Yaglom limit for stable processes in cones[*]

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Dedicated to Krzysztof Burdzy on the occasion of his 60th birthday

Abstract

We give the asymptotics of the tail of the distribution of the first exit time of the isotropic α-stable Lévy process from the Lipschitz cone in $\mathbb{R}^d$. We obtain the Yaglom limit for the killed stable process in the cone. We construct and estimate entrance laws for the process from the vertex into the cone. For the symmetric Cauchy process and the positive half-line we give a spectral representation of the Yaglom limit.

Our approach relies on the scalings of the stable process and the cone, which allow us to express the temporal asymptotics of the distribution of the process at infinity by means of the spatial asymptotics of harmonic functions of the process at the vertex; on the representation of the probability of survival of the process in the cone as a Green potential; and on the approximate factorization of the heat kernel of the cone, which secures compactness and yields a limiting (Yaglom) measure by means of Prokhorov's theorem.

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1 Introduction

Let $0 < \alpha < 2$, $d = 1, 2, \ldots$, and let $X = \{X_t, t \geq 0\}$ be the isotropic α-stable Lévy process in $\mathbb{R}^d$. We denote by $\mathbb{P}_x$ and $\mathbb{E}_x$ the probability and expectation for the process starting from any $x \in \mathbb{R}^d$, see Section 2 for details. Let $\Gamma \subset \mathbb{R}^d$ be an arbitrary Lipschitz cone with vertex at the origin 0. We define

$$\tau_\Gamma = \inf\{t > 0 : X_t \notin \Gamma\}, \tag{1.1}$$

the time of the first exit of X from Γ. The following measure μ will be called the Yaglom limit for X and Γ.

Theorem 1.1. *There is a probability measure μ concentrated on Γ such that for every Borel set $A \subset \mathbb{R}^d$,*

$$\lim_{t \to \infty} \mathbb{P}_x \left(\frac{X_t}{t^{1/\alpha}} \in A \big| \tau_\Gamma > t \right) = \mu(A), \quad x \in \Gamma. \tag{1.2}$$

The above condition $\tau_\Gamma > t$ means that X stays, or survives, in Γ for time longer than t. Theorem 1.1 asserts that, given its survival, X_t rescaled by $t^{1/\alpha}$ has a limiting distribution independent of the starting point. We note that rescaling is essential for the limit to be nontrivial. The Yaglom limit μ corresponds with the idea of 'quasi-stationarity', as expressed by Bartlett [6]:

> It still may happen that the time to extinction is so long that it is still of more relevance to consider the effectively ultimate distribution (called a quasi-stationary distribution) [...]

Namely, μ is a quasi-stationary distribution for $(t+1)^{-1/\alpha} X_t$ in the following sense.

Proposition 1.2. *Let $\mathbb{P}_\mu(\cdot) = \int_\Gamma \mathbb{P}_y(\cdot) \, \mu(\mathrm{d}y)$. For every Borel set $A \subset \mathbb{R}^d$,*

$$\mathbb{P}_\mu \left(\frac{X_t}{(t+1)^{1/\alpha}} \in A \big| \tau_\Gamma > t \right) = \mu(A), \quad t \geq 0. \tag{1.3}$$

Note that $Y_t = (t+1)^{-1/\alpha} X_t$ is a time-inhomogenous Markov process and under $\mathbb{P}_\mu$, the law of Y_0 is μ.

This is the first paper where the Yaglom limit is identified for the multi-dimensional α-stable Lévy processes. For the one-dimensional self-similar processes, including the symmetric α-stable Lévy process in the one-dimensional cone $\Gamma = (0, \infty)$, Yaglom limits similar to (1.2), and also using rescaling, were given by Haas and Rivero [47]. Their proofs rely on precise estimates for the tail distribution of exponential functionals of non-increasing Lévy processes and are completely different from ours. As we will see below, the Yaglom limit may be obtained from the asymptotics (i.e. limits) of the survival probability $\mathbb{P}_x(\tau_\Gamma > t)$. We note that such asymptotics were studied for the multi-dimensional Brownian motion by DeBlassie [40]. Bañuelos and Smits [5] gave the asymptotics of the heat kernel of the cone in terms of the orthonormal eigenfunctions of the Laplace-Beltrami operator on the cone's spherical cap. Denisov and Wachtel [43] derived a result similar to Theorem 1.1 for multidimensional random walks by using coupling with the Brownian motion. The tail distribution of τ_Γ for the isotropic α-stable Lévy process and wedges $\Gamma \subset \mathbb{R}^2$ was estimated by DeBlassie [41]. Bañuelos and Bogdan [3] also provided estimates but not asymptotics for general cones in $\mathbb{R}^d$. They used the boundary Harnack principle (BHP), which turns out to be very useful also in our situation, because it, in fact, yields the asymptotics of the survival probability $\mathbb{P}_x(\tau_\Gamma > 1)$ for $x \to 0$, as we show below. These asymptotics are given in Theorem 3.1, and they lead to Theorem 1.1, to sharp estimates for the density function of the Yaglom limit μ, and to

the existence and estimates of laws of excursions of the stable process from the vertex into the cone, which we give in Theorem 3.3.

Information on quasi-stationary (QS) distributions for time-homogeneous Markov processes can be found in the classical works of Seneta and Vere-Jones [72], Tweedie [77], Jacka and Roberts [50]. The bibliographic database of Pollet [70] gives detailed history of QS distributions. In particular, Yaglom [79] was the first to explicitly identify QS distributions for the subcritical Bienaymé-Galton-Watson branching process. Part of the results on QS distributions concern Markov chains on positive integers with an absorbing state at the origin [38, 44, 46, 72, 78, 82]. Other objects of study are the extinction probabilities for continuous-time branching process and the Fleming-Viot process [1, 45, 65]. A separate topic is the one-dimensional Lévy processes exiting from the positive half-line. Here the case of the Brownian motion with drift was resolved by Martinez and San Martin [67], complementing the result for random walks obtained by Iglehart [48]. The case of jump Lévy processes was studied by E. Kyprianou [64], A. Kyprianou and Palmowski [63] and Mandjes et al. [66]. These papers are based on the Wiener-Hopf factorization and Tauberian theorems. They are intrinsically one-dimensional and they do not use the boundary asymptotics of harmonic functions or rescaling to obtain the limiting distribution. We also note in passing that these results relate to the behavior of the one-dimensional Lévy processes and random walks conditioned to stay positive, for which we refer the reader to Bertoin [10], Bertoin and Doney [9], and Chaumont and Doney [33].

On a general level our development depends on a compactness argument based on recent sharp estimates of the heat kernel of cones and on a formula expressing the survival probability $\mathbb{P}_x(\tau_\Gamma > t)$ as a Green potential. The latter allows us to obtain the spatial asymptotics of the survival probability at the vertex of the cone Γ in terms of the cone's Martin kernel with the pole at infinity, and is a consequence of BHP. By scaling we then obtain the asymptotics of the survival probability as $t \to \infty$. The construction allows for the identification of the limiting boundary behavior of the heat kernel at the vertex of the cone. Such asymptotics are completely new, and may be regarded as a culmination of the study of the Dirichlet fractional Laplacian, which started with boundary estimates and asymptotics of harmonic functions, developed into estimates and asymptotics of the Green function, gave the Martin representation of harmonic functions, and resolved into sharp estimates of the heat kernel. The development was initiated by Bogdan [18] and Song and Wu [74] with proofs of BHP for the fractional Laplacian. Then Jakubowski [52] gave sharp estimates of the Green function. Bogdan et al. [27] gave the boundary limits of ratios of harmonic functions and Bogdan et al. [23] gave sharp estimates of the Dirichlet heat kernel. Related works on the Dirichlet problem in cones are given by DeBlassie [41], Kulczycki [57], Kulczycki and Burdzy [31], Méndez-Hernández [68], Bogdan and Jakubowski [25], Michalik [69], Kulczycki and Siudeja [59], and Bogdan and Grzywny [22]. For smooth domains we refer to the pioneering works by Kulczycki [56], Song and Chen [36] and Kim et al. [53, 54]. Historically, the results for cones often preceded and informed generalizations to Lipschitz and arbitrary open sets. We expect similar generalizations for the asymptotics of heat kernels. The present paper only resolves the asymptotics of the heat kernel of the fractional Laplacian in the Lipschitz cone at the vertex, so there is much more work left to do.

