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Path large deviations for interacting diffusions with local mean-field interactions in random environment *

Patrick E. Müller †

Abstract

We consider a system of N^d spins in random environment with a random local mean-field type interaction. Each spin has a fixed spatial position on the torus $\mathbb{T}^d$, an attached random environment and a spin value in $\mathbb{R}$ that evolves according to a space and environment dependent Langevin dynamic. The interaction between two spins depends on the spin values, the spatial distance and the random environment of both spins. We prove the path large deviation principle from the hydrodynamic (or local mean-field McKean-Vlasov) limit and derive different expressions of the rate function for the empirical process and for the empirical measure of the paths. To this end we generalize an approach of Dawson and Gärtner.

Keywords: large deviations; interacting diffusion; interacting particle systems; local mean-field McKean-Vlasov equation.
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1 Introduction

The model. We consider a system of N^d interacting spins with spin values $\underline{\theta}_t^N \in \mathbb{R}^{N^d}$ at time $t > 0$, that evolve according to the following Langevin dynamics

$$\begin{aligned} \mathrm{d}\theta_t^{k,N} &= b\left(\frac{k}{N}, w^{k,N}, \theta_t^{k,N}, \mu_t^N\right) \mathrm{d}t + \sigma \mathrm{d}B_t^{k,N}, \\ \theta_0^{k,N} &\sim \nu_{\frac{k}{N}} \in \mathbb{M}_1(\mathbb{R}), \end{aligned} \tag{1.1}$$

for $k \in \mathbb{T}_N^d = \mathbb{Z}^d/N\mathbb{Z}^d$, the periodic d-dimensional lattice of length N. The $B^{k,N}$ are independent Brownian motions, $\sigma \in \mathbb{R}$, and the empirical measure μ_t^N is defined as

$$\mu_t^N := \frac{1}{N^d} \sum_{k \in \mathbb{T}_N^d} \delta_{\left(\frac{k}{N}, w^{k,N}, \theta_t^{k,N}\right)} \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right). \tag{1.2}$$

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†Institut für Angewandte Mathematik, Rheinische Friedrich-Wilhelms-Universität, Endenicher Allee 60, 53115 Bonn, Germany. E-mail: patrickm@uni-bonn.de

The $w^{k,N}$, appearing in (1.1) and (1.2), are independent random variables with values in $\mathcal{W} \subset \mathbb{R}$. Each of these random variable is distributed according to $\zeta_{\frac{k}{N}} \in \mathbb{M}_1(\mathcal{W})$ and is frozen over time. We consider very general drift coefficients $b : \mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \times \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right) \to \mathbb{R}$ (see Assumption 3.1 for details). With $\mathbb{M}_1(Y)$ we denote the space of probability measures on a polish space Y (see Section 2.1 for precise definitions of the all the spaces).

There are three sources of randomness in this system. The spin values at time zero are independently distributed. Moreover, the random variables $w^{k,N}$ represent a random environment that influences the drift. Finally, each spin value is subject to independent stochastic fluctuations, given by independent Brownian motions $B^{k,N}$.

For each $k \in \mathbb{T}_N^d$, we call $\frac{k}{N}$ the (fixed normalised spatial) position of the spin with value $\theta_t^{k,N}$. The evolution of the spin value depends, through the drift coefficient, on its position $\frac{k}{N}$ on the torus, on the random variable $w^{k,N}$ attached to this spin and on the current spin value. Moreover it depends through the empirical measure on the spatial positions, the random environment and the spin values of the other sites. This dependence models the space dependent interaction between the spins.

Given a realisation $\underline{\theta}_{[0,T]}^N = \left\{ t \mapsto \underline{\theta}_t^N \right\}$ of the solution of (1.1) and a realisation of the random environment $\underline{w}^N$, let us denote by $\mu_{[0,T]}^N$ the empirical process, that is the time evolution of the empirical measures μ_t^N defined in (1.2),

$$\mu_{[0,T]}^N \coloneqq \left\{ t \mapsto \mu_t^N \right\} \in \mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right), \tag{1.3}$$

and by L^N the empirical measure on $\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$

$$L^N = L^N\left(\underline{w}^N, \underline{\theta}_{[0,T]}\right) \coloneqq \frac{1}{N^d} \sum_{k \in \mathbb{T}_N^d} \delta_{\left(\frac{k}{N}, w^{k,N}, \theta_{[0,T]}^{k,N}\right)} \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right). \tag{1.4}$$

Special case. The following special case is covered by the more general model (1.1). Let the diffusion coefficient σ be equal to 1, take $\mathcal{W} \subset \mathbb{R}$ compact and choose the drift coefficient as

$$b(x,w,\theta,\mu) = -\partial_\theta \Psi(w,\theta) + \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} J(x-x', w, w')\,\theta' \mu(\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta'), \tag{1.5}$$

for $(x,w,\theta,\mu) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \times \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$, with Ψ a single spin potential and J a weight function of the spatial distance between the spins. For example Ψ can be chosen as $\Psi(w,\theta) = \theta^4 + w\theta$ or $\theta^2 + w\theta$ or $\theta^4 - \theta^2 + w\theta$. Then the first coordinate of the random environment w_1 represents a random chemical potential. With these coefficients, the SDE (1.1) is given by

$$\mathrm{d}\theta_t^{k,N} = -\partial_\theta \Psi\left(w^{k,N}, \theta_t^{k,N}\right)\mathrm{d}t + \frac{1}{N^d} \sum_{j \in \mathbb{T}_N^d} J\left(\frac{k-j}{N}, w^{k,N}, w^{j,N}\right) \theta_t^{j,N} \mathrm{d}t + \mathrm{d}B_t^{k,N}. \tag{1.6}$$

We are mainly interested in this specific model.

Motivation. One can show for the local mean-field system (1.6), that the empirical process $\mu_{[0,T]}^N$ converges to a deterministic continuous trajectory on $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ when the number of spins N tends to infinity (for example see [31] (no random environment) or [27] (slightly different model with bounded interaction)). This is called hydrodynamic limit. Without the random environment, we show in [31] that each measure on this trajectory has a density ξ_t with respect to the Lebesgue measure. Moreover,

$\xi \in \mathsf{C}^{1,0,2}\big((0,T] \times \mathbb{T}^d \times \mathbb{R}\big)$ and the time evolution of ξ is the classical solution of the following PDE (that we call local mean-field McKean-Vlasov equation),

$$\partial_t \xi_t(x,\theta) = \partial_\theta \left(\left(\Psi'(\theta) - \int_{\mathbb{T}^d \times \mathbb{R}} J(x'-x)\,\theta' \xi_t(x',\theta')\,\mathrm{d}\theta' \mathrm{d}x' \right) \xi_t(x,\theta) \right) + \frac{1}{2} \partial_\theta^2 \xi_t(x,\theta). \tag{1.7}$$

The aim of the current paper is to investigate the large deviations from the hydrodynamic limit for the general system (1.1). Large deviation principles are one main ingredient to understand metastability through Freidlin-Wentzell theory (see [18] and the introduction of [13]).

The main novelty of this paper is the investigation of spatially extended versions of mean-field models. Such space-dependent systems are important in many different contexts. This includes, for example, spatial versions of the Kuramoto model (to model systems of oscillators, e.g. [32], [28], [23]), neuronal science ([3], [27], [7], [30] and reference therein), chemical kinetics ([33]) or finance ([19]).

Main results. We prove in this paper large deviation principles of the families of random elements $\left\{\mu^N_{[0,T]}\right\}$ and $\left\{L^N\right\}$. We derive different representations of the corresponding rate functions. Moreover, we show relations between the two principles, the rate functions and the minimizer of the rate function.

To state the rate function of $\left\{\mu^N_{[0,T]}\right\}$, we need the following norm.

Definition 1.1. *For a measure $\pi \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ and ξ a distribution on the space of test functions $\mathsf{C}_c^\infty\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$, define*

$$\begin{aligned} |\xi|_\pi^2 &:= \frac{1}{2} \sup_{f \in \mathbb{D}_\pi} \frac{|\langle \xi, f\rangle|^2}{\sigma^2 \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \left(\partial_\theta f(x,w,\theta)\right)^2 \pi(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)} \\ &= \sup_{f \in \mathsf{C}_c^\infty(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})} \left\{ \langle \xi, f\rangle - \frac{\sigma^2}{2} \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \left(\partial_\theta f(x,w,\theta)\right)^2 \pi(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) \right\}, \end{aligned} \tag{1.8}$$

with $\mathbb{D}_\pi := \left\{ f \in \mathsf{C}_c^\infty\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big) : \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \left(\partial_\theta f(x,w,\theta)\right)^2 \pi(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) \neq 0 \right\}$.

With abuse of notation we also use the symbol $|\xi|_\pi$ for $\pi \in \mathbb{M}_1(\mathbb{R})$ and ξ a distribution on the space of test functions $\mathsf{C}_c^\infty(\mathbb{R})$.

For suitable $\mu \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ and all $(x,w) \in \mathbb{T}^d \times \mathcal{W}$, we denote by $\left(\mathbb{L}_{\mu,x,w}\right)^*$ the formal adjoint of the following operator

$$\mathbb{L}_{\mu,x,w} f(\theta) := \frac{\sigma^2}{2} \partial_{\theta^2}^2 f(\theta) + b(x,w,\theta,\mu)\, \partial_\theta f(\theta), \tag{1.9}$$

acting on $f \in \mathsf{C}_b^2(\mathbb{R})$, i.e. for a $\nu \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$

$$\int f \left(\mathbb{L}_{\mu,x,w}\right)^* \nu := \int \left(\mathbb{L}_{\mu,x,w} f\right) \nu. \tag{1.10}$$

Now we are in the position to state the main results of this paper. Under regularity assumptions on b, stated in Assumption 3.1, and usual assumptions on ν_x and ζ_x, we prove the following theorems in this paper.

Theorem 1.2. *The family $\left\{\mu^N_{[0,T]}\right\}$ satisfies on $\mathsf{C}\big([0,T], \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)\big)$ the large deviation principle with rate function*

$$S_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \int_0^T \left| \partial_t \mu_t - \left(\mathbb{L}_{\mu_t,\cdot,\cdot}\right)^* \mu_t \right|_{\mu_t}^2 \mathrm{d}t + \mathcal{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right), \tag{1.11}$$

for suitable $\mu_{[0,T]} \in \mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)$*. Here* $\mathcal{H}\left(.|.\right)$ *is the relative entropy.*

Hence the rate function $S_{\nu,\zeta}$ measures the deviation from the hydrodynamic equation in a suitable way. The precise statement of this theorem and of the assumptions are given in Section 3.

Theorem 1.3. *The family* $\left\{L^N\right\}$ *satisfies on* $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ *the large deviation principle with a good rate function* I*. This rate function has for suitable* $Q \in \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ *the following expression*

$$I\left(Q\right) = \mathcal{H}\left(Q \middle| P^{I,\Pi(Q)}\right), \tag{1.12}$$

where the measure $P^{I,\Pi(Q)} \in \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ *is defined in a suitable way through diffusions similar to* (1.1) *where a fixed external field depending on* Q *is used instead of the interaction.*

A precise statement of this theorem, including assumptions and the precise definition of $P^{I,\Pi(Q)}$, may be found in Section 5.

Note that all results of this paper can be extended to multidimensional diffusions in $\mathbb{R}^p$ for $p \geq 2$ and random variables in a subspace of $\mathbb{R}^m$ for $m \geq 2$. Multidimensionality in these spaces would only be of notational nature and do not require any new arguments besides those developed in this paper.

Remark 1.4. From the large deviation principles that we prove in this paper, one can easily infer the corresponding large deviation principles for the annealed measures. To do so, use at first the canonical projection of $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right) \to \mathbb{M}_1\left(\mathbb{T}^d \times \mathsf{C}([0,T])\right)$ and finally apply the contraction principle.

Note, that already Theorem 1.2 and 1.3 are annealed large deviations principles, in the sense that they hold under the joint law of the Brownian motions and disorder.

Sufficient assumptions for the special case (1.6)**.** For the special case characterised by (1.6), the following assumptions imply the validity of Theorem 1.2 and 1.3.

Assumption 1.5. *The family of initial distributions* $\{\nu_x\}_{x\in\mathbb{T}^d} \subset \mathbb{M}_1(\mathbb{R})$ *is Feller continuous, i.e.* $\nu_{x^{(n)}}$ *converges to* ν_x *when* $x^{(n)} \to x$*, or equivalently the map* $x \mapsto \int_{\mathbb{R}} f\left(\theta\right)\nu_x\left(\mathrm{d}\theta\right)$ *is continuous for all* $f \in \mathsf{C}_b(\mathbb{R})$*.*

This assumption is for example satisfied, when the ν_x are all the same (i.e. initial spin values are i.i.d.), or when there is a function $g \in \mathsf{C}\left(\mathbb{T}^d\right)$ such that $\nu_x = \delta_{g(x)}$, or when each ν_x is the Gaussian measure associated with the normal distributed with mean $g\left(x\right)$ and variance one.

Assumption 1.6. *The family of distributions of the random environment* $\{\zeta_x\}_{x\in\mathbb{T}^d} \subset \mathbb{M}_1(\mathcal{W})$ *is Feller continuous.*

The regularity assumptions on b for the general case (1.1), simplify for the special case (1.6) to the following assumptions.

Assumption 1.7. *Let* $\mathcal{W}$ *be a compact subset of* $\mathbb{R}$*. The interaction weight* J *is in* $L^2\left(\mathbb{T}^d, \mathsf{C}(\mathcal{W} \times \mathcal{W})\right)$ *and satisfies the following conditions:*

- *There is a* $\overline{J} \in L^2\left(\mathbb{T}^d\right)$*, such that* $\sup_{(w,w')\in\mathcal{W}\times\mathcal{W}} \left|J\left(x,w,w'\right)\right| < \overline{J}\left(x\right)$ *for all* $x \in \mathbb{T}^d$*.*
- J *is even on* $\mathbb{T}^d$*, i.e.* $J\left(x,w,w'\right) = J\left(-x,w,w'\right)$ *for all* $x \in \mathbb{T}^d$ *and* $w,w' \in \mathcal{W}$*.*
- $$\frac{1}{N^d}\sum_{i\in\mathbb{T}_N^d}\sup_{w,w'\in\mathcal{W}}\left|J\left(\frac{i}{N},w,w'\right) - N^d\int_{\Delta_{i,N}} J\left(x,w,w'\right)\mathrm{d}x\right|^2 \to 0, \tag{1.13}$$

  when $N \to \infty$*, with* $\Delta_{i,N} \coloneqq \left\{x \in \mathbb{T}^d : \left|x - \frac{i}{N}\right| < \frac{1}{2N}\right\}$*.*

Example 1.8. This assumption is in particular satisfied in the following cases:

- J is continuous in all variables.
- $J(x,w,w) = J_1(x) J_2(w,w')$ or $J(x,w,w) = J_1(x) + J_2(w,w')$. In both situations:
  - $J_2 \in \mathsf{C}(\mathcal{W}\times\mathcal{W})$, for example $J_2(w,w') = ww'$ or $J_2(w,w') = w - w'$.
  - $J_1 \in L^2(\mathbb{T}^d)$ is even and
    - either continuous, or
    - $J_1 = 1_A$ for $A \subset \mathbb{T}^d$ a rectangle, or
    - J_1 can even have a singularity like $J_1(x) = |x|^{-\frac{1}{2}+\epsilon}$, $\epsilon \in \left(0,\frac{1}{2}\right)$ with $J_1(0) = 0$.

Assumption 1.9. $\Psi(\theta,w) = \overline{\Psi}(\theta) + w_1\theta$, *for* $(w,\theta)\in\mathcal{W}\times\mathbb{R}$, *where* $\overline{\Psi}$ *is a polynomial of even degree* ≥ 2, *with positive leading coefficient. Define*

$$c_\Psi := \liminf_{|\theta|\to\infty} \frac{\overline{\Psi}(\theta)}{|\theta|^2}, \tag{1.14}$$

with $c_\Psi = \infty$ *if the degree of* $\overline{\Psi}$ *is greater than 2. Assume that*

$$c_\Psi > \left\|\overline{J}\right\|_{L^1}. \tag{1.15}$$

For example Ψ can be chosen as $\Psi(w,\theta) = \theta^4 + w_1\theta$ or $\theta^2 + w_1\theta$ or $\theta^4 - \theta^2 + w_1\theta$.

Assumption 1.10.

$$\sup_{x\in\mathbb{T}^d} \int_{\mathbb{R}} e^{2\overline{\Psi}(\theta)} \nu_x(\mathrm{d}\theta) < \infty. \tag{1.16}$$

1.1 Historical overview and discussion of results

Dynamical large deviation principles for models similar to (1.6) and (1.1) are considered by many authors. These models differ in one or more of the following three properties:

1. Various authors consider models with mean-field interaction (like the Curie-Weiss model), e.g. [35], [13], [6], [11], [16]. In these models the spatial structure of the spins is not relevant.
2. As in this paper, a random environment variable is attached to each site for example in [11]. Whereas in [7], [8], random pair interactions are considered.
3. A different dynamic of the spins is used instead of the Langevin dynamic. For example, in [10] the spins evolve according to a Glauber dynamic with values ± 1. The proof of the large deviation result depends crucially on the jump dynamic.

For these different models, the following four strategies are used to prove the large deviation principle. In this paper, we generalize the following Strategy (S.1) to be applicable to the system (1.1) and emphasise in the following list the necessary changes and difficulties.

(S.1) For a model with irrelevance of the spatial structure and without random environment, the dynamical large deviation principle for empirical processes is derived in [13]. This principle is used in [12] to connect the quasi potential with the free energy function. The idea of the approach in [13] is to fix an empirical process in the drift coefficient to get a system of N^d independent, time inhomogeneous diffusions. For this independent system the large deviation principle is derived. Finally, this LDP is transferred to the LDP for the interacting system. The main

difficulty is to show that the rate function has the particular form (similar to (1.11)). In this paper we generalize the approach of [13] to the space and random environment dependent empirical processes $\left\{\mu^N_{[0,T]}\right\}$ and to the empirical measures $\left\{L^N\right\}$. Changes in the proof are required due to this dependence in the drift coefficient, in the empirical process and in the initial data. Moreover, we consider the space of continuous functions on the usual space of probability measures $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)$, equipped with the usual topologies (the uniform topology and the weak convergence) and not, as in [13], a subset of this space with a stronger topology. We explain the approach and changes compared to [13], in detail in Section 3.

(S.2) In [35] the large deviation principle for the empirical measure $\frac{1}{N^d}\sum_{k\in\mathbb{T}^d_N}\delta_{\theta^{k,N}_{[0,T]}}$ in $\mathbb{M}_1(\mathsf{C}([0,T]))$ is derived. In [11], a mean-field interaction with random environment is considered. In both models, the authors assume that the drift coefficient b is bounded and does not depend on the spatial positions of the spins. Due to this boundedness, it is possible to transfer the large deviation principle for N^d independent Wiener process to the large deviation principle for the interacting system by an application of Varadhan's lemma. With the contraction principle, the authors easily infer the large deviation principle for the empirical process. However, the rate function does not have an explicit expression as in (1.11). In [11], the authors try to show that the rate function has such an expression. Unfortunately, there is a circular reasoning in the proof of this result (in Step 4 of the proof of Theorem 3). Therefore, only for some trajectories of measures the equality of these two expressions is proven. For the same subsets of trajectories this equality is also proven in [6].

We generalise in [31] the approach of [11] to prove the large deviation principle for the empirical measure $\left\{L^N\right\}$ for the example (1.6). From this we infer the large deviation principle for the empirical process $\mu^N_{[0,T]}$.

In [7], [8], [4] and [21] similar large deviation principles are derived for spin-glass-type dynamics. One of the main differences of these dynamics to the dynamics considered here, is that the disorder is on the connections between two particles, not on the particles themselves. In the cited papers, the authors prove large deviation principles for the empirical measures defined similar to $\left\{L^N\right\}$ and characterise the minima of the rate function.

(S.3) In [16] a third strategy is used to prove the LDP for mean-field systems as in [13] with slightly more restrictive assumptions. The authors connect the LDP with a variational problem arising from control theory (see Example 1.14, Section 13.3 and Theorem 13.37 of [16]).

(S.4) A direct approach to derive the large deviation principle for the empirical process is used in [24] for independent Brownian motions. This approach requires that the hydrodynamic limit has a unique weak solution (see also [22] page 40 and the discussion in [31] Section 0.5).

1.2 Outline of the paper

This paper is organised as follows. In Section 2, we state some preliminaries that are required in the subsequent sections. At first this comprises some definitions and notations (Section 2.1). Then, in Section 2.2, we generalise Sanov's Theorem to vectors of empirical measures that are space-($\mathbb{T}^d$), random environment-($\mathcal{W}$) and spin value-($\mathbb{R}$) dependent. Then we state a generalisation of the Arzelá Ascoli theorem for sets and measures on $\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$ (Section 2.3), and we generalise the definitions and results on distribution-valued functions of the Section 4.1 of [13] to the space $\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$ (Section 2.4). Finally in Section 2.5, we discuss the relation of the spaces on which L^N and $\mu^N_{[0,T]}$ are defined.

In Section 3 we state a precise version of Theorem 1.2 and prove it. We first (Section 3.1) derive a LDP of the empirical process for the independent system and finally transfer this LDP to an LDP for the interacting system (Section 3.2). Most of Section 3.1 is dedicated to showing that the rate function actually has the form (1.11). In particular, in the proof of a lower bound on the rate function of the independent system (Section 3.1.2), we need a solution to a PDE that is continuous in the space and environment variables. We prove the existence and uniqueness of such a solution in Section 3.1.2. We show in Section 3.3, that the special case (1.6) with Assumption 1.5-1.10 is covered by the weaker assumptions that we assume in Section 3.

In Section 4, we state different representations of the rate function for the empirical process.

In Section 5, we show that the same approach as in Section 3 can be used to derive a large deviation principle for the family $\left\{L^N\right\}$, provided that this family is exponentially tight.

In Section 6, we show at first (Theorem 6.1), that the minimizer of the rate functions of $\left\{\mu^N_{[0,T]}\right\}$ and $\left\{L^N\right\}$ are one to one related. In the remainder of Section 6, we discuss a second method to obtain a large deviation principle for $\left\{\mu^N_{[0,T]}\right\}$ using the contraction principle.

2 Preliminaries

2.1 Definitions and notations

We use the following notation.

Notation 2.1. *Let Y be a Polish space. We denote by $\mathbb{M}_1(Y)$ the space of probability measures on Y equipped with the topology of weak convergence.*

We write $\mathbb{M}^L_1\left(\mathbb{T}^d \times Y\right)$ for the subset of $\mathbb{M}_1\left(\mathbb{T}^d \times Y\right)$, that consists of those measures, that have the Lebesgue measure as projection to $\mathbb{T}^d$.

The measures in $\mathbb{M}^L_1\left(\mathbb{T}^d \times Y\right)$ are also called Young measures (see [2] Definition 4.3.1).

Definition 2.2. *We denote the space of continuous functions from $[0,T]$ into the space of probability measures $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ by*

$$\mathscr{C} := \mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right), \tag{2.1}$$

and its subspace with values in $\mathbb{M}^L_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ by

$$\mathscr{C}^L := \mathsf{C}\left([0,T], \mathbb{M}^L_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right). \tag{2.2}$$

For the rest of this paper, fix a non-negative $\varphi \in \mathsf{C}^2(\mathbb{R})$, that satisfies $\lim_{|\theta|\to\infty} \varphi(\theta) = \infty$.

Definition 2.3. *We denote the subset of $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ of measures, whose integral with respect to a $\varphi \in \mathsf{C}(\mathbb{R})$ is bounded by $R > 0$ by*

$$\mathbb{M}_{\varphi,R} := \left\{\mu \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right) : \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \varphi(\theta)\, \mu(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \le R\right\}. \tag{2.3}$$

Moreover, we denote the subset of $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$, with finite integral with respect to φ by

$$\mathbb{M}_{\varphi,\infty} := \bigcup_{R>0} \mathbb{M}_{\varphi,R} = \left\{ \mu \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big) : \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \varphi\left(\theta\right) \mu\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta\right) < \infty \right\}. \tag{2.4}$$

With abuse of notation we use also the symbol $\mathbb{M}_{\varphi,R}$ for the appropriate subspace of $\mathbb{M}_1\big(\mathbb{T}^d \times \mathbb{R}\big)$.

Definition 2.4. *We denote the subset of $\mathscr{C}$, that consists of the paths which are everywhere in $\mathbb{M}_{\varphi,R}$, for a $R > 0$, by*

$$\mathscr{C}_{\varphi,R} := \left\{ \mu_{[0,T]} \in \mathscr{C} : \sup_{t\in[0,T]} \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \varphi\left(\theta\right) \mu_t\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta\right) \leq R \right\} \subset \mathscr{C}. \tag{2.5}$$

For the union of these sets we use the symbol

$$\mathscr{C}_{\varphi,\infty} := \bigcup_{R=1}^{\infty} \mathscr{C}_{\varphi,R} = \left\{ \mu_{[0,T]} \in \mathscr{C} : \sup_{t\in[0,T]} \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \varphi\left(\theta\right) \mu_t\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta\right) < \infty \right\}. \tag{2.6}$$

We endow the spaces $\mathbb{M}_{\varphi,R}$, $\mathbb{M}_{\varphi,\infty}$, $\mathscr{C}_{\varphi,R}$ and $\mathscr{C}_{\varphi,\infty}$ with the subspace topology of $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ and $\mathscr{C}$ respectively. By this property these spaces differ from the definition used in [20] and [13]. There the authors equip the spaces with a stronger topology.

Definition 2.5. *For a measure $\mu \in \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$, we denote by $\mu_x \in \mathbb{M}_1(\mathcal{W} \times \mathbb{R})$ the regular conditional probability measures such that $\mu = \mathrm{d}x \otimes \mu_x$.*

For the projection of μ_x on the environment coordinate $\mathcal{W}$, we use the symbol $\mu_{x,\mathcal{W}}$ and for the corresponding regular conditional probability measures $\mu_{x,w} \in \mathbb{M}_1(\mathbb{R})$. Then $\mu = \mathrm{d}x \otimes \mu_{x,\mathcal{W}}\left(\mathrm{d}w\right) \otimes \mu_{x,w}$.

Definition 2.6. *We define the relative entropy between two probability measures $\mu, \nu \in \mathbb{M}_1(Y)$ on a Polish space Y, by*

$$\mathcal{H}\left(\mu|\nu\right) := \begin{cases} \int_Y \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\right)\mu & \text{if } \mu << \nu \\ \infty & \text{otherwise.} \end{cases} \tag{2.7}$$

Definition 2.7.

- *For each $N \in \mathbb{N}$, we denote by $\nu^N := \bigotimes_{k\in\mathbb{T}_N^d} \nu_{\frac{k}{N}} \in \mathbb{M}_1\left(\mathbb{R}^{N^d}\right)$ the initial distribution of the N^d-dimensional spin system.*
- *We define the product measure of the random environment by $\zeta^N := \bigotimes_{k\in\mathbb{T}_N^d} \zeta_{\frac{k}{N}} \in \mathbb{M}_1\left(\mathcal{W}^{N^d}\right)$.*

Notation 2.8. *We use the following notation.*

- *With x, y, z we usually denote macroscopic coordinates, i.e. positions on the torus $\mathbb{T}^d$. Whereas by i, j, k we denote microscopic coordinates, i.e. positions on the discrete torus $\mathbb{T}_N^d$. These two coordinate systems are related by $x = \frac{i}{N}$.*
- *As time variables we use the letters s, t, u.*
- *We use the letters θ, η for the spin values. With $\theta_{[0,T]}$ we denote the whole path of the spin value, i.e. an element of $\mathsf{C}([0,T])$. With $\theta_t \in \mathbb{R}$ we denote the spin value at time $t \in [0,T]$.*

- *For a N^d-dimensional vector of spin values, numbered by $k \in \mathbb{T}_N^d$, we use the symbol $\underline{\theta}^N$ and analogue $\underline{\theta}_{[0,T]}^N$, $\underline{\theta}_t^N$. We write $\theta^{k,N}$ for the element at position $k \in \mathbb{T}_N^d$ in this vector.*
- *We use the letter w for a value of the random environment. Again $\underline{w}^N$ is the N^d-dimensional vector of the environment and $w^{k,N}$ the specific value of the environment associate with the position $k \in \mathbb{T}_N^d$.*
- *We use lower-case letters, mostly μ, ν, π for measures on $\mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})$, $\mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$ or $\mathbb{M}_1(\mathbb{R})$ (ν is usually the distribution of the initial values). For the path on measures, i.e. for an element in $\mathscr{C}$, we write $\mu_{[0,T]}$. For the measure at time $t \in [0,T]$ of the path $\mu_{[0,T]}$ we write μ_t.*
- *We use upper-case letters, e.g. Q or Γ, for measures on $\mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T]))$.*
- *We denote the spaces of continuous functions from X to Y by $\mathsf{C}(X,Y)$. For its subset of bounded functions we use the notation $\mathsf{C}_b(X,Y)$, of functions that vanish at the boundary $\mathsf{C}_0(X,Y)$ and of functions with compact support $\mathsf{C}_c(X,Y)$. With a superscript like in $\mathsf{C}^k(X,Y)$ we state the k-times continuous differentiability. To shorten the notation we often skip Y if $Y = \mathbb{R}$, i.e. $\mathsf{C}(X) = \mathsf{C}(X,\mathbb{R})$.*

2.2 A Sanov type result

Let $Y_1, ..., Y_r$ be Polish spaces for $r \geq 1$ and let $\{Q_{x,w} : (x,w) \in \mathbb{T}^d \times \mathcal{W}\}$ be a family of probability measures on $Y = Y_1 \times ... \times Y_r$.

We generalise in this section the Sanov type Theorem 3.5 of [13] to the setting we consider here (Lemma 2.10). More precisely, we add the position on $\mathbb{T}^d$ and the random environment in the vector of the empirical measure, i.e. for $(y^i)_{i \in \mathbb{T}_N^d} \in Y^{N^d}$ and $(w^{i,N}) \in \mathcal{W}^{N^d}$ we define the vector $L_r^N \in \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times Y_1) \times ... \times \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times Y_r)$ by

$$L_r^N := \left(N^{-d} \sum_{i \in \mathbb{T}_N^d} \delta_{\left(\frac{i}{N}, w^{i,N}, y_1^i\right)}, ..., N^{-d} \sum_{i \in \mathbb{T}_N^d} \delta_{\left(\frac{i}{N}, w^{i,N}, y_r^i\right)} \right). \tag{2.8}$$

Moreover, we prove (Lemma 2.11), that the rate function can be expressed as a relative entropy.

The following assumption implies in particular that the integrals in Lemma 2.10 are well defined and that we get a suitable convergence of the logarithmic moment generating function.

