the whole set $\mathbf{N}^0$ (in contrast to the random partition of a random set described above). Note however that it is sufficient that $\theta_f$ be defined $\mathbb{P}_\Phi$-almost surely and hence it may be undefined for some elements of $\mathbf{N}^0$ of null measure for $\mathbb{P}_\Phi$.

Here are some observations on the flow-adapted partitions $\mathcal{C}^F$ and $\mathcal{L}^F$. These two partitions do not depend on $\mathbb{P}$ at all (since they are defined on realizations). In particular, they do not depend on whether the point process is considered under $\mathbb{P}$ or $\mathbb{P}_\Phi$.

The elements of each of these two partitions can be enumerated in a natural way following the method discussed just before Definition 5.

The dichotomy of Section 2.4 applies: there are cases where $\mathcal{L}^F$ (resp. $\mathcal{C}^F$) is a markable partition and cases where it is not[5]. A simple instance of the latter case is obtained when $F$ is bijective; then the foil partition coincides with the singleton partition, which is not a markable partition.

The following solidarity properties hold:

**Proposition 20.** If the foil partition $\mathcal{L}^F$ is markable, so is the component partition $\mathcal{C}^F$. Conversely, if $C$ is a component which is the support of a flow-adapted point process $\Xi$, then either the foil partition of $\Xi$, $\mathcal{L}^{F_\Xi}_\Xi$ is markable or there is no stationary point process with a positive intensity having for support a foil of $\mathcal{L}^{F_\Xi}_\Xi$.

[5]The partition $\mathcal{L}^{F_\Phi}_\Phi$ gathers points whose marks are in the same equivalence class w.r.t. some equivalence relation. This does not mean that this partition is markable.

*Proof.* The first assertion is immediate. The proof of the converse leverages the foil order introduced in Subsection 3.2. First observe that $\Xi$ has a single component. Assume that, for some foil $L$ of $\Xi$, the point process $\Psi(L)$ with support $L$ is a stationary point process. Then $\Psi(L_+) = \Psi(F(L))$ is a stationary point process with non empty support (as $L$ is non empty) and hence with positive intensity. Hence all foils that are senior to $L$ are stationary point processes with a positive intensity. Similarly, either $L_-$ is empty, and then there is no foil junior to $L$, or $\Psi(L_-) = \Psi(F^{-1}(L))$ is also a stationary point process with a positive intensity. It then follows that the foil partition of $\Xi$ is markable. $\square$

**Remark 21.** Under $\mathbb{P}_\Phi$, the foil order leads to a natural enumeration of the foils of the component of the origin. The foil of the origin is numbered 0 and will be denoted by $L^F(0) = L_0$, the foil senior (resp. junior) to it will be numbered 1 and will be denoted by $L_1$ (resp. $L_{-1}$ if non empty), and so on. Note that this enumeration is not flow-adapted.

## 5 Point-map cardinality classification

In the rest of this work $(\Phi, \mathbb{P})$ is a stationary point process such that $\Phi(\mathbb{R}^d) = \infty$, $\mathbb{P}$ a.s. Its Palm version will be denoted by $(\Phi, \mathbb{P}_\Phi)$. Note that the point process is always assumed to have a finite and positive intensity.

The foliation $\mathcal{L}^F$ partitions the support of the point process $\Phi$ into a discrete set of connected components; each component is in turn decomposed in a discrete set of $F$-foils, and each foil in a set of points. The present subsection proposes a classification of point-maps based on the cardinality of these sets (Theorem 22).

### 5.1 Connected components

The cardinality classification of connected components of the two dynamics differ.

The partition $\mathcal{C}_\Phi^{F_\Phi}$ is countable. Its cardinality is a random variable with support on the positive integers and possibly infinite. If $(\Phi, \mathbb{P})$ is ergodic, this is a positive constant or $\infty$ almost surely.

As already mentioned, in contrast, the partition $\mathcal{C}^{\theta_f}$ is deterministic and non-countable in general.

### 5.2 Inside connected components

In view of Corollary 17, the cardinality classification of the foils belonging to a given connected component is the same for $\mathcal{L}_\Phi^{F_\Phi}$ and for $\mathcal{L}^{\theta_f}$.

It is easy to see that the cardinality of the set of foils of a component is a random variable with support in $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. The same holds true for the set of points of a non-empty foil. The following theorem shows that only a few combinations are however possible:

**Theorem 22** (Cardinality classification of a connected component)**.** Let $(\Phi, \mathbb{P})$ be a stationary point process. Then $\mathbb{P}$ almost surely, each connected component $C$ of $G^F(\Phi)$ is in one of the three following classes:

**Class $\mathcal{F}/\mathcal{F}$:** $C$ is finite, and hence so is each of its $F$-foils. In this case, when denoting by $1 \le n = n(C) < \infty$ the number of its foils:

- $C$ has a unique cycle of length $n$;
- $F^\infty(C)$ is the set of vertices of this cycle.

**Class $\mathcal{I}/\mathcal{F}$:** $C$ is infinite and each of its $F$-foils is finite. In this case:

- $C$ is acyclic;
- Each foil has a junior foil, i.e., a predecessor for the order of Definition 15;

- $F^{\infty}(C)$ is a unique *bi-infinite path*, i.e., a sequence of points $(x_n)_{n\in\mathbb{Z}}$ of $\phi$ such that $F_\phi(x_n) = x_{n+1}$ for all $n$.

**Class $\mathcal{I}/\mathcal{I}$:** $C$ is infinite and all its $F$-foils are infinite. In this case:

- $C$ is acyclic;
- $F^{\infty}(C) = \emptyset$.

The following definitions will be used:

**Definition 23.** Let $C$ be a connected component of $G^F(\Phi)$. The point-shift *evaporates* $C$ if $F^{\infty}_{\mathbf{G}}(C) = \emptyset$ almost surely.

It follows from Theorem 22 that:

**Corollary 24.** The point-shift $F$ evaporates $C$ if and only if $C$ is of Class $\mathcal{I}/\mathcal{I}$.

Before giving the proof of Theorem 22, a collection of preliminary results (Proposition 25 to Corollary 30) is presented.

**Proposition 25.** Let $(\Phi, \mathbb{P})$ be a stationary point process with Palm probability $\mathbb{P}_\Phi$. Let $d_n(0)$ and $d(0)$ be as in Definition 16 for the $F$-foliation of $\phi = \Phi(\omega)$. One has

$$\forall n \geq 0 : \mathbb{E}_\Phi \left[d_n(0)\right] = 1. \tag{5.1}$$

In particular, for all $n$, $d_n(0)$ is $\mathbb{P}_\Phi$-almost surely finite. If, in addition, $G^F(\Phi)$ is $\mathbb{P}_\Phi$-almost surely acyclic, then

$$\mathbb{E}_\Phi \left[d(0)\right] = \infty.$$

*Proof.* The map

$$w(\phi, x, y) := \mathbf{1}\{F^n_\phi(x) = y\}$$

is a flow-adapted mass transport (see [11]). The first statement is hence an immediate consequence of Proposition 51. For the second part, when $G^F$ is acyclic, the $D_n$-s form a partition of $D$ and hence

$$\mathbb{E}_\Phi \left[d(0)\right] = \sum_{n=1}^{\infty} \mathbb{E}_\Phi \left[d_n(0)\right] = \infty. \qquad \square$$

**Remark 26.** Note that $G^F(\Phi)$ is $\mathbb{P}_\Phi$-almost surely acyclic if and only if $G^F(\Phi)$ is $\mathbb{P}$-almost surely acyclic.

