Stochastic differential equations with sticky reflection and boundary diffusion

Martin Grothaus [*][†][‡]   Robert Voßhall [*][†][§]

Abstract

We construct diffusion processes in bounded domains Ω with sticky reflection at the boundary Γ in use of Dirichlet forms. In particular, the occupation time on the boundary is positive. The construction covers a static boundary behavior and an optional diffusion along Γ. The process is a solution to a given SDE for q.e. starting point. Using regularity results for elliptic PDE with Wentzell boundary conditions we show strong Feller properties and characterize the constructed process even for every starting point in $\overline{\Omega}\backslash\Xi$, where Ξ is given explicitly by the involved densities. By a time change we obtain pointwise solutions to SDEs with immediate reflection under weak assumptions on Γ and the drift. A non-trivial extension of the construction yields N-particle systems with the stated boundary behavior and singular drifts. Finally, the setting is applied to a model for particles diffusing in a chromatography tube with repulsive interactions.

1 Introduction

In the first part of the present paper, we construct via Dirichlet form techniques diffusions on $\overline{\Omega}$ for bounded domains Ω of $\mathbb{R}^d$, $d \geq 1$, with boundary Γ, and identify them as weak solutions of SDEs given by

$$d\mathbf{X}_t = \mathbb{1}_\Omega(\mathbf{X}_t)\Big(dB_t + \frac{1}{2}\frac{\nabla\alpha}{\alpha}(\mathbf{X}_t)dt\Big) - \mathbb{1}_\Gamma(\mathbf{X}_t)\frac{1}{2}\frac{\alpha}{\beta}(\mathbf{X}_t)\,n(\mathbf{X}_t)dt$$

$$+ \delta\,\mathbb{1}_\Gamma(\mathbf{X}_t)\Big(dB_t^\Gamma + \frac{1}{2}\frac{\nabla_\Gamma\beta}{\beta}(\mathbf{X}_t)dt\Big), \tag{1.1}$$

$$dB_t^\Gamma = P(\mathbf{X}_t) \circ dB_t,$$

$$\mathbf{X}_0 = x,$$

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[*]University of Kaiserslautern, P.O.Box 3049, 67653 Kaiserslautern, Germany.

[†]http://www.mathematik.uni-kl.de/fuana/

[‡]E-mail: grothaus@mathematik.uni-kl.de

[§]E-mail: vosshall@mathematik.uni-kl.de

for q.e. $x \in \overline{\Omega}$ under weak assumptions on the drifts given by α and β, where $n(y)$ is the outward normal at $y \in \Gamma$, P is the projection on the tangent space and $\delta \in \{0, 1\}$. In the case $\delta = 1$ we additionally assume that $d \geq 2$.

A solution to (1.1) can be characterized as Brownian motion with drift inside Ω and if the process reaches Γ, Brownian motion with drift along Γ may take place, while a further drift term in normal direction is directed back into the interior of Ω. In addition, the Brownian motion $B^\Gamma = (B_t^\Gamma)_{t \geq 0}$ on Γ is the projection of the d-dimensional Brownian motion $B = (B_t)_{t \geq 0}$ onto the manifold Γ (in the sense of a Stratonovich SDE). In this situation, the boundary behavior is called sticky and is connected to so-called Wentzell boundary conditions. In contrast to reflecting (Neumann) boundary conditions which provide an immediate reflection, Wentzell boundary conditions yield sojourn on Γ. The infinitesimal generator and semigroup associated to such kind of diffusions were first investigated in [Fel52] and in [Wen59] on $[0, \infty)$. This kind of diffusion is also considered in [IW89, Chap. IV, Sect. 7] on the half-space $\mathbb{R}_+^d := \{x \in \mathbb{R}^d : x_d \geq 0\}$, $d \geq 2$, with Lipschitz continuous, bounded drifts. In [Car09] the author uses a Dirichlet form approach in order to construct Brownian motion with boundary diffusion in a similar setting to ours with the essential difference that the boundary behavior is not sticky and also a drift does not occur, i.e., only constant densities are admissible. More precisely, the considered approach corresponds to ordinary reflecting boundary conditions (with the d-dimensional Lebesgue measure as reference measure) instead of a sticky boundary behavior. Moreover, in [VV03] and [MR06] diffusion operators on $\overline{\Omega}$ with sticky boundary behavior are considered, but without introducing a boundary diffusion operator on Γ. This is in accordance with our setting for $\delta = 0$, but the authors assume stronger conditions and in particular, a drift is not included. Furthermore, we also construct and analyze the underlying dynamics. Additionally, we deduce regularity properties of the associated semigroup and resolvent in use of the regularity results given in [Nit11], [War12] and [War13]. In application, the required additional conditions on the density can be verified in use of a criterion by [Stu95]. Neither in [Car09] nor in [VV03] and [MR06] Feller properties of the associated semigroup are investigated. Moreover, for $\delta = 0$ it is possible to use a random time change in order to obtain solutions to SDEs with immediate reflection from sticky reflection for a Lipschitz boundary Γ. In this generality, the existence result seems also new.

In the second part of the present paper, we use the previous results as well as tensor products and Girsanov transformations of Dirichlet forms in order to construct a solution of the system of SDEs given by

$$
\begin{aligned}
d\mathbf{X}_t^i =& \mathbb{1}_\Omega(\mathbf{X}_t^i)\Big(dB_t^i + \frac{1}{2}\big(\frac{\nabla_i \alpha_i}{\alpha_i}(\mathbf{X}_t^i) + \frac{\nabla_i \phi}{\phi}(\mathbf{X}_t)\big)dt\Big) - \mathbb{1}_\Gamma(\mathbf{X}_t^i)\frac{1}{2}\frac{\alpha_i}{\beta_i}(\mathbf{X}_t^i)\, n(\mathbf{X}_t^i)dt \\
&+ \delta\ \mathbb{1}_\Gamma(\mathbf{X}_t^i)\Big(dB_t^{\Gamma,i} + \frac{1}{2}\big(\frac{\nabla_{\Gamma,i}\beta_i}{\beta_i}(\mathbf{X}_t^i) + \frac{\nabla_{\Gamma,i}\phi}{\phi}(\mathbf{X}_t)\big)dt\Big), \quad i = 1, \dots, N \qquad (1.2) \\
dB_t^{\Gamma,i} =& P(\mathbf{X}_t^i) \circ dB_t^i \\
\mathbf{X}_0 =& x \in \overline{\Omega}^N,
\end{aligned}
$$

where $(B_t)_{t \geq 0}$, $B_t = (B_t^1, \dots, B_t^N)$, is an Nd-dimensional standard Brownian motion. The particle interaction is given by $\frac{\nabla_i \phi}{\phi}$ and $\frac{\nabla_{\Gamma,i}\phi}{\phi}$, $i = 1, \dots, N$, where the subindex in ∇_i and $\nabla_{\Gamma,i}$ refers to the i-th component. The Dirichlet form construction takes place under extremely weak assumptions on the density. In order to analyze the constructed process, the densities α_i and β_i, $i = 1, \dots, N$, are only assumed to fulfill the conditions assumed for (1.1) whereas ϕ is C^1. Nevertheless, in general ϕ is allowed to vanish on a set of measure zero. Hence, it is possible to consider singular interactions. This property of our construction is important in order to model interacting particle systems in a physically

reasonable way, since naturally two particles are not allowed to be located at the same position at the same time. For example, a Gibbs measure defined by the Lennard-Jones potential is an admissible choice and models the repulsion of the particles. Thus, our results prove the existence of solutions to systems of singular SDEs with sticky boundary. In the case of a bounded drift, it is not possible to model repulsive behavior.

In [Gra88] a sticky diffusion is constructed by probabilistic methods with regard to a propagation of chaos result in the follow-up paper [GM89]. The constructed process coincides with our setting in special cases, but the considered domain is determined by the zero set of a $C^2(\mathbb{R}^d)$-function and the drift is assumed to be Lipschitz continuous and bounded which is quite restrictive, especially with regard to singular interactions. The author uses the constructed diffusion to model a system of particles interacting at the boundary. This interacting particle system in turn is used to model the behavior of molecules in a chromatography tube. We apply our results to this kind of application.

In [EP14] the authors analyze Brownian motion on $[0, \infty)$ which is sticky in 0. They show that strong solutions do not exist and that the sticky Brownian motion is the limit of time scaled reflected Brownian motions. This suggests that a strong solution in our framework also does not exist and hence, the solutions we construct in this paper are optimal in this sense.

The main novelties summarize as follows:

- Existence of a weak solution to the SDE (1.1) with Lipschitz boundary Γ for $\delta = 0$, C^2 boundary for $\delta = 1$ and possibly unbounded, non-Lipschitz drifts. (Theorem 3.17)

- Ergodicity of the solution to (1.1). (Theorem 3.23)

- Regularity properties of the associated semigroup and resolvent as well as existence of a solution of (1.1) for an explicitly known set of starting points and singular drifts. (Section 3.3, Theorem 3.38)

- Existence of weak solutions to SDEs with immediate reflection at Lipschitz boundaries Γ and with singular drifts for an explicitly known set of starting points. (Corollary 3.39)

- Existence of a weak solution to the N-particle SDE (1.2). (Theorem 4.22)

- Application of the concepts to N-particle dynamics in a chromatography tube with singular interactions. (Section 4.3)

2 Preliminaries

Throughout this paper, $\Omega \subset \mathbb{R}^d$, $d \geq 1$, denotes a non-empty bounded Lipschitz domain, λ the Lebesgue measure on Ω and σ the surface measure on $\Gamma := \partial\Omega$. In the case $\delta = 1$ we assume that $d \geq 2$. The standard scalar product on $\mathbb{R}^d$ is given by $(\cdot, \cdot)$ and norms on $\mathbb{R}^d$ by $|\cdot|$ (in particular, for the modulus in $\mathbb{R}$; eventually labeled by a lower index in order to distinguish norms). Similarly, $\|\cdot\|$ denotes norms on function spaces. The metric on $\mathbb{R}^d$ induced by the euclidean metric is denoted by d_{euc}.

For a vector $x \in \overline{\Omega}^N$, $N \in \mathbb{N}$, we use the representation $x = (x^1, \dots, x^N)$, where $x^i \in \overline{\Omega}$, $i = 1, \dots, N$, is represented in the form $x^i = (x^i_1, \dots, x^i_d)$. We denote by ∇ the gradient of a smooth function and by $\partial_{x^i_k}$, $i = 1, \dots, N$, $k = 1, \dots, d$, its partial derivatives. In the case $N = 1$ we simply write ∂_k for $k = 1, \dots, d$. By ∇_i, $i = 1, \dots, N$, we denote the d-dimensional vector given by the partial derivatives with respect to the coordinates x^i_k, $k = 1, \dots, d$. Moreover, ∇^2 denotes the Hessian for functions mapping from subsets of

$\mathbb{R}^d$ to $\mathbb{R}$ and $\Delta = \mathrm{Tr}(\nabla^2)$ the Laplacian. ∇_i^2 and Δ_i are defined analogously. In the case of Sobolev functions we use the same notations in the weak sense.

In order to fix the notation we make the following definitions:

Definition 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. The boundary Γ of Ω is said to be Lipschitz continuous (respectively C^k-smooth) if for every $x \in \Gamma$ there exists a neighborhood V of x in $\mathbb{R}^d$ such that $\Gamma \cap V$ is the graph of a Lipschitz continuous (respectively C^k-smooth) function and $\Omega \cap V$ is located at one side of the graph, i.e., it exist new orthogonal coordinates $(y_1, \ldots, y_d)$ (given by an orthogonal map T), a reference point $z \in \mathbb{R}^{d-1}$, real numbers $r, h > 0$ and a Lipschitz continuous (respectively C^k-smooth) function $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that in the new coordinates it holds*

(i)
$$V = \{y = (y_1, \ldots, y_d) \in \mathbb{R}^d | \ |y_{,d} - z| < r, \ |y_d - \varphi(y_{,d})| < h\},$$

where $y_{,d} := (y_1, \ldots, y_{d-1})$ for $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$,

(ii)
$$\Omega \cap V = \{y \in V | \ -h < y_d - \varphi(y_{,d}) < 0\},$$
$$\Gamma \cap V = \{y \in V | \ y_d = \varphi(y_{,d})\}.$$

So Γ is Lipschitz continuous (respectively C^k-smooth) if Ω is locally below the graph of a Lipschitz continuous (respectively C^k-) function and the graph coincides with Γ. In this case, we also simply say that Γ is Lipschitz (respectively C^k) or that Ω has Lipschitz boundary (respectively C^k-boundary). Note that in the case of a Lipschitz continuous boundary Γ, each φ is almost everywhere differentiable by Rademacher's theorem. Moreover, it is possible to find a finite open cover $(V_i)_{i=1,\ldots,l}$ of Γ such that the assumptions in Definition 2.1 are fulfilled for each V_i, $i = 1, \ldots, l$, since Γ is compact.

Definition 2.2. *Let Ω be open and bounded with Lipschitz continuous boundary Γ. Then we define for $y = (y_1, \ldots, y_d) \in V$*

$$\tilde{n}(y) := \frac{(-\nabla\varphi(y_{,d}), 1)}{\sqrt{|\nabla\varphi(y_{,d})|^2 + 1}}$$

supposed that φ is differentiable at $y_{,d} := (y_1, \ldots, y_{d-1})$. Let $x \in \Gamma$ and $T \in \mathbb{R}^{d \times d}$ be the orthogonal coordinate transformation from Definition 2.1. Then define the (outward) normal vector at x by

$$n(x) := T^{-1} \tilde{n}(Tx).$$

Remark 2.3. Note that the definition of n also makes sense in a neighborhood of x and n is differentiable near x if Γ is C^2.

Definition 2.4. *Let $x \in \Gamma$ be such that $n(x)$ exists in the sense of Definition 2.2. Define*

$$P(x) := E - n(x)n(x)^t \in \mathbb{R}^{d \times d},$$

where E is the $d \times d$ identity matrix. We call $P(x)$ the orthogonal projection on the tangent space at x. Note that $P(x)z = z - (n(x), z)\, n(x)$ for $z \in \mathbb{R}^d$.

Definition 2.5. *Let $f \in C^1(\overline{\Omega})$ and $x \in \Gamma$. Then we define (whenever Γ is sufficiently smooth at x) the gradient of f at x along Γ by*

$$\nabla_\Gamma f(x) := P(x)\nabla f(x)$$

and if $f \in C^2(\overline{\Omega})$ the Laplace-Beltrami of f at x by

$$\Delta_\Gamma f(x) := Tr(\nabla_\Gamma^2 f(x)) = div_\Gamma \nabla_\Gamma f(x) = Tr(P(x)\nabla(P(x)\nabla f(x))),$$

where $div_\Gamma \Phi := Tr(P\nabla\Phi)$ for $\Phi = (\Phi_1, \ldots, \Phi_d) \in C^1(\overline{\Omega}; \mathbb{R}^d)$ with $\nabla\Phi = (\nabla\Phi_1 | \ldots | \nabla\Phi_d)$.
Analogously, we define higher derivatives of order $k \in \mathbb{N}$. Let $C^k(\Gamma)$ be the space of k-times continuously differentiable functions on Γ. As usual, set $C^\infty(\Gamma) := \cap_{k \in \mathbb{N}} C^k(\Gamma)$. As before, denote by ∇_Γ the gradient on Γ and by div_Γ the divergence operator. Moreover, in the case that n is differentiable at x we define the mean curvature of Γ at x by

$$\kappa(x) := div_\Gamma\ n(x).$$

In the above way, it is possible to obtain from functions in $C^1(\overline{\Omega})$ and $C^2(\overline{\Omega})$ elements in $C^1(\Gamma)$ and $C^2(\Gamma)$ respectively by restriction.

We have the following relation for the mean curvature κ:

Lemma 2.6. *Assume that Γ is C^2-smooth. Then*

$$(P\nabla)^t P = -\kappa n,$$

where $((P\nabla)^t P)_i := \sum_{k,j} P_{jk} \partial_j P_{ik}$ for $i = 1, \ldots, d$.

Proof. Fix $i \in \{1, \ldots d\}$. It holds

$$
\begin{aligned}
\left((P\nabla)^t P\right)_i &= \sum_{k,j} P_{jk} \partial_j P_{ik} \\
&= -\sum_{k,j} P_{jk} \partial_j (n_i n_k) \\
&= -\sum_{k,j} (1 - n_j n_k)(\partial_j n_i n_k + n_i \partial_j n_k) \\
&= -\big(\sum_{k,j} (1 - n_j n_k) \partial_j n_k\big) n_i - \sum_{k,j} (1 - n_j n_k) \partial_j n_i n_k \\
&= -\mathrm{Tr}(P\nabla n) n_i - (n, P\nabla n_i) = -\kappa n_i - (n, P\nabla n_i).
\end{aligned}
$$

Using that P is the orthogonal projection on $\big(\mathrm{span}(n)\big)^\perp$, we get that $(n, P\nabla n_i) = 0$ and therfore, the assertion holds true. $\qquad\square$

Definition 2.7. *The Sobolev space $H^{1,k}(\Gamma)$, $k \geq 1$, is defined by $\overline{C^1(\Gamma)}^{\|\cdot\|_{H^{1,k}(\Gamma)}} \subset L^k(\Gamma; \sigma)$, i.e., the closure $C^1(\Gamma)$ with respect to the norm*

$$\| \cdot \|_{H^{1,k}(\Gamma)} := \big(\| \cdot \|_{L^k(\Gamma;\sigma)}^k + \|\nabla_\Gamma \cdot \|_{L^k(\Gamma;\sigma)}^k\big)^{\frac{1}{k}}.$$

Remark 2.8. For a vector valued C^1-function Φ on Γ and $g \in H^{1,k}(\Gamma)$, we have the divergence theorem

$$\int_\Gamma (\Phi, \nabla_\Gamma g)\ d\sigma = -\int_\Gamma div_\Gamma \Phi\ g\ d\sigma \tag{2.1}$$

in view of [Tay11, Chap. 2, Proposition 2.2].

We shortly recall some facts about Brownian motion on Γ. For details about stochastic analysis on manifolds, we refer to [HT94], [Hsu02] and [IW89]:

By definition, Brownian motion $(B_t^\Gamma)_{t \geq 0}$ on a smooth boundary Γ is a Γ-valued stochastic process that is generated by $\frac{1}{2}\Delta_\Gamma$, in analogy to Brownian motion on $\mathbb{R}^d$, in the sense that $(B_t^\Gamma)_{t \geq 0}$ solves the martingale problem for $(\frac{1}{2}\Delta_\Gamma, C^\infty(\Gamma))$. We recall the following:

Lemma 2.9. *Let Γ be a smooth submanifold of $\mathbb{R}^d$ as in Definition 2.1. Then a solution of the Stratonovich SDE*

$$d\mathbf{X}_t = P(\mathbf{X}_t) \circ dB_t, \quad \mathbf{X}_0 \in \Gamma,$$

is a Brownian motion on Γ, where $(B_t)_{t \geq 0}$ is a Brownian motion in $\mathbb{R}^d$.

Proof. See [Hsu02, Chap. 3, Sect. 2]. □

Remark 2.10. Note that the dimension of the driving Brownian motion $(B_t)_{t\geq 0}$ is strictly larger than the dimension of the submanifold Γ and hence, according to [Hsu02] the driving Brownian motion contains some extra information beyond what is usually provided by a Brownian motion on Γ. Furthermore, a solution of the above SDE is naturally Γ-valued, since $P(x)z$ is tangential to Γ at x for every $x \in \Gamma$ and $z \in \mathbb{R}^d$. In our application, it is natural to construct a Brownian motion on Γ by means of a d-dimensional Brownian motion, since a Brownian motion on $\mathbb{R}^d$ is involved anyway.

3 Sticky reflected diffusions on $\overline{\Omega}$

3.1 The Dirichlet form and the associated Markov process

Condition 3.1. Γ *is Lipschitz continuous. Moreover,* $\alpha \in L^1(\Omega; \lambda)$, $\alpha > 0$ λ*-a.e., and* $\beta \in L^1(\Gamma; \sigma)$, $\beta > 0$ σ*-a.e..*

Define
$$\varrho := \mathbb{1}_\Omega \, \alpha + \mathbb{1}_\Gamma \, \beta$$
as well as
$$\mu := \varrho \, (\lambda + \sigma) = \alpha\lambda + \beta\sigma.$$

Note that the condition $\alpha \in L^1(\Omega; \lambda)$, $\alpha > 0$ λ-a.e., and $\beta \in L^1(\Gamma; \sigma)$, $\beta > 0$ σ-a.e. is equivalent to $\varrho \in L^1(\overline{\Omega}; \lambda + \sigma)$, $\varrho > 0$ $(\lambda + \sigma)$-a.e.. μ is a Borel measure on $\overline{\Omega}$. Hence, we can conclude the following:

Proposition 3.2. *Under Condition 3.1 we have that* $C^\infty(\overline{\Omega})$ *is dense in* $L^2(\overline{\Omega}; \mu)$.

Let the symmetric and positive definite bilinear form $(\mathcal{E}, \mathcal{D})$ be given by

$$\mathcal{E}(f, g) := \frac{1}{2} \int_\Omega (\nabla f, \nabla g) \, \alpha d\lambda + \frac{\delta}{2} \int_\Gamma (\nabla_\Gamma f, \nabla_\Gamma g) \, \beta d\sigma \ \text{ for } f, g \in \mathcal{D} := C^1(\overline{\Omega}), \qquad (3.1)$$

where $(\cdot, \cdot)$ denotes the euclidean scalar product in $\mathbb{R}^d$ and $\delta \in \{0, 1\}$. In addition, let

$$\mathcal{E}_\Omega(f, g) := \frac{1}{2} \int_\Omega (\nabla f, \nabla g) \, \alpha d\lambda \ \text{ for } f, g \in \mathcal{D}_\Omega := C^1(\overline{\Omega})$$

as well as

$$\mathcal{E}_\Gamma(f, g) := \frac{1}{2} \int_\Gamma (\nabla_\Gamma f, \nabla_\Gamma g) \, \beta d\sigma \ \text{ for } f, g \in \mathcal{D}_\Gamma := C^1(\Gamma).$$

Note that $e(\mathcal{D}) = e(\mathcal{D}_\Omega) \subset \mathcal{D}_\Gamma$, where $e : C^1(\overline{\Omega}) \to C^1(\Gamma)$ is defined by the restriction of functions to Γ. In these terms, for $f, g \in \mathcal{D}$ we get

$$\mathcal{E}(f, g) = \mathcal{E}_\Omega(f, g) + \delta \, \mathcal{E}_\Gamma(f, g).$$

In order to prove closability of $(\mathcal{E}, \mathcal{D})$, we need an additional assumption on the density ϱ. Define

$$R_\alpha(\Omega) := \{x \in \Omega : \int_{\{y \in \Omega : |x-y| < \varepsilon\}} \alpha^{-1} d\lambda < \infty \ \text{ for some } \varepsilon > 0\}$$

and analogously $R_\beta(\Gamma)$ with Ω replaced by Γ and λ replaced by σ.