Assumption 2.9. *$\{Q_{x,w} : (x,w) \in \mathbb{T}^d \times \mathcal{W}\} \subset \mathbb{M}_1(Y)$ is Feller continuous.*

With these $\{Q_{x,w}\}$, define the product measures $Q_{\underline{w}^N}^N := \bigotimes_{i \in \mathbb{T}_N^d} Q_{\frac{i}{N}, w^{i,N}} \in \mathbb{M}_1\left(Y^{N^d}\right)$ and the joint measures $Q^N := \zeta^N\left(\mathrm{d}\underline{w}^N\right) \otimes Q_{\underline{w}^N}^N \in \mathbb{M}_1\left(\mathcal{W}^{N^d} \times Y^{N^d}\right)$ for each $N \in \mathbb{N}$.

Lemma 2.10 (compare to [13] Theorem 3.5 for mean-field LDP)**.** *If Assumption 1.6 and Assumption 2.9 hold, then the family $\{L_r^N, Q^N\}$ satisfies the large deviation principle on the space $\mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times Y_1) \times ... \times \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times Y_r)$ with good rate function*

$$\begin{aligned} L_{\nu,\zeta}\left(\Gamma^1, ..., \Gamma^r\right) = \sup_{\substack{f_1 \in \mathsf{C}_b(\mathbb{T}^d \times \mathcal{W} \times Y_1) \\ \vdots \\ f_r \in \mathsf{C}_b(\mathbb{T}^d \times \mathcal{W} \times Y_r)}} & \left\{ \sum_{\ell=1}^r \int_{\mathbb{T}^d \times \mathcal{W} \times Y_\ell} f_\ell(x,w,y_\ell) \Gamma^\ell(\mathrm{d}x, \mathrm{d}w, \mathrm{d}y_\ell) \right. \\ & \left. - \int_{\mathbb{T}^d} \log\left(\int_{\mathcal{W}} \int_Y e^{\sum_{\ell=1}^r f_\ell(x,w,y_\ell)} Q_{x,w}(\mathrm{d}y_1, ..., \mathrm{d}y_r) \zeta_x(\mathrm{d}w) \right) \mathrm{d}x \right\} \end{aligned} \tag{2.9}$$

for $\Gamma^\ell \in \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times Y_\ell)$.

In the case when $r = 1$, i.e. $Y = Y_1$, we can express the rate function as a relative entropy.

Lemma 2.11. *If $r = 1$ then for $\Gamma = \mathrm{d}x \otimes \Gamma_x \in \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times Y\big)$*

$$\begin{aligned} L_{\nu,\zeta}(\Gamma) &= \mathcal{H}\left(\Gamma | \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w}\right) = \int_{\mathbb{T}^d} \mathcal{H}\left(\Gamma_x | \zeta_x(\mathrm{d}w) \otimes Q_{x,w}\right) \mathrm{d}x \\ &= \int_{\mathbb{T}^d} \int_{\mathcal{W}} \mathcal{H}\left(\Gamma_{x,w} | Q_{x,w}\right) \Gamma_{x,\mathcal{W}}(\mathrm{d}w)\, \mathrm{d}x + \int_{\mathbb{T}^d} \mathcal{H}\left(\Gamma_{x,\mathcal{W}} | \zeta_x\right) \mathrm{d}x. \end{aligned} \tag{2.10}$$

Otherwise $L_{\nu,\zeta}(\Gamma) = \infty$. Here $\Gamma_{x,\mathcal{W}} \in \mathbb{M}_1(\mathcal{W})$ is defined as in Definition 2.5.

Before proving these two lemmas in Section 2.2.2, we state in Section 2.2.1 some immediate consequences of the assumptions, needed in the proof of Lemma 2.10 and 2.11.

2.2.1 Preliminaries for the proof of the Sanov type result

We infer from Assumption 2.9, the following stronger continuity result.

Lemma 2.12. *Under Assumption 2.9, the map $x, w \mapsto \int f(x,w,y)\, Q_{x,w}(\mathrm{d}y)$ is continuous for each $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times Y\big)$.*

Proof. Fix an arbitrary sequence $\big(x^{(n)}, w^{(n)}\big) \to (x,w) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$. Then

$$\begin{aligned} &\left| \int f\left(x^{(n)}, w^{(n)}, y\right) Q_{x^{(n)},w^{(n)}}(\mathrm{d}y) - \int f(x,w,y)\, Q_{x,w}(\mathrm{d}y) \right| \\ &\le \left| \int f\left(x^{(n)}, w^{(n)}, y\right) - f(x,w,y)\, Q_{x^{(n)},w^{(n)}}(\mathrm{d}y) \right| \\ &\quad + \left| \int f(x,w,y) \left(Q_{x^{(n)},w^{(n)}}(\mathrm{d}y) - Q_{x,w}(\mathrm{d}y)\right) \right| =: 1 + 2\,. \end{aligned} \tag{2.11}$$

By the Feller continuity of $Q_{x,w}$ (Assumption 2.9), the sequence $Q_{x^{(n)},w^{(n)}}$ is tight (Prokhorov 's theorem). Hence for each $\epsilon > 0$, there is a compact set $K^\epsilon \subset Y$, such that

$$1 \le \sup_{y \in K^\epsilon} \left| f\left(x^{(n)}, w^{(n)}, y\right) - f(x,w,y) \right| + 2\,|f|_\infty\, Q_{x^{(n)},w^{(n)}}(Y \backslash K^\epsilon) \le \epsilon, \tag{2.12}$$

by the continuity of f and the compactness of K^ϵ for n large enough. From the Feller continuity (Assumption 2.9), we infer moreover that 2 is bounded by ϵ for n large enough. ☐

Now we show that Assumption 1.6 and Assumption 2.9 imply the following convergence.

Lemma 2.13. *Let Assumption 1.6 and Assumption 2.9 be satisfied. Then for all $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times Y\big)$, that satisfy $f \ge c$ for some fixed $c > 0$,*

$$\begin{aligned} &\frac{1}{N^d} \sum_{k \in \mathbb{T}_N^d} \log \left(\int_{\mathcal{W} \times Y} f\left(\frac{k}{N}, w, y\right) Q_{\frac{k}{N},w}(\mathrm{d}y)\, \zeta_{\frac{k}{N}}(\mathrm{d}w) \right) \\ &\quad \to \int_{\mathbb{T}^d} \log \left(\int_{\mathcal{W} \times Y} f(x,w,y)\, Q_{\frac{k}{N},w}(\mathrm{d}y)\, \zeta_x(\mathrm{d}w) \right) \mathrm{d}x. \end{aligned} \tag{2.13}$$

Proof. Fix an $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times Y\big)$, that satisfies $f \ge c$ for an arbitrary $c > 0$. By Lemma 2.12 and the Feller continuity of ζ_x (Assumption 1.6), the function

$$x \mapsto H_f(x) := \int_{\mathcal{W}} \int_Y f(x,w,y)\, Q_{\frac{k}{N},w}(\mathrm{d}y)\, \zeta_x(\mathrm{d}w) \tag{2.14}$$

is continuous. This can be shown by the same arguments, used to prove Lemma 2.12. Then H_f is, as a continuous function, also Riemann integrable.

By the continuity of $\log$ on $[c, |f|_\infty] \subset \mathbb{R}$, also $x \mapsto \log H_f(x)$ is Riemann integrable. This Riemann integrability implies the convergence of the sums in Lemma 2.13. □

Lemma 2.14. *Let Assumption 2.9 and Assumption 1.6 hold. Then* $\mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w}$*, defined by*

$$(\mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w})[A_1 \times A_2 \times A_3] = \int_{A_1}\int_{A_2}\int_{A_3} Q_{x,w}(\mathrm{d}y)\,\zeta_x(\mathrm{d}w)\,\mathrm{d}x, \tag{2.15}$$

for $A_1 \subset \mathbb{T}^d$*,* $A_2 \subset \mathcal{W}$*,* $A_3 \subset Y$*, is a well defined probability measure in* $\mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times Y)$*.*

Proof. We show at first that $(\zeta_x(\mathrm{d}w) \otimes Q_{x,w})$ is well defined for each $x \in \mathbb{T}^d$, by constructing a probability kernel. For each $f \in \mathsf{C}_b(Y)$,

$$\mathbb{T}^d \times \mathcal{W} \ni (x,w) \mapsto \overline{H}_f(x,w) \coloneqq \int_Y f(y)\,Q_{x,w}(\mathrm{d}y) \tag{2.16}$$

is continuous by Assumption 2.9. Therefore, $\overline{H}_f$ is also Borel-measurable, for all non negative $f \in \mathsf{C}_b(Y)$. Then for each open set $B \subset Y$, $\overline{H}_f$ is also Borel-measurable when $f = \mathbb{1}_B$, by a pointwise approximation of $\mathbb{1}_B$ with continuous function. Then $\overline{H}_{\mathbb{1}_B}$ is also Borel measurable for all Borel measurable $B \subset Y$ (as pointwise limits). Hence, $P(x,w,A) = \int_Y \mathbb{1}_A(y)\,Q_{x,w}(\mathrm{d}y)$ is a probability kernel. Therefore $(\zeta_x(\mathrm{d}w) \otimes Q_{x,w}) \in \mathbb{M}_1(\mathcal{W} \times Y)$ is well defined for all $x \in \mathbb{T}^d$.

By the same argument, also $P(x,B) = \int_{\mathcal{W}\times Y} \mathbb{1}_B(w,y)\,Q_{x,w}(\mathrm{d}y)\,\zeta_x(w)$ is a probability kernel. This requires Assumption 1.6. Therefore, $(\mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w})$ is well defined. □

2.2.2 Proof of Lemma 2.10 and Lemma 2.11

Proof of Lemma 2.10. The log moment generating function can be calculated for each vector $f = (f_1, ..., f_r) \in \mathsf{C}_b(\mathbb{T}^d \times \mathcal{W} \times Y_1) \times ... \times \mathsf{C}_b(\mathbb{T}^d \times \mathcal{W} \times Y_r)$ by

$$\begin{aligned}
\Gamma_{\nu,\zeta}(f) &= \lim_{N\to\infty} N^{-d} \log \int_{\mathcal{W}^{N^d} \times Y^{N^d}} e^{N^d \int f L_r^N(\mathrm{d}x,\mathrm{d}w,\mathrm{d}y)} \zeta^N\left(\mathrm{d}\underline{w}^N\right) \otimes Q^N_{\underline{w}^N}\left(\mathrm{d}\underline{y}\right) \\
&= \lim_{N\to\infty} N^{-d} \log \prod_{k\in\mathbb{T}^d_N} \int_{\mathcal{W}}\int_Y e^{\sum_{\ell=1}^r f_\ell\left(\frac{k}{N},w,y_\ell\right)} Q_{\frac{k}{N},w}(\mathrm{d}y_1,...,\mathrm{d}y_r)\,\zeta_{\frac{k}{N}}(\mathrm{d}w) \\
&= \lim_{N\to\infty} N^{-d} \sum_{k\in\mathbb{T}^d_N} \log \int_{\mathcal{W}}\int_Y e^{\sum_{\ell=1}^r f_\ell\left(\frac{k}{N},w,y_\ell\right)} Q_{\frac{k}{N},w}(\mathrm{d}y_1,...,\mathrm{d}y_r)\,\zeta_{\frac{k}{N}}(\mathrm{d}w) \\
&= \int_{\mathbb{T}^d} \log\left(\int_{\mathcal{W}}\int_Y e^{\sum_{\ell=1}^r f_\ell(x,w,y_\ell)} Q_{x,w}(\mathrm{d}y_1,...,\mathrm{d}y_r)\,\zeta_x(\mathrm{d}w)\right)\mathrm{d}x.
\end{aligned} \tag{2.17}$$

In the last equality we use Lemma 2.13. Note that by Lemma 2.12 and by H_f (defined in (2.14)) being continuous, all integrals in (2.17) are well defined.

The right hand side of (2.17) is finite and Gateaux differentiable. Also as in [13] we can show if $L_{\nu,\zeta}\left(\Gamma^1, ..., \Gamma^r\right) < \infty$, then $\Gamma^i \in \mathbb{M}_1(\mathbb{T}^d \times Y_i)$. Therefore, all conditions of Theorem 3.4 in [13] are satisfied and the claims of Lemma 2.10 are proven. □

Proof of Lemma 2.11. By Lemma 2.10, we know that $\left\{L_r^N\right\}$ satisfies under $\left\{Q^N_{v_N}\right\}$ a LDP with rate function $L_{\nu,\zeta}(\Gamma)$. Now we show that the rate function $L_{\nu,\zeta}$ has the claimed representation (2.10). The measure $(\mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w})$ in the relative entropy is well defined by Lemma 2.14.

***Step 1: If** $L_{\nu,\zeta}(\Gamma) < \infty$ **then** $\Gamma \in \mathbb{M}_1^L(\mathbb{T}^d \times \mathcal{W} \times Y)$.*

Fix $\Gamma \in \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times Y)$ with $L_{\nu,\zeta}(\Gamma) < \infty$. Then $\int_{\mathbb{T}^d \times \mathcal{W} \times Y} f(x)\,\Gamma(\mathrm{d}x, \mathrm{d}w, \mathrm{d}y) = \int_{\mathbb{T}^d} f(x)\,\mathrm{d}x$ for all $f \in \mathsf{C}_b(\mathbb{T}^d)$. Indeed, assume there were a $f \in \mathsf{C}_b(\mathbb{T}^d)$ for which this is not satisfied. Then for all $\lambda \in \mathbb{R}$,

$$L_{\nu,\zeta}(\Gamma) \geq \lambda \int_{\mathbb{T}^d \times \mathcal{W} \times Y} f(x)\,\Gamma(\mathrm{d}x, \mathrm{d}w, \mathrm{d}y) - \lambda \int_{\mathbb{T}^d} f(x)\,\mathrm{d}x \neq 0. \tag{2.18}$$

Because λ is arbitrary, this is a contradiction to $L_{\nu,\zeta}(\Gamma) < \infty$.

For each open $A \subset \mathbb{T}^d$, we can find a sequence of $f_n \in \mathsf{C}_b(\mathbb{T}^d)$, such that $f_n \geq 0$, $f_n \nearrow \mathbb{1}_A$ (see e.g. [1] A6). Therefore, we get by the dominated convergence theorem that the projection of Γ on $\mathbb{T}^d$ has to be the Lebesgue measure. The disintegration theorem for measures on a product space (see [2] Theorem 4.2.4) states that $\Gamma = \mathrm{d}x \otimes \Gamma_x$ with $\Gamma_x \in \mathbb{M}_1(\mathcal{W} \times Y)$.

***Step 2:** $L_{\nu,\zeta}(\Gamma) \leq \mathcal{H}(\mathrm{d}x \otimes \Gamma_x | \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w})$ **for** $\Gamma \in \mathbb{M}_1^L(\mathbb{T}^d \times \mathcal{W} \times Y)$.*

Fix $\Gamma \in \mathbb{M}_1^L(\mathbb{T}^d \times \mathcal{W} \times Y)$, such that $\mathcal{H}(\mathrm{d}x \otimes \Gamma_x | \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w}) < \infty$. Hence $\mathrm{d}x \otimes \Gamma_x$ is absolutely continuous with respect to $\mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w}$ with density ρ:

$$\mathrm{d}x \otimes \Gamma_x(\mathrm{d}w, \mathrm{d}y) = \rho(x, w, y)\,\mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w}(\mathrm{d}y). \tag{2.19}$$

Because $\Gamma \in \mathbb{M}_1^L(\mathbb{T}^d \times \mathcal{W} \times Y)$, $\int_{\mathcal{W}} \int_Y \rho(x, w, y)\,Q_{x,w}(\mathrm{d}y)\,\zeta_x(\mathrm{d}w) = 1$ for all $x \in \mathbb{T}^d$. The claimed upper bound on $L_{\nu,\zeta}(\Gamma)$, follows from finally by the same steps as in the second point of the proof of Theorem 3.1 in [29].

***Step 3:** $L_{\nu,\zeta}(\Gamma) \geq \mathcal{H}(\mathrm{d}x \otimes \Gamma_x | \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w})$ **for** $\Gamma \in \mathbb{M}_1^L(\mathbb{T}^d \times \mathcal{W} \times Y)$.*

This is just an application of Jensen's inequality to the convex function $-\log$ in $L_{\nu,\zeta}$ and the variation formula of the relative entropy.

Step 4: Second representation of rate function.

The second representation of the rate function follows from Theorem C.3.1 in [15]□

Remark 2.15. We could exchange the space $\mathbb{T}^d$ by an arbitrary compact Polish spaces X. If adjusted assumptions hold for X, then we would get the same large deviation result. We need the Lemma 2.10 in the sequel only with the space $\mathbb{T}^d$. To simplify the comprehensibility, we state it here not in its most general form.

2.3 Extended Arzelá-Ascoli theorem

We give a mild generalisation of the Arzelá-Ascoli theorem to subsets of $\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$. By the compactness of $\mathbb{T}^d$ we basically only have to take care of the projections of a set $A \subset \mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$ to the $\mathcal{W}$ and the $\mathsf{C}([0,T])$ component. For the latter projection we can use the conditions of the original Arzelá-Ascoli theorem.

Lemma 2.16 (Extended Arzelá-Ascoli Theorem)**.**

(i) $A \subset \mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$ *is relatively compact if and only if*

$$\mathrm{Proj}_{\mathsf{C}}[A] = \left\{\theta_{[0,T]} \in \mathsf{C}([0,T]) : \exists (x, w) \in \mathbb{T}^d \times \mathcal{W} : \left(x, w, \theta_{[0,T]}\right) \in A\right\} \tag{2.20}$$

is equibounded and equicontinuous and $\mathrm{Proj}_{\mathcal{W}}[A]$ *is relatively compact.*

(ii) *A sequence* $\{Q^{(n)}\} \subset \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T]))$ *is tight if and only if*

(a) for each $\eta > 0$ *there exists an* $a > 0$ *such that for all* $n > 0$ *and* $t \in [0,T]$

$$Q^{(n)}\left[\left(x, w, \theta_{[0,T]}\right) \in \mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T]) : |\theta_0| \geq a\right] \leq \eta \quad \text{and} \tag{2.21}$$

(b) *for each $\kappa, \eta > 0$ there exists $\delta \in (0,1)$ such that for all $n > 0$*

$$Q^{(n)}\left[\left(x, w, \theta_{[0,T]}\right) \in \mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T]) : \sup_{|t-s|\leq\delta} |\theta_t - \theta_s| \geq \kappa\right] \leq \eta \quad \text{and} \tag{2.22}$$

(c) *for each $\eta > 0$ there exists an $M > 0$ such that for all $n > 0$*

$$Q^{(n)}\left[\left(x, w, \theta_{[0,T]}\right) \in \mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T]) : |w| \geq M\right] \leq \eta. \tag{2.23}$$

Proof. ***(i)*** We claim that the relative compactness of A is equivalent to the relative compactness of $\mathrm{Proj}_{\mathsf{C}}[A]$ and the relative compactness of $\mathrm{Proj}_{\mathcal{W}}[A]$. Then (i) follows from the Arzelá-Ascoli theorem (see for example [5] Theorem 7.2).

If A is relatively compact, then, for each ϵ, there are $\left(x^{(\ell)}, w^{(\ell)}, \theta^{(\ell)}_{[0,T]}\right)_{\ell=1}^n \subset \mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$ for a $n = n(\epsilon) \in \mathbb{N}$, such that $A \subset \bigcup_{\ell=1}^n B_\epsilon\left(\left(x^{(\ell)}, w^{(\ell)}, \theta^{(\ell)}_{[0,T]}\right)\right)$. Then

$$\mathrm{Proj}_{\mathsf{C}}\left[B_\epsilon\left(\left(x^{(\ell)}, w^{(\ell)}, \theta^{(\ell)}_{[0,T]}\right)\right)\right] = B_\epsilon\left(\theta^{(\ell)}_{[0,T]}\right), \tag{2.24}$$

and therefore $\mathrm{Proj}_{\mathsf{C}}[A] \subset \bigcup_{i=1}^n B_\epsilon\left(\theta^{(\ell)}_{[0,T]}\right)$. Hence we found a finite open cover of $\mathrm{Proj}_{\mathsf{C}}[A]$, i.e. $\mathrm{Proj}_{\mathsf{C}}[A]$ is totally bounded and therefore relatively compact. By the same argument $\mathrm{Proj}_{\mathcal{W}}[A]$ is relative compact.

Let $\mathrm{Proj}_{\mathsf{C}}[A]$ and $\mathrm{Proj}_{\mathcal{W}}[A]$ be relatively compact. Then $\mathrm{Proj}_{\mathsf{C}}[A] \subset \bigcup_{\ell=1}^n B_\epsilon\left(\theta^{(\ell)}_{[0,T]}\right)$ and $\mathrm{Proj}_{\mathcal{W}}[A] \subset \bigcup_{i=1}^{n'} B_\epsilon\left(w^{(i)}\right)$. Hence A is totally bounded with open cover $A \subset \bigcup_{\ell=1}^n \bigcup_{i=1}^{n'} \bigcup_{k\in\mathbb{T}^d_{\frac{1}{\epsilon}}} B_{4\epsilon}\left(\left(k\epsilon, w^{(i)}, \theta^{(\ell)}_{[0,T]}\right)\right)$.

(ii) This claim follows by applying part (i), as in the proof of [5] Theorem 7.3. ☐

2.4 Distribution-valued functions

In this section we state the definitions and results of Section 4.1 of [13] transferred to the space-dependent setting considered here.

Definition 2.17.
- *We denote by $\mathbb{D} = \mathsf{C}_c^\infty\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ the space of test functions having compact support and continuous derivatives of all orders with the usual inductive topology.*
- *For a compact set $K \subset \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$, let $\mathbb{D}_K$ be the subset of $\mathbb{D}$ of functions with support in K.*
- *By $\mathbb{D}'$ and $\mathbb{D}'_K$, we denote the space of real distributions on $\mathbb{D}$ respectively on $\mathbb{D}_K$.*
- *Moreover, we write $\langle \xi, f\rangle$ for the application of $\xi \in \mathbb{D}'$ to $f \in \mathbb{D}$.*

Definition 2.18 (Variation of Definition 4.1 in [13])**.** *A map $\xi : [0,T] \to \mathbb{D}'$ is called absolutely continuous if for each compact set $K \subset \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$, there exist a neighbourhood U_K of 0 in $\mathbb{D}_K$ and an absolutely continuous function $H_K : [0,T] \to \mathbb{R}$ such that*

$$|\langle \xi(u), f\rangle - \langle \xi(v), f\rangle| \leq |H_K(u) - H_K(v)|, \tag{2.25}$$

for all $u, v \in I$ and $f \in U_K$.

Lemma 2.19 (Lemma 4.2 in [13])**.** *If $\xi : [0,T] \to \mathbb{D}'$ is absolutely continuous, then $\langle \xi(.), f\rangle : [0,T] \to \mathbb{R}$ is also absolutely continuous for each $f \in \mathbb{D}$.*

Moreover, the time derivative of ξ in the distributions sense

$$\partial_t \xi(t) = \lim_{h\to 0} h^{-1}\left(\xi(t+h) - \xi(t)\right) \tag{2.26}$$

exists for almost all $t \in [0,T]$.

Lemma 2.20 (Lemma 4.3 in [13],integration by parts)**.** *For all absolutely continuous map* $\xi : [0,T] \to \mathbb{D}'$*, each* $f \in \mathsf{C}_c^\infty\big([0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ *and* $s < t$

$$\langle \xi(t), f(t) \rangle - \langle \xi(s), f(s) \rangle = \int_s^t \langle \partial_t \xi(u), f(u) \rangle \, \mathrm{d}u + \int_s^t \langle \xi(u), \partial_t f(u) \rangle \, \mathrm{d}u \,. \tag{2.27}$$

The proofs of these two lemmas are analogue to the one of Lemma 4.2 in [13] respectively Lemma 4.3 in [13]. The crucial property of $\mathbb{D}$ and $\mathbb{D}_K$ for the proofs is their separability. This is the case for the spaces considered here as well as in [13].

Remark 2.21. We apply the results of this section later to probability measure valued functions in $\mathscr{C}$. This is possible because each measure in $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ is a Radon measure and hence also an element of $\mathbb{D}'$.

2.5 Relation between the spaces of the empirical measures and empirical processes

We are looking at two different levels of large deviation principles. The higher level are the empirical measures L^N in $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. The second level are the empirical processes $\mu^N_{[0,T]}$ in $\mathscr{C}$. Both elements are defined (see (1.4) and (1.3)) as images of the paths of the spins on the space $\mathsf{C}([0,T])^{N^d}$ and of the random environment $\underline{w}^N \in \mathcal{W}^{N^d}$.

Let us now define a map $\Pi : \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big) \to \mathscr{C}$, which maps L^N to $\mu^N_{[0,T]}$ for each $N \in \mathbb{N}$.

Definition 2.22. *For* $Q \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$ *we define* $\Pi(Q)_{[0,T]} \in \mathscr{C}$ *for each* $t \in [0,T]$ *by*

$$\begin{aligned}\Pi(Q)_t(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) &= Q\left[\big(y_x, y_w, y_{[0,T]}\big) \in \mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T]) : (y_x, y_w, y_t) \in \mathrm{d}x\mathrm{d}w\mathrm{d}\theta\right] \\ &= Q \circ \left(id_{\mathbb{T}^d}, id_{\mathcal{W}}, \theta_t\right)^{-1}(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)\end{aligned} \tag{2.28}$$

for $(x, w, \theta) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$.

The measure $\Pi(Q)_t$ is the one-dimensional distribution at time $t \in [0,T]$ of the measure $Q \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. Let us show that $\Pi(Q)_{[0,T]}$ of Definition 2.22 is actually an element of the space $\mathscr{C}$.

Lemma 2.23. *The function* Π *is well defined.*

Proof. Fix a $Q \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. We have to show that $\Pi(Q)_{[0,T]}$ is in $\mathscr{C}$. By the definition of Π, we know already that $\Pi(Q)_t \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ for all $t \in [0,T]$. Now we prove the continuity in time. Take a bounded L_f-Lipschitz continuous function $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ and $s, t \in [0,T]$ with $|s - t| < \delta$, then

$$\begin{aligned}&\left|\int f(y,w,\theta)\left(\Pi(Q)_t - \Pi(Q)_s\right)\right| = \left|\int f(y,w,\theta_t) - f(y,w,\theta_s)\, Q\left(\mathrm{d}y, \mathrm{d}w, \mathrm{d}\theta_{[0,T]}\right)\right| \\ &\leq \int \left|f(y,w,\theta_t) - f(y,w,\theta_s)\right| \mathbb{1}_{|\theta_t - \theta_s| < \kappa} Q\left(\mathrm{d}y, \mathrm{d}w, \mathrm{d}\theta_{[0,T]}\right) + 2\,|f|_\infty\, Q\left[|\theta_t - \theta_s| \geq \kappa\right] \\ &\leq L_f \kappa + 2\,|f|_\infty\, Q\left[\sup_{|u-v|<\delta} |\theta_u - \theta_v| \geq \kappa\right] \leq \epsilon,\end{aligned} \tag{2.29}$$

when $\kappa = \frac{\epsilon}{2L_f}$ and δ is small enough (by the extended Arzelá-Ascoli Lemma 2.16 (ii)). Hence $\Pi(Q)_{t_n} \to \Pi(Q)_t$ weakly in $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ if $t_n \to t$ by the Portmanteau theorem. □

Moreover, we show now that Π is a continuous function.

Lemma 2.24. *The function Π is continuous.*

Proof. The proof of this lemma follows the ideas in the proof of [13] Lemma 4.6 for the mean-field model.

Take a sequence $Q^{(n)} \to Q$ in $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. This implies that for each $t \in [0,T]$ and each $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$, that is Lipschitz continuous,

$$\left| \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f(x,w,\theta) \left(\Pi\left(Q^{(n)}\right)_t - \Pi(Q)_t \right) (\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \right| \to 0. \tag{2.30}$$

The topology on $\mathscr{C}$ is the topology of uniform convergence. Therefore, we have to show that the convergence (2.30) is uniform in t. The weak convergence of $Q^{(n)}$ implies tightness (Prokhorov's theorem), because $\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$ is a separable metric space. Moreover, we can split the absolute value in (2.30) into the following summands.

$$\begin{aligned} (2.30) \leq & \left| \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f(x,w,\theta) \left(\Pi\left(Q^{(n)}\right)_s - \Pi(Q)_s \right) (\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \right| \\ & + \left| \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f(x,w,\theta) \left(\Pi\left(Q^{(n)}\right)_t - \Pi\left(Q^{(n)}\right)_s \right) (\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \right| \qquad (2.31) \\ & + \left| \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f(x,w,\theta) \left(\Pi(Q)_t - \Pi(Q)_s \right) (\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \right| =: 1 + 2 + 3 . \end{aligned}$$

The terms 2 and 3 are bounded by ϵ for all $t,s \in [0,T]$ with $|t-s| < \delta$ for a δ small enough. This can be shown as in (2.29). Moreover, this bound is uniform in $n \in \mathbb{N}$, because the analogue of (2.29) is bounded uniformly in n by Lemma 2.16 (ii).

For each $k \in \left\{1, ..., \frac{T}{\delta}\right\}$, there is a $N_k \in \mathbb{N}$, such that 1 is bounded by ϵ for all $n > N_k$. Therefore, we conclude that for all $n > \max_{k=0}^{\frac{T}{\delta}} N_k$

$$\sup_t \left| \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f(x,w,\theta) \left(\Pi\left(Q^{(n)}\right)_t - \Pi(Q)_t \right) (\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \right| \leq 3\epsilon, \tag{2.32}$$

i.e. the uniform (in $t \in [0,T]$) convergence of (2.30). □

Notation 2.25. *With abuse of notation, we use the symbol Π also for:*

- *The analogously defined function $\mathbb{M}_1(\mathsf{C}([0,T])) \to \mathsf{C}([0,T], \mathbb{M}_1(\mathbb{R}))$. Then $\Pi(q)_{[0,T]}$ takes values in $\mathsf{C}([0,T], \mathbb{M}_1(\mathbb{R}))$ for $q \in \mathbb{M}_1(\mathsf{C}([0,T]))$.*
- *The analogously defined function $\mathbb{M}_1(\mathcal{W} \times \mathsf{C}([0,T])) \to \mathsf{C}([0,T], \mathbb{M}_1(\mathcal{W} \times \mathbb{R}))$.*

In the following lemma we state that the projection of $\Pi(Q)$ to $\mathbb{T}^d$ is the Lebesgue measure, if this is the case for Q. Moreover, we show that the projection of Π to the environment coordinate is frozen over time.