**Corollary 27.** Almost surely, if $l^F(x) = \infty$, then for all positive $n$, $l^F(F^n_\phi(x)) = \infty$.

*Proof.* Proposition 25 implies that the degrees of all vertices in $G^F(\Phi)$ are a.s. finite. Hence the claim. $\square$

**Corollary 28.** The point-shift $F_\phi$ is almost surely surjective on the support of $\phi$ if and only if it is almost surely injective.

*Proof.* If $F_\phi$ is surjective (resp. injective), then almost surely $d_1(0) \geq 1$ (resp. $d_1(0) \leq 1$). Since $\mathbb{E}_\Phi[d_1(0)] = 1]$, almost surely $d_1(0) = 1$, and hence the point-shift is bijective. $\square$

**Proposition 29.** The connected component $C$ of $G^F(\Phi)$ is acyclic if and only if it is infinite. Hence $G^F(\Phi)$ is acyclic if and only if it has no finite connected component.

*Proof.* According to Lemma 9 each connected component of $G^F(\phi)$ has at most one cycle. If the latter is finite, it possesses exactly one cycle. This proves the first statement.

Let $n = n(\Phi)$ be the number of connected components of $G^F(\Phi)$ which are infinite and possess a cycle. Let $\mathcal{T} = \{T_i\}_{i=1}^n$ denote the collection of such components. Note that $n$ may be infinite. According to Lemma 9, each $T_i$ has exactly one cycle. This cycle is a finite inclusion of $T_i$, which contradicts Theorem 2. Therefore almost surely, there is no such component. □

The next corollary follows from Lemma 9.

**Corollary 30.** If $G^F(\Phi)$ is almost surely connected, it is almost surely a tree.

*Proof of Theorem 22.* The result for connected components of Class $\mathcal{F}/\mathcal{F}$ is an immediate consequence of Lemma 9.

Assume $C$ is an infinite component. According to Proposition 29, $C$ is acyclic. Consider the collection of all connected components with both finite and infinite foils. Denote this collection by $\mathcal{T} = \{T_i\}_{i=1}^n$, where $n$ may be infinity. Corollary 27 implies that each $T_i$ should have a largest finite foil, say $R_i$, where the order is that based on seniority (Definition 15). Therefore, $R_i$ is a finite inclusion of $T_i$ which contradicts Theorem 2. So, almost surely, there is no connected component with both finite and infinite foils, which proves that each acyclic component is either of Class $\mathcal{I}/\mathcal{F}$ or $\mathcal{I}/\mathcal{I}$.

Let $C$ be a connected component of Class $\mathcal{I}/\mathcal{F}$. Almost surely, $C$ cannot have a smallest foil. Otherwise the latter would again be a finite flow-adapted inclusion of the infinite connected component $C$, which contradicts Theorem 2. This proves the second assertion on the foils of $C$ in this case. Now let $L_0$ be an arbitrary foil of $C$ and, for all integers $i$, let $L_i$ be the foil containing $F_\phi^i(L_0)$. Since $L_0$ is finite, there exists a least non-negative integer $n$ such that $F_\phi^n(L_0)$ is a single point. Let

$$C_0 := \{F_\phi^m(L_0),\ -\infty < m < n\},$$

The graph $G^F(C_0)$ is infinite, connected and all its vertices are of finite degree. It hence follows from König's infinity lemma [10] that $G^F(C_0)$ has an infinite path (or branch) $\{x_i\}_{i\le 0}$. For $i > 0$, define $x_i := F_\phi^i(x_0)$. Then $(x_i)_{i\in\mathbb{Z}}$ is a bi-infinite path. Clearly $(x_i)_{i\in\mathbb{Z}} \subset F_\phi^\infty(C)$. Since all edges of $G^F(C)$ are from a foil $L$ to the foil $L_+$, $(x_i)_{i\in\mathbb{Z}}$ has exactly one vertex in each foil. To prove $F_\phi^\infty(C) \subset (x_i)_{i\in\mathbb{Z}}$, assume there exists $y \in F_\phi^\infty(C)\backslash(x_i)_{i\in\mathbb{Z}}$. Then there exist at least two bi-infinite paths. Since all bi-infinite paths of $C$ intersect $L(y)$, which is finite, there is a point of $\phi$ through which all the bi-infinite paths pass. Since there are at least two infinite paths, this point is well-defined and is a finite inclusion of $C$, which contradicts Theorem 2. This completes the proof of the properties of Class $\mathcal{I}/\mathcal{F}$.

Consider now $C$ of Class $\mathcal{I}/\mathcal{I}$ and assume that $F_\phi^\infty(C)$ is not empty. If $x \in F_\phi^\infty(C)$, then $G^F(D(x))$ is an infinite connected graph with vertices of finite degree and hence it possesses an infinite path, which in turn gives a bi-infinite path using the same construction as what was described above for the Class $\mathcal{I}/\mathcal{F}$. Hence, in order to prove that $F_\phi^\infty(C)$ is empty, it is sufficient to show that $C$ has no bi-infinite path. If $C$ has finitely many bi-infinite paths, then the intersections of these bi-infinite paths with each foil of $C$ give a collection of finite inclusion of infinite sets, which contradicts Theorem 2. Consider now the case where $C$ has infinitely-many bi-infinite paths. Since $C$ is connected, each two bi-infinite paths should intersect at some point. Let $J = J(C)$ be the set of all points $x \in C$ such that at least two bi-infinite paths join at $x$. It is now shown that, almost surely, the intersection of a bi-infinite path and $J$ has neither a first nor a last point for the order induced by $F$. If it has a first (resp. last) point, then the part of the path before the first (resp. after the last) point is an infinite flow-adapted set with a finite flow-adapted inclusion, which contradicts Theorem 2. Therefore, for each point $x \in J$, there is a smallest positive integer $n(J,x)$ such that $F_\phi^{n(J,x)} \in J$. Now define a point-shift $h$ on the whole point process as follows

$$h(x) = \begin{cases} F_\phi^{n(J,x)} & x \in J(C) \\ x & \text{otherwise.} \end{cases}$$

Since the intersection of any bi-infinite path with $J$ does not have a first point, $h$ is almost surely surjective. But from the very definition of $J$, all points of this set have at least two pre-images, which contradicts Corollary 28. Hence the situation with infinitely-many bi-infinite paths is not possible either, which concludes the proof. □

In graph theoretic terms, one can summarize the results discussed in the last proof as follows:

**Corollary 31.** A Class $\mathcal{I}/\mathcal{I}$ component has one (positive) end. A Class $\mathcal{I}/\mathcal{F}$ component has two ends (a positive and a negative one).