Condition 3.3 (Hamza condition). $\alpha = 0$ λ*-a.e. on* $\Omega \setminus R_\alpha(\Omega)$ *and additionally* $\beta = 0$ σ*-a.e. on* $\Gamma \setminus R_\beta(\Gamma)$ *if* $\delta = 1$.

Lemma 3.4. *Assume that Condition 3.1 and Condition 3.3 are fulfilled. Then the densely defined, symmetric bilinear forms* $(\mathcal{E}_\Omega, \mathcal{D}_\Omega)$ *and* $(\mathcal{E}_\Gamma, \mathcal{D}_\Gamma)$ *(if* $\delta = 1$*) are closable on* $L^2(\overline{\Omega}; \alpha\lambda)$ *and on* $L^2(\Gamma; \beta\sigma)$ *respectively. Moreover, the closures* $(\mathcal{E}_\Omega, D(\mathcal{E}_\Omega))$ *and* $(\mathcal{E}_\Gamma, D(\mathcal{E}_\Gamma))$ *are conservative, strongly local, regular, symmetric Dirichlet forms.*

Proof. The symmetric densely defined bilinear forms are closable and its closures are symmetric Dirichlet forms by [MR92, Chap. 2, Sect. 2, Example a)] (see in particular Remark 2.3 of the reference). The remaining properties follow exactly like in the following proofs for the closure of $(\mathcal{E}, \mathcal{D})$. □

Proposition 3.5. *Suppose that Condition 3.1 and Condition 3.3 are satisfied. Then $(\mathcal{E}, \mathcal{D})$ is closable on $L^2(\overline{\Omega}; \mu)$. We denote its closure by $(\mathcal{E}, D(\mathcal{E}))$.*

Proof. Let $(f_k)_{k \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{D} = C^1(\overline{\Omega})$ with respect to $\mathcal{E}$, i.e.,

$$\mathcal{E}(f_k - f_l, f_k - f_l) \to 0 \ \text{ as } k, l \to \infty.$$

Moreover, assume that $(f_k)_{k \in \mathbb{N}}$ converges to 0 in $L^2(\overline{\Omega}; \mu)$. We have to show that $\mathcal{E}(f_k, f_k) \to 0$ as $k \to \infty$.

Since $(f_k)_{k \in \mathbb{N}}$ is a Cauchy sequence with respect to $\mathcal{E}$, it is also a Cauchy sequence with respect to $\mathcal{E}_\Omega$ (and $\mathcal{E}_\Gamma$ if $\delta = 1$). Moreover, the convergence of $(f_k)_{k \in \mathbb{N}}$ to 0 in $L^2(\overline{\Omega}; \mu)$ implies by definition the convergence to 0 in $L^2(\overline{\Omega}; \alpha\lambda)$ and $L^2(\Gamma; \beta\sigma)$. Therefore, we get $\mathcal{E}_\Omega(f_k, f_k) \to 0$ (and $\mathcal{E}_\Gamma(f_k, f_k) \to 0$ if $\delta = 1$) as $k \to \infty$ by Lemma 3.4. Hence,

$$\mathcal{E}(f_k, f_k) = \mathcal{E}_\Omega(f_k, f_k) + \delta \, \mathcal{E}_\Gamma(f_k, f_k) \to 0 \ \text{ as } k \to \infty. \qquad □$$

Proposition 3.6. *Suppose that Condition 3.1 and Condition 3.3 are satisfied. Then $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric, regular Dirichlet form.*

Proof. The Markov property follows as in [MR92, Chap.2, Sect. 2, Example c)] by [MR92, Chap. 1, Proposition 4.10] and the chain rule. Hence, $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet Form. By the Stone-Weierstraß theorem, it holds that $C^\infty(\overline{\Omega})$ is dense in $C(\overline{\Omega})$ with respect to $\| \cdot \|_{\sup}$. Furthermore, $\mathcal{D}$ is dense in $D(\mathcal{E})$ with respect to the $\mathcal{E}_1$-norm. Since $C^\infty(\overline{\Omega}) \subset \mathcal{D} \subset D(\mathcal{E}) \cap C(\overline{\Omega})$, we obtain that $(\mathcal{E}, D(\mathcal{E}))$ is also regular. □

Proposition 3.7. *Suppose that Condition 3.1 and Condition 3.3 are satisfied. Then the symmetric, regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is strongly local and recurrent.*

Proof. Using [FOT11, Theo. 3.1.1] and [FOT11, Exercise 3.1.1] it is sufficient to show the strong local property for elements in $\mathcal{D}$. Therefore, let $f, g \in \mathcal{D}$ such that g is constant on some open neighborhood U of $\mathrm{supp}(f)$ (in the trace topology of $\overline{\Omega}$). Then

$$
\begin{aligned}
\mathcal{E}(f, g) =& \frac{1}{2} \int_\Omega (\nabla f, \nabla g) \, \alpha d\lambda + \frac{\delta}{2} \int_\Gamma (\nabla_\Gamma f, \nabla_\Gamma g) \, \beta d\sigma \\
=& \frac{1}{2} \int_{\Omega \cap \mathrm{supp}(f)} (\nabla f, \nabla g) \, \alpha d\lambda + \frac{1}{2} \int_{\Omega \setminus \mathrm{supp}(f)} (\nabla f, \nabla g) \, \alpha d\lambda \\
& + \frac{\delta}{2} \int_{\Gamma \cap \mathrm{supp}(f)} (\nabla_\Gamma f, \nabla_\Gamma g) \, \beta d\sigma + \frac{\delta}{2} \int_{\Gamma \setminus \mathrm{supp}(f)} (\nabla_\Gamma f, \nabla_\Gamma g) \, \beta d\sigma \\
=& \, 0,
\end{aligned}
$$

because each summand is zero, since the integrals are defined over sets where either f or g is constant. Hence, $(\mathcal{E}, D(\mathcal{E}))$ is strongly local. Clearly, $\mathbb{1}_{\overline{\Omega}} \in \mathcal{D} \subset D(\mathcal{E})$ and $\mathcal{E}(\mathbb{1}_{\overline{\Omega}}, \mathbb{1}_{\overline{\Omega}}) = 0$. Therefore, $(\mathcal{E}, D(\mathcal{E}))$ is also recurrent. □

By summarizing the preceding results, we get the following theorem:

Theorem 3.8. *Assume Condition 3.1 and Condition 3.3. Then the symmetric and positive definite bilinear form $(\mathcal{E}, D)$ is densely defined and closable on $L^2(\overline{\Omega}; \mu)$. Its closure $(\mathcal{E}, D(\mathcal{E}))$ is a recurrent, strongly local, regular, symmetric Dirichlet form on $L^2(\overline{\Omega}; \mu)$.*

By the theory of Dirichlet forms, we obtain immediately the following theorem. For details see e.g. [MR92, Chap. V, Theorem 1.11] or [FOT11, Theorem 7.2.2 and Exercise 4.5.1]. We remark that the definitions of capacities (and hence, of exceptional sets) used in the textbooks [FOT11] and [MR92] are introduced in different ways, but that the defintions coincide in our setting (see [MR92, Chap. III, Remark 2.9 and Exercise 2.10]). $(T_t)_{t>0}$ denotes the sub-Markovian strongly continuous contraction semigroup on $L^2(\overline{\Omega}; \mu)$ corresponding to $(\mathcal{E}, D(\mathcal{E}))$.

Theorem 3.9. *Suppose that Condition 3.1 and Condition 3.3 are satisfied. Then there exists a conservative diffusion process (i.e. a strong Markov process with continuous sample paths and infinite life time)*

$$\mathbf{M} := \big(\boldsymbol{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\mathbf{X}_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\overline{\Omega}}\big)$$

with state space $\overline{\Omega}$ which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$, i.e., for all ($\mu$-versions of) $f \in \mathcal{B}_b(\overline{\Omega}) \subset L^2(\overline{\Omega}; \mu)$ and all $t > 0$ the function

$$\overline{\Omega} \ni x \mapsto p_t f(x) := \mathbb{E}_x\big(f(\mathbf{X}_t)\big) := \int_{\overline{\Omega}} f(\mathbf{X}_t) d\mathbf{P}_x \in \mathbb{R}$$

is a quasi continuous version of $T_t f$. $\mathbf{M}$ is up to μ-equivalence unique. In particular, $\mathbf{M}$ is μ-symmetric, i.e.,

$$\int_{\overline{\Omega}} p_t f \ g \ d\mu = \int_{\overline{\Omega}} f \ p_t g \ d\mu \ \ \text{for all } f, g \in \mathcal{B}_b(\overline{\Omega}) \ \text{ and all } t > 0,$$

and has μ as invariant measure, i.e.,

$$\int_{\overline{\Omega}} p_t f \ d\mu = \int_{\overline{\Omega}} f \ d\mu \ \ \text{for all } f \in \mathcal{B}_b(\overline{\Omega}) \ \text{ and all } t > 0.$$

Remark 3.10. Note that $\mathbf{M}$ is canonical, i.e., $\boldsymbol{\Omega} = C(\mathbb{R}_+, \overline{\Omega})$ and $\mathbf{X}_t(\omega) = \omega(t)$, $\omega \in \boldsymbol{\Omega}$. For each $t \geq 0$ we denote by $\Theta_t : \boldsymbol{\Omega} \to \boldsymbol{\Omega}$ the shift operator defined by $\Theta_t(\omega) = \omega(\cdot + t)$ for $\omega \in \boldsymbol{\Omega}$ such that $\mathbf{X}_s \circ \Theta_t = \mathbf{X}_{s+t}$ for all $s \geq 0$. We take into account to extend the setting to $C(\mathbb{R}_+, \mathbb{R}^d)$ by neglecting paths leaving $\overline{\Omega}$.

3.2 Analysis of the Markov process

3.2.1 Generators and boundary conditions

By Friedrichs representation theorem we have the existence of a unique self-adjoint generator $(L, D(L))$ corresponding to $(\mathcal{E}, D(\mathcal{E}))$.

Proposition 3.11. *Suppose that Condition 3.1 and Condition 3.3 are satisfied. Then there exists a unique, positive, self-adjoint, linear operator $(L, D(L))$ on $L^2(\overline{\Omega}; \mu)$ such that*

$$D(L) \subset D(\mathcal{E}) \ \text{ and } \ \mathcal{E}(f, g) = (-Lf, g)_{L^2(\overline{\Omega};\mu)} \ \text{ for all } f \in D(L), \ g \in D(\mathcal{E}).$$

In order to determine the generator on a subspace of $D(L)$ we assume the following condition:

Condition 3.12. Γ is Lipschitz continuous. Moreover, $\alpha, \beta \in C(\overline{\Omega})$, $\alpha > 0$ λ-a.e. on Ω, $\beta > 0$ σ-a.e. on Γ such that $\sqrt{\alpha} \in H^{1,2}(\Omega)$ and additionally, Γ is C^2-smooth and $\sqrt{\beta} \in H^{1,2}(\Gamma)$ if $\delta = 1$.

Remark 3.13. Note that Condition 3.12 implies Condition 3.1 and Condition 3.3. In particular, Condition 3.12 holds if $\alpha, \beta \in C^1(\overline{\Omega})$, $\alpha, \beta > 0$.

Proposition 3.14. *Suppose that Condition 3.12 is satisfied. Then, $C^2(\overline{\Omega}) \subset D(L)$ and*

$$Lf = \tilde{L}f := \frac{1}{2}\Big(\mathbb{1}_\Omega \left(\Delta f + (\frac{\nabla\alpha}{\alpha}, \nabla f)\right) - \mathbb{1}_\Gamma \frac{\alpha}{\beta} (n, \nabla f) + \delta \, \mathbb{1}_\Gamma \left(\Delta_\Gamma f + (\frac{\nabla_\Gamma\beta}{\beta}, \nabla_\Gamma f)\right)\Big)$$

for $f \in C^2(\overline{\Omega})$.

Proof. Let $f \in C^2(\overline{\Omega})$ and $g \in \mathcal{D} = C^1(\overline{\Omega})$. Then we get by the divergence theorem on Ω and (2.1):

$$\begin{aligned}
\mathcal{E}(f,g) &= \frac{1}{2}\int_\Omega (\nabla f, \nabla g)\, \alpha d\lambda + \frac{\delta}{2}\int_\Gamma (\nabla_\Gamma f, \nabla_\Gamma g)\, \beta d\sigma \\
&= \frac{1}{2}\int_\Omega (\alpha\nabla f, \nabla g)\, d\lambda + \frac{\delta}{2}\int_\Gamma (\beta\nabla_\Gamma f, \nabla_\Gamma g)\, d\sigma \\
&= -\frac{1}{2}\int_\Omega g\, \mathrm{div}(\alpha\nabla f)\, d\lambda + \frac{1}{2}\int_\Gamma g\, (\nabla f, n)\, \alpha d\sigma - \frac{\delta}{2}\int_\Gamma g\, \mathrm{div}_\Gamma(\beta\nabla_\Gamma f)\, d\sigma \\
&= -\frac{1}{2}\int_\Omega g\, (\Delta f \alpha + (\nabla\alpha, \nabla f))\, d\lambda + \frac{1}{2}\int_\Gamma g\, \frac{\alpha}{\beta}(\nabla f, n)\, \beta d\sigma \\
&\qquad - \frac{\delta}{2}\int_\Gamma g(\beta\Delta_\Gamma f + (\nabla_\Gamma\beta, \nabla_\Gamma f))\, d\sigma \\
&= -\frac{1}{2}\int_\Omega g\, (\Delta f + (\frac{\nabla\alpha}{\alpha}, \nabla f))\, \alpha d\lambda + \frac{1}{2}\int_\Gamma g\, \frac{\alpha}{\beta}(\nabla f, n)\, \beta d\sigma \\
&\qquad - \frac{\delta}{2}\int_\Gamma g(\Delta_\Gamma f + (\frac{\nabla_\Gamma\beta}{\beta}, \nabla_\Gamma f))\, \beta d\sigma \\
&= (-\tilde{L}f, g)_{L^2(\overline{\Omega};\mu)}.
\end{aligned}$$

By density of $\mathcal{D}$ in $D(\mathcal{E})$ with respect to the $\mathcal{E}_1$-norm, the claim follows. $\square$

We can define the operator L_Ω and the boundary operator L_Γ by

$$L_\Omega f := \frac{1}{2}\big(\Delta f + (\frac{\nabla\alpha}{\alpha}, \nabla f)\big) \ \text{ and } \ L_\Gamma f := \frac{1}{2}\big(\delta\, \Delta_\Gamma f + \delta\, (\frac{\nabla_\Gamma\beta}{\beta}, \nabla_\Gamma f) - \frac{\alpha}{\beta}(n, \nabla f)\big)$$

for $f \in C^2(\overline{\Omega})$. Then the generator L has the representation $Lf = \mathbb{1}_\Omega\, L_\Omega f + \mathbb{1}_\Gamma\, L_\Gamma f$. The associated Cauchy problem for $g \in C^2(\overline{\Omega})$ has the form

$$\begin{cases}
\frac{\partial}{\partial t}u_t = \frac{1}{2}\big(\Delta u_t + (\frac{\nabla\alpha}{\alpha}, \nabla u_t)\big), & \text{on } \overline{\Omega}, t > 0 \\
\Delta u_t + (\frac{\nabla\alpha}{\alpha}, \nabla u_t) - \delta\, \Delta_\Gamma u_t - \delta\, (\frac{\nabla_\Gamma\beta}{\beta}, \nabla_\Gamma u_t) + \frac{\alpha}{\beta}(n, \nabla u_t) = 0, & \text{on } \Gamma, t > 0, \\
u_0 = g & \text{on } \overline{\Omega}.
\end{cases} \quad (3.2)$$

The condition in (3.2) is called Wentzell boundary condition. Note that if we multiply (3.2) for $\delta = 0$ by β and then set β to zero, the equation reduces to the Neumann boundary condition.

For $h \in C^1(\overline{\Omega})$, we have by definition and calculation $(\nabla_\Gamma h, \nabla_\Gamma f) = (P\nabla h, \nabla f)$ and $\Delta_\Gamma f = \mathrm{Tr}(P\nabla^2 f) - (n, \nabla f)\, \mathrm{Tr}(P\nabla n) = \mathrm{Tr}(P\nabla^2 f) - (\kappa n, \nabla f)$. Hence, we get with

$$A := \mathbb{1}_\Omega E + \delta\, \mathbb{1}_\Gamma P \tag{3.3}$$

as well as

$$b := \frac{1}{2}\Big(\mathbb{1}_\Omega \frac{\nabla\alpha}{\alpha} + \mathbb{1}_\Gamma\big(\delta\, P\frac{\nabla\beta}{\beta} - (\frac{\alpha}{\beta} + \kappa)n\big)\Big) \tag{3.4}$$

the representation

$$Lf = \frac{1}{2}\mathrm{Tr}(A\nabla^2 f) + (b, \nabla f). \tag{3.5}$$

Note that $AA^t = A^2 = A$.

3.2.2 Solution to the martingale problem and SDE

Theorem 3.15. *The diffusion process* $\mathbf{M}$ *from Theorem 3.9 is up to* μ-*equivalence the unique diffusion process having* μ *as symmetrizing measure and solving the martingale problem for* $(L, D(L))$, *i.e., for all* $g \in D(L)$

$$\tilde{g}(\mathbf{X}_t) - \tilde{g}(\mathbf{X}_0) - \int_0^t (Lg)(\mathbf{X}_s)ds, \ t \geq 0,$$

is an $\mathcal{F}_t$-*martingale under* $\mathbf{P}_x$ *for quasi all* $x \in \overline{\Omega}$. *Here* $\tilde{g}$ *denotes a quasi-continuous version of* g *(for the definition of quasi-continuity see e.g. [MR92, Chap. IV, Proposition 3.3]).*

Proof. See e.g. [AR95, Theorem 3.4 (i)]. □

By the explicit calculation of L given in Proposition 3.14 and the notation in (3.5), we obtain the following corollary:

Corollary 3.16. *Assume that Condition 3.12 is fulfilled. Let* $g \in C^2(\overline{\Omega})$ *and let* $\mathbf{M}$ *be the diffusion process from Theorem 3.9. Then*

$$g(\mathbf{X}_t) - g(\mathbf{X}_0) - \int_0^t \frac{1}{2}Tr\big(A(\mathbf{X}_s)\nabla^2 g(\mathbf{X}_s)\big) + \big(b(\mathbf{X}_s), \nabla g(\mathbf{X}_s)\big)ds, \ t \geq 0,$$

is an $\mathcal{F}_t$-*martingale under* $\mathbf{P}_x$ *for quasi every* $x \in \overline{\Omega}$, *where* A *and* b *are defined as in (3.3) and (3.4).*

Due to the connection of martingale problems and SDEs we get for the coefficients given by A and b as defined above the following (see [Kal97, Theorem 18.7]):

Theorem 3.17. $\mathbf{M}$ *is a solution to the SDE*

$$d\mathbf{X}_t = \mathbb{1}_\Omega(\mathbf{X}_t)\Big(dB_t + \frac{1}{2}\frac{\nabla\alpha}{\alpha}(\mathbf{X}_t)dt\Big) - \mathbb{1}_\Gamma(\mathbf{X}_t)\frac{1}{2}\frac{\alpha}{\beta}(\mathbf{X}_t)\,n(\mathbf{X_t})dt$$
$$+ \delta\,\mathbb{1}_\Gamma(\mathbf{X}_t)\Big(dB_t^\Gamma + \frac{1}{2}\frac{\nabla_\Gamma\beta}{\beta}(\mathbf{X}_t)dt\Big),$$
$$dB_t^\Gamma = P(\mathbf{X}_t) \circ dB_t,$$
$$\mathbf{X}_0 = x,$$

for quasi every starting point $x \in \overline{\Omega}$, *where* $(B_t)_{t \geq 0}$ *is a* d-*dimensional standard Brownian motion, i.e.,*

$$\mathbf{X}_t = x + \int_0^t \mathbb{1}_\Omega(\mathbf{X}_s)dB_s + \int_0^t \mathbb{1}_\Omega(\mathbf{X}_s)\frac{1}{2}\frac{\nabla\alpha}{\alpha}(\mathbf{X}_s)ds$$
$$+ \delta\int_0^t \mathbb{1}_\Gamma(\mathbf{X}_s)P(\mathbf{X}_s)dB_s - \delta\int_0^t \mathbb{1}_\Gamma(\mathbf{X}_s)\frac{1}{2}\kappa(\mathbf{X}_s)n(\mathbf{X}_s)ds \qquad (3.6)$$
$$+ \delta\int_0^t \mathbb{1}_\Gamma(\mathbf{X}_s)\frac{1}{2}\frac{\nabla_\Gamma\beta}{\beta}(\mathbf{X}_s)ds - \int_0^t \mathbb{1}_\Gamma(\mathbf{X}_s)\frac{1}{2}\frac{\alpha}{\beta}(\mathbf{X}_s)n(\mathbf{X}_s)ds$$

almost surely under $\mathbf{P}_x$ *for quasi every* $x \in \overline{\Omega}$.

Remark 3.18. A Fukushima decomposition of $\mathbf{M}$ (see [FOT11, Chap. 5]) yields the same result as in Theorem 3.17. We would like to mention that the argument used here in order to get a solution to the SDE (1.1) does not work for reflecting (Neumann) boundary conditions, since in this case the reflection is not given by a drift term. However, a Fukushima decomposition is still valid (see e.g. [Tru03]), because in this case it is also possible to assign an additive functional to the surface measure σ. The advantage in our situation is that we are able to express the boundary behavior in terms of the generator.

