

A comparison of a nonlinear sigma model with general pinning and pinning at one point

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Abstract

We study the nonlinear supersymmetric hyperbolic sigma model introduced by Zirnbauer in 1991. This model can be related to the mixing measure of a vertex-reinforced jump process. We prove that the two-point correlation function has a probabilistic interpretation in terms of connectivity in rooted random spanning forests. Using this interpretation, we dominate the two-point correlation function for general pinning, e.g. for uniform pinning, with the corresponding correlation function with pinning at one point. The result holds for a general finite graph, asymptotically as the strength of the pinning converges to zero. Specializing this to general ladder graphs, we deduce in the same asymptotic regime exponential decay of correlations for general pinning.

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1 Introduction

History of the model. The nonlinear sigma model that we consider here was introduced by Zirnbauer in [8] as a toy model inspired by random matrices in the context of disordered materials. It can be seen as a statistical mechanical model where spins are replaced by vectors with both real and Grassmann components. We associate with each point two real and two Grassmann variables parametrizing a supersymmetric extension of the hyperbolic plane. Therefore the model is often denoted by $H^{2|2}$. In dimension three and higher, a phase transition between a localized (disordered) and an extended (ordered) phase was proved by Disertori, Spencer, and Zirnbauer in [6] and [5].

After integrating out the Grassmann variables in the nonlinear sigma model, the corresponding marginal is a probability measure. It was shown by Sabot and Tarrès in

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[7] that this marginal has an interpretation as a mixing measure for a vertex-reinforced jump process and can also be related to linearly edge-reinforced random walk. Exploiting the former relation, the results in [6] and [5] were used by Sabot and Tarrès in [7] to deduce recurrence for vertex-reinforced jump processes in all dimensions for small initial weights and transience in dimensions $d \geq 3$ for large initial weights. For linearly edge-reinforced random walks, Sabot and Tarrès proved recurrence in all dimensions for small initial weights. An alternative proof, without using the connection to $H^{2|2}$, was given by Angel, Crawford, and Kozma in [2]. In dimensions $d \geq 3$, Disertori, Sabot, and Tarrès showed in [4] transience for linearly edge-reinforced random walks for large initial weights using techniques similar to the one used in [6]. In [3], we proved recurrence for vertex-reinforced jump processes on general ladder graphs with arbitrary constant initial weights using the connection to $H^{2|2}$ just mentioned and a transfer operator method applied to $H^{2|2}$.

The role of pinning. Without a regularization the model $H^{2|2}$ is ill-defined. On a lattice, the most natural choice is to introduce a translationally invariant regularization. This is equivalent to introduce a constant “magnetic field” ε in the corresponding statistical mechanics model. This magnetic field at point j can be interpreted as a “pinning”, forcing the spin at point j to remain near a certain value. Then, a constant magnetic field can be seen as uniform pinning. Another possibility is to take an inhomogeneous magnetic field, the easiest choice being pinning at a single point.

In this paper, we consider the model on a finite connected undirected graph \mathcal{G} , rather than only lattices. The pinning can also be seen as a modification of the underlying graph as follows. We augment \mathcal{G} by an additional vertex ρ in two different ways. In the case of uniform pinning, ρ is connected to every other vertex. In the case of pinning at one point, ρ is only connected to a single vertex in \mathcal{G} . When \mathcal{G} is a lattice or a ladder graph, the first graph has a nonlocal structure since the graph distance between any two vertices is bounded by 2, whereas the second graph remains local.

In the case of ladder graphs, the local structure for pinning at one point allowed us to prove exponential decay of correlations for arbitrary inverse temperature β ; see [3]. However, due to the nonlocal structure of the augmented graph, a similar method is not directly applicable for uniform pinning.

Aim and organization of this paper. The aim of this paper is to bound the expectation of the Green’s function in the case of uniform (or general) pinning with the corresponding Green’s function for pinning at one point, asymptotically as $\varepsilon \rightarrow 0$, for any inverse temperature $\beta > 0$. This result holds for general finite graphs. Specializing it down to ladder graphs, it allows us to transfer known bounds for pinning at one point to the case of the Green’s function for uniform pinning.

In Section 2, the model is formally defined and the results are stated and discussed. In Subsection 3.1, we relate the Green’s function with a probability concerning certain random spanning trees. Subsection 3.2 contains the proof of the comparison between the different pinnings. The model $H^{2|2}$ with pinning at one point has a product structure when passing to gradient variables, that the model $H^{2|2}$ with uniform pinning does not exhibit. This product structure is explained in the appendix.

