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Finitely dependent insertion processes

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Abstract

A q-coloring of $\mathbb{Z}$ is a random process assigning one of q colors to each integer in such a way that consecutive integers receive distinct colors. A process is k-dependent if any two sets of integers separated by a distance greater than k receive independent colorings. Holroyd and Liggett constructed the first stationary k-dependent q-colorings by introducing an insertion algorithm on the complete graph K_q. We extend their construction from complete graphs to weighted directed graphs. We show that complete multipartite analogues of K_3 and K_4 are the only graphs whose insertion process is finitely dependent and whose insertion algorithm is consistent. In particular, there are no other such graphs among all unweighted graphs and among all loopless complete weighted directed graphs. Similar results hold if the consistency condition is weakened to eventual consistency. Finally we show that the directed de Bruijn graphs of shifts of finite type do not yield k-dependent insertion processes, assuming eventual consistency.

Keywords: proper coloring; directed graph; weighted graph; m-dependence; stationary process; shift of finite type.
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1 Introduction

A **proper q-coloring of** $\mathbb{Z}$ is a sequence of colors $(x_i)_{i\in\mathbb{Z}}$ with $x_i \in [q] := \{1, \dots, q\}$ such that $x_i \neq x_{i+1}$ for all i. A random q-coloring $(X_i)_{i\in\mathbb{Z}}$ is **stationary** if $(X_i)_{i\in\mathbb{Z}}$ and $(X_{i+1})_{i\in\mathbb{Z}}$ are equal in law. A stationary q-coloring is ***k*-dependent** if $(X_i)_{i<0}$ and $(X_i)_{i\geq k}$ are independent, and **finitely dependent** if it is k-dependent for some $k \geq 0$.

The simplest examples of stationary finitely dependent processes are the **block factors**. These are stochastic processes of the form $\{f(Y_i, \dots, Y_{i+k})\}_{i\in\mathbb{Z}}$ where f is deterministic and $\{Y_i\}_{i\in\mathbb{Z}}$ are an i.i.d. sequence. In the 1960s, Ibragimov and Linnik first suggested that there may exist non-block factor stationary finitely dependent processes [15, 16]. Since then, examples of such processes have been constructed by several authors in the course of studying properties of finitely dependent processes

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[1, 7, 9, 2, 17]. Until recently, it has been believed that most 'natural' finitely dependent processes are block factors [6].

Yet block factors have subtle limitations. For example, these processes are never supported on proper colorings [3]. It turns out that finitely dependent processes do not have this limitation, although this fact is highly non-obvious and remains to be fully understood. This was discovered by Holroyd and Liggett in a recent breakthrough [13], in which they disproved a conjecture of Schramm [14] by showing that stationary finitely dependent colorings of the integers exist. These are perhaps the first natural non-block factor finitely dependent processes.

Specifically, Holroyd and Liggett constructed symmetric 3- and 4-colorings with these properties. It is remarkable that, while their construction produces a q-coloring for each integer $q \geq 2$, only when $q \in \{3, 4\}$ is the coloring finitely dependent. Using a more complicated construction, these authors later obtained symmetric q-colorings for all $q \geq 4$ [12].

As described in [13], the q-colorings therein have the following characterization. For each integer $q \geq 2$, let $Z = (Z_1, \ldots, Z_n)$ be a sequence of independent random variables each taking the values $1, 2, \ldots, q$ with equal probability. Let σ be an independent uniformly random permutation of $1, \ldots, n$, which we interpret as meaning that the symbol $Z_{\sigma(i)}$ arrives at time i. Let E be the event that, for every time $t = 1, \ldots, n$, the subsequence of Z formed by those symbols that arrived up to time t (ordered as in the original sequence Z) forms a proper coloring (i.e. no two consecutive elements in the subsequence are equal). Then the conditional law of Z given E equals the law of $(X_1, \ldots, X_n)$, where X is the q-coloring constructed in [13].

It was observed by Holroyd (personal communication) that the proper coloring condition in the previous paragraph may be replaced by a graph adjacency condition. The case of a q-coloring corresponds to the complete graph with vertex set $\{1, \ldots, q\}$, denoted K_q. A general graph will encode which pairs of vertices may appear consecutively. See e.g. [5, Example 2.5] for more on this perspective. Since few stationary finitely dependent colorings are currently known, it is natural to pursue this generalization in the search for new finitely dependent processes.

Fix a finite graph G containing at least one edge. Let $Z = (Z_1, \ldots, Z_n)$ be a sequence of independent uniformly random vertices of G. Let σ be an independent uniformly random permutation of $1, \ldots, n$, which we interpret as meaning that the vertex $Z_{\sigma(i)}$ arrives at time i. Let E be the event that, for every time $t = 1, \ldots, n$, the subsequence of Z formed by those vertices that arrived up to time t (ordered as in the original sequence Z) forms a path in G. Then let $(Y_1, \ldots, Y_n)$ denote a random tuple whose law equals the conditional law of Z given E. This generalizes the construction of the q-coloring in [13].

To produce a stochastic process $(Y_i)_{i \in \mathbb{Z}}$ from these random tuples, we must apply a limiting procedure. Let P_n denote the probability mass function of $(Y_1, \ldots, Y_n)$. If for all n the mass functions of $(Y_1, \ldots, Y_{n-1})$ and $(Y_2, \ldots, Y_n)$ equal P_{n-1}, then by the Kolmogorov Extension Theorem [10] there exists a unique stochastic process $(\tilde{Y}_i)_{i \in \mathbb{Z}}$ such that $(\tilde{Y}_1, \ldots, \tilde{Y}_n)$ has mass function P_n for all $n \geq 1$. In this case we say that $\{P_n\}_{n=1}^{\infty}$ is **consistent** and that G satisfies property (C). More generally, we say that the mass functions are **eventually consistent** and that G satisfies property (EC) if the preceding condition holds for all sufficiently large n. In this case, we construct a process $(\tilde{Y}_i)_{i \in \mathbb{Z}}$ in the same manner as before by taking a projective limit [18] over sufficiently large n. We call $(\tilde{Y}_i)_{i \in \mathbb{Z}}$ the **insertion process** associated to G.

Holroyd and Liggett showed that for all $q \geq 2$, the complete graph K_q has property (C) [13, Proposition 10]. Furthermore, they discovered that the insertion process associated to K_4 is 1-dependent and the insertion process associated to K_3 is 2-dependent and that these are the *unique* values of q for which the insertion process is finitely dependent [13, Proposition 13].

One may embellish these examples in our general setting. Replacing each of the 4 (resp. 3) colors with r copies of itself yields the complete multipartite graphs $K_{r,r,r,r}$ (resp. $K_{r,r,r}$). Both of these graphs are easily seen to have property (C) and to have a 1- (resp. 2-)dependent insertion process. We establish the remarkable fact that these are essentially the *only* (EC) graphs with a finitely dependent insertion process.

Theorem 1. *Let G be a finite graph satisfying property (C). Then the insertion process associated to G is k-dependent if and only if either: $G = K_{r,r,r}$ and $k \geq 2$, or $G = K_{r,r,r,r}$ and $k \geq 1$.*

If G satisfies property (EC), then the associated insertion process is k-dependent if and only if it is a disjoint union of one of the above graphs with a set of isolated vertices.

In the hope of uncovering other finitely dependent insertion processes, we consider weighted graph variants. A **weighted digraph** on a vertex set V is simply a non-negative function $w\colon V^2 \to \mathbb{R}_{\geq 0}$. For $i, j \in V$, we regard $w(i,j)$ as the weight of the directed edge from i to j, with 0 signifying that no edge is present. Suppose a weighted digraph with vertex set V is given. Then consider the random sequence

$$X_1^1,\ (X_1^2, X_2^2),\ \dots,\ (X_1^n, \dots, X_n^n),\ \dots \in V \times V^2 \times \cdots \times V^n \times \cdots$$

produced by the following algorithm (where, for all $n \geq 1$, we set $X_0^n = X_{n+1}^n = \Delta$ to be a symbol not in V and where we define $w(\Delta, i) = w(i, \Delta) = w(\Delta, \Delta) = 1$ for all $i \in V$).