Lemma 2.26. *For $Q \in \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$, $\Pi(Q)_t \in \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ for all $t \in [0,T]$. Moreover, $\Pi(Q)_{t,x,\mathcal{W}} = \Pi(Q)_{0,x,\mathcal{W}} = Q_{x,\mathcal{W}}$ (see Definition 2.5) for all $t \in [0,T]$.*

Proof. Fix a $Q \in \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$ and a $t \in [0,T]$. Then $Q = \mathrm{d}x \otimes Q_x$ and it is easy to see that $\Pi(Q)_t = \mathrm{d}x \otimes \Pi(Q_x)_t$. Moreover, $Q = \mathrm{d}x \otimes Q_{x,\mathcal{W}}(\mathrm{d}w) \otimes Q_{x,w}$. Then for all $t \in [0,T]$

$$\begin{aligned}\int_{\mathbb{T}^d\times\mathcal{W}} f(x,w)\, Q_{x,\mathcal{W}}(\mathrm{d}w)\,\mathrm{d}x &= \int_{\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])} f(x,w)\, Q = \int_{\mathbb{T}^d\times\mathcal{W}} f(x,w)\, Q_{x,\mathcal{W}}(\mathrm{d}w)\,\mathrm{d}x \\ &= \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f(x,w)\, \Pi(Q)_t = \int_{\mathbb{T}^d\times\mathcal{W}} f(x,w)\, \Pi(Q)_{t,x,\mathcal{W}}(w)\,\mathrm{d}x,\end{aligned} \tag{2.33}$$

what we wanted to show. □

3 The LDP of the empirical process

In this section we state and prove the large deviation principle for the family of empirical processes $\left\{\mu^N_{[0,T]}\right\}$ defined in (1.3). We examine the N^d dimensional system of interacting spins defined by (1.1), with drift coefficient $b : \mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\times\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)\to\mathbb{R}$ and diffusion coefficient $\sigma > 0$.

We define the N^d-dimensional diffusion generator corresponding to (1.1) for fixed environment $\underline{w}^N$, acting on $f \in \mathsf{C}^2_b\left(\mathbb{R}^{N^d}\right)$ by

$$\mathbb{L}^N_{\underline{w}^N} f\left(\underline{\theta}^N\right) := \sum_{k\in\mathbb{T}^d_N} \mathbb{L}_{\mu^N,\frac{k}{N},w^{k,N}}\ f\left(\underline{\theta}^N\right), \tag{3.1}$$

where $\mathbb{L}_{\mu^N,\frac{k}{N},w^{k,N}}$ is the operator defined in (1.9) with derivatives in the $\theta^{k,N}$ direction and with drift coefficient $b\left(\frac{k}{N}, w^{k,N}, \cdot, \mu^N\right) : \mathbb{R}\to\mathbb{R}$, and $\mu^N \in \mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)$ is the empirical measure defined as in (1.2) with $\underline{\theta}^N$ and $\underline{w}^N$.

For the proof of the large deviation principle, we require that the drift coefficient b is chosen in such a way that the following assumption is satisfied.

Assumption 3.1. *There is a non-negative function $\varphi \in \mathsf{C}^2(\mathbb{R})$ with $\lim_{|\theta|\to\infty}\varphi(\theta) = \infty$, such that:*

a) *The function $b : \mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\times\mathbb{M}_{\varphi,\infty}\to\mathbb{R}$ satisfies:*

  a.i) *The restriction of b to $\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\times\left(\mathbb{M}_{\varphi,R}\cap\mathbb{M}^L_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)\right)\to\mathbb{R}$ is continuous for all $R > 0$.*

  a.ii) *For all $N\in\mathbb{N}$ and all $\underline{w}^N \in \mathcal{W}^{N^d}$, $b^N : \mathbb{R}^{N^d}\to\mathbb{R}^{N^d}$, defined by*

$$b^N\left(\underline{\theta}^N\right) := \left(b\left(\frac{k}{N}, w^{k,N}, \theta_k, \mu^N\right)\right)_{k\in\mathbb{T}^d_N}, \tag{3.2}$$

  is a locally bounded measurable function.

b) *There is a constant $\lambda > 0$ and a $\overline{N}\in\mathbb{N}$, such that for all $N > \overline{N}$ and all empirical measures μ^N (defined by $\underline{\theta}^N \in \mathbb{R}^{N^d}$ and $\underline{w}^N \in \mathcal{W}^{N^d}$),*

$$\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \mathbb{L}_{\mu^N,x,w}\varphi(\theta) + \frac{\sigma^2}{2}\left|\partial_\theta\varphi(\theta)\right|^2 \mu^N(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) \le \lambda \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \varphi(\theta)\,\mu^N(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta). \tag{3.3}$$

c) *For each $\mu_{[0,T]} \in \mathscr{C}_{\varphi,\infty}\cap\mathscr{C}^L$, there is a constant $\lambda\left(\mu_{[0,T]}\right) > 0$ such that*

$$\mathbb{L}_{\mu_t,x,w}\varphi(\theta) + \frac{\sigma^2}{2}\left|\partial_\theta\varphi(\theta)\right|^2 \le \lambda\left(\mu_{[0,T]}\right)\varphi(\theta), \tag{3.4}$$

  for all $(t,x,w,\theta) \in [0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}$.

d) *For each $R > 0$ and each $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,R} \cap \mathscr{C}^L$,*

$$\int_0^T \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \sigma^2 \left| b\left(x, w, \theta, \mu_t^{(n)}\right) - b\left(x, w, \theta, \overline{\mu}_t\right) \right|^2 \mu_t^{(n)}\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta\right) \mathrm{d}t \to 0, \tag{3.5}$$

for $n \to \infty$, when $\mu_{[0,T]}^{(n)} \to \overline{\mu}_{[0,T]}$, for a sequence $\left\{\mu_{[0,T]}^{(n)}\right\} \subset \left(\mathscr{C}_{\varphi,R} \cap \mathscr{C}^L\right)$ or a sequence

$$\left\{\mu_{[0,T]}^{(n)}\right\} \subset \left\{\mu_{[0,T]} \in \mathscr{C}_{\varphi,R} : \mu_{[0,T]} = \mu_{[0,T]}^N \text{ is a empirical process for a } N \in \mathbb{N}\right\}. \tag{3.6}$$

Example 3.2. We show in Section 3.3, that the example (1.6) of a local mean-field model satisfies Assumption 3.1, if Assumption 1.7 and 1.9 hold.

Remark 3.3. For each given environment $\underline{w}^N \in \mathcal{W}^{N^d}$, the martingale problem for the generator $\mathbb{L}_{\underline{w}^N}^N$ is well posed by Assumption 3.1 a.ii) and b). Indeed, from Theorem 10.1.2 of [34] and Theorem 7.2.1 of [34], we infer the uniqueness of the solution to the martingale problem, because the drift coefficient is locally bounded and measurable (Assumption 3.1 a.ii)). For the existence of a solution of the martingale Problem, we apply Theorem 10.2.1 of [34] with $\varphi\left(\underline{\theta}^N\right) := \frac{1}{N^d}\sum \varphi\left(\theta^{k,N}\right)$. The conditions of this theorem are satisfied by Assumption 3.1 b). We denote by $P^N_{\underline{w}^N,\underline{\theta}^N} \in \mathbb{M}_1\left(\mathsf{C}([0,T])^N\right)$ the unique solution of this martingale problem.

Notation 3.4.

- *We denote by $P^N_{\underline{w}^N} := \int_{\mathbb{R}^{N^d}} P^N_{\underline{w}^N,\underline{\theta}^N} \nu^N\left(\mathrm{d}\underline{\theta}^N\right) \in \mathbb{M}_1\left(\mathsf{C}([0,T])^{N^d}\right)$, the law of the paths of the N^d-dimensional spin system with a given environment $\underline{w}^N \in \mathcal{W}$ and with initial distribution ν^N.*
- *We use the symbol $P^N = \zeta^N(\mathrm{d}w) \otimes P^N_{\underline{w}^N} \in \mathbb{M}_1\left(\mathcal{W}^{N^d} \times \mathsf{C}([0,T])^{N^d}\right)$ for the joint distribution of the random environment and the paths of the spin system.*

Besides Assumption 1.5 on the Feller continuity of the initial distribution $\{\nu_x\}$, we require that these measures satisfy the following uniform integration condition.

Assumption 3.5. *There is a $\ell > 1$ such that*

$$\sup_{x \in \mathbb{T}^d} \int_{\mathbb{R}} e^{\ell\varphi(\theta)} \nu_x(\mathrm{d}\theta) < C. \tag{3.7}$$

The following large deviation principle is the main result of this section and is the precise version of Theorem 1.2.

Theorem 3.6. *Let Assumption 1.5, Assumption 1.6, Assumption 3.1 and Assumption 3.5 hold. Then the family $\left\{\mu_{[0,T]}^N, P^N\right\}$ satisfies on $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)$ the large deviation principle with good rate function*

$$S_{\nu,\zeta}\left(\mu_{[0,T]}\right) := \begin{cases} \int_0^T \left|\partial_t \mu_t - \left(\mathbb{L}_{\mu_t,\cdot,\cdot}\right)^* \mu_t\right|^2_{\mu_t} \mathrm{d}t + \mathcal{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right) & \text{if } \mu_{[0,T]} \in \mathbb{A} \cap \mathscr{C}_{\varphi,\infty}, \\ \infty & \text{otherwise,} \end{cases} \tag{3.8}$$

where the norm $|.|_{\mu_t}$ is defined in Definition 1.1 and where

$$\mathbb{A} := \left\{\mu \in \mathscr{C}^L : \mu_{[0,T]} \text{ is absolutely continuous in the sense of Definition 2.18}\right\}. \tag{3.9}$$

Moreover, the integral with respect to $\mathbb{T}^d \times \mathcal{W}$ *and the supremum in the norm in* $S_{\nu,\zeta}$ *can be interchanged, i.e.* $S_{\nu,\zeta}\left(\mu_{[0,T]}\right) = S_{\nu,\zeta}^{\mathbb{T}^d\times\mathcal{W}}\left(\mu_{[0,T]}\right)$*, defined by*

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathcal{W}} \left|\partial_t \mu_{t,x,w} - \left(\mathbb{L}_{\mu_t,x,w}\right)^* \mu_{t,x,w}\right|^2_{\mu_{t,x,w}} \mu_{0,x,\mathcal{W}}\left(\mathrm{d}w\right) \mathrm{d}x\mathrm{d}t + \mathcal{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right) \tag{3.10}$$

if $\mu_{[0,T]} \in \mathbb{A} \cap \mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$ *and* $S_{\nu,\zeta}^{\mathbb{T}^d\times\mathcal{W}}\left(\mu_{[0,T]}\right) = \infty$ *otherwise.*

To prove this theorem, we generalise the proof of the large deviation principle for the mean-field model of [13], to the space and random environment dependent setting we consider here. Therefore, the structure of the proof of Theorem 3.6 is similar to the structure of the corresponding proof in [13]. However, there are three main differences to [13]. The main difference is that the drift coefficient b and the empirical process $\mu^N_{[0,T]}$ depend on $x \in \mathbb{T}^d$ and on the random environment $w \in \mathcal{W}$. Moreover, in [13] the spins take fixed initial values, whereas in the model we consider, the spins are initially randomly distributed. Last but not least, we show the large deviation principle on the space $\mathscr{C}$ (and not, as in [13], on $\mathscr{C}_{\varphi,\infty}$ with another topology than the subspace topology).

Due to these differences, changes are necessary in the proofs (compared to the approach in [13]). Many of these changes are of technical nature. We point out at the beginning of each proof of the partial results, how the proof differs from the corresponding proof in [13]. Then we state the proofs with emphasis on these necessary modifications.

The proof of Theorem 3.6 is organised as follows.

1.) At first (Section 3.1), we prove the large deviation principle for a system of independent spins (see Theorem 3.10) and show that the rate function has the representation $S^I_{\nu,\zeta}$ (defined in (3.13)), that is similar to $S_{\nu,\zeta}$. We infer this large deviation principle from the generalised Sanov-type large deviation result derived in Section 2.2. The rest of this Section 3.1 is dedicated to showing that the rate function has the representation $S^I_{\nu,\zeta}$.

   1.1.) To show the form of the rate function, we derive at first two different representations $S^{I,1}_{\nu,\zeta}$ and $S^{I,2}_{\nu,\zeta}$ of the rate function (Section 3.1.1). For both representations we use the Sanov-type large deviation result derived in Section 2.2. These proofs are formally almost equal to the corresponding proofs in [13]. The space and random environment dependence only leads to formal changes in the notation. However, the applied results of Section 2.2 are different from the Sanov-type results used in [13], due to these new dependence.

   1.2.) Next, we show that $S^{I,1}_{\nu,\zeta}$ ($S^{I,2}_{\nu,\zeta}$) is an upper (lower) bound on the claimed form $S^{I,\mathbb{T}^d}_{\nu,\zeta}$ ($S^I_{\nu,\zeta}$) of the rate function (Section 3.1.1).

   In the proof of the upper bound (Section 3.1.2), we generalise an approach used in [11], which is partially based on approaches of [17] and [6]. In contrast to [11], we consider the space dependence $x \in \mathbb{T}^d$ in addition to the random environment $w \in \mathcal{W}$.

   Note that the proof of the lower bound given in [11] unfortunately has a gap and cannot be used. We give a proof of the lower bound in (Section 3.1.2) that generalises the ideas used in [13]. The proof requires the existence of a solution to a boundary value partial differential equation, which has to be continuous in the space variable $x \in \mathbb{T}^d$ and the environment variable $w \in \mathcal{W}$. This condition is obviously not needed in [13]. Therefore, we show in Section 3.1.2, that there exist such a solution.

1.3.) Finally, we derive another formula for $S^I_{\nu,\zeta}$. This is (again modulo changes due to the space dependence) similar to the corresponding proof in [13]. However, in [13] this formula is used to derive the large deviation upper bound. We do not use it in the proof of the large deviation upper bound, because it only bounds $S^I_{\nu,\zeta}$ (see the beginning of Section 3.1.2 for more details). However, we need this result in Section 3.2 to show that the rate function $S_{\nu,\zeta}$ is actually lower semi-continuous.

2.) In Section 3.2, we infer from this large deviation principle for independent spins, a local large deviation principle for the interacting spin system (Theorem 3.28). To do this, we define the independent generator $\mathbb{L}^I_{t,x,w} := \mathbb{L}_{\overline{\mu}_t,x,w}$ for fixed $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$. For the empirical process defined by the spins that evolve according to the Langevin dynamics with this generator, we get by Section 3.1 the large deviation principle. From this principle, we infer the local large deviation principle under $\left\{P^N\right\}$, with the help of exponential bounds (that we show in Section 3.2.1). This is again a generalisation of [13]. Moreover, we give a new proof of the local large deviation principle around $\overline{\mu}_{[0,T]}$ that are not in $\mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$ (see Section 3.2.3). This is necessary because we assume the continuity of b only on a subset of $\mathbb{M}_{\varphi,R}$ (see Assumption 3.1 a.i)). Also with the mentioned exponential bounds, we prove the exponential tightness of $\left\{\mu^N_{[0,T]}, P^N\right\}$ (Theorem 3.29). Finally, we infer from the exponential tightness and the local large deviation principle, Theorem 3.6.

We finish this section with a short discussion how Assumption 3.1 enter into this approach.

Remark 3.7. As explained in Remark 3.3, we use Assumption 3.1 a.ii) and b), to infer that the martingale problem for the generator $\mathbb{L}^N_{\underline{w}^N}$ is well defined. Moreover, Assumption 3.1 b) implies the exponential bounds in Section 3.2. We get analogue results for the independent system defined by the generator $L^I_{t,x,w}$ due to Assumption 3.1 a.i) and c). Finally, we require Assumption 3.1 d) to show that $S_{\nu,\zeta}$ is a good rate function (here we need the sequences in $\mathscr{C}^L$) and to connect the independent system with the interacting system when deriving the local large deviation principle in Section 3.2 (here we need the sequences of empirical processes).

3.1 Independent spins

In this section we investigate the large deviation principle for the empirical process for systems of independent spins. As explained, we derive such a system by fixing the interaction between the spins in the SDE (1.1). Therefore, we consider a drift coefficient $b^I : [0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \to \mathbb{R}$ here that depends not any more on the empirical measure but on the time.

For each $x \in \mathbb{T}^d$, $w \in \mathcal{W}$ and $t \in [0,T]$, define the time-dependent diffusion generator

$$\mathbb{L}^I_{t,x,w} := \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial^2\theta} + b^I\left(t,x,w,\cdot\right)\frac{\partial}{\partial\theta}, \tag{3.11}$$

which is the infinitesimal generator of

$$\mathrm{d}\theta^x_t = b^I\left(t,x,w,\theta^x_t\right)\mathrm{d}t + \sigma \mathrm{d}B^x_t. \tag{3.12}$$

Let us assume that b^I is chosen such that the following assumptions are satisfied.

Assumption 3.8. a) *b^I is continuous on $[0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$.*

b) *For each $x \in \mathbb{T}^d$ and each $w \in \mathcal{W}$, the martingale problem for $\mathbb{L}^I_{t,x,w}$ is well posed, with $\left\{P^I_{t,x,w,\theta} \in \mathbb{M}_1(\mathsf{C}([t,T])), (t,\theta) \in [0,T] \times \mathbb{R}\right\}$ being the corresponding family of probability measures.*

We interpret $P^I_{t,x,w,\theta}$ as the measure of the path of the spins at the position $x \in \mathbb{T}^d$ with initial value $\theta \in \mathbb{R}$ at time $s \in [0,T]$ and fixed environment $w \in \mathcal{W}$, that evolves according to (3.11). We use the shorter notation $P^I_{x,w,\theta}$, when $t = 0$. By (3.11), the spins at position $x, y \in \mathbb{T}^d$ evolve mutually independent for $x \neq y$.

Notation 3.9. *We write $P^I_{x,w}$ for the distribution of the path of the spin at the position $x \in \mathbb{T}^d$ with fixed environment $w \in \mathcal{W}$ and with initial distribution ν_x at time 0, i.e. $P^I_{x,w} = \int_{\mathbb{R}} P^I_{x,w,\theta} \nu_x(\mathrm{d}\theta)$.*

Similar to Notation 3.4, we define $P^{I,N}_{\underline{w}^N}$ and $P^{I,N}$ (using now $P^I_{x,w,\theta}$).

The following large deviation principle for independent spins is the main result of this subsection.

Theorem 3.10. *Let Assumption 1.5, 1.6 and 3.8 hold. Then the family $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ satisfies on $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)$ the large deviation principle with good rate function*

$$S^I_{\nu,\zeta}\left(\mu_{[0,T]}\right) := \begin{cases} \int_0^T \left|\partial_t \mu_t - \left(\mathbb{L}^I_{t,\cdot,\cdot}\right)^* \mu_t\right|^2_{\mu_t} \mathrm{d}t + \mathcal{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right) & \text{if } \mu_{[0,T]} \in \mathbb{A}, \\ \infty & \text{otherwise,} \end{cases} \tag{3.13}$$

with $\mathbb{A}$ defined in (3.9).

Moreover $S^I_{\nu,\zeta}\left(\mu_{[0,T]}\right) = S^{I,\mathbb{T}^d}_{\nu,\zeta}\left(\mu_{[0,T]}\right)$, defined by

$$\begin{aligned} S^{I,\mathbb{T}^d}_{\nu,\zeta}\left(\mu_{[0,T]}\right) := &\int_0^T \int_{\mathbb{T}^d} \int_{\mathcal{W}} \left|\partial_t \mu_{t,x,w} - \left(\mathbb{L}^I_{t,x,w}\right)^* \mu_{t,x,w}\right|^2_{\mu_{t,x,w}} \mu_{0,x,\mathcal{W}}(\mathrm{d}w)\, \mathrm{d}x \mathrm{d}t \\ &+ \mathcal{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right) \end{aligned} \tag{3.14}$$

if $\mu_{[0,T]} \in \mathbb{A}$ and $S^{I,\mathbb{T}^d}_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \infty$ otherwise.

Throughout the remainder of this subsection, we assume the validity of the assumptions of Theorem 5.5, without further mentioning.

Remark 3.11. The rate functions $S_{\nu,\zeta}$ (of Theorem 3.6) and $S^I_{\nu,\zeta}$ (of Theorem 3.10) are related to each other. Set $\mathbb{L}^I_{t,x,w} = \mathbb{L}_{\overline{\mu}_t,x,w}$ for a $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty}$. And let $S^I_{\nu,\zeta}$ be the rate function defined by (3.13) corresponding to this generator. Then $S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) = S^I_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right)$. We use this relation in Section 3.2.

Proof of Theorem 3.10*.* It is easy to see that the family $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ satisfies the large deviation principle by Lemma 2.10 and the contraction principle (see the proof of Lemma 3.12).

The main difficulty of the proof of Theorem 3.10 is to show that the rate function $S_{\nu,\zeta}$ has the form (3.13). To prove this, we generalise the approach used to prove Theorem 4.5 in [13] to the setting we consider here.

As in [13], we derive two different representations, $S^{I,1}_{\nu,\zeta}$ and $S^{I,2}_{\nu,\zeta}$, of the rate function and show that these provide a lower bound on $S^I_{\nu,\zeta}$ and an upper bound on $S^{I,\mathbb{T}^d}_{\nu,\zeta}$, respectively

To get the first representation, we use the contraction principle and transfer the LDP for $\left\{L^N, P^{I,N}\right\}$, that we get by Lemma 2.10, to the LDP for $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$.

Lemma 3.12 (see [13] Lemma 4.6 for the mean-field case)**.** *The family $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ satisfies on $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)$ the large deviation principle with rate function*

$$S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \inf_{Q \in \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])) : \Pi(Q)_{[0,T]} = \mu_{[0,T]}} L^1_{\nu,\zeta}\left(Q\right), \tag{3.15}$$

for $\mu_{[0,T]} \in \mathscr{C}$, with

$$\begin{aligned} L^1_{\nu,\zeta}\left(Q\right) &= \int_{\mathbb{T}^d} \int_{\mathcal{W}} \mathcal{H}\left(Q_{x,w} \middle| P^I_{x,w}\right) Q_{x,\mathcal{W}}\left(\mathrm{d}w\right) \mathrm{d}x + \int_{\mathbb{T}^d} \mathcal{H}\left(Q_{x,\mathcal{W}} \middle| \zeta_x\right) \mathrm{d}x \\ &= \sup_{f \in \mathsf{C}_b(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T]))} \left\{ \int_{\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])} f\left(x, w, \theta_{[0,T]}\right) Q\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta_{[0,T]}\right) \right. \\ &\qquad \left. - \int_{\mathbb{T}^d} \log\left(\int_{\mathcal{W}} \int_{\mathsf{C}([0,T])} e^{f\left(x,w,\theta_{[0,T]}\right)} P^I_{x,w}\left(\mathrm{d}\theta_{[0,T]}\right) \zeta_x\left(w\right) \right) \mathrm{d}x \right\}, \end{aligned} \tag{3.16}$$

for $Q \in \mathbb{M}^L_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ and $L^1_{\nu,\zeta}\left(Q\right) = \infty$ otherwise.

In particular, $S^{I,1}\left(\mu_{[0,T]}\right)$ is only finite if $\mu_t \in \mathbb{M}^L_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ for all $t \in [0,T]$ and if $\mu_{t,x,\mathcal{W}} = \mu_{0,x,\mathcal{W}}$ for all $t \in [0,T]$ and almost all $x \in \mathbb{T}^d$.

To derive the second representation, we define for $0 \le s \le t \le T$ the operator acting on $f \in \mathsf{C}_b\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ by

$$U_{s,t} f\left(x, w, \theta\right) := \int_{\mathsf{C}([s,T])} f\left(x, w, \theta_t\right) P^I_{s,x,w,\theta}\left(\mathrm{d}\theta_{[s,T]}\right). \tag{3.17}$$

With this operator we get the following representation of the rate function.

Lemma 3.13 (see [13] Lemma 4.7 for the mean-field case)**.** *The family $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ satisfies on $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)$ the large deviation principle with rate function*

$$S^{I,2}_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \sup_{r \in \mathbb{N}, 0 \le t_1 < \ldots < t_r \le T} L^{t_1,\ldots,t_r}_{\nu,\zeta}\left(\mu_{t_1}, \ldots, \mu_{t_r}\right) \text{ for } \mu_{[0,T]} \in \mathscr{C}, \tag{3.18}$$

where for $\mu_i \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$, $L^{t_1,\ldots,t_r}_{\nu,\zeta}\left(\mu_1, \ldots, \mu_r\right)$ is defined by

$$\begin{aligned} &\sup_{f \in \mathsf{C}^\infty_c(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})} \left\{ \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f\left(x, w, \theta\right) \mu_1 - \int_{\mathbb{T}^d} \log\left(\int_{\mathcal{W} \times \mathbb{R}} U_{0,t_1} e^f\left(x, w, \theta\right) \nu_x\left(\mathrm{d}\theta\right) \zeta_x\left(\mathrm{d}w\right) \right) \mathrm{d}x \right\} \\ &+ \sum_{i=2}^r \sup_{f \in \mathsf{C}^\infty_c(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})} \left\{ \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f\left(x, w, \theta\right) \mu_i - \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \log U_{t_{i-1},t_i} e^f\left(x, w, \theta\right) \mu_{i-1} \right\}, \end{aligned} \tag{3.19}$$

where the μ_i integrate with respect to the variables $\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta$.

Finally, we show that $S^I_{\nu,\zeta}$, respectively $S^{I,\mathbb{T}^d}_{\nu,\zeta}$, is bounded by these two rate functions.

Lemma 3.14. *For all $\mu_{[0,T]} \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$*

$$S^{I,2}_{\nu,\zeta}\left(\mu_{[0,T]}\right) \le S^I_{\nu,\zeta}\left(\mu_{[0,T]}\right) \le S^{I,\mathbb{T}^d}_{\nu,\zeta}\left(\mu_{[0,T]}\right) \le S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right). \tag{3.20}$$

Moreover, $S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right) < \infty$ implies that $\mu_{[0,T]}$ is weakly differentiable.

From these three lemmas, we conclude Theorem 3.10 by the uniqueness of the rate function of large deviation principles.

We postpone the proofs of the lemmas to the following subsections. $\square$

3.1.1 Proof of the two representation of the rate function (Lemma 3.12 and Lemma 3.13)

Proof of Lemma 3.12. We apply the Sanov type Lemma 2.10 with $r = 1$, $Y = \mathsf{C}([0,T])$ to conclude that the family $\{L^N, P^{I,N}\}$ satisfies on $\mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T]))$ the large deviation principle with rate function $L_{\nu,\zeta}$. Applying Lemma 2.10 requires Assumption 1.6 and the Feller continuity of $\{P^I_{x,w}\}$ (defined in Notation 3.9). The Feller continuity follows from Assumption 1.5 and the following lemma.

Lemma 3.15. *Assumption 3.8 implies that the family* $\left\{P^I_{x,w,\theta} : (x,w,\theta) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right\}$ *is Feller continuous.*

Before proving Lemma 3.15, we finish the proof of Lemma 3.12. The map Π (defined in Definition 2.22) is continuous (Lemma 2.24). It maps each probability measure on $\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$ to a continuous measure valued trajectories in $\mathscr{C}$. Moreover, for each fixed vector $\underline{\theta}^N_{[0,T]}$ and each $\underline{w}^N$, the image of the corresponding empirical path measure L^N under Π is the corresponding empirical process $\mu^N_{[0,T]}$. Therefore, the contraction principle implies the large deviation principle for $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ with the rate function $S^{I,1}_{\nu,\zeta}$.

The right hand side of (3.15) is only finite if there is a $Q \in \mathbb{M}_1^L(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T]))$ with $\Pi(Q)_{[0,T]} = \mu_{[0,T]}$. This implies that $\mu_t \in \mathbb{M}_1^L(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})$ for all $t \in [0,T]$, and that $\mu_{t,x,\mathcal{W}} = \mu_{0,x,\mathcal{W}}$ for all $t \in [0,T]$ and almost all $x \in \mathbb{T}^d$, by Lemma 2.26. □

Proof of Lemma 3.15. Fix a convergent sequence $(x^{(n)}, w^{(n)}, \theta^{(n)}) \to (x,w,\theta) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$. We define $a_n := a \equiv \sigma$ and $b^{I,(n)}(t,\eta) := b^I(t, w^{(n)}, x^{(n)}, \eta)$, $b^I(t,\eta) = b^I(t,x,w,\eta)$ for $(t,\eta) \in [0,T] \times \mathbb{R}$. These functions are continuous by Assumption 3.8 a). Moreover, we know, by Assumption 3.8 b), that $P^I_{x^{(n)},w^{(n)},\theta^{(n)}}$ is the solution to the martingale problem corresponding to the drift coefficient $b^{I,(n)}$.

The Theorem 11.1.4 in [34] implies that the solutions to the martingale problem $P^I_{x^{(n)},w^{(n)},\theta^{(n)}}$ converge weakly to $P^I_{x,w,\theta}$. The conditions of Theorem 11.1.4 of [34] are satisfied by Assumption 3.8. Therefore, $P^I_{x,w,\theta}$ is Feller continuous. □

Proof of Lemma 3.13. This proof is a generalisation of the proof of [13] Lemma 4.6 and we use the ideas of this proof. At first we prove a LDP for the finite dimensional distributions of $\left\{\mu^N_{[0,T]}\right\}$ (i.e. the distribution of $\mu^N_{[0,T]}$ at a finite number of times) and in a second step we transfer this LDP to the LDP for $\left\{\mu^N_{[0,T]}\right\}$ by using the projective limit approach.

Step 1: LDP for the finite dimensional distributions of $P^{I,N}$.

Fix $N \geq 1$, $r \in \mathbb{N}$, $0 = t_0 \leq t_1 < \dots < t_r \leq T$. We define the random elements

$$\mu^N_{t_1,\dots,t_r} := \left(\mu^N_{t_1}, \dots, \mu^N_{t_r}\right) \in \left(\mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})\right)^r. \tag{3.21}$$

Then $\mu^N_{t_1,\dots,t_r}$ depends only on the spins at the times $t_1, \dots, t_r$, i.e. on $\underline{\theta}^N_{t_1}, \dots, \underline{\theta}^N_{t_r}$ and not any more on the whole path.