Theorem 22 also has the following corollary:

**Corollary 32.** For all stationary point processes $(\Phi, \mathbb{P})$, for all point-shifts $F$, there exist three stationary point processes $(\Phi_{\mathcal{F}/\mathcal{F}}, \mathbb{P})$, $(\Phi_{\mathcal{I}/\mathcal{F}}, \mathbb{P})$ and $(\Phi_{\mathcal{I}/\mathcal{I}}, \mathbb{P})$ (which may be empty with positive probability), all defined on the same probability space, and such that

$$\Phi = \Phi_{\mathcal{F}/\mathcal{F}} + \Phi_{\mathcal{I}/\mathcal{F}} + \Phi_{\mathcal{I}/\mathcal{I}}.$$

All connected components of $G^F(\Phi_i)$ are of Class $i$, $i \in \{\mathcal{F}/\mathcal{F}, \mathcal{I}/\mathcal{F}, \mathcal{I}/\mathcal{I}\}$. If $(\Phi, \mathbb{P})$ is ergodic, then each of these point processes is also ergodic.

*Proof.* The statement is an immediate consequence of Theorem 22 and the fact that being a connected component of Class $i$ is a flow-adapted property. Thus if $\Phi_i$ is defined as the set of all points in components of Class $i$, $\Phi_i$ is flow-adapted. Stationarity and ergodicity depend only on $\mathbb{P}$ and the flow on the probability space and they are being carried to new point processes. □

### 5.3 Comments and examples

Here are a few observations on the cardinality classification.

A point process $(\Phi, \mathbb{P})$ can have a mix of components of all three classes.

If $(\Phi, \mathbb{P})$ only has $\mathcal{F}/\mathcal{F}$ components, then it should have an infinite number of connected components. An example of this situation is provided by the Mutual Nearest Neighbor Point-Shift $N$ on Poisson point process on $\mathbb{R}^2$ (see below).

If $(\Phi, \mathbb{P})$ only has $\mathcal{I}/\mathcal{F}$ components, then the cardinality of $\mathcal{C}_\Phi^{F_\Phi}$ may be finite or infinite. An example of the first situation is provided by the Royal Line of Succession Point-Shift on Poisson point processes on $\mathbb{R}^2$ (see below). An example of the latter is provided by the Strip Point-Shift $S$ on the Bernoulli grid of dimension 2.

If $(\Phi, \mathbb{P})$ only has $\mathcal{I}/\mathcal{I}$ components, then the cardinality of $\mathcal{C}_\Phi^{F_\Phi}$ may again be finite or infinite. An example of the first situation is provided by the Strip Point-Shift $S$ on Poisson point processes on $\mathbb{R}^2$. An example with infinite cardinality is provided in Subsection 8.3 of the appendix.

The end of this section gathers examples of the three classes.

#### 5.3.1 Class $\mathcal{F}/\mathcal{F}$

For the Mutual Nearest Neighbor Point-Shift $N$ on a Poisson point process, there is no evaporation and there is an infinite number of connected components, all of class $\mathcal{F}/\mathcal{F}$. The order of the foils is that of $\mathbb{Z}$ mod 2 or $\mathbb{Z}$ mod 1. The foil partition is the singleton partition, hence not markable. The connected component partition is not markable either.

#### 5.3.2 Class $\mathcal{I}/\mathcal{F}$

An example of this class is provided by the Strip point-shift on a stationary Poisson point process $(\Phi, \mathbb{P})$ of $\mathbb{R}^2$ and $\mathbb{R}^3$. This point-shift has a single connected component [6]. The RLS ordering (see the proof of Proposition 36) hence defines a total order on $(\Phi, \mathbb{P})$, which is equivalent to that of $\mathbb{Z}$. This allows one to define the RLS Point-Shift $F_{\text{rls}}$ which associates to $x \in \Phi$ the unique point $y \in \Phi$ such that $x$ comes next to $y$ in this total order. This point-shift is clearly bijective. Hence the foil of $x$ is $\{x\}$. The unique connected component of this point-shift is thus of class $\mathcal{I}/\mathcal{F}$. The unique connected component has two ends. Its foliation has the same order as $\mathbb{Z}$. It is not markable.

Note that these two examples are bijective point-shifts. But there are cases of type $\mathcal{I}/\mathcal{I}$ which are not bijective.

#### 5.3.3 Class $\mathcal{I}/\mathcal{I}$

Here are three examples illustrating that this class of point-shifts can have either markable or non markable foliations.

**Proposition 33.** The Drainage point-shift on the 2-dimensional Bernoulli grid with $0 < p < 1$, has a single connected component of type $\mathcal{I}/\mathcal{I}$. Foils of this connected component are not markable.

*Proof.* For $x \in \Phi$, let $\pi_1(x) := x_1$ and $\pi_2(x) := x_2$ denote the first and the second coordinate of $x$, respectively. It is clear that all points of a same foil share the same second coordinate. The claim is that all points with the same second coordinate are included in a single foil.

If $x \in \Phi$, one has $D(x) = (x_1 + K, x_2 - 1)$, where $K + 1$ is distributed as a geometric random variable with parameter $p$. Assume $x = (x_1, m)$ and $y = (y_1, m)$ are two points of $\Phi$ such that $x_1 < y_1$. Let $(K_n)_{n\in\mathbb{Z}}, (K'_n)_{n\in\mathbb{Z}}$ be two sequences of i.i.d. random variables distributed as geometric random variable with parameter $p$, minus 1. If $\delta_n = D^n(y) - D^n(x)$, one has $\delta_0 = y_1 - x_1$ and, for $n \in \mathbb{N}$, one can consider

$$\delta_{n+1} = \begin{cases} \delta_n - K_n + K'_n & K_n < \delta_n, \\ 0 & K_n \geq \delta_n. \end{cases}$$

This means $(\delta_n)_{n\in\mathbb{N}}$ behaves like a random walk on $\mathbb{Z}$ starting from $y_1 - x_1$, before hitting the set of non-positive integers. It is zero as soon as the random walk hits the set of non-positive integers. Since the random walk on $\mathbb{Z}$ with mean zero steps is recurrent, $\delta_n$ hits zero almost surely, which proves that $x$ and $y$ are in the same connected component. Therefore all foils are infinite and the connected component is of type $\mathcal{I}/\mathcal{I}$. If foils of this connected component were markable, the partition of the point process into vertical lines form a collection of infinite disjoint subsets of the Bernoulli grid, where the intersection of these subsets with a some fixed foil is a finite non-empty inclusion, which contradicts Theorem 2. □

**Proposition 34.** The Condenser point-shift on the Poisson point process in $\mathbb{R}$ has a single $\mathcal{I}/\mathcal{I}$ connected component and the foliation is markable.

*Proof.* Clearly points of a single foil have the same mark. So in order to prove the statement, it is sufficient to show that all the points with the same mark belong to the same foil.