Algorithm 1.

i. Let X_1^1 be a uniformly random element of V. Set $n = 1$.

ii. If $\displaystyle\sum_{i=0}^{n} \sum_{v \in V} w(X_i^n, v) \cdot w(v, X_{i+1}^n) > 0$, let $I \in \{0, \dots, n\}$ and $\mathcal{V} \in V$ be random with

$$\mathbb{P}(I = i, \mathcal{V} = v \mid X_1^n, \dots, X_n^n) = \frac{w(X_i^n, v) \cdot w(v, X_{i+1}^n)}{\sum_{i,v} w(X_i^n, v) \cdot w(v, X_{i+1}^n)}, \quad i \in \{0, \dots, n\}, \quad \mathcal{V} \in V,$$

and proceed to (iii). Otherwise, halt.

iii. Set $(X_1^{n+1}, \dots, X_{n+1}^{n+1}) = (X_1^n, \dots, X_I^n, \mathcal{V}, X_{I+1}^n, \dots, X_n^n)$, increment n, and go to (ii).

The weighted digraph is said to satisfy property (EC) if this algorithm does not halt and if the law of $(X_1^n, \dots, X_n^n)$ is eventually consistent as $n \to \infty$. In this case, we associate to the weighted digraph a stochastic process $Y = (Y_i)_{i=1}^{\infty}$ such that $(Y_1, \dots, Y_n)$ and $(X_1^1, \dots, X_n^n)$ are equal in law for all large enough n. We call Y the insertion process.

A weighted digraph is **uniform of weight w** if $w(i,j) \in \{0, w\}$ for all vertices i and j, and if in addition $w(i,j) = w(j,i)$ and $w(i,i) = 0$. We remark that uniform weight graphs underly a model of endpoint-weighted insertion introduced in [11, §4].

Theorem 2. *Let G be a finite weighted digraph of uniform weight satisfying property (C). Then the insertion process associated to G is k-dependent if and only if either: $G = K_{r,r,r}$ and $k \geq 2$, or $G = K_{r,r,r,r}$ and $k \geq 1$.*

If G satisfies property (EC), then the associated insertion process is k-dependent if and only if it is a disjoint union of one of the above graphs with a set of isolated vertices.

Theorems 1 and 2 demonstrate that the requirement of satisfying property (EC) and having a finitely dependent associated insertion process is extremely restrictive on a graph. This continues to hold for a wide range of weighted digraphs.

Theorem 3. *Let G be a weighted digraph satisfying property (C) such that $w(i,i) = 0$ and $w(i,j) > 0$ for all distinct vertices i and j. Then the insertion process associated to G is k-dependent if and only if G is unweighted and either: $G = K_3$ and $k \geq 2$; or $G = K_4$ and $k \geq 1$.*

Another way to generalize graph coloring is to consider shifts of finite type [19, 14]. A **loopless shift of finite type** is a set of the form

$$S = \Big\{ x \in [q]^{\mathbb{Z}} \colon (x_{i+1}, \ldots, x_{i+n}) \in W \quad \forall i \in \mathbb{Z} \Big\},$$

for some $W \subset [q]^n \setminus \{(i, \ldots, i) \colon i \in [q]\}$. The **de Bruijn graph** [8] associated to S has vertex set W and edge set

$$E = \Big\{ \big((x_1, \ldots, x_n), (x_2, \ldots, x_{n+1})\big) \colon (x_1, \ldots, x_n), (x_2, \ldots, x_{n+1}) \in W \Big\}.$$

Theorem 4. *Consider a loopless shift of finite type with de Bruijn graph G. Suppose that G has property (EC). The insertion process associated to G extends to a process supported on the shift of finite type, but this process is never finitely dependent.*

When $n = 2$, shifts of finite type correspond to edge sets of directed graphs. In analogy with Theorems 1 to 3, one might expect K_3 and K_4 to appear in Theorem 4. The reason this is not the case is that Theorems 1 to 3 pertain to insertion processes on the vertex set, whereas Theorem 4 describes insertion processes on the edge set.

Open questions

In our quest to uncover new finitely dependent processes, we considered a class of insertion processes arising from graphs (or weighted and directed variants thereof) with vertex set V. Such an insertion process arose as a limit of a random tuple in V^n as $n \to \infty$. However, our method of extracting such a limit required delicate combinatorial identities to hold in the graph: the sequence of laws was required to be eventually consistent. Thus it is natural to ask whether similar results continue to hold after weakening this requirement further.

Question 1.1. Does Theorem 1 continue to hold if we replace property (EC) with the assumption that sequence of laws converges weakly in the product topology?

In Theorem 3, we do not even know if property (C) can be weakened to property (EC), let alone to weak limits.

Question 1.2. Does Theorem 3 continue to hold for strongly connected weighted digraphs if we replace property (C) with property (EC)?

Unweighted digraphs already pose challenges for our techniques.

Question 1.3. Does Theorem 1 continue to hold for strongly connected unweighted digraphs?

The hypotheses of Theorem 3 require that the weighted digraph is strictly positive. Is this assumption necessary?

Question 1.4. Let G be a weighted digraph satisfying property (C) such that $w(i,i) = 0$ for all i. Suppose that the insertion process associated to G is k-dependent for some integer k. Does it follows that G is unweighted and either $G = K_3$ for $k \geq 2$ or $G = K_4$ for $k \geq 1$?

Overview

Section 2 presents a combinatorial analysis of insertion processes associated to weighted digraphs. Theorems 1 and 2 are proven in Section 3. Uniform weight graphs

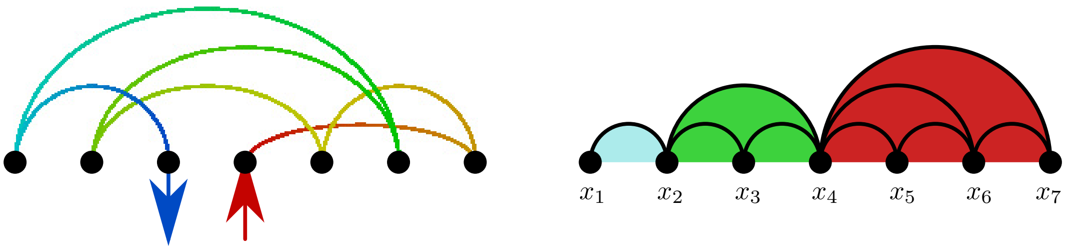

Figure 1: Left: The arrival order for $\sigma = 4752613$. Right: Multiply the edge weights to get $w(x_1 \cdots x_7; \sigma)$.

appear in Subsection 3.1, complete multipartite graphs in Subsection 3.2, and the proofs of Theorems 1 and 2 are in Subsection 3.3. Section 4 combines Theorem 2 with some additional arguments to deduce Theorem 3. Lastly, we deduce Theorem 4 in Section 5 from Lemma 2.4 in Subsection 2.2.

2 Weighted insertion

Let V be a finite alphabet. A **word** (of length n) is a finite sequence $x = (x_1, x_2, \dots, x_n) \in V^n$, which we sometimes abbreviate to $x_1 x_2 \cdots x_n$. The word of length 0 is denoted by $\emptyset$. Let S_n be the symmetric group of all permutations of $1, \dots, n$. Let $x \in V^n$ be a word and let $\sigma \in S_n$ be a permutation. We interpret σ as meaning that at time $t = 1, \dots, n$ symbol $x_{\sigma(t)}$ arrives in (relative) position $\sigma(t)$.