2 Model and results

2.1 Formal definition

Let $\mathcal{G} = (V, E)$ be a finite connected graph with vertex set V and edge set E , consisting of undirected edges $i \sim j$. We extend \mathcal{G} by adding an extra vertex ρ and edges $i \sim \rho$

connecting it to every other vertex i . The extended graph is denoted by $\mathcal{G}_\rho := (V_\rho, E_\rho)$ with $V_\rho := V \cup \{\rho\}$ and $E_\rho := E \cup \{(i \sim \rho) : i \in V\}$.

We attach to every edge $i \sim j$ of \mathcal{G} an edge weight $\beta_{ij} = \beta_{ji} > 0$. Moreover, for $i \in V$, the edge $i \sim \rho$ gets a weight $\beta_{i\rho} = \beta_{\rho i} = \varepsilon_i \geq 0$. We assume that $\varepsilon_i > 0$ for at least one vertex $i \in V$. To every vertex $i \in V$, we associate two real variables t_i and s_i , and abbreviate $t := (t_i)_{i \in V}$ and $s := (s_i)_{i \in V}$. Furthermore, we set $t_\rho := 0$ and $s_\rho := 0$. For $i, j \in V_\rho$, we define

$$B_{ij}(t, s) := \cosh(t_i - t_j) + \frac{1}{2}(s_i - s_j)^2 e^{t_i + t_j}. \tag{2.1}$$

In particular, for $i \in V$, we have

$$B_{i\rho}(t, s) = \cosh(t_i) + \frac{1}{2}s_i^2 e^{t_i}. \tag{2.2}$$

In the following, we study an equivalent version of the model $H^{2|2}$, where the contribution from Grassmann variables is replaced by a sum over spanning trees T of \mathcal{G}_ρ . Let \mathcal{T} denote the set of spanning trees of \mathcal{G}_ρ . We identify every tree with its edge set. The spanning trees T of \mathcal{G}_ρ are in a natural one-to-one correspondence with rooted spanning forests of \mathcal{G} as follows. Given $T \in \mathcal{T}$, the corresponding spanning forest has the edge set

$$F(T) := T \cap E \tag{2.3}$$

and the set of roots

$$R(T) := \{i \in V : (i \sim \rho) \in T\}. \tag{2.4}$$

Using this notation, we have for $t \in \mathbb{R}^V$ and $T \in \mathcal{T}$

$$\prod_{(i \sim j) \in T} \beta_{ij} e^{t_i + t_j} = \prod_{(i \sim j) \in F(T)} \beta_{ij} e^{t_i + t_j} \prod_{i \in R(T)} \varepsilon_i e^{t_i}. \tag{2.5}$$

In this representation, $H^{2|2}$ is described by the following probability measure on $\mathbb{R}^V \times \mathbb{R}^V \times \mathcal{T}$

$$\begin{aligned} \mu^\varepsilon(dt ds dT) &:= \prod_{j \in V} \frac{dt_j ds_j e^{-t_j}}{2\pi} dT \prod_{(i \sim j) \in E_\rho} e^{-\beta_{ij}(B_{ij}(t,s)-1)} \prod_{(i \sim j) \in T} \beta_{ij} e^{t_i + t_j} \\ &= \prod_{j \in V} \frac{dt_j ds_j e^{-t_j}}{2\pi} dT \prod_{(i \sim j) \in E} e^{-\beta_{ij}(B_{ij}(t,s)-1)} \prod_{i \in V} e^{-\varepsilon_i(B_{i\rho}(t,s)-1)} \prod_{(i \sim j) \in F(T)} \beta_{ij} e^{t_i + t_j} \prod_{i \in R(T)} \varepsilon_i e^{t_i}, \end{aligned} \tag{2.6}$$

where dt_j and ds_j , $j \in V$, denote the Lebesgue measure on \mathbb{R} and dT is the counting measure on \mathcal{T} . As is shown in [6], using supersymmetry and the equivalent description in terms of Grassmann variables, μ^ε is a probability measure.