Assume that for some $m \in \mathbb{N}$, there exist $x, y \in \Phi$ such that $m_c(x) = m_c(y) = m$, $x \leq y$, and both $x$ and $y$ are not in the same component of $G^F$. For all $z \in \Phi$ with $m_c(z) = m$ and $y \leq z$, one has, $F^n(x) \leq F^n(y) \leq F^n(z)$, for all $n \in \mathbb{N}$. Therefore, the fact that $x \nsim_F y$ implies $x \nsim_F z$ as well. This implies that the points of $C^F(x)$ with mark $m$ are less than $y$. Hence, there is a largest point of $C^F(x)$ with mark $m$. For $t \in \Phi$, Let $R(t)$ be the largest point $C^F(t)$ with mark $m$ if it exists and $C^F(t)$ otherwise. $R(x)$ is a finite inclusion of $C^F(x)$, which is clearly infinite. This contradicts Theorem 2.

Therefore each foil consists of all points with the same mark $m_c$. The foliation has the order of $\mathbb{N}$. □

Here are further examples of this class. The authors in [6] prove that the Strip point-shift $S$ on the Poisson point process in $\mathbb{R}^2$, has a single connected component which is one ended. Therefore this connected component evaporates under the action of $f_s$ and is of Class $\mathcal{I}/\mathcal{I}$. On a Poisson point process, the expander point-shift also has a single connected component which evaporate under the action of the point-shift. This component is of Class $\mathcal{I}/\mathcal{I}$.

## 6 Partition preserving point-maps

It is well-known that all bijective point-shifts preserve the Palm probability of stationary point processes and that this property characterizes the Palm probabilities of stationary point processes [12].

This section features a fixed point-shift $F$ and considers the class of bijective point-shifts which preserve the two partitions $\mathcal{C}^F = \mathcal{C}_\Phi^{F_\Phi}$ and $\mathcal{L}^F = \mathcal{L}_\Phi^{F_\Phi}$ of a stationary point process $\Phi$. Such point-shifts will be instrumental to derive quantitative results in Section 7.

### 6.1 Bijective point-shifts preserving components

Let $\Gamma_\mathcal{C}^F := \Gamma_{\mathcal{C}^F}(\Phi)$ be the $\mathcal{C}^F$-stable group as defined in Definition 7. As mentioned in Definition 7, $\Gamma_\mathcal{C}^F$ is a subgroup of the symmetric group on the support of $\Phi$.

**Proposition 35.** For each point-shift $F$ and each stationary point process $(\Phi, \mathbb{P})$, there exists a $\mathcal{C}^F$-dense element (see Definition 7) of the $\mathcal{C}^F$-stable group; i.e., there exists $H \in \Gamma_\mathcal{C}^F$ such that for all $x \in \Phi$,

$$\{H^i(x); i \in \mathbb{Z}\} = C_\Phi^F(x).$$

There is no uniqueness in general.

*Proof.* The construction of $H$ is different for each of the three classes of components identified in Theorem 22. In each case, the first step is the construction of a total order on the points of $C$ which is flow-adapted and the second one is the definition of a dense and bijective point-shift preserving $C$.

If $C = C_\Phi^F(x)$ is of $\mathcal{F}/\mathcal{F}$ class, then it is easy to create a total order which is translation invariant on the points of $C$ as it is a finite set (e.g. using lexicographic order) with points that can be numbered $0, 1, \ldots, n-1$ for some integer $n = n(x) \geq 1$. A flow-adapted bijection $H$ preserving $C$ is then easy to build by taking $H = M_n$ with $M_n(k) = k + 1$ mod $n$.

If $C = C_\Phi^F(x)$ is of $\mathcal{I}/\mathcal{F}$ class, then, the existence of a single bi-infinite path in $G^F(C)$ (see Theorem 22) and the finiteness of the foils can be used to construct a total order. Let $\{x_n\}_{n\in\mathbb{Z}}$ be the bi-infinite path in question and let $L_n$ denote the foil of $x_n$. Since $L_n$ is finite for all $n$, one can use the lexicographic order to create a total order between its points. The total order is then obtained by saying that all points of $L_n$ have precedence over those of $L_{n-1}$. This total order, which is that of $\mathbb{Z}$, is flow-adapted. The bijective point-shift is that associating to a point $x$ of $C$ its direct successor for this order. This point-shift will be referred to as the Bi-Infinite Path Point-Shift $B$, with associated point-map $b$. On such a component, one takes $H = B$.

If $C = C_\Phi^F(x)$ is of $\mathcal{I}/\mathcal{I}$ class, then the construction uses a total order on the nodes of $G^F(C)$ known as RLS (Royal Line of Succession). The latter order is based on two ingredients:

1. A local (total) order among the sons of a given node in $G^F(C)$. This can be done as follows: for a given point $x$ of $C$ let $B^F(x) = B_\phi^F(x)$ be the set of its brothers; i.e.,

$$B^F(x) := \{y \in \phi; F_\phi(x) = F_\phi(y)\}.$$

The elements of $B^F(x)$ can then be ordered in a flow-adapted manner using the lexicographic order of the Euclidean space.

2. The Depth First Search (DFS - see Appendix 8.2) pre-order on rooted trees.

The RLS order on a rooted tree is a total order on a finite tree obtained by combining (1) and (2): DFS is used throughout and the sons of any given node are visited in the order prescribed by (1), with priority given to the older son.

It is now explained how this also creates a total order on the nodes of $C$. For $x, y \in C$, there exist positive integers $m$ and $n$ such that $F_\phi^m(x) = F_\phi^n(y)$. One says that $x \geq_r y$ if $x$ has RLS priority over $y$ in the rooted tree of descendants of $F_\phi^m(x)$. This tree is a.s. finite because in the $\mathcal{I}/\mathcal{I}$ case, there is evaporation of $C$ by the point-shift, which this in turn implies that for all points $z \in \phi$, the total number of descendents of $z$ is a.s. finite. The DFS preordering on descendants of a node in a tree forms an interval of this preordering. This implies that $\geq_r$ is a well-defined order on $C$ and also that it orders elements of $C$ in the same linear order as that of an interval in $\mathbb{Z}$. Furthermore, since $C$ is infinite, this order on $C$ cannot have a greatest element or a least element. Otherwise the greatest and the least elements would be a finite flow-adapted inclusion of $C$ or the foil, which contradicts Theorem 2. Therefore the order on $C$ as well as its restriction to a foil is a linear order, similar to that of $\mathbb{Z}$.

On such a components, one defines $H = R$ where $R$ denotes the RLS point-shift, namely the point-shift that associates to each point its successor in the RLS order, which is bijective and translation invariant. □

Let $h$ denote the point-map of the point-shift $H$ defined in the last theorem. Notice that since $H$ is bijective, the dynamical system $(\mathbf{N}^0, \theta_h)$ preserves $\mathbb{P}_\Phi$.

### 6.2 Bijective point-shifts preserving foils

The results of this subsection parallel those of the last subsection, with an important refinement which is that of *order preservation*.

Let $\Gamma_L^F := \Gamma_L^F(\Phi)$ denote the set of all bijective and $\mathcal{L}_\Phi^F$-preserving point-shifts. This group, which is called the *$\mathcal{L}^F$-stable group*, is a subgroup of the $\mathcal{C}^F$-stable group.