Let $\mathrm{Min}(\sigma) := \{1 \le t \le n \colon \sigma(t) = \min_{s \le t} \sigma(s)\}$ denote the running minima of σ. Let

$$\mathrm{Max}(\sigma) := \{1 \le t \le n \colon \sigma(t) = \max_{s \le t} \sigma(s)\}$$

denote the set of running maxima. If $\sigma(t)$ is neither a running minimum nor maximum, then at time t the symbol $x_{\sigma(t)}$ is inserted between $x_{\sigma(t^-)}$ and $x_{\sigma(t^+)}$ where

$$\sigma(t^-) = \max_{s<t}\{\sigma(s) \colon \sigma(s) < \sigma(t)\} \text{ and } \sigma(t^+) = \min_{s<t}\{\sigma(s) \colon \sigma(s) > \sigma(t)\}.$$

A **weighted digraph** with vertex set V is a function $w \colon V^2 \to \mathbb{R}_{\ge 0}$. We define the **weight** of the pair $(x, \sigma) \in V^n \times S_n$ to be

$$w(x;\sigma) := \prod_{t \notin \mathrm{Min}(\sigma)} w\big(x_{\sigma(t^-)}, x_{\sigma(t)}\big) \prod_{t \notin \mathrm{Max}(\sigma)} w\big(x_{\sigma(t)}, x_{\sigma(t^+)}\big). \tag{2.1}$$

For example, when $\sigma = \mathrm{id}$ is the identity permutation of $1, \dots, n$ we have $w(x; \mathrm{id}) = w(x_1, x_2) \cdots w(x_{n-1}, x_n)$. We denote this quantity by $w(x)$. When the length of x is at most 1, we have that $w(x) = 1$.

If we imagine building the word dynamically using σ, then when $x_{\sigma(t)}$ is inserted between $x_{\sigma(t^-)}$ and $x_{\sigma(t^+)}$ a multiplicative weight of $w(x_{\sigma(t^-)}, x_{\sigma(t)})\ w(x_{\sigma(t)}, x_{\sigma(t^+)})$ is incurred in (2.1).

Definition 1. Given a weighted digraph with vertex set V and a word $x \in V^n$, define

$$B(x) := \sum_{\sigma \in S_n} w(x;\sigma).$$

This is a generalization of the building number defined in [13], which is the special case consisting of the weighted digraph w with vertex set $\{1, \dots, q\}$ and weight function $w(i,j) = \mathbf{1}[i \neq j]$.

Clearly $B(x) > 0$ if and only if there exists $\sigma \in S_n$ such that $w(x;\sigma) > 0$. In this case $\sigma = \mathrm{id}$ works. Word x has **positive weight** if either of the equivalent conditions $B(x) > 0$ or $w(x) > 0$ holds.

For a word $x = x_1 x_2 \cdots x_n$, we write $\widehat{x}_i = x_1 \cdots x_{i-1} x_{i+1} \cdots x_n$.

Lemma 2.1. *Suppose that $x \in V^n$. Then $B(x)$ is equal to*

$$B(\widehat{x}_1)w(x_1,x_2) + \sum_{i=2}^{n-1} w(x_{i-1},x_i)\; B(\widehat{x}_i)\; w(x_i,x_{i+1}) + w(x_{n-1},x_n)B(\widehat{x}_n).$$

Proof. Fix an index i. For any σ which satisfies $i = \sigma(n)$,

$$w(x;\sigma) = w(x_{i-1},x_i)\; w(\widehat{x}_i;\widehat{\sigma}_i)\; w(x_i,x_{i+1}),$$

with the obvious modifications when $i = 1, n$. Summing over all $\sigma \in S_n$ yields the result. ☐

2.1 Eventual consistency

Let G be a weighted digraph with (finite) vertex set V. Say that G is **recurrent** if there exist arbitrarily long words of positive weight. We remark that this is equivalent to there existing positive-weight words of *every* length, since if $w(x_1x_2\cdots x_n) > 0$ then also $w(x_1x_2\cdots x_k) > 0$ for all $1 \le k \le n$. Furthermore, it is easy to see that G is recurrent if and only if

$$\sum_{y\in V^n} B(y) > 0 \qquad \text{for all } n \ge 0. \tag{2.2}$$

If G is a recurrent digraph, then for all $n \ge 0$ we define a probability mass function P_n on V^n by setting

$$P_n(x) = \frac{B(x)}{\sum_{y\in V^n} B(y)}, \qquad x \in V^n. \tag{2.3}$$

It is a simple consequence of Lemma 2.1 that the random element of V^n computed by Algorithm 1 (from the introduction) has mass function P_n. Moreover, Algorithm 1 halts if and only if G is not recurrent.

If G is recurrent and the mass functions $\{P_n\}_{n=0}^{\infty}$ satisfy

$$P_n(x) = \sum_{v\in V} P_{n+1}(xv) = \sum_{v\in V} P_{n+1}(vx) \qquad \text{for all } n \ge 0, \tag{2.4}$$

we say that G satisfies **property (C)** and that the mass functions are **consistent**. Likewise if G is recurrent and (2.4) holds for all sufficiently large n, we say that G satisfies **property (EC)** and that the mass functions are **eventually consistent**.

Clearly, equation (2.4) is equivalent to the existence of constants $C_n > 0$ such that (2.5) holds for all $x \in V^n$:

$$\sum_{v\in V} B(xv) = \sum_{v\in V} B(vx) = C_n B(x). \tag{2.5}$$

Thus, a recurrent graph satisfies property (C) if and only if there exists $C_n > 0$ such that (2.5) holds for all $n \ge 0$. Similarly, a recurrent graph satisfies property (EC) if and only if for all sufficiently large n, there exists $C_n > 0$ such that (2.5) holds.

When the mass functions $\{P_n\}$ are eventually consistent, it follows from the Kolmogorov extension theorem [10] that there is a unique process $(X_i)_{i\in\mathbb{Z}}$ satisfying

$$\mathbb{P}\big((X_{i+1},\dots,X_{i+n}) = x\big) = P_n(x)$$

for all sufficiently large n. This is the **insertion process** associated to an (EC) weighted digraph. It follows by construction that the insertion process is stationary.

Lemma 2.2. *Suppose that G has property (EC). Then the associated insertion process is k-dependent if and only if for all $n, m \in \mathbb{N}$ sufficiently large, there are positive constants $C_{n,m} > 0$ for which*

$$\sum_{W \in V^k} B(x\ W\ y) = C_{n,m} B(x) B(y), \qquad x \in V^n \text{ and } y \in V^m. \tag{2.6}$$

Proof. By normalizing (2.6) we see it is equivalent to

$$\sum_{W \in V^k} P_{n+k+m}(x\ W\ y) = P_n(x) P_m(y).$$

This is equivalent to the associated insertion process $(X_n)_{n \in \mathbb{Z}}$ having the property that $(X_i)_{i \in I}$ and $(X_j)_{j \in J}$ are independent for any pair of sufficiently large intervals I and J separated by a distance of k or greater. Since restriction preserves independence, we may remove the adjective 'sufficiently large' from the previous sentence. □

Holroyd and Liggett [13] proved that the (unweighted, undirected) complete graph K_q has property (C) for all $q \geq 2$. Furthermore, they showed that the associated insertion process is k-dependent if and only if $q = 3$ and $k \geq 2$ or $q = 4$ and $k \geq 1$. In Section 3 we show that apart from minor modifications, these are the only unweighted and undirected graphs with property (EC) for which the associated insertion process is finitely dependent.

2.2 Necessary conditions for k-dependence

Consider a weighted digraph G satisfying property (EC). In this subsection, we present necessary conditions for the associated insertion process to be finitely dependent. We show that such digraphs must contain directed triangles (Lemma 2.4) and are not too far from being strongly connected (Lemma 2.5).

A **directed triangle on** (a, b) is a triple of vertices $(a, b, c) \in V^3$ satisfying

$$w(a, b)\ w(b, c)\ w(a, c) > 0.$$

In the next lemma, we consider weighted digraphs that lack directed triangles. Equivalently, all insertions in Algorithm 1 must occur at the endpoints.