For convenience, we suppress the dependence of μ^ε on \mathcal{G} and β in the notation. The symbols t , s , and T are used in two slightly different ways: On the one hand, they denote the canonical projections on $\mathbb{R}^V \times \mathbb{R}^V \times \mathcal{T}$. On the other hand, the same symbols are used as integration (or summation) variables.

Consider the matrix $A(t) \in \mathbb{R}^{V \times V}$ defined by

$$A(t)_{ij} := \begin{cases} -\beta_{ij} e^{t_i + t_j} & \text{for } (i \sim j) \in E, \\ \sum_{\substack{k \in V: \\ (k \sim j) \in E}} \beta_{kj} e^{t_k + t_j} & \text{for } i = j, \\ 0 & \text{else.} \end{cases} \tag{2.7}$$

Let $A^\rho(t)$ denote the analog of $A(t)$ when the underlying graph is \mathcal{G}_ρ instead of \mathcal{G} and let $\hat{\varepsilon}$ be the following diagonal matrix:

$$\hat{\varepsilon} := \text{diag}(\varepsilon_i e^{t_i}, i \in V). \tag{2.8}$$

Deleting the row and the column indexed by ρ from $A^\rho(t)$, we get the matrix $A^\rho(t)_{\rho^c \rho^c} = A(t) + \hat{\varepsilon}(t)$. Hence, by the well-known matrix tree theorem (see [1] for a generalized version),

$$\det(A(t) + \hat{\varepsilon}) = \det(A^\rho(t)_{\rho^c \rho^c}) = \sum_{T \in \mathcal{T}} \prod_{(i \sim j) \in T} \beta_{ij} e^{t_i + t_j}. \tag{2.9}$$

Consequently, the (t, s) -marginal of μ^ε is precisely the supersymmetric sigma model studied e.g. in [5].

2.2 Results

The proofs of all results stated here are given in Section 3. Let $x, y \in V$ be two different vertices and $\pi_i \geq 0, i \in V$, be fixed numbers with $\pi_x, \pi_y > 0$. Given $\varepsilon > 0$, we set

$$\varepsilon = (\varepsilon_i)_{i \in V} = (\pi_i \varepsilon)_{i \in V}. \tag{2.10}$$

We are interested in the following Green's function

$$G_{xy}^\varepsilon := e^{t_x + t_y} (A(t) + \hat{\varepsilon}(t))_{xy}^{-1}. \tag{2.11}$$

It has the following probabilistic interpretation in terms of random weighted spanning trees:

Lemma 2.1. *There is a version $P_{\beta, t}$ of the conditional law of μ^ε given t that fulfills*

$$P_{\beta, t}(T = S) = \frac{\prod_{e \in S} \beta_e(t)}{\sum_{S' \in \mathcal{T}} \prod_{e' \in S'} \beta_{e'}(t)}, \quad \text{for } S \in \mathcal{T}, \tag{2.12}$$

with $\beta_e(t) := \beta_{ij} e^{t_i + t_j}$ for $e = (i \sim j)$. Writing $x \overset{T}{\leftrightarrow} y$ if x and y are connected in the spanning tree T through a path which does not use ρ , we have

$$G_{xy}^\varepsilon = \frac{e^{t_x + t_y}}{\varepsilon_x e^{t_x} + \varepsilon_y e^{t_y}} P_{\beta, t}(\{(x \sim \rho) \in T \text{ and } x \overset{T}{\leftrightarrow} y\} \cup \{(y \sim \rho) \in T \text{ and } x \overset{T}{\leftrightarrow} y\}). \tag{2.13}$$

We define

$$\mathcal{O}_{xy}^\pi := \frac{e^{t_x + t_y}}{\pi_x e^{t_x} + \pi_y e^{t_y}} 1_{\{(x \sim \rho) \in T \text{ and } x \overset{T}{\leftrightarrow} y\}}. \tag{2.14}$$

As a consequence of Lemma 2.1, one has

$$G_{xy}^\varepsilon = \varepsilon^{-1} E_{\mu^\varepsilon}[\mathcal{O}_{xy}^\pi + \mathcal{O}_{yx}^\pi | t] = E_{\mu^\varepsilon}[\mathcal{O}_{xy}^\varepsilon + \mathcal{O}_{yx}^\varepsilon | t]. \tag{2.15}$$

The expression $E_{\mu^\varepsilon}[\cdot | t]$ stands for the conditional expectation given t .