As above, an element $H$ of the $\mathcal{L}^F$-stable group is said to be $\mathcal{L}_\Phi^F$-dense if

$$\{H_\phi^i(x); i \in \mathbb{Z}\} = L_\phi^F(x).$$

Each $\mathcal{L}^F_\Phi$-dense element $H$ of the $\mathcal{L}^F$-stable group induces a total order $\preceq_H$ on the elements of each infinite foil of $C$ by

$$x \preceq_H H(x). \tag{6.1}$$

This total order is flow-adapted. It is said to be preserved by $F$ if

$$x \preceq_H y \Rightarrow F(x) \preceq_H F(y).$$

The following proposition uses the fact (proved in Theorem 22) that in a connected component $C$ of $G^F(\Phi)$, either all foils of $C$ have finite cardinality or all foils have infinite cardinality.

**Proposition 36.** For each stationary point process $(\Phi, \mathbb{P})$, and each point-shift $F$, there exists a $\mathcal{L}^F_\Phi$-dense element $F_\perp$ of the $\mathcal{L}^F$-stable group. In addition $F_\perp$ can be chosen such that the $\preceq_{F_\perp}$ order is preserved by $F$ on components with all its foils with infinite cardinality. There is no uniqueness in general.

*Proof.* If the connected component $C$ of a realization $\phi$ has finite foils, the following construction can be used: $F_\perp(x)$ is the element coming next to $x$ in the lexicographic order. This rule is applied to all elements of a foil except the greatest element for this order, whereas $F_\perp$ of the greatest element is the least element.

For a connected component $C$ with all its foils with infinite cardinality, the construction uses the RLS total order on the nodes of $G^F(C)$.

One defines $F_\perp(x)$ as the next element in $L(x)$, i.e., the greatest element of $L(x)$ which is less than $x$, makes $F_\perp$ a bijection, and the orbit of each point $x$ of $L(x)$ is the foil $L(x)$.

The property that $\preceq_{F_\perp}$ is preserved by $F$ follows from the fact that if $x$ has priority over $y$ for DFS, then the father of $x$ also has priority over the father of $y$ for DFS. $\square$

Let $f_\perp$ denote the point-map of the point-shift $F_\perp$ defined in the last theorem. For the same reasons as above, the dynamical system $(\mathbf{N}^0, \theta_{f_\perp})$ preserves $\mathbb{P}_\Phi$.

In the next definition and below, in order to simplify notation, $\preceq_{F_\perp}$ (defined in (6.1)) is often replaced by $\preceq_\perp$.

**Definition 37.** For two elements $x$ and $y$ of the same foil s.t. $x \preceq_\perp y$, let

$$\Delta(x, y) := \mathrm{Card}\{z : x \preceq_\perp z \prec_\perp y\}.$$

By convention let $\Delta(y, x) := -\Delta(x, y)$.

It is easy to verify that for all $x$ and $y$ in the same foil,

$$F_\perp^{\Delta(x,y)}(x) = y. \tag{6.2}$$

### 6.3 Point foils and components

This subsection discusses some properties of the foil and the component of the origin, seen as point processes. In particular, it provides conditions under which these objects are *not* stationary point processes.

For all countable sets $S$ of points of $\mathbb{R}^d$ without accumulation, let $\Psi(S)$ denote the counting measure with support $S$.

Let $L_0$ (resp. $C_0$) denote the foil (resp. component) of the origin under $\mathbb{P}_\Phi$. The counting measure $\Psi(L_0)$ under $\mathbb{P}_\Phi$ (resp. $\Psi(C_0)$ under $\mathbb{P}_\Phi$) will be called the *point foil* (resp. the *point component*) of $\Phi$ w.r.t. the point-shift $F$.

The terms point foil and point component are used to stress that these random counting measures are not always Palm versions of stationary point processes. More precisely, let $\mathcal{Q}_0$ (resp. $\mathcal{R}_0$) denote the distribution of the point foil $\Psi(L_0)$ (resp. $\Psi(C_0)$). The following proposition summarizes the connections between foils/components and stationary point processes. Its proof follows from the definition of markability.

**Proposition 38.** If the foliation of $C_0$ is not markable, then $\mathcal{Q}_0$ is not the Palm distribution of a stationary point process (see Subsection 2.4). Similarly, if $C_0$ is not markable, then the distribution $\mathcal{R}_0$ of $\Psi(C_0)$ is not the Palm distribution of a stationary point process.

It follows from the above considerations that both in the markable and the non-markable cases, $\mathcal{Q}_0$ (resp. $\mathcal{R}_0$) is preserved by $\theta_{f_\perp}$ (resp. $\theta_h$). This invariance property is of course classical in the markable case.

The fact that it holds in general can be phrased as follows: for all (non–necessarily measure-preserving) dynamics $f$ on a stationary point process, there exists a dynamics $f_\perp$ on the typical leaf of the stable manifold of $f$, which is bijective, dense (has the whole leaf as orbit), and which preserves the law of the leaf.

# 7 Statistical properties of point-map foils

This section is devoted to quantitative complements to the results obtained hitherto on the structure of foils and components. By quantitative results, one means here properties pertaining to the mean values of certain random variables associated with these objects, e.g., intensities or *relative intensities.*

## 7.1 Foil cardinalities

This subsection establishes a connection between the Palm-distribution of $l_n(0)$ (the cardinality of the set of $F$-cousins of 0 with the same $n$-th order ancestor) and the distribution of $d_n(0)$ (the cardinality of set of $F$ descendants of generation $n$ w.r.t. $0$ – see Definition 11). As a byproduct of this useful result, one can conclude:

1. If $F$ evaporates $\Phi$, namely if $\Phi_{\mathcal{F}/\mathcal{F}}$ and $\Phi_{\mathcal{I}/\mathcal{F}}$ are empty, then each $F$-foil of $\Phi$ has an infinite number of points (Corollary 42), and the typical point has a number of descendants which is a.s. finite (Corollary 41) but with infinite mean (Proposition 25), and hence heavy tailed. See Subsection 2.3.1 for an example.
2. If $\Phi_{\mathcal{I}/\mathcal{I}}$ is empty, then each $F$-foil of $\Phi$ has an a.s. finite number of points (Theorem 22); the typical point has descendants of all orders with a positive probability[6] (Corollary 41), and hence an infinite number of descendants. However, the expected number of descendants of order $n$ does not diverge in mean (Corollary 44) as $n$ tends to infinity. If in addition $\Phi_{\mathcal{I}/\mathcal{F}}$ is empty, then the set of descendants of the typical point looks like a "finite star with a loop at the center". See Subsection 2.3.2 for an example. If in place $\Phi_{\mathcal{F}/\mathcal{F}}$ is empty, then the set of descendants of the typical point is either finite or looks like an "infinite path with finite trees attached to it". The points in this infinite path constitute a sub-stationary point process. This point process always has a positive intensity. (For instance, for the RLS point-shift on the Poisson point process in $\mathbb{R}^2$, this is the whole point process. There exist cases where $F$ is not bijective and the connected components are of type $\mathcal{I}/\mathcal{F}$ and hence such that this sub-point process is not the whole point process.