Lemma 2.3. *For a weighted digraph lacking directed triangles, any $x \in V^n$ satisfies $B(x) = 2^{n-1} w(x)$.*

Proof. For any $1 < i < n$, since $w(x_{i-1}, x_i) > 0$ and $w(x_i, x_{i+1}) > 0$ we must have that $w(x_{i-1}, x_{i+1}) = 0$. Applying Lemma 2.1 yields that

$$B(x) = w(x_1, x_2) \cdot B(\widehat{x}_1) + w(x_{\ell-1}, x_\ell) \cdot B(\widehat{x}_\ell),$$

which implies the claim by induction on the length of the word. □

Lemma 2.4. *If a weighted digraph satisfies property (EC) and the associated insertion process is finitely dependent, then it contains a directed triangle.*

Proof. Suppose there was a weighted digraph G lacking directed triangles that satisfies property (EC) and whose associated insertion process is k-dependent for some $k \geq 0$. Let V denote its vertex set. Applying Lemma 2.2, it follows that for all sufficiently large m and n and for all vertices $i, j \in V$ we have that

$$\sum_{P \in V^m, W \in V^k, Q \in V^m} B(iPWQj) = C_{m,n} \sum_{P \in V^m} B(i\ P) \sum_{Q \in V^n} B(Q\ j). \tag{2.7}$$

Let A denote the adjacency matrix of G, which is the $V \times V$ matrix with $A_{ij} = w(i,j)$. Using Lemma 2.3 to express the left hand side of (2.7) in terms of the adjacency matrix we obtain that

$$(A^{m+k+n+1})_{ij} = 2^{-m-k-n+1} \sum_{P \in V^m, W \in V^k, Q \in V^m} B(iPWQj). \tag{2.8}$$

Combining (2.7) with (2.8) implies that $\operatorname{rank}(A^N) \leq 1$ for $N = m+k+n+1$. Let $\lambda_1, \dots, \lambda_{|V|}$ denote the multiset of (possibly complex) eigenvalues of A, listed according to algebraic multiplicity. Then $\lambda_1^N, \dots, \lambda_{|V|}^N$ comprise the multiset of eigenvalues of A^N. But since $\operatorname{rank}(A^N) \leq 1$, there can be at most 1 nonzero eigenvalue.

In fact, all eigenvalues must vanish. To see why, consider $\operatorname{Tr} A = \sum_{i \in V} w(i,i)$. Since (i,i,i) is a directed triangle, our hypothesis that G lacks directed triangles implies that $w(i,i) = 0$ and thus $\operatorname{Tr} A = 0$. Combined with the previous paragraph, this implies that all eigenvalues vanish and thus A is nilpotent.

From this it follows that all but finitely many words in $\bigcup_{\ell \geq 0} V^\ell$ have weight zero. Thus G is not recurrent, contradicting our assumption that G satisfies property (EC). $\square$

A weighted digraph G is **strongly connected** if for every pair of vertices $i, j \in V$, there is a positive-weight word $x \in \bigcup_{\ell \geq 1} V^\ell$ that begins with i and ends with j. We denote the existence of such an x by writing $i \to j$. The **strongly connected components** (SCCs) of a weighted digraph are the equivalence classes of the relation

$$\big\{(i,j) \in V^2 \colon i \to j \text{ and } j \to i\big\}.$$

By convention $i \to i$ since singleton words are defined to have weight 1 (see the remark prior to Definition 1). Also, recall from our discussion following Definition 1 that a word has positive weight if and only if $w(x) > 0$, which occurs if and only if $B(x) > 0$.

Recall that a weighted digraph is a function $w \colon V^2 \to \mathbb{R}_{\geq 0}$. A subset $U \subseteq V$ induces a subdigraph by restriction of w to U^2. Say that an SCC is **recurrent** if the induced subdigraph of G is recurrent. By our remarks at the beginning of Subsection 2.1, an SCC with vertex set $\mathcal{C}$ is recurrent if and only if there are arbitrarily large n such that there exists a positive-weight word $x \in \mathcal{C}^n$.

Note that any SCC with at least two vertices is recurrent, for if a and b are any two such vertices, then each of the words $(ab)^n \in \mathcal{C}^{2n}$ have positive weight. Also observe that for a directed acyclic graph, the SCCs are singletons, none of which is recurrent. Moreover, a singleton SCC is recurrent if and only if its vertex has a self-loop. In general, each directed cycle is contained in a recurrent SCC.

For the remainder of this section, we restrict attention to weighted digraphs satisfying property (EC) and whose associated insertion process is finitely dependent. We will show in Lemma 2.5 that such digraphs are not too far from being strongly connected.

Lemma 2.5. *Let G be a weighted digraph with property (EC) whose associated insertion process is finitely dependent.*

- *i. G has a unique recurrent SCC.*
- *ii. Vertices not in the recurrent SCC belong to singleton SCCs that have no directed path to or from the recurrent SCC.*
- *iii. The subdigraph of G induced by the recurrent SCC also has property (EC). Moreover, its associated insertion process coincides with that of G.*
- *iv. If moreover G has property (C), then it is strongly connected.*

In particular, it follows from Lemma 2.5 that the family of finitely dependent insertion processes associated to weighted digraphs is unchanged if one restricts attention to strongly connected weighted digraphs.

Proof. Let $(X_i)_{i\in\mathbb{Z}}$ denote the insertion process on G. Suppose without loss of generality that it is $(k+1)$-dependent, for some $k \geq 0$. In particular, we have the i.i.d. subsequence $(X_{ik})_{i\in\mathbb{Z}}$.

Let $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ denote the strongly connected components of G. The weighted digraph G induces the weighted digraph G_{SCC} on the set of SCCs, denoted $V_{\mathrm{SCC}} := \{\mathcal{C}_1, \dots, \mathcal{C}_\ell\}$, given by the weight

$$w_{\mathrm{SCC}}(\mathcal{C}_i, \mathcal{C}_j) := \sum_{u\in\mathcal{C}_i, v\in\mathcal{C}_j} w(u,v).$$

Observe that $\mathcal{C}_i \to \mathcal{C}_j$ in G_{SCC} if and only if for some $u \in \mathcal{C}_i$ and $v \in \mathcal{C}_j$, there is a positive-weight word x composed of vertices of G that begins with u and ends with v. Thus if $\mathcal{C}_i \to \mathcal{C}_j$ and $\mathcal{C}_j \to \mathcal{C}_i$, then $\mathcal{C}_i \cup \mathcal{C}_j$ is also an equivalence class, whereupon $\mathcal{C}_i = \mathcal{C}_j$.

Consider the function $f \colon V \to V_{\mathrm{SCC}}$ that assigns to each vertex its strongly connected component. Then the sequence $\big(f(X_i)\big)_{i\in\mathbb{Z}}$ has the property that for all $i < j$, we have $f(X_i) \to f(X_j)$ in G_{SCC} almost surely. But if $f(X_i)$ occurs once in the sequence, it a.s. occurs infinitely many times (by passing to an i.i.d. subsequence). Thus there exists $\ell > j$ such that $f(X_\ell) = f(X_i)$. Consequently the sequence

$$\big(f(X_i)\big)_{i\in\mathbb{Z}} = (\dots, \mathcal{C}_{\mathrm{rec}}, \mathcal{C}_{\mathrm{rec}}, \dots) \tag{2.9}$$

is a.s. constant. Furthermore, the unique value $\mathcal{C}_{\mathrm{rec}}$ that it takes must be a recurrent SCC, since the marginals of the process $(X_i)_{i\in\mathbb{Z}}$ then yield arbitrarily long positive-weight words in $\bigcup_{\ell\geq 1} \mathcal{C}_{\mathrm{rec}}^\ell$.

On the other hand, each vertex $v \in V$ belonging to a recurrent SCC occurs infinitely often in the sequence $(X_i)_{i\in\mathbb{Z}}$. Indeed, we can fix a sufficiently long positive-weight word containing v and observe that it occurs with positive density in $(X_i)_{i\in\mathbb{Z}}$ by finite dependence. Thus by the previous paragraph, it follows that $\mathcal{C}_{\mathrm{rec}}$ is the a unique recurrent SCC of G, establishing property (i).

Property (ii) follows by combining our previous observation that an SCC containing distinct vertices is recurrent with the following argument. If there was a directed path joining $\mathcal{C}_{\mathrm{rec}}$ to another strongly connected component denoted $\mathcal{C}'$, then there would be arbitrary long positive-weight words that contain vertices in $\mathcal{C}'$. But then $\mathcal{C}'$ appears in the sequence $\big(f(X_i)\big)_{i\in\mathbb{Z}}$, by consideration of a sufficiently long marginal of the insertion process. This contradicts (2.9), proving (ii).