We denote by $\varepsilon_x \delta_x \in \mathbb{R}^V$ the vector with coordinate ε_x at x and coordinates 0 at all other locations. Our main theorem can now be phrased as follows.

Theorem 2.2. *Let $\pi_i \geq 0, i \in V$, be fixed numbers with $\pi_x, \pi_y > 0$. We have the following asymptotic comparison between the supersymmetric sigma model with arbitrary pinning $\varepsilon = (\varepsilon_i)_{i \in V} = (\pi_i \varepsilon)_{i \in V}$ and with pinning at one point:*

$$0 < \lim_{\varepsilon \downarrow 0} \varepsilon E_{\mu^\varepsilon}[G_{xy}^\varepsilon] \leq \lim_{\varepsilon \downarrow 0} (E_{\mu^{\varepsilon_x \delta_x}}[\mathcal{O}_{xy}^\pi] + E_{\mu^{\varepsilon_y \delta_y}}[\mathcal{O}_{yx}^\pi]) < \infty \tag{2.16}$$

In particular, the two limits exist.

Remark. It is important that in (2.16) we keep the random variable \mathcal{O}_{xy}^π with the original π . Indeed, if we replaced \mathcal{O}_{xy}^π by $\mathcal{O}_{xy}^{\pi_x \delta_x}$ we would get

$$E_{\mu^{\varepsilon_x \delta_x}} [\mathcal{O}_{xy}^{\pi_x \delta_x}] = E_{\mu^{\varepsilon_x \delta_x}} \left[\frac{e^{ty}}{\pi_x} \right] = \frac{1}{\pi_x} \tag{2.17}$$

by formula (A.7) in the appendix. This bound would give no information on eventual decay in $|y - x|$.

Ladder graphs. In the special case of quasi-one-dimensional graphs this theorem implies exponential decay of the expectation of G_{xy}^ε . More precisely, consider a finite undirected graph $\Gamma_0 = (V_0, E_0)$ with vertex set V_0 and edge set E_0 . Let $\mathcal{G} = (V, E)$ be the “ladder” built of copies $\Gamma_n = (V_n, E_n)$, $n \in \mathbb{Z}$, of Γ_0 . More precisely, we take the vertex sets $V := \mathbb{Z} \times V_0$ and $V_n := \{n\} \times V_0$, where the copy at level 0 is identified with Γ_0 . The edge set E consists of “vertical” edges $e_n := ((n, v) \sim (n, v'))$ with $n \in \mathbb{Z}$ and $e = (v \sim v') \in E_0$ and “horizontal” edges $v_{n+1/2} := ((n, v) \sim (n + 1, v))$ with $n \in \mathbb{Z}$ and $v \in V_0$. For $\underline{L}, \bar{L} \in \mathbb{N}$, we set $L := (-\underline{L}, \bar{L})$ and denote by \mathcal{G}_L the subgraph of \mathcal{G} consisting of the vertex set $V_L := \{-\underline{L}, \dots, \bar{L}\} \times V_0$ and the edge set E_L containing all edges $e \in E$ connecting two vertices in V_L . We associate with every edge $e \in E$ a weight $\beta_e > 0$. We assume that the family of weights $\beta := (\beta_e)_{e \in E}$ is translation invariant in the sense $\beta_{e_n} = \beta_{e_0}$ and $\beta_{v_{n+1/2}} = \beta_{v_{1/2}}$ for all $n \in \mathbb{Z}$, $e \in E_0$, and $v \in V_0$. For $x = (n, v)$ and $y = (m, w)$ in V , their horizontal distance is defined by $|x - y| := |n - m|$. Let μ_L^ε denote the distribution of the supersymmetric sigma model on the graph \mathcal{G}_L with pinning $\varepsilon = (\varepsilon_i)_{i \in V_L} = (\varepsilon \pi_i)_{i \in V_L}$, where $\pi_i \geq 0$, $i \in V_L$, with at least one $\pi_i > 0$.

Corollary 2.3. *There exist constants $c_1, c_2 > 0$ depending only on Γ_0 and β such that for any L , any two different vertices $x, y \in V_L$ with $\pi_x, \pi_y > 0$, and $c_3(\pi) := \min\{\pi_x, \pi_y\}^{-1}$ one has*

$$0 < \lim_{\varepsilon \downarrow 0} \varepsilon E_{\mu_L^\varepsilon} [G_{xy}^\varepsilon] \leq c_1 c_3 e^{-c_2|x-y|}. \tag{2.18}$$

For sufficiently small $\beta > 0$, the methods from [5] can be used to prove a version of (2.18) not only for the limit as $\varepsilon \downarrow 0$, but also for given $\varepsilon > 0$ small enough. In contrast, our result holds for arbitrary $\beta > 0$, but only asymptotically for $\varepsilon \downarrow 0$.