3.2.3 Ergodicity and occupation time

Throughout this section we assume that Condition 3.1 and Condition 3.3 are fulfilled and denote by $\mathbf{M}$ the process constructed in Theorem 3.9. Given the process $\mathbf{M}$, we can define via its transition semigroup $(p_t)_{t \geq 0}$ a Dirichlet form and by construction of $\mathbf{M}$ this form is $(\mathcal{E}, D(\mathcal{E}))$ again. Recall that the sub-Markovian strongly continuous contraction semigroup on $L^2(\overline{\Omega}; \mu)$ of $(\mathcal{E}, D(\mathcal{E}))$ is denoted by $(T_t)_{t \geq 0}$. We use the results provided in [FOT11, Chap. 4.7] in order to prove an ergodic theorem for $\mathbf{M}$. To do this, we restrict to invariant subsets of $\overline{\Omega}$ and show the part of the form $(\mathcal{E}, D(\mathcal{E}))$ on the invariant set is irreducible recurrent. This allows to determine the occupation time of the process on Γ and, as a consequence, to show that the boundary behavior is indeed sticky. The main result of this section is Theorem 3.23. In order to avoid confusion, we label the capacity of a set by the underlying Dirichlet form. For the sake of convenience, we state all proofs for the case $\delta = 1$, which can easily be modified to hold for $\delta = 0$.

First, we define the notion of parts of Dirichlet forms:

Definition 3.19 (*part of a Dirichlet form*)**.** *Let $(\mathcal{G}, D(\mathcal{G}))$ be an arbitrary regular Dirichlet form on some locally compact, separable metric space X, m a positive Radon measure on X with full topological support and G an open subset of X. Then we define by $\mathcal{G}^G(f, g) := \mathcal{G}(f, g)$ for $f, g \in D(\mathcal{G}^G) := \{f \in D(\mathcal{G}) \mid \tilde{f} = 0 \ \mathcal{G}\text{-q.e. on } X \backslash G\}$ the part of the form $(\mathcal{G}, D(\mathcal{G}))$ on G, where $\tilde{f}$ denotes an $\mathcal{G}$-quasi-continuous version of f. Indeed, this defines a regular Dirichlet form on $L^2(G; m)$ (see [FOT11, Theorem 4.4.3]).*

Throughout this section, suppose that Condition 3.12 is satisfied and denote by

$$\mathbf{M} := \big(\boldsymbol{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbf{X}_t)_{t \geq 0}, (\Theta_t)_{t \geq 0}, (\mathbf{P}_x)_{x \in \overline{\Omega}}\big)$$

the process constructed in Section 3.1. Furthermore, for an open subset G of $\overline{\Omega}$

$$\mathbf{M}^G := \big(\boldsymbol{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbf{X}_t^0)_{t \geq 0}, (\Theta_t)_{t \geq 0}, (\mathbf{P}_x)_{x \in G_\Delta}\big)$$

is called the part of the process $\mathbf{M}$ on G, where $\mathbf{X}_t^0(\omega)$ results from $\mathbf{X}_t(\omega)$ by killing the path upon leaving G for $\omega \in \boldsymbol{\Omega}$. By [FOT11, Theorem 4.4.2] the process $\mathbf{M}^G$ is associated to $(\mathcal{E}^G, D(\mathcal{E}^G))$.

Let $\mathcal{C}$ be the set of all connected components of $\overline{\Omega}_1 := \overline{\Omega} \backslash \Xi$, where

$$\Xi := \{x \in \overline{\Omega} \mid \alpha(x) = 0 \text{ or } (x \in \Gamma \text{ and } \beta(x) = 0)\} = \{\varrho = 0\} \cup \{x \in \Gamma \mid \alpha(x) = 0\}.$$

Moreover, for $G \in \mathcal{C}$ let $G_\Gamma := G \cap \Gamma$.

Condition 3.20. $cap_{\mathcal{E}}(\Xi) = 0$ and $\alpha, \beta \in C(\overline{\Omega})$.

Note that Condition 3.20 implies Condition 3.3.

Lemma 3.21. *Assume that Condition 3.20 is fulfillded. Then*

(i) *$cap_{\mathcal{E}_\Omega}(\Xi) = 0$ and $cap_{\mathcal{E}_\Gamma}(\Xi \cap \Gamma) = 0$.*

(ii) *Each $G \in \mathcal{C}$ is open in $\overline{\Omega}$ and quasi closed with respect to $\mathcal{E}$. In particular, G is T_t-invariant.*

(iii) *The assertion in (ii) holds accordingly for G and G_Γ with respect to $\mathcal{E}_\Omega$ and $\mathcal{E}_\Gamma$ respectively.*

Proof. (i) Note that $D(\mathcal{E})$ is a subset of $D(\mathcal{E}_\Omega)$ and $D(\mathcal{E}_\Gamma)$ by restriction and $\mathcal{E}_{\Omega,1}, \mathcal{E}_{\Gamma,1} \leq \mathcal{E}_1$ on this set. Let $\varepsilon > 0$. Then there exists an open set U in $\overline{\Omega}$ which contains Ξ such that $cap_{\mathcal{E}}(U) < \varepsilon$. By definition of the capacity, we get also $cap_{\mathcal{E}_\Omega}(U) < \varepsilon$ and $cap_{\mathcal{E}_\Gamma}(U \cap \Gamma) < \varepsilon$. Hence, the assertion holds true.

(ii) Let $G \in \mathcal{C}$. G is open by definition. Let $\varepsilon > 0$. We show that $\overline{\Omega} \backslash G$ is quasi open. Since $\mathrm{cap}_{\mathcal{E}}(\Xi) = 0$, there exists an open subset U of $\overline{\Omega}$ which contains Ξ such that $\mathrm{cap}_{\mathcal{E}}(U) < \varepsilon$. The set $\tilde{U} := \bigcup_{\tilde{G} \in \mathcal{C} \backslash \{G\}} \tilde{G} \cup U$ is open and contains $\overline{\Omega} \backslash G$. Moreover,

$$\mathrm{cap}_{\mathcal{E}}(\tilde{U} \backslash (\overline{\Omega} \backslash G)) \leq \mathrm{cap}_{\mathcal{E}}(U) < \varepsilon.$$

Hence, G is quasi closed.

(iii) Note that G and G_Γ are open in $\overline{\Omega}$ and Γ respectively. The remaining part of the statement follows by (i) with the same arguments as in (ii). $\qquad\square$

Remark 3.22. Let $G \in \mathcal{C}$.

   (i) Due to [FOT11, Lemma 4.6.3], the preceding lemma implies that there exists a properly exceptional set N such that $G \backslash N$ is $\mathbf{M}$-invariant in the sense that

$$\mathbf{P}_x(\mathbf{X}_t \in (G \backslash N)_\Delta \text{ for all } t \geq 0) = 1 \text{ for all } x \in G \backslash N.$$

  (ii) It is possible that $G_\Gamma = G \cap \Gamma$ is not connected in $\Gamma \backslash \Xi$. Therefore, we denote by $\mathcal{C}_G$ the set containing all connected components of G_Γ. In particular, $\bigcup_{G \in \mathcal{C}} \mathcal{C}_G$ is the set of all connected components of $\Gamma \backslash \Xi$.

 (iii) Define $F_k := \{x \in G \mid d_{\mathrm{euc}}(x, \Xi) > \frac{1}{k}\}$. This yields a sequence of open subsets of G increasing to G. For $\alpha, \beta \in C(\overline{\Omega})$, it follows that $\gamma_k := \mathrm{ess\,inf}_{x \in F_k} \varrho > 0$, $k = 1, 2, \ldots$, (with respect to the measure λ). Similarly, we define F_k for sets in $\mathcal{C}_G$. More precisely, for $A_G \in \mathcal{C}_G$ let $F_k := \{x \in A_G \mid d_{\mathrm{euc}}(x, \Xi \cap \Gamma) > \frac{1}{k}\}$ and define γ_k with respect to σ.

 (iv) By a similar argument as in (iii), L^p-norms on K with respect to the measures μ and λ (or σ) respectively are equivalent for some compact set K properly contained in some G (or A_G).

Theorem 3.23. *Suppose that Condition 3.20 is fulfilled. Then for all $G \in \mathcal{C}$ and $f \in L^1(G; \mu)$ it holds*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(\mathbf{X}_s) ds = \frac{\int_G f \, d\mu}{\mu(G)}$$

almost surely under $\mathbf{P}_x$ for quasi all $x \in G$.

Proof. Fix $G \in \mathcal{C}$. Due to [FOT11, Theorem 4.7.3(iii)], the definition of $\mathbf{M}^G$ and Remark 3.22 (i) it is sufficient to show that $(\mathcal{E}^G, D(\mathcal{E}^G))$ is irreducible recurrent. In order to deduce recurrence of $(\mathcal{E}, D(\mathcal{E}))$, by [FOT11, Theorem 1.6.3] it is enough to observe that $\mathbb{1}_{\overline{\Omega}} \in D(\mathcal{E})$ and $\mathcal{E}(\mathbb{1}_{\overline{\Omega}}, \mathbb{1}_{\overline{\Omega}}) = 0$. Hence, $\mathbb{1}_G = \mathbb{1}_G \mathbb{1}_{\overline{\Omega}} \in D(\mathcal{E}^G)$ by T_t-invariance of G and $\mathcal{E}^G(\mathbb{1}_G, \mathbb{1}_G) = 0$, since

$$0 = \mathcal{E}(\mathbb{1}_{\overline{\Omega}}, \mathbb{1}_{\overline{\Omega}}) = \mathcal{E}(\mathbb{1}_G, \mathbb{1}_G) + \mathcal{E}(\mathbb{1}_{\overline{\Omega} \backslash G}, \mathbb{1}_{\overline{\Omega} \backslash G}).$$

This implies recurrence of $(\mathcal{E}^G, D(\mathcal{E}^G))$ by [FOT11, Theorem 1.6.3]. Taking into account that the considered form is recurrent, irreducibility is equivalent to the condition that every $f \in D(\mathcal{E}^G)$ with $\mathcal{E}^G(f, f) = 0$ is μ-a.e. constant (on G) by [CF11, Theorem 2.1.11]. Let $A_G \in \mathcal{C}_G$ and denote by $(\mathcal{E}_\Gamma^{A_G}, D(\mathcal{E}_\Gamma^{A_G}))$ the part of the form $(\mathcal{E}_\Gamma, D(\mathcal{E}_\Gamma))$ on A_G. Moreover, denote by $(\mathcal{E}_\Omega^G, D(\mathcal{E}_\Omega^G))$ the part of the form $(\mathcal{E}_\Omega, D(\mathcal{E}_\Omega))$ on G. $(\mathcal{E}_\Gamma^{A_G}, D(\mathcal{E}_\Gamma^{A_G}))$ is the closure of $(\mathcal{E}_\Gamma, C^1(A_G))$ by [FOT11, Theorem 4.4.3] and thus, it is irreducible. Indeed, the closure of the pre-Dirichlet form

$$\int_{A_G} (\nabla_\Gamma f, \nabla_\Gamma g) \, d\sigma, \ f, g \in C^1(\overline{A_G})$$

on $L^2(A_G; \sigma)$ yields reflecting Brownian motion which is irreducible (see e.g. [CF11, p.128]). Hence, the closure of the form defined for functions in $C^1(A_G)$ on $L^2(A_G; \sigma)$ is also irreducible in view of [CF11, Theorem 2.1.11]. Hence, it follows by [FOT11, Corollary 4.6.4] and Remark 3.22 (iii) that $(\mathcal{E}_\Gamma^{A_G}, D(\mathcal{E}_\Gamma^{A_G}))$ is irreducible. Similarly, it holds that $(\mathcal{E}_\Omega^G, D(\mathcal{E}_\Omega^G))$ is irreducible.

Let $f \in D(\mathcal{E}^G)$ and choose a sequence $(f_k)_{k \in \mathbb{N}}$ in $C^1(G)$ such that $f_k \to f$ with respect to $\sqrt{\mathcal{E}_1^G}$. Then the restriction to Γ is by definition $\mathcal{E}_\Gamma$-Cauchy and converges to the restriction of f in $L^2(\Gamma; \beta\sigma)$. Therefore, the convergence holds also in $D(\mathcal{E}_\Gamma)$. An analogous statement holds in $D(\mathcal{E}_\Omega)$. Thus,

$$\mathcal{E}^G(f,f) = \mathcal{E}(f,f) = \lim_{k\to\infty} \mathcal{E}(f_k, f_k) = \lim_{k\to\infty}(\mathcal{E}_\Omega(f_k,f_k) + \mathcal{E}_\Gamma(f_k,f_k)) = \mathcal{E}_\Omega(f,f) + \mathcal{E}_\Gamma(f,f)$$

by definition. By invariance it holds

$$\mathcal{E}^G(f,f) = \mathcal{E}_\Omega^G(\mathbb{1}_G f, \mathbb{1}_G f) + \sum_{A_G \in \mathcal{C}_G} \mathcal{E}_\Gamma^{A_G}(\mathbb{1}_{A_G} f, \mathbb{1}_{A_G} f).$$

Therefore, $\mathcal{E}^G(f,f) = 0$ implies that each summand on the right hand side vanishes and hence, $f = c_G \; \alpha\lambda$-a.e. on $G \cap \Omega$ for some constant c_G and $f = c_{A_G} \; \beta\sigma$-a.e. on A_G for some constant c_{A_G} by [CF11, Theorem 2.1.11] and irreducibility. Thus, we can conclude

$$f = c_G \mathbb{1}_{G \cap \Omega} + \sum_{A_G \in \mathcal{C}_G} c_{A_G} \mathbb{1}_{A_G}.$$

It rests to show that $c_G = c_{A_G}$ for all $A_G \in \mathcal{C}_G$. Fix a point $z \in A_G$. Then there exists a neighborhood U of z in $\overline{\Omega}$ such that $\overline{U} \subset G$ and $\overline{U} \cap \Gamma \subset A_G$. Choose a C^∞-cutoff function η defined on $\overline{\Omega}$ which is constantly one near z and has support properly contained in U. Then it is easy to see that $\eta f \in D(\mathcal{E}^G)$ and $(\eta f_k)_{k \in \mathbb{N}}$ is an approximation for ηf whenever $(f_k)_{k \in \mathbb{N}}$ is a sequence of $C^1(G)$-functions which approximates f in $D(\mathcal{E}^G)$. In particular, this implies convergence in $L^2(U \cap \Gamma; \sigma)$ and even in $L^2(\partial(U \cap \Omega); \sigma)$. Since ηc_G is the unique continuous extension of $f|_{U \cap \Omega}$ to U, it is clear that $\eta f \in H^{1,2}(U \cap \Omega) \cap C(U \cap \Omega)$ and $\mathrm{Tr}(\eta f) = \eta c_G$, where $\mathrm{Tr} : H^{1,2}(U \cap \Omega) \to L^2(U \cap \Gamma; \sigma)$ is the (restricted) trace operator. Thus,

$$\eta c_G = \mathrm{Tr}(\eta f) = L^2(U \cap \Gamma; \sigma) - \lim_{k\to\infty} \mathrm{Tr}(\eta f_k) = L^2(U \cap \Gamma; \sigma) - \lim_{k\to\infty}(\eta f_k)|_{U \cap \Gamma} = \eta c_{A_G}.$$

Hence, $\eta c_{A_G} = \eta c_G \; \sigma$-a.e. on $U \cap \Gamma$ and therefore, $c_{A_G} = c_G$. $\qquad\square$

Corollary 3.24. *Suppose that Condition 3.20 is fulfilled. Fix a component G of $\overline{\Omega}_1$ which intersects Γ. Then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{1}_\Gamma(\mathbf{X}_s) ds = \frac{\mu(G \cap \Gamma)}{\mu(G)} \tag{3.7}$$

almost surely under $\mathbf{P}_x$ for quasi all $x \in G$.

Remark 3.25. Note that the right hand side of (3.7) is strictly positive if $\mu(G \cap \Gamma) > 0$ and there exists always some $G \in \mathcal{C}$ such that $\mu(G \cap \Gamma) > 0$, since $\mu(\Gamma) > 0$. This implies that the process sojourns arbitrarily long on Γ.

For the subsequent proposition and example we need the notion of a strongly regular Dirichlet form (see also [Stu94] and [Stu95]):

Definition 3.26 (*strong regularity*)**.** *A regular Dirichlet form $(\mathcal{G}, D(\mathcal{G}))$ on $L^2(X; m)$, where X is a connected, locally compact, separable Hausdorff space and m is a positive Radon measure with full support, is called* strongly regular, *if the topology induced by the intrinsic metric*

$$d(x,y) := \sup\{f(x) - f(y)| \; f \in D(\mathcal{G}) \cap C(X) \text{ with } \nu_{\langle f \rangle} \leq m\}, \quad x, y \in X,$$

coincides with the original topology on X. Here $\nu_{\langle f \rangle} \le m$ means that the so-called energy measure of f is absolutely continuous with respect to m and its Radon-Nikodym derivative $\frac{d\nu_{\langle f \rangle}}{dm}$ is almost everywhere less or equal than one.

Lemma 3.27. *Let $\emptyset \ne U$ be an open subset of $\overline{\Omega} \backslash \Xi$ such that $\overline{U} \subset \overline{\Omega} \backslash \Xi$. Then the restriction maps $i_1 : f \mapsto f|_{U \cap \Omega}$ and $i_2 : f \mapsto f|_{U \cap \Gamma}$ (under the condition that $U \cap \Gamma \ne \emptyset$) are continuous maps from $D(\mathcal{E})$ to $H^{1,2}(U \cap \Omega)$ and $H^{1,2}(U \cap \Gamma)$ respectively. In particular, there exists a constant $C_1 = C_1(\varrho, U) < \infty$ such that $\|f\|_{H^{1,2}(U \cap \Omega)}, \|f\|_{H^{1,2}(U \cap \Gamma)} \le C_1 \sqrt{\mathcal{E}_1(f,f)}$ for $f \in D(\mathcal{E})$.*

Proof. By continuity of α and β, there exist constants $0 < \varrho^-$ and $\varrho^+ < \infty$ such that $\varrho^- \le \varrho \le \varrho^+$ on $\overline{U}$. Let $f \in \mathcal{D}$. Then

$$\int_{U \cap \Omega} (f^2 + |\nabla f|^2) \, d\lambda \le \frac{1}{\varrho^-} \int_{U \cap \Omega} (f^2 + |\nabla f|^2) \, \alpha d\lambda$$
$$\le \frac{1}{\varrho^-} \int_{\Omega} (f^2 + |\nabla f|^2) \, \alpha d\lambda$$
$$\le \frac{1}{\varrho^-} \mathcal{E}_1(f,f) < \infty.$$

Similary, we obtain

$$\int_{U \cap \Gamma} (f^2 + |\nabla_\Gamma f|^2) \, d\sigma \le \frac{1}{\varrho^-} \mathcal{E}_1(f,f) < \infty.$$

Hence, $i_1 : \mathcal{D} \to H^{1,2}(U \cap \Omega)$ and $i_1 : \mathcal{D} \to H^{1,2}(U \cap \Omega)$ are well-defined and continuous. Therefore, the maps admit a continuous extension to $D(\mathcal{E})$. Let $f \in D(\mathcal{E})$. Then the image of f is simply the restriction of f to the respective set (see also Remark 3.22 (iv)) and thus, the restriction is an element of the corresponding Sobolev space. The last statement holds with $C_1 := \frac{1}{\varrho^-}$. $\qquad\square$

Lemma 3.28. *Let $f \in D(\mathcal{E}) \cap C(\overline{\Omega})$ and choose a sequence $(f_k)_{k \in \mathbb{N}}$ in $\mathcal{D}$ whiches converges to f with respect to $\mathcal{E}_1$. Then*

$$\nu_{\langle f_k \rangle} = |\nabla f_k|^2 \, \alpha\lambda + |\nabla_\Gamma f_k|^2 \, \beta\sigma$$

and $|\nabla_\Gamma f_k|^2 = |\nabla f_k|^2 - |nn^t \nabla f_k|^2$ for each $k \in \mathbb{N}$. Moreover, $(\nabla f_k)_{k \in \mathbb{N}}$ has the limit ∇f in $L^2(\overline{\Omega}; \alpha\lambda)$ and similarly, $(\nabla_\Gamma f_k)_{k \in \mathbb{N}}$ has the limit $\nabla_\Gamma f \in L^2(\Gamma; \beta\sigma)$. In particular the convergence holds in $L^2_{loc}(\Omega \backslash \Xi; \lambda)$ and $L^2_{loc}(\Gamma \backslash \Xi; \sigma)$. The energy measure of f is given by

$$\nu_{\langle f \rangle} = |\nabla f|^2 \, \alpha\lambda + |\nabla_\Gamma f|^2 \, \beta\sigma.$$

Proof. Let $f \in \mathcal{D}$. Define $\nu := |\nabla f|^2 \, \alpha\lambda + |\nabla_\Gamma f|^2 \, \beta\sigma$. We have to show that

$$2 \, \mathcal{E}(fg, f) - \mathcal{E}(f^2, g) = \int_{\overline{\Omega}} g \, d\nu$$

for all $g \in D(\mathcal{E}) \cap C(\overline{\Omega})$. Then the result follows by uniqueness of $\nu_{\langle f \rangle}$. Since also $\mathcal{D}$ is dense in $C(\overline{\Omega})$ with respect to $\|\cdot\|_{\sup}$, it is enough to restrict to functions $g \in \mathcal{D}$. In this case,