For all $n > |V|$, every word $x \in V^n$ contains some vertex at least twice. Thus by (ii), it follows that if such a word satisfies $B(x) > 0$, then necessarily $x \in \mathcal{C}_{\mathrm{rec}}^n$. Hence

$$\sum_{x\in V^n} B(x) = \sum_{x\in\mathcal{C}_{\mathrm{rec}}^n} B(x),$$

and it follows that all sufficiently long marginals of the insertion processes associated to G and the subdigraph induced by $\mathcal{C}_{\mathrm{rec}}$ coincide. This yields (iii).

Finally, (iv) follows from the observation that $B(x) = 1$ for all words of length 1. Thus if G satisfies property (C), the random variable X_0 in the associated insertion process is uniformly random on the vertex set V. But by (2.9) we have that $X_0 \in \mathcal{C}_{\mathrm{rec}}$ almost surely, implying that $V = \mathcal{C}_{\mathrm{rec}}$ and therefore G is strongly connected. □

3 Uniform weight graphs

We establish Theorem 2 in this section, and deduce Theorem 1 as a corollary. The plan of attack is as follows:

i. Subsection 3.1 defines the special class of weighted digraphs to which Theorem 2 applies, which we call uniform weight graphs. We deduce necessary structural properties for a uniform weight graph to satisfy property (EC).

ii. Subsection 3.2 is devoted to complete multipartite graphs of uniform weight. It is shown that understanding finite dependence for such graphs reduces to the analysis of complete graphs, which has already been undertaken in [13].

iii. Subsection 3.3 proves Theorem 2 by using a combinatorial argument to show that the structural properties in Subsection 3.1 imply that the graph is complete multipartite, then applying the results of Subsection 3.2.

3.1 Uniform weights

Uniform weight graphs are weighted digraphs that are undirected and of constant weight. Their associated insertion processes are sufficiently general to encompass the end-weighted insertion processes of [11, Section 4] as a special case.

Definition 2. A **uniform weight graph** with weight $w > 0$ is a weighted digraph whose weight function is such that for all $i, j \in V$,

i. either $w(i,j) = 0$ or $w(i,j) = w$, and

ii. $w(i,j) = w(j,i)$, and

iii. $w(i,i) = 0$.

We regard uniform weight graphs as being undirected. As such, we refer to their strongly connected components simply as connected components (when applying Lemma 2.5, for instance). For uniform weight graphs, we say that vertices i and j are **adjacent** if and only if $w(i,j) > 0$, in which case $w(i,j) = w$.

Applying (2.5) to a positive-weight word in a uniform weight graph yields that

$$B(x) = wB(\widehat{x}_1) + w^2 \sum_{i=2}^{n-1} B(\widehat{x}_i) + wB(\widehat{x}_n), \qquad x \in V^n. \tag{3.1}$$

For any positive integer n, we denote by $(ab)^{n/2}$ the unique alternating word in $\{a,b\}^n$ beginning with a (in particular, note that n may be odd).

Lemma 3.1. *If $w(a,b) > 0$ and $w(b,v) > 0$, then*

$$B\left((ab)^{n/2}\, v\right) = \begin{cases} (2w)^n, & w(a,v) = 0 \\ \left[2w^2(w+w^2)^{n-1} - (2w)^n\right]/(w-1), & w(a,v) = w \text{ and } w \neq 1 \\ 2^{n-1}(n+1), & w(a,v) = w \text{ and } w = 1. \end{cases} \tag{3.2}$$

Proof. First suppose that $w(a,v) = 0$. By (3.1) we have that

$$B\left((ab)^{n/2}\, v\right) = w \cdot B\left((ba)^{(n-1)/2}\, v\right) + w \cdot B\left((ab)^{n/2}\right).$$

Applying this inductively yields the first case of (3.2).

Next, suppose that $w(a,v) = w$ and $w \neq 1$. Again by (3.1),

$$\begin{aligned} B\left((ab)^{n/2}\, v\right) &= w \cdot B\left((ba)^{(n-1)/2}\, v\right) + w^2 \cdot B\left((ab)^{(n-1)/2}\, v\right) + w \cdot B\left((ab)^{n/2}\right) \\ &= (w+w^2) \cdot B\left((ba)^{(n-1)/2}\, v\right) + w \cdot (2w)^{n-1}. \end{aligned}$$

Now the second case of (3.2) follows by induction, starting from the base case $B(av) = 2w$. The final case of (3.2) follows from a simplification of the previous calculation. □

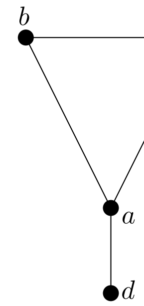

Figure 2: The kite $(abc;d)$

Lemma 3.2. *If G is a non-empty connected uniform weight graph satisfying property (EC), then the following conditions hold.*

i. The graph G is d-regular for some $d \geq 1$.

ii. For some $t \geq 0$, there are t triangles on every edge of G.

Proof. Let w be the common weight of the edges in G. Fix a vertex b. Let d denote the degree of b. By hypothesis, there exists an edge (a,b). Let t denote the number of triangles on the edge (a,b). First suppose that $w \neq 1$. By Lemma 3.1,

$$\sum_v B\left((ab)^{n/2}\ v\right) = (2w)^n(d-t) + \left[\frac{2w^2(w+w^2)^{n-1} - (2w)^n}{w-1}\right] t.$$

Recall that $B\big((ab)^{n/2}\big) = (2w)^{n-1}$. Thus by (2.5), it follows that

$$C_n = 2w(d-t) + \left[\frac{w(\frac{w+1}{2})^{n-1} - 1}{w-1}\right](2wt) \tag{3.3}$$

for all sufficiently large n if $w \neq 1$. Likewise if $w = 1$, for all sufficiently large n we have that $C_n = 2(d-t) + (n+1)t$. In either case, both d and t are determined by the values of C_n for n sufficiently large. Thus d and t take the same value, for all vertices b and all edges (a,b). □

Consider the graph that appears in Figure 2.

Definition 3. An undirected graph on the vertices a,b,c,d is a **kite** if (a,b,c) form a triangle and d is adjacent to a and to no other vertex.

Lemma 3.3. *Suppose that G has uniform weight and satisfies property (EC). Then there is no kite which occurs as an induced subgraph of G.*

Proof. Suppose to the contrary that G contains a kite $(abc;d)$. For all $n \geq 3$, we will construct words x and y satisfying

$$\sum_{v\in V} \frac{B(xv)}{B(x)} > \sum_{v \in V} \frac{B(yv)}{B(y)},$$

from which it will follow that G cannot satisfy property (EC).

Let $x = (ba)^{n/2}$ and let $y = d(ab)^{(n-1)/2}$. Simple casework reveals that for each $\sigma \in S_n$ we have $w(x;\sigma) = w(y;\sigma)$, and consequently $B(x) = B(y)$. Note that this common value is positive (by Lemma 3.1, for instance). Thus we need only establish that

$$\sum_v B(xv) > \sum_v B(yv).$$

First observe that we have the weaker inequality

$$\sum_v B(xv) \geq \sum_v B(yv). \tag{3.4}$$

Indeed, each $\sigma \in S_n + 1$ with $w(yv;\sigma) > 0$ is seen to satisfy $w(xv;\sigma) > 0$ as well. Since the graph has uniform weight, it therefore follows that $w(xv;\sigma) \geq w(yv;\sigma)$ and the claim follows upon summation.

To obtain strict inequality, observe that if $\sigma \in S_{n+1}$ satisfies $\sigma(\{1,2\}) = \{1, n+1\}$ then $w(xc;\sigma) > w(yc;\sigma) = 0$. When combined with the previous inequality, it follows that $B(xc) > B(yc)$. □

3.2 Complete multipartite graphs

In this section we relate property (EC) for complete multipartite graphs to property (EC) for complete graphs. This allows us to extend results of Holroyd and Liggett [13] to handle complete multipartite graphs.

We use the following notation: K_q denotes the complete graph on q vertices, $K_{r,\dots,r}$ denotes the complete multipartite graph with q parts each of size r, and $w \cdot K_{r,\dots,r}$ denotes the uniform weight graph in which each edge of the corresponding complete multipartite graph has weight $w > 0$. In all cases, we take the vertex set to be $[qr] := \{1, \dots, qr\}$ (with $r = 1$ in the case of K_q), and we take the edge set to be $\{(i,j) \colon i \not\equiv j \mod q\}$. In the case of $w \cdot K_{r,\dots,r}$, the weight function is given by $w(i,j) = w \cdot \mathbf{1}[i \not\equiv j \mod q]$.