2.3 Discussion

Meaning and some properties of the Green’s function. In the original representation of the $H^{2|2}$ -model the spin at a point j is given by a vector consisting of three even elements x_j, y_j, z_j and two odd elements ξ_j, η_j in a real Grassmann algebra subject to the constraints $1 + x_j^2 + y_j^2 - z_j^2 + 2\xi_j \eta_j = 0$ and $z_j > 0$; see [6]. Introducing horospherical coordinates $t_j, s_j, \bar{\psi}_j, \psi_j$ as in section 2.2 of [6], one has in particular $y_j = s_j e^{t_j}$. Since the Gaussian part in the measure μ^ε is given by $\exp[-\frac{1}{2} s^t (A(t) + \hat{\varepsilon}(t)) s]$, the Green’s function defined in (2.11) is given by the conditional two-point correlation function

$$G_{ij}^\varepsilon = E_{\mu^\varepsilon} [s_i s_j e^{t_i + t_j} | t] = E_{\mu^\varepsilon} [y_i y_j | t] = \text{Cov}_{\mu^\varepsilon} (y_i, y_j | t). \tag{2.19}$$

Moreover, if we denote by $\langle \cdot \rangle_\varepsilon$ the superintegration measure for $H^{2|2}$ including the Grassmann variables, we get the unconditional two-point correlation function

$$E_{\mu^\varepsilon} [G_{ij}^\varepsilon] = E_{\mu^\varepsilon} [s_i s_j e^{t_i + t_j}] = E_{\mu^\varepsilon} [y_i y_j] = \langle y_i y_j \rangle_\varepsilon = \langle x_i x_j \rangle_\varepsilon, \tag{2.20}$$

where the last identity is due to rotational symmetry in the xy -plane.

The following sum rule for the expected Green's function holds:

$$\sum_{j \in V} \varepsilon_j E_{\mu^\varepsilon} [G_{ij}^\varepsilon] = 1. \tag{2.21}$$

For the special case of uniform pinning this equals formula (4.6) in [6]. As explained in appendix B of that paper, this formula is a consequence of the Lorentz boost symmetry in the yz -plane. The same argument proves also the generalization (2.21) as follows. The Lorentz boost symmetry generator L_2 fulfills $L_2 y_j = z_j$ and $L_2 z_j = y_j$ for any $j \in V$. Let $A_{\beta, \varepsilon}$ denote the action of the $H^{2|2}$ -model in x, y, z coordinates as given in formula (2.6) of [6]. With explanations given below, we obtain for the observable $F = y_i$

$$\begin{aligned} 1 = \langle z_i \rangle_\varepsilon &= \langle L_2 F \rangle_\varepsilon = \langle F L_2 A_{\beta, \varepsilon} \rangle_\varepsilon = \left\langle F \sum_{j \in V} \varepsilon_j L_2 z_j \right\rangle_\varepsilon \\ &= \sum_{j \in V} \varepsilon_j \langle y_i y_j \rangle_\varepsilon = \sum_{j \in V} \varepsilon_j E_{\mu^\varepsilon} [G_{ij}^\varepsilon]. \end{aligned} \tag{2.22}$$

The first equality is contained in (B.3) from [6]. The third equality follows from (B.2) in [6]. Finally, the fourth equality holds since $\sum_j \varepsilon_j z_j$ is the only contribution in the action $A_{\beta, \varepsilon}$ not invariant under L_2 .