By Lemma 2.10 (with $Y_1 = \dots = Y_r = \mathbb{R}$), the family $\{\mu^N_{t_1,\dots,t_r}, P^{I,N}\}$ satisfies the large deviation principle on $\left(\mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})\right)^r$ with rate function

$$L^{t_1,\dots,t_r}_{\mu_0}(\mu_1, \dots, \mu_r) = \sup_{\substack{f_1,\dots,f_r \\ \in \mathsf{C}_b(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})}} \left[\sum_{\ell=1}^r \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f_\ell(x,w,\theta)\, \mu_\ell(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) - H(f_1, \dots, f_r)\right] \tag{3.22}$$

for $\mu_\ell \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$, where

$$H\left(f_1, ..., f_r\right) := \int_{\mathbb{T}^d} \log \left(\int_{\mathcal{W}} \int_{\mathsf{C}([0,T])} e^{\sum_{\ell=1}^{r} f_\ell\left(x,w,\theta_{t_\ell}\right)} P^I_{x,w}\left(\mathrm{d}\theta_{[0,T]}\right) \zeta_x\left(\mathrm{d}w\right) \right) \mathrm{d}x. \tag{3.23}$$

To show that this function coincides with (3.19), we first get by the Markov property of $\{P_{t,x,w,\theta}\}$ that

$$\begin{aligned} H\left(f_1, ..., f_r\right) = \int_{\mathbb{T}^d} \log \Bigg(\int_{\mathcal{W}} \int_{\mathsf{C}([0,T])} \int_{\mathsf{C}([0,T])} & e^{f_r\left(y,w,\theta_{t_r}\right)} P^I_{t_{r-1},x,w,\theta_{t_{r-1}}}\left(\mathrm{d}\theta_{[0,T]}\right) \\ & e^{\sum_{\ell=1}^{r-1} f_\ell\left(y,w,\theta_{t_\ell}\right)} P^I_{x,w}\left(\mathrm{d}\theta_{[0,T]}\right) \zeta_x\left(\mathrm{d}w\right) \Bigg) \mathrm{d}x \end{aligned} \tag{3.24}$$

$$= \int_{\mathbb{T}^d} \log \left(\int_{\mathcal{W}} \int_{\mathbb{R}} U_{t_0,t_1}\left(e^{f_1} ... U_{t_{r-1},t_r} e^{f_r}\right)\left(x,w,\theta\right) \nu_x\left(\mathrm{d}\theta\right) \zeta_x\left(\mathrm{d}w\right) \right) \mathrm{d}x.$$

Now performing formally (by pushing through the space dependence) the same calculation as Dawson and Gärtner in [13] page 275, we can transfer the right hand side of (3.24) to the right hand side of (3.19) with the supremum taken over all $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$. But the operators $U_{s,t}$ are continuous linear operators, hence the supremum over $\mathsf{C}_c^\infty\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ equals the supremum over $\mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$.

Step 2: Transfer of the LDP for $\{\mu^N_{t_1,...,t_r}\}$ to the LDP for $\left\{\mu^N_{[0,T]}\right\}$.

An LDP for $\left\{\mu^N_{[0,T]}\right\}$ follows from the LDP for the finite dimensional marginals of the first step, by the projective limit approach. In [13] on page 276 this is done for the mean-field model. This proof can be almost directly used in the setting we consider here. To have a complete picture, we state nevertheless the idea here.

To have a projective system corresponding to $\big(\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)\big)^r$ with order relation $\subseteq$ for $\{t_1, ..., t_r\}$, we embed the space $\mathscr{C}$ into $\mathbb{M}_1^{[0,T]}\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big) := \{f : [0,T] \to \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)\}$ furnished with the product topology.

We know by Lemma 3.12 already that $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ satisfies the large deviation principle on $\mathscr{C}$. Then $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ satisfies also the large deviation principle on $\mathbb{M}_1^{[0,T]}\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$, by the contraction principle. We denote its rate function by $\widehat{S}^2$. But this LDP can also be identified with the projective limit of the finite dimensional LDPs derived above. Hence by the projective limit theorem ([14] Theorem 4.6.1, [13] Theorem 3.3) we see that $\widehat{S}^2$ has the desired form (3.18) on $\mathbb{M}_1^{[0,T]}\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$.

Moreover, $\widehat{S}^2$ is infinite on $\mathbb{M}_1^{[0,T]}\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big) \setminus \mathscr{C}$ and the random variables $\mu^N_{[0,T]}$ under $P^{I,N}$ are concentrated on $\mathscr{C}$. Hence we can reduce the LDP to an LDP on $\mathscr{C}$ by Lemma 4.1.5 (b) in [14]. This finishes the proof of Lemma 3.13. □

3.1.2 Coincidence of the two representations with $S^I_{\nu,\zeta}$ (proof of Lemma 3.14)

In this section we prove Lemma 3.14. Therefore, we show at first an upper bound on $S^{I,\mathbb{T}^d}_{\nu,\zeta}$ and then a lower bound on $S^I_{\nu,\zeta}$.

Upper bound on $S^{I,\mathbb{T}^d}_{\nu,\zeta}$ We show in this section that $S^{I,\mathbb{T}^d}_{\nu,\zeta} \leq S^{I,1}_{\nu,\zeta}$. As mentioned, the proof we state here, is based on an approach in [11].

Lemma 3.16. *If $S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right) < \infty$ for a $\mu_{[0,T]} \in \mathscr{C}$, then*

$$S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right) = S^{I,\mathbb{T}^d}_{\nu,\zeta}\left(\mu_{[0,T]}\right), \tag{3.25}$$

and $t \mapsto \mu_{t,x,w}$ *is weakly differentiable for almost all* $(x,w) \in \mathbb{T}^d \times \mathcal{W}$.

In particular $S^{I,1}_{\nu,\zeta} \geq S^{I,\mathbb{T}^d}_{\nu,\zeta} \geq S^{I}_{\nu,\zeta}$.

Remark 3.17. Note that the lemma only states the equality of $S^{I,1}_{\nu,\zeta}$ and $S^{I,\mathbb{T}^d}_{\nu,\zeta}$, when $S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right)$ is finite, i.e. when there is a $Q \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$ with $L^1_{\nu,\zeta}(Q) < \infty$ and $\Pi(Q)_{[0,T]} = \mu_{[0,T]}$. In [17], $\mu_{[0,T]}$ that satisfy this condition are called admissible. Therefore, this result is not enough to show the claimed equality in Theorem 3.10 and we are bound to also prove a lower bound (in Section 3.1.2). Note that for Nelson-Processes a similar upper bound is shown in [9].

Proof of Lemma 3.16. Fix a $\mu_{[0,T]} \in \mathscr{C}$ with $S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right) < \infty$.

The idea of this proof is based on the steps 1-3 of the proof of Theorem 3 in [11], that are partly based on [17] and [6]. The proof is organised as follows. We show in Step 1, that there is a $\overline{Q} \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$, which is a minimizer of the right hand side of (3.15) for $S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right)$. In Step 2 we derive another representation of $S^{I,1}_{\mu}\left(\mu_{[0,T]}\right)$, by applying a result of [17]. Finally in Step 3, we show that the new representation of $S^{I,1}_{\mu}\left(\mu_{[0,T]}\right)$ equals $S^{I,\mathbb{T}^d}_{\nu,\zeta}$.

Step 1: There is a $\overline{Q}$ ***with*** $L^1_{\nu,\zeta}\left(\overline{Q}\right) = S^{I,1}_{\mu}\left(\mu_{[0,T]}\right)$ ***and nice properties.***

We restrict the infimum in (3.15) to the set

$$A_{\mu,C} := \left\{Q : \Pi(Q)_{[0,T]} = \mu_{[0,T]}\right\} \cap \left\{Q : L^1_{\nu,\zeta}(Q) \leq C\right\} \subset \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)\,, \tag{3.26}$$

for a $C > 0$ large enough. This set is non empty and compact (the last set is compact because $L^1_{\nu,\zeta}$ is a good rate function and the first set is closed). Hence by the lower semi continuity of $L^1_{\nu,\zeta}$, there exists a $\overline{Q} \in A_{\mu,C}$ that is a minimiser of $L^1_{\nu,\zeta}$ in $A_{\mu,C}$. This implies that $L^1_{\nu,\zeta}\left(\overline{Q}\right) = S^{I,1}_{\mu}\left(\mu_{[0,T]}\right)$.

Hence, $L^1_{\nu,\zeta}\left(\overline{Q}\right) = \mathcal{H}\left(\overline{Q}\middle|\mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes P^I_{x,w}\right) < \infty$ and $\overline{Q} \in \mathbb{M}^L_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. Let us write $\overline{Q} = \mathrm{d}x \otimes \overline{Q}_x$ for $\overline{Q}_x \in \mathbb{M}_1(\mathcal{W} \times \mathsf{C}([0,T]))$ and $Q_x = \overline{Q}_{x,\mathcal{W}} \otimes \overline{Q}_{x,w}$ for $\overline{Q}_{x,w} \in \mathbb{M}_1(\mathsf{C}([0,T]))$, $\overline{Q}_{x,w} \in \mathbb{M}_1(\mathcal{W})$. Then for almost all $x \in \mathbb{T}^d$ and $\overline{Q}_{x,\mathcal{W}}$-almost all $w \in \mathcal{W}$, $\mathcal{H}\left(\overline{Q}_{x,w}\middle|P^I_{x,w}\right) < \infty$, $\mathcal{H}\left(\overline{Q}_{x,\mathcal{W}}\middle|\zeta_x\right) < \infty$ and $\Pi\left(\overline{Q}_{x,w}\right)_t = \mu_{t,x,w}$. Moreover, $\Pi\left(\overline{Q}\right)_t = \mathrm{d}x \otimes \overline{Q}_{x,\mathcal{W}} \otimes \Pi\left(\overline{Q}_{x,w}\right)_t = \mu_t \in \mathbb{M}^L_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$.

Step 2: Another representation of $S^{I,1}_{\mu}\left(\mu_{[0,T]}\right)$***.***

By these properties, we get, for almost all $x \in \mathbb{T}^d$, as in [17] Theorem II.1.31 and Remark II.1.3 (see also [26] Chapter 7 (in particular Theorem 7.11)), that there is a map $b^{x,w} : [0,T] \times \mathbb{R} \to \mathbb{R}$ such that $\overline{Q}_{x,w}$ is the law of $\theta^{x,w}_{[0,T]}$ described by the following SDE

$$\mathrm{d}\theta^{x,w}_t = \left(\sigma b^{x,w}\left(t,\theta^{x,w}_t\right) - b^I\left(t,x,w,\theta^{x,w}_t\right)\right)\mathrm{d}t + \sigma \mathrm{d}B^{\overline{Q}_{x,w}}_t\,, \tag{3.27}$$

with $\theta^{x,w}_0 \sim \mu_{0,x,w}$ and

$$\frac{\mathrm{d}\overline{Q}_{x,w}}{\mathrm{d}P^I_{x,w}} = e^{\int_0^T b^{x,w}(t,\cdot)\mathrm{d}B^{\overline{Q}_{x,w}}_t + \frac{1}{2}\int_0^T b^{x,w}(t,\cdot)^2\mathrm{d}t}\,\frac{\mathrm{d}\mu_{0,x,w}}{\mathrm{d}\nu_x}\,. \tag{3.28}$$

Here $B^{\overline{Q}_{x,w}}_t$ is a Wiener process under $\overline{Q}_{x,w}$. Inserting this derivative in the relative entropy, we get

$$\begin{aligned}\mathcal{H}\left(\overline{Q}_{x,w}\middle|P^I_{x,w}\right) - \mathcal{H}\left(\mu_{0,x,w}\middle|\nu_x\right) &= \frac{1}{2}\int_{\mathsf{C}([0,T])}\int_0^T \left(b^{x,w}\left(t,\theta_t\right)\right)^2\mathrm{d}t\; Q_{x,w}\left(\mathrm{d}\theta_{[0,T]}\right)\\ &= \frac{1}{2}\int_0^T\int_{\mathbb{R}} \left(b^{x,w}\left(t,\theta_t\right)\right)^2 \mu_{t,x,w}\left(\mathrm{d}\theta\right)\,\mathrm{d}t.\end{aligned} \tag{3.29}$$

Integrating over $\mu_{0,x,\mathcal{W}} = \overline{Q}_{x,\mathcal{W}} \in \mathbb{M}_1(\mathcal{W})$ and then over $x \in \mathbb{T}^d$ implies that

$$\begin{aligned} S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right) = L^1_{\nu,\zeta}\left(\overline{Q}\right) =& \frac{1}{2}\int_{\mathbb{T}^d}\int_{\mathcal{W}}\int_0^T\int_{\mathbb{R}} \left(b^{x,w}\left(t,\theta\right)\right)^2 \mu_{t,x,w}\left(\mathrm{d}\theta\right)\mathrm{d}t\,\mu_{0,x,\mathcal{W}}\left(\mathrm{d}w\right)\mathrm{d}x \\ &+ \int_{\mathbb{T}^d}\int_{\mathcal{W}} \mathcal{H}\left(\mu_{0,x,w}|\nu_x\right)\mu_{0,x,\mathcal{W}}\left(\mathrm{d}w\right)\mathrm{d}x + \int_{\mathbb{T}^d}\mathcal{H}\left(\mu_{0,x,\mathcal{W}}|\zeta_x\right)\mathrm{d}x. \end{aligned} \tag{3.30}$$

Step 3: The new representation of $S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right)$ equals $S^{I,\mathbb{T}^d}_{\nu,\zeta}$.

To finish the proof, we only need that for almost all $t \in [0,T]$, almost all $x \in \mathbb{T}^d$ and $Q_{x,\mathcal{W}}$-almost all $w \in \mathcal{W}$

$$\frac{1}{2}\int_{\mathbb{R}} \left(b^{x,w}\left(t,\theta\right)\right)^2 \mu_{t,x,w}\left(\mathrm{d}\theta\right) = \left|\partial_t\mu_{t,x,w} - \left(\mathbb{L}^I_{t,x,w}\right)^*\mu_{t,x,w}\right|^2_{\mu_{t,x,w}}, \tag{3.31}$$

with $\mathbb{L}^I_{t,x,w}$ defined in (3.11). Equation (3.31) can be shown as in the Steps 2 and 3 in the proof of Theorem 3 in [11]. Therefore, we sketch the proof here only.

The measure $\overline{Q}_{x,w}$ is the law of (3.27) and by construction $\mu_{t,x,w}$ is the evolution of the time marginal of this law. Hence $\mu_{t,x,w}$ is a weak solution of the Fokker-Plank equation

$$\partial_t\mu_{t,x,w} = -\partial_\theta\left(\left[\sigma b^{x,w}\left(t,.\right) - b^I\left(t,x,w,\cdot\right)\right]\mu_{t,x,w}\right) + \frac{\sigma^2}{2}\partial^2_{\theta^2}\mu_{t,x,w}. \tag{3.32}$$

From this, we subtract now the generator $\left(\mathbb{L}^I_{t,x,w}\right)^*$

$$\partial_t\mu_{t,x,w} - \left(\mathbb{L}^I_{t,x,w}\right)^*\mu_{t,x,w} = -\partial_\theta\left(\sigma b^{x,w}\left(t,.\right)\mu_{t,x,w}\right), \tag{3.33}$$

what leads to

$$\begin{aligned} \left|\partial_t\mu_{t,x,w} - \left(\mathbb{L}^I_{t,x,w}\right)^*\mu_{t,x,w}\right|^2_{\mu_{t,x,w}} &= \frac{1}{2}\sup_{f\in\mathbb{D}_{\mu_{t,x,w}}} \frac{\left|\int_{\mathbb{R}}\sigma b^{x,w}\left(t,\theta\right)\partial_\theta f\left(\theta\right)\mu_{t,x,w}\left(\mathrm{d}\theta\right)\right|^2}{\sigma^2\int_{\mathbb{R}}\left(\partial_\theta f\left(\theta\right)\right)^2\mu_{t,x,w}\left(\mathrm{d}\theta\right)} \\ &\leq \frac{1}{2}\int_{\mathbb{R}}\left(b^{x,w}\left(t,\theta\right)\right)^2\mu_{t,x,w}\left(\mathrm{d}\theta\right), \end{aligned} \tag{3.34}$$

with $\mathbb{D}_{\mu_{t,x,w}} := \left\{f \in \mathsf{C}^\infty_c(\mathbb{R}) : \int_{\mathbb{R}}\left(\partial_\theta f\left(\theta\right)\right)^2\mu_{t,x,w}\left(\mathrm{d}\theta\right) > 0\right\}$.

To conclude (3.31), we have to show that the last inequality is actually an equality. This can be done as in Step 3 of the proof of Theorem 3 in [11], by showing that $\left\{\partial_\theta f : f \in \mathbb{D}_{\mu_{t,x,w}}\right\}$ is dense in $L^2\left(\mathbb{R},\mu_{t,x,w}\right)$. Then we take a approximating sequence $f_n \in \mathbb{D}_{\mu_{t,x,w}}$, $\partial_\theta f_n \to b^{x,w}_t$ and get the corresponding lower bound. □

Remark 3.18. Instead of Lemma 3.16, we could also show similarly as in Lemma 4.9 in [13], that $S^{I,1}_{\nu,\zeta} \geq S^I_{\nu,\zeta}$, by using a representation of $S^I_{\nu,\zeta}$, that we derive in Lemma 3.27. This would require some changes (compared to [13]), due to the space dependence and the initial distribution of the spins that we consider here. However, the advantage of Lemma 3.16 is that it bounds also $S^{I,\mathbb{T}^d}_{\nu,\zeta}$. This could be archived also by a variation of Lemma 4.9 in [13] and a variation of Lemma 3.27, i.e. by moving the integral with respect to $x \in \mathbb{T}^d$ out of the supremum in (3.66).

Lower bound on $S^I_{\nu,\zeta}$ We prove in this section the following lower bound on $S^I_{\nu,\zeta}$. The proof is a generalisation of the corresponding proof in [13]. The most important difference to the original proof is that we derive for solutions of the arising PDE (see the proof of Lemma 3.20) also regularity in the space variable and the random environment variable.

Lemma 3.19 (compare to Lemma 4.10 in [13] for the mean-field case)**.** $S^{I,2}_{\nu,\zeta} \leq S^I_{\nu,\zeta}$.

Proof. It suffices to show, by (3.19), (3.13) and the second formula of the norm in Definition 1.1, that

$$\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f(x,w,\theta)\,\mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) - \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \log U_{s,t}e^f(x,w,\theta)\,\mu_s(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)$$
$$\leq \int_s^t \sup_{\substack{h\in\\ \mathsf{C}_c^\infty(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})}} \left(\langle \partial_u\mu_u, h\rangle - \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \mathbb{L}^I_{u,x} h(x,w,\theta) + \frac{\sigma^2}{2}\left(\partial_\theta h(x,w,\theta)\right)^2 \mu_u(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\right)\mathrm{d}u \tag{3.35}$$

for all $f \in \mathsf{C}_c^\infty(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})$, $0\leq s<t\leq T$, $\nu\in\mathbb{M}_1^L(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})$ and $\mu_{[0,T]}\in\mathscr{C}$ with $S^I_{\nu,\zeta}(\mu_{[0,T]})<\infty$. Indeed, by (3.35), we bound separately each summand of the sum on the right hand side of (3.19). For the first summand on the right hand side of (3.19), we have to differentiate between the cases $t_1=0$ and $t_1>0$ in the supremum in (3.18). If $t_1>0$, then apply first the Jensen inequality to the this summand of the right hand side of (3.19) before using (3.35). In the case $t_1=0$, the first summand on the right hand side of (3.19) equals to $\mathcal{H}(\mu_0|\mathrm{d}x\otimes\zeta_x\otimes\nu_x)$, which appears in formula (3.13) of $S^I_{\nu,\zeta}$ (by a similar estimate as used in the proof of Lemma 2.11).

To restrict the analysis to compact sets (see Lemma 3.20), we define a new semi group corresponding to the diffusion processes which is killed when leaving the ball $B_R = \{(x,w,\theta)\in\mathbb{T}^d\times\mathcal{W}\times\mathbb{R} : |\theta|<R\}$ by

$$U^R_{s,t}f(x,w,\theta) = \int_{\mathsf{C}([s,T])} f(x,w,\theta_t)\,\mathbb{1}_{\tau^s_R>t}P_{s,x,w,\theta}\left(\mathrm{d}\theta_{[s,T]}\right), \tag{3.36}$$

for $f\in\mathsf{C}_b(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T]))$, with $\tau^s_R(\theta_{[s,T]}) = \min\{t\in[s,T] : |\theta_t|\geq R\}$.

Lemma 3.20 (compare to Lemma 4.11 in [13])**.** *Given a $\mu_{[0,T]}\in\mathscr{C}$ with $S^I_{\nu,\zeta}(\mu_{[0,T]})<\infty$, then for all $R>0$, $0\leq s<t\leq T$ and $f\in\mathsf{C}_c^\infty(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})$ with $f\leq 0$ and* $\operatorname{supp}(f)\subset B_R$.

$$\int f(x,w,\theta)\,\mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) - \int \log\left[1+U^R_{s,t}\left(e^f-1\right)\right](x,w,\theta)\,\mu_s(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)$$
$$\leq \int_s^t \sup_{\substack{h\in\\ \mathsf{C}_c^\infty(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})}} \left(\langle \partial_u\mu_u, h\rangle - \int \mathbb{L}^I_{u,x,w} h(x,w,\theta) + \frac{\sigma^2}{2}\left(\partial_\theta h(x,w,\theta)\right)^2 \mu_u(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\right)\mathrm{d}u, \tag{3.37}$$

where the integrals without bounds integrate over the space $\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}$.

This lemma implies (3.35) by the same approximation approach given in [13] after Lemma 4.11. Hence once we prove Lemma 3.20, the proof of Lemma 3.19 is finished. □

Proof of Lemma 3.20. In this proof we generalise the proof of Lemma 4.11 in [13] to the model considered here. In contrast to [13] we do not assume that the drift coefficient b is locally Hölder continuous. However, we need this assumption to get the existence of a solution to a PDE (see Step 1.1). Therefore, we assume at first (Step 1), that b^I is Hölder continuous in time and spin. Finally, in Step 2, we show how to generalise this to general drift coefficients.

Fix an $R>0$, an arbitrary $f\in\mathsf{C}_c^\infty(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})$ with $f\leq 0$ and $\operatorname{supp}(f)\subset B_R$ and arbitrary $0\leq s<t\leq T$. Let $\mathcal{W}_f\subset\mathcal{W}$ be a compact subset such that the projection on $\mathcal{W}$ of the support of f is contained in $\mathcal{W}_f$

Step 1: The drift coefficient is Hölder continuous.

Let us assume that b^I is $\frac{1}{4}$-Hölder continuous in time and $\frac{1}{2}$-Hölder continuous in $\theta \in B_R$ on the subset $[0,T] \times \mathbb{T}^d \times \mathcal{W}_f \times B_R$. Moreover, let b^I be continuous on $[0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$. To generalise the ideas of [13] to the space and random environment dependent model, we need in particular the existence of a unique solution to an initial boundary value problem. This solution has to be moreover continuous in the space variable $x \in \mathbb{T}^d$ and in the random environment variable. We prove the existence and uniqueness of such a solution in Theorem 3.24. We follow the lines of the proof in [13] with focus on the extensions needed to treat the space and random environment dependence.

Step 1.1: Construction of a (non smooth) function that solves a PDE.

By Theorem 3.24, there is a unique classical solution g^* to the terminal boundary value problem

$$\begin{aligned} \partial_s g(s,x,w,\theta) &= -\mathbb{L}^I_{s,x,w} g(s,x,w,\theta) && (s,x,w,\theta) \in [0,t) \times \mathbb{T}^d \times \mathcal{W}_f \times B_R, \\ g(t,x,w,\theta) &= e^{f(x,w,\theta)} - 1 && (x,w,\theta) \in \mathbb{T}^d \times \mathcal{W}_f \times B_R, \\ g(s,x,w,\theta) &= 0 && (s,x,w,\theta) \in [0,t) \times \mathbb{T}^d \times \mathcal{W}_f \times \partial B_R. \end{aligned} \tag{3.38}$$

This implies that $g^*(s,x,w,\theta) = 0$ for $(s,x,w,\theta) \in [0,t] \times \mathbb{T}^d \times \partial\mathcal{W}_f \times B_R$. We define g^* to be zero for $w \notin \mathcal{W}_f$ or $\theta \notin B_R$.

The function g^* satisfies for $(s,x,w,\theta) \in [0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$

$$\begin{aligned} g^*(s,x,w,\theta) &= \int_{\mathsf{C}([s,T])} g^*(t \wedge \tau_R, x, w, \theta_{t\wedge\tau_R}) P_{s,x,w,\theta}\left(\mathrm{d}\theta_{[s,T]}\right) \\ &= \int_{\mathsf{C}([s,T])} \left(e^{f(x,w,\theta_t)} - 1\right) \mathbb{1}_{\tau_R > t} P_{s,x,w,\theta}\left(\mathrm{d}\theta_{[s,T]}\right) = U^R_{s,t}\left(e^f - 1\right)(x,w,\theta). \end{aligned} \tag{3.39}$$

The first equality is true because $g^*(t\wedge\tau_R, x, w, \theta(t\wedge\tau_R))$ is a $P_{s,x,w,\theta}$ martingale for all (s,x,w,θ) in $[0,t] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$ by Assumption 3.8 b). The next equality is due to the boundary and the initial condition in (3.38), respectively the chosen continuation of g^*. Note that the equality of g^* and the third representation is the corresponding Feynman-Kac formula (for fixed $(x,w) \in \mathbb{T}^d \times \mathcal{W}_f$).

Define the function $h^* := \log(g^* + 1)$. This function solves

$$\begin{aligned} &\partial_t h = -\mathbb{L}^I_{t,x,w} h - \frac{\sigma^2}{2}(\partial_\theta h)^2 \qquad \text{on } [0,T] \times \mathbb{T}^d \times \mathcal{W}_f \times B_R \qquad \text{and} \\ &h(t,\cdot,\cdot,\cdot)\big|_{\mathbb{T}^d \times \mathcal{W} \times B_R} = f(\cdot,\cdot,\cdot) \quad \text{and} \quad h\big|_{\partial B_R} = 0. \end{aligned} \tag{3.40}$$

If we could use the function h on the right hand side of (3.37), then the integration by parts Lemma 2.20 would prove Lemma 3.20. Unfortunately the function h^* is not in $\mathsf{C}^\infty_c\left([0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$. By its construction and the compactness of f, the support of g^* and thus of h^* is compact, but h^* is not smooth.

Step 1.2: Smoothing of g^*.

The last part of the proof consists of approaching g^* with smooth functions g_ϵ, defined by

$$g_\epsilon := k_\epsilon *_{x,w,\theta} g^*, \tag{3.41}$$

with $k_\epsilon(x,w,\theta) = k^1_\epsilon(x) k^2_\epsilon(w) k^3_\epsilon(\theta)$. Here k^1_ϵ is a Dirac sequence (approximation to the identity) in $\mathbb{T}^d$ such that $k^1_\epsilon(x) = \epsilon^{-d} k^1\left(\epsilon^{-1} x\right)$ and $k^1 \in \mathsf{C}^\infty_c\left(\mathbb{T}^d\right)$, $k^1 \geq 0$ and $\int_{\mathbb{T}^d} k^1(x)\,\mathrm{d}x = 1$. Analogue we define k^2_ϵ and k^3_ϵ as a Dirac sequence on $\mathcal{W}$ and $\mathbb{R}$ respectively.

Then $g_\epsilon \in \mathsf{C}_c^\infty\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ but it does not satisfy any more (3.38) and

$$h_\epsilon := \log\left(1 + g_\epsilon\right) \tag{3.42}$$

does not satisfy any more (3.40). Therefore, we can not use directly the integration by parts Lemma 2.20 to show (3.37).

Step 1.3: Smoothed function almost satisfies (3.37)**.**

Nevertheless, we prove in the following that h_ϵ used on the right hand side of (3.37) (instead of the supremum) almost satisfies (3.37), with an error that vanishes as $\epsilon \to 0$.

Indeed, by the integration by parts Lemma 2.20

$$\begin{aligned}
&\text{L} := \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} h_\epsilon\left(t,x,w,\theta\right)\mu_t\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right) - \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} h_\epsilon\left(s,x,w,\theta\right)\mu_s\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right)\\
&= \int_s^t \left\langle \partial_u\mu_u, h_\epsilon\left(u\right)\right\rangle + \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \partial_u h_\epsilon\left(u,x,w,\theta\right)\mu_u\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right) \quad \mathrm{d}u\\
&= \int_s^t \left\langle \partial_u\mu_u, h_\epsilon\left(u\right)\right\rangle - \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \mathbb{L}^I_{u,x,w} h_\epsilon\left(u,x,\theta\right) + \frac{\sigma^2}{2}\left|\partial_\theta h_\epsilon\left(u,x,w,\theta\right)\right|^2 \mu_u\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right)\\
&\quad + \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \frac{\left(\partial_u + \mathbb{L}^I_{u,x,w}\right) g_\epsilon\left(u,x,w,\theta\right)}{1+g_\epsilon\left(u,x,w,\theta\right)}\mu_u\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right) \quad \mathrm{d}u =: \text{R1} - \text{R2} + \text{R3}\,,
\end{aligned} \tag{3.43}$$

because $\partial_u h_\epsilon = \frac{\partial_u g_\epsilon}{1+g_\epsilon}$ and $\mathbb{L}^I_{u,x,w} h_\epsilon = \frac{\mathbb{L}^I_{u,x,w} g_\epsilon}{1+g_\epsilon} - \frac{\sigma^2}{2}\left|\partial_\theta h_\epsilon\right|^2$.

The L converges to the left hand side of (3.37), because $g_\epsilon\left(s\right) \to g^*\left(s\right)$ uniformly on $\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$. Indeed

$$\left|g_\epsilon\left(s,x,w,\theta\right) - g^*\left(s,x,w,\theta\right)\right| \leq \sup_{(y,\eta)\in\operatorname{supp}\{k_\epsilon\}} \left|g^*\left(s,x+y,w,\theta+\eta\right) - g^*\left(s,x,w,\theta\right)\right|, \tag{3.44}$$

and $g^*\left(s\right)$ is uniformly continuous (as a continuous function with compact support). Therefore, $h_\epsilon\left(t\right) \to f$ and $h_\epsilon\left(s\right) \to \log\left(1 + g^*\left(s\right)\right)$ uniformly.

The integrals R1 and R2 are smaller or equal to the right hand side of (3.37). We interpret R3 as an error and show in the next step that it can be bounded from a above by a vanishing function.

Step 1.4: A vanishing upper bound on R3.

By the following lemma we get a vanishing upper bound on the last integral R3 of (3.43).

Lemma 3.21 (compare to Lemma 4.12 in [13])**.** *For $\epsilon > 0$ small enough, there exists a continuous function r_ϵ on $[0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$, such that*

$$\left(\partial_u + \mathbb{L}^I_{u,x,w}\right) g_\epsilon\left(u,x,w,\theta\right) \leq r_\epsilon\left(u,x,w,\theta\right) \text{ for } \left(u,x,w,\theta\right) \in [0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \tag{3.45}$$

and $r_\epsilon \to 0$ uniformly on $[0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$ for $\epsilon \to 0$.

We state the proof of this lemma after we have finished the proof of Lemma 3.20. By Lemma 3.21

$$\text{R3} \leq \int_s^t \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \frac{r_\epsilon\left(u,x,w,\theta\right)}{1+g_\epsilon\left(u,x,w,\theta\right)}\mu_u\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right)\,\mathrm{d}u. \tag{3.46}$$

The right hand side vanishes for $\epsilon \to 0$, because $r_\epsilon \to 0$ uniformly and $e^{-|f|_\infty} \leq 1 + g_\epsilon \leq 1$ (by (3.39)).

Hence we conclude that (3.37) holds for Hölder continuous drift coefficients.