[6]With probability 1 iff $F$ is bijective.

**Proposition 39.** For all point-shifts $F$, for all stationary point processes $(\Phi, \mathbb{P})$, for all $h : \mathbb{N} \to \mathbb{R}^+$,

$$\mathbb{E}_\Phi\left[h(d_n(0))\right] = h(0)\mathbb{P}_\Phi\left[0 \notin F^n_\Phi(\Phi)\right] + \mathbb{E}_\Phi\left[\frac{h(l_n(0))}{l_n(0)}\right]. \tag{7.1}$$

*Proof.* By sending a total mass of $h(l_n(x))$ to $F^n_\phi(x)$, the total mass mass received by point $y$ is easily seen to be $h(d_n(y))$. Let

$$w(\phi, x, y) = \mathbf{1}\{F^n_\phi(x) = y\}\frac{h(l_n(x))}{l_n(x)},$$

where $\phi = \Phi(\omega)$. For all $x$ and $y$ in $\phi$,

$$w^+(x) = \frac{h(l_n(x))}{l_n(x)}, \quad w^-(y) = \mathbf{1}\{d_n(y) \neq 0\}h(d_n(y)),$$

and therefore using Lemma 51,

$$\begin{aligned}
\mathbb{E}_\Phi\left[\frac{h(l_n(0))}{l_n(0)}\right] &= \mathbb{E}_\Phi\left[\mathbf{1}\{d_n(0) \neq 0\}h(d_n(0))\right] \\
&= \mathbb{E}_\Phi\left[h(d_n(0))\right] - \mathbb{E}_\Phi\left[\mathbf{1}\{d_n(0) = 0\}h(d_n(0))\right] \\
&= \mathbb{E}_\Phi\left[h(d_n(0))\right] - \mathbb{P}_\Phi\left[d_n(0) = 0\right]h(0) \\
&= \mathbb{E}_\Phi\left[h(d_n(0))\right] - \mathbb{P}_\Phi\left[0 \notin F^n_\Phi(\Phi)\right]h(0). \quad \square
\end{aligned}$$

The announced quantitative results are given in the following corollaries of Proposition 39.

If in (7.1) $h(x)$ is replaced by $xh(x)$, one gets:

**Corollary 40.** For all $n \geq 0$,

$$\mathbb{E}_\Phi\left[h(l_n(0))\right] = \mathbb{E}_\Phi\left[d_n(0)h(d_n(0))\right]. \tag{7.2}$$

**Corollary 41.** For all $n \geq 0$,

$$\mathbb{P}_\Phi\left[0 \in F^n_\Phi(\Phi)\right] = \mathbb{E}_\Phi\left[\frac{1}{l_n(0)}\right]. \tag{7.3}$$

In addition

$$\mathbb{P}_\Phi\left[0 \in F^\infty(\Phi)\right] = \mathbb{E}_\Phi\left[\frac{1}{l_\infty(0)}\right]. \tag{7.4}$$

*Proof.* The first result is obtained by putting $h \equiv 1$ in (7.1). Equation (7.4) is obtained when letting $n \to \infty$ in Equation (7.3) and when using monotone convergence. $\square$

Equation (7.4) immediately proves:

**Corollary 42.** $F$ evaporates $(\Phi, \mathbb{P})$ if and only if the $F$-foil of $0$ is $\mathbb{P}_\Phi$ a.s. infinite[7].

This is consistent with the result of Corollary 24 since the property that the foil of $0$ is infinite a.s. is equivalent to having all connected components of Class $\mathcal{I}/\mathcal{I}$, or equivalently to having $(\Phi_{\mathcal{F}/\mathcal{F}}, \mathbb{P})$ and $(\Phi_{\mathcal{I}/\mathcal{F}}, \mathbb{P})$ almost surely empty.

**Corollary 43.** For all $n \geq 0$,

$$\mathbb{E}_\Phi\left[d_n(0) | 0 \in F^n_\Phi(\Phi)\right] = 1/\mathbb{E}_\Phi\left[\frac{1}{l_n(0)}\right]. \tag{7.5}$$

[7] Equivalently, the iterated images of $C$, seen as counting measures, converge to 0 for the vague topology.

*Proof.* Taking $h$ the identity in (5.1) and (7.1) gives

$$\begin{aligned} 1 = \mathbb{E}_\Phi\left[d_n(0)\right] &= \mathbb{E}_\Phi\left[d_n(0)|0 \notin F^n_\Phi(\Phi)\right]\mathbb{P}_\Phi\left[0 \notin F^n_\Phi(\Phi)\right] \\ &\quad + \mathbb{E}_\Phi\left[d_n(0)|0 \in F^n_\Phi(\Phi)\right]\mathbb{P}_\Phi\left[0 \in F^n_\Phi(\Phi)\right] \\ &= \mathbb{E}_\Phi\left[d_n(0)|0 \in F^n_\Phi(\Phi)\right]\mathbb{P}_\Phi\left[0 \in F^n_\Phi(\Phi)\right]. \end{aligned}$$

Replacing $\mathbb{P}_\Phi\left[0 \in F^n_\Phi(\Phi)\right]$ using (7.3) implies the result. ☐

**Corollary 44.** If $f$ does not evaporate $(\Phi, \mathbb{P})$, then

$$\mathbb{E}_\Phi\left[d_n(0)|0 \in F^n(\Phi)\right] \uparrow_{n\to\infty} 1/\mathbb{E}_\Phi\left[\frac{1}{l_\infty(0)}\right] < \infty \tag{7.6}$$

and

$$\limsup_{n\to\infty} \mathbb{E}_\Phi\left[d_n(0)|0 \in F^\infty(\Phi)\right] \le 1/\mathbb{E}_\Phi\left[\frac{1}{l_\infty(0)}\right] < \infty. \tag{7.7}$$

*Proof.* The first assertion follows from Equation (7.5). The second follows from Equation (7.5) and simple monotonicity arguments. ☐

### 7.2 Foil intensities

This subsection is focused on the intensity of the $F$-foils. From Proposition 20, either all foils of a markable component are stationary point processes, or none of them are. The notion of intensity only makes sense in the former case. The notion of *relative intensity* defined in the next subsection allows one to discuss the “density” of foils in whole generality, namely regardless of the above dichotomy.

The goal of the section is to complement the structural results discussed hitherto by more quantitative results. The main result is an expression for the average number of different points in the foil $L^F_+(0)$ per point in the foil of 0, $L^F(0)$, when the point process is under its Palm distribution.

#### 7.2.1 Relative intensities

Below, when considering a component of Class $\mathcal{I}/\mathcal{I}$, it is assumed that $F_\perp$ is an $\mathcal{L}^F_\Phi$-dense element of the $\mathcal{L}^F_\Phi$-stable group and that $\preceq_{F_\perp}$ is $F$-compatible.