$2 \, \mathcal{E}(fg, f) - \mathcal{E}(f^2, g)$

$$= \int_{\Omega} (\nabla(fg), \nabla f)\alpha d\lambda + \int_{\Gamma} (\nabla_\Gamma(fg), \nabla_\Gamma f)\beta d\sigma - \frac{1}{2} \int_{\Omega} (\nabla f^2, \nabla g)\alpha d\lambda - \frac{1}{2} \int_{\Gamma} (\nabla_\Gamma f^2, \nabla_\Gamma g)\beta d\sigma$$

$$= \int_{\Omega} (\nabla(fg), \nabla f)\alpha d\lambda + \int_{\Gamma} (\nabla_\Gamma(fg), \nabla_\Gamma f)\beta d\sigma - \int_{\Omega} (\nabla f, f \, \nabla g)\alpha d\lambda - \int_{\Gamma} (\nabla_\Gamma f, f \, \nabla_\Gamma g)\beta d\sigma$$

$$= \int_{\Omega} g(\nabla f, \nabla f) \, \alpha d\lambda + \int_{\Gamma} g(\nabla_\Gamma f, \nabla_\Gamma f) \, \beta d\sigma$$

$$= \int_{\overline{\Omega}} g \, d\nu.$$

Note that $|\nabla_\Gamma f|^2 = |(E - nn^t)\nabla f|^2 = |\nabla f|^2 - |nn^t \nabla f|^2$. Replacing f by f_k yields the first statement. By [FOT11, p.124] it holds

$$\left| \left(\int_{\overline{\Omega}} g d\nu_{\langle f \rangle} \right)^{\frac{1}{2}} - \left(\int_{\overline{\Omega}} g d\nu_{\langle f_k \rangle} \right)^{\frac{1}{2}} \right| \leq \left(\int_{\overline{\Omega}} g d\nu_{\langle f - f_k \rangle} \right)^{\frac{1}{2}} \leq \sqrt{2\|f\|_{\sup} \, \mathcal{E}(f - f_k, f - f_k)}.$$

Hence,

$$\int_{\overline{\Omega}} g \, d\nu_{\langle f \rangle} = \lim_{k \to \infty} \int_{\overline{\Omega}} g \, d\nu_{\langle f_k \rangle}$$

$$= \lim_{k \to \infty} \left(\int_\Omega g \, |\nabla f_k|^2 \, \alpha d\lambda + \int_\Gamma g \, |\nabla_\Gamma f_k|^2 \, \beta d\sigma \right).$$

Define $G_j := \overline{\Omega} \backslash \overline{B_{\frac{1}{j}}(\Xi)}$ for $j \in \mathbb{N}$. Then each G_j fulfills the assumptions of Lemma 3.27 and $G_j \uparrow \overline{\Omega} \backslash \Xi$ as $j \to \infty$. This yields a weak gradient ∇f and $\nabla_\Gamma f$ on each set G_j and $G_j \cap \Gamma$ respectively. Therfore, we can define ∇f and $\nabla_\Gamma f$ globally outside Ξ and

$$\int_\Omega |\nabla f|^2 \, \alpha d\lambda \leq \liminf_{j \to \infty} \int_\Omega \mathbb{1}_{G_j} |\nabla f|^2 \, \alpha d\lambda \leq \mathcal{E}_\Omega(f, f),$$

since the last inequality holds for fixed $j \in \mathbb{N}$. The statement holds similarly for $\nabla_\Gamma f$. Applying this to $f - f_k$ finishes the proof. $\qquad \square$

Proposition 3.29. $(\mathcal{E}, D(\mathcal{E}))$ *is strongly regular.*

Proof. We show that the intrinsic metric d is equivalent to the euclidean metric d_{euc}. First, let $f_i(x) := x_i$, $x \in \overline{\Omega}$, for $i = 1, \ldots, d$. Then $f_i \in \mathcal{D}$ with $\frac{d\nu_{\langle f_i \rangle}}{d\mu} \leq 1$ a.e. and for $x, y \in \overline{\Omega}$ holds (by eventually replacing f_i by $-f_i$)

$$d(x, y) \geq \max_{i=1,\ldots,d} \left(f_i(x) - f_i(y) \right) = \max_{i=1,\ldots,d} |x_i - y_i| \geq \tilde{C}_1 \, d_{\text{euc}}(x, y)$$

for some constant $\tilde{C}_1 = \tilde{C}_1(d) < \infty$. Moreover, by Lemma 3.28

$$d(x, y) \leq \sup\{f(x) - f(y)| \, f \in D(\mathcal{E}) \cap C(\overline{\Omega}) \text{ with } \nu_{\langle f \rangle} \leq \mu\}$$

$$\leq \sup\{f(x) - f(y)| \, f \in H^{1,\infty}(\Omega) \cap C(\overline{\Omega}) \text{ with } |\nabla f| \leq 1 \text{ a.e.}\}$$

and the last expression is locally bounded by d_{euc}. Indeed, by the proof of [Alt06, Satz 8.5] every $f \in H^{1,\infty}(\Omega)$ has a unique continuous version in $C^{0,1}(\overline{\Omega})$ and there is some constant $\tilde{C}_2 = \tilde{C}_2(\Omega) < \infty$ such that

$$f(x) - f(y) \leq \tilde{C}_2 \, \|\nabla f\|_{L^\infty(\Omega)} \, d_{\text{euc}}(x, y). \qquad \square$$

Example 3.30. Assume additionally to Condition 3.1 that $\alpha, \beta \in C(\overline{\Omega})$ and the following property:

$$\mu(B_r(\Xi)) \leq C \, r^2 \text{ as } r \to 0. \tag{3.8}$$

Then, as a consequence of strong regularity, $\text{cap}_\mathcal{E}(\Xi) = 0$ by [Stu95, Theorem 3] and therefore, Theorem 3.23 applies.

3.3 $\mathcal{L}^p$-strong Feller properties

The diffusion process constructed in Section 3.2.2 has the drawback that the main result given in Theorem 3.17 only holds for quasi every starting point $x \in \overline{\Omega}$ and it is not explicitly known how this set of admissible starting points looks like. In the following, we prove regularity properties of the associated L^p-resolvent and conclude that the results

of Theorem 3.17 even hold for every starting point $x \in \overline{\Omega}_1 := \overline{\Omega} \backslash \Xi$ under additional conditions on the density, where as before

$$\Xi := \{x \in \overline{\Omega} | \ \alpha(x) = 0 \text{ or } (x \in \Gamma \text{ and } \beta(x) = 0)\} = \{\varrho = 0\} \cup \{x \in \Gamma | \ \alpha(x) = 0\}.$$

More precisely, we show the sufficient conditions given in [BGS13, Condition 1.3] in use of a regularity result from [Nit11] for $\delta = 0$ and from [War13] (see also [War12]) for $\delta = 1$. Then, we apply [BGS13, Theorem 1.4]. Moreover, we use the connection of parts of processes and parts of Dirichlet forms in order to identify the Dirichlet form of the new process with state space $\overline{\Omega}_1$. This allows to proceed again as in Section 3.2.2, but now without a set of starting points we have to exclude. Note that $\overline{\Omega}_1$ is not closed in $\mathbb{R}^d$ if $\Xi \neq \emptyset$. We use this notation in order to be consistent with [BGS13].

We denote by $(T_t)_{t \geq 0}$ the strongly continuous contraction semigroup, by $(G_\lambda)_{\lambda > 0}$ the strongly continuous contraction resolvent and by $(L, D(L))$ the generator corresponding to $(\mathcal{E}, D(\mathcal{E}))$. By the Beurling-Deny theorem there exists an associated strongly continuous contraction semigroup $(T_t^r)_{t > 0}$ on $L^r(\overline{\Omega}; \mu)$ with generator $(L_r, D(L_r))$ and resolvent $(G_\lambda^r)_{\lambda > 0}$ for every $1 \leq r < \infty$, see [LS96, Proposition 1.8] and [LS96, Remark 1.3]. If $r > 1$ then $(T_t^r)_{t > 0}$ is the restriction of an analytic semigroup by [LS96, Remark 1.2]. In this context associated means that for $f \in L^1(\overline{\Omega}; \mu) \cap L^\infty(\overline{\Omega}; \mu) = L^\infty(\overline{\Omega}; \mu)$, it holds that $T_t f = T_t^r f$ for every $t > 0$. With this notation we also have $T_t = T_t^2$ for $t \geq 0$, $G_\lambda = G_\lambda^2$ for $\lambda > 0$ and $L_2 = L$.

Assume that Condition 3.12 is fulfilled. In order to prove the required regularity result we assume additionally the following property:

Condition 3.31. *There exists $p \geq 2$ with $p > \frac{d}{2}$ and $p > d$ if $\delta = 0$ such that*

$$\frac{|\nabla \alpha|}{\alpha} \in L_{loc}^p(\overline{\Omega} \backslash \Xi; \alpha\lambda) \text{ and additionally } \frac{|\nabla_\Gamma \beta|}{\beta} \in L_{loc}^p(\Gamma \backslash \Xi; \beta\sigma) \text{ if } \delta = 1$$

or equivalently

$$\mathbb{1}_\Omega \frac{|\nabla \alpha|}{\alpha} + \delta \ \mathbb{1}_\Gamma \frac{|\nabla_\Gamma \beta|}{\beta} \in L_{loc}^p(\overline{\Omega}_1; \mu).$$

In the following, we assume Condition 3.12, Condition 3.31 and again that

(i) $\text{cap}_\mathcal{E}(\Xi) = 0$ (i.e., Condition 3.20),

which is e.g. fulfilled under the condition (3.8) given in Example 3.30.

We prove that

(ii) there exists $p > 1$ such that $D(L_p) \hookrightarrow C(\overline{\Omega}_1)$ and the embedding is locally continuous, i.e., for $x \in \overline{\Omega}_1$ there exists a $\overline{\Omega}_1$-neighborhood U and a constant $C = C(U) < \infty$ such that

$$\sup_{y \in U} |\tilde{u}(y)| \leq C \|u\|_{D(L_p)} \text{ for all } u \in D(L_p),$$

where $\tilde{u}$ denotes the continuous version of u (on $\overline{\Omega}_1$),

(iii) for each point $x \in \overline{\Omega}_1$ exists a sequence of functions $(u_n)_{n \in \mathbb{N}}$ in $D(L_p)$ such that for every $y \neq x$, $y \in \overline{\Omega}_1$, exists a u_n with $u_n(y) = 0$ and $u_n(x) = 1$.

We say that a sequence $(u_n)_{n \in \mathbb{N}}$ as in (iii) is point separating in x.

Then, as a consequence of [BGS13, Theorem 1.4], there exists a diffusion process

$$\mathbf{M} := \left(\mathbf{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbf{X}_t)_{t \geq 0}, (\Theta_t)_{t \geq 0}, (\mathbf{P}_x)_{x \in \overline{\Omega}}\right)$$

with state space $\overline{\Omega}$ which leaves $\overline{\Omega}_1$ $\mathbf{P}_x$-a.s., $x \in \overline{\Omega}_1$, invariant. The Dirichlet form associated to $\mathbf{M}$ is given by $(\mathcal{E}, D(\mathcal{E}))$ and the transition semigroup $(p_t)_{t > 0}$ of $\mathbf{M}$ is $\mathcal{L}^p$-strong Feller, i.e., $p_t(\mathcal{L}^p(\overline{\Omega}; \mu)) \subset C(\overline{\Omega}_1)$. Moreover, it solves the $(L_p, D(L_p))$ martingale problem for every point $x \in \overline{\Omega}_1$.

Lemma 3.32. *For p as in Condition 3.31 and $f \in C_c^2(\overline{\Omega}_1)$ holds*

$$L_p f = L_2 f = L f$$

and in this case Lf is explicitly given by (3.5). Moreover, if the additional stronger condition

$$\frac{|\nabla \alpha|}{\alpha} \in L^p(\overline{\Omega}; \alpha\lambda) \quad \text{and additionally} \quad \frac{|\nabla_\Gamma \beta|}{\beta} \in L^p(\Gamma; \beta\sigma) \text{ if } \delta = 1$$

is fulfilled, the statement holds even for every $f \in C^2(\overline{\Omega})$.

Proof. The statement for $p = 2$ has been proven in Proposition 3.14. Then, the general statement follows by the assumptions on α and β similar to [BG14, Lemma 2.3], since f and Lf are elements of $L^p(\overline{\Omega}; \mu)$ for $f \in C_c^2(\overline{\Omega}_1)$. Under the additional condition we even have $f, Lf \in L^p(\overline{\Omega}; \mu)$ for $f \in C^2(\overline{\Omega})$ and the statement extends to the larger class. □

In a similar way as in the case of Neumann boundary conditions (see [BG14, Section 4]) we get the following:

Theorem 3.33. *Assume that Condition 3.12 is fulfilled. Let U be an open subset of $\overline{\Omega}$ in the subspace topology. The following holds:*

(i) *$C_c^1(U) \hookrightarrow \mathcal{D} \hookrightarrow D(\mathcal{E})$.*

(ii) *Assume additionally that $\overline{U} \subset \overline{\Omega}_1$. The restriction maps i_Ω and i_Γ (supposed that $\delta = 1$ and $U \cap \Gamma \neq \emptyset$), which restrict functions from $\overline{\Omega}$ to $U \cap \Omega$ and $U \cap \Gamma$ respectively, are continuous mappings from $D(\mathcal{E})$ to $H^{1,2}(U \cap \Omega)$ and $H^{1,2}(U \cap \Gamma)$ respectively. Moreover, it holds*

$$\mathcal{E}(u, v) = \frac{1}{2} \int_{U \cap \Omega} (\nabla u, \nabla v) \, \alpha d\lambda + \frac{\delta}{2} \int_{U \cap \Gamma} (\nabla_\Gamma u, \nabla_\Gamma v) \, \beta d\sigma \tag{3.9}$$

and there exists a constant $C_2 = C_2(\alpha, \beta, d, G) < \infty$ such that

$$\|u\|_{H^{1,2}(U \cap \Omega)}^2 + \delta \|u\|_{H^{1,2}(U \cap \Gamma)}^2 \leq C_2 \mathcal{E}_1(u, u) \tag{3.10}$$

for $u \in D(\mathcal{E})$ and $v \in C_c^1(U)$.

(iii) *Let $2 \leq p < \infty$, $\gamma > 0$. Let $x \in \overline{\Omega}$ and let $U := B_R(x) = \{y \in \overline{\Omega} | \, d_{euc}(x, y) < R\}$ be an open ball around x in $\overline{\Omega}$ with radius $R > 0$ such that $\overline{U} \subset \overline{\Omega}_1$. For all $f \in L^p(\overline{\Omega}; \mu)$, we have $G_\gamma^p f \in H^{1,2}(U \cap \Omega)$ and $G_\gamma^p f \in H^{1,2}(U \cap \Gamma)$ for $\delta = 1$, whenever $U \cap \Gamma$ is non-empty. Moreover, with $u := G_\gamma^p f$ it holds*

$$\gamma \int_U uv \, d\mu + \frac{1}{2} \int_{U \cap \Omega} (\nabla u, \nabla v) \, \alpha d\lambda + \frac{\delta}{2} \int_{U \cap \Gamma} (\nabla_\Gamma u, \nabla_\Gamma v) \, \beta d\sigma = \int_U fv \, d\mu \tag{3.11}$$

for all $v \in C_c^1(U)$.

Additionally, for $R_0 > R$ such that $\overline{U_0} \subset \overline{\Omega} \backslash \{\varrho = 0\}$, where $U_0 := B_{R_0}(x)$, we have the norm inequalities

$$\|u\|_{H^{1,2}(U \cap \Omega)}^2 + \delta \|u\|_{H^{1,2}(U \cap \Gamma)}^2 \leq C_3 (\|f\|_{L^p(U_0; \lambda + \sigma)} + \|u\|_{L^p(U_0; \lambda + \sigma)})^2 \tag{3.12}$$

and

$$\|u\|_{H^{1,2}(U \cap \Omega)}^2 + \delta \|u\|_{H^{1,2}(U \cap \Gamma)}^2 \leq C_4 \|f\|_{L^p(\overline{\Omega}; \mu)}^2 \tag{3.13}$$

with constants $C_3 = C_3(\alpha, \beta, R, R_0, d, p) < \infty$ and $C_4 = 2\, C_3 < \infty$.

Proof. (i) is clear. The first part of (ii) and inequality (3.10) hold by Lemma 3.27 (the result for $\delta = 0$ holds similarly).

$$\mathcal{E}(u,v) = \frac{1}{2} \int_{U \cap \Omega} (\nabla u, \nabla v) \, \alpha d\lambda + \frac{\delta}{2} \int_{U \cap \Gamma} (\nabla_\Gamma u, \nabla_\Gamma v) \, \beta d\sigma$$

is evident for $u \in \mathcal{D}$ and $v \in C_c^1(U) \subset \mathcal{D} = C^1(\overline{\Omega})$. Fix $v \in C_c^1(U)$. Then $\mathcal{E}(\cdot, v)$ is a continuous linear functional on $D(\mathcal{E})$ with respect to the $\mathcal{E}_1^{\frac{1}{2}}$-norm. Moreover,

$$F(u) := \frac{1}{2} \int_{U \cap \Omega} (\nabla u, \nabla v) \, \alpha d\lambda + \frac{\delta}{2} \int_{U \cap \Gamma} (\nabla_\Gamma u, \nabla_\Gamma v) \, \beta d\sigma$$

is continuous on $D(\mathcal{E})$ (or rather on the space obtained by restricting functions to U) with respect to the norm given by $\|u\|_{H^{1,2}(U \cap \Omega)}^2 + \delta \|u\|_{H^{1,2}(U \cap \Gamma)}^2$, since α and β are bounded from above and from below away from zero on U (by continuity). Thus, it is also continuous with respect to the $\mathcal{E}_1^{\frac{1}{2}}$-norm in view of (3.10) and therefore, F has to coincide with $\mathcal{E}(\cdot, v)$ by uniqueness, since the equality holds on the dense subset $\mathcal{D}$. Therefore, (3.9) is established.

Next, we prove (iii). Let R and R_0 be as stated. First, we show (3.12) for $p = 2$. Choose a cutoff function η which is constantly one in $B_{R'}(x)$ for some $R_0 > R' > R$ and has compact support in $B_{R_0}(x)$. For $f \in L^2(\overline{\Omega}; \mu)$ we have $u := G_\gamma^2 f \in D(\mathcal{E})$ and it is easy to see that also $\eta u \in D(\mathcal{E})$, since ηu_n converges to ηu as $n \to \infty$ if $(u_n)_{n \in \mathbb{N}}$ approximates u in $D(\mathcal{E})$. As in (ii) it can be shown that for fixed $v \in D(\mathcal{E})$ holds

$$\mathcal{E}(v, \eta u) = \frac{1}{2} \int_{U_0 \cap \Omega} (\nabla v, \nabla(\eta u)) \, \alpha d\lambda + \frac{\delta}{2} \int_{U_0 \cap \Gamma} (\nabla_\Gamma v, \nabla_\Gamma(\eta u)) \, \beta d\sigma.$$

Note that η^2 is again a cutoff function with the properties we supposed for η. We have by calculation

$$\begin{aligned}
\mathcal{E}_\gamma(\eta u, \eta u) =& \gamma \int_{U_0} (\eta u)^2 d\mu + \frac{1}{2} \int_{U_0 \cap \Omega} (\nabla(\eta u), \nabla(\eta u)) \, \alpha d\lambda + \frac{\delta}{2} \int_{U_0 \cap \Gamma} (\nabla_\Gamma(\eta u), \nabla_\Gamma(\eta u)) \, \beta d\sigma \\
=& \mathcal{E}_\gamma(u, \eta^2 u) - \frac{1}{2} \int_{U_0 \cap \Omega} \eta u (\nabla u, \nabla \eta) \, \alpha d\lambda - \frac{\delta}{2} \int_{U_0 \cap \Gamma} \eta u (\nabla_\Gamma u, \nabla_\Gamma \eta) \, \beta d\sigma \\
& + \frac{1}{2} \int_{U_0 \cap \Omega} u (\nabla \eta, \nabla(\eta u)) \, \alpha d\lambda + \frac{\delta}{2} \int_{U_0 \cap \Gamma} u (\nabla_\Gamma \eta, \nabla_\Gamma(\eta u)) \, \beta d\sigma \\
=& \int_{U_0} f \eta^2 u \, d\mu - \frac{1}{2} \int_{U_0 \cap \Omega} \eta u (\nabla u, \nabla \eta) \, \alpha d\lambda - \frac{\delta}{2} \int_{U_0 \cap \Gamma} \eta u (\nabla_\Gamma u, \nabla_\Gamma \eta) \, \beta d\sigma \\
& + \frac{1}{2} \int_{U_0 \cap \Omega} u (\nabla \eta, \nabla(\eta u)) \, \alpha d\lambda + \frac{\delta}{2} \int_{U_0 \cap \Gamma} u (\nabla_\Gamma \eta, \nabla_\Gamma(\eta u)) \, \beta d\sigma. \quad (3.14)
\end{aligned}$$

We get with the inequality $ab \leq \frac{\varepsilon}{2} b^2 + \frac{1}{2\varepsilon} a^2$ for $\varepsilon > 0$, $a, b \geq 0$:

$$\begin{aligned}
\left| \int_{U_0 \cap \Omega} \eta u (\nabla u, \nabla \eta) \, \alpha d\lambda \right| &\leq K_1 \int_{U_0 \cap \Omega} |\eta \nabla u| |\nabla \eta| |u| \, d\lambda \\
&\leq K_2 \int_{U_0 \cap \Omega} |\eta \nabla u| |u| \, d\lambda \\
&\leq K_2 \Big(\int_{U_0 \cap \Omega} |\nabla(\eta u)| |u| \, d\lambda + \int_{U_0 \cap \Omega} |u| |\nabla \eta| |u| \, d\lambda \Big) \\
&\leq K_3 \big(\|\nabla(\eta u)\|_{L^2(U_0 \cap \Omega; \lambda)} \|u\|_{L^2(U_0 \cap \Omega; \lambda)} + \|u\|_{L^2(U_0 \cap \Omega; \lambda)}^2 \big) \\
&\leq \frac{\varepsilon}{2} \|\eta u\|_{H^{1,2}(U_0 \cap \Omega)}^2 + \big(\frac{K_3^2}{2\varepsilon} + K_3 \big) \|u\|_{L^2(U_0 \cap \Omega; \lambda)}^2
\end{aligned}$$

for suitable constants $K_1 \leq K_2 \leq K_3 < \infty$. Similarly, we get (by eventually increasing K_3) that

$$\left| \int_{U_0 \cap \Omega} u(\nabla \eta, \nabla(\eta u)) \, \alpha d\lambda \right| \leq \frac{\varepsilon}{2} \|\eta u\|_{H^{1,2}(U_0 \cap \Omega)}^2 + (\frac{K_3^2}{2\varepsilon} + K_3) \|u\|_{L^2(U_0 \cap \Omega; \lambda)}^2.$$

For the two corresponding terms in (3.14) on $U_0 \cap \Gamma$, the similar statement follows by the same arguments. Moreover, we have

$$\left| \int_{U_0} f \eta^2 u \, d\mu \right| \leq \|f\|_{L^2(U_0; \mu)} \|\eta u\|_{L^2(U_0; \mu)}$$

$$\leq \frac{1}{2} \big(\|f\|_{L^2(U_0; \mu)}^2 + \|\eta u\|_{L^2(U_0; \mu)}^2 \big)$$

$$\leq K_4 \big(\|f\|_{L^2(U_0; \lambda + \sigma)}^2 + \|\eta u\|_{L^2(U_0; \lambda + \sigma)}^2 \big)$$

for a constant $K_4 < \infty$. Together with (3.10) and (3.14) follows that there exists a constant $K_5 < \infty$ such that

$$\|\eta u\|_{H^{1,2}(U_0 \cap \Omega)}^2 + \delta \|\eta u\|_{H^{1,2}(U_0 \cap \Gamma)}^2$$
$$\leq K_5 \big(\|f\|_{L^2(U_0; \lambda + \sigma)}^2 + (1 + \frac{1}{\varepsilon}) \|u\|_{L^2(U_0; \lambda + \sigma)}^2 + \varepsilon (\|\eta u\|_{H^{1,2}(U_0 \cap \Omega)}^2 + \delta \|\eta u\|_{H^{1,2}(U_0 \cap \Gamma)}^2)) \big)$$

Choosing $\varepsilon = \frac{1}{2K_5}$ yields a constant $K_6 < \infty$ such that

$$\|u\|_{H^{1,2}(U \cap \Omega)}^2 + \delta \|u\|_{H^{1,2}(U \cap \Gamma)}^2 \leq K_6 \big(\|f\|_{L^2(U_0; \lambda + \sigma)}^2 + \|u\|_{L^2(U_0; \lambda + \sigma)}^2 \big).$$

For arbitrary $p \geq 2$ note that for $W := L^1(\overline{\Omega}; \mu) \cap L^\infty(\overline{\Omega}; \mu) \subset L^2(\overline{\Omega}; \mu) \cap L^p(\overline{\Omega}; \mu)$, W is dense in $L^p(\overline{\Omega}; \mu)$ and $G_\gamma^p f = G_\gamma^2 f$ for $f \in W$. For $f \in W$ inequality (3.12) applies, since the L^2-norm on U_0 can be estimated by the L^p-norm. Then (3.12) holds also for each $f \in L^p(\overline{\Omega}; \mu)$ by a density argument and continuity of G_γ^p. (3.13) is a direct consequence of (3.12) and the fact that G_γ^p is a contraction.