Step 2: General drift coefficient b^I***.***

Last but not least we show now that Lemma 3.20 also holds for general (non-Hölder continuous) drift coefficients provided that Assumption 3.8 is satisfied. Therefore, we approximate at first (Step 2.1) the drift coefficient b^I by a sequence of Hölder continuous functions $b^{I,(n)}$, that converge to b^I on $\mathsf{C}\big([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\big)$. Then we show that Step 1 can be applied for all $b^{I,(n)}$ (Step 2.2), i.e. that (3.37) holds for each $b^{I,(n)}$. Finally, we justify that we can take the limit on both sides of (3.37) such that this inequality also holds for b^I. To this end we only need to show that the left hand side of (3.37) for $b^{I,(n)}$ is in the limit greater than the corresponding one for b^I and an analogue result for the right hand side (Step 2.3 and Step 2.4). For that matter we follow the ideas of Dawson and Gärtner in Section 4.5 of [13] and generalise their proof to the setting we consider here.

Step 2.1: Approximation of b^I***.***

Denote by $\mathcal{W}_{f,2}$ the open set of all points in $\mathcal{W}$ with distance at most 1 from $\mathcal{W}_f$. We approximate the continuous drift coefficient b^I by functions $b^{I,(n)}\in\mathsf{C}\big([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\big)$. These functions are chosen such that $b^{I,(n)}$ is on $[0,T]\times\mathbb{T}^d\times\mathcal{W}_f\times B_R$ also $\frac{1}{4}$-Hölder continuous in time and $\frac{1}{2}$-Hölder continuous in B_R. Moreover, $b^{I,(n)}=b^I$ outside of $[0,T]\times\mathbb{T}^d\times\mathcal{W}_{f,2}\times B_{2R}$ and $b^{I,(n)}\to b^I$ uniformly. Finding such a sequence is for example possible by the Stone-Weierstrass Theorem (on the compact set $\overline{\mathcal{W}}_{f,2}$) and the Urysohn's Lemma (with $\mathcal{W}_f$ and $\mathcal{W}_{f,2}$).

Step 2.2: (3.37) ***holds for each*** $b^{I,(n)}$***.***

One has to prove, that the martingale problem for the generator $\mathbb{L}^{I,(n)}_{s,x,w}$ with drift coefficient $b^{I,(n)}$ is well posed. But this we get from the (Cameron-Martin-) Girsanov theorem ([34] Theorem 6.4.2) because the difference between $b^{I,(n)}$ and b^I is at most ϵ for n large enough by the uniform convergence. We call the corresponding solution $P^{I,(n)}_{s,x,w,\theta}$ and its semi-group $U^{R,(n)}_{s,t}$. Hence by Step 1, (3.37) holds with $U^{R,(n)}_{s,t}$ and $\mathbb{L}^{I,(n)}$.

Step 2.3: The LHS of (3.37) ***for*** $b^{I,(n)}$ ***is in the limit greater than the LHS for*** b^I***.***

Fix $(s,t,x,w,\theta)\in[0,T]\times[0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}$. By [34] Theorem 11.1.4, $P^{I,(n)}_{s,x,w,\theta}\to P^I_{s,x,w,\theta}$, and by [34] Theorem 11.1.2, $\theta_{[s,T]}\mapsto\tau_R^s\left(\theta_{[s,T]}\right)$ is lower semi-continuous. Hence $\{\tau_R^s>t\}$ is an open set and $\mathbb{1}_{\tau_R^s>t}$ is lower semi-continuous. The function $\left(e^f-1\right)$ is non positive and continuous, what implies that $\left(e^{f\left(x,w,\theta_{[0,T]}\right)}-1\right)\mathbb{1}_{\tau_R^s>t}$ is upper semi continuous. By the Portmanteau theorem

$$\begin{aligned}\limsup_{n\to\infty}U^{R,(n)}_{s,t}\left(e^f-1\right)(x,w,\theta)&=\limsup_{n\to\infty}\int\limits_{\mathsf{C}([0,T])}\left(e^{f\left(x,w,\theta_{[0,T]}\right)}-1\right)\mathbb{1}_{\tau_R^s>t}P^{I,(n)}_{s,x,w,\theta}\left(\mathrm{d}\theta_{[0,T]}\right)\\&\le\int_{\mathsf{C}([0,T])}\left(e^{f\left(x,w,\theta_{[0,T]}\right)}-1\right)\mathbb{1}_{\tau_R^s>t}P^{I}_{s,x,w,\theta}\left(\mathrm{d}\theta_{[0,T]}\right)=U^R_{s,t}\left(e^f-1\right)(x,w,\theta)\,.\end{aligned}\tag{3.47}$$

Due to the compactness of f, there is a $c\in(-1,0)$, such that $c\le U^{R,(n)}_{s,t}\left(e^f-1\right)\le 0$ for all n. Hence we conclude with the Fatou-Lebesgue theorem

$$\begin{aligned}&\limsup_{n\to\infty}\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}}\log\left[1+U^{R,(n)}_{s,t}\left(e^f-1\right)\right](x,w,\theta)\,\mu_s\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right)\\&\le\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}}\limsup_{n\to\infty}\log\left[1+U^{R,(n)}_{s,t}\left(e^f-1\right)\right](x,w,\theta)\,\mu_s\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right)\\&\le\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}}\log\left[1+U^{R}_{s,t}\left(e^f-1\right)\right](x,w,\theta)\,\mu_s\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right).\end{aligned}\tag{3.48}$$

Step 2.4: The RHS of (3.37) ***for*** $b^{I,(n)}$ ***is in the limit smaller than the RHS for*** b^I***.***

By the triangle inequality we get

$$\left|\partial_u \mu_u - \left(\mathbb{L}^{I,(n)}_{u,\cdot,\cdot}\right)^* \mu_u\right|^2_{\mu_u} \leq \left|\partial_u \mu_u - \left(\mathbb{L}^{I}_{u,\cdot,\cdot}\right)^* \mu_u\right|^2_{\mu_u} + \left|\left(\mathbb{L}^{I}_{u,\cdot,\cdot} - \mathbb{L}^{I,(n)}_{u,\cdot,\cdot}\right)^* \mu_u\right|^2_{\mu_u}. \tag{3.49}$$

The last term is smaller than $\frac{\sigma^2}{2}\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \left|b^{I,(n)}(x,w,\theta) - b^I(x,w,\theta)\right|^2 \mu_u(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)$, what vanishes when $n\to\infty$ by the uniform convergence.

Step 2.5: Conclusion.

Hence we conclude

$$\begin{aligned}
&\int\limits_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f(x,w,\theta)\,\mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) - \int\limits_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \log\left[1+U^R_{s,t}\left(e^f-1\right)\right](x,w,\theta)\,\mu_s(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\\
&\leq \liminf_{n\to\infty}\left\{\int f(x,w,\theta)\,\mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) - \int \log\left[1+U^{R,(n)}_{s,t}\left(e^f-1\right)\right](x,w,\theta)\,\mu_s(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\right\}\\
&\leq \liminf_{n\to\infty}\int_s^t \left|\partial_u\mu_u - \left(\mathbb{L}^{I,(n)}_{u,\cdot}\right)^*\mu_u\right|^2_{\mu_u}\mathrm{d}u \leq \int_s^t \left|\partial_u\mu_u - \left(\mathbb{L}^{I}_{u,\cdot}\right)^*\mu_u\right|^2_{\mu_u}\mathrm{d}u.
\end{aligned} \tag{3.50}$$

□

Proof of Lemma 3.21. Fix $(s,x,w,\theta)\in[0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}$. We get by the integration by parts formula (and the same argument as in [13] in the proof of Lemma 4.12 to bound the derivatives at the boundary ∂B_R),

$$\begin{aligned}
&\left(\partial_s + \mathbb{L}^I_{s,x,w}\right) g_\epsilon(s,x,w,\theta)\\
&\leq \int k_\epsilon(x-x',w-w',\theta-\theta')\Bigg(\partial_s g(s,x',w',\theta') + \frac{\sigma^2}{2}\partial^2_{\theta'\theta'} g(s,x',w',\theta')\\
&\qquad\qquad + b^I(s,x,w,\theta)\,\partial_{\theta'} g(s,x',w',\theta')\Bigg)\mathrm{d}\theta'\mathrm{d}w'\mathrm{d}x'\\
&= \int k_\epsilon(x-x',w-w',\theta-\theta')\left(b^I(s,x,w,\theta) - b^I(s,x',w',\theta')\right)\partial_{\theta'} g(s,x',w',\theta')\,\mathrm{d}\theta'\mathrm{d}w'\mathrm{d}x',
\end{aligned} \tag{3.51}$$

where the two integrals are over the space $\mathbb{T}^d\times\mathcal{W}_f\times B_R$. In the last equality we use that g is a solution to (3.38). We denote the right hand side of (3.51) by $r_\epsilon(s,x,w,\theta)$.

For each ϵ, the integrand in r_ϵ is continuous and uniformly bounded, because b^I and $\partial_{\theta'} g$ are continuous and we consider a compact set. This implies that r_ϵ is continuous.

For all $(s,x,w,\theta)\in[0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}$

$$|r_\epsilon(s,x,w,\theta)| \leq \sup_{\substack{x',x''\in\mathbb{T}^d;w'\in\mathcal{W};w''\in\mathcal{W}_f;\theta',\theta''\in B_{2R}\\ |x'-x''|<\epsilon,|w'-w''|<\epsilon,|\theta'-\theta''|<\epsilon}} \left|b^I(s,x',w',\theta') - b^I(s,x'',w'',\theta'')\right| \left|\partial_{\theta'} g\right|_\infty, \tag{3.52}$$

for ϵ small enough. The derivative $\partial_{\theta'} g$ is bounded and b^I is uniform continuous on the compact set $[0,T]\times\mathbb{T}^d\times\overline{\mathcal{W}}_{f,2}\times B_{2R}$. Hence r_ϵ converges uniformly to 0. □

PDE preliminaries In this section we prove (see Theorem 3.24) the uniqueness and the existence of a Hölder continuous (in time and spin) solution of the terminal boundary value problem (3.38), that is moreover continuous on $\mathbb{T}^d$ and on a connected subset $\widehat{\mathcal{W}}\subset\mathcal{W}$. We did not find such a result in the literature due to the non-ellipticity in the $\mathbb{T}^d\times\mathcal{W}$-directions.

In the proof of this result, we look at first at the PDE (3.38) with fixed $(x,w)\in\mathbb{T}^d\times\widehat{\mathcal{W}}$. For each of these PDEs, we get by a result of [25] (that we repeat in Theorem 3.25) the existence and uniqueness of a solution $g_{x,w}$ on $[0,T]\times B_R$. The main part of the proof then consists of showing that these solutions are continuous in $x\in\mathbb{T}^d$ and $w\in\widehat{\mathcal{W}}$.

We define the Hölder space, on which we derive the solution. We refer to the page 7 in [25] for this definition (without the dependence on $\mathbb{T}^d$).

Definition 3.22. *We denote by $H^{\ell/2,0,0,\ell}\left([0,T]\times\mathbb{T}^d\times\widehat{\mathcal{W}}\times B_R\right)$ the Banach space of continuous functions on $[0,T]\times\mathbb{T}^d\times\widehat{\mathcal{W}}\times B_R$, which have continuous derivatives $\partial_t^r\partial_\theta^s$, with $2r+s\le\ell$, and with finite norm*

$$|u|_{H^{\ell/2,0,\ell}}=\sum_{2r+s\le\lfloor\ell\rfloor}|\partial_t^r\partial_\theta^s u|_\infty+\sum_{2r+s=\lfloor\ell\rfloor}|\partial_t^r\partial_\theta^s u|_{\ell-\lfloor\ell\rfloor,\theta}+\sum_{2r+s\in\{\lfloor\ell\rfloor-1,\lfloor\ell\rfloor\}}|\partial_t^r\partial_\theta^s u|_{\frac{2r+s}{2},t}\,, \tag{3.53}$$

where $|.|_{\ell-\lfloor\ell\rfloor,\theta}$ and $|.|_{\ell-\lfloor\ell\rfloor,t}$ are the usual Hölder norms in $\theta\in\mathbb{R}$ and $t\in[0,T]$ respectively.

The space $H^{\ell/2,\ell}\left([0,T]\times B_R\right)$ is defined analogously, just without the dependence on $\mathbb{T}^d\times\widehat{\mathcal{W}}$.

Remark 3.23. For $\ell\in(0,1)$, the norm $|u|_{H^{\ell/2,0,\ell}}$ is simply $|u|_\infty+|u|_{\ell,\theta}+|u|_{\frac{\ell}{2},t}$.

Theorem 3.24. *Let $\ell>0$ be a non integer number. Assume that the drift coefficient of $\mathbb{L}^I$ (see (3.11)) $b^I\in H^{\ell/2,0,0,\ell}\left([0,T]\times\mathbb{T}^d\times\widehat{\mathcal{W}}\times B_R\right)$ and that $i\in H^{0,0,\ell+2}\left(\mathbb{T}^d\times\widehat{\mathcal{W}}\times B_R\right)$. Then for each $R\in\mathbb{R}$, there is a unique solution $g^*\in H^{\ell/2+1,0,0,\ell+2}\left([0,t]\times\mathbb{T}^d\times\widehat{\mathcal{W}}\times\overline{B_R}\right)$ of the following terminal boundary value problem*

$$\begin{aligned}
\partial_s g(s,x,w,\theta) &= -\mathbb{L}^I_{t,x,w}g(s,x,w,\theta) && (s,x,w,\theta)\in[0,t)\times\mathbb{T}^d\times\widehat{\mathcal{W}}\times B_R,\\
g(t,x,w,\theta) &= i(x,w,\theta) && (x,w,\theta)\in\mathbb{T}^d\times\widehat{\mathcal{W}}\times B_R,\\
g(s,x,w,\theta) &= 0 && (s,x,w,\theta)\in[0,t)\times\mathbb{T}^d\times\widehat{\mathcal{W}}\times\partial B_R(0).
\end{aligned} \tag{3.54}$$

In the proof of this theorem, we use the following version of Theorem 5.2 in Chapter IV of [25]. Because we need it only for a specific class of PDEs, it is not as general as the original version of the theorem.

Theorem 3.25 ([25] Chapter IV Theorem 5.2). *Let $\ell>0$ be a non integer number and $\bar{i}\in H^{\ell+2}(B_R)$ and $\bar{b}^I,w\in H^{\ell/2,\ell}([0,t]\times B_R)$. Then for each $R>0$, there is a unique classical solution $g^*\in H^{\ell/2+1,\ell+2}\left([0,t]\times\overline{B_R}\right)$ of the following terminal boundary value problem*

$$\begin{aligned}
\partial_s g(s,\theta) &= -\left(\frac{\sigma^2}{2}\partial^2_{\theta^2}+\bar{b}^I(s,\theta)\,\partial_\theta\right)g(s,\theta)+w(\theta,s) && (s,\theta)\in[0,t)\times B_R,\\
g(t,\theta) &= \bar{i}(\theta) && \theta\in B_R,\\
g(s,\theta) &= 0 && (s,\theta)\in[0,t)\times\partial B_R(0).
\end{aligned} \tag{3.55}$$

Moreover, the solution g^* satisfies

$$|g^*|_{H^{\ell/2+1,\ell+2}\left([0,t]\times\overline{B_R}\right)}\le C\left(|w|_{H^{\ell/2,\ell}\left([0,t]\times\overline{B_R}\right)}+|i|_{H^{\ell+2}(B_R)}\right), \tag{3.56}$$

for a constant $C>0$ independent of w and i.

For a proof of this Theorem 3.25 we refer to [25]. Now we prove Theorem 3.24.

Proof of Theorem 3.24.* *Step 1: Existence and regularity.

The PDE (3.54) corresponds for a fixed tuple $(x,w) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$ to the PDE (3.55) with $w \equiv 0$, $\bar{i}(\theta) = i(x,w,\theta)$, $\bar{b}^I(s,\theta) = b^I(s,x,w,\theta)$, due to the independence in $x \in \mathbb{T}^d$ and $w \in \widehat{\mathcal{W}}$ of the operator $\mathbb{L}^I_{t,x,w}$. Therefore, we know by Theorem 3.25, that there is a unique solution $g^*_{x,w} \in H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right)$ of the corresponding PDE (3.55), for each $(x,w) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$. Set $g^*(.,x,w,\cdot) \coloneqq g_{x,w}$. The function g^* is a solution of (3.54). To show the claimed regularity of this solution, we need to show that $(x,w) \mapsto g^*_{x,w}$ is a continuous map $\mathbb{T}^d \times \widehat{\mathcal{W}} \to H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right)$.

Fix an arbitrary tuple $(x_0,w_0) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$. The proof of the continuity at (x_0,w_0) is organised as follows: In Step 1.1, we define an operator $I_{x,w} : H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right) \to H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right)$ for each $(x,w) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$. Then in Step 1.2, we show that $I_{x,w}$ is a contraction, when $|x-x_0|$ and $|w-w_0|$ are small enough. Next in Step 1.3, we show that the sequence $(I_{x,w})^n\left(g^*_{x_0,w_0}\right)$ converges to $g^*_{x,w}$ (also for $|x-x_0|$ and $|w-w_0|$ small enough). Finally in Step 1.4, we conclude from the previous steps the continuity of $g^*_{x,w}$ at $(x_0,w_0) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$.

Step 1.1: Define the operator

$$T_{s,x,w} \coloneqq \mathbb{L}^I_{s,x_0,w_0} - \mathbb{L}^I_{s,x,w} = \left(b^I(s,x_0,w_0,\cdot) - b^I(s,x,w,\cdot)\right)\partial_\theta. \tag{3.57}$$

With this operator, $\mathbb{L}^I_{s,x,w}$ can be seen as a perturbation of $\mathbb{L}^I_{s,x_0,w_0}$, by $\mathbb{L}^I_{s,x,w} = \mathbb{L}^I_{s,x_0,w_0} - T_{s,x,w}$. Moreover, we define the operator

$$I_{x,w} : H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right) \to H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right), \tag{3.58}$$

as the map that sends a function $v \in H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right)$ to the (unique) solution of

$$\begin{aligned}
\partial_s g(s,\theta) &= -\mathbb{L}^I_{s,x_0,w_0} g(s,\theta) + T_{s,x,w} v && (s,\theta) \in [0,t) \times B_R, \\
g(t,\theta) &= i(x,w,\theta) && \theta \in B_R, \\
g(s,\theta) &= 0 && (s,\theta) \in [0,t) \times \partial B_R.
\end{aligned} \tag{3.59}$$

We get the existence and the uniqueness of a solution to this PDE from Theorem 3.25.

Step 1.2: We show now that $I_{x,w}$ is a contraction.

Fix arbitrary $u_1,u_2 \in H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right)$. By the definition, $I_{x,w}(u_1) - I_{x,w}(u_2)$ is the unique classical solution to

$$\begin{gathered}
\left(\partial_s + \mathbb{L}^I_{s,x_0,w_0}\right)\left(I_{x,w}(u_1) - I_{x,w}(u_2)\right) = T_{s,x,w}(u_1-u_2), \\
\text{with } 0 \text{ terminal and } 0 \text{ boundary condition.}
\end{gathered} \tag{3.60}$$

Then by (3.56), for $|x_0-x|$ and $|w_0-w|$ small enough,

$$\begin{aligned}
\left|I_{x,w}(u_1) - I_{x,w}(u_2)\right|_{H^{\ell/2+1,\ell+2}} &\le C\left|T_{.,x,w}(u_1-u_2)\right|_{H^{\ell/2,\ell}} \\
&\le C\left|b^I(.,x_0,w_0,\cdot) - b^I(.,x,w,\cdot)\right|_{H^{\ell/2,\ell}}\left|\partial_\theta(u_1-u_2)\right|_{H^{\ell/2,\ell}} \\
&\le \epsilon\left|u_1-u_2\right|_{H^{\ell/2+1,\ell+2}}.
\end{aligned} \tag{3.61}$$

In the last inequality we use that $b^I \in H^{\ell/2,0,0,\ell}\left([0,T] \times \mathbb{T}^d \times \widehat{\mathcal{W}} \times B_R\right)$. This implies that $I_{x,w}$ is a contraction. Note that the ϵ is independent of $(x,w) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$, as long as $|x_0-x|$ and $|w_0-w|$ are small enough, because the constant C depends only on $\mathbb{L}^I_{.,x_0,w_0}$.

Step 1.3: Define the sequence $\left\{(I_{x,w})^n\left(g^*_{x_0,w_0}\right)\right\}_n$, where $g^*_{x_0,w_0}$ is the solution of (3.54) at (x_0, w_0). Then by (3.61)

$$\begin{aligned} &\left|(I_{x,w})^{n+1}\left(g^*_{x_0,w_0}\right) - (I_{x,w})^{n}\left(g^*_{x_0,w_0}\right)\right|_{H^{\ell/2+1,\ell+2}} \\ &\leq \epsilon \left|(I_{x,w})^{n}\left(g^*_{x_0,w_0}\right) - (I_{x,w})^{n-1}\left(g^*_{x_0,w_0}\right)\right|_{H^{\ell/2+1,\ell+2}} \\ &\leq \epsilon^n \left|I_{x,w}\left(g^*_{x_0,w_0}\right) - g^*_{x_0,w_0}\right|_{H^{\ell/2+1,\ell+2}} . \end{aligned} \tag{3.62}$$

Therefore, $\left\{(I_{x,w})^n\left(g^*_{x_0,w_0}\right)\right\}_n$ is a Cauchy sequence. The Hölder spaces are complete, hence there is a $u^*_{x,w} \in H^{\ell/2+1,\ell+2}$ such that $(I_{x,w})^n\left(g^*_{x_0,w_0}\right) \to u^*_{x,w}$. The continuity of $I_{x,w}$ implies that also $I_{x,w}\left((I_{x,w})^n\left(g^*_{x_0,w_0}\right)\right) \to I_{x,w}\left(u^*_{x,w}\right)$. Therefore, $u^*_{x,w} = I_{x,w}\left(u^*_{x,w}\right)$. By the definition of $I_{x,w}$ and the uniqueness of Theorem 3.25, we conclude $u^*_{x,w} = g^*_{x,w}$.
Step 1.4: Then by (3.62)

$$\begin{aligned} \left|g^*_{x,w} - g^*_{x_0,w_0}\right|_{H^{\ell/2+1,\ell+2}} &\leq \sum_{n=0}^{\infty} \left|(I_{x,w})^{n+1}\left(g^*_{x_0,w_0}\right) - (I_{x,w})^{n}\left(g^*_{x_0,w_0}\right)\right|_{H^{\ell/2+1,\ell+2}} \\ &\leq \left|I_{x,w}\left(g^*_{x_0,w_0}\right) - g^*_{x_0,w_0}\right|_{H^{\ell/2+1,\ell+2}} \frac{1}{1-\epsilon}. \end{aligned} \tag{3.63}$$

We show now that the right hand side is bounded by a $\epsilon_1 > 0$ for $(x,w) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$ with $|x_0 - x|$ and $|w_0 - w|$ small enough. By construction $I_{x,w}\left(g^*_{x_0,w_0}\right) - g^*_{x_0,w_0}$ is the solution to the PDE $\partial_s g = -\mathbb{L}^I_{s,x_0,w_0} g + T_{s,x,w} g^*_{x_0,w_0}$ with $i(x,w,\cdot) - i(x_0,w_0,\cdot)$ boundary condition. Hence by (3.56)

$$\begin{aligned} &\left|I_{x,w}\left(g^*_{x_0,w_0}\right) - g^*_{x_0,w_0}\right|_{H^{\ell/2+1,\ell+2}} \\ &\leq C\left(\left|T_{t,x,w} g^*_{x_0,w_0}\right|_{H^{\ell/2+1,\ell+2}} + \left|i(x,w,\cdot) - i(x_0,w_0,\cdot)\right|_{H^{\ell/2+1,\ell+2}}\right). \end{aligned} \tag{3.64}$$

Then as in (3.61) and finally by applying again (3.56) for $g^*_{x_0,w_0}$, we get that the right hand side of (3.64) is smaller or equal to

$$C\left(\epsilon \left|g^*_{x_0,w_0}\right|_{H^{\ell/2+1,\ell+2}} + \epsilon\right) \leq \epsilon C\left(\left|i(x_0,w_0,\cdot)\right|_{H^{\ell+2}} + 1\right) \leq \epsilon_1, \tag{3.65}$$

because $i(x_0,w_0,\cdot) \in H^{\ell+2}$. Therefore, $\left|g^*_{x,w} - g^*_{x_0,w_0}\right|_{H^{\ell/2+1,\ell+2}} < \epsilon_1$ for $|x_0 - x|$ and $|w_0 - w|$ small enough, by (3.63).

This is the claimed regularity of the solution $g^*_{x,w}$ at $(x,w) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$.

Step 2: Uniqueness.

Let g^* be a solution of (3.54). Then, for each tuple $(x,w) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$, $g^*_{x,w}$ has to be the unique solution of (3.55) with $w \equiv 0$, $\bar{i}(\theta) = i(x,w,\theta)$, $\bar{b}^I(s,\theta) = b^I(s,x,w,\theta)$. Therefore, there is at most one solution of (3.54) in $H^{\ell/2+1,0,0,\ell+2}\left([0,t] \times \mathbb{T}^d \times \widehat{\mathcal{W}} \times \overline{B_R}\right)$. □

Remark 3.26. Using the calculation in (3.64) and in (3.65), we could show even higher regularity than continuity of the solution in $\mathbb{T}^d \times \widehat{\mathcal{W}}$, if we assume higher regularity of b and i in $\mathbb{T}^d \times \widehat{\mathcal{W}}$.

3.1.3 Another representation of the rate function $S^I_{\nu,\zeta}$

We state in the next lemma another representation of the rate function $S^I_{\nu,\zeta}$. This representation is not used in the proof of Theorem 3.10. As explained in Remark 3.18 we could use it to show an upper bound on S^I. Nevertheless, we prove this lemma here, because we need it in Section 3.2 when showing that the rate function of the interacting system is actually lower semi-continuous

Lemma 3.27 (see [13] Lemma 4.8 for the mean-field case)**.** *Take a* $\nu \in \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ *and a* $\mu \in \mathscr{C}$*. Then*

$$S^I_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \mathcal{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right) + \sup_{f \in \mathsf{C}_c^{1,0,2}([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})} I\left(\mu_{[0,T]}, f\right), \tag{3.66}$$

where

$$\begin{aligned} I\left(\mu_{[0,T]}, f\right) = &\int\limits_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f\left(T,x,w,\theta\right)\mu_T\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right) - \int\limits_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f\left(0,x,w,\theta\right)\mu_0\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right) \\ &- \int_0^T \int\limits_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \left(\frac{\partial}{\partial t} + \mathbb{L}^I_{t,x,w}\right) f\left(t,x,\theta\right) - \frac{\sigma^2}{2}\left(\partial_\theta f\left(t,x,\theta\right)\right)^2 \mu_t\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right)\mathrm{d}t. \end{aligned} \tag{3.67}$$

Proof. Most parts of this proof are almost equal (modulo additional integrals with respect to $\mathbb{T}^d$ and $\mathcal{W}$) to the proof of Lemma 4.8 in [13]. Therefore, we only state the ideas and point out where things have to be changed due to the space and random environment dependence.

Fix a $\mu_{[0,T]} \in \mathscr{C}$ with $\mathcal{H}\left(\mu_0|\nu\right) < \infty$.

Step 1: We define for $f \in \mathsf{C}_c^{1,0,2}\left([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)$

$$\begin{aligned} \ell_{s,t}\left(f\right) = &\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f\left(t,x,w,\theta\right)\mu_t\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right) - \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f\left(s,x,w,\theta\right)\mu_s\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right) \\ &- \int_s^t \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \left(\partial_u + \mathbb{L}^I_{u,x,w}\right) f\left(u,x,w,\theta\right)\mu_u\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right)\mathrm{d}t. \end{aligned} \tag{3.68}$$

Note that this is equal to $I\left(\mu,f\right)$ without the $\left(\partial_\theta f\left(t,\cdot,\cdot,\cdot\right)\right)^2$ part and with the restriction to the time interval $[s,t]$. Analogue to (4.26) of [13], we can prove that

$$\begin{aligned} &\left|\ell_{s,t}\left(f\right)\right|^2 \\ &\leq \int_0^T \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \sigma^2\left(\partial_\theta f\left(t,x,w,\theta\right)\right)^2 \mu_t\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right)\mathrm{d}t \sup_{\substack{g\in \\ \mathsf{C}_c^{1,0,2}\left([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)}} I\left(\mu_{[0,T]},g\right). \end{aligned} \tag{3.69}$$

Step 2: As in the second step in [13] we can show that

$$I\left(\mu_{[0,T]},g\right) \leq S^I_{\nu,\zeta}\left(\mu_{[0,T]}\right) - \mathcal{H}\left(\mu_0|\mathrm{d}x\otimes\zeta_x\otimes\nu_x\right), \tag{3.70}$$

for each $g \in \mathsf{C}_c^{1,0,2}\left([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)$, by the integration by parts Lemma 2.20.

Step 3: We may assume that $\sup_{g\in\mathsf{C}_c^{1,0,2}([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})} I\left(\mu_{[0,T]},g\right) < \infty$. Denote by $\widehat{L}^2_{\mu_{[0,T]}}\left(s,t\right)$ the Hilbert space of all measurable maps $h : [s.t]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\to\mathbb{R}$, with finite norm

$$\left|h\right|^2_{\mu_{[0,T]}} := \int_s^t \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \frac{\sigma^2}{2}\left(h\left(u,x,w,\theta\right)\right)^2 \mu_u\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right)\mathrm{d}u. \tag{3.71}$$

Moreover, let $L^2_{\mu_{[0,T]}}\left(s,t\right)$ be the closure in $\widehat{L}^2_{\mu_{[0,T]}}\left(s,t\right)$ of the subset consisting of the maps $(t,x,\theta)\mapsto\partial_\theta h\left(t,x,\theta\right)$ with $h\in\mathsf{C}_c^{1,0,2}\left([s,t]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)$.

Similar as in the third step of the proof in [13] (but now with the additional dependence on the space $\mathbb{T}^d$), we can use this space to prove that there is a $h^{\mu_{[0,T]}} \in \widehat{L}^2_{\mu_{[0,T]}}(s,t)$, such that

$$\ell_{0,t}(f) = \int_0^t \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \sigma^2 h^{\mu_{[0,T]}}(u,x,w,\theta)\,\partial_\theta f(u,x,w,\theta)\,\mu_u(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\,\mathrm{d}u. \tag{3.72}$$

The existence of such an $h^{\mu_{[0,T]}}$, origins from applying the Riesz representation theorem for ℓ. Then the same arguments as in [13] lead to

$$\sup_{f\in \mathsf{C}_c^{1,0,2}([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})} I\left(\mu_{[0,T]}, f\right) = \frac{1}{2}\int_0^T \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \frac{\sigma^2}{2}\left(h^{\mu_{[0,T]}}(t,x,w,\theta)\right)^2 \mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\,\mathrm{d}t. \tag{3.73}$$

Step 4: In this last part, one uses the right hand side of (3.73) to show the equation (3.66). This follows again from the same arguments as in [13], by showing that $\mu_{[0,T]}$ is absolutely continuous as a map from $[0,T]\to\mathbb{D}'$ and finally by applying the Lemma 2.19. □

3.2 From independent to interacting spins

In this section, we finish the proof of Theorem 3.6 by generalising the proofs given in Section 5 of [13]. As explained subsequent to Theorem 3.6, we use the following local version of an LDP (Theorem 3.28) and exponential tightness result (Theorem 3.29), to prove Theorem 3.6.