Let $f_\perp$ denote the point-map of $F_\perp$ and let $\theta_{f_\perp}$ denote its related shift on $\mathbf{N}^0$. Equations (4.1) and (6.2) give that for $\mathbb{P}_\Phi$-almost all $\phi \in \mathbf{N}^0$,

$$\theta^n_{f_\perp}\phi = \theta_{F^n_\perp(\phi,0)}\phi. \tag{7.8}$$

Hence if $\phi \sim_{\theta_f} \psi$, with abuse of notation, one can define $\Delta(\phi, \psi)$ as the unique integer $n$ such that $\theta^n_{f_\perp}\phi = \psi$.

Consider the dynamical system $(\mathbf{N}^0, \theta_{f_\perp})$. The fact that $F_\perp$ is bijective implies that $\theta_{f_\perp}$ preserves $\mathbb{P}_\Phi$.

**Theorem 45.** Let $(\Phi, \mathbb{P})$ be a stationary point process, $F$ be an arbitrary point-shift and $F_\perp$ and $\Delta$ be as in Definition 37. Let $\mathcal{P}_0$ denote the distribution of $\Phi$ under $\mathbb{P}_\Phi$. Then, for $\mathcal{P}_0$ almost-all realizations $\phi$ for which the connected component of the origin is $\mathcal{I}/\mathcal{I}$, the limit

$$\lambda_+(\phi) := \lim_{n\to\infty}\frac{\Delta\left(F_\phi(0), F_\phi \circ F^n_\perp(0)\right)}{\Delta\left(0, F^n_\perp(0)\right)} = \lim_{n\to\infty}\frac{\Delta\left(F_\phi(0), F_\phi \circ F^n_\perp(0)\right)}{n} \tag{7.9}$$

exists, is positive and in $L_1(\mathcal{P}_0)$. In addition, $\lambda_+(\phi)$ is a function of the foil of $0$ only; i.e., if $0 \sim_F x$, $\lambda_+(\theta_x\phi) = \lambda_+(\phi)$ and $\lambda_+$ is independent of the choice $F_\perp$ as far as it satisfies the properties in Proposition 36.

**Remark 46.** Note that the denominator of (7.9) is equal to $n$. The reason for which (7.9) is written in this way is to stress the fact that numerator counts the number of elements of $L^F_+(0)$ (which are older than $F_\phi(0)$) as $n$ tends to infinity, whereas the denominator counts the number of elements of $L^F(0)$ (which are older than 0). Note that $\lambda_+$ can be also defined as

$$\lim_{n\to\infty} \frac{\Delta\left(F_\phi \circ F_\perp^{-n}(0), F_\phi \circ F_\perp^{n}(0)\right)}{\Delta\left(F_\perp^{-n}(0), F_\perp^{n}(0)\right)}.$$

But for sake of simplicity the latter is not used.

**Remark 47.** In the case where the connected component of the origin is of type $\mathcal{I}/\mathcal{F}$ or $\mathcal{F}/\mathcal{F}$, define

$$\lambda_+(\phi) = \frac{l_\infty(F_\phi(0))}{l_\infty(0)}. \tag{7.10}$$

The existence of the non-degenerate limit in (7.9) and the non-degenerate value in (7.10) can be seen as a proof of the fact that all foils of a connected component have the "same dimension". This fact justifies the use of the term "foliation" within this context (see e.g. [1]).

*Proof.* The main idea is to distribute the mass from all brothers in a family to their father and all older "close" individuals in the generation of their father without a descendant.

If $x$ is in a connected component of $G^F(\phi)$ which is $\mathcal{F}/\mathcal{F}$ or $\mathcal{I}/\mathcal{F}$, all statements follow from finiteness of the foils. Hence assume $C(0)$ is $\mathcal{I}/\mathcal{I}$. Hence it is sufficient to show that, for $\mathcal{P}_0$ almost all $\phi$, the limit

$$\lambda_+(\phi) = \lim_{n\to\infty} \frac{\Delta\left(\theta_f\phi, \theta_f \circ \theta^n_{f_\perp}\phi\right)}{\Delta\left(\phi, \theta^n_{f_\perp}\phi\right)}, \tag{7.11}$$

exists and is positive, finite and constant on the foil $L^{\theta_f}(\phi)$ provided the latter is infinite. Let $\Delta_+(\phi) := \Delta(\theta_f\phi, \theta_f \circ \theta_{f_\perp}\phi)$. Now consider the following mass transport:

$$w(\phi, x, y) = \begin{cases} 1 & y \in L^F_+(x) \text{ and } 0 \le \Delta(F_\phi(x), y) < \Delta(F_\phi(x), F_\phi \circ F_\perp(x)) \\ 0 & \text{otherwise.} \end{cases}$$

One has, for all points $x, y \in \phi$, $w^-(y) \le 1$ and $w_+(x) = \Delta_+(\theta_x\phi)$. Therefore

$$\mathbb{E}_\Phi\left[\Delta_+\right] = \mathbb{E}_\Phi\left[w^+(0)\right] = \mathbb{E}_\Phi\left[w^-(0)\right] \le 1.$$

Since the denominator in (7.11) is equal to $n$, one has

$$\begin{aligned} \lim_{n\to\infty} \frac{\Delta\left(\theta_f\phi, \theta_f \circ \theta^n_{f_\perp}\phi\right)}{\Delta\left(\phi, \theta^n_{f_\perp}\phi\right)} &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \Delta\left(\theta_f \circ \theta^{i-1}_{f_\perp}\phi, \theta_f \circ \theta_{f_\perp} \circ \theta^{i-1}_{f_\perp}\phi\right) \\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \Delta_+\left(\theta^i_{f_\perp}\phi\right). \end{aligned}$$

Since $F_\perp$ is $\mathbb{P}$-almost surely bijective, $\mathbb{P}_\Phi$ is $\theta_{f_\perp}$-invariant. Therefore if one denotes by $\mathcal{I}$ the invariant $\sigma$-field of $\theta_{f_\perp}$, because of the finiteness of $\mathbb{E}_\Phi[\Delta_+]$ the conditions of Birkhoff's ergodic theorem are satisfied. This implies that the last limit exists for $\mathbb{P}_\Phi$-almost all $\phi$ and it is equal to $\mathbb{E}_\Phi[\Delta_+|\mathcal{I}]$, which is finite and invariant under the action of $\theta_{f_\perp}$; i.e., it is a function of $L^{\theta_f}(\phi)$.

To prove that $\lambda_+(\phi)$ is a.s. positive, note that, if $Y$ is the event of being the youngest son of the family, then $\Delta_+ \ge \mathbf{1}_Y$. Hence if, with positive probability, $\lambda_+(\phi) = 0$, this means that, with positive probability, $\mathbb{E}_\Phi[\mathbf{1}_Y|\mathcal{I}]$ is zero. But since $\mathbb{E}_\Phi$ is $\theta_{f_\perp}$-invariant, this means that, with positive probability, there is no youngest son on the foil of $L^{\theta_f}(\phi)$, which means that all points of $L^{\theta_f}(\phi)$ are brothers. Since $C^{\theta_f}(\phi)$ is infinite, this contradicts the a.s. finiteness of $d_1(F_\phi(0))$.