It rests to prove (3.11). For $f \in W$ and $v \in C_c^1(U)$ holds $\mathcal{E}_\gamma(G_\gamma^2 f, v) = (f, v)_{L^2(\overline{\Omega}; \mu)}$, i.e.,

$$\gamma \int_U G_\gamma^2 f v \, d\mu + \frac{1}{2} \int_{U \cap \Omega} (\nabla G_\gamma^2 f, \nabla v) \, \alpha d\lambda + \frac{\delta}{2} \int_{U \cap \Gamma} (\nabla_\Gamma G_\gamma^2 f, \nabla_\Gamma v) \, \beta d\sigma = \int_U f v \, d\mu$$

by (ii). Fix $v \in C_c^1(U)$ and let $f \in L^p(\overline{\Omega}; \mu)$. Then we can approximate f in $L^p(\overline{\Omega}; \mu)$ by functions from W due to density. Using (3.13) and continuity of the considered functionals, this proves (3.11). $\qquad \square$

Corollary 3.34. *Assume that Condition 3.12 is fulfilled. Let $2 \leq p < \infty$, $\gamma > 0$. Furthermore, let $x \in \overline{\Omega}$ and $U := B_R(x) = \{y \in \overline{\Omega} | \, d_{euc}(x, y) < R\}$ be an open ball around x in $\overline{\Omega}$ with radius $R > 0$ such that $\overline{U} \subset \overline{\Omega}_1$. For $u := G_\gamma^p f$ holds*

$$\gamma \int_U uv \, d\mu + \frac{1}{2} \int_{U \cap \Omega} (\nabla u, \nabla v) \, \alpha d\lambda + \frac{\delta}{2} \int_{U \cap \Gamma} (\nabla_\Gamma u, \nabla_\Gamma v) \, \beta d\sigma = \int_U f v \, d\mu \qquad (3.15)$$

for all $v \in \mathcal{K}$, where $\mathcal{K}$ is defined as the closure of $C_c^1(U)$ with respect to the norm given by

$$\| \cdot \|_\mathcal{K}^2 := \| \cdot \|_{H^{1,2}(U \cap \Omega)}^2 + \delta \| \cdot \|_{H^{1,2}(U \cap \Gamma)}^2.$$

Proof. We fix $f \in L^p(\overline{\Omega}; \mu)$ and $u = G_\gamma^p f$. Then, (3.15) yields continuous linear functionals on $\mathcal{K}$ and therefore, the assertion holds by density and (3.11). $\qquad \square$

Remark 3.35. We want to deduce from (3.15) that $G_\gamma^p f$ is continuous on $\overline{\Omega}_1$ for p as in Condition 3.31. Note that for interior points $x \in \Omega \backslash \Xi = \Omega \backslash \{\alpha = 0\}$ it is possible to choose R small enough such that $B_R(x) \cap \Gamma = \emptyset$. In this case, (3.15) reduces to

$$\gamma \int_U uv \, \alpha d\lambda + \frac{1}{2} \int_U (\nabla u, \nabla v) \, \alpha d\lambda = \int_U fv \, \alpha d\lambda \qquad (3.16)$$

for all $v \in H_0^{1,2}(U)$ with $f \in L^p(G; \lambda)$, i.e., this is the weak formulation of an elliptic PDE on G with Dirichlet boundary conditions. Then it is well-known by the theory of DeGiorgi-Nash-Moser that u is Hölder continuous near x for $p > \frac{d}{2}$ (see e.g. [GT01], [Sta63] or [HL97]). Thus, $x \in \Gamma \backslash \Xi$ is the case of main interest.

By Corollary 3.34 $u := G_\gamma^p f$ solves the equation (3.15). Therefore, the following theorem holds by [Nit11, Theorem 3.14] for $\delta = 0$ and [War13, Theorem 3.2] (see also [War12]) for $\delta = 1$:

Theorem 3.36. *Assume that Condition 3.12 is fulfilled. Let $p > \frac{d}{2}$, $p \geq 2$, $p > d$ if $\delta = 0$ and $\gamma > 0$ and $f \in L^p(\overline{\Omega}; \mu)$. Then $u := G_\gamma^p f \in C(\overline{\Omega}_1)$ and for every $x \in \overline{\Omega}_1$ exists a neighborhood U with $\overline{U} \subset \overline{\Omega}_1$ and a constant $C_4 = C_4(U, \alpha, \beta, d, p, \gamma) < \infty$ such that*

$$\sup_{y \in U} |\tilde{u}(y)| \leq C_4 \|f\|_{L^p(\overline{\Omega}; \mu)},$$

where $\tilde{u}$ denotes the continuous version of u on $\overline{\Omega}_1$.

Proof. For $x \in \overline{\Omega}_1$ let U be an open ball in $\overline{\Omega}$ around x such that $\overline{U} \subset \overline{\Omega}_1$. This is possible, since $\overline{\Omega}_1$ is open in $\overline{\Omega}$ by Condition 3.12. u solves (3.15) and therefore, u possesses a Hölder continuous version $\tilde{u}$ on U in view of [Nit11, Theorem 3.14] for $\delta = 0$ and [War13, Theorem 3.2] for $\delta = 1$. Moreover, the aforementioned results yield the stated norm estimate, since $u \in L^p(\overline{\Omega}; \mu)$, G_γ^p is a contraction, α, β are bounded and the $L^p(U)$-norm as well as the $L^p(U \cap \Gamma)$-norm can be estimated by the $L^p(\overline{\Omega}; \mu)$-norm.

Indeed, [War13, Theorem 3.2] is proven directly by a generalization of de Girogi's method to the Wentzell setting and is formulated for the special case that α is strictly positive and C^1 as well as β is a positive constant. Nevertheless, by the ideas of the proof of [GT01, Theorem 8.24] the proof generalizes to our setting, since the densities α and β are assumed to be continuous and therefore, they are locally on $\overline{\Omega}_1$ bounded from below away from zero. Moreover, the localization in Section 4.3 of [War13] shows that it is sufficient to consider test functions in $\mathcal{K}$, i.e., functions vanishing on $\partial U \cap \Omega$, in order to obtain the required regularity result. Hence, the claim follows by (3.15). Note that [War13, Theorem 3.2] applies directly in use of the localization to U if we assume additionally to Condition 3.12 that $\alpha \in C^1(\overline{\Omega})$ and β is a positive constant.

In [Nit11, Theorem 3.14] the problem is reduced to the interior case mentioned in Remark 3.35 and [GT01, Theorem 8.24] is used. Thus, this approach also includes a localization which implies that it is sufficient to consider test functions in $C_c^1(U)$ in order to get a local result. Hence, the claim follows again by (3.15). $\qquad \square$

Lemma 3.37. *For each point $x \in \overline{\Omega}_1$ exists a sequence $(u_n)_{n \in \mathbb{N}}$ in $C_c^\infty(\overline{\Omega}_1) \subset D(L_p)$, $p > 1$, that is point separating in x.*

Proof. Fix $x \in \overline{\Omega}_1$ and $n \in \mathbb{N}$. Then it is clear that we can find a function $\tilde{u}_n$ in $C_c^\infty(\mathbb{R}^d)$ such that $\tilde{u}_n(x) = 1$ and $\text{supp}(\tilde{u}_n) \subset B_{\frac{1}{n}}(x)$. Define $u_n := \tilde{u}_{n+m}|_{\overline{\Omega}}$ for m large enough. $\qquad \square$

Theorem 3.38. *Assume that Condition 3.12, Condition 3.20 and Condition 3.31 are fulfilled. Then there exists a conservative diffusion process*

$$\mathbf{M} = \left(\mathbf{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbf{X}_t)_{t \geq 0}, (\Theta_t)_{t \geq 0}, (\mathbf{P}_x)_{x \in \overline{\Omega}_1} \right)$$

with state space $\overline{\Omega}_1$ such that

$$\mathbf{X}_t = x + \int_0^t \mathbb{1}_\Omega(\mathbf{X_s})dB_s + \int_0^t \mathbb{1}_\Omega(\mathbf{X}_s)\frac{1}{2}\frac{\nabla\alpha}{\alpha}(\mathbf{X}_s)ds$$

$$+ \delta \int_0^t \mathbb{1}_\Gamma(\mathbf{X}_s)P(\mathbf{X}_s)dB_s - \delta \int_0^t \mathbb{1}_\Gamma(\mathbf{X}_s)\frac{1}{2}\kappa(\mathbf{X}_s)n(\mathbf{X}_s)ds$$

$$+ \delta \int_0^t \mathbb{1}_\Gamma(\mathbf{X}_s)\frac{1}{2}\frac{\nabla_\Gamma\beta}{\beta}(\mathbf{X}_s)ds - \int_0^t \mathbb{1}_\Gamma(\mathbf{X}_s)\frac{1}{2}\frac{\alpha}{\beta}(\mathbf{X}_s)n(\mathbf{X}_s)ds$$

almost surely under $\mathbf{P}_x$ for every $x \in \overline{\Omega}_1$. Moreover, its Dirichlet form is given by $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\overline{\Omega}_1; \mu)$ and the transition semigroup $(p_t)_{t>0}$ of $\mathbf{M}$ is $\mathcal{L}^p$-strong Feller, i.e., $p_t(\mathcal{L}^p(\overline{\Omega}_1; \mu)) \subset C(\overline{\Omega}_1)$. In particular, $(p_t)_{t>0}$ it strong Feller, i.e., $p_t(\mathcal{B}_b(\overline{\Omega}_1)) \subset C(\overline{\Omega}_1)$. Furthermore, Theorem 3.23 holds for every starting point in $\overline{\Omega}_1$. In particular, $\mathbf{M}$ has a sticky boundary behavior, i.e.,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{1}_\Gamma(\mathbf{X}_s)ds > 0$$

$\mathbf{P}_x$-a.s. for every $x \in \overline{\Omega}_1$ such that x is in a component of $\overline{\Omega}_1$ intersecting Γ.

Proof. First, we have to check the assumptions of [BGS13, Theorem 1.4], namely that $D(L_p) \hookrightarrow C(\overline{\Omega}_1)$, that the embedding is locally continuous and the point separating property. It holds $D(L_p) = G_\gamma^p L^p(\overline{\Omega}; \mu)$ and hence, we have $D(L_p) \hookrightarrow C(\overline{\Omega}_1)$ and moreover, for $u = G_1^p f \in D(L_p)$ it holds locally

$$\sup_{y \in U} |\tilde{u}(y)| \le C_4 \|f\|_{L^p(\overline{\Omega}; \mu)}$$
$$= C_4 \|(1 - L_p)u\|_{L^p(\overline{\Omega}; \mu)} \le C_4(\|u\|_{L^p(\overline{\Omega}; \mu)} + \|L_p u\|_{L^p(\overline{\Omega}; \mu)}) = C_4 \|u\|_{D(L_p)}.$$

The existence of a point spearating sequence for each point $x \in \overline{\Omega}_1$ follows by Lemma 3.32 and Lemma 3.37. This assures the existence of a process $\mathbf{M}$ with state space $\overline{\Omega}$ as stated at the beginning of this section such that $\overline{\Omega}_1$ is invariant for all starting points in $\overline{\Omega}_1$ and its transition semigroup is $\mathcal{L}^p$-strong Feller. In particular, the process $\mathbf{M}$ solves the $(L_p, D(L_p))$ martingale problem and L_p is given as in Proposition 3.14 for functions in $C_c^2(\overline{\Omega}_1)$ (see also Lemma 3.32). Since $\mathcal{B}_b(\overline{\Omega}) \subset L^p(\overline{\Omega}; \mu)$, it follows that the process is also strong Feller in the sense that the transition semigroup maps $\mathcal{B}_b(\overline{\Omega})$ into $C(\overline{\Omega}_1)$. By admitting only starting points in $\overline{\Omega}_1$ and invariance, we obtain a process $\mathbf{M}$ as stated. Then, this process is ($\mathcal{L}^p$-)strong Feller. In particular, the absolute continuity condition given in [FOT11, (4.2.9)] is fulfilled. The associated Dirichlet form is given by $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\overline{\Omega}_1; \mu)$ by Definition 3.19 and the following remark on parts of processes. Similarly, the generator is also given by $(L, D(L))$ considered as an operator on $L^2(\overline{\Omega}_1; \mu)$. For this reason, $\mathbf{M}$ solves the $(L, D(L))$-martingale problem under $\mathbf{P}_x$ for every $x \in \overline{\Omega}_1$. By Proposition 3.14 we even get that $C^2(\overline{\Omega}) \subset D(L)$. Hence, by the same arguments as in Theorem 3.17 we can conclude that $\mathbf{M}$ solves the given SDE and furthermore, the ergodicity result holds accordingly for every starting point $x \in \overline{\Omega}_1$, since the required properties directly transfer from the $L^2(\overline{\Omega}; \mu)$ to the $L^2(\overline{\Omega}_1; \mu)$ setting. $\square$

In the case $\delta = 0$, we obtain the following corollary on distorted Brownian motion in Ω with immediate reflection in normal direction at Lipschitz boundaries Γ and singular drifts:

Corollary 3.39. *Assume that Γ is Lipschitz continuous, $\alpha \in C(\overline{\Omega})$, $\alpha > 0$ λ-a.e., $\sqrt{\alpha} \in H^{1,2}(\Omega)$,*

$$\frac{|\nabla\alpha|}{\alpha} \in L^p_{loc}(\overline{\Omega}\backslash\{\alpha = 0\}; \alpha\lambda) \quad \text{for some } p > d, \; p \ge 2,$$

and $cap_{\mathcal{E}}(\{\alpha = 0\}) = 0$. *Then, there exists a diffusion process*

$$\tilde{\mathbf{M}} = \left(\mathbf{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, (\mathbf{Y}_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\overline{\Omega}\setminus\{\alpha=0\}}\right)$$

with state space $\overline{\Omega}\setminus\{\alpha = 0\}$ *such that*

$$\mathbf{Y}_t = x + \tilde{B}_t + \int_0^t \frac{1}{2}\frac{\nabla\alpha}{\alpha}(\mathbf{Y}_s)\,ds - \int_0^t \frac{1}{2}\alpha(\mathbf{Y}_s)n(\mathbf{Y}_s)\,dl_s^{\mathbf{Y}}$$

almost surely under $\mathbf{P}_x$ *for every* $x \in \overline{\Omega}\setminus\{\alpha = 0\}$*, where* $(\tilde{B}_t)_{t\geq 0}$ *is a* d*-dimensional standard Brownian motion and* $(l_t^{\mathbf{Y}})_{t\geq 0}$ *is the boundary local time of* $\tilde{\mathbf{M}}$*, i.e.,* $l_0^{\mathbf{Y}} = 0$*,* $(l_t^{\mathbf{Y}})_{t\geq 0}$ *is non-decreasing and*

$$\int_0^t \mathbb{1}_{\Gamma}(\mathbf{Y}_s)\,dl_s^{\mathbf{Y}} = l_t^{\mathbf{Y}}, \quad t \geq 0.$$

Proof. Set $\beta(x) := 1$ for every $x \in \overline{\Omega}$. Then, the assumptions of Theorem 3.38 are fulfilled with $\delta = 0$ and it exists a process $\mathbf{M}$ as stated. In particular, we have $\Xi = \{\alpha = 0\}$. Similarly as in [EP14, Theorem 5], define

$$\tau(t) := \int_0^t \mathbb{1}_{\Omega}(\mathbf{X}_s)\,ds, \quad t \geq 0.$$

It holds $\tau(0) = 0$, $\tau(t) \to \infty$ as $t \to \infty$ due to the ergodicity and $(\tau(t))_{t\geq 0}$ is non-decreasing. Thus, the right-inverse $(A_t)_{t\geq 0}$ of $(\tau(t))_{t\geq 0}$ exists and we define the time changed process

$$\mathbf{Y}_t := \mathbf{X}_{A_t}, \quad t \geq 0,$$

with underlying time changed σ-algebra $\tilde{\mathcal{F}}$ and filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$. It holds

$$\langle \int_0^{A_\cdot} \mathbb{1}_{\Omega}(\mathbf{X}_s)dB_s\rangle_t = \int_0^{A_t} \mathbb{1}_{\Omega}(\mathbf{X}_s)\,ds = \int_0^{A_t} d\tau(s) = \tau(A_t) = t.$$

Hence, there exists a standard Brownian motion $(\tilde{B}_t)_{t\geq 0}$ such that

$$\int_0^{A_t} \mathbb{1}_{\Omega}(\mathbf{X}_s)dB_s = \tilde{B}_t, \quad t \geq 0.$$

Moreover, we have

$$\int_0^{A_t} \mathbb{1}_{\Omega}(\mathbf{X}_s)\frac{1}{2}\frac{\nabla\alpha}{\alpha}(\mathbf{X}_s)ds = \int_0^{A_t} \frac{1}{2}\frac{\nabla\alpha}{\alpha}(\mathbf{X}_s)d\tau(s) = \int_0^t \frac{1}{2}\frac{\nabla\alpha}{\alpha}(\mathbf{X}_{A_s})ds = \int_0^t \frac{1}{2}\frac{\nabla\alpha}{\alpha}(\mathbf{Y}_s)ds$$

for $t \geq 0$. Set $l_t^{\mathbf{X}} := \int_0^t \mathbb{1}_{\Gamma}(\mathbf{X}_s)\,ds$ and define $l_t^{\mathbf{Y}} := l_{A_t}^{\mathbf{X}}$ for $t \geq 0$. It holds

$$t = \tau(A_t) = \int_0^{A_t} \mathbb{1}_{\Omega}(\mathbf{X}_s)\,ds = A_t - l_t^{\mathbf{Y}}, \quad t \geq 0.$$

Hence, $A_t = t + l_t^{\mathbf{Y}}$ for $t \geq 0$. Moreover,

$$\int_0^t \mathbb{1}_{\Gamma}(\mathbf{Y}_s)\,ds = \int_0^{A_t} \mathbb{1}_{\Gamma}(\mathbf{X}_s)\,d\tau(s) = \int_0^{A_t} \mathbb{1}_{\Gamma}(\mathbf{X}_s)\,\mathbb{1}_{\Omega}(\mathbf{X}_s)\,ds = 0$$

for $t \geq 0$. Consequently,

$$l_t^{\mathbf{Y}} = \int_0^{A_t} \mathbb{1}_{\Gamma}(\mathbf{X}_s)\,ds = \int_0^t \mathbb{1}_{\Gamma}(\mathbf{Y}_s)\,dA_s = \int_0^t \mathbb{1}_{\Gamma}(\mathbf{Y}_s)\,dl_s^{\mathbf{Y}}. \qquad \square$$

Remark 3.40. Note that Γ is only assumed to be Lipschitz continuous in Theorem 3.38 for $\delta = 0$ and Corollary 3.39. This weak assumption is possible, since we do not need to assume an additional boundary condition in order to identify elements in the domain of the L^p-generator in the sticky boundary setting. The Wentzell boundary condition is rather contained in the measure μ in terms of the surface measure σ. Therefore, we can specify a point separating sequence in Lemma 3.37. As a consequence, the outward normal direction $n(x)$ is only well-defined for σ-a.e. $x \in \Gamma$. Nevertheless, it is reasonable that the constructed processes do not hit such non-smooth boundary points for $t > 0$ (set of Hausdorff dimension $d - 2$) and starting in a non-smooth boundary point does not require to define a normal direction. A similar result to Corollary 3.39 can e.g. be found in [FT95]. Nevertheless, as far as we know Corollary 3.39 extends previous results, since we provide an additional singular drift resulting from α.