Theorem 3.28 (compare to Theorem 5.2 in [13] for the mean-field version)**.** *If the assumptions of Theorem 3.6 hold, the following statements are true for fixed* $\overline{\mu}_{[0,T]}\in\mathscr{C}$.

(i) *For all open neighbourhoods* $V\subset\mathscr{C}$ *of* $\overline{\mu}_{[0,T]}$

$$\liminf_{N\to\infty} N^{-d}\log P^N\left[\mu^N_{[0,T]}\in V\right] \geq -S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right). \tag{3.74}$$

(ii) *For each* $\gamma>0$, *there is an open neighbourhood* $V\subset\mathscr{C}$ *of* $\overline{\mu}_{[0,T]}$ *such that*

$$\limsup_{N\to\infty} N^{-d}\log P^N\left[\mu^N_{[0,T]}\in V\right] \leq \begin{cases} -S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right)+\gamma & \text{if } S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right)<\infty,\\ -\gamma & \text{otherwise.}\end{cases} \tag{3.75}$$

Theorem 3.29 (compare to Theorem 5.3 in [13] for the mean-field version)**.** *If the assumptions of Theorem 3.6 hold, there is, for all* $s>0$, *a compact set* $\mathcal{K}_s\subset\mathscr{C}$, *with* $\mathcal{K}_s\subset\mathscr{C}_{\varphi,R}$ *for a* R *large enough, such that*

$$\limsup_{N\to\infty} N^{-d}\log P^N\left[\mu^N_{[0,T]}\in\mathscr{C}\backslash\mathcal{K}_s\right]\leq -s. \tag{3.76}$$

We state the proofs of these two theorems in Section 3.2.3. and Section 3.2.2.

Before inferring from these results Theorem 3.6, let us briefly state the idea of the proofs of these two theorems and explain how the rest of this section is organised.

1.) In Section 3.2.1, we show some preliminary lemmas. At first (in Section 3.2.1) we show that the operator $\mathbb{L}^I_{t,x,w}=\mathbb{L}_{\overline{\mu}_t,x,w}$ satisfies the assumptions of Section 3.1, for all $\overline{\mu}_{[0,T]}\in\mathscr{C}_\varphi\cap\mathscr{C}^L$. This implies the validity of the results of Section 3.1 for the independent system with fixed effective field $\overline{\mu}_{[0,T]}$.

Then we show (in Section 3.2.1), that $\mu_{[0,T]}^N$ is in $\mathscr{C}_{\varphi,\infty}$ almost surely under P^N, for all $N \in \mathbb{N}$.

Finally (Section 3.2.1), we derive exponential small bounds for P^N. For example we show that the probability of being outside of $\mathscr{C}_{\varphi,R}$ is exponentially small. The proofs of these results are for fixed initial data formally the same as the proofs in [13], at least after applying the result of Section 3.2.1. However, due to the different initial distribution, some new estimates are required. Here Assumption 3.5 is needed.

2.) Next, we prove in Section 3.2.2 Theorem 3.29, by combining in a suitable way the exponential bounds. The approach of this proof does not differ from the corresponding proof in [13].

3.) Finally, we prove Theorem 3.28 in Section 3.2.3. Here we separate the proof in the cases when $\overline{\mu}_{[0,T]}$ is in $\mathscr{C}_{\varphi,\infty}$, in $\mathscr{C}^L$ and when it is not in these sets. The part of the proof when $\overline{\mu}_{[0,T]}$ is in $\mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$ is formally similar to the proof in [13]. We use, in this part, the exponential bounds derived in Section 3.2.1 as well as the large deviation principle for independent spins (derived in Section 3.1). The other case, i.e. when $\overline{\mu}_{[0,T]}$ is not in $\mathscr{C}^L$ or not in $\mathscr{C}_{\varphi,\infty}$, are new here. When $\overline{\mu}_{[0,T]} \notin \mathscr{C}^L$, we show that in a small neighbourhood around $\overline{\mu}_{[0,T]}$, there is no empirical process for N large enough. From this we conclude the local large deviation result. For the case that $\overline{\mu}_{[0,T]}$ is not in $\mathscr{C}_{\varphi,\infty}$, we infer the local large deviation result from the exponential bounds.

Remark 3.30. All the results of this section can be transferred to hold also on $\mathscr{C}_{\varphi,\infty}$ with the stronger topology considered in [13]. The proofs would formally be the same.

Proof of Theorem 3.6. This proof of Theorem 3.6 is similar to the proof of the corresponding mean-field theorem in [13]. Despite these similarities we state the proof here, because it illustrates how Theorem 3.28 and Theorem 3.29 are applied. Differences to [13] arise only in the proof that $S_{\nu,\zeta}$ is a good rate function. This is mainly due to the space and random environment dependence and because the spins do not start at fixed positions (as considered in [13]), but are initially distributed according to ν.

Step 1: The large deviation lower bound.

Let $G \subset \mathscr{C}$ be a open set. The large deviation lower bound follows directly by applying Theorem 3.28 (i) with $V = G$ for all $\mu_{[0,T]} \in G$.

Step 2: The large deviation upper bound.

Let $F \subset \mathscr{C}$ be a closed set. We assume that $\inf_{\mu \in F} S_{\nu,\zeta}(\mu) = \overline{s} < \infty$. The case when the infimum is not finite can be treated similarly.

By Theorem 3.29 we know that there is compact set $\mathcal{K} \subset \mathscr{C}$ such that (3.76) is satisfied with $s = \overline{s}$. We further know by Theorem 3.28 (ii) that for a fixed $\gamma > 0$ and for each $\mu_{[0,T]} \in F \cap \mathcal{K}$, there is an open neighbourhood $V_{\mu_{[0,T]}}$ of $\mu_{[0,T]}$ such that (3.75) is satisfied for $\mu_{[0,T]}$. Because $F \cap \mathcal{K}$ is compact, it is covered by a finite number of these neighbourhoods. Combining these results we get

$$\begin{aligned}
&\limsup_{N\to\infty} N^{-d} \log P^N \left[\mu_{[0,T]}^N \in F\right] \\
&\leq \max\left\{\limsup_{N\to\infty} N^{-d} \log P^N \left[\mu_{[0,T]}^N \in F \cap \mathcal{K}\right], \limsup_{N\to\infty} N^{-d} \log P^N \left[\mu_{[0,T]}^N \notin \mathcal{K}\right]\right\} \\
&\leq -\overline{s} + \gamma.
\end{aligned} \tag{3.77}$$

Because the parameter γ is arbitrary, this proves the large deviation upper bound.

Step 3: $S_{\nu,\zeta}$ ***is a good rate function.***

To show that $S_{\nu,\zeta}$ is a good rate function, we have to show that the level sets

$$\mathcal{L}^{\leq s}\left(S_{\nu,\zeta}\right) := \left\{\mu_{[0,T]} \in \mathscr{C} : S_{\nu,\zeta}\left(\mu_{[0,T]}\right) \leq s\right\} \tag{3.78}$$

are compact in $\mathscr{C}$, for each $s \geq 0$. We show at first that the level set $\mathcal{L}^{\leq s}\left(S_{\nu,\zeta}\right)$ is relatively compact and then that it is closed.

Step 3.1: $\mathcal{L}^{\leq s}\left(S_{\nu,\zeta}\right)$ ***is relatively compact.***

By Theorem 3.29 we know that there is a compact set $\mathcal{K}_{s+\epsilon} \subset \mathscr{C}_{\varphi,R} \subset \mathscr{C}$, for $R > 0$ large enough, such that (3.76) holds for $s+\epsilon$. We claim that $\mathcal{L}^{\leq s}\left(S_{\nu,\zeta}\right) \subset \mathcal{K}_{s+\epsilon}$. Let us assume that there is a $\mu_{[0,T]} \in \mathcal{L}^{\leq s}\left(S_{\nu,\zeta}\right)$ that is not in $\mathcal{K}_{s+\epsilon}$. Then we know by (3.76) and Theorem 3.28 (i) (because $\mathscr{C}\backslash\mathcal{K}_{s+\epsilon}$ is an open neighbourhood of $\mu_{[0,T]}$), that $s+\epsilon \leq S_{\nu,\zeta}\left(\mu_{[0,T]}\right)$, a contradiction.

Step 3.2: $\mathcal{L}^{\leq s}\left(S_{\nu,\zeta}\right)$ ***is closed.***

Let $I\left(\mu_{[0,T]}, f\right)$ be defined as in (3.67). By Lemma 3.27 we know that

$$S_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \mathcal{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right) + \sup_{f \in \mathsf{C}_c^{1,0,2}([0,T]\times\mathbb{T}^d\times\mathbb{R})} I^{\mathbb{L}_{\mu_{[0,T]},\cdot,\cdot}}\left(\mu_{[0,T]}, f\right). \tag{3.79}$$

Moreover, we know by the previous step and the definition of $S_{\nu,\zeta}$ that $\mathcal{L}^{\leq s}\left(S_{\nu,\zeta}\right) \subset \mathscr{C}_{\varphi,R} \cap \mathscr{C}^L$, for a R large enough. Therefore, $\mathcal{L}^{\leq s}\left(S_{\nu,\zeta}\right) = \bigcap_{f \in \mathsf{C}_c^{1,0,2}([0,T]\times\mathbb{T}^d\times\mathbb{R})} \mathcal{L}^{\leq s}_{f,R}\left(S_{\nu,\zeta}\right)$ with

$$\mathcal{L}^{\leq s}_{f,R}\left(S_{\nu,\zeta}\right) := \left\{\mu_{[0,T]} \in \mathscr{C}_{\varphi,R} \cap \mathscr{C}^L : I^{\mathbb{L}_{\mu_{[0,T]},\cdot,\cdot}}\left(\mu_{[0,T]}, f\right) + \mathcal{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right) \leq s\right\}. \tag{3.80}$$

Hence, it is enough to show that the set $\mathcal{L}^{\leq s}_{f,R}\left(S_{\nu,\zeta}\right)$ is closed for each function $f \in \mathsf{C}_c^{1,0,2}\big([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\big)$. The map $\mu_{[0,T]} \mapsto I^{\mathbb{L}_{\mu_{[0,T]},\cdot,\cdot}}\left(\mu_{[0,T]}, f\right)$ is continuous as a function $\mathscr{C}_{\varphi,R} \cap \mathscr{C}^L \to \mathbb{R}$ for all $R \in \mathbb{R}_+$ and for all $f \in \mathsf{C}_c^{1,0,2}\big([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\big)$. This follows from Assumption 3.1 d). Moreover, $\mu^{(n)} \to \mu$ implies that $\mu_0^{(n)} \to \mu_0$, and $\mu_0 \mapsto \mathcal{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right)$ is lower semi continuous. Hence, the set $\mathcal{L}^{\leq s}_{f,R}\left(S_{\nu,\zeta}\right)$ is closed in $\mathscr{C}_{\varphi,R} \cap \mathscr{C}^L$. Due to $\mathscr{C}_{\varphi,R} \cap \mathscr{C}^L$ being closed in $\mathscr{C}$, this implies that $\mathcal{L}^{\leq s}_{f,R}\left(S_{\nu,\zeta}\right)$ is also closed in $\mathscr{C}$. □

3.2.1 Preliminaries

The assumptions of the corresponding independent systems are satisfied Fix a $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$. Define the function $b^I\left(t,x,w,\theta\right) := b\left(x,w,\theta,\overline{\mu}_t\right)$. We show now that Assumption 3.8 is satisfied for the independent spin system (given by (3.12)) with this drift coefficient b^I, i.e. $\mathbb{L}^I_{t,x,w} := \mathbb{L}_{\overline{\mu}_t,x,w}$.

a) The Assumption 3.8 a) is satisfied because of Assumption 3.1 a.i) and $\overline{\mu}_t \in \mathbb{M}_{\varphi,R}$ for all $t \in [0,T]$ and for a R large enough.

b) We infer from Theorem 10.1.2 of [34] the uniqueness of the martingale problem for each tuple $(x,w) \in \mathbb{T}^d \times \mathcal{W}$, because the drift coefficient is continuous (by a)). To apply this theorem, let G_n be a set with compact closure in R^{N^d} and define a continuous and bounded function $b^{I,(n)} : [0,T]\times\mathbb{R}$ to equal $b^I\left(.,x,\cdot\right)$ on G_n. Then Theorem 7.2.1 of [34] gives that for each n the martingale problem corresponding to $b^{I,(n)}$ is well defined. To show the existence, we apply Theorem 10.2.1 of [34]. The conditions of this theorem are satisfied by Assumption 3.1 c), because $\mathbb{L}^I_{t,x,w} = \mathbb{L}_{\overline{\mu}_t,x,w}$.

Therefore, the martingale problem is well defined, i.e. Assumption 3.8 b) is satisfied.

The empirical process is with probability one in $\mathscr{C}_{\varphi,\infty}$

Lemma 3.31. (i) *Let Assumption 3.5 hold. Then for all $N \in \mathbb{N}$,*

$$\sup_{\underline{w}^N \in \mathcal{W}^{N^d}} P^N_{\underline{w}^N}\left[\mu^N_{[0,T]} \in \mathscr{C}\backslash\mathscr{C}_{\varphi,\infty}\right] = 0. \tag{3.81}$$

(ii) *For any $r > 0$ and for all $N \in \mathbb{N}$,*

$$\sup_{\underline{w}^N \in \mathcal{W}^{N^d}} \sup_{\underline{\theta}^N \in \mathbb{R}^{N^d} : \mu^N_{\underline{\theta}^N} \in \mathbb{M}_{r,\varphi}} P^N_{\underline{w}^N,\underline{\theta}^N}\left[\mu^N_{[0,T]} \in \mathscr{C}\backslash\mathscr{C}_{\varphi,\infty}\right] = 0, \tag{3.82}$$

where $P^N_{\underline{w}^N,\underline{\theta}^N} \in \mathbb{M}_1\left(\mathbb{R}^{N^d}\right)$ is defined as $P^N_{\underline{w}^N}$ (see Notation 3.4) with fixed initial values $\underline{\theta}^N$.

Proof. ***(i)*** For all $R > 0$ and $\underline{w}^N \in \mathcal{W}^{N^d}$

$$\begin{aligned} P^N_{\underline{w}^N}\left[\mu^N_{[0,T]} \in \mathscr{C}\backslash\mathscr{C}_{\varphi,\infty}\right] &\leq P^N_{\underline{w}^N}\left[\underline{\theta}^N_{[0,T]} : \sup_{t\in[0,T]} \frac{1}{N^d}\sum_{k\in\mathbb{T}^d_N}\varphi\left(\theta^{k,N}_t\right) > R\right] \\ &= P^N_{\underline{w}^N}\left[\underline{\theta}^N_{[0,T]} : \sup_{t\in[0,T]} \log\left(1 + \frac{1}{N^d}\sum_{k\in\mathbb{T}^d_N}\varphi\left(\theta^{k,N}_t\right)\right) > \log(R+1)\right]. \end{aligned} \tag{3.83}$$

We want to show that the right hand side converge to zero when R tends to infinity. To do this, we use an approach that is for example used in the proof of Theorem 1.5 in [20] and apply it to the setting we consider here.

Fix $\underline{w}^N \in \mathcal{W}^{N^d}$. Applying Itô's lemma to $h\left(\underline{\theta}^N_t\right) := \log\left(1 + \frac{1}{N^d}\sum_{k\in\mathbb{T}^d_N}\varphi\left(\theta^{k,N}_t\right)\right)$, we get

$$\begin{aligned} h\left(\underline{\theta}^N_t\right) &\leq h\left(\underline{\theta}_0\right) + \int_0^t \frac{1}{1 + \frac{1}{N^d}\sum_{k\in\mathbb{T}^d_N}\varphi\left(\theta^{k,N}_s\right)} \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \mathbb{L}_{\mu^N_s,x,w}\varphi(\theta)\,\mu^N_s(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\,\mathrm{d}s + M_t \\ &\leq h\left(\underline{\theta}_0\right) + T + M_t, \end{aligned} \tag{3.84}$$

by Assumption 3.1 b), where μ^N_s is the empirical measure defined by $\underline{w}^N$ and $\underline{\theta}^N$. The M_t is a continuous local $P^N_{\underline{w}^N}$ martingale with $M_0 = 0$. Define the non negative $P^N_{\underline{w}^N}$ supermartingale

$$S^R_t := \min\left\{h\left(\underline{\theta}_0\right) + T + M_t, \log(R)\right\}. \tag{3.85}$$

By the Doob supermartingale inequality

$$\begin{aligned} &P^N_{\underline{w}^N}\left[\sup_{t\in[0,T]} h\left(\underline{\theta}^N_t\right) > \log(R+1)\right] \leq P^N_{\underline{w}^N}\left[\sup_{t\in[0,T]} S^R_t > \log(R+1)\right] \\ &\leq \frac{1}{\log(R+1)} E_{P^N_{\underline{w}^N}}\left[S^R_0\right] \leq \left(\log(R+1)\right)^{-\frac{1}{2}} + \nu^N\left[h\left(\underline{\theta}\right) > \left(\log(R+1)\right)^{\frac{1}{2}} - T\right]. \end{aligned} \tag{3.86}$$

To bound the probability, we apply the Chebyshev inequality,

$$\nu^N\left[h\left(\underline{\theta}\right) > \left(\log(R+1)\right)^{\frac{1}{2}} - T\right] \leq e^{-\kappa N^d\left(e^{(\log(R+1))^{\frac{1}{2}} - T} - 1\right)} \prod_{i\in\mathbb{T}^d_N}\int_{\mathbb{R}} e^{\kappa\varphi(\theta)}\nu_{\frac{i}{N}}(\mathrm{d}\theta). \tag{3.87}$$

By Assumption 3.5, the integral is bounded by a constant. Therefore, the right hand side of (3.86) converges to zero uniformly for all $\underline{w}^N$, when R tends to infinity. Combining this with (3.83), implies (i).

(ii) We get by the same arguments as in (i) ((3.83) to (3.86))

$$\sup_{\underline{\theta}^N \in \mathbb{R}^{N^d}: \mu^N_{\underline{\theta}^N} \in \mathbb{M}_{r,\varphi}} P^N_{\underline{w}^N, \underline{\theta}^N} \left[\mu^N_{[0,T]} \in \mathscr{C} \backslash \mathscr{C}_{\varphi,\infty} \right] \leq \left(\log\left(R+1\right)\right)^{-\frac{1}{2}}, \tag{3.88}$$

for all $R > 0$ large enough, when r is fixed. □

Exponential bounds In the next two lemmas we show that it is exponentially unlikely that an empirical process leaves the sets $\mathscr{C}_{\varphi,R}$. At first we show this uniformly for fixed initial conditions in $\mathbb{M}_{r,\varphi}$ (Lemma 3.32), then for initial conditions distributed according to ν (Lemma 3.33).

Lemma 3.32 (compare to Lemma 5.5 in [13] for the mean-field case)**.** *For any $r > 0$, $R > 0$ and for all $N \in \mathbb{N}$,*

$$\sup_{\underline{w}^N \in \mathcal{W}^{N^d}} \sup_{\underline{\theta}^N \in \mathbb{R}^{N^d}: \mu^N_{\underline{\theta}^N} \in \mathbb{M}_{r,\varphi}} P^N_{\underline{w}^N, \underline{\theta}^N} \left[\mu^N_{[0,T]} \in \mathscr{C} \backslash \mathscr{C}_{\varphi,R} \right] \leq e^{-N^d R_T}, \tag{3.89}$$

with $R_T = Re^{-\lambda T} - r$, where λ is defined in Assumption 3.1 b).

Proof. First note that by Lemma 3.31 (ii), it is enough to show for each $\underline{w}^N \in \mathcal{W}^{N^d}$

$$\sup_{\underline{\theta}^N \in \mathbb{R}^{N^d}: \mu^N \in \mathbb{M}_{r,\varphi}} P^N_{\underline{w}^N, \underline{\theta}^N} \left[\mu^N_{\underline{\theta}^N} \in \mathscr{C}_{\varphi,\infty} \backslash \mathscr{C}_{\varphi,R} \right] \leq e^{-N^d R_T}. \tag{3.90}$$

This bound can be proven (at least formally) exactly as the proof of Lemma 5.5 in [13]. Therefore, we do not state it here. Neither the different topology on $\mathscr{C}_{\varphi,\infty}$ considered in that paper nor the space dependence, is crucial in the proof. The proof requires Assumption 3.1 b). □

Lemma 3.33. *Let Assumption 3.5 hold. For all $s > 0$, there is a $R = R_s > 0$, such that for all $N \in \mathbb{N}$*

$$\sup_{\underline{w}^N \in \mathcal{W}^{N^d}} P^N_{\underline{w}^N} \left[\mu^N_{[0,T]} \in \mathscr{C} \backslash \mathscr{C}_{\varphi,R} \right] \leq e^{-N^d s}. \tag{3.91}$$

Proof. For all $R > 0$, $\underline{w}^N \in \mathcal{W}^{N^d}$

$$\begin{aligned} P^N_{\underline{w}^N} \left[\mu^N_{[0,T]} \in \mathscr{C} \backslash \mathscr{C}_{\varphi,R} \right] &= \int_{\mathbb{R}^{N^d}} P^N_{\underline{w}^N, \underline{\theta}^N} \left[\mu^N_{[0,T]} \in \mathscr{C} \backslash \mathscr{C}_{\varphi,R} \right] \nu^N \left(\mathrm{d}\underline{\theta}^N \right) \\ &\leq \sum_{k=0}^{\infty} e^{-N^d R e^{-\lambda T} + N^d (k+1)} \nu^N \left[\mathbb{M}_{k+1,\varphi} \backslash \mathbb{M}_{k,\varphi} \right], \end{aligned} \tag{3.92}$$

where we use Lemma 3.32 in the inequality. For the probability of the right hand side we use the exponential Chebyshev inequality with $\ell > 1$

$$\begin{aligned} \nu^N \left[\mathbb{M}_{k+1,\varphi} \backslash \mathbb{M}_{k,\varphi} \right] &\leq \nu^N \left[\sum_{k \in \mathbb{T}^d_N} \varphi\left(\theta^{k,N} \right) > N^d k \right] \\ &\leq e^{-\ell N^d k} \prod_{i \in \mathbb{T}^d_N} \int_{\mathbb{R}} e^{\ell \varphi(\theta)} \nu_{\frac{i}{N}} \left(\mathrm{d}\theta \right) \leq e^{-\ell N^d k} C^{N^d}, \end{aligned} \tag{3.93}$$

by Assumption 3.5. Then

$$\begin{aligned} P^N_{\underline{w}^N}\left[\mu^N_{[0,T]} \in \mathscr{C}\backslash\mathscr{C}_{\varphi,R}\right] &\leq C^{N^d} e^{-N^d R e^{-\lambda T} + N^d} \sum_{k=0}^{\infty} e^{N^d k(1-\ell)} \\ &\leq C^{N^d} e^{-N^d R e^{-\lambda T} + N^d} \frac{1}{1 - e^{N^d(1-\ell)}} \leq e^{-N^d R e^{-\lambda T} \frac{1}{2}}, \end{aligned} \tag{3.94}$$

for R large enough. □

For Theorem 3.29, we need compact subsets of $\mathscr{C}$. These sets are characterised in the following lemma.

Lemma 3.34 (Lemma 1.3 in [20])**.** *Let $\{f_n\}_n$ be an arbitrary countable dense subset of $\mathsf{C}_c\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$. A set $\mathcal{K}$ is relatively compact in $\mathscr{C}$ if and only if*

$$\mathcal{K} \subset \mathcal{K}_K \cap \bigcap \mathcal{K}_n, \tag{3.95}$$

with

$$\mathcal{K}_K = \left\{\mu_{[0,T]} \in \mathscr{C} : \mu_t \in K \text{ for all } t \in [0,T]\right\}, \tag{3.96}$$

$$\mathcal{K}_n = \left\{\mu_{[0,T]} \in \mathscr{C} : \left\{t \mapsto \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f_n\left(x,w,\theta\right) \mu_t\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta\right)\right\} \in K_n\right\}, \tag{3.97}$$

where $K \subset \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ and $K_n \subset \mathsf{C}([0,T])$ are compact.

For a proof of this lemma, see Lemma 1.3 in [20].

The next lemma states an exponential bound on the probability that the empirical process is outside of a subset of $\mathscr{C}$, that is defined via the projection to $\mathsf{C}([0,T])$. We use this set in Theorem 3.29 as the set $\mathcal{K}_n$, defined in Lemma 3.34 in the characterisation of relative compact subset of $\mathscr{C}$.

Lemma 3.35 (compare to Lemma 5.6 in [13] for the mean-field case)**.** *For all $R > 0, s > 0$ and $f \in \mathsf{C}_c^\infty\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$, there exists a compact set $K \subset \mathsf{C}([0,T])$, such that for all $N \in \mathbb{N}$*

$$\sup_{\underline{w}^N \in \mathcal{W}^{N^d}} \sup_{\underline{\theta}^N \in \mathbb{R}^{N^d}} P^N_{\underline{w}^N, \underline{\theta}^N}\left[\mu^N_{[0,T]} \in \mathscr{C}_{\varphi,R} \backslash \mathcal{K}_f\right] \leq e^{-N^d s}, \tag{3.98}$$

with $\mathcal{K}_f = \left\{\mu_{[0,T]} \in \mathscr{C} : \left\{t \mapsto \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f\left(x,w,\theta\right) \mu_t\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta\right)\right\} \in K\right\}$.

Proof. This proof is formally exactly the proof of Lemma 5.6 in [13] for each $\underline{w}^N \in \mathcal{W}^{N^d}$. Indeed, in the proof only uses the function $\left\{t \mapsto \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f\left(x,w,\theta\right) \mu_t\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta\right)\right\}$, which is (here as in [13]) a function in $\mathsf{C}([0,T], \mathbb{R})$ and one does not have to care about the structure within the integral. Moreover, the topology of $\mathscr{C}$ is not relevant in the proof. The proof requires Assumption 3.1 a.ii). □

3.2.2 Proof of Theorem 3.29

Proof of Theorem 3.29. This proof equals the proof of Theorem 5.3 in [13], besides formal changes due to the space dependence. The only generalisation is that we consider random initial data here.

By Lemma 3.34 it is enough to define compact sets $K \subset \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ and $K_n \subset \mathsf{C}([0,T])$ to get a compact set in $\mathscr{C}$. We set $K = \mathbb{M}_{\varphi,R}$ and therefore $\mathcal{K}_K = \mathscr{C}_{\varphi,R}$. Moreover, we choose by Lemma 3.35 for each n a $K_n \subset \mathsf{C}([0,T])$, such that

$$\sup_{\underline{w}^N \in \mathcal{W}^{N^d}} \sup_{\underline{\theta}^N \in \mathbb{R}^{N^d}} P^N_{\underline{w}^N, \underline{\theta}^N}\left[\mu^N_{[0,T]} \in \mathscr{C}_{\varphi,R} \backslash \mathcal{K}_n\right] \leq e^{-nN^d s}. \tag{3.99}$$

Define the compact set $\mathcal{K} := \overline{\mathscr{C}_{\varphi,R} \cap \bigcap \mathcal{K}_n}$. This is a subset of $\mathscr{C}_{\varphi,R}$, because $\mathscr{C}_{\varphi,R}$ is closed in $\mathscr{C}$.

By Lemma 3.33 and (3.99) we conclude for all $N \in \mathbb{N}$ and R large enough

$$\begin{aligned} &P^N\left[\mu^N_{[0,T]} \in \mathscr{C}\backslash\mathcal{K}\right] \\ &\leq P^N\left[\mu^N_{[0,T]} \in \mathscr{C}\backslash\mathscr{C}_{\varphi,R}\right] + \sum_{n=1}^{\infty} \sup_{\underline{w}^N \in \mathcal{W}^{N^d}} \sup_{\underline{\theta}^N \in \mathbb{R}^{N^d}} P^N_{\underline{w}^N,\underline{\theta}^N}\left[\mu^N_{[0,T]} \in \mathscr{C}_{\varphi,R}\backslash\mathcal{K}_n\right] \\ &\leq e^{-N^d s} + \sum_{n=1}^{\infty} e^{-nN^d s}. \end{aligned} \tag{3.100}$$

□

3.2.3 Proof of Theorem 3.28

We prove in this section Theorem 3.28. In the proof, we investigate separately the cases, when $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty}$ (Case 1 and Case 2), and when it is not in this space (Case 3). Moreover, we divide the first case in the subcases that $\overline{\mu}_t \in \mathscr{C}^L$ (Case 1), and when this is not true (Case 2). The ideas of the proofs of the three cases are as follows.

Case 1: .

For $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$, we reduce the claims of Theorem 3.28 to large deviation upper and lower bounds for a system of independent SDEs. For this independent system these large deviation bounds hold by Theorem 3.10. To reduce the claims we choose at first (Step 1.1), for each $N \in \mathbb{N}$, a system of spins, that evolve mutually independent, with the constraint that their empirical process should be close to $\overline{\mu}_{[0,T]}$ with high probability. Therefore, we choose the drift coefficient $\overline{b}^I(x, w, \theta, t) := b(x, w, \theta, \overline{\mu}_t)$. We regard the empirical process of interacting diffusions, in a small neighbourhood of $\overline{\mu}_{[0,T]}$, as a small perturbation of the empirical process for the independent diffusions with drift coefficient $\overline{b}^I$. Then, in Step 1.2, we apply the (Cameron-Martin-) Girsanov theorem and receive a density between the measures of the solution to the original SDE and the one of the SDE with drift coefficient $\overline{b}^I$. Using this density, we reduce in Step 1.3 and Step 1.4 the claims of Theorem 3.28 to large deviation bounds for the independent system. We get these bounds by Theorem 3.10, which is applicable by Section 3.2.1.

The proof of this first case is very similar to the one in [13] in Section 5.4 for the mean-field setting. However, differences arise due to the space and random environment dependence. Moreover, we show the large deviation principle on the space $\mathscr{C}$ and not like in [13] on $\mathscr{C}_{\varphi,\infty}$ equipped even with another topology than the subspace topology.

Case 2: . When $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty}$ with $\overline{\mu}_t \notin \mathbb{M}^L_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})$ for some $t \in [0,T]$, we show that there is no empirical process within an ϵ-ball around $\overline{\mu}_{[0,T]}$ for N large enough. From this we infer the claims of Theorem 3.28.[1]

Case 3: . When $\overline{\mu}_{[0,T]}$ is not in $\mathscr{C}_{\varphi,\infty}$, then the first statement of Theorem 3.28 is obviously satisfied and the second statement follows from Lemma 3.33.

Proof. Fix an arbitrary $\overline{\mu}_{[0,T]} \in \mathscr{C}$.