The paper is organized as follows. In Section 2 we give basic notation and facts. In Section 3 we present our main results, which complement Theorem 1.1 and Proposition 1.2. Most of the proofs are given in Section 4. In Section 5 we discuss in detail the Cauchy process on the positive half-line and we give a spectral decomposition of its Yaglom limit.

2 Preliminaries

As defined in the introduction, $X = \{X_t,\ t > 0\}$ is the isotropic α-stable Lévy process on the Euclidean space $\mathbb{R}^d$. The process is determined by the jump measure with the density function

$$\nu(y) = \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|}|y|^{-d-\alpha}, \quad y \in \mathbb{R}^d, \tag{2.1}$$

where $0 < \alpha < 2$, $d = 1, 2, \ldots$. The coefficient in (2.1) is chosen so that

$$\int_{\mathbb{R}^d} [1 - \cos(\xi \cdot y)]\,\nu(y)\mathrm{d}y = |\xi|^\alpha, \quad \xi \in \mathbb{R}^d, \tag{2.2}$$

for convenience. Here $\xi \cdot y$ is the Euclidean scalar product and $|\xi|$ is the Euclidean norm. We always assume in this paper that the considered sets, measures and functions are Borel. The process X is Markovian with the following time-homogeneous transition probability

$$P_t(x, A) = \int_A p_t(x, y)\mathrm{d}y, \quad t > 0,\ x \in \mathbb{R}^d,\ A \subset \mathbb{R}^d,$$

where $p_t(x, y) := p_t(x - y)$ and p_t is the smooth real-valued function on $\mathbb{R}^d$ with the Fourier transform:

$$\int_{\mathbb{R}^d} p_t(x)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}x = \mathrm{e}^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d. \tag{2.3}$$

In particular, if $\alpha = 1$, then X is a Cauchy process, and we have

$$p_t(x) = \Gamma((d+1)/2)\pi^{-(d+1)/2}\frac{t}{\left(|x|^2 + t^2\right)^{(d+1)/2}},$$

see [21, 75]. For every $\alpha \in (0, 2)$, the infinitesimal generator of X is the fractional Laplacian,

$$\Delta^{\alpha/2}\varphi(x) = \lim_{\varepsilon \downarrow 0} \int_{|y|>\varepsilon} [\varphi(x+y) - \varphi(x)]\,\nu(y)\mathrm{d}y, \quad x \in \mathbb{R}^d, \tag{2.4}$$

defined at least on smooth compactly supported functions $\phi \in C_c^\infty(\mathbb{R}^d)$, cf. [8, 19, 21, 30, 51, 71, 81, 60]. The following scaling property is a consequence of (2.3),

$$p_t(x) = t^{-d/\alpha}p_1(t^{-1/\alpha}x), \quad x \in \mathbb{R}^d,\ t > 0. \tag{2.5}$$

Furthermore,

$$c^{-1}\left(\frac{t}{|x|^{d+\alpha}} \wedge t^{-d/\alpha}\right) \le p_t(x) \le c\left(\frac{t}{|x|^{d+\alpha}} \wedge t^{-d/\alpha}\right), \quad x \in \mathbb{R}^d,\ t > 0, \tag{2.6}$$

see [11, 24, 30] for the explicit constant c. Below we will use the notation $f \approx g$ when functions $f, g \ge 0$ are *comparable* i.e. their ratio is bounded between two positive constants (uniformly on the whole domain of the functions). In particular we can rewrite (2.6) as follows:

$$p_t(x) \approx t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}, \quad x \in \mathbb{R}^d,\ t > 0. \tag{2.7}$$

We will also write $\lim f(x)/g(x) = 1$ as $f(x) \sim g(x)$.

As stated, Γ denotes a generalized Lipschitz cone in $\mathbb{R}^d$ with vertex 0, that is, an open Lipschitz set $\Gamma \subset \mathbb{R}^d$ such that $0 \in \partial\Gamma$, and if $y \in \Gamma$ and $r > 0$ then $ry \in \Gamma$. Recall that an open set $D \subset \mathbb{R}^d$ is called Lipschitz if there exist $R > 0$ and $\Lambda > 0$ such that for every $Q \in \partial D$, there exist a Lipschitz function $\phi_Q\colon \mathbb{R}^{d-1} \to \mathbb{R}$ with Lipschitz

constant not greater than Λ and an orthonormal coordinate system CS_Q such that if $y = (y_1, \ldots, y_{d-1}, y_d)$ in CS_Q coordinates, then

$$D \cap B(Q, R) = \{y : \ y_d > \phi_Q(y_1, \ldots, y_{d-1})\} \cap B(Q, R),$$

where $B(Q, R) = \{z \in \mathbb{R}^d : \ |z - Q| < R\}$, the Euclidean ball of radius R centered at Q. We note that the trivial cones $\Gamma = \mathbb{R}^d$ and $\Gamma = \emptyset$ are excluded from our considerations because we require $0 \in \partial\Gamma$, and the Lipschitz condition excludes, e.g., $\mathbb{R}^d \setminus \{0\}$. In particular for $d = 1$, Γ is necessarily a half-line. We note that the cone Γ is characterized by its intersection with the unit sphere $\mathbf{S}_{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$.

The first exit time from Γ, as defined in (1.1), yields the heat kernel $p_t^\Gamma(x, y)$ of the cone,

$$p_t^\Gamma(x, y) := p_t(x, y) - \mathbb{E}_x[\tau_\Gamma < t; \ p_{t-\tau_\Gamma}(X_{\tau_\Gamma}, y)], \quad x, \ y \in \mathbb{R}^d, \ t > 0, \tag{2.8}$$

where $\mathbb{E}_x[\tau_\Gamma < t; \ p_{t-\tau_\Gamma}(X_{\tau_\Gamma}, y)] = \int\limits_{\{\tau_\Gamma < t\}} p_{t-\tau_\Gamma}(X_{\tau_\Gamma}, y)\mathrm{d}\mathbb{P}_x$. For bounded or nonnegative functions f we have

$$P_t^\Gamma f(x) := \mathbb{E}_x[f(X_t^\Gamma)] = \mathbb{E}_x[\tau_\Gamma < t; \ f(X_t)] = \int_{\mathbb{R}^d} f(y) p_t^\Gamma(x, y)\mathrm{d}y,$$

cf. [37, Section 2]. We also note that

$$0 \le p_t^\Gamma(x, y) = p_t^\Gamma(y, x) \le p_t(y - x), \tag{2.9}$$

and p^Γ satisfies the Chapman-Kolmogorov equations:

$$\int p_s^\Gamma(x, y) p_t^\Gamma(y, z)\mathrm{d}y = p_{t+s}^\Gamma(x, z), \quad s, \ t > 0, \quad x, \ z \in \mathbb{R}^d, \tag{2.10}$$

see, e.g., [19, 16, 34]. Since Γ is Lipschitz, by the exterior cone condition and Blumenthal 0-1 law, $\mathbb{P}_x(\tau_\Gamma = 0) = 0$ if $x \in \Gamma^c$, in particular $p_t^\Gamma(x, y) = 0$ whenever x or y are outside of Γ. We note that

$$\mathbb{P}_x(\tau_\Gamma > t) = \int_\Gamma p_t^\Gamma(x, y)\mathrm{d}y. \tag{2.11}$$