4 Interacting particle systems with sticky boundary

4.1 The Dirichlet form and the associated Markov process

4.1.1 General setting

Assume that $\Gamma := \partial \Omega$ is Lipschitz continuous. Let $(\mathcal{G}, D(\mathcal{G}))$ be the recurrent, strongly local, regular, symmetric Dirichlet form on $L^2(\overline{\Omega}; \lambda + \sigma)$ in accordance with Theorem 3.8 for $\alpha = \beta = \mathbb{1}_{\overline{\Omega}}$. Set $\Lambda := \overline{\Omega}^N$. Note that $\Lambda \subset \mathbb{R}^{Nd}$ is connected and compact. In the following we use the product measure $\prod_{i=1}^N \mu_i$ on Λ, where $\mu_i := \lambda_i + \sigma_i$ is defined on $\overline{\Omega}$ and the index i gives reference to the corresponding coordinate. For functions $f, g \in C^1(\Lambda)$, $i \in I$ and $x^j \in \overline{\Omega}$ for $j \in I$, $j \neq i$, define

$$\mathcal{E}^i(f,g)(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^N)$$
$$:= \mathcal{G}(f(x^1, \ldots, x^{i-1}, \cdot, x^{i+1}, \ldots, x^N), g(x^1, \ldots, x^{i-1}, \cdot, x^{i+1}, \ldots, x^N)).$$

Define the symmetric bilinear form $(\tilde{\mathcal{E}}, \mathcal{D})$ by

$$\tilde{\mathcal{E}}(f,g) := \sum_{i=1}^N \int_{\overline{\Omega}^{N-1}} \mathcal{E}^i(f,g) \prod_{j \neq i} d\mu_j \quad \text{for } f, g \in \mathcal{D} := C^1(\Lambda). \tag{4.1}$$

Using the definition of the form $\mathcal{G}$ yields

$$\tilde{\mathcal{E}}(f,g) = \frac{1}{2} \int_\Lambda \underbrace{\sum_{i=1}^N \left(\mathbb{1}_{\Lambda^{i,\Omega}} \left(\nabla_i f, \nabla_i g \right) + \delta \, \mathbb{1}_{\Lambda^{i,\Gamma}} \left(\nabla_{\Gamma,i} f, \nabla_{\Gamma,i} g \right) \right)}_{=:\Gamma(f,g)} \prod_{j=1}^N d\mu_j, \tag{4.2}$$

where $\Lambda^{i,\Omega} := \{x = (x_1, \ldots, x_N) \in \Lambda | \, x_i \in \Omega\}$ and $\Lambda^{i,\Gamma} := \{x = (x_1, \ldots, x_N) \in \Lambda | \, x_i \in \Gamma\}$. In particular, $\Lambda^{i,\Omega} \, \dot{\cup} \, \Lambda^{i,\Gamma} = \Lambda$ for every $i = 1, \ldots, N$. Here, the subindex $i = 1, \ldots, N$ in ∇_i and $\nabla_{\Gamma,i}$ refers to the gradient with respect to the i-th component in $x = (x^1, \ldots, x^N) \in \Lambda$ with $x^i \in \overline{\Omega}$. In the same way, we use the notation Δ_i, $i = 1, \ldots, N$, for the Laplacian with respect to the i-th component.

Condition 4.1. $\varrho \in L^1(\Lambda; \prod_{i=1}^N \mu_i)$, $\varrho > 0 \prod_{i=1}^N \mu_i$-a.e..

Define μ by $\mu := \varrho \prod_{j=1}^N \mu_j$ and $(\mathcal{E}, \mathcal{D})$ by

$$\mathcal{E}(f,g) := \frac{1}{2} \int_\Lambda \sum_{i=1}^N \left(\mathbb{1}_{\Lambda^{i,\Omega}} \left(\nabla_i f, \nabla_i g \right) + \delta \, \mathbb{1}_{\Lambda^{i,\Gamma}} (\nabla_{\Gamma,i} f, \nabla_{\Gamma,i} g) \right) \varrho \prod_{j=1}^N d\mu_j \qquad (4.3)$$

$$= \frac{1}{2} \int_\Lambda \Gamma(f,g) \, \varrho \prod_{j=1}^N d\mu_j$$

$$= \frac{1}{2} \int_\Lambda \Gamma(f,g) \, d\mu$$

for $f, g \in \mathcal{D}$. Note that the case $\delta = 0$ corresponds to the setting of a system of particles which has a sticky but static boundary behavior. Then, the bilinear form $(\mathcal{E}, \mathcal{D})$ can be written in the simpler form

$$\mathcal{E}(f,g) = \frac{1}{2} \int_\Lambda \sum_{i=1}^N \mathbb{1}_{\Lambda^{i,\Omega}} (\nabla_i f, \nabla_i g) \, \varrho \prod_{j=1}^N d\mu_j \quad \text{for } f, g \in \mathcal{D}.$$

Remark 4.2. In the case of immediately reflecting (Neumann) boundary condition the invariant measure for the corresponding diffusion on $\overline{\Omega}$ (for $N = 1$) is given by $\varrho \lambda$. For this reason, the invariant measure for an interacting N particle system, $N \in \mathbb{N}$, is absolutely continuous with respect to the Lebesgue measure on $\overline{\Omega}^N$. Thus, the cases $N = 1$ and $N > 1$ can be unified. In the case of a sticky boundary behavior, this is not possible anymore, since the surface measure σ is involved.

By the fact that μ is a Borel measure on Λ we get again the following result:

Proposition 4.3. *Under Condition 4.1 we have that $C^\infty(\Lambda)$ is dense in $L^2(\Lambda; \mu)$.*

Although $(\mathcal{E}, \mathcal{D})$ is given in (4.3) by a square field operator, it is sometimes useful to rewrite $(\mathcal{E}, \mathcal{D})$ as sum of bilinear forms. Define $\Lambda_B := \{x \in \Lambda | \, x_i \in \Omega \text{ for } i \in B, \, x_i \in \Gamma \text{ for } i \in I \backslash B\}$ and $\nu_B := \prod_{i \in B} \lambda_i \prod_{i \in I \backslash B} \sigma_i$. Then

$$\Lambda = \dot{\bigcup}_{B \subset I} \Lambda_B \quad \text{and} \quad \mu = \sum_{B \subset I} \underbrace{\varrho \, \nu_B}_{=: \mu_B}.$$

In this terms it holds

$$\mathcal{E}(f,g) = \sum_{\emptyset \neq B \subset I} \mathcal{E}_B(f,g) \quad \text{for } f, g \in \mathcal{D},$$

where

$$\mathcal{E}_B(f,g) := \frac{1}{2} \int_{\Lambda_B} \sum_{i \in B} (\nabla_i f, \nabla_i g) + \delta \sum_{i \in I \backslash B} (\nabla_{\Gamma,i} f, \nabla_{\Gamma,i} g) \, d\mu_B.$$

Moreover, define for $x \in \Gamma^{N-|B|}$, $B \neq \emptyset$, and $\varrho \in L^1(\Lambda; \prod_{i=1}^N \mu_i)$

$$R_\varrho^\Omega(B, x) := \{y \in \Omega^{|B|} | \int_{\{z \in \Omega^{|B|} | \, |z-y| < \varepsilon\}} \varrho^{-1} \prod_{i \in B} \lambda_i < \infty \text{ for some } \varepsilon > 0\}.$$

The dependence of x is given in the sense that the variables of ϱ given by the index set $I \backslash B$ are fixed by the components of x. Since ϱ is an element of $L^1(\Lambda_B; \mu_B)$, $R_\varrho^\Omega(B, x)$ is only defined for $\prod_{i \in I \backslash B} \sigma_i$-a.e. $x \in \Gamma^{N-|B|}$. Similarly, for $y \in \Omega^{|B|}$ let $R_\varrho^\Gamma(B, y)$ be given by

$$R_\varrho^\Gamma(B, y) := \{x \in \Gamma^{N-|B|} | \int_{\{z \in \Gamma^{N-|B|} | \, |z-x| < \varepsilon\}} \varrho^{-1} \prod_{i \in I \backslash B} \sigma_i < \infty \text{ for some } \varepsilon > 0\}.$$

In this case, the variables of ϱ given by the index set B are fixed by the components of y and $R_\varrho^\Gamma(B, y)$ is only defined for $\prod_{i \in B} \lambda_i$-a.e. $y \in \Omega^{|B|}$. Note that in both cases B determines the components which are *not* at the boundary.

The following condition is a generalized version of the usual Hamza condition (see e.g. [MR92, Chapter II, (2.4)]):

Condition 4.4 (Hamza condition)**.** *It holds*

(H1) $\varrho = 0$ $\prod_{i \in B} \lambda_i$*-a.e. on* $\Omega^{|B|} \setminus R_\varrho^\Omega(B, x)$ *for* $\prod_{i \in I \setminus B} \sigma_i$*-a.e. $x \in \Gamma^{N-|B|}$ for every* $\emptyset \neq B \subset I$

and if $\delta = 1$ additionally

(H2) $\varrho = 0$ $\prod_{i \in I \setminus B} \sigma_i$*-a.e. on* $\Gamma^{N-|B|} \setminus R_\varrho^\Gamma(B, y)$ *for* $\prod_{i \in B} \lambda_i$*-a.e. $y \in \Omega^{|B|}$ for every $B \subsetneq I$.*

(For $B = I$ the condition (H1) and for $B = \emptyset$ the condition (H2) reduce to the ordinary Hamza condition.)

Remark 4.5.   (i) Condition 4.4 is a natural generalization of the ordinary Hamza condition, since in the present setting of sticky particles we are also interested in dynamics whenever one (or several) particles are located at the boundary. The set B determines the components inside Ω and its complement $I \setminus B$ the components on Γ. Thus, *(H1)* ensures that the Hamza condition for the components inside Ω is fulfilled, wherever the remaining components stick on Γ. Since we are also interested in dynamics on Γ if $\delta = 1$, *(H2)* is the corresponding condition in this case.

   (ii) For $\Omega = (0, \infty)$ *(H1)* of Condition 4.4 coincides with [FGV16, Condition 2.7] (disregarding that $(0, \infty)$ is unbounded), since in this case the surface measure on Γ reduces to the case of the point measure in 0.

Remark 4.6. If ϱ is e.g. continuous on Λ and positive $\prod_{i=1}^N \mu_i$-a.e., then ϱ is outside the set $\{\varrho = 0\}$ locally bounded away from zero and hence, $R_\varrho^\Omega(B, x) = \{y \in \Omega^{|B|} |\ \varrho(z_{(B,x,y)}) > 0\}$ and $R_\varrho^\Gamma(B, y) = \{x \in \Gamma^{N-|B|} |\ \varrho(z_{(B,x,y)}) > 0\}$, where

$$z_{(B,x,y)}^i = \begin{cases} y^{\gamma_B(i)}, & \text{if } i \in B \\ x^{i - \gamma_B(i)}, & \text{if } i \in I \setminus B \end{cases}$$

with $\gamma_B : I \to \{1, \ldots, |B|\}, i \mapsto |\{1 \leq j \leq i |\ j \in B\}|$. Hence,

$$\Omega^{|B|} \setminus R_\varrho^\Omega(B, x) = \{y \in \Omega^{|B|} |\ \varrho(z_{(B,x,y)}) = 0\}$$

and

$$\Gamma^{N-|B|} \setminus R_\varrho^\Gamma(B, x) = \{y \in \Gamma^{N-|B|} |\ \varrho(z_{(B,x,y)}) = 0\}$$

for every $x \in \Gamma^{N-|B|}$, $y \in \Omega^{|B|}$ and Condition 4.4 is fulfilled.

Lemma 4.7. *Suppose that Condition 4.1 and Condition 4.4 are satisfied. Then the bilinear form $(\mathcal{E}, \mathcal{D})$ is closable on $L^2(\Lambda; \mu)$.*

Proof. Let $(f_k)_{k \in \mathbb{N}}$ be an $\mathcal{E}$-Cauchy sequence in $\mathcal{D}$ such that $f_k \to 0$ in $L^2(\Lambda; \mu)$ as $k \to \infty$. In particular, $(f_k)_{k \in \mathbb{N}}$ is $\mathcal{E}_B$-Cauchy and converges to 0 in $L^2(\Lambda_B; \mu_B)$ for every $\emptyset \neq B \subset I$. Thus, by definition of $\mathcal{E}_B$ we have that $(\partial_j f_k)_{k \in \mathbb{N}}$ is Cauchy in $L^2(\Lambda_B; \mu_B)$ for every $j = d(i-1) + l$, where $i \in B$ and $l \in \{1, \ldots, d\}$ and hence, $\partial_j f_k \to h_j \in L^2(\Lambda_B; \mu_B)$ as $k \to \infty$. In other words,

$$\int_{\Gamma^{N-|B|}} \int_{\Omega^{|B|}} (\partial_j f_k - h_j)^2 \varrho \prod_{i \in B} \lambda_i \prod_{i \in I \setminus B} \sigma_i \to 0 \quad \text{as } k \to \infty. \tag{4.4}$$

Therefore, it exists a subsequence $(\partial_j f_{k_l})_{l \in \mathbb{N}}$ such that $\partial_j f_{k_l} \to h_j$ as $l \to \infty$ in the space $L^2(\Omega^{|B|}; \varrho \prod_{i \in B} \lambda_i) \prod_{i \in I \setminus B} \sigma_i$-a.e. and similarly, $f_{k_l} \to 0$ as $l \to \infty$ in $L^2(\Omega^{|B|}; \varrho \prod_{i \in B} \lambda_i)$

$\prod_{i \in I \setminus B} \sigma_i$-a.e.. This implies that $h_j = 0$ on $\Omega^{|B|} \varrho \prod_{i \in B} \lambda_i$-a.e. $\prod_{i \in I \setminus B} \sigma_i$-a.e. by *(H1)* of Condition 4.4 (see [MR92, Chapter II, Section 2a)]) and hence, $h_j = 0$ μ_B-a.e. on Λ_B. In the case $\delta = 1$, we obtain a similar statement for the components of $\nabla_{\Gamma, i} f_k$, $i \in I \setminus B$, $k \in \mathbb{N}$, by considering the term in analogy to (4.4) and integrating first with respect to $\prod_{i \in I \setminus B} \sigma_i$ and afterwards with respect to $\prod_{i \in B} \lambda_i$. By Fatou's lemma holds

$$\mathcal{E}(f_k, f_k) \leq \liminf_{l \to \infty} \mathcal{E}(f_k - f_{k_l}, f_k - f_{k_l}) \to 0 \quad \text{as } k \to \infty. \qquad \square$$

We denote the closure of $(\mathcal{E}, \mathcal{D})$ on $L^2(\Lambda; \mu)$ by $(\mathcal{E}, D(\mathcal{E}))$.

Proposition 4.8. *Suppose that Condition 4.1 and Condition 4.4 are satisfied. Then $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric, regular Dirichlet form.*

Proof. By Proposition 4.3 and Lemma 4.7 $(\mathcal{E}, \mathcal{D})$ is symmetric, densely defined and closable with closure $(\mathcal{E}, D(\mathcal{E}))$ which is also symmetric. Moreover, by [MR92, Chapter I, Prop. 4.10] and the representation (4.3), $(\mathcal{E}, D(\mathcal{E}))$ possesses the Markov property. Finally, $C^\infty(\Lambda) \subset C^1(\Lambda) \subset D(\mathcal{E}) \cap C(\Lambda)$ implies that $D(\mathcal{E}) \cap C(\Lambda)$ is dense in $D(\mathcal{E})$ with respect to the $\mathcal{E}_1^{\frac{1}{2}}$-norm as well as in $C(E)$ with respect to the sup-norm. Hence, $(\mathcal{E}, D(\mathcal{E}))$ is regular. $\qquad \square$

By the same arguments as in Proposition 3.7 it holds:

Proposition 4.9. *Suppose that Condition 4.1 and Condition 4.4 are satisfied. Then the symmetric, regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is strongly local and recurrent.*

We summarize the preceding results in the following theorem:

Theorem 4.10. *Assume that Condition 4.1 and Condition 4.4 are fulfilled. Then the symmetric and positive definite bilinear form $(\mathcal{E}, \mathcal{D})$ is densely defined and closable on $L^2(\Lambda; \mu)$. Its closure $(\mathcal{E}, D(\mathcal{E}))$ is a recurrent, strongly local, regular, symmetric Dirichlet form on $L^2(\Lambda; \mu)$.*

In analogy to Theorem 3.9 we can conclude the following:

Theorem 4.11. *Suppose that Condition 4.1 and Condition 4.4 are satisfied. Then there exists a conservative diffusion process (i.e. a strong Markov process with continuous sample paths and infinite life time)*

$$\mathbf{M} := \big(\mathbf{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbf{X}_t)_{t \geq 0}, (\Theta_t)_{t \geq 0}, (\mathbf{P}_x)_{x \in \Lambda} \big)$$

with state space Λ which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$, i.e., for all ($\mu$-versions of) $f \in \mathcal{B}_b(\Lambda) \subset L^2(\Lambda; \mu)$ and all $t > 0$ the function

$$\Lambda \ni x \mapsto p_t f(x) := \mathbb{E}_x \big(f(\mathbf{X}_t) \big) := \int_\Lambda f(\mathbf{X}_t) d\mathbf{P}_x \in \mathbb{R}$$

is a quasi continuous version of $T_t f$. $\mathbf{M}$ is up to μ-equivalence unique. In particular, $\mathbf{M}$ is μ-symmetric (μ is stationary), i.e.,

$$\int_\Lambda p_t f \ g \ d\mu = \int_\Lambda f \ p_t g \ d\mu \quad \text{for all } f, g \in \mathcal{B}_b(\Lambda) \ \text{ and all } t > 0,$$

and has μ as invariant measure (μ is reversible), i.e.,

$$\int_\Lambda p_t f \ d\mu = \int_\Lambda f \ d\mu \quad \text{for all } f \in \mathcal{B}_b(\Lambda) \ \text{ and all } t > 0.$$

Remark 4.12. Note that $\mathbf{M}$ is canonical, i.e., $\mathbf{\Omega} = C(\mathbb{R}_+, \Lambda)$ and $\mathbf{X}_t(\omega) = \omega(t)$, $\omega \in \mathbf{\Omega}$. For each $t \geq 0$ we denote by $\Theta_t : \mathbf{\Omega} \to \mathbf{\Omega}$ the shift operator defined by $\Theta_t(\omega) = \omega(\cdot + t)$ for $\omega \in \mathbf{\Omega}$ such that $\mathbf{X}_s \circ \Theta_t = \mathbf{X}_{s+t}$ for all $s \geq 0$. We take into account to extend the setting to $C(\mathbb{R}_+, \mathbb{R}^{Nd})$ by neglecting paths leaving Λ.

4.1.2 Densities with product structure

We introduce a special case of the setting given in Section 4.1.1 which will be of particular importance later on.

Condition 4.13. *Assume that ϱ is of the form*

$$\varrho(x) = \phi(x) \prod_{i=1}^{N} \varrho_i(x^i) \quad \text{for } x = (x^1, \ldots, x^N) \in \Lambda. \tag{4.5}$$

$\varrho_i \in L^1(\overline{\Omega}; \lambda + \sigma)$, $i = 1, \ldots, N$, *is given as in Condition 3.1 for some* $\alpha_i \in L^1(\Omega; \lambda)$, $\alpha_i > 0$ *λ-a.e. and* $\beta_i \in L^1(\Gamma; \sigma)$, $\beta_i > 0$ *σ-a.e. such that the respective Hamza conditions are fulfilled (see Condition 3.3). Moreover,* ϕ *is a* $\prod_{i=1}^{N}(\alpha_i \lambda_i + \beta_i \sigma_i)$*-a.e. positive, real valued, measurable function on* Λ *such that* $\phi \in L^1(\Lambda; \prod_{i=1}^{N}(\alpha_i \lambda_i + \beta_i \sigma_i))$. *Furthermore, we assume that* ϕ *fulfills Condition 4.4.*

Remark 4.14. Note that Condition 4.13 implies Condition 4.1 and Condition 4.4.

Under these conditions it is also possible to consider the form defined in (4.3) from a different point of view. Define the form $(\mathcal{E}^i, D(\mathcal{E}^i))$, $i = 1, \ldots, N$, as the closure of the bilinear form

$$\frac{1}{2} \int_{\Omega} (\nabla f, \nabla g) \, \alpha_i d\lambda + \frac{\delta}{2} \int_{\Gamma} (\nabla_\Gamma f, \nabla_\Gamma g) \, \beta_i d\sigma \quad \text{for } f, g \in C^1(\overline{\Omega}) \tag{4.6}$$

on $L^2(\overline{\Omega}; \alpha_i \lambda + \beta_i \sigma)$ and set $\mu_i := \alpha_i \lambda_i + \beta_i \sigma_i$. Then, it is possible to define $(\tilde{\mathcal{E}}, \mathcal{D})$ and $(\mathcal{E}, \mathcal{D})$ as in (4.2) and (4.3) respectively with ϱ replaced by ϕ. This construction yields the same bilinear form $(\mathcal{E}, \mathcal{D})$ on $L^2(\Lambda; \mu)$, where $\mu = \varrho \prod_{i=1}^{N}(\lambda_i + \sigma_i)$.

Roughly speaking, the first definition of $(\mathcal{E}, \mathcal{D})$ in Section 4.1.1 corresponds to a Girsanov transformation of N independent sticky Brownian motions on $\overline{\Omega}$ with constant stickyness along Γ (each associated to the form $(\mathcal{G}, D(\mathcal{G}))$) such that the transformed process has a drift given by $\frac{\nabla \varrho}{\varrho}$. In the present section, the form $(\mathcal{E}^i, D(\mathcal{E}^i))$, $i = 1, \ldots, N$, describes a *distorted* sticky Brownian motion on $\overline{\Omega}$ with drift $\frac{\nabla \alpha_i}{\alpha_i}$ inside Ω and the stickyness along Γ is given by $\frac{\alpha_i}{\beta_i}$ as well as a drift along Γ given by $\frac{\nabla_\Gamma \beta_i}{\beta_i}$. Then, the Girsanov transformation by ϕ yields an additional drift $\frac{\nabla \phi}{\phi}$. Note that the resulting form and process (up to equivalence) are the same, since the pre-Dirichlet forms on $C^1(\Lambda)$ coincide.