Vertex-reinforced jump processes for general pinning. According to [7], the mixing measure for the vertex-reinforced jump process (VRJP) with starting point x on the graph $\mathcal{G} = (V, E)$ with edge weights β_{ij} can be described in terms of the model $H^{2|2}$ on the same graph with the same edge weights and pinning at the single point x . Here is an alternative description, which is well-suited for the generalization to general pinning. Consider the augmented graph \mathcal{G}_ρ from Section 2.1 with the additional edge weights $\beta_{x\rho} = \varepsilon_x > 0$ but $\beta_{i\rho} = 0$ for $i \in V \setminus \{x\}$. The VRJP on \mathcal{G} with starting point x can be obtained from the VRJP on \mathcal{G}_ρ with starting point ρ by removing all time intervals where the random walker stays in ρ . Corresponding to this fact, the $H^{2|2}$ model on \mathcal{G}_ρ with reference point ρ , i.e. $t_\rho = s_\rho = 0$, encodes both, the mixing measure for VRJP with starting point ρ on the graph \mathcal{G}_ρ and for VRJP with starting point x on the graph \mathcal{G} .

For general pinning, we drop the condition $\beta_{i\rho} = 0$ for $i \in V \setminus \{x\}$, taking again the augmented graph \mathcal{G}_ρ , now with arbitrary edge weights $\beta_{i\rho} = \varepsilon_i$, $i \in V$, $\varepsilon_x > 0$. In this general setup, the easy connection between VRJP on \mathcal{G}_ρ with start in ρ and VRJP on \mathcal{G} with start in x gets lost. However, the description of the mixing measure for VRJP on \mathcal{G}_ρ with start in ρ in terms of the $H^{2|2}$ model on \mathcal{G}_ρ with reference point ρ but arbitrary edge weights remains valid.

Order of limits. In spin models like the Ising model or the Heisenberg model, having a compact symmetry space, one usually takes the thermodynamic limit $|V| \rightarrow \infty$ first and then the limit of vanishing symmetry breaking field $h \rightarrow 0$, the other order of limits $\lim_{|V| \rightarrow \infty} \lim_{h \rightarrow 0}$ being trivial.

In contrast to this, the $H^{2|2}$ models have a non-compact symmetry space. Then the order of limits $\lim_{|V| \rightarrow \infty} \lim_{\varepsilon \rightarrow 0}$ is ill-defined unless one takes care of the $\frac{1}{\varepsilon}$ divergence, which is already visible in the sum rule (2.21). In the case of uniform pinning, the order of limits $\lim_{\varepsilon \rightarrow 0} \lim_{|V| \rightarrow \infty}$ is less relevant than the coupled limit $|V| \rightarrow \infty$ with $\varepsilon_i = O(|V|^{-1})$ for all $i \in V$. For a general pinning this corresponds to require $\sum_{j \in V} \varepsilon_j = O(1)$. From the point of view of the VRJP this means that the total jump rate starting from the reference point ρ should remain bounded if the volume $|V|$ of the graph diverges.

Here, we rescale the $\frac{1}{\epsilon}$ divergence and consider the limit $\epsilon \rightarrow 0$ before the limit $|V| \rightarrow \infty$. Theorem 2.2 shows for any finite graph that the limit

$$g_{ij} := \lim_{\epsilon \downarrow 0} \epsilon_j E_{\mu^\epsilon} [G_{ij}^\epsilon] \tag{2.23}$$

exists. By the sum rule (2.21), it fulfills

$$\sum_{j \in V} g_{ij} = 1. \tag{2.24}$$

For strips, g_{ij} decays exponentially in $|i - j|$ uniformly in the volume by Corollary 2.3.

The order of limits $\lim_{|V| \rightarrow \infty} \lim_{\epsilon \rightarrow 0}$ studied in this paper is relevant for the localized regime, where one expects $\epsilon_j E_{\mu^\epsilon} [G_{ij}^\epsilon]$ to decay exponentially in $|i - j|$, uniformly in the volume and ϵ . In this case, the rescaled expected Green’s function has a summable dominating function. In particular, if the infinite volume limit $\lim_{|V| \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \epsilon_j E_{\mu^\epsilon} [G_{ij}^\epsilon]$ exists, the sum rule (2.24) survives in the infinite volume limit in this regime.

It remains an open problem to analyze the coupled limit of $\epsilon_j E_{\mu^\epsilon} [G_{ij}^\epsilon]$ as $|V| \rightarrow \infty$ and $\epsilon_i = |V|^{-1}$ for all $i \in V$.