Case 1: $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$**.**

Step 1.1: Definition of a system of diffusions with a fixed effective field. We set $b^I(x, w, \theta, t) := b(x, w, \theta, \overline{\mu}_t)$ and use this function as drift coefficient to define the time dependent diffusion generator $\mathbb{L}^I_{t,x,w}$ (defined as in (3.11)). Then $\mathbb{L}^I_{t,x,w} = \mathbb{L}_{\overline{\mu}_t,x,w}$. Moreover, we define the measures $P^{I,N} \in \mathbb{M}_1\left(\mathsf{C}([0,T])^{N^d}\right)$ as in Notation 3.9.

[1] If we assumed in Assumption 3.1 a.i) that the continuity of b holds on $\mathbb{M}_{\varphi,R}$ and not only on $\mathbb{M}_{\varphi,R} \cap \mathbb{M}^L_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})$, then we could handle this case as in the previous step. However, to keep the assumption more general, we have to use a new approach.

As shown in Section 3.2.1, Assumption 3.1 implies the Assumptions 3.8 for the generator $\mathbb{L}^I_{t,x,w}$. Therefore, Theorem 3.10 is applicable for $P^{I,N}$.

Step 1.2: Comparison of the two processes with help of the Girsanov theorem. We claim that for each $\underline{w}^N \in \mathcal{W}^{N^d}$, $P^N_{\underline{w}^N}$ is absolutely continuous with respect to $P^{I,N}_{\underline{w}^N}$, with Radon-Nikodym derivative

$$\frac{\mathrm{d}P^N_{\underline{w}^N}}{\mathrm{d}P^{I,N}_{\underline{w}^N}} = e^{M^N_{\underline{w}^N,T} \; -\frac{1}{2}\langle\langle M^N_{\underline{w}^N}\rangle\rangle_T}, \tag{3.101}$$

for all $\underline{\theta}^N \in \mathbb{R}^{N^d}$. Here $M^N_{\underline{w}^N,t}$ is a continuous local $P^{I,N}_{\underline{w}^N}$ martingale with quadratic variation

$$\langle\langle M^N_{\underline{w}^N}\rangle\rangle_t \left(\underline{\theta}^N_{[0,T]}\right) = N^d \int_0^t \int\limits_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \sigma^2 \left| b\left(x,w,\theta,\mu^N_u\right) - b\left(x,w,\theta,\overline{\mu}_u\right)\right|^2 \mu^N_u\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right)\mathrm{d}u, \tag{3.102}$$

where μ^N_u is the empirical measure defined by $\underline{\theta}^N_u$ and $\underline{w}^N$. This can be shown by a spatial localisation argument. The generators $\mathbb{L}^N_{\cdot}$ and $\mathbb{L}^{I,N}_{\cdot}$ only differ in their drift coefficients. The martingale problems corresponding to both generators are well defined. Moreover, b^N (defined in Assumption 3.1 a.ii)) and b^I (as continuous function) are both locally bounded. By spatial localisation (see [34] Theorem 10.1.1) it is hence enough to consider bounded drift coefficients. For bounded drift coefficients, we know by [34] Theorem 6.4.2 the claimed representation of the Radon-Nikodym formula.

Step 1.3: The proof of (i). For $S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) = \infty$, (i) is obviously satisfied. Therefore, assume that $S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) < \infty$. Fix an open neighbourhood $V \subset \mathscr{C}$ of $\overline{\mu}_{[0,T]}$ and an arbitrary $\gamma > 0$.

The Lemma 3.33 can also be applied to $P^{I,N}$ instead of P^N by Assumption 3.1 c). This lemma then states (with $s = S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) + \gamma$), that there is a $R > 0$ such that

$$P^{I,N}\left[\mu^N_{[0,T]} \in \mathscr{C}\backslash\mathscr{C}_{\varphi,R}\right] \le e^{-N^d S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right)} e^{-N^d\gamma}. \tag{3.103}$$

Assume that this R is so large that $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,R}$. We choose now two constants $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and a $\delta > 0$ such that

$$\frac{1}{2}\left(1+\frac{p}{q}\right)\delta + pS_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) \le S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) + \gamma. \tag{3.104}$$

By Assumption 3.1 d) and (3.102), there is a open neighbourhood $W \subset \mathscr{C}$ of $\overline{\mu}_{[0,T]}$ such that $W \cap \mathscr{C}_{\varphi,R} \subset V$ and $\langle\langle M^N_{\underline{w}^N}\rangle\rangle_T \left(\underline{\theta}^N_{[0,T]}\right) \le N^d\delta$ for $\underline{w}^N \in \mathcal{W}^{N^d}$ and $\underline{\theta}^N_{[0,T]} \in \mathsf{C}([0,T])^{N^d}$ when the corresponding empirical processes $\mu^N_{[0,T]} \in W \cap \mathscr{C}_{\varphi,R}$. With the same arguments as in [13] we can show by using the Radon-Nikodym derivative (3.101) that for each $\underline{w}^N \in \mathcal{W}$

$$\begin{aligned} P^N_{\underline{w}^N}\left[\mu^N_{[0,T]} \in V\right] &\ge P^N_{\underline{w}^N}\left[\mu^N_{[0,T]} \in W \cap \mathscr{C}_{\varphi,R}\right] \\ &\ge e^{-\frac{1}{2}\left(1+\frac{p}{q}\right)\delta N^d} \left(P^{I,N}_{\underline{w}^N}\left[\mu^N_{[0,T]} \in W \cap \mathscr{C}_{\varphi,R}\right]\right)^p. \end{aligned} \tag{3.105}$$

We integrate (3.105) with respect to ζ^N and apply the Jensen inequality,

$$P^N\left[\mu^N_{[0,T]} \in V\right] \ge e^{-\frac{1}{2}\left(1+\frac{p}{q}\right)\delta N^d} \left(P^{I,N}\left[\mu^N_{[0,T]} \in W \cap \mathscr{C}_{\varphi,R}\right]\right)^p. \tag{3.106}$$

Moreover

$$P^{I,N}\left[\mu^N_{[0,T]} \in W \cap \mathscr{C}_{\varphi,R}\right] \geq P^{I,N}\left[\mu^N_{[0,T]} \in W\right]\left(1 - e^{-N^d\frac{\gamma}{2}}\right), \tag{3.107}$$

for N large enough. Indeed, (3.107) holds, by the triangle inequality and

$$P^{I,N}\left[\mu^N_{[0,T]} \notin \mathscr{C}_{\varphi,R}\right] \leq e^{-N^d S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right)} e^{-N^d\gamma} \leq e^{-N^d\frac{\gamma}{2}} P^{I,N}\left[\mu^N_{[0,T]} \in W\right], \tag{3.108}$$

by (3.103) and because W is an open set and $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ satisfies the large deviation principle (Theorem 3.10).

Combine (3.106) and (3.107), we get

$$\begin{aligned} &\liminf_{N\to\infty} N^{-d}\log P^N\left[\mu^N_{[0,T]} \in V\right] \\ &\geq -\frac{1}{2}\left(1+\frac{p}{q}\right)\delta + p\liminf_{N\to\infty} N^{-d}\log P^{I,N}\left[\mu^N_{[0,T]} \in W\right]. \end{aligned} \tag{3.109}$$

Finally, we conclude by the large deviation principle for $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ (Theorem 3.10) and (3.104)

$$(3.109) \geq -\frac{1}{2}\left(1+\frac{p}{q}\right)\delta - pS_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) \geq -S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) - \gamma. \tag{3.110}$$

This inequality holds for all $\gamma > 0$. Hence we have proven (i) for this case.

Step 1.4: The proof of (ii). We assume $S_{\nu,\zeta}(\overline{\mu}) < \infty$. The case when it is not finite can be treated analogue. Fix a $\gamma > 0$. Due to Lemma 3.33 it is sufficient to find for $R > 0$ large enough with $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,R}$, an open neighbourhood $V \subset \mathscr{C}$ of $\overline{\mu}_{[0,T]}$ such that

$$\limsup_{N\to\infty} N^{-d}\log P^N\left[\mu^N_{[0,T]} \in V \cap \mathscr{C}_{\varphi,R}\right] \leq -S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) + \gamma. \tag{3.111}$$

Fix again $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and a $\delta > 0$, such that

$$\frac{p-1}{2}\delta + \frac{1}{q}\left(-S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) + \frac{\gamma}{2}\right) \leq -S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) + \gamma. \tag{3.112}$$

By Assumption 3.1 d) and (3.102) and by Theorem 3.10, there is a small open neighbourhood $V \subset \mathscr{C}$ of $\overline{\mu}_{[0,T]}$, such that $\langle\langle M^N_{\underline{w}^N}\rangle\rangle_T\left(\underline{\theta}^N_{[0,T]}\right) \leq N^d\delta$ for $\underline{w}^N \in \mathcal{W}^{N^d}$ and $\underline{\theta}^N_{[0,T]} \in \mathsf{C}([0,T])^{N^d}$ when the corresponding empirical processes $\mu^N_{[0,T]} \in V \cap \mathscr{C}_{\varphi,R}$, and such that

$$\limsup_{N\to\infty} N^{-d}\log P^{I,N}\left[\mu^N_{[0,T]} \in V\right] \leq -S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) + \frac{\gamma}{2}. \tag{3.113}$$

In the last inequality we use that $S^I_{\nu,\zeta}$ is lower semi-continuous and $S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) = S^I_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right)$ As in [13], we can show, by using the Radon-Nikodym derivative (3.101), that for all $\underline{w}^N \in \mathcal{W}$

$$P^N_{\underline{w}^N}\left[\mu^N_{[0,T]} \in V \cap \mathscr{C}_{\varphi,R}\right] \leq e^{\frac{p-1}{2}\delta N}\left(P^{I,N}_{\underline{w}^N}\left[\mu^N_{[0,T]} \in V\right]\right)^{\frac{1}{q}}. \tag{3.114}$$

To conclude (3.111), integrate both sides with respect to ζ^N, apply the Jensen inequality and finally use (3.112) and (3.113). Hence we showed (ii) for this case.

Case 2: $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty}$ *and* $\overline{\mu}_{[0,T]} \notin \mathscr{C}^L$.

Fix an arbitrary $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty}$ with $\overline{\mu}_{[0,T]} \notin \mathscr{C}^L$. Then $S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) = \infty$, by the definition of the rate function. This implies that (i) of Theorem 3.28 is obviously satisfied.

Now we prove that (ii) of Theorem 3.28 holds. At first we fix an open ball around $\overline{\mu}_{[0,T]}$, that does not intersect $\mathscr{C}^L$ (Step 2.1). Then we show that in such an open ball there is no empirical process with N large enough (Step 2.2). From this we conclude (ii) (in Step 2.3).

Step 2.1: A open ball around $\overline{\mu}_{[0,T]}$**.** The set $\mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ is closed in the space $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ (see e.g. [2] Proposition 4.3.1). This implies that also the set $\mathscr{C}^L$ is closed in $\mathscr{C}$. Hence there is a $\epsilon > 0$ such that

$$\operatorname{dist}\left\{\overline{\mu}_{[0,T]}, \mathscr{C}^L\right\} = \inf_{\pi \in \mathscr{C}^L} \left\{\sup_{t \in [0,T]} \rho^{Lip}\left(\overline{\mu}_t, \pi_t\right)\right\} > 2\epsilon, \tag{3.115}$$

where ρ^{Lip} is the bounded Lipschitz norm on $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$.

Define the open ϵ ball $B_\epsilon\left(\overline{\mu}_{[0,T]}\right)$ around $\overline{\mu}_{[0,T]}$ in this norm.

Step 2.2: No empirical process in the open ball for N large enough. Assume that we could find a sequence $N_\ell \nearrow \infty$ in $\mathbb{N}$, such that for each N_ℓ there is an empirical process $\mu_{[0,T]}^{N_\ell} \in B_\epsilon\left(\overline{\mu}\right)$, with a $\underline{\theta}_{[0,T]}^{N_\ell} \subset \mathsf{C}([0,T])^{N_\ell^d}$ and a $\underline{w}^N \in \mathcal{W}^{N^d}$. We claim that this leads to a contradiction. For each N_ℓ in the sequence, define $\mu_{t,x}^{(\ell)} = \delta_{\left(w^{k,N_\ell}, \theta_t^{k,N_\ell}\right)}$ when $\left|x - \frac{k}{N_\ell}\right| < \frac{1}{2N_\ell}$. Then $\left\{t \mapsto \mu_t^{(\ell)} := \mathrm{d}x \otimes \mu_{t,x}^{(\ell)}\right\} \in \mathscr{C}^L$. For each $f \in \mathsf{C}\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ that is Lipschitz continuous with $|f|_\infty + |f|_{Lip} \leq 1$,

$$\begin{aligned}
&\left|\int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f(x,w,\theta)\, \mu_t^{N_\ell}(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) - \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f(x,w,\theta)\, \mu_t^{(\ell)}(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)\right| \\
&= \sum_{k \in \mathbb{T}_{N_\ell}^d} \left|\frac{1}{N_\ell} f\left(\frac{k}{N_\ell}, w^{k,N_\ell}, \theta_t^{k,N_\ell}\right) - \int_{\Delta_{k,N_\ell}} f\left(x, w^{k,N_\ell}, \theta_t^{k,N_\ell}\right) \mathrm{d}x\right| \\
&\leq \sum_{k \in \mathbb{T}_N^d} |f|_{Lip} \left(\frac{1}{N_\ell}\right)^2 \leq \frac{1}{N_\ell},
\end{aligned} \tag{3.116}$$

with Δ_{k,N_ℓ} defined as in Assumption 1.7. Hence the distance between $\mu_{[0,T]}^{N_\ell}$ and $\mathscr{C}^L$ vanishes, a contraction. Therefore, we can fix an $\overline{N} \in \mathbb{N}$, such that there is no empirical process $\mu_{[0,T]}^N$ in $B_\epsilon\left(\overline{\mu}_{[0,T]}\right)$ when $N > \overline{N}$.

Step 2.3: Conclusion of (ii). From the previous step we infer that for $N > \overline{N}$,

$$P^N\left[\mu_{[0,T]}^N \in B_\epsilon\left(\overline{\mu}_{[0,T]}\right)\right] = 0. \tag{3.117}$$

This implies (ii) of Theorem 3.28 for this case.

Case 3: $\overline{\mu}_{[0,T]} \notin \mathscr{C}_{\varphi,\infty}$.

Because $\overline{\mu}_{[0,T]} \notin \mathscr{C}_{\varphi,\infty}$, $S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) = \infty$. Therefore, the condition (i) of Theorem 3.28 is obviously satisfied. To prove (ii) of Theorem 3.28, note that for each $R > 0$, the open set $\mathscr{C} \backslash \mathscr{C}_{\varphi,R}$ is a neighbourhood of $\overline{\mu}_{[0,T]}$. By Lemma 3.33, there is for each γ and R such that

$$P^N\left[\mu_{[0,T]}^N \in \mathscr{C} \backslash \mathscr{C}_{\varphi,R}\right] \leq e^{-N^d \gamma}. \tag{3.118}$$

This implies the claimed condition (ii) of Theorem 3.28 in this case. □

3.3 The special case (1.6) of a local mean-field model

In this section we show the following lemma.

Lemma 3.36. *When $\sigma = 1$ and Assumption 1.7, Assumption 1.9 hold, then the special case* (1.6) *of a local mean-field model satisfies Assumption 3.1.*

Denote by b the drift coefficient of the SDE (1.6).

Proof. Fix $\varphi(\theta) := 1 + \theta^2$. We show now separately that each item of Assumption 3.1 is satisfied.

Step 1: Assumption 3.1 a.i).

The function $\partial_\theta \Psi$ is continuous by Assumption 1.9. Hence the drift coefficient is continuous on $\mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \times \left(\mathbb{M}_{\varphi,R} \cap \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)$ if the map

$$(x, w, \mu) \mapsto \beta(x, w, \mu) := \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} J(x - x', w, w') \, \theta' \mu(\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta') \tag{3.119}$$

is continuous on this space. This holds if for $R > 0$ and each sequence $\left(x^{(n)}, w^{(n)}, \mu^{(n)}\right) \to (x, w, \mu)$ in $\mathbb{T}^d \times \mathcal{W} \times \left(\mathbb{M}_{\varphi,R} \cap \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)$, the following absolute value vanishes

$$\begin{aligned}\left|\beta\left(x^{(n)}, w^{(n)}, \mu^{(n)}\right) - \beta(x, w, \mu)\right| \leq &\left|\beta\left(x, w, \mu^{(n)}\right) - \beta\left(x^{(n)}, w^{(n)}, \mu^{(n)}\right)\right| \\ &+ \left|\beta\left(x, w, \mu^{(n)}\right) - \beta(x, w, \mu)\right| =: 1 + 2 .\end{aligned} \tag{3.120}$$

We show now that 1 and 2 vanish when n tends to infinity.

Step 1.1: Bound on 1 . There is a sequence of functions $J_\ell \in \mathsf{C}\left(\mathbb{T}^d \times \mathcal{W} \times \mathcal{W}\right)$, such that $J_\ell \to J$ in $L^2\left(\mathbb{T}^d, \mathsf{C}(\mathcal{W} \times \mathcal{W})\right)$, because $J \in L^2\left(\mathbb{T}^d, \mathsf{C}(\mathcal{W} \times \mathcal{W})\right)$. This implies that for all $\overline{x} \in \mathbb{T}^d$, $\overline{w} \in \mathcal{W}$ and $n \in \mathbb{N}$

$$\begin{aligned}&\left|\int_{\mathbb{T}^d \times \mathcal{W}} (J - J_\ell)(\overline{x} - x', \overline{w}, w') \int_{\mathbb{R}} \theta' \mu^{(n)}_{x', w'}(\mathrm{d}\theta') \, \mu^{(n)}_{x', \mathcal{W}}(\mathrm{d}w') \, \mathrm{d}x'\right| \\ &\leq \left(\int_{\mathbb{T}^d} \left(\sup_{w', w'' \in \mathcal{W}} \left|(J - J_\ell)(x, w'', w')\right|\right)^2 \mathrm{d}x\right)^{\frac{1}{2}} R,\end{aligned} \tag{3.121}$$

because $\mu^{(n)} \in \mathbb{M}_{\varphi,R}$. Therefore 1 is lesser or equal to

$$\sup_{x' \in \mathbb{T}^d, w' \in \mathcal{W}} \left|J_\ell\left(x^{(n)} - x', w^{(n)}, w'\right) - J_\ell(x - x', w, w')\right| (1 + R) + 2\left\|J - J_\ell\right\| R \leq \epsilon, \tag{3.122}$$

for $k \in \mathbb{N}$ and $n \in \mathbb{N}$ large enough, because J_ℓ is uniformly continuous on the compact set $\mathbb{T}^d \times \mathcal{W} \times \mathcal{W}$.

Step 1.2: Bound on 2. To bound 2, define the function $\chi_M(\theta) := (\theta \wedge M) \vee -M$ and approximate J by J_ℓ as in the previous step. Then 2 is lesser or equal to

$$\begin{aligned}&\left|\int_{\mathbb{T}^d \times \mathcal{W}} (J - J_\ell)(x - x', w, w') \int_{\mathbb{R}} \theta' \mu^{(n)}_{x'}(\mathrm{d}\theta') \, \mu^{(n)}_{x', \mathcal{W}}(\mathrm{d}w') \, \mathrm{d}x'\right| + \text{(this term with } \mu) \\ &+ \left|\int J_\ell(x - x', w, w') \left(\theta' - \chi_M(\theta')\right) \mu^{(n)}(\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta')\right| + \text{(this term with } \mu) \\ &+ \left|\int J_\ell(x - x', w, w') \, \chi_M(\theta') \left(\mu^{(n)} - \mu\right)(\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta')\right| := A + B + C + D + E .\end{aligned} \tag{3.123}$$

The (A) and (B) are bounded by ϵ, when k and n are large enough as shown in (3.121). We bound (C) by

$$\begin{aligned} \text{(C)} &\leq |J_\ell|_\infty \int |\theta'| \mathbb{1}_{|\theta'|>M} \mu^{(n)}(\mathrm{d}x', \mathrm{d}w, \mathrm{d}\theta') \\ &\leq |J_\ell|_\infty \mu^{(n)} \left[(x, w, \theta') : |\theta'| > M\right]^{\frac{1}{2}} \left(\int (\theta')^2 \mu^{(n)}(\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta')\right)^{\frac{1}{2}}. \end{aligned} \tag{3.124}$$

For an arbitrary fixed $k \in \mathbb{N}$ and for all $n \in \mathbb{N}$, the right hand side is bounded by ϵ for M large enough, because $\mu^{(n)} \in \mathbb{M}_{\varphi,R}$ and by the tightness of $\left\{\mu^{(n)}\right\}_n$ (as a converging sequence). The same arguments show (D) is bounded by ϵ. The (E) converges to zero when $n \to \infty$, for arbitrary fixed k and M, because the integrand is bounded and continuous.

Therefore, we fix at first a $k \in \mathbb{N}$, then an $M > 0$. Then for $n \in \mathbb{N}$ large enough, (2) is bounded by ϵ.

We have hence shown that (3.120) vanishes when n tends to infinity, i.e. that β is continuous.

Step 2: Assumption 3.1 a.ii).

Fix an arbitrary $N \in \mathbb{N}$ and an arbitrary $\underline{w}^N \in \mathcal{W}$. The function $b^N : \mathbb{R}^{N^d} \to \mathbb{R}^{N^d}$ is continuous, by Assumption 1.9 and because $\frac{1}{N^d} \sum_{j \in \mathbb{T}_N^d} J\left(\frac{i-j}{N}, w^{i,N}, w^{j,N}\right) \theta^{j,N}$ is continuous (because $J\left(\frac{i-j}{N}, w, w'\right)$ is finite for all $i, j \in \mathbb{T}_N^d$ and all $w, w' \in \mathcal{W}$ by Assumption 1.7). Hence b^N is locally bounded.

Step 3: Assumption 3.1 b).

Let $\mathbb{L}^{\mathrm{LMF}}_{\mu^N,\cdot,\cdot}$ be the generator of the local mean-field model, defined as (1.9), with μ^N the empirical measure corresponding to $\underline{\theta}^N \in \mathbb{R}^{N^d}$ and $\underline{w}^N \in \mathcal{W}$. Then for N large enough

$$\begin{aligned} &\int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \mathbb{L}^{\mathrm{LMF}}_{\mu^N,x,w} \varphi(\theta) + \frac{1}{2} \left|\partial_\theta \varphi(\theta)\right|^2 \mu^N(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \\ &= 2 + 2 \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \left(-\overline{\Psi}'(\theta)\theta - w\theta^2 + \theta^2\right) \mu^N(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) + 2B^N_{\underline{w}^N}\left(\underline{\theta}^N\right), \end{aligned} \tag{3.125}$$

where

$$\begin{aligned} B^N_{\underline{w}^N}\left(\underline{\theta}^N\right) &:= \frac{1}{N^{2d}} \sum_{i,j \in \mathbb{T}_N^d} J\left(\frac{i-j}{N}, w^{i,N}, w^{j,N}\right) \theta^{i,N} \theta^{j,N} \\ &\leq \left(\sum_{i \in \mathbb{T}_N^d} \sup_{w,w' \in \mathcal{W}} \left|\frac{1}{N^d} J\left(\frac{i}{N}, w, w'\right) - \int_{\Delta_{i,N}} J(x, w, w') \mathrm{d}x\right| + \left\|\overline{J}\right\|_{L^1}\right) \frac{1}{N^d} \sum_{j \in \mathbb{T}_N^d} \left(\theta^{j,N}\right)^2 \\ &\leq \left(\delta + \left\|\overline{J}\right\|_{L^1}\right) \frac{1}{N^d} \sum_{j \in \mathbb{T}_N^d} \left(\theta^{j,N}\right)^2, \end{aligned} \tag{3.126}$$

with $\delta > 0$ if $N > \overline{N}_\delta$ by Assumption 1.7. With this upper bound on B^N, Ψ being a polynomial of even degree with positive leading coefficient (Assumption 1.9) and $\mathcal{W}$ being compact, we conclude that

$$\begin{aligned} (3.125) &\leq C + 2 \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \left(|w| + 1 + \left\|\overline{J}\right\|_{L^1} + \delta\right) \theta^2 \mu^N(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \\ &\leq \lambda \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \varphi(\theta) \mu^N(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta). \end{aligned} \tag{3.127}$$

Here the constant λ only depends on Ψ and J for N large enough but not on μ^N. Hence Assumption 3.1 b) is satisfied.

Step 4: Assumption 3.1 c).

Fix an arbitrary $\mu_{[0,T]} \in \mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$. We know by Step 1, that $(x, w, t) \mapsto \beta(x, w, \mu_t)$ is continuous. And the set $\mathbb{T}^d \times \mathcal{W} \times \{\mu_t\}_{t\in[0,T]}$ is compact in $\mathbb{T}^d \times \mathcal{W} \times \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})$, by Prokhorov's theorem. Hence β is bounded on this set by a constant C_β. Then for all $(t, x, w, \theta) \in [0, T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$

$$\begin{aligned} \mathbb{L}^{\mathrm{LMF}}_{\mu_t,x,w}\varphi(\theta) + \frac{1}{2}\left|\partial_\theta\varphi(\theta)\right|^2 &= -2\partial_\theta\overline{\Psi}(\theta)\,\theta - 2w\theta^2 + 2\theta\beta(x, \mu_t) + 2 + 2\theta^2 \\ &\leq -2\partial_\theta\overline{\Psi}(\theta)\,\theta + 2|w|\,\theta^2 + 2|\theta|\,C_\beta + 2 + 2\theta^2 \leq \lambda\left(\mu_{[0,T]}\right)\varphi(\theta), \end{aligned} \tag{3.128}$$

because $\overline{\Psi}$ is a polynomial of even degree (Assumption 1.9) and $\mathcal{W}$ is compact.

Step 5: Assumption 3.1 d).

Fix an $R > 0$ and a $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,R} \cap \mathscr{C}^L$. Take an arbitrary sequence $\left\{\mu^{(n)}_{[0,T]}\right\}$ from one of the sets given in Assumptions 3.1 d), such that $\mu^{(n)}_{[0,T]} \to \overline{\mu}_{[0,T]}$. We show in the subsequent steps that

$$\int_0^T \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \left|\beta\left(x, w, \mu^{(n)}_t\right) - \beta(x, w, \overline{\mu}_t)\right|^2 \mu^{(n)}_t(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)\,\mathrm{d}t \to 0. \tag{3.129}$$

Step 5.1: Case of sequence $\left\{\mu^{(n)}_{[0,T]}\right\}$ in $\mathscr{C}^L$.

Assume at first that $\mu^{(n)}_{[0,T]} \in \mathscr{C}_{\varphi,R} \cap \mathscr{C}^L$ for all $n \in \mathbb{N}$. For each $t \in [0, T]$, $\mu^{(n)}_t \to \overline{\mu}_t$ in $\mathbb{M}_{\varphi,R}$ by the uniform topology on $\mathscr{C}$. Therefore, the set $U_t := \left\{\mu^{(n)}_t\right\}_n \cup \{\overline{\mu}_t\}$ is compact. $\mathbb{T}^d \times \mathcal{W} \times U_t \ni (x, w, \mu) \mapsto |\beta(x, w, \mu) - \beta(x, w, \overline{\mu}_t)|$ is uniformly continuous (we show the continuity in Step 1). Hence for each $t \in [0, T]$, the absolute value in (3.129) converges uniformly in $(x, w) \in \mathbb{T}^d \times \mathcal{W}$ to zero, when n tends to infinity. Moreover, this absolute value is uniformly bounded, because for all $(x, w) \in \mathbb{T}^d \times \mathcal{W}$

$$\left|\int\limits_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} J(x - x', w, w')\,\theta'\mu^{(n)}_t(\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta')\right|^2 \leq \left\|\overline{J}\right\|^2_{L^2} \int\limits_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} |\theta'|^2\,\mu^{(n)}_t(\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta'). \tag{3.130}$$

The right hand side is bounded by $\left\|\overline{J}\right\|^2_{L^2} R$, because all $\mu^{(n)}_{[0,T]}$ are in $\mathscr{C}_{\varphi,R}$. This implies the convergence (3.129) for sequences in $\mathscr{C}^L$.

Step 5.2: Case of sequence $\left\{\mu^{(n)}_{[0,T]}\right\}$ consists of empirical processes.

Fix a sequence of empirical processes $\left\{\mu^{(n)}_{[0,T]}\right\}_n \subset \mathscr{C}_{\varphi,R}$, such that $\mu^{(n)}_{[0,T]} \to \overline{\mu}_{[0,T]}$. Fix $N_n \in \mathbb{N}$, $\theta^{i,N_n}_{[0,T]} \in \mathsf{C}([0, T])$, $w^{i,N_n} \in \mathcal{W}$ such that $\mu^{(n)}_{[0,T]} = \frac{1}{N_n^d}\sum_{i\in\mathbb{T}^d_{N_n}} \delta_{\left(\frac{i}{N_n}, w^{i,N_n}, \theta^{i,N_n}_t\right)}$. Note that we do not get for this sequence the continuity of β at $t \in [0, T]$ from Step 1.

For each $t \in \mathbb{T}^d_N$ and $n \in \mathbb{N}$, the inner integral in (3.129) is given by

$$\frac{1}{N_n^d}\sum_{j\in\mathbb{T}^d_{N_n}} \left|\beta\left(\frac{j}{N_n}, w^{j,N_n}, \mu^{(n)}_t\right) - \beta\left(\frac{j}{N_n}, w^{j,N_n}, \overline{\mu}_t\right)\right|^2. \tag{3.131}$$

We show in the following that this sum converges for each $t \in [0, T]$ pointwise to zero (Step 5.2.1). Moreover, we show that this sum is uniformly bounded (Step 5.2.2). From these two results we conclude (3.129) by the dominated convergence theorem.