Densities with product structure as presented in the present section have the advantage that we can handle the densities ϱ_i, $i = 1, \ldots, N$, by considering the forms $(\mathcal{E}^i, D(\mathcal{E}^i))$, $i = 1, \ldots, N$, as given in (4.6). For this type of Dirichlet form it is possible to use the (regularity) results of Section 3.3. In this way the assumptions imposed on ϱ_i, $i = 1, \ldots, N$, are not very restrictive. Only for the interaction part ϕ it is necessary to demand stronger requirements.

It is desirable to prove similar results as in Section 3.3 for the present setting and general densities ϱ which are not necessarily strictly positive. Unfortunately, we do not have a suitable regularity result at hand.

4.2 Analysis of the Markov process

4.2.1 Generators and boundary conditions

Proposition 4.15. *Suppose that Condition 4.1 and Condition 4.4 are satisfied. Then there exists a unique, self-adjoint, linear operator $(L, D(L))$ on $L^2(\Lambda; \mu)$ such that*

$$D(L) \subset D(\mathcal{E}) \quad \text{and} \quad \mathcal{E}(f, g) = (-Lf, g)_{L^2(\Lambda; \mu)} \text{ for all } f \in D(L), \ g \in D(\mathcal{E}).$$

In order to determine $(L, D(L))$ for a suitable class of functions we need the following additional condition on ϱ and Γ:

Condition 4.16. *Assume that Γ is C^2-smooth. ϱ is given as in Section 4.1.2. Moreover, it holds $\phi \in C^1(\Lambda)$ such that $\frac{\nabla \phi}{\phi} \in L^2(\Lambda; \mu)$. $\alpha_i, \beta_i \in C(\overline{\Omega})$, $\sqrt{\alpha_i} \in H^{1,2}(\Omega)$ and if $\delta = 1$ $\sqrt{\beta_i} \in H^{1,2}(\Gamma)$ for $i = 1, \ldots, N$ (,i.e., Condition 3.12).*

Remark 4.17. Note that if α_i, β_i, $i = 1, \ldots, N$, and ϕ are a.e. positive and the additional conditions of Condition 4.16 are fulfilled, Condition 4.1 and Condition 4.4 are implied in view of Remark 4.6.

Proposition 4.18. *Suppose that Condition 4.16 is satisfied. Then, $C^2(\Lambda) \subset D(L)$ and*

$$Lf = \sum_{i=1}^{N} \left(\mathbb{1}_{\Lambda^{i,\Omega}} \left(L^{i,\Omega} f + L_\phi^{i,\Omega} f \right) + \mathbb{1}_{\Lambda^{i,\Gamma}} \left(L^{i,\Gamma} f + L_\phi^{i,\Gamma} f \right) \right) \quad \text{for } f \in C^2(\Lambda), \qquad (4.7)$$

where $L^{i,\Omega} f$, $L^{i,\Gamma} f$, $L_\phi^{i,\Omega} f$ and $L_\phi^{i,\Gamma} f$ for $i = 1, \ldots, N$ are given by

$$L^{i,\Omega} f = \frac{1}{2} \left(\Delta_i f + \left(\frac{\nabla_i \alpha_i}{\alpha_i}, \nabla_i f \right) \right),$$

$$L^{i,\Gamma} f = -\frac{1}{2} \frac{\alpha_i}{\beta_i} \left(n^i, \nabla_i f \right) + \frac{\delta}{2} \left(\Delta_{\Gamma,i} f + \left(\frac{\nabla_{\Gamma,i} \beta_i}{\beta_i}, \nabla_{\Gamma,i} f \right) \right),$$

$$L_\phi^{i,\Omega} f = \frac{1}{2} \left(\frac{\nabla_i \phi}{\phi}, \nabla_i f \right),$$

$$L_\phi^{i,\Gamma} f = \frac{\delta}{2} \left(\frac{\nabla_{\Gamma,i} \phi}{\phi}, \nabla_{\Gamma,i} f \right),$$

where n^i is the outward normal for the i-th particle.

Proof. Let $f \in C^2(\Lambda)$ and $g \in \mathcal{D} = C^1(\Lambda)$. By integration by parts, (4.5) and Proposition 3.14 follows

$$\mathcal{E}(f, g) = \frac{1}{2} \int_\Lambda \sum_{i=1}^{N} \left(\mathbb{1}_{\Lambda^{i,\Omega}} (\nabla_i f, \nabla_i g) + \delta \, \mathbb{1}_{\Lambda^{i,\Gamma}} (\nabla_{\Gamma,i} f, \nabla_{\Gamma,i} g) \right) \varrho \prod_{j=1}^{N} (d\lambda_j + d\sigma_j)$$

$$= \frac{1}{2} \sum_{i=1}^{N} \int_{\Lambda^{i,\Omega}} (\nabla_i f, \nabla_i g) \, \varrho \prod_{j=1}^{N} (d\lambda_j + d\sigma_j) + \frac{\delta}{2} \sum_{i=1}^{N} \int_{\Lambda^{i,\Gamma}} (\nabla_{\Gamma,i} f, \nabla_{\Gamma,i} g) \, \varrho \prod_{j=1}^{N} (d\lambda_j + d\sigma_j)$$

$$= \frac{1}{2} \sum_{i=1}^{N} \int_{\overline{\Omega}^{N-1}} \left(\int_\Omega (\nabla_i f, \nabla_i g) \, \varrho \, d\lambda_i \right) \prod_{j \neq i} (d\lambda_j + d\sigma_j)$$

$$+ \frac{\delta}{2} \sum_{i=1}^{N} \int_{\overline{\Omega}^{N-1}} \left(\int_\Gamma (\nabla_{\Gamma,i} f, \nabla_{\Gamma,i} g) \, \varrho \, d\sigma_i \right) \prod_{j \neq i} (d\lambda_j + d\sigma_j)$$

$$= \frac{1}{2} \sum_{i=1}^{N} \int_{\overline{\Omega}^{N-1}} \left(-\int_\Omega \left(\Delta_i f + \left(\frac{\nabla_i \varrho}{\varrho}, \nabla_i f \right) \right) g \, \varrho d\lambda_i + \int_\Gamma \frac{\alpha_i}{\beta_i} \left(n^i, \nabla_i f \right) g \, \varrho d\sigma_i \right) \prod_{j \neq i} (d\lambda_j + d\sigma_j)$$

$$+ \frac{\delta}{2} \sum_{i=1}^{N} \int_{\overline{\Omega}^{N-1}} \left(-\int_\Gamma \left(\Delta_{\Gamma,i} f + \left(\frac{\nabla_{\Gamma,i} \varrho}{\varrho}, \nabla_{\Gamma,i} f \right) \right) \varrho \, d\sigma_i \right) \prod_{j \neq i} (d\lambda_j + d\sigma_j).$$

Note that on $\Lambda^{i,\Omega}$ holds $\frac{\nabla_i \varrho}{\varrho} = \frac{\nabla_i \alpha_i}{\alpha_i} + \frac{\nabla_i \phi}{\phi}$ due to the product structure of ϱ and similarly, if $\delta = 1$ and $x^i \in \Gamma$ holds $\frac{\nabla_{\Gamma,i} \varrho}{\varrho} = \frac{\nabla_{\Gamma,i} \beta_i}{\beta_i} + \frac{\nabla_{\Gamma,i} \phi}{\phi}$ due to (4.5). Hence,

$$\mathcal{E}(f, g) = \int_\Lambda -\sum_{i=1}^{N} \left(\mathbb{1}_{\Lambda^{i,\Omega}} \left(L^{i,\Omega} + L_\phi^{i,\Omega} f \right) + \mathbb{1}_{\Lambda^{i,\Gamma}} \left(L^{i,\Gamma} f + L_\phi^{i,\Gamma} f \right) \right) g \, d\mu$$

and therefore, the assertion holds true, since $\mathcal{D}$ is dense in $D(\mathcal{E})$. $\qquad \square$

Remark 4.19. Note that the drift in normal direction increases if the factor $\frac{\alpha_i}{\beta_i}$ increases. Hence, it is justifiable to say that the boundary is less sticky for the i-th particle at a point $x \in \Gamma$ if $\beta_i(x)$ decreases. This property can also be discovered in a similar way by Corollary 3.24, since as a consequence of this ergodicity theorem the particle spends less time on the boundary if $\int_\Gamma \beta_i d\sigma$ decreases (compare also to [FGV16, Corollary 5.7]).

Define for $i = 1, \ldots, N$

$$A_i := \mathbb{1}_{\Lambda^{i,\Omega}} E + \delta \, \mathbb{1}_{\Lambda^{i,\Gamma}} P^i$$

as well as

$$b_i := \frac{1}{2} \Big(\mathbb{1}_{\Lambda^{i,\Omega}} \big(\frac{\nabla_i \alpha_i}{\alpha_i} + \frac{\nabla_i \phi}{\phi} \big) + \mathbb{1}_{\Lambda^{i,\Gamma}} \big(- \frac{\alpha_i}{\beta_i} \, n^i + \delta \, \frac{\nabla_{\Gamma,i} \beta_i}{\beta_i} + \delta \, \frac{\nabla_{\Gamma,i} \phi}{\phi} \big) \Big),$$

where E denotes the $d \times d$ identity matrix and P^i is the projection onto the tangent space for the i-th particle. Then, set

$$A := \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \ldots & 0 & A_N \end{pmatrix} \quad \text{and} \quad b := \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \tag{4.8}$$

Using this notation, we get for $f \in C^2(\Lambda)$ the representation

$$Lf = \frac{1}{2} \text{Tr}(A\nabla^2 f) + (b, \nabla f). \tag{4.9}$$

Note that $AA^t = A^2 = A$.

4.2.2 Solution to the martingale problem and SDE

Theorem 4.20. *The diffusion process* **M** *from Theorem 4.11 is up to μ-equivalence the unique diffusion process having μ as symmetrizing measure and solving the martingale problem for $(L, D(L))$, i.e., for all $g \in D(L)$*

$$\tilde{g}(\mathbf{X}_t) - \tilde{g}(\mathbf{X}_0) - \int_0^t (Lg)(\mathbf{X}_s)ds, \ t \geq 0,$$

is an $\mathcal{F}_t$-martingale under $\mathbf{P}_x$ for quasi all $x \in \Lambda$. Here $\tilde{g}$ denotes a quasi-continuous version of g (for the definition of quasi-continuity see e.g. [MR92, Chap. IV, Proposition 3.3]).

Proof. See e.g. [AR95, Theorem 3.4 (i)]. $\qquad\qquad\square$

By Proposition 4.18 L is explicitly known on the set $C^2(\Lambda)$. Using the representation given in (4.9), we obtain the following corollary:

Corollary 4.21. *Assume that Condition 4.16 is fulfilled. Let $g \in C^2(\Lambda)$ and let* **M** *be the diffusion process from Theorem 4.11. Then*

$$g(\mathbf{X}_t) - g(\mathbf{X}_0) - \int_0^t \frac{1}{2} Tr(A(\mathbf{X}_s)\nabla^2 g(\mathbf{X}_s)) + (b(\mathbf{X}_s), \nabla g(\mathbf{X}_s))ds, \ t \geq 0,$$

is an $\mathcal{F}_t$-martingale under $\mathbf{P}_x$ for quasi every $x \in \Lambda$, where A and b are defined in (4.8).

Consequently, we obtain the following theorem in analogy to Theorem 3.17:

Theorem 4.22. M *is a solution to the SDE*

$$
\begin{aligned}
d\mathbf{X}_t^i =& \mathbb{1}_\Omega(\mathbf{X}_t^i)\Big(dB_t^i + \frac{1}{2}\big(\frac{\nabla_i\alpha_i}{\alpha_i}(\mathbf{X}_t^i) + \frac{\nabla_i\phi}{\phi}(\mathbf{X}_t)\big)dt\Big) - \mathbb{1}_\Gamma(\mathbf{X}_t^i)\frac{1}{2}\frac{\alpha_i}{\beta_i}(\mathbf{X}_t^i)\,n(\mathbf{X}_t^i)dt \\
&+ \delta\,\mathbb{1}_\Gamma(\mathbf{X}_t^i)\Big(dB_t^{\Gamma,i} + \frac{1}{2}\big(\frac{\nabla_{\Gamma,i}\beta_i}{\beta_i}(\mathbf{X}_t^i) + \frac{\nabla_{\Gamma,i}\phi}{\phi}(\mathbf{X}_t)\big)dt\Big), \quad i = 1,\ldots,N \qquad (4.10)
\end{aligned}
$$

$$dB_t^{\Gamma,i} = P(\mathbf{X}_t^i) \circ dB_t^i$$

$$\mathbf{X}_0 = x,$$

for quasi every starting point $x \in \Lambda$, *where* $(B_t)_{t\geq0}$, $B_t = (B_t^1,\ldots,B_t^N)$, *is an* Nd-*dimensional standard Brownian motion.*

Remark 4.23. As before, this results can also be deduced by a Fukushima decomposition of M the argument used here in order to get a solution to the SDE (4.10) does not work in this way for reflecting (Neumann) boundary conditions.

Remark 4.24. In the proof of Corollary 3.39 we constructed a diffusion with immediate reflection from a diffusion with sticky reflection by a random time change. For $\delta = 0$ an evident idea would be to construct an interacting particle system with instantaneous reflection and to transform this system of SDEs to a solution of (4.10) by a random time change. However, this seems not possible. The canonical Dirichlet form is given by the closure of

$$\frac{1}{2}\int_\Lambda (\nabla f, \nabla g)\varrho d\lambda^N \quad \text{for } f,g \in C^1(\Lambda) \text{ on } L^2(\Lambda;\varrho\lambda^N), \qquad (4.11)$$

where λ^N denotes the Lebesgue measure on Λ. For this kind of Dirichlet form we have a well-known regularity theory at hand which enables us to construct solutions to the underlying SDE even for singular drifts for every starting point in a specified set of admissible initial values (see e.g. [FG08], [BG14] and [FT95]). Usually, only starting points in $\{\varrho = 0\}$ and in the corners of Λ (two or more particles at the boundary of Ω) are not admissible, since the boundary is not sufficiently smooth at these points. Nevertheless, such kind of dynamics do not diffuse on the boundary of Λ and hence, a time changed process will also not have this property. Therefore, it is not possible to construct an interacting particle system with sticky reflection via time change in use of the closure of (4.11), since a particle which reaches Γ is expected to sojourn a positive amount of time on Γ and meanwhile, the remaining particles keep on moving undelayed. This implies a diffusion on the boundary of Λ.

4.2.3 Solutions by Girsanov transformations

Assume that Condition 4.16 is fulfilled and set

$$\Xi_i := \{x \in \overline{\Omega}|\ \alpha_i(x) = 0 \text{ or } (x \in \Gamma \text{ and } \beta_i(x) = 0)\} = \{\varrho_i = 0\} \cup \{x \in \Gamma|\ \alpha_i(x) = 0\}.$$

Condition 4.25. *For every* $i = 1,\ldots,N$, *there exists* $p_i \geq 2$ *with* $p_i > \frac{d}{2}$ *and* $p_i > d$ *if* $\delta = 0$ *such that*

$$\frac{|\nabla\alpha_i|}{\alpha_i} \in L_{loc}^{p_i}(\overline{\Omega}\backslash\Xi_i;\alpha_i\lambda) \quad \text{and additionally} \quad \frac{|\nabla_\Gamma\beta_i|}{\beta_i} \in L_{loc}^{p_i}(\Gamma\backslash\Xi_i;\beta_i\sigma) \text{ if } \delta = 1$$

or equivalently

$$\mathbb{1}_\Omega\frac{|\nabla\alpha_i|}{\alpha_i} + \delta\,\mathbb{1}_\Gamma\frac{|\nabla\beta_i|}{\beta_i} \in L_{loc}^{p_i}(\overline{\Omega}\backslash\Xi_i;\mu_i).$$

Moreover, $cap_{\mathcal{E}^i}(\Xi_i) = 0$.

Define $\overline{\Omega}_i := \overline{\Omega} \backslash \Xi_i$. Assume that Condition 4.16 and Condition 4.25 are fulfilled. According to Theorem 3.38 there exists for every $i = 1, \ldots, N$ a diffusion process

$$\mathbf{M}^i := \left(\boldsymbol{\Omega}^i, \mathcal{F}^i, (\mathcal{F}^i_t)_{t \geq 0}, (\mathbf{X}^i_t)_{t \geq 0}, (\Theta^i_t)_{t \geq 0}, (\mathbf{P}^i_x)_{x \in \overline{\Omega}_i} \right)$$

with strong Feller transition semigroup $(p^i_t)_{t > 0}$ and transition function $(p^i_t(x, \cdot))_{t > 0}$, $x \in \overline{\Omega}_i$. The processes $\mathbf{M}^i$, $i = 1, \ldots, N$, is associated to the form $(\mathcal{E}^i, D(\mathcal{E}^i))$ on $L^2(\overline{\Omega}_i; \mu_i)$, where $\mu_i = \alpha_i \lambda + \beta_i \sigma$. In particular, $(p^i_t)_{t > 0}$ is absolutely continuous with respect to μ_i, i.e., for every $t > 0$ and $x \in \overline{\Omega}_i$, there exists a non-negative, measurable function $p^i_t(x, y)$, $y \in \overline{\Omega}_i$, such that

$$p^i_t(x, A) = \int_A p^i_t(x, y) d\mu_i(y) \quad \text{for every } A \in \mathcal{B}(\overline{\Omega}_i).$$

Let $\mathbf{M}$ be given by

$$\mathbf{M} := \left(\times_{i=1}^N \boldsymbol{\Omega}^i, \otimes_{i=1}^N \mathcal{F}^i, (\otimes_{i=1}^N \mathcal{F}^i_t)_{t \geq 0}, (\mathbf{X}_t)_{t \geq 0}, (\Theta_t)_{t \geq 0}, (\otimes_{i=1}^N \mathbf{P}^i_{x^i})_{x=(x^1, \ldots, x^N) \in \tilde{\Lambda}} \right),$$

where $\tilde{\Lambda} := \times_{i=1}^N \overline{\Omega}_i$ as well as

$$\mathbf{X}_t(\omega) := (\mathbf{X}^1_t(\omega_1), \ldots, \mathbf{X}^N_t(\omega_N)) \quad \text{and} \quad \Theta_t(\omega) := (\Theta^1_t(\omega_1), \ldots, \Theta^N_t(\omega_N))$$

for $\omega = (\omega_1, \ldots, \omega_N) \in \times_{i=1}^N \boldsymbol{\Omega}^i$. Set $\mathbf{P}_x := \otimes_{i=1}^N \mathbf{P}^i_{x^i}$ for $x = (x^1, \ldots, x^N) \in \tilde{\Lambda}$.
Denote by $(p_t)_{t > 0}$ the transition semigroup and by $(p_t(x, \cdot))_{t > 0}$, $x \in \tilde{\Lambda}$, the transition function of $\mathbf{M}$. Then, it holds for every $A = A_1 \times \cdots \times A_N \in \times_{i=1}^N \mathcal{B}(\overline{\Omega}_i) \subset \mathcal{B}(\tilde{\Lambda})$

$$p_t(x, A) = \int_{\times_{i=1}^N \boldsymbol{\Omega}^i} \mathbb{1}_A(\mathbf{X}_t(\omega)) \, d\mathbf{P}_x(\omega)$$

$$= \int_{\times_{i=1}^N \boldsymbol{\Omega}^i} \prod_{i=1}^N \mathbb{1}_{A_i}(\mathbf{X}^i_t(\omega_i)) \, d\mathbf{P}_x(\omega)$$

$$= \prod_{i=1}^N \int_{\boldsymbol{\Omega}^i} \mathbb{1}_{A_i}(\mathbf{X}^i_t(\omega_i)) \, d\mathbf{P}^i_{x^i}(\omega_i) = \prod_{i=1}^N p^i_t(x^i, A_i)$$

by definition of $\mathbf{P}_x$. Since $\times_{i=1}^N \mathcal{B}(\overline{\Omega}_i)$ generates $\mathcal{B}(\tilde{\Lambda})$, it holds

$$p_t(x, A) = \int_A \prod_{i=1}^N p^i_t(x^i, y^i) \prod_{i=1}^N d\mu_i(y^i) \quad \text{for every } A \in \mathcal{B}(\tilde{\Lambda}).$$

As a consequence, $p_t(x, \cdot)$, $t > 0$, $x \in \tilde{\Lambda}$, is absolutely continuous with respect to $\prod_{i=1}^N \mu_i$ and

$$p_t f(x^1, \ldots, x^N) = \hat{p}^N_t \ldots \hat{p}^1_t f(x^1, \ldots, x^N) \quad \text{for every } f \in \mathcal{B}_b(\tilde{\Lambda}), \tag{4.12}$$

where

$$\hat{p}^i_t f(x^1, \ldots, x^N) := p^i_t f(x^1, \ldots, x^{i-1}, \cdot, x^{i+1}, \ldots, x^N)(x^i)$$

and the order of the $\hat{p}^i_t$, $i = 1, \ldots, N$, is arbitrary.
Consider the symmetric bilinear form on $L^2(\tilde{\Lambda}; \prod_{i=1}^N \mu_i)$ given by

$$\left(\otimes_{i=1}^N \mathcal{E}^i \right)(f, g) := \sum_{i=1}^N \int_{\times_{j \neq i} \overline{\Omega}_j} \mathcal{E}^i(f, g) \prod_{j \neq i} d\mu_j,$$

where

$$f, g \in D(\otimes_{i=1}^N \mathcal{E}^i) := \{ f \in L^2(\tilde{\Lambda}; \prod_{i=1}^N \mu_i) \big| \text{ for each } i = 1, \ldots, N \text{ and for } \prod_{j \neq i} \mu_j - a.e.$$

$$(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^N) \in \times_{j \neq i} \overline{\Omega}_j : f(x^1, \ldots, x^{i-1}, \cdot, x^{i+1}, \ldots, x^n) \in D(\mathcal{E}^i) \}$$

and $\mathcal{E}^i(f,g)$ denotes $\mathcal{E}^i$ acting on the i-th variable of f and g. Due to [BH91, Chapter V, Section 2.1] $(\otimes_{i=1}^N \mathcal{E}^i, D(\otimes_{i=1}^N \mathcal{E}^i))$ is a Dirichlet form on $L^2(\tilde{\Lambda}; \prod_{i=1}^N \mu_i)$. Obviously, this Dirichlet form extends the pre-Dirichlet form $(\tilde{\mathcal{E}}, \mathcal{D})$ defined in (4.1).