3 Proofs

3.1 Probabilistic interpretation of the Green’s function

Let \mathcal{G} and \mathcal{G}_ρ be the graphs introduced in Section 2.1. Without loss of generality, we assume $V = \{1, \dots, n\}$ throughout this subsection. Recall that \mathcal{T} denotes the set of spanning trees of \mathcal{G}_ρ . We will use the definitions (2.3) of the forest $F(T)$ and (2.4) of the set of roots $R(T)$. Finally, recall that $x \overset{T}{\leftrightarrow} y$ iff x and y are connected in $F(T)$. For any $x, y \in V$, let

$$\mathcal{T}_{xy} := \{T \in \mathcal{T} : x \in R(T) \text{ and } x \overset{T}{\leftrightarrow} y\}. \tag{3.1}$$

For the proof of Lemma 2.1, we need the following result.

Lemma 3.1. *Consider $x, y \in V$ and a real symmetric matrix $M \in \mathbb{R}^{V \times V}$ with $M_{ij} = 0$ whenever $i \neq j$ and there is no edge between i and j . Then, the determinant of the minor of M obtained by taking away the column x and the row y is given by*

$$\det M_{y^c x^c} = (-1)^{x+y} \sum_{T \in \mathcal{T}_{xy}} \left[\prod_{j \in R(T) \setminus \{x\}} \left(\sum_{i \in V} M_{ij} \right) \right] \prod_{(i \sim j) \in F(T)} (-M_{ij}). \tag{3.2}$$

Proof. In the case that \mathcal{G} is the *complete* undirected graph with vertex set V , this is the special case of Theorem 1 of Abdesselam’s article [1] for a *symmetric* matrix M when the index sets I and J are replaced by *singletons* $\{y\}, \{x\} \subseteq \{1, \dots, n\}$, respectively. Note that the sign $\epsilon(\mathcal{F})$ appearing in Abdesselam’s formula equals 1 in our special case because $I = \{y\}$ and $J = \{x\}$ are singletons. Since $M_{ij} = 0$ whenever $i \neq j$ and there is no edge between i and j , formula (3.2) remains literally true if we replace the complete graph by the given graph \mathcal{G} . □

Proof of Lemma 2.1. The tree dependent part of the density of μ^ϵ in (2.6) is given by $\prod_{e \in T} \beta_e(t)$; note that s and T are conditionally independent given t . Consequently, formula (2.12) describes indeed the law of T conditional on t .

By definition (2.11), it holds

$$e^{-(t_x+t_y)} G_{xy}^\varepsilon = (A(t) + \hat{\varepsilon}(t))_{xy}^{-1} = (-1)^{x+y} \frac{\det(A(t) + \hat{\varepsilon}(t))_{y^c x^c}}{\det(A(t) + \hat{\varepsilon}(t))}. \quad (3.3)$$

For the numerator, we use Lemma 3.1 with $M = A(t) + \hat{\varepsilon}$. Clearly, all columns of $A(t)$ sum up to 0. Consequently, for all $j \in V$, one has

$$\sum_{i \in V} M_{ij} = \sum_{i \in V} (A_{ij}(t) + \hat{\varepsilon}_{ij}) = \hat{\varepsilon}_{jj} = \varepsilon_j e^{t_j}. \quad (3.4)$$

Furthermore, for $(i \sim j) \in F(T)$, $T \in \mathcal{T}$, one has $-M_{ij} = \beta_{ij} e^{t_i+t_j}$. Using this, formula (3.2) yields

$$(-1)^{x+y} \det(A(t) + \hat{\varepsilon}(t))_{y^c x^c} = \sum_{T \in \mathcal{T}_{xy}} \left[\prod_{j \in R(T) \setminus \{x\}} \varepsilon_j e^{t_j} \right] \prod_{(i \sim j) \in F(T)} \beta_{ij} e^{t_i+t_j}. \quad (3.5)$$

Multiplying this equation by $\varepsilon_x e^{t_x}$ and using formula (2.5), we get

$$\begin{aligned} (-1)^{x+y} \varepsilon_x e^{t_x} \det(A(t) + \hat{\varepsilon}(t))_{y^c x^c} &= \sum_{T \in \mathcal{T}_{xy}} \left[\prod_{j \in R(T)} \varepsilon_j e^{t_j} \right] \prod_{(i \sim j) \in F(T)} \beta_{ij} e^{t_i+t_j} \\ &= \sum_{T \in \mathcal{T}_{xy}} \prod_{(i \sim j) \in T} \beta_{ij} e^{t_i+t_j}. \end{aligned} \quad (3.6)$$