Finally to prove that $\lambda_+(\phi)$ is independent of the choice of $F_\perp$, it is sufficient to show that $\mathbb{E}_\Phi[\Delta_+|\mathcal{I}]$ depends only on $f$. To do so, it is enough to prove that for all $A \in \mathcal{I}$, $\mathbb{E}_\Phi[\Delta_+\mathbf{1}_A]$ depends only on $f$. Since $A \in \mathcal{I}$, with abuse of notation, one has $\mathbf{1}_A(\phi) = \mathbf{1}_A(L^{\theta_f}(\phi))$. Let

$$\begin{aligned} A_+ &= \{\theta_f\phi; \phi \in A\}, \\ L_-^{\theta_f}(A) &= \{\phi \in \mathbf{N}^0; L_-^{\theta_f}(\phi) \neq \emptyset \text{ and } L_-^{\theta_f}(\phi) \in A\}. \end{aligned}$$

It is easy to see that $L_-^{\theta_f}(A) \in \mathcal{I}$. Let

$$u_\phi(x,y) = \Delta_+(\theta_x\phi)\mathbf{1}_{\{y=F_\phi(x)\}}\mathbf{1}_A(\theta_x\phi),$$

which is a flow-adapted transport kernel. If $A_+$ denotes $\{\theta_f\phi; \phi \in A\}$, by the mass transport principle,

$$\mathbb{E}_\Phi\left[\Delta_+\mathbf{1}_A\right] = \mathbb{E}_\Phi\left[w^+(0)\right] = \mathbb{E}_\Phi\left[w^-(0)\right] = \mathbb{E}_\Phi\left[\Delta_\perp\mathbf{1}_{A_+}\right], \tag{7.12}$$

where $\Delta_\perp(\phi)$ is the smallest $i > 0$ such that $F_\perp^i(x)$ has a child and zero otherwise. Note that all elements of $A_+$ have at least one child and therefore $\mathbf{1}_{A_+}(\phi)$ is zero whenever $\phi$ has no child. Let

$$v_\phi(x,y) = \begin{cases} 1 & \theta_x(\phi) \in A_+ \text{ and } y = f_\perp^i(x) \text{ for some } 0 \le i < \Delta_\perp(x), \\ 0 & \text{otherwise.} \end{cases}$$

Since $A \in \mathcal{I}$, (7.12) and the mass transport principle give

$$\begin{aligned} \mathbb{E}_\Phi\left[\Delta_+\mathbf{1}_A\right] &= \mathbb{E}_\Phi\left[\Delta_\perp\mathbf{1}_{A_+}\right] = \mathbb{E}_\Phi\left[v^+(0)\right] \\ &= \mathbb{E}_\Phi\left[v^-(0)\right] = \mathbb{E}_\Phi\left[\mathbf{1}_{L_-^{\theta_f}(A)}\right] = \mathbb{P}_\Phi\left[L_-^{\theta_f}(A)\right]. \end{aligned}$$

Clearly the latter depends only on $f$ and not on the choice of $F_\perp$ which completes the proof. $\square$

**Corollary 48.** Letting $A = \Omega$ in the last proof gives

$$\mathbb{E}_\Phi[\Delta_+] = \mathbb{P}_\Phi[L_-^{\theta_f}(\Omega)],$$

where the R.H.S. is the probability that 0 is not in the first foil (if there is any) of its component.

**Definition 49.** The quantity $\lambda_+(\Phi)$, defined $\mathbb{P}_\Phi$ a.s., counts the average number of different points in the foil $L_+^F(0)$ per point in the foil of 0, $L^F(0)$, and is hence called the *relative intensity of* $L_+^F(0)$ *with respect to* $L^F(0)$ in $\Phi$. This notion extends to the relative intensity

$$\Lambda_+(x,\Phi) = \lambda_+(\theta_x\Phi)$$

of $L_+^F(x)$ with respect to $L^F(x)$ for all $x \in \Phi$.

#### 7.2.2 Intensities

In the particular case where foils are markable, one gets back the following classical result as a direct corollary of Theorem 45:

**Proposition 50.** Assume that $(\mathbb{P}, \theta_t)$ is ergodic. Assume $L^F(0)$ is markable, so that it is the support of a point process. Let $\beta$ (resp. $\beta_+$) denote the intensity of $\Psi(L^F(0))$ (resp. $\Psi(L^F_+(0))$). Then $\beta_+ = \beta\Lambda_+$, where $\Lambda_+ = \mathbb{E}_0(\Lambda_+(0,\Phi))$.

Note that it follows from $\beta_+ = \beta\Lambda_+$ that $\Lambda_+ < 1$.

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## 8 Appendix

### 8.1 Mass transport

Let $w$ be a point-shift and $(\Phi, \mathbb{P})$ be a stationary point process. Let $G^w(\Phi)$ be the directed graph of Definition 8. Define $w^+(\Phi, 0)$ (ref. $w^-(\Phi, 0)$) to be the out-degree (in-degree) of node $0$ under $\mathbb{P}_\Phi$. The following is classical:

**Lemma 51** (Mass transport principle)**.** If $w$ is a mass transport and $(\Phi, \mathbb{P})$ is a stationary point process then

$$\mathbb{E}_\Phi\left[w^+(\Phi, 0)\right] = \mathbb{E}_\Phi\left[w^-(\Phi, 0)\right]. \tag{8.1}$$

### 8.2 Depth first search

DFS is a recursive algorithm prescribing a class of ways to traverse a rooted tree. Nodes belong to two categories: visited and unvisited. The algorithm starts from the root, with the latter visited and all other nodes unvisited. From a given visited node, the node visited next is one of its yet unvisited sons. If all its sons have already been visited (in particular if it has no sons), then the algorithm moves to the father of the given node to search for the next unvisited node.

### 8.3 Multi type strip point-shift

Consider the following variant of the Strip Point-Shift. To each point $x_i$ of the Poisson point process $\Phi$, one associates an independent mark $m_i$, which is a Poisson point process of intensity 1 on a circle of radius 1. Consider the (Poisson cluster) point process

$$\Psi = \Phi + \sum_i x_i + m_i.$$

Each realization of $\Psi$ determines the points $x_i$ of $\Phi$ and the associated cluster $x_i + m_i$. It hence allows one to classify the points of $\Phi$ in types taking their values in $\mathbb{N}$, with the type of $x_i$ being the cardinality of $m_i$. The Multi Type Strip Point-Shift $f$ maps $y \in x_i + m_i$ to $x_i$ and uses the $f_s$ map within points of type $k \in \mathbb{N}$ with $\Phi$.

On $\Psi$, this point-shift admits an infinite number of connected components (one per type). It follows from the results of Subsection 2.3.1 (and from the fact that the points of type $k$ in $\Phi$ form a stationary Poisson point process of positive intensity that each connected component has properties similar that that of the unique component of Subsection 2.3.1; in particular, it is of Class $\mathcal{I}/\mathcal{I}$ and evaporates under the action of $f$.

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