According to (2.5) and the fact that $t^{-1/\alpha}\Gamma = \Gamma$, the following scaling property holds:

$$p_t^\Gamma(x, y) = t^{-d/\alpha} p_1^\Gamma(t^{-1/\alpha}x, t^{-1/\alpha}y), \quad x, \ y \in \mathbb{R}^d, \ t > 0. \tag{2.12}$$

By (2.11), a similar scaling holds for the survival probability:

$$\mathbb{P}_x(\tau_\Gamma > t) = \mathbb{P}_{t^{-1/\alpha}x}(\tau_\Gamma > 1).$$

We define the Green function of Γ:

$$G_\Gamma(x, y) = \int_0^\infty p_t^\Gamma(x, y)\mathrm{d}t, \quad x, \ y \in \mathbb{R}^d,$$

and the Green operator:

$$G_\Gamma f(x) = \int_\Gamma G_\Gamma(x, y) f(y) \, \mathrm{d}y,$$

for integrable or nonnegative functions f. By (2.10),

$$P_t^\Gamma \left(G_\Gamma(\cdot, y)\right)(x) \le G_\Gamma(x, y), \quad t > 0, \ x, \ y \in \mathbb{R}^d. \tag{2.13}$$

We should note that G_Γ is always locally integrable because $\Gamma \neq \mathbb{R}$ and $\Gamma \neq \mathbb{R} \setminus \{0\}$, cf. [17].

For $r \in (0, \infty)$ we let $B_r = \{|x| < r\}$ and we define the truncated cone

$$\Gamma_r = \Gamma \cap B_r \,.$$

By the strong Markov property, for $t > 0$ and x, $y \in \mathbb{R}^d$,

$$p_t^\Gamma(x, y) = p_t^{\Gamma_r}(x, y) + \mathbb{E}_x \left[\tau_{\Gamma_r} < t;\, p_{t-\tau_{\Gamma_r}}^\Gamma (X_{\tau_{\Gamma_r}}, y) \right] \,. \tag{2.14}$$

Integrating the above identity against $\mathrm{d}t$ (see [27, eq. (15)]), we obtain

$$G_\Gamma(x, y) = G_{\Gamma_r}(x, y) + \mathbb{E}_x \left[G_\Gamma(X_{\tau_{\Gamma_r}}, y) \right] \,. \tag{2.15}$$

In particular, $x \to G_\Gamma(x, y)$ is regular α-harmonic on Γ_r if $|y| > r$. We recall that a function $u : \mathbb{R}^d \to \mathbb{R}$ is called regular harmonic with respect to X on an open set $\Gamma \subset \mathbb{R}^d$ if

$$u(x) = \mathbb{E}_x \left[\tau_\Gamma < \infty;\, u(X_{\tau_\Gamma}) \right] \,, \quad x \in \Gamma \,,$$

where we assume that the expectation is finite.

Two more facts are crucial in our development. First, if $I \subset (0, \infty)$ and $A \subset \Gamma$, $B \subset (\overline{\Gamma})^c$, then

$$\mathbb{P}_x[\tau_\Gamma \in I,\ X_{\tau_\Gamma -} \in A,\ X_{\tau_\Gamma} \in B] = \int_I \int_{B-y} \int_A p_u^\Gamma(x, \mathrm{d}y)\nu(z)\mathrm{d}z\mathrm{d}u \,. \tag{2.16}$$

This identity is called the Ikeda-Watanabe formula [49], and gives the joint distribution of $(\tau_\Gamma, X_{\tau_\Gamma -}, X_{\tau_\Gamma})$, see also [4, Lemma 1], [15], [35, Appendix A], [42, VII.68] or [73, Theorem 2.4].

Second, we use the following boundary Harnack principle (BHP) from Bogdan [18]: There is a constant $C = C(\Gamma, \alpha) > 0$ such that if $r > 0$ and functions $u, v \geq 0$ are regular harmonic in Γ_{2r} with respect to X and vanish on $\Gamma^c \cap B_{2r}$, then

$$u(x)v(y) \leq Cu(y)v(x) \,, \quad x,\ y \in \Gamma_r \,. \tag{BHP}$$

Generalizations of (BHP) can be found in [27, 74] for the fractional Laplacian and in [28, 53, 54] for more general jump Markov processes. Without essential loss of generality, in what follows we assume that

$$\mathbf{1} := (0, \ldots, 0, 1) \in \Gamma \,.$$

By [3, Theorem 3.2], there is a unique nonnegative function M on $\mathbb{R}^d$, called the Martin kernel with the pole at infinity for Γ, such that $M(\mathbf{1}) = 1$, $M = 0$ on Γ^c and for every $r > 0$, M is regular harmonic with respect to X on Γ_r. The function is locally bounded on $\mathbb{R}^d$ and homogeneous of degree $\beta = \beta(\Gamma, \alpha)$, that is,

$$M(x) = |x|^\beta M(x/|x|) \,, \quad x \neq 0 \,. \tag{2.17}$$

Furthermore, $0 < \beta < \alpha$. The exponent β is decreasing in Γ and it delicately depends on the geometry of Γ. When Γ is a right-circular cone, a rather explicit estimate for M is available [69, Theorem 3.13], expressed in terms of β. More information on β for narrow right-circular cones is given in [29]. As we shall see below, using (BHP) and M we can capture the boundary asymptotics of harmonic functions and some Green potentials. Following [3] we consider the Kelvin transformation K of M:

$$K(x) = |x|^{\alpha-d} M(x/|x|^2) = |x|^{\alpha-d-\beta} K(x/|x|) \,, \quad x \neq 0 \,, \tag{2.18}$$

see also Bogdan and Żak [17] for a general discussion of the Kelvin transform. The function K is called the Martin kernel at 0 for Γ. Clearly, $K(\mathbf{1}) = 1$ and $K = 0$ on Γ^c. By [3, Theorem 3.4],

$$K(x) = \mathbb{E}_x\left[\tau_B < \infty; K(X_{\tau_B})\right], \quad x \in \mathbb{R}^d,$$

for every open set $B \subset \Gamma$ with $\mathrm{dist}\,(0, B) > 0$. In particular K is α-harmonic in Γ. We note that

$$K(y) = \lim_{\Gamma \ni x \to 0} \frac{G_\Gamma(x, y)}{G_\Gamma(x, \mathbf{1})}, \quad y \in \mathbb{R}^d \setminus \{0\}. \tag{2.19}$$

This follows from the uniqueness of K in [3, Theorem 3.4] and from [27, (77)] for $M_D(x, 0)$ therein with $D = \Gamma$. By Kelvin transform,

$$M(y) = \lim_{\Gamma \ni x, |x| \to \infty} \frac{G_\Gamma(x, y)}{G_\Gamma(x, \mathbf{1})}, \quad y \in \mathbb{R}^d,$$

see [27, (81)] and (2.18); see also [69, Theorem 3.13] for the special case of the right-circular cones. We note in passing that the existence of the limit (2.19) is due to the phenomenon of oscillation-reduction for ratios of harmonic functions, which is essentially a consequence of (BHP). The oscillation-reduction argument is given for Lipschitz open sets in [3, Lemma 16]. It is detailed in [27, Theorem 3.2] for cones at the vertex and in [27, Lemma 8] for arbitrary open sets. More general Lévy processes are studied by Kim et al. [55], and Markov processes are considered by Juszczyszyn and Kwaśnicki [62]. The reader may also consult [7, Lemma 4.2] to see oscillation-reduction in the much simpler case when the process has no jumps. It is fair to remark that the behavior of harmonic functions at infinity usually requires a specialized approach [54, Lemma 4.7 and Theorem 4.9] but for our isotropic α-stable Lévy process it reduces to the behavior at the origin thanks to the Kelvin transform [17], see also Kwaśnicki [61, Corollary 3].