Note that the graph $K_{r,\dots,r}$ is a Turán graph [4].

Lemma 3.4. *For any $r \geq 1$ and $k \geq 0$, the graph $w \cdot K_{r,\dots,r}$ satisfies property (EC) and its associated insertion process is k-dependent if and only if the same holds for $w \cdot K_q$.*

Proof. Consider the mapping $f \colon [qr] \to [q]$ given by $f(i) = i \mod q$. For any word $x \in [qr]^n$, let $f(x)$ denote the word $f(x_1)f(x_2)\cdots f(x_n) \in [q]^n$. Observe that

$$w(i,j) = w\big(f(i), f(j)\big) \quad \forall i,j \in [qr],$$

from which it follows that $B(x) = B\big(f(x)\big)$. Thus the following are equivalent:

i. $\displaystyle\sum_{v \in [qr]} B(xv) = rC_n B(x)$

ii. $\displaystyle\sum_{v' \in [q]} B\big(f(x)v'\big) = C_n B\big(f(x)\big)$

By (2.5), it follows that $w \cdot K_q$ satisfies property (EC) if and only if $w \cdot K_{r,\dots,r}$ satisfies property (EC).

Similarly, the following are equivalent:

i. $\displaystyle\sum_{W \in [qr]^k} B(x\ W\ y) = r^k C_{nm} B(x) B(y)$

ii. $\displaystyle\sum_{W' \in [q]^k} B\big(f(x)\ W'\ f(y)\big) = C_{nm} B\big(f(x)\big) B\big(f(y)\big)$

The result now follows by Lemma 2.2. □

Lemma 3.5. *Suppose that $w \neq 1$ and $q \geq 3$. Then the graph $w \cdot K_q$ does not satisfy property (EC).*

Proof. Consider an integer $n \geq 3$. Fix distinct vertices $a, b, c \in K_q$. Keeping in mind our notation for alternating words, set

$$x = (ba)^{n/2} \in \{a, b\}^n, \qquad y = c\ (ab)^{(n-1)/2} \in \{a, b, c\}^n$$

(regardless of the parity of n). Consider the quantities

$$Q_n = \sum_{v \in K_q} \frac{B(xv)}{B(x)}, \qquad R_n = \sum_{v} \frac{B(yv)}{B(y)}.$$

We will prove by induction on n that $Q_n > R_n$ for all $w > 1$ and $Q_n < R_n$ for all $w \in (0, 1)$.

From (3.1) we deduce the recurrences $B(xv) = wB(\widehat{x}_1 v) + w^2 B(\widehat{x}_n v) + wB(x)$ and $B(yv) = wB(\widehat{y}_1 v) + w^2 B(\widehat{y}_2 v) + w^2 B(\widehat{y}_n v) + wB(y)$, yielding that

$$\begin{aligned} Q_n &= \left(\frac{w+1}{2}\right) Q_{n-1} + qw - w(w+1) \\ R_n &= \frac{wB(\widehat{x}_1)}{B(y)} Q_{n-1} + \frac{2w^2 B(\widehat{y}_2)}{B(y)} R_{n-1} + qw - w(w+1). \end{aligned}$$

Subtracting the previous equations and substituting the recurrence $B(y) = wB(\widehat{y}_1) + w^2 B(\widehat{y}_2) + wB(\widehat{y}_n)$ yields that $2B(y)(Q_n - R_n)/w$ equals

$$\left[(w-1)B(\widehat{x}_1) + (w+1)^2 B(\widehat{y}_2)\right] Q_{n-1} - 4wB(\widehat{y}_2) R_{n-1} \tag{3.5}$$

When $w > 1$, we leave out the first term, using $(w+1)^2 \geq 4w$ to obtain

$$2B(y)(Q_n - R_n)/w > 4w(Q_{n-1} - R_{n-1}).$$

Since the right side vanishes for $n = 3$, it follows that $Q_n > R_n$ by induction.

Next suppose that $w < 1$. This case requires a tighter bound. We begin by establishing that for all $m \geq 3$,

$$B\left((ba)^{m/2}\right) > (1-w) B\left(c\ (ba)^{(m-1)/2}\right).$$

Indeed, we deduce the bound inductively from

$$\begin{aligned} B\left((ba)^{m/2}\right) &= 2wB\left((ab)^{(m-1)/2}\right) \\ &= [(1-w)w + w^2 + w] B\left((ab)^{(m-1)/2}\right) \\ \text{(induction)} \qquad &> (1-w)\Big[wB\left((ab)^{(m-1)/2}\right) + w^2 B\left(c\ (ba)^{(m-2)/2}\right) \\ &\qquad + wB\left(c\ (ba)^{(m-2)/2}\right)\Big] \\ &= (1-w) B\left(c\ (ab)^{(m-1)/2}\right). \end{aligned}$$

Consequently

$$(1-w)B(\widehat{x}_1) > (1-w)^2 B(\widehat{y}_2) \tag{3.6}$$

Now we plug the bound (3.6) into (3.5) to obtain a bound which is suitable for induction:

$$\begin{aligned} 2B(y)(R_n - Q_n)/w &= 4wB(\widehat{y}_2) R_{n-1} \\ &\qquad + \left[(1-w)B(\widehat{x}_1) - (w+1)^2 B(\widehat{y}_2)\right] Q_{n-1} \\ \text{using (3.6)} \qquad &> 4wB(\widehat{y}_2)(R_{n-1} - Q_{n-1}). \end{aligned}$$

As before, when $n = 3$ the right side vanishes. Thus it follows by induction that if $0 < w < 1$, we have $R_n > Q_n$ for all $n \geq 3$. Combined with the previous case, we conclude that when $w \neq 1$ and $q \geq 3$, the graph $w \cdot K_q$ does not satisfy property (EC). $\square$

Combining several results allows us to determine which uniform weight complete multipartite graphs satisfy property (EC) and have a finitely dependent associated insertion process.

Lemma 3.6. *The graph $w \cdot K_{r,\dots,r}$ for $q \geq 3$ and $r \geq 1$ has property (EC) and has a k-dependent associated insertion process if and only if $w = 1$ and either: $q = 3$ and $k \geq 2$; or $q = 4$ and $k \geq 1$.*

Proof. By Lemma 3.4, it suffices to consider $w \cdot K_q$. By Lemma 3.5, $w = 1$. Applying Propositions 10 and 13 of [13], the insertion process on K_q is k-dependent if and only if $q = 3$ ($k \geq 2$) or $q = 4$ ($k \geq 1$). □

3.3 Extension to uniform weight graphs

We now combine the results of Subsections 3.1 and 3.2 to extend the conclusion of Lemma 3.6 to all uniform weight graphs. We establish that the only uniform weight graphs satisfying property (EC) that have a k-dependent associated insertion process are $K_{r,r,r}$ and $K_{r,r,r,r}$, and unions thereof with isolated vertices. Note that these graphs have weight $w = 1$, even though we allow $w > 0$ to be arbitrary a priori. These graphs are closely related to the graphs K_3 and K_4 corresponding to the colorings discovered by Holroyd and Liggett; in fact, the graphs $K_{r,r,r}$ and $K_{r,r,r,r}$ are obtained from K_3 (resp. K_4) by replacing each vertex with r copies of itself. Similarly, the insertion process on $K_{r,r,r}$ (resp. $K_{r,r,r,r}$) is obtained from the corresponding process on K_3 (resp. K_4) by replacing each instance of vertex i with an i.i.d. choice of one of its r copies in $K_{r,r,r}$ (resp. $K_{r,r,r,r}$).

The following graph-theoretic lemma allows us to reduce the general uniform weight case to that of the complete multipartite graphs treated in Subsection 3.2. We will use Lemma 3.7 to show that any uniform weight graph either contains a kite, or is complete multipartite.