Lemma 4.26. $C^1(\Lambda)$ *is dense in* $D(\otimes_{i=1}^N \mathcal{E}^i)$ *w.r.t.* $\left(\otimes_{i=1}^N \mathcal{E}^i\right)_1^{\frac{1}{2}}$, *i.e.*, $(\otimes_{i=1}^N \mathcal{E}^i, D(\otimes_{i=1}^N \mathcal{E}^i))$ *is the closure of* $(\tilde{\mathcal{E}}, \mathcal{D})$ *on* $L^2(\tilde{\Lambda}; \prod_{i=1}^N \mu_i)$.

Proof. First, note that $C^1(\Lambda) \subset D(\otimes_{i=1}^N \mathcal{E}^i)$ by definition of $D(\otimes_{i=1}^N \mathcal{E}^i)$. For simplicity, we only consider the case $N = 2$. The statement for arbitrary $N \in \mathbb{N}$ follows by the same arguments. By [BH91, Proposition 2.1.3b)] $D(\mathcal{E}^1) \otimes D(\mathcal{E}^2)$ is dense in $D(\mathcal{E}^1 \otimes \mathcal{E}^2)$. Thus, it is sufficient to show that $C^1(\Lambda)$ is dense in $D(\mathcal{E}^1) \otimes D(\mathcal{E}^2)$. Then the assertion follows by a diagonal sequence argument. Let $h \in D(\mathcal{E}^1) \otimes D(\mathcal{E}^2)$ such that $h(x^1, x^2) = f(x^1)g(x^2)$ for $\prod_{i=1}^2 \mu_i$-a.e. $(x^1, x^2) \in \tilde{\Lambda}$, where $f \in D(\mathcal{E}^1)$ and $g \in D(\mathcal{E}^2)$. Since $C^1(\overline{\Omega})$ is dense in $D(\mathcal{E}^1)$ and $D(\mathcal{E}^2)$, we can choose sequences $(f_k)_{k \in \mathbb{N}}$ and $(g_k)_{k \in \mathbb{N}}$ in $C^1(\overline{\Omega})$ such that $f_k \to f$ in $D(\mathcal{E}^1)$ and $g_k \to g$ in $D(\mathcal{E}^2)$ as $k \to \infty$. Define $h_k(x^1, x^2) := f_k(x^1)g_k(x^2)$ for $x^1, x^2 \in \overline{\Omega}$. Then it follows easily by the prodcut structure of the underlying measure that the sequence $(h_k)_{k \in \mathbb{N}}$, $h_k \in C^1(\Lambda)$, converges in $L^2(\tilde{\Lambda}; \prod_{i=1}^2 \mu_i)$ to h and moreover, the sequence is $\mathcal{E}^1 \otimes \mathcal{E}^2$-Cauchy. $\qquad \square$

Denote by $(T_t^i)_{t>0}$ the $L^2(\overline{\Omega}_i; \mu_i)$-semigroup of $(\mathcal{E}^i, D(\mathcal{E}^i))$, $i = 1, \ldots, N$. By [BH91, Chapter V, Proposition 2.1.3] the $L^2(\tilde{\Lambda}; \prod_{i=1}^N \mu_i)$-semigroup $(T_t)_{t>0}$ associated to the product Dirichlet form $(\otimes_{i=1}^N \mathcal{E}^i, D(\otimes_{i=1}^N \mathcal{E}^i))$ is given by

$$T_t f = \hat{T}_t^N \cdots \hat{T}_t^1 f \quad \text{for } f \in L^2(\tilde{\Lambda}; \prod_{i=1}^N \mu_i),$$

where

$$\hat{T}_t^i f(x^1, \ldots, x^n) := T_t^i f(x^1, \ldots, x^{i-1}, \cdot, x^{i+1}, \ldots, x^n)(x^i)$$

for $x = (x^1, \ldots, x^n) \in \tilde{\Lambda}$. Since $\mathbf{M}^i$ is associated to the form $(\mathcal{E}^i, D(\mathcal{E}^i))$ for $i = 1, \ldots, N$, it follows by (4.12) and Lemma 4.26 the following:

Proposition 4.27. *The Dirichlet form associated to* $\mathbf{M}$ *is given by the closure of* $(\tilde{\mathcal{E}}, C^1(\Lambda))$ *on* $L^2(\tilde{\Lambda}; \prod_{i=1}^N \mu_i)$.

Additionally to Condition 4.16 and Condition 4.25 we assume the following:

Condition 4.28. ϕ *is strictly positive.*

Under these conditions on ϕ it is possible to perform a Girsanov transformation of $\mathbf{M}$. Consider the multiplicative functional $(Z_t)_{t \geq 0}$, $Z_t = \exp(M_t - \frac{\langle M \rangle_t}{2})$, given by

$$M_t := \int_0^t \nabla \ln \phi(\mathbf{X}_t) dB_t, \ \ t \geq 0.$$

Note that $\nabla \ln \phi(\mathbf{X}_t) = \frac{\nabla \phi}{\phi}(\mathbf{X}_t)$ and B_t, $t \geq 0$, are $\mathbb{R}^{Nd}$ valued and also that $\nabla \ln \phi$ is bounded due to Condition 4.28.

In view of Remark 3.10 (applied to $(\mathcal{E}^i, D(\mathcal{E}^i))$, $i = 1, \ldots, N$), it holds $\times_{i=1}^N \boldsymbol{\Omega}^i = C(\mathbb{R}_+, \tilde{\Lambda})$ and $\otimes_{i=1}^N \mathcal{F}^i = \mathcal{B}(C(\mathbb{R}_+, \tilde{\Lambda}))$. Thus, $(\times_{i=1}^N \boldsymbol{\Omega}^i, \otimes_{i=1}^N \mathcal{F}^i)$ is a standard measurable space (see [IW89, Chapter I, Definition 3.3]) and hence, by [IW89, Chapter IV, Section 4] there exists for every $x \in \tilde{\Lambda}$ a probability measure $\mathbf{P}_x^\phi$ such that $\left(\mathbf{P}_x^\phi\right)_{|\otimes_{i=1}^N \mathcal{F}_t^i} = \mathbf{P}_{x,t}^\phi$, where

$$\mathbf{P}_{x,t}^\phi(A) := \int_A Z_t(\omega) d\mathbf{P}_x(\omega) \quad \text{for } A \in \otimes_{i=1}^N \mathcal{F}_t^i.$$

Let

$$\mathbf{M}^\phi := \left(\times_{i=1}^N \boldsymbol{\Omega}^i, \otimes_{i=1}^N \mathcal{F}^i, (\otimes_{i=1}^N \mathcal{F}_t^i)_{t \geq 0}, (\mathbf{X}_t)_{t \geq 0}, (\Theta_t)_{t \geq 0}, (\mathbf{P}_x^\phi)_{x \in \tilde{\Lambda}} \right).$$

Then, the transition function $(p_t^\phi(x, \cdot))_{t>0}$ of $\mathbf{M}^\phi$ is absolutely continuous with respect to μ for every $x \in \tilde{\Lambda}$. Indeed, by the previous considerations the transition function $(p_t(x, \cdot))_{t>0}$ is absolutely continuous with respect to $\prod_{i=1}^N \mu_i$. Assume that $A \in \mathcal{B}(\tilde{\Lambda})$ is given such that $\mu(A) = 0$. Since ϕ is bounded from above and from below away from zero in view of Condition 4.28 and the continuity of ϕ, it also holds that $\prod_{i=1}^N \mu_i(A) = 0$ and hence, $p_t(x, A) = 0$ for every $t > 0$ and $x \in \tilde{\Lambda}$, i.e.,

$$\int_{\times_{i=1}^N \mathbf{\Omega}^i} \mathbb{1}_A(\mathbf{X}_t) \, d\mathbf{P}_x = 0 \quad \text{for every } t > 0 \text{ and } x \in \tilde{\Lambda}.$$

Therefore, we also have

$$p_t^\phi(x, A) = \int_{\times_{i=1}^N \mathbf{\Omega}^i} \mathbb{1}_A(\mathbf{X}_t) \, d\mathbf{P}_x^\phi = \int_{\times_{i=1}^N \mathbf{\Omega}^i} \mathbb{1}_A(\mathbf{X}_t) \, d\mathbf{P}_{x,t}^\phi = \int_{\times_{i=1}^N \mathbf{\Omega}^i} Z_t \, \mathbb{1}_A(\mathbf{X}_t) \, d\mathbf{P}_x = 0$$

We summarize the results of this section in the following theorem:

Theorem 4.29. $\mathbf{M}^\phi$ *is a solution to the SDE*

$$
\begin{aligned}
d\mathbf{X}_t^i =&\mathbb{1}_\Omega(\mathbf{X}_t^i)\Big(dB_t^i + \frac{1}{2}\big(\frac{\nabla_i \alpha_i}{\alpha_i}(\mathbf{X}_t^i) + \frac{\nabla_i \phi}{\phi}(\mathbf{X}_t)\big)dt\Big) - \mathbb{1}_\Gamma(\mathbf{X}_t^i)\frac{1}{2}\frac{\alpha_i}{\beta_i}(\mathbf{X}_t^i)\, n(\mathbf{X}_t^i)dt \\
&+ \delta\, \mathbb{1}_\Gamma(\mathbf{X}_t^i)\Big(dB_t^{\Gamma,i} + \frac{1}{2}\big(\frac{\nabla_{\Gamma,i}\beta_i}{\beta_i}(\mathbf{X}_t^i) + \frac{\nabla_{\Gamma,i}\phi}{\phi}(\mathbf{X}_t)\big)dt\Big), \quad i = 1, \ldots, N \qquad (4.13) \\
dB_t^{\Gamma,i} =& P(\mathbf{X}_t^i) \circ dB_t^i \\
\mathbf{X}_0 =& x,
\end{aligned}
$$

for every starting point $x \in \tilde{\Lambda}$, where $(B_t)_{t\geq 0}$, $B_t = (B_t^1, \ldots, B_t^N)$, is an Nd-dimensional standard Brownian motion. Moreover, the Dirichlet form associated to $\mathbf{M}^\phi$ is given by $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\tilde{\Lambda}; \mu)$ and its transition function $(p_t^\phi(x, \cdot))_{t>0}$ is absolutely continuous with respect to μ for every $x \in \tilde{\Lambda}$.

Proof. Due to Theorem 3.38 every $\mathbf{M}^i$ solves the respective d-dimensional SDE for every starting point in $\overline{\Omega}_i$, $i = 1, \ldots, N$. Hence, the process $\mathbf{M}$ solves the SDE for N independent particles, i.e., it solves (4.13) for ϕ given by the indicator function on Λ. As a consequence $\mathbf{M}^\phi$ solves (4.13) by the Girsanov transformation theorem (see [IW89, Chapter IV, Section 4]). Moreover, by the same arguments as in [GV17] the Dirichlet form of the transformed process $\mathbf{M}^\phi$ is given by $(\mathcal{E}, D(\mathcal{E}))$. $\square$

4.3 Application to particle systems with singular interactions

In [Gra88] the author investigates a martingale problem with Wentzell boundary conditions in a very general form. In particular, the relation to SDEs is developed and an existence result is shown. As an application the author constructs a system of interacting particles in a domain with sticky boundary. This particle system gives a model for particles diffusing in a chromatography tube. More precisely, the considered domain is given by $\Theta := \{x \in \mathbb{R}^d | \, x_1 > 0\}$ and the investigated SDE on $\overline{\Theta}$ reads as follows:

$$
\begin{aligned}
d\mathbf{X}_t &= \sigma(\mathbf{X}_t)dN_t + b(\mathbf{X}_t)(dt - \rho(\mathbf{X}_t)dK_t) + \gamma(\mathbf{X}_t)dK_t + \tau(\mathbf{X}_t)dC_{K_t}, \\
\mathbf{X}_0 &= x \in \overline{\Theta},
\end{aligned}
$$

where $(\mathbf{X}_t)_{t\geq 0}$ is a continuous, $\overline{\Theta}$-valued process, $(C_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion, $(N_t)_{t\geq 0}$ is a d-dimensional continuous martingale and $(K_t)_{t\geq 0}$ is given such that $K_0 = 0$, K_t is increasing, $dK_t = \mathbb{1}_{\partial\Theta}(\mathbf{X}_t)dK_t$, and

$$\langle N^i, N^j \rangle_t = \delta_{ij}\big(t - \int_0^t \rho(\mathbf{X}_s)dK_s\big).$$

Here, the main focus is placed on the very general form of the martingale problem and SDE as well as the assumptions on σ and $a = \sigma\sigma^T$, which is not necessarily strictly elliptic. In former results (see e.g. [IW89, Chapter IV, Section 7]), it is assumed amongst other things that $a_{11} \geq c > 0$. In [Gra88] it is shown that the martingale problem with the sojourn condition $\rho(\mathbf{X}_t)dK_t \leq \mathbb{1}_{\partial\Theta}(\mathbf{X}_t)dt$ has a solution if and only if the above SDE has a weak solution. Sufficient conditions are $\tau = 0$, σ and b are uniformly Lipschitz continuous and bounded, $\gamma = n$ is the inward normal vector and ρ is bounded, measurable and positive. Nevertheless, the smoothness conditions on b are rather strong. If we assume additionally that $a_{11} > 0$ (e.g. if σ is given by the identity matrix), it holds that

$$\rho(\mathbf{X}_t)dK_t = \mathbb{1}_{\partial\Theta}(\mathbf{X}_t)dt.$$

In the case of the identity matrix, the underlying SDE is given by

$$d\mathbf{X}_t = \mathbb{1}_\Theta(\mathbf{X}_t)dB_t + b(\mathbf{X}_t)\mathbb{1}_\Theta(\mathbf{X}_t)dt + \frac{1}{\rho(\mathbf{X}_t)} \, n(\mathbf{X}_t)dt,$$

$$\mathbf{X}_0 = x \in \overline{\Theta},$$

where $(B_t)_{t\geq 0}$ is a d-dimensional standrad Brownian motion. This setting corresponds to the one considered in Section 3 for $\delta = 0$. The corresponding system of interacting particles is given by

$$d\mathbf{X}_t^i = \mathbb{1}_\Theta(\mathbf{X}_t^i)dB_t^i + b^i(\mathbf{X}_t)\mathbb{1}_\Theta(\mathbf{X}_t^i)dt + \frac{1}{\rho^i(\mathbf{X}_t)} \, n(\mathbf{X}_t^i)dt, \quad i = 1, \ldots, N,$$

$$\mathbf{X}_0 = x \in \overline{\Theta}^N,$$

where $\mathbf{X}_t = (\mathbf{X}_t^1, \ldots, \mathbf{X}_t^N)$. According to [Gra88] an application for this system of SDEs is a model for molecules diffusing in a chromatography tube. The particles are pushed by a flow of gas and are absorbed and released by a liquid state deposited on the boundary of the tube. Hence, it is resonable to suppose a sticky boundary behavior. However, it is physically unreasonable that two molecules are located at the same position in $\overline{\Theta}$ at the same time. In order to avoid this kind of behavior it is necessary to consider a singular drift b^i, $i = 1, \ldots, N$, which causes a strong repulsion if two particles get close to each other. The construction of such kind of stochastic dynamics via Dirichlet forms has already been realized for absorbing and reflecting boundary conditions.

In analogy to [FG08, Section 5], a continuous pair potential (without hard core) is a continuous function $\zeta : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ such that $\zeta(-x) = \zeta(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d\backslash\{0\}$. ζ is said to be repulsive if there exists a continuous decreasing function $\eta : (0, \infty) \to [0, \infty)$ with $\lim_{t\to 0}\eta(t) = \infty$ and $R > 0$ such that

$$\zeta(x) \geq \eta(|x|) \quad \text{for } |x| \leq R.$$

In particular, $\zeta(0) = \infty$. For $N \in \mathbb{N}$ and a repulsive continuous pair potential ζ we consider the function

$$\phi(x) := \exp(-\sum_{\substack{1\leq i,j\leq N \\ i\neq j}} \zeta(x^i - x^j)) \quad \text{for } x = (x^1, \ldots, x^N) \in \Lambda = \overline{\Omega}^N.$$

Note that $\phi(x) = 0$ if there exist $i, j \in \{1, \ldots, N\}$ such that $x^i = x^j$.

Let Γ be C^2-smooth. We assume that ζ is a repulsive, continuous pair potential such that $\phi \in C^1(\Lambda)$ and moreover, we assume that

$$\frac{\nabla\phi}{\phi} \in L^2(\Lambda; \mu) \quad \text{with } \mu = \phi \prod_{i=1}^N (\alpha_i\lambda_i + \beta_i\sigma_i),$$

where α_i and β_i are continuous and a.e. positive such that $\sqrt{\alpha_i} \in H^{1,2}(\Omega)$ and $\sqrt{\beta_i} \in H^{1,2}(\Gamma)$ for $i = 1, \ldots, N$. Then Condition 4.16 is fulfilled and Theorem 4.22 can be applied, i.e., there exists a solution to the SDE

$$d\mathbf{X}_t^i = \mathbb{1}_\Omega(\mathbf{X}_t^i)\Big(dB_t^i + \frac{1}{2}\big(\frac{\nabla_i\alpha_i}{\alpha_i}(\mathbf{X}_t^i) - \sum_{j\neq i}\nabla_i\zeta(\mathbf{X}_t^i - \mathbf{X}_t^j))dt\big) - \frac{1}{2}\mathbb{1}_\Gamma(\mathbf{X}_t^i)\frac{\alpha_i}{\beta_i}(\mathbf{X}_t^i)\,n(\mathbf{X}_t^i)dt$$

$$+ \delta\,\mathbb{1}_\Gamma(\mathbf{X}_t^i)\Big(dB_t^{\Gamma,i} + \frac{1}{2}\big(\frac{\nabla_{\Gamma,i}\beta_i}{\beta_i}(\mathbf{X}_t^i) - \sum_{j\neq i}\nabla_{\Gamma,i}\zeta(\mathbf{X}_t^i - \mathbf{X}_t^j))dt\big),\quad i = 1, \ldots, N$$

$$dB_t^{\Gamma,i} = P(\mathbf{X}_t^i) \circ dB_t^i$$

$$\mathbf{X}_0 = x,$$

for quasi every starting point $x \in \Lambda$.

Example 4.30. A possible example is given by the Lennard-Jones potential

$$\zeta(x) = 4\varepsilon\,\big((\frac{c}{|x|})^{12} - (\frac{c}{|x|})^6\big),$$

where ε and c are positive constants. It holds

$$\frac{\nabla_i\phi}{\phi}(x) = -\sum_{j\neq i}\nabla_i\zeta(x^i - x^j)$$

$$= \frac{24\varepsilon}{c^2}\sum_{j\neq i}\Big(2\,\big(\frac{c}{|x^i - x^j|}\big)^{14} - \big(\frac{c}{|x^i - x^j|}\big)^8\Big)\big(x^i - x^j\big).$$

With $f(r) := \frac{24\varepsilon}{c^2}\Big(2\,\big(\frac{c}{r}\big)^{14} - \big(\frac{c}{r}\big)^8\Big)$ we get

$$\nabla_i \ln\phi(x) = \sum_{j\neq i}f(|x^i - x^j|)\big(x^i - x^j\big).$$

Thus, the absolute value of the acting force obviously depends only on the distance of the respective particles. Moreover, let $\alpha_i, \beta_i \in C^1(\overline{\Omega})$ be strictly positive for $i = 1, \ldots, N$. In this case, the corresponding system of SDEs is given by

$$d\mathbf{X}_t^i = \mathbb{1}_\Omega(\mathbf{X}_t^i)\Big(dB_t^i + \frac{1}{2}\big(\frac{\nabla_i\alpha_i}{\alpha_i}(\mathbf{X}_t^i) - \sum_{j\neq i}f(|\mathbf{X}_t^i - \mathbf{X}_t^j|)(\mathbf{X}_t^i - \mathbf{X}_t^j))dt\big)$$

$$- \frac{1}{2}\mathbb{1}_\Gamma(\mathbf{X}_t^i)\frac{\alpha_i}{\beta_i}(\mathbf{X}_t^i)\,n(\mathbf{X}_t^i)dt$$

$$+ \delta\,\mathbb{1}_\Gamma(\mathbf{X}_t^i)\Big(dB_t^{\Gamma,i} + \frac{1}{2}\big(\frac{\nabla_{\Gamma,i}\beta_i}{\beta_i}(\mathbf{X}_t^i) - \sum_{j\neq i}f(|\mathbf{X}_t^i - \mathbf{X}_t^j|)\,P(\mathbf{X}_t^i)(\mathbf{X}_t^i - \mathbf{X}_t^j))dt\big),$$

$$dB_t^{\Gamma,i} = P(\mathbf{X}_t^i) \circ dB_t^i,\quad i = 1, \ldots, N$$

$$\mathbf{X}_0 = x,$$

where $(B_t)_{t\geq 0}$, $B_t = (B_t^1, \ldots, B_t^N)$, is an Nd-dimensional standard Brownian motion. Note that if a particle sticks on Γ and $\delta = 1$, the acting force causes a drift along Γ until the particle is reflected. Moreover, it is natural to obtain in this case only a solution for quasi every starting point, since points in Λ which describe configurations where two or more particles are at the same position in $\overline{\Omega}$ are naturally not admissible in view of the singularity of ζ in 0. An appropriate regularity results regarding the elliptic PDE associated to the form $(\mathcal{E}, D(\mathcal{E}))$ would allow to apply the results of [BGS13]. In this case, for strictly positive α_i and β_i, $i = 1, \ldots, N$, a process on $\Lambda\backslash\{\phi = 0\} = \{x = (x^1, \ldots, x^N) \in \Lambda|\ x^i \neq x^j$ for every $i \neq j\}$ can be constructed which is a solution to the above SDE for every starting point in $\Lambda\backslash\{\phi = 0\}$.

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