3 Full picture

Theorem 1.1 and Proposition 1.2 are manifestations of phenomena which we present below in this section.

By (BHP), a finite positive limit

$$C_0 = \lim_{\Gamma \ni x \to 0} G_\Gamma(x, \mathbf{1})/M(x) \tag{3.1}$$

exists. We denote

$$\kappa_\Gamma(z) = \int_{\Gamma^c} \nu(z - y)\mathrm{d}y, \tag{3.2}$$

where ν is a jump measure defined in (2.1), and we define

$$C_1 = C_0 \int_\Gamma \int_\Gamma K(y)p_1^\Gamma(y, z)\kappa_\Gamma(z)\mathrm{d}z\mathrm{d}y. \tag{3.3}$$

Theorem 3.1. *We have $0 < C_1 < \infty$ and*

$$\lim_{\Gamma \ni x \to 0} \frac{\mathbb{P}_x(\tau_\Gamma > 1)}{M(x)} = C_1. \tag{3.4}$$

Here is a reformulation using the scaling property (2.17) of X and β-homogeneity (2.5) of M.

Corollary 3.2. *If $t > 0$, $x \in \Gamma$ and $t^{-1/\alpha}x \to 0$, then*

$$\mathbb{P}_x(\tau_\Gamma > t) \sim C_1 M(x)t^{-\beta/\alpha}. \tag{3.5}$$

We note in passing that Corollary 3.2 refines Lemma 4.2 of Bañuelos and Bogdan [3]. The proofs of Theorem 3.1 and the following results of this section are mostly deferred to Section 4, to allow for a streamlined presentation. The next theorem is the main result of the paper.

Theorem 3.3. *The following limit exists,*

$$n_t(y) = \lim_{\Gamma \ni x \to 0} \frac{p_t^\Gamma(x,y)}{\mathbb{P}_x(\tau_\Gamma > 1)}, \quad (t,y) \in (0,\infty) \times \Gamma. \tag{3.6}$$

It is a finite strictly positive jointly continuous function of t and y, and for $0 < s, t < \infty$, $y \in \Gamma$,

$$n_t(y) = t^{-(d+\beta)/\alpha} n_1(t^{-1/\alpha} y), \tag{3.7}$$

$$n_1(y) \approx \frac{\mathbb{P}_y(\tau_\Gamma > 1)}{(1+|y|)^{d+\alpha}}, \tag{3.8}$$

$$n_{t+s}(y) = \int_\Gamma n_t(z) p_s^\Gamma(z,y) \mathrm{d}z. \tag{3.9}$$

We will see in the proofs of Theorem 3.3 and 1.1 that n_1 is the density for the Yaglom limit μ with respect to the Lebesgue measure:

$$\mu(A) = \int_A n_1(z)\,\mathrm{d}z, \quad A \subset \Gamma. \tag{3.10}$$

In view of (3.9), $n_t(y)\mathrm{d}y$ defines an entrance law of excursions from 0 into Γ, cf. Rivero and Haas [47], Blumenthal [14, page 104] and Bañuelos et al. [2].

We note that Yano [80] studies excursions of symmetric Lévy processes into $\mathbb{R} \setminus \{0\}$, a situation not discusses in this paper. We however note that in our situation

$$\int_\Gamma n_t(x)\mathrm{d}x = t^{-\beta/\alpha}, \quad t > 0,$$

which nicely corresponds with [80, Example 1.1], because $\beta = \alpha - 1$ for $\Gamma = \mathbb{R} \setminus \{0\}$ and $\alpha \in (1,2)$, see [3].

Example 3.4. If $d = 1$, Γ is the half-line $(0,\infty)$, then $M(x) = x^{\alpha/2}$ for $x > 0$ [3, Example 3.2]. By [23, Theorem 2],

$$\mathbb{P}_x(\tau_\Gamma > 1) \approx x^{\alpha/2} \wedge 1, \quad x > 0.$$

By (3.8),

$$n_1(x) \approx x^{\alpha/2} \wedge x^{-d-\alpha}, \quad x > 0.$$

Therefore by (3.7),

$$n_t(x) \approx (x^{\alpha/2} t^{-1-1/\alpha}) \wedge (x^{-1-\alpha} t^{1/2}), \quad t, x > 0. \tag{3.11}$$

The first expression gives the minimum if $x^\alpha \leq t$ (small space), and the second – if $x^\alpha > t$ (short time). Estimates of $n_t(x)$ for half-spaces in $\mathbb{R}^d$ may be obtained in a similar way.

Some of the other objects we study can also be expressed in terms of n. Namely we have

$$\begin{aligned}
K(y) &= \lim_{\Gamma \ni x \to 0} \frac{G_\Gamma(x,y)}{\mathbb{P}_x(\tau_\Gamma > 1)} \frac{\mathbb{P}_x(\tau_\Gamma > 1)}{M(x)} \frac{M(x)}{G_\Gamma(x,\mathbf{1})} \\
&= \frac{C_1}{C_0} \lim_{\Gamma \ni x \to 0} \int_0^\infty \frac{p_t^\Gamma(x,y)}{\mathbb{P}_x(\tau_\Gamma > 1)} \mathrm{d}t \\
&= \frac{C_1}{C_0} \int_0^\infty n_t(y) \mathrm{d}t.
\end{aligned} \tag{3.12}$$

Therefore,

$$\frac{C_0}{C_1} = \int_0^\infty n_t(\mathbf{1}) \mathrm{d}t \tag{3.13}$$

may be interpreted as occupation time density at $\mathbf{1}$ for the excursions from the vertex into Γ.

We note in passing that the spatial asymptotics of the heat kernel at infinity was given in the works of Blumenthal and Getoor [11, 12] (see also [39]), who showed that $p_t(x) \sim t\nu(x)$ as $t|x|^{-\alpha} \to 0$. More results of this type for unimodal Lévy processes can be found in recent works of Tomasz Grzywny et al., including [39], however, the above papers only concern $\Gamma = \mathbb{R}^d$.

Our approach to Theorem 3.3 depends on three properties. First, the scaling (2.5) yields

$$\mathbb{P}_x\left(\tau_\Gamma > t, \frac{X_t}{t^{1/\alpha}} \in A\right) = \mathbb{P}_{t^{-1/\alpha}x}\left(\tau_\Gamma > 1, X_1 \in A\right) \tag{3.14}$$

and

$$\mathbb{P}_x\left(\tau_\Gamma > t\right) = \mathbb{P}_{t^{-1/\alpha}x}\left(\tau_\Gamma > 1\right). \tag{3.15}$$

Then the Ikeda-Watanabe formula (2.16) gives the representation

$$\mathbb{P}_x(\tau_\Gamma > 1) = G_\Gamma P_1^\Gamma \kappa_\Gamma(x). \tag{3.16}$$

Recall that $\kappa_\Gamma(x)$ may be considered as the killing intensity because it is the intensity of jumps of X from x to Γ^c. Similarly, $P_1^\Gamma \kappa_\Gamma(x)$ may be interpreted as the intensity of killing precisely one unit of time from now.