Lemma 3.7. *Let G be a kite-free connected loopless graph containing a triangle abc. Then every vertex $d \in V$ is adjacent to at least two of $\{a, b, c\}$.*

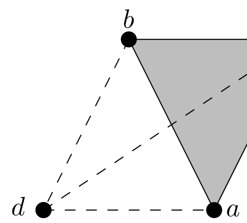

Figure 3: The problematic triangle abc in Lemma 3.7

Proof. By Lemma 3.3, no vertex $d \in V$ is adjacent to exactly one of abc. Hence it suffices to show that every vertex d is adjacent to abc.

Suppose to the contrary that some $d \in V$ is non-adjacent to abc (Figure 3). Choose a minimal path joining d to abc. We show that a kite is present near the intersection of the path with abc, from which we obtain the desired contradiction.

Let d' denote the path vertex adjacent to abc. There are two cases to consider: either d' is adjacent to a single vertex of abc (left half of Figure 4), or it is adjacent to more than one vertex (right half of Figure 4).

If d' is adjacent to a single vertex, then $(abc; d')$ is a kite. Now suppose that d' is adjacent to more than one vertex. Without loss of generality, suppose that d' is adjacent to a and c. Let d'' denote a neighbor of d' on the path from d to d'. By minimality of the path, d'' is non-adjacent to abc. Consequently $(d'ac; d'')$ is a kite. □

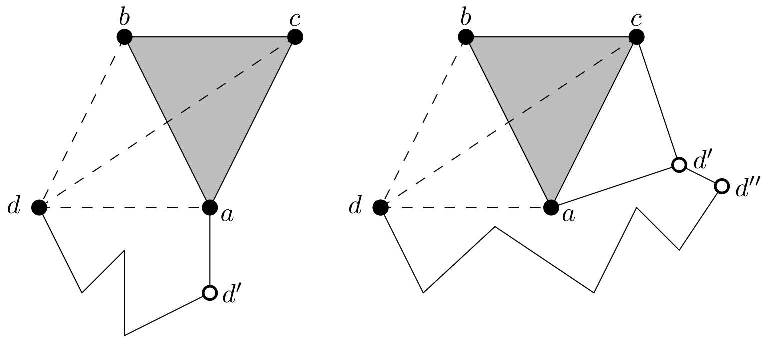

Figure 4: Reaching a contradiction

We are now in a position to prove Theorem 2.

Proof of Theorem 2. Suppose that G is a uniform weight graph with property (EC) and suppose that the insertion process associated to G is k-dependent. We will deduce that either $G = K_{r,r,r,r}$ and $k \geq 2$, or $G = K_{r,r,r}$ and $k \geq 1$, or G is a disjoint union of one of these graphs with a collection of isolated vertices.

Since uniform weight graphs are undirected, Lemma 2.5 takes on a simpler form in the present context. Indeed, it implies that both the (EC) property and the insertion process are unchanged by deletion of isolated vertices, and moreover it implies that the resulting graph is connected. Hence it suffices to consider the connected case.

We show in this case that G is complete multipartite. Consider the relation

$$\big\{(i,j) \in V^2 \colon w(i,j) = 0\big\}. \tag{3.7}$$

This relation is reflexive and symmetric by definition of a uniform weight graph. Once we establish transitivity, it will follow that the graph G is complete multipartite with partite sets given by the equivalence classes of this relation.

Suppose to the contrary that transitivity did not hold. Then there would exist vertices a, b, d such that $w(a,b) > 0$ yet $w(a,d) = w(b,d) = 0$. By Lemma 2.4 and Lemma 3.2, there are $t \geq 1$ triangles on every edge. In particular, we may complete the edge (a,b) into a triangle abc. By Lemma 3.3, G lacks kites, and by assumption it is connected. Moreover by definition of uniform weight, G is loopless.

Thus we have verified the conditions of Lemma 3.7. Consequently, the vertex d is adjacent to at least two of a, b, c. This contradicts the hypothesis that $w(a,d) = w(a,b) = 0$. Thus the relation (3.7) is transitive, and we deduce that it is an equivalence relation.

Decompose the vertex set into equivalence classes of (3.7). Then G is complete multipartite, with partite sets are given by the equivalence classes of (3.7). By Lemma 3.2, G is a regular graph and thus the parts have equal sizes. Thus $G = w \cdot K_{r,\ldots,r}$, for some $w > 0$ and $r \geq 1$.

Applying Lemma 2.4 again, we see that G contains a triangle and therefore $q \geq 3$. The result now follows when G has property (EC) by Lemma 3.6.

Finally, observe that if G also satisfies property (C), then there can be no isolated vertices by Lemma 2.5(iv). ☐

4 Complete weighted digraphs with property (C)

The results in this section apply to loopless complete weighted digraphs satisfying property (C). That is, the digraphs under consideration satisfy $w(i,i) = 0$ and $w(i,j) > 0$

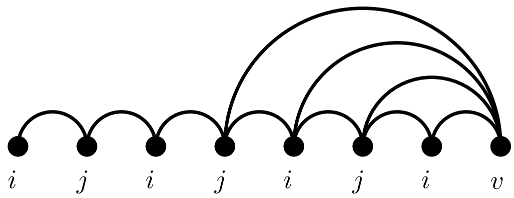

Figure 5: Let $x \in \{i,j\}^n$ be the unique alternating word ending in i. For $\sigma \in S_{n+1}$, let ℓ be as in (4.3). Consider the nearest neighbor graph on $\{1,\dots,n+1\}$, except we modify vertex $n+1$ to be adjacent to its nearest ℓ neighbors. This graph is drawn above with vertex i labeled with the i^{th} symbol of the word xv. Then $w(xv;\sigma) = \prod_{e=(i,j)} w\big((xv)_i,(xv)_j\big)$, where the product is taken over the edges of the graph we have just described.

for all distinct vertices i and j, as well as property (C). We will establish Theorem 3, which states that the only such graphs for which the associated insertion process is k-dependent are the (unweighted) graphs K_3 (for $k \geq 1$) and K_4 (for $k \geq 2$). We will establish this result by reducing it to the case of a uniform weight graph and applying results from Section 3.

For distinct indices $i,j \in V$, we define the quantity

$$T_n(i,j) = \sum_{v \in V} w(i,v)^{\lceil n/2 \rceil} w(j,v)^{\lfloor n/2 \rfloor},$$

where we use the convention $0^0 := 1$.

Lemma 4.1. *Fix a loopless complete weighted digraph satisfying property (C) and fix an integer $n \geq 1$. Then the value of $T_n(i,j)$ is constant over all vertices i and j with $w(i,j) > 0$.*

Proof. Let $x \in \{i,j\}^n$ be the unique alternating word ending in i. By Definition 1,

$$B(xv) = \sum_{\sigma \in S_n} w(xv;\sigma). \tag{4.1}$$

Since $w(i,i) = w(j,j) = 0$ by assumption, for all permutations with $w(xv;\sigma) > 0$ we have

$$w(xv;\sigma) = w(x) w(i,v)^{\lceil \ell/2 \rceil} w(j,v)^{\lfloor \ell/2 \rfloor}, \tag{4.2}$$

where ℓ is given by the formula

$$\ell = n + 1 - \max_{t < \sigma^{-1}(n+1)} \sigma(t). \tag{4.3}$$

This is straightforward to verify from the definitions, as indicated in Figure 5.

By (4.3), we see that ℓ ranges over $\{1,\dots,n\}$ as σ ranges over S_{n+1}. Thus substituting (4.2) into (4.1) implies that there are integers $d_1,\dots,d_n > 0$ whose values depend only on n such that

$$B(xv) = w(x) \sum_{\ell=1}^{n} d_\ell \, w(i,v)^{\lceil \ell/2 \rceil} w(j,v)^{\lfloor \ell/2 \rfloor}.$$

Summing over all vertices $v \in V$, we obtain

$$\sum_{v \in V} B(xv) = w(x) \sum_{\ell=1}^{n} d_\ell \, T_\ell(i,j).$$

By (2.5) we have that $\sum_{v\in V} B(xv) = C_n B(x)$. Furthermore $B(x) = 2^{n-1}w(x)$, by Lemma 2.3 applied to the subgraph induced by $\{i,j\}$. As $w(x)>0$, we have that

$$\sum_{\ell=1}^{n} d_\ell \, T_\ell(i,j) = 2^{n-1}C_n.$$

Since $d_1,\dots,d_n>0$, we may explicitly solve this system of equations to obtain that

$$\begin{pmatrix} T_1(i,j) \\ \vdots \\ T_n(i,j) \end{pmatrix} = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ d_1 & d_2 & \cdots & 0 \\ d_1 & d_2 & \ddots & 0 \\ d_1 & d_2 & \cdots & d_n \end{pmatrix}^{-1} \begin{pmatrix} C_1 \\ 2C_2 \\ \vdots \\ 2^{n-1}C_n \end{pmatrix}.$$

In particular, it follows that $T_n(i,j)$ is independent of the pair (i,j). $\square$

In light of Lemma 4.1, we write T_n in place of $T_n(i,j)$ from now on.