Step 5.2.1: (3.131) ***vanishes pointwise.*** To show that (3.131) vanishes, we divide the absolute value as in (3.123) into five summands. Fix an arbitrary small $\epsilon > 0$. By fixing $k \in \mathbb{N}$ and $M > 0$ large enough, the B, C and D of these summands are smaller than ϵ for all $(x, w) \in \mathbb{T}^d \times \mathcal{W}$ for fixed k and all $n \in \mathbb{N}$ large enough, by the same arguments that we use in Step 1.2. Hence to bound (3.131) we only need to bound the following two summands

$$\begin{aligned} \text{A} &\coloneqq \frac{1}{N_n^d} \sum_{j \in \mathbb{T}^d_{N_n}} \left| \frac{1}{N_n^d} \sum_{i \in \mathbb{T}^d_{N_n}} \theta_t^{i,N_n} \left(J - J_\ell \right) \left(\frac{j-i}{N_n}, w^{j,N_n}, w^{i,N_n} \right) \right|, \\ \text{E} &\coloneqq \frac{1}{N_n^d} \sum_{j \in \mathbb{T}^d_{N_n}} \left| \int J_\ell \left(\frac{j}{N_n} - x', w^j, w^i \right) \chi_M \left(\theta' \right) \left(\mu_t^{(n)} - \overline{\mu}_t \right) (\mathrm{d}x', \mathrm{d}\theta') \right|. \end{aligned} \tag{3.132}$$

We prove now that A and E are smaller than ϵ when n is large enough (Step 5.2.1.3 and Step 5.2.1.2). Both proofs require that N_n converges to infinity. We show in Step 5.2.1.1, that this is a consequence of the convergence of $\mu_t^{(n)}$ to a measures in $\mathbb{M}_1^L(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})$.
Step 5.2.1.1: The sequence $N_n \to \infty$***.*** Assume that this were not the case, i.e. that there is a subsequence $\{N_{n_\ell}\}_{\ell=1}^\infty$ such that $N_{n_\ell} \le \overline{N} < \infty$. This is a contradiction to the convergence of $\mu_t^{(n)}$ to $\overline{\mu}_t$. Indeed, choose $f \in \mathsf{C}_b(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})$ such that $f(x, w, \theta) = f(x) \ge 0$ for all $(x, w, \theta) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$, $\int_{\mathbb{T}^d} f(x)\, \mathrm{d}x > 0$ and $f\left(\frac{k}{N}\right) = 0$ for all $N \le \overline{N}$, $k \in \mathbb{T}^d_N$. Then $\int f(x)\, \mu_t^{(n_\ell)} = 0$ for all $\ell \in \mathbb{N}$, but $\int f(x)\, \overline{\mu}_t > 0$. A contradiction.
Step 5.2.1.2: E. The function J_ℓ is uniformly continuous on $\mathbb{T}^d \times \mathcal{W}$. By the compactness of $\mathbb{T}^d \times \mathcal{W}$, there are finitely many $\{x_a\}_{a \in A} \subset \mathbb{T}^d$ and finitely many $\{w_{a'}\}_{a' \in A'} \subset \mathcal{W}$, such that

$$\text{E} \le 2\epsilon M + \max_{a \in A, a' \in A'} \left| \int J_\ell \left(x_a - x', w_{a'}, w' \right) \chi_M \left(\theta' \right) \left(\mu_t^{(n)} - \overline{\mu}_t \right) (\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta') \right|. \tag{3.133}$$

The maximum is only over a finite number of values, hence the convergence of $\mu_t^{(n)}$ to $\overline{\mu}_t$ implies that for n large enough, the maximum is bounded by ϵ.
Step 5.2.1.3: A. We bound A by ϵ through a similar estimate as in Step 1.1. In particular we use the following estimate instead of (3.121). For all $j \in \mathbb{T}^d_{N_n}$, A is less or equal to

$$\begin{aligned} &\left| \sum_{i \in \mathbb{T}^d_{N_n}} \theta_t^{i,N_n} \int_{\Delta_{i,N_n}} \left(J - J_\ell \right) \left(\frac{j}{N_n} - x', w^{j,N_n}, w^{i,N_n} \right) \mathrm{d}x' \right| \\ &+ \left| \sum_{i \in \mathbb{T}^d_{N_n}} \theta_t^{i,N_n} \left(\frac{1}{N_n^d} J_\ell \left(\frac{j-i}{N_n}, w^{j,N_n}, w^{i,N_n} \right) - \int_{\Delta_{i,N_n}} J_\ell \left(\frac{j}{N_n} - x', w^{j,N_n}, w^{i,N_n} \right) \mathrm{d}x' \right) \right| \\ &+ \left| \sum_{i \in \mathbb{T}^d_{N_n}} \theta_t^{i,N_n} \left(\frac{1}{N_n^d} J \left(\frac{j-i}{N_n}, w^{j,N_n}, w^{i,N_n} \right) - \int_{\Delta_{i,N_n}} J \left(\frac{j}{N_n} - x', w^{j,N_n}, w^{i,N_n} \right) \mathrm{d}x' \right) \right|. \end{aligned} \tag{3.134}$$

We denote the three summands by A1, A2 and A3 and we bound them separately. By applying twice the Hölder inequality

$$\text{A1} \le \left(\frac{1}{N_n^d} \sum_{i \in \mathbb{T}^d_{N_n}} \left| \theta_t^{i,N_n} \right|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^d} \left(\sup_{w,w' \in \mathcal{W}} \left| (J - J_\ell)(x', w, w') \right| \right)^2 \mathrm{d}x' \right)^{\frac{1}{2}} \le R\epsilon, \tag{3.135}$$

for k large enough.

$$\text{A2} \leq \frac{1}{N_n^d} \sum_{i \in \mathbb{T}^d_{N_n}} \left|\theta_t^{i,N_n}\right| \sup_{|y-y'| \leq \frac{1}{N_n}} \sup_{w,w' \in \mathcal{W}} \left|J_\ell\left(y',w,w'\right) - J_\ell\left(y,w,w'\right)\right| \leq R\epsilon, \tag{3.136}$$

for each k, when n (and hence N_n) is large enough. Last but not least, by a change of variables

$$\left(\text{A3}\right)^2 \leq \frac{1}{N_n^d} \sum_{i \in \mathbb{T}^d_{N_n}} \left|\theta_t^{i,N_n}\right|^2 \sum_{i \in \mathbb{T}^d_{N_n}} \sup_{w,w' \in \mathcal{W}} \left|\int_{\Delta_{i,N_n}} J\left(\frac{i}{N_n},w,w'\right) - J\left(x',w,w'\right) \mathrm{d}x'\right|^2, \tag{3.137}$$

which is also bounded by $R\epsilon$, when n is large enough by Assumption 1.7.

Step 5.2.2: (3.131) ***is uniformly (in*** $t \in [0,T]$***) bounded.***

We show that each summand of (3.131) is bounded uniformly in $t \in [0,T]$, $j \in \mathbb{T}^d_{N_n}$, $n \in \mathbb{N}$. By applying the Hölder inequality we get

$$\begin{aligned}\left|\beta\left(\frac{j}{N_n}, w^{j,N_n}, \mu_t^{(n)}\right)\right|^2 \leq \frac{1}{N_n^d} \sum_{i \in \mathbb{T}^d_{N_n}} \left|\theta_t^{i,N_n}\right|^2 \\ \left(\sum_{i \in \mathbb{T}^d_N} \sup_{w,w' \in \mathcal{W}} \left|\frac{1}{N_n^d} J\left(\frac{i}{N_n},w,w'\right) - \int_{\Delta_{i,N_n}} J\left(x,w,w'\right) \mathrm{d}x\right|^2 + \left\|\overline{J}\right\|_{L^2}\right).\end{aligned} \tag{3.138}$$

This is bounded by $R\left(\left\|\overline{J}\right\|_{L^2} + \delta\right)$ for a $\delta > 0$, when N_n is large enough, by Assumption 1.7. Moreover, we get a uniform upper bound on $\left|\beta\left(\frac{j}{N_n}, w^{j,N_n}, \underline{\mu}_t\right)\right|$ as in (3.130).

We have hence proven Assumption 3.1 d).

Summarized, the example (1.6) of a local mean-field model satisfies Assumption 3.1 if the Assumption 1.7, Assumption 1.9 hold. □

Remark 3.37. When considering only continuous J, the proofs are much simpler. However, also interaction weights that are not continuous are of particular interest (for some examples see Example 1.8).

4 Representations of the rate function for the LDP of the empirical process

In this section, we state three other representations of the rate function $S_{\nu,\zeta}$, besides the two given in Theorem 3.6. These expressions might be useful when working on the mentioned long time behaviour (see also [12] in the mean-field case), in particular when the model is not reversible.

To state these representations we need the following notation.

Notation 4.1. *For* $\mu_{[0,T]} \in \mathscr{C}_{\varphi,\infty}$ *and* $(t,x,w,\theta) \in [0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$*, set*

$$b^{I,\mu_{[0,T]}}\left(t,x,w,\theta\right) := b\left(x,w,\theta,\mu_t\right). \tag{4.1}$$

With $b^{I,\mu_{[0,T]}}$ *as drift coefficient, define the generator* $\mathbb{L}^{I,\mu_{[0,T]}}_{t,x,w}$ *as in* (3.11)*. For this system, Assumption 3.8 are satisfied if the assumptions of Theorem 3.6 hold (as shown in Section 3.2.1). In particular the corresponding martingale problem has for each* $(x,w,\theta) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$ *a unique solution, which we denote by* $P^{I,\mu_{[0,T]}}_{x,w,\theta}$*. Then we define* $P^{I,\mu_{[0,T]}}_{x,w} \in \mathbb{M}_1(\mathsf{C}([0,T]))$*,* $P^{I,N,\mu_{[0,T]}}_{\underline{w}^N} \in \mathbb{M}_1\left(\mathsf{C}([0,T])^{N^d}\right)$ *and* $P^{I,N,\mu_{[0,T]}} \in \mathbb{M}_1\left(\mathcal{W}^{N^d} \times \mathsf{C}([0,T])^{N^d}\right)$ *as in Notation 3.9.*

Moreover, we denote by $U_{s,t}^{\mu_{[0,T]}}$ *the operator* $U_{s,t}$ *defined in* (3.17) *with* P^I *replaced by* $P^{I,\mu_{[0,T]}}$.

Theorem 4.2. *Let the assumptions of Theorem 3.6 hold.* $S_{\nu,\zeta}$ *has the following representations for* $\mu_{[0,T]} \in \mathscr{C}$, *with* $S_{\nu,\zeta}\left(\mu_{[0,T]}\right) < \infty$.

(i)

$$S_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \inf_{\substack{Q \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right) \\ \Pi(Q)_{[0,T]} = \mu_{[0,T]}}} \mathcal{H}\left(Q \middle| \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes P_{x,w}^{I,\mu_{[0,T]}} \right) \tag{4.2}$$

(ii) $S_{\nu,\zeta}\left(\mu_{[0,T]}\right)$ *is equal to*

$$\begin{aligned} \sup_{\substack{r \in \mathbb{N}, \\ 0 \le t_1 < \ldots < t_r \le T}} \Bigg[\sup_f \Bigg\{ \int\limits_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f \mu_{t_1} - \int_{\mathbb{T}^d} \log \Bigg(\int\limits_{\mathcal{W} \times \mathbb{R}} U_{0,t_1}^{\mu_{[0,T]}} e^f (x,w,\theta)\, \nu_x(\mathrm{d}\theta)\, \zeta_x(\mathrm{d}w) \Bigg) \mathrm{d}x \Bigg\} \\ + \sum_{i=2}^r \sup_f \Bigg\{ \int\limits_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f \mu_{t_i} - \int\limits_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \log U_{t_{i-1},t_i}^{\mu_{[0,T]}} e^f (x,w,\theta)\, \mu_{t_{i-1}} \Bigg\} \Bigg], \end{aligned} \tag{4.3}$$

where the μ_{t_i} *integrate with respect to the variables* $\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta$ *and the functions* f *in the suprema are in the set* $\mathsf{C}_c^\infty\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$.

(iii) *There is a function* $h^{\mu_{[0,T]}} \in \widehat{L}^2_{\mu_{[0,T]}}(0,T)$ *(this space is defined in the Step 3 of the proof of Lemma 3.27), such that* $S_{\nu,\zeta}\left(\mu_{[0,T]}\right)$ *is equal to*

$$\frac{1}{2} \int_0^T \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \frac{\sigma^2}{2} \left(h^{\mu_{[0,T]}}(t,x,w,\theta)\right)^2 \mu_t(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)\, \mathrm{d}t + \mathcal{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right). \tag{4.4}$$

Moreover, $\mu_{[0,T]}$ *satisfies in a weak sense (i.e. when integrated against an arbitrary function in* $\mathsf{C}_c^{1,0,2}\left([0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$*) the PDE*

$$\partial_t \mu_t = \left(\mathbb{L}_{\mu_t,\cdot,\cdot}\right)^* \mu_t + \sigma^2 \partial_\theta \left(\mu_t h^{\mu_{[0,T]}}(t)\right). \tag{4.5}$$

Proof. When $S_{\nu,\zeta}\left(\mu_{[0,T]}\right) < \infty$, then $\mu_{[0,T]} \in \mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$. Therefore, we know by Section 3.2.1, that the measure $P_{x,w}^{I,\mu_{[0,T]}}$ is well defined. Moreover, all the results of Section 3.1 hold for the independent spin system with the drift coefficient b^I of Notation 4.1.

The representations (i) and (ii) follow directly from Lemma 3.12 and Lemma 3.13.

The representation (iii), follows from Lemma 3.27 and the proof of this lemma, in particular (3.73) in Step 3 of this proof. That $\mu_{[0,T]}$ is a weak solution of the PDE (4.5), follows from (3.72) and (3.68). □

5 The LDP of the empirical measure

In this section we show the large deviation principle for the empirical measures L^N under the assumptions of Section 3 and the following exponential tightness assumption.

Assumption 5.1. *The family* $\left\{L^N, P^N\right\}$ *is exponential tight, i.e. for all* $s > 0$, *there is a compact set* $\mathcal{K}_s \subset \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$, *such that*

$$\limsup_{N \to \infty} N^{-d} \log P^N \left[L^N \notin \mathcal{K}_s\right] \le -s. \tag{5.1}$$

We prove in [31] Section V.5.2 the exponential tightness of the empirical measure L^N for the special case (1.6) of a local mean-field model.

To state the large deviation principle result, we need the following definitions and notations.

Definition 5.2. *We say $Q \in \mathcal{M}_{\varphi,R}$ if and only if $\Pi(Q)_{[0,T]} \in \mathbb{M}_{\varphi,R}$, for $R \in (0,\infty]$.*

For fixed $x \in [0,T]$ and $Q \in \mathcal{M}_{\varphi,R} \cap \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$, we define $b^{I,\Pi(Q)}$, $\mathbb{L}_{t,x,w}^{I,\Pi(Q)}$ and the measures $P_{x,w}^{I,\Pi(Q)} \in \mathbb{M}_1(\mathsf{C}([0,T]))$ and $P^{I,N,\Pi(Q)} \in \mathbb{M}_1\left(\mathcal{W}^{N^d} \times \mathsf{C}([0,T])^{N^d}\right)$ as in Notation 4.1.

Theorem 5.3. *If the assumptions of Theorem 3.6 and Assumption 5.1 hold, then the family of empirical measures $\left\{L^N, P^N\right\}$ satisfies on $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ the large deviation principle with good rate function*

$$I(Q) := \begin{cases} \mathcal{H}\left(Q \middle| P^{I,\Pi(Q)}\right) & \text{if } Q \in \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right) \cap \mathcal{M}_{\varphi,\infty}, \\ \infty & \text{otherwise.} \end{cases} \tag{5.2}$$

where $P^{I,\Pi(Q)} := \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes P_{x,w}^{I,\Pi(Q)} \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$.

To prove this theorem, we use the same approach as for the proof of the large deviation principle for the empirical process $\mu_{[0,T]}^N$, given in Section 3. Only the proof that I is a good rate function uses another strategy than the corresponding proof in Section 3. Here, we require the exponential tightness of $\left\{L^N\right\}$.

We show in the next lemma, that the measure in the relative entropy in (5.2) is actually a probability measure.

Lemma 5.4. *For each $Q \in \mathcal{M}_{\varphi,\infty} \cap \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$, the measure $P^{I,\Pi(Q)}$ is well defined.*

Proof of Lemma 5.4. Fix a $Q \in \mathcal{M}_{\varphi,\infty} \cap \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$. The function $b^{I,\Pi(Q)}$ is continuous. Indeed, $t \mapsto \Pi(Q)_t$ is continuous (Lemma 2.23), $\Pi(Q)_t \in \mathbb{M}_{\varphi,R} \cap \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ for all $t \in [0,T]$ and the function b is continuous on $\mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \times \left(\mathbb{M}_{\varphi,R} \cap \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)$ (Assumption 3.1 a.i)). Therefore, we can apply [34] Theorem 11.1.4 to get the continuity of $(x,w,\theta) \mapsto P_{x,w,\theta}^{I,\Pi(Q)}$ (see also Lemma 3.15). By this continuity, Assumption 1.5 and Assumption 1.6, we conclude (as in Lemma 2.14) that the measure $P^{I,\Pi(Q)}$ is well defined. □

5.1 Independent spins

Fix a $Q \in \mathcal{M}_{\varphi,\infty} \cap \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$. We get, as in the proof of Lemma 3.12, the following large deviation principle for the independent system (by Lemma 2.10 and Lemma 2.11 with $r = 1$, $Y = \mathsf{C}([0,T])$).

Lemma 5.5. *The family $\left\{L^N, P^{I,N,\Pi(Q)}\right\}$ satisfies on $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ the large deviation principle with rate function*

$$I^Q(\Gamma) := \mathcal{H}\left(\Gamma \middle| P^{I,\Pi(Q)}\right), \tag{5.3}$$

for $\Gamma \in \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ and infinity otherwise.

5.2 From independent to interacting spins

As in Section 3.2, we show at first the following local version of the LDP.

Lemma 5.6. *Under the same assumptions as in Theorem 3.6, the following statements are true, for each $\overline{Q} \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$.*

(i) *For all open neighbourhoods* $V \subset \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$ *of* $\overline{Q}$

$$\liminf_{N\to\infty} N^{-d} \log P^N \left[L^N \in V\right] \geq -I\left(\overline{Q}\right). \tag{5.4}$$

(ii) *For each* $\gamma > 0$*, there is an open neighbourhood* $V \subset \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$ *of* $\overline{Q}$ *such that*

$$\limsup_{N\to\infty} N^{-d} \log P^N \left[L^N \in V\right] \leq \begin{cases} -I\left(\overline{Q}\right) + \gamma & \text{if } I\left(\overline{Q}\right) < \infty, \\ -\gamma & \text{otherwise.} \end{cases} \tag{5.5}$$

The Lemma 5.6 can be proven as Theorem 3.28. This proof requires Lemma 5.5 and the following exponential bound instead of Lemma 3.33..

Lemma 5.7. *For all* $s > 0$*, there is a* $R = R_s > 0$*, such that for all* $N \in \mathbb{N}$

$$\sup_{\underline{w}^N \in \mathcal{W}^{N^d}} P^N_{\underline{w}^N}\left[L^N \notin \mathcal{M}_{\varphi,R}\right] \leq e^{-N^d s}. \tag{5.6}$$

The Lemma 5.7 follows directly from Lemma 3.33, because $L^N \in \mathcal{M}_{\varphi,R}$ if and only if $\Pi\left(L^N\right)_{[0,T]} \in \mathscr{C}_{\varphi,R}$, i.e.

$$P^N_{\underline{w}^N}\left[L^N \in \mathcal{M}_{\varphi,R}\right] = P^N_{\underline{w}^N}\left[\mu^N_{[0,T]} \in \mathscr{C}_{\varphi,R}\right]. \tag{5.7}$$

Then Lemma 5.6 and Assumption 5.1 imply the lower and upper large deviation bound with the good rate function I. Indeed, we show in the next lemma that I is a good rate function. This finishes the proof of Theorem 5.3.

Lemma 5.8. *The function* $Q \mapsto I(Q)$ *is a good rate function.*

Proof. We show at first that the level set $\mathcal{L}^{\leq s}(I)$ is relatively compact and then that it is closed.

Step 1: $\mathcal{L}^{\leq s}(I)$ ***is relatively compact.***

By Assumption 5.1 and Lemma 5.7 we know that there is a compact set $\mathcal{K}_{s+\epsilon} \subset \mathcal{M}_{\varphi,R}$, for $R > 0$ large enough, such that (5.1) holds. We claim that $\mathcal{L}^{\leq s}(I) \subset \mathcal{K}_{s+\epsilon}$. Assume that there is a $Q \in \mathcal{L}^{\leq s}(I)$ that is not in $\mathcal{K}_{s+\epsilon}$. Then we know by (5.1) and Theorem 5.6 (i) (because $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big) \setminus \mathcal{K}_{s+\epsilon}$ is an open neighbourhood of Q), that $s+\epsilon \leq I(Q)$, a contradiction.

Step 2: $\mathcal{L}^{\leq s}(I)$ ***is closed.***

By the definition of I and the previous step, $\mathcal{L}^{\leq s}(I) \subset \mathcal{K}_{s+\epsilon} \cap \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. Fix an arbitrary converging sequence $\left\{Q^{(n)}\right\}_n \subset \mathcal{L}^{\leq s}(I)$. The limit point Q^* of this sequence is in $\mathcal{K}_{s+\epsilon} \cap \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. We prove now that $Q \in \mathcal{L}^{\leq s}(I)$.

This follows if we knew that for all $(x,w) \in \mathbb{T}^d \times \mathcal{W}$, $P^{I,\Pi\left(Q^{(n)}\right)}_{x,w} \to P^{I,\Pi(Q^*)}_{x,w}$. Indeed, this implies that also $\mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes P^{I,\Pi\left(Q^{(n)}\right)}_{x,w} \to \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes P^{I,\Pi(Q^*)}_{x,w}$. Then we conclude the lower semi-continuity of I, from the lower semi-continuity of the relative entropy in both variables.

The convergence of $P^{I,\Pi\left(Q^{(n)}\right)}_{x,w}$ follows from [34] Theorem 11.1.4. This theorem is applicable if for each $M \in \mathbb{R}$

$$\lim_{n\to\infty} \int_0^T \sup_{|\theta|\leq M} \left| b\left(x,w,\theta,\Pi\left(Q^{(n)}\right)_t\right) - b\left(x,w,\theta,\Pi\left(Q^*\right)_t\right)\right| \mathrm{d}t = 0. \tag{5.8}$$

This convergence follows if

$$\sup_{t\in[0,T]} \sup_{|\theta|\leq M} \left| b\left(x,w,\theta,\Pi\left(Q^{(n)}\right)_t\right) - b\left(x,w,\theta,\Pi\left(Q^*\right)_t\right)\right| \to 0. \tag{5.9}$$

Let us show that (5.9) holds. The function

$$(t,x,w,\theta,Q)\mapsto b\left(x,w,\theta,\Pi\left(Q\right)_t\right) \tag{5.10}$$

is continuous on $[0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\times\left(\mathcal{M}_{\varphi,R}\cap\mathbb{M}_1^L\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)\right)$, as composition of continuous functions (Assumption 3.1 a.i)). Moreover, only the compact sets $[0,T]$, $|\theta|\le M$ and $Q^{(n)},Q^*\in\mathcal{K}_{s+\epsilon}$ are considered in (5.9). From the uniform convergence of b on this set, we conclude (5.9). Hence $\mathcal{L}^{\le s}\left(I\right)$ is closed. □

6 Comparison of the LDPs of the empirical measure and of the empirical process

In this section we state at first (Section 6.1) a one-to-one relation between the minimizer of the rate functions I (of $\left\{L^N,P^N\right\}$ derived in Theorem 5.3) and $S_{\nu,\zeta}$ (of $\left\{\mu^N_{[0,T]},P^N\right\}$ derived in Theorem 3.6). Then we explain how one can easily infer from the large deviation principle for the empirical measure $\left\{L^N\right\}$, the large deviation principle for the empirical process $\left\{\mu^N_{[0,T]}\right\}$ in $\mathscr{C}$. This follows by a simple application of the contraction principle (see Theorem 6.2). However, the derived rate function does not have the expression $S_{\nu,\zeta}$ defined in (3.10). We show in Section 6.3 that the derived rate function is at least an upper bound on $S_{\nu,\zeta}$.

6.1 Relation between the minimiser of the rate function

We know by Theorem 4.2 (i) and (5.2) the following relation between $S_{\nu,\zeta}$ and I

$$\begin{aligned} S_{\nu,\zeta}\left(\mu_{[0,T]}\right) &= \inf_{\substack{Q\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)\\ \Pi(Q)_{[0,T]}=\mu_{[0,T]}}} \mathcal{H}\left(Q\,\middle|\,\mathrm{d}x\otimes\zeta_x\left(\mathrm{d}w\right)\otimes P^{I,\mu_{[0,T]}}_{x,w}\right)\\ &= \inf_{\substack{Q\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)\\ \Pi(Q)_{[0,T]}=\mu_{[0,T]}}} I\left(Q\right). \end{aligned} \tag{6.1}$$

We show in the next theorem a one-to-one relation between the minimizer of I and $S_{\nu,\zeta}$. Note that in general there can be two $Q,Q'\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)$ with the same projection $\Pi\left(Q\right)=\Pi\left(Q'\right)$ and with $I\left(Q\right)=I\left(Q'\right)$. However, when $S_{\nu,\zeta}\left(\Pi\left(Q\right)\right)=0$, then this is not the case.

Theorem 6.1. (i) *If $I\left(Q\right)=0$, then $S_{\nu,\zeta}\left(\Pi\left(Q\right)_{[0,T]}\right)=0$.*

(ii) *If $S_{\nu,\zeta}\left(\mu_{[0,T]}\right)=0$, then there is exactly one $Q\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)$ with $\Pi\left(Q\right)_{[0,T]}=\mu_{[0,T]}$ and $I\left(Q\right)=0$. This Q equals $\mathrm{d}x\otimes\zeta_x\left(\mathrm{d}w\right)\otimes P^{I,\mu_{[0,T]}}_{x,w}$.*

Proof. By (6.1), (i) is obviously satisfied. To prove (ii), fix a $\mu_{[0,T]}\in\mathscr{C}$ with $S_{\nu,\zeta}\left(\mu_{[0,T]}\right)=0$. Assume that there is a $Q\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)$, with $\Pi\left(Q\right)_{[0,T]}=\mu_{[0,T]}$ and $I\left(Q\right)=0$. Then $Q=\mathrm{d}x\otimes\zeta_x\left(\mathrm{d}w\right)\otimes P^{I,\mu_{[0,T]}}_{x,w}$ and hence there is at most one minimizer with $\Pi\left(Q\right)_{[0,T]}=\mu_{[0,T]}$ and $I\left(Q\right)$. Hence, we only have to show the existence of such a minimizer. By Section 3.2.1 the large deviation results of Section 3.1 hold for the SDE with fixed interaction $\mu_{[0,T]}$. Therefore, we get by the beginning of Step 1 of the proof of Lemma 3.16, that there is a $Q\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)$, with $\Pi\left(Q\right)_{[0,T]}=\mu_{[0,T]}$ and with $I\left(Q\right)=0$. □

6.2 From the LDP of the empirical measure to the LDP of the empirical process

We infer now the large deviation principle for the empirical process $\left\{\mu^N_{[0,T]},P^N\right\}$ from the large deviation principle for the empirical measure $\left\{L^N,P^N\right\}$, in the following theorem. This is a simple application of the contraction principle. This theorem requires only the large deviation principle for $\left\{L^N\right\}$. However, the rate function for the empirical processes is only described via a minimizing problem.

Theorem 6.2. *If the assumptions of Theorem 5.3 hold, then the family of the empirical processes $\left\{\mu^N_{[0,T]}, P^N\right\}$ satisfies on $\mathscr{C}$ the large deviation principle with rate function*

$$j\left(\mu_{[0,T]}\right) := \inf_{Q \in \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])) : \Pi(Q)_{[0,T]} = \mu_{[0,T]}} I\left(Q\right). \tag{6.2}$$

Proof. The family $\left\{L^N, P^N\right\}$ satisfies by Theorem 5.3 on $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ the large deviation principle with rate function I. The map $\Pi : \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right) \to \mathscr{C}$ is continuous (Lemma 2.24). Hence, the contraction principle implies the LDP of $\left\{\mu^N_{[0,T]}, P^N\right\}$ with rate function j. □

6.3 An upper bound on the rate function $S_{\nu,\zeta}$

By Theorem 6.2, j is the rate function of the large deviation principle for $\left\{\mu^N_{[0,T]}, P^N\right\}$. Moreover, by Theorem 3.6 and the uniqueness of rate functions, j has to be equal to $S_{\nu,\zeta}$ and $S^{\mathbb{T}^d \times \mathcal{W}}_{\nu,\zeta}$. We show now that j is equal to $S^{\mathbb{T}^d \times \mathcal{W}}_{\nu,\zeta}$ at least when j is finite, without using Theorem 3.6 (we need only Lemma 3.12 and Lemma 3.16). However, j is not everywhere finite (see also Remark 3.17 for the concept of admissible flows). Therefore, this is only an upper bound on $S^{\mathbb{T}^d \times \mathcal{W}}_{\nu,\zeta}$. Nevertheless, the upper bound on $S^{\mathbb{T}^d \times \mathcal{W}}_{\nu,\zeta}$, implies at least a large deviation upper bound with $S^{\mathbb{T}^d \times \mathcal{W}}_{\nu,\zeta}$ as rate function. For the large deviation lower bound (and another proof of the upper bound) we refer to Section 3.

Lemma 6.3. *Let the assumptions of Theorem 5.3 hold.*

If $j\left(\mu_{[0,T]}\right) < \infty$ for a $\mu_{[0,T]} \in \mathscr{C}$, then $j\left(\mu_{[0,T]}\right) = S^{\mathbb{T}^d \times \mathcal{W}}_{\nu,\zeta}\left(\mu_{[0,T]}\right)$.

In particular, this implies $j\left(\mu_{[0,T]}\right) \geq S^{\mathbb{T}^d \times \mathcal{W}}_{\nu,\zeta}\left(\mu_{[0,T]}\right) \geq S_{\nu,\zeta}\left(\mu_{[0,T]}\right)$.

Remark 6.4. In [11] a proof of the equality between the counterparts of j and $S^{\mathbb{T}^d \times \mathcal{W}}_{\nu,\zeta}$ is given. However, in that proof the authors accidentally use a circular reasoning (in the equality (2.24) in [11]). We are also not able to prove the missing lower bound on $S^{\mathbb{T}^d \times \mathcal{W}}_{\nu,\zeta}$, without using Theorem 3.6,.

Proof of Lemma 6.3. Fix a $\mu_{[0,T]} \in \mathscr{C}$ with $j\left(\mu_{[0,T]}\right) < \infty$. Then there is a $R > 0$, such that $\mu_{[0,T]} \in \mathscr{C}_{\varphi,R}$, because there has to be a $Q \in \mathcal{M}_{\varphi,\infty}$ with $I\left(Q\right) < \infty$ and $\Pi\left(Q\right)_{[0,T]} = \mu_{[0,T]}$. By the same argument $\mu_{[0,T]} \in \mathscr{C}^L$.

Define $b^{I,\mu_{[0,T]}}\left(t,x,w,\theta\right) := b\left(x,w,\theta,\mu_t\right)$ as in Notation 4.1. With this $b^{I,\mu_{[0,T]}}$, we can define a system of independent SDEs as in (3.12). This system satisfies Assumption 3.8 as shown in Section 3.2.1. Then the Lemma 3.12 is applicable and we denote the rate function (3.15) by $S^{I,1,\mu_{[0,T]}}_{\nu,\zeta}$, i.e.

$$j\left(\mu_{[0,T]}\right) = \inf_{Q \in \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])) : \Pi(Q)_{[0,T]} = \mu_{[0,T]}} I\left(Q\right) = S^{I,1,\mu_{[0,T]}}_{\nu,\zeta}\left(\mu_{[0,T]}\right). \tag{6.3}$$

From this equality and Lemma 3.16 (which is applicable for the same reasons), we conclude the Lemma 6.3. □

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