To actually prove the existence of n_t in (3.6), we use the asymptotics of Green potentials at the vertex 0.

Lemma 3.5. *If f is a measurable function bounded on Γ_1 and $G_\Gamma|f|(\mathbf{1}) < \infty$, then $\int_\Gamma K(y)|f(y)|\mathrm{d}y < \infty$ and*

$$\lim_{\Gamma \ni x \to 0} \frac{G_\Gamma f(x)}{M(x)} = C_0 \int_\Gamma K(y)f(y)\mathrm{d}y. \tag{3.17}$$

4 Proofs

4.1 Proof of Lemma 3.5

If $G_\Gamma|f|(x) < \infty$ for some $x \in \Gamma$, then by [20, Lemma 5.1], $G_\Gamma|f|(x) < \infty$ for almost all $x \in \mathbb{R}^d$. Let $0 < \delta < 1$. Choose $x_1 \in \Gamma_{\delta/2}$ so that $G_\Gamma|f|(x_1) < \infty$. By (BHP),

$$\frac{G_\Gamma(x,y)}{G_\Gamma(x,\mathbf{1})} \le c_1 \frac{G_\Gamma(x_1,y)}{G_\Gamma(x_1,\mathbf{1})}, \quad x, y \in \Gamma, \quad |x| < \delta/2, \quad |y| \ge \delta \tag{4.1}$$

for some constant c_1. By (2.19), (4.1), and Fatou's lemma we see that the right-hand side of (3.17) is finite. Observe that we also have

$$\int_{\Gamma \setminus \Gamma_\delta} G_\Gamma(x_1,y)|f(y)|\mathrm{d}y \le \int_\Gamma G_\Gamma(x_1,y)|f(y)|\mathrm{d}y < \infty. \tag{4.2}$$

By (2.19), (4.1), (4.2) and the dominated convergence theorem,

$$\lim_{\Gamma \ni x \to 0} \int_{\Gamma \setminus \Gamma_\delta} \frac{G_\Gamma(x,y)}{G_\Gamma(x,\mathbf{1})} f(y)\mathrm{d}y = \int_{\Gamma \setminus \Gamma_\delta} K(y)f(y)\mathrm{d}y. \tag{4.3}$$

Next we consider the integral over Γ_δ. By our assumptions on f, a change of variables, and the scaling property $G_\Gamma(\delta x, \delta y) = \delta^{-d+\alpha}G_\Gamma(x,y)$ we can conclude that for some

constant c_2,

$$\int_{\Gamma_\delta} G_\Gamma(x,y)|f(y)|\mathrm{d}y \leq c_2\delta^d \int_{\Gamma_1} G_\Gamma(x,\delta z)\mathrm{d}z = c_2\delta^\alpha \int_{\Gamma_1} G_\Gamma(\delta^{-1}x,z)\mathrm{d}z\,, \quad x \in \Gamma_\delta\,. \quad (4.4)$$

Now, by (BHP), for some contant c_3,

$$\frac{G_\Gamma(v,y)}{M(y)} \leq c_3 \frac{G_\Gamma(v,\mathbf{1})}{M(\mathbf{1})} = c_3 G_\Gamma(v,\mathbf{1}), \quad v \in \Gamma \setminus \Gamma_2\,, \; y \in \Gamma_1\,. \quad (4.5)$$

Indeed, by the symmetry of G_Γ, the regular harmonicity of $y \mapsto G_\Gamma(v,y)$ on Γ_1 follows from (2.15). Furthermore, the continuity of α-harmonic functions allows us to use $y = \mathbf{1} \in \partial\Gamma_1$ in the inequality (4.5). By (2.15) and (4.5) we have

$$\begin{aligned} G_\Gamma(x,y) &= G_{\Gamma_2}(x,y) + \mathbb{E}_x\left[G_\Gamma(X_{\tau_{\Gamma_2}},y)\right] \leq G_{\Gamma_2}(x,y) + c_3\mathbb{E}_x\left[G_\Gamma(X_{\tau_{\Gamma_2}},\mathbf{1})\right]M(y) \\ &\leq G_{\Gamma_2}(x,y) + c_3 G_\Gamma(x,\mathbf{1})M(y)\,, \quad x\,, \; y \in \Gamma_1\,. \end{aligned} \quad (4.6)$$

By identities (4.4), (4.6) and the local boundedness of M, we have

$$\int_{\Gamma_\delta} G_\Gamma(x,y)|f(y)|\mathrm{d}y \leq c_2\delta^\alpha \int_{\Gamma_1} G_{\Gamma_2}(\delta^{-1}x,z)\mathrm{d}z + c_2 c_3\delta^\alpha G_\Gamma(\delta^{-1}x,\mathbf{1}) \quad (4.7)$$

for every $x \in \Gamma_\delta$. Let

$$c_4 = \inf_{z\in\Gamma_1}\int_{\Gamma\setminus\Gamma_2}\nu(z-w)\mathrm{d}w\,.$$

Clearly, $c_4 > 0$. By the Ikeda–Watanabe formula (2.16) and (BHP), for $x \in \Gamma_\delta$,

$$\begin{aligned} \int_{\Gamma_1} G_{\Gamma_2}(\delta^{-1}x,z)\mathrm{d}z &\leq c_4^{-1}\int_{\Gamma\setminus\Gamma_2}\int_{\Gamma_1} G_{\Gamma_2}(\delta^{-1}x,z)\frac{\mathcal{A}_{d,\alpha}}{|w-z|^{d+\alpha}}\mathrm{d}z\mathrm{d}w \\ &\leq c_4^{-1}\mathbb{P}_{\delta^{-1}x}\left(X_{\tau_{\Gamma_2}} \in \Gamma\right) \\ &\leq c_5 G_\Gamma(\delta^{-1}x,\mathbf{1})\,. \end{aligned}$$

Again by (BHP) we have

$$G_\Gamma(\delta^{-1}x,\mathbf{1}) \approx M(\delta^{-1}x) = \delta^{-\beta}M(x) \approx \delta^{-\beta}G_\Gamma(x,\mathbf{1})$$

for $x \in \Gamma_{\delta/2}$. In view of (4.7),

$$\int_{\Gamma_\delta} \frac{G_\Gamma(x,y)}{G_\Gamma(x,\mathbf{1})}|f(y)|\mathrm{d}y \leq c_6\delta^{\alpha-\beta}\,, \quad x \in \Gamma_{\delta/2} \quad (4.8)$$

for some constant c_6. From (4.3), Fatou's lemma and (4.8) it follows that

$$\int_{\Gamma\setminus\Gamma_\delta} K(y)f(y)\mathrm{d}y \leq \liminf_{\Gamma\ni x\to 0}\frac{G_\Gamma f(x)}{G_\Gamma(x,\mathbf{1})} \leq \limsup_{\Gamma\ni x\to 0}\frac{G_\Gamma f(x)}{G_\Gamma(x,\mathbf{1})} \leq \int_{\Gamma\setminus\Gamma_\delta} K(y)f(y)\mathrm{d}y + c_6\delta^{\alpha-\beta}\,.$$

Taking the limit in the above identity as $\delta \to 0$ and using the fact that $\alpha > \beta$, we establish that

$$\lim_{\Gamma\ni x\to 0}\frac{G_\Gamma f(x)}{G_\Gamma(x,\mathbf{1})} = \int_\Gamma K(y)f(y)\mathrm{d}y\,. \quad (4.9)$$