Lemma 4.2. *For all pairs of distinct indices (i,j) and (i',j'), we have $w(i,j)=w(i',j')$.*

Proof. Let $z_1>z_2>\cdots$ denote the set of distinct positive values attained by $w(i,v)w(j,v)$ as v ranges over V. Let V_ℓ denote the vertices which contribute to z_ℓ, given by

$$V_\ell = \{v\in V\colon w(i,v)w(j,v)=z_\ell\},$$

and let $a_\ell = \sum_{v\in V_\ell} w(i,v)$. Rewriting the expression for T_{2n+1} yields

$$T_{2n+1} = \sum_\ell a_\ell z_\ell^n.$$

Next, observe that $z_1 = \inf_{n\to\infty}(T_{2n+1})^{1/n}$ and $a_1 = \inf_{n\to\infty}\frac{T_{2n+1}}{z_1^n}$. Hence the parameters a_1 and z_1 can be reconstructed given the sequence $\{T_n\}$. Applying the same procedure to $T_{2n+1}-a_1z_1^n$ allows us to iteratively reconstruct all of the parameters. Thus a_ℓ and z_ℓ are uniquely determined by the $\{T_n\}$, so they are independent of the choice (i,j). Finally, observe that $\sum_\ell a_\ell = \sum_{v\neq j} w(i,v)$. Therefore

$$w(i,j) = T_1 - \sum_\ell a_\ell,$$

which shows that for $i\neq j$, the value of $w(i,j)$ is constant. $\square$

Combining Lemma 4.2 with the results of Section 3 allows us to deduce Theorem 3.

Proof of Theorem 3. Suppose that G is a loopless weighted digraph such that $w(i,j)>0$ for all distinct vertices i and j. Moreover, suppose that G has property (C) and that the associated insertion process is k-dependent process. We show that either: $G=K_3$ and $k\geq 2$; or $G=K_4$ and $k\geq 1$.

By Lemma 4.2, the graph G has uniform weight, and by Lemma 2.5 it is strongly connected. Applying Theorem 2, we conclude that either $G=K_{r,r,r}$ and $k\geq 2$, or $G=K_{r,r,r,r}$ and $k\geq 1$. Since $w(i,j)>0$ for all $i\neq j$, it follows that $r=1$. Thus G is a complete graph, and applying [13, Proposition 13] we deduce that G is either K_3 (for $k\geq 2$) or K_4 (for $k\geq 1$).

Conversely, by the main result of [13] the graphs K_3 and K_4 are k-dependent for $k\geq 2$ and $k\geq 1$ respectively. $\square$

5 Shifts of finite type

We turn to the proof of Theorem 4, which states that if the de Bruijn graph of a loopless shift of finite type satisfies property (EC), then the associated insertion process on the shift of finite type is not finitely dependent.

Proof of Theorem 4. Let G denote the de Bruijn graph of the shift of finite type and let $\{Y_\ell\}_{\ell\in\mathbb{Z}}$ denote the insertion process associated to G. We write

$$Y_\ell = (x_\ell, \ldots, x_{\ell+n-1}) \in \{1, \ldots, q\}^n.$$

Since $\{Y_\ell\}$ is almost surely a path in G, the overlapping elements in adjacent tuples Y_ℓ and $Y_{\ell+1}$ almost surely coincide. We extend the insertion process from G to the shift of finite type by considering the random sequence $\{x_\ell\}_{\ell\in\mathbb{Z}}$.

Suppose to the contrary that for some k, this process is k-dependent. Then the tuples $Y_\ell = (x_\ell, \ldots, x_{\ell+n-1})$ form an $(n+k-1)$-dependent sequence. Applying Lemma 2.4 to the insertion process on G, it follows that G has a directed triangle (a, b, c). Hence we may write

$$a = (x_1, x_2, \ldots, x_n), \quad b = (x_2, \ldots, x_{n+1}), \quad c = (x_3, \ldots, x_{n+2}).$$

Since the edge (a, c) is present in G, we must have

$$(x_2, x_3, \ldots, x_n) = (x_3, \ldots, x_{n+1}).$$

Thus $x_2 = x_3 = \cdots = x_{n+1}$, so b is a constant sequence. This contradicts our assumption that the vertex set of G lacks elements of the form $(i, \ldots, i)$. Therefore the associated insertion process is not k-dependent. $\square$

References

[1] J. Aaronson, D. Gilat, and M. Keane. On the structure of 1-dependent Markov chains. *J. Theoret. Probab.*, 5(3):545–561, 1992. MR-1176437

[2] J. Aaronson, D. Gilat, M. Keane, and V. de Valk. An algebraic construction of a class of one-dependent processes. *Ann. Probab.*, 17(1):128–143, 1989. MR-0972778

[3] N. Alon and O. N. Feldheim. A note on general sliding window processes. *Electron. Commun. Probab.*, 19(66), 7, 2014. MR-3262072

[4] B. Bollobás. *Modern graph theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998. MR-1633290

[5] C. Borgs, J. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi. Counting graph homomorphisms. In *Topics in discrete mathematics*, volume 26 of *Algorithms Combin.*, pages 315–371. Springer, Berlin, 2006. MR-2249277

[6] A. Borodin, P. Diaconis, and J. Fulman. On adding a list of numbers (and other one-dependent determinantal processes). *Bull. Amer. Math. Soc. (N.S.)*, 47(4):639–670, 2010. MR-2721041

[7] R. M. Burton, M. Goulet, and R. Meester. On 1-dependent processes and k-block factors. *Ann. Probab.*, 21(4):2157–2168, 1993. MR-1245304

[8] N. G. de Bruijn. A combinatorial problem. *Nederl. Akad. Wetensch., Proc.*, 49:758–764 = Indagationes Math., 8:461–467, 1946. MR-0018142

[9] V. de Valk. *One-dependent processes: two-block factors and non-two-block factors*, volume 85 of *CWI Tract*. Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1994. MR-1269540

[10] R. Durrett. *Probability: theory and examples*, volume 31 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, fourth edition, 2010. MR-2722836

[11] A. E. Holroyd. One-dependent coloring by finitary factors, arXiv:1411.1463

[12] A. E. Holroyd and T. M. Liggett. Symmetric 1-dependent colorings of the integers. *Electron. Commun. Probab.*, 20(31), 8, 2015. MR-3327870

[13] A. E. Holroyd and T. M. Liggett. Finitely dependent coloring. *Forum Math. Pi*, 4:e9, 43, 2016. MR-3570073

[14] A. E. Holroyd, O. Schramm, and D. B. Wilson. Finitary coloring, arXiv:1412.2725

[15] I. A. Ibragimov and Y. V. Linnik. *Independent and stationarily connected variables*. Izdat. "Nauka", Moscow, 1965. MR-0202176

[16] I. A. Ibragimov and Y. V. Linnik. *Independent and stationary sequences of random variables*. Wolters-Noordhoff Publishing, Groningen, 1971. With a supplementary chapter by I. A. Ibragimov and V. V. Petrov, Translation from the Russian edited by J. F. C. Kingman. MR-0322926

[17] S. Janson. Runs in m-dependent sequences. *Ann. Probab.*, 12(3):805–818, 1984. MR-0744235

[18] O. Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002. MR-1876169

[19] D. Lind and B. Marcus. *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge, 1995. MR-1369092

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