We then apply (3.1), which completes the proof. $\qquad\square$

4.2 Proof of (3.16)

We observe that by the Lipschitz condition and Sztonyk [76, Theorem 1] we have $\mathbb{P}_x\left(X_{\tau_\Gamma} \in \partial\Gamma\right) = 0$ for every $x \in \Gamma$. Thus,

$$\mathbb{P}_x\left(X_{\tau_\Gamma-} = X_{\tau_\Gamma}\right) = 0. \tag{4.10}$$

Now for $x \in \Gamma^c$ we have $\tau_\Gamma = 0$ $\mathbb{P}^x$-a.s. and (3.16) holds true. To prove (3.16) for $x \in \Gamma$, observe that by (2.16) and (4.10),

$$\begin{aligned}
\mathbb{P}_x(\tau_\Gamma > 1) &= \int_1^\infty \int_\Gamma p_s^\Gamma(x,z)\kappa_\Gamma(z)\mathrm{d}z\mathrm{d}s \\
&= \int_0^\infty \int_\Gamma \int_\Gamma p_s^\Gamma(x,w)p_1^\Gamma(w,z)\mathrm{d}w\,\kappa_\Gamma(z)\mathrm{d}z\mathrm{d}s \\
&= G_\Gamma P_1^\Gamma \kappa_\Gamma(x).
\end{aligned}$$

$\square$

4.3 Proof of Theorem 3.1

We will prove that $f(x) = P_1^\Gamma \kappa_\Gamma(x)$ satisfies the assumptions of Lemma 3.5, i.e. $G_\Gamma|f|(\mathbf{1}) < \infty$ and f is bounded on Γ_1. Then (3.4) follows from the representation (3.16) and the identity (3.17). By the proof of (3.16), $G_\Gamma P_1^\Gamma \kappa_\Gamma(\mathbf{1}) \le 1$. It suffices to prove

$$P_1^\Gamma \kappa_\Gamma(x) \approx \mathbb{P}_x(\tau_\Gamma > 1) \quad \text{for } x \in \Gamma_1, \tag{4.11}$$

which clearly gives boundedness of $P_1^\Gamma \kappa_\Gamma$. If $|x| \le 1$, then by (2.7),

$$p_1(x,y) \approx 1 \wedge |x-y|^{-d-\alpha} \approx (1+|y|)^{-d-\alpha}. \tag{4.12}$$

Furthermore, by [22] or [23, Theorem 2], the following approximate factorization holds,

$$p_1^\Gamma(x,y) \approx \mathbb{P}_x(\tau_\Gamma > 1)\mathbb{P}_y(\tau_\Gamma > 1)p_1(x,y), \quad x,y \in \Gamma. \tag{4.13}$$

Thus,

$$P_1^\Gamma \kappa_\Gamma(x) \approx \mathbb{P}_x(\tau_\Gamma > 1) \int_\Gamma \mathbb{P}_y(\tau_\Gamma > 1)(1+|y|)^{-d-\alpha}\kappa_\Gamma(y)\mathrm{d}y, \quad x \in \Gamma_1. \tag{4.14}$$

By [56] we have $G_\Gamma(x,w) > 0$ for all $x,w \in \Gamma$. Since $G_\Gamma P_1^\Gamma \kappa_\Gamma(\mathbf{1}) < \infty$, we have that $P_1^\Gamma \kappa_\Gamma$ is finite almost everywhere, in particular the integral in (4.14) is finite. This finishes the proof of (4.11). $\square$

As we noted, Corollary 3.2 is a reformulation of Theorem 3.1 using (2.17) and (2.5).

4.4 Proof of Theorem 3.3

Consider the family of measures

$$\mu_x(A) = \frac{\int_A p_1^\Gamma(x,y)\mathrm{d}y}{\mathbb{P}_x(\tau_\Gamma > 1)}, \quad x \in \Gamma, \ A \subset \mathbb{R}^d. \tag{4.15}$$

We start by proving that the family $\{\mu_x : x \in \Gamma_1\}$ is tight. Indeed, by (4.13) and (4.12) we can bound their densities by a fixed integrable function:

$$\frac{p_1^\Gamma(x,y)}{\mathbb{P}_x(\tau_\Gamma > 1)} \approx \mathbb{P}_y(\tau_\Gamma > 1)(1+|y|)^{-d-\alpha}, \quad x \in \Gamma_1, \ y \in \mathbb{R}^d. \tag{4.16}$$

We will prove now that the measures μ_x converge weakly to a probability measure μ on Γ as $\Gamma \ni x \to 0$:

$$\mu_x \Rightarrow \mu \qquad \text{as } \Gamma \ni x \to 0\,. \tag{4.17}$$

To prove (4.17), we consider an arbitrary sequence $\{x_n\}$ such that $\Gamma_1 \ni x_n \to 0$. By the tightness of the family of measures $\{\mu_x : x \in \Gamma_1\}$ there exists a subsequence $\{x_{n_k}\}$ such that $\mu_{x_{n_k}}$ converge weakly to a probability measure μ as $k \to \infty$.

Let $\phi \in C_c^\infty(\Gamma)$ and $u_\phi = -\Delta^{\alpha/2}\phi$. The function u_ϕ is bounded and continuous and $G_\Gamma u_\phi(x) = \phi(x)$ for $x \in \mathbb{R}^d$, see [26, Eq. (19)] and [27, Eq. (11)] for more details. Furthermore, if $\text{dist}\,(x, \sup \phi) \geq 1$, then for some constants c_7, c_8,

$$\left|\Delta^{\alpha/2}\phi(x)\right| = \left|\int_{\mathbb{R}^d} \phi(y)\nu(y-x)\mathrm{d}y\right| \leq c_7 \int_{\mathbb{R}^d} |\phi(y)|\,(1+|x|)^{-d-\alpha}\,\mathrm{d}y \leq c_8 \left(1 \wedge |x|^{-d-\alpha}\right)\,.$$

By (2.7), $|u_\phi(x)| \leq c_9 p_1(x, 0)$. Then,

$$P_1^\Gamma |u_\phi|(x) \leq c_9 p_2(x)\,, \tag{4.18}$$

for some constant c_9 and for every $x \in \Gamma$,

$$G_\Gamma P_1^\Gamma |u_\phi|(x) \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_\Gamma(x, y) p_1(y, z)|u_\phi(z)|\mathrm{d}z\mathrm{d}y \leq c_9 \int_{\mathbb{R}^d} G_\Gamma(x, y) p_2(y, 0)\mathrm{d}y < \infty\,,$$

see [27, Eq. (74)]. By Fubini's theorem,

$$G_\Gamma P_1^\Gamma u_\phi(x) = P_1^\Gamma G_\Gamma u_\phi(x) = P_1^\Gamma \phi(x)\,. \tag{4.19}$$

It follows from Lemma 3.5 and (4.18) that

$$\lim_{\Gamma \ni x \to 0} \frac{P_1^\Gamma \phi(x)}{M(x)} = \lim_{\Gamma \ni x \to 0} \frac{G_\Gamma P_1^\Gamma u_\phi(x)}{M(x)} = C_0 \int_\Gamma K(y) P_1^\Gamma u_\phi(y)\mathrm{d}y\,.$$

Denoting $\mu_x(\phi) = \int_\Gamma \phi(y)\,\mu_x(\mathrm{d}y)$ and applying Theorem 3.1 we get a finite limit

$$\lim_{\Gamma \ni x \to 0} \mu_x(\phi) = \lim_{\Gamma \ni x \to 0} \frac{P_1^\Gamma \phi(x)}{\mathbb{P}_x(\tau_\Gamma > 1)} = \frac{\int_\Gamma K(y) P_1^\Gamma u_\phi(y)\mathrm{d}y}{\int_\Gamma K(y) P_1^\Gamma \kappa_\Gamma(y)\mathrm{d}y}\,.$$

In particular, $\mu(\phi) = \lim_{k \to \infty} \mu_{x_{n_k}}(\phi)$ does not depend on the choice of the subsequence $\{x_{n_k}\}$. Thus, μ_x weakly converges to this μ as $\Gamma \ni x \to 0$.

We are now in a position to prove that n_1 is the density of the Yaglom limit μ appearing in (4.17) and that n_t is well-defined. By the Chapman-Kolmogorov equation applied to $\phi_y(\cdot) = p_1^\Gamma(\cdot, y) \in C_0(\mathbb{R}^d)$,

$$p_2^\Gamma(x, y) = \int_\Gamma p_1^\Gamma(x, z) p_1^\Gamma(z, y)\mathrm{d}z = P_1^\Gamma \phi_y(x)\,, \quad x, y \in \Gamma\,. \tag{4.20}$$

Thus, for all $y \in \Gamma$,

$$\frac{p_2^\Gamma(x, y)}{\mathbb{P}_x(\tau_\Gamma > 1)} = \frac{P_1^\Gamma \phi_y(x)}{\mathbb{P}_x(\tau_\Gamma > 1)} = \mu_x(\phi_y) \to \mu(\phi_y) < \infty\,, \tag{4.21}$$

as $\Gamma \ni x \to 0$, see Theorem 3.1 and (4.17). This proves the existence of the limit n_t defined in (3.6) for $t = 2$. Using the existence of this limit, the scaling property and Corollary 3.2 we can conclude now that for any $(t, y) \in (0, \infty) \times \Gamma$ the following holds true

$$\begin{aligned}
n_t(y) &= \lim_{\Gamma \ni x \to 0} \frac{p_t^\Gamma(x, y)}{\mathbb{P}_x(\tau_\Gamma > 1)} \\
&= (t/2)^{-d/\alpha} \lim_{\Gamma \ni x \to 0} \frac{p_2^\Gamma \left((t/2)^{-1/\alpha}x, (t/2)^{-1/\alpha}y\right)}{\mathbb{P}_{(t/2)^{-1/\alpha}x}(\tau_\Gamma > 1)} \lim_{\Gamma \ni x \to 0} \frac{\mathbb{P}_x(\tau_\Gamma > t/2)}{\mathbb{P}_x(\tau_\Gamma > 1)} \\
&= (t/2)^{-(d+\beta)/\alpha} n_2((t/2)^{-1/\alpha}y)\,.
\end{aligned} \tag{4.22}$$

This proves the existence of the limit $n_t(y)$ for general $t > 0$, and the equation (3.7). By (4.16) we get (3.8). By the weak convergence (4.17), and by (4.16) along with the dominated convergence theorem, we get that for every bounded continuous function ϕ on Γ we have

$$\mu(\phi) = \lim_{\Gamma \ni x \to 0} \int_\Gamma \frac{p_1^\Gamma(x, y)}{\mathbb{P}_x(\tau_\Gamma > 1)} \phi(y) \mathrm{d}y = \int_\Gamma n_1(y)\phi(y)\mathrm{d}y \,.$$

This proves (3.10). Note that (3.9) follows directly from the Chapman-Kolmogorov equation and the dominated convergence theorem:

$$n_{t+s}(y) = \lim_{\Gamma \ni x \to 0} \int_\Gamma \frac{p_t^\Gamma(x, z)}{\mathbb{P}_x(\tau_\Gamma > 1)} p_s^\Gamma(z, y)\mathrm{d}z = P_s^\Gamma n_t(y) \,.$$

To end the proof we show that $n_t(y)$ is jointly continuous on $(0, \infty) \times \Gamma$. Indeed, $p_1^\Gamma(z, y)/p_1^\Gamma(z, y_1) \approx 1$ for $z \in \Gamma$, if $y, y_1 \in \Gamma$ are close to each other. The continuity of $n_2(y)$ follows from the dominated convergence theorem and the continuity of p_1^Γ. The joint continuity of $n_t(y)$ follows from the scaling property. $\qquad \square$

4.5 Relatively uniform convergence

Lemma 4.1. *If* $0 \leq c^{-1}f_n \leq f_m \leq cf_n$ *for all* m, n, *and* $f = \lim f_n$, *then* $\lim \int f_n \mathrm{d}\eta = \int f\mathrm{d}\eta$.

This is true because if the integral $\int f \mathrm{d}\eta$ is finite, then the dominated convergence theorem applies.

4.6 Proof of Theorem 1.1

By the scaling property of X_t we have

$$\mathbb{P}_x \left(\frac{X_t}{t^{1/\alpha}} \in A \middle| \tau_\Gamma > t \right) = \frac{\mathbb{P}_x \left(\tau_\Gamma > t, \frac{X_t}{t^{1/\alpha}} \in A \right)}{\mathbb{P}_x(\tau_\Gamma > t)} = \frac{\mathbb{P}_{t^{-1/\alpha}x}(\tau_\Gamma > 1, X_1 \in A)}{\mathbb{P}_{t^{-1/\alpha}x}(\tau_\Gamma > 1)}$$

$$= \frac{\int_A p_1^\Gamma(t^{-1/\alpha}x, y)\mathrm{d}y}{\mathbb{P}_{t^{-1/\alpha}x}(\tau_\Gamma > 1)}, \quad x \in \Gamma, \ t > 0 \,.$$

By Theorem 3.3 and (4.16), $p_1^\Gamma(t^{-1/\alpha}x, y)/\mathbb{P}_{t^{-1/\alpha}x}(\tau_\Gamma > 1)$ converges relatively uniformly to $n_1(y)$ as $t \to \infty$, in the sense of the condition in Lemma 4.1. Therefore, $\mathbb{P}_x\left(\frac{X_t}{t^{1/\alpha}} \in A | \tau_\Gamma > t\right) \to \int_A \eta_1(y)\mathrm{d}y = \mu(A)$. $\qquad \square$

4.7 Proof of Proposition 1.2

Recall that by (3.10) we have $\mu(\mathrm{d}z) = n_1(z)\,\mathrm{d}z$. By using (3.9) and (3.7), for $A \subset \mathbb{R}^d$ we get

$$\mathbb{P}_\mu \left(\frac{X_t}{(t+1)^{1/\alpha}} \in A, \tau_\Gamma > t \right) = \int_\Gamma \int_{(t+1)^{1/\alpha}A} n_1(x)p_t^\Gamma(x, y)\mathrm{d}y\mathrm{d}x$$

$$= \int_{(t+1)^{1/\alpha}A} n_{t+1}(y)\mathrm{d}y$$

$$= \int_{(t+1)^{1/\alpha}A} (t+1)^{-(d+\beta)/\alpha} n_1\left((t+1)^{-1/\alpha}y\right) \mathrm{d}y$$

$$= (t+1)^{-\beta/\alpha} \int_A n_1(y)\mathrm{d}y = (t+1)^{-\beta/\alpha}\mu(A) \,.$$

In particular, $\mathbb{P}_\mu(\tau_\Gamma > t) = (t+1)^{-\beta/\alpha}$, which ends the proof. $\qquad \square$

5 Symmetric Cauchy process on half-line

Let $d = \alpha = 1$ and $\Gamma = (0, \infty)$. Then X_t is the symmetric Cauchy process on $\mathbb{R}$ and

$$\tau_\Gamma = \inf\{t \geq 0 : X_t < 0\},$$

which is sometimes called the ruin time. For this particular situation we can add specific spectral information on the Yaglom limit μ. Following [58], for $x > 0$, we let

$$r(x) = \frac{\sqrt{2}}{2\pi} \int_0^\infty \frac{t}{(1+t^2)^{5/4}} \exp\left(\frac{1}{\pi} \int_0^t \frac{\log s}{1+s^2} ds\right) e^{-tx} dt \tag{5.1}$$

and

$$\psi(x) = \sin\left(x + \frac{\pi}{8}\right) - r(x). \tag{5.2}$$

Theorem 5.1. *If X_t is the symmetric Cauchy process on $\mathbb{R}$ and $\Gamma = (0, \infty)$, then μ has the density function*

$$n_1(y) = \lim_{x \to 0+} \frac{p_1^\Gamma(x, y)}{\mathbb{P}_x(\tau_\Gamma > 1)} = \sqrt{\frac{\pi}{2}} \int_0^\infty \lambda^{1/2} \psi(\lambda y) e^{-\lambda} d\lambda, \quad y > 0. \tag{5.3}$$

Proof. By [3, Example 3.2 and 3.4], we have

$$M(x) = (x \vee 0)^{1/2}, \quad K(x) = (x \vee 0)^{-1/2}, \quad x \in \mathbb{R}.$$

The Green function $G_\Gamma(x, y)$ is given by the well-known Riesz's formula:

$$G_\Gamma(x, y) = \frac{1}{\pi} \arcsin \sqrt{\frac{4xy}{(x-y)^2}}, \quad x, y > 0; \tag{5.4}$$

see [13] or [32, Theorem 3.3] with $m = 0$. Thus the constant C_0 defined in (3.1) is given by

$$C_0 = \lim_{x \to 0+} \frac{G_\Gamma(x, 1)}{M(x)} = \frac{2}{\pi}. \tag{5.5}$$

For $t > 0$ we define

$$\xi(t) = \frac{1}{\pi} \frac{t}{(1+t^2)^{5/4}} \exp\left(-\frac{1}{\pi} \int_0^t \frac{\log s}{1+s^2} ds\right).$$

Note that $\int_0^\infty \log s/(1+s^2) ds = 0$. Thus

$$\xi(t) \sim \frac{1}{\pi} t^{-3/2} \quad \text{as } t \to \infty. \tag{5.6}$$

It follows from [58, Theorem 5] and the line following that theorem that

$$\mathbb{P}_x(\tau_\Gamma \in dt) = \frac{1}{t} \xi\left(\frac{t}{x}\right) dt. \tag{5.7}$$

Therefore

$$\mathbb{P}_x(\tau_\Gamma > 1) = \int_1^\infty \frac{1}{t} \xi\left(\frac{t}{x}\right) dt = \int_{x^{-1}}^\infty \frac{\xi(t)}{t} dt \sim \frac{2}{\pi} x^{1/2} = \frac{2}{\pi} M(x) \tag{5.8}$$

as $x \to 0+$ and hence $C_1 = C_0$, where C_1 is defined in (3.3).

By [58, (5.7)] we have that $0 \leq r(x) \leq r(0) = \sin(\pi/8)$ and $|\psi(x)| \leq 2$ for $x \geq 0$, where $r(x)$ and $\psi(x)$ are defined in (5.1) and (5.2), respectively. Further, by [58, Theorem 2]

the function $\psi_\lambda(x) = \psi(\lambda x)$ is an eigenfunction of the semigroup P_t^Γ acting on $C(\Gamma)$, with the eigenvalue $\mathrm{e}^{-\lambda t}$. Thus

$$p_t^\Gamma(x,y) = \frac{2}{\pi} \int_0^\infty \psi_\lambda(x)\psi_\lambda(y)\mathrm{e}^{-\lambda t}\mathrm{d}\lambda \, ; \tag{5.9}$$

see [58, (7.4)]. Note that

$$r'(x) = -\frac{\sqrt{2}}{2\pi} \int_0^\infty \frac{t^2}{(1+t^2)^{5/4}} \exp\left(\frac{1}{\pi}\int_0^t \frac{\log s}{1+s^2}\mathrm{d}s\right)\mathrm{e}^{-tx}\mathrm{d}t \, .$$

Since

$$-\int_0^t \frac{\log s}{1+s^2}\mathrm{d}s = \int_t^\infty \frac{\log s}{1+s^2}\mathrm{d}s$$

is positive for all $t > 0$ and it is regularly varying at ∞ of index -1, the following estimates hold for $x > 0$:

$$\int_0^1 \frac{t^2}{(1+t^2)^{5/4}} \exp\left(\frac{1}{\pi}\int_0^t \frac{\log s}{1+s^2}\mathrm{d}s\right)\mathrm{e}^{-tx}\mathrm{d}t \le 1 \, ,$$

$$\left| \int_1^\infty \frac{t^2}{(1+t^2)^{5/4}} \exp\left(\frac{1}{\pi}\int_0^t \frac{\log s}{1+s^2}\mathrm{d}s\right)\mathrm{e}^{-tx}\mathrm{d}t - \int_1^\infty t^{-1/2}\mathrm{e}^{-tx}\mathrm{d}t \right|$$

$$\le \int_1^\infty \left| \frac{t^2}{(1+t^2)^{5/4}} - t^{-1/2} \right|\mathrm{d}t + \int_1^\infty t^{-1/2} \left| 1 - \exp\left(\frac{1}{\pi}\int_0^t \frac{\log s}{1+s^2}\mathrm{d}s\right) \right|\mathrm{d}t$$

$$\le 1 + \frac{1}{\pi}\int_1^\infty t^{-1/2}\int_t^\infty \frac{\log s}{1+s^2}\mathrm{d}s\mathrm{d}t < \infty$$

and

$$\left| \int_1^\infty t^{-1/2}\mathrm{e}^{-tx}\mathrm{d}t - \sqrt{\pi}x^{-1/2} \right| \le \int_0^1 t^{-1/2}\mathrm{e}^{-tx}\mathrm{d}t \le 2 \, .$$

Thus there exists a constant $c_{10} > 0$ such that for $x > 0$,

$$\left| r(x) - r(0) - \sqrt{\frac{2}{\pi}}x^{1/2} \right| \le \int_0^x \left| r'(s) + \frac{1}{\sqrt{2\pi}}s^{-1/2} \right|\mathrm{d}s \le c_{10}x \tag{5.10}$$

and

$$\left| \psi(x) - \sqrt{\frac{2}{\pi}}x^{1/2} \right| \le \left| \sin\left(x + \frac{\pi}{8}\right) - \sin\frac{\pi}{8} \right| + \left| r(x) - r(0) - \sqrt{\frac{2}{\pi}}x^{1/2} \right| \le c_{10}x \, . \tag{5.11}$$

The inequality (5.11) with x being replaced by λx and the identity (5.9) imply that

$$\left| p_1^\Gamma(x,y) - \frac{2}{\pi}\sqrt{\frac{2}{\pi}}x^{1/2}\int_0^\infty \lambda^{1/2}\psi(\lambda y)\mathrm{e}^{-\lambda}\mathrm{d}\lambda \right| \le c_{10}x \int_0^\infty \lambda|\psi(\lambda y)|\mathrm{e}^{-\lambda}\mathrm{d}\lambda \, . \tag{5.12}$$

The identity (5.3) now follows from (5.8) and (5.12). Since we have (3.10), the proof is complete. □

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