Hence, using (2.9) for the denominator, we obtain

$$\begin{aligned} \varepsilon_x e^{t_x} (A(t) + \hat{\varepsilon}(t))_{xy}^{-1} &= (-1)^{x+y} \varepsilon_x e^{t_x} \frac{\det(A(t) + \hat{\varepsilon}(t))_{y^c x^c}}{\det(A(t) + \hat{\varepsilon}(t))} \\ &= \frac{\sum_{T \in \mathcal{T}_{xy}} \prod_{(i \sim j) \in T} \beta_{ij} e^{t_i+t_j}}{\sum_{T \in \mathcal{T}} \prod_{(i \sim j) \in T} \beta_{ij} e^{t_i+t_j}} \\ &= P_{\beta,t}(T \in \mathcal{T}_{xy}) = P_{\beta,t}((x \sim \rho) \in T \text{ and } x \xleftrightarrow{T} y). \end{aligned} \quad (3.7)$$

Exchanging x and y and using the symmetry of $A(t) + \hat{\varepsilon}(t)$, we get

$$\varepsilon_y e^{t_y} (A(t) + \hat{\varepsilon}(t))_{xy}^{-1} = P_{\beta,t}(T \in \mathcal{T}_{yx}) = P_{\beta,t}((y \sim \rho) \in T \text{ and } x \xleftrightarrow{T} y). \quad (3.8)$$

Since $x \neq y$, the sets \mathcal{T}_{xy} and \mathcal{T}_{yx} are disjoint. Hence, $P_{\beta,t}(T \in \mathcal{T}_{xy}) + P_{\beta,t}(T \in \mathcal{T}_{yx}) = P_{\beta,t}(T \in \mathcal{T}_{xy} \cup \mathcal{T}_{yx})$. Finally, we add (3.7) and (3.8) and insert them into (3.3) to obtain the claim (2.13). \square

3.2 Comparing different pinnings

We write the random variable \mathcal{O}_{xy}^π defined in (2.14) as a sum:

$$\mathcal{O}_{xy}^\pi = \mathcal{O}_{xy}^\pi 1_{\{R(T)=\{x\}\}} + \mathcal{O}_{xy}^\pi 1_{\{|R(T)|>1\}}. \quad (3.9)$$

Note that we have $\mathcal{O}_{xy}^\pi 1_{\{|R(T)|=1\}} = \mathcal{O}_{xy}^\pi 1_{\{R(T)=\{x\}\}}$ because \mathcal{O}_{xy}^π contains the indicator function of the event $\{(x \sim \rho) \in T\}$.

The proof of Theorem 2.2 is based on Lemmas 3.2 and 3.4, below, dealing with the first and second summand in (3.9), respectively. Surprisingly, the main mass contributing to the expectation of the first and second summand in (3.9) comes from quite different locations; see also the explanations following formula (3.31), below. On the one hand, values (t, s) with $t_j \approx -\log \epsilon$ carry most of the mass for the expectation of $\mathcal{O}_{xy}^\pi 1_{\{R(T)=\{x\}\}}$.

On the other hand, the main contribution to the expectation of $\mathcal{O}_{xy}^\pi 1_{\{|R(T)|>1\}}$ comes from values (t, s) with $t_j \approx +\log \epsilon$. Thus, for small $\epsilon > 0$, in the two expectations the main masses sit at opposite ends. As we shall see, the main contribution comes from the term with precisely one root at x . We examine this contribution first.

Lemma 3.2 (Contribution of one root). *For all $\epsilon > 0$,*

$$E_{\mu^\epsilon} [\mathcal{O}_{xy}^\pi 1_{\{R(T)=\{x\}\}}] \leq e^{\sum_{i \in V \setminus \{x\}} \epsilon^i} E_{\mu^{\epsilon x \delta_x}} [\mathcal{O}_{xy}^\pi]. \tag{3.10}$$

Furthermore,

$$\begin{aligned} 0 < \lim_{\epsilon \downarrow 0} E_{\mu^\epsilon} [\mathcal{O}_{xy}^\pi 1_{\{R(T)=\{x\}\}}] &= \lim_{\epsilon \downarrow 0} E_{\mu^{\epsilon x \delta_x}} \left[\mathcal{O}_{xy}^\pi \sqrt{\frac{\pi_x e^{t_x}}{\sum_{i \in V} \pi_i e^{t_i}}} \prod_{i \in V \setminus \{x\}} e^{-\frac{1}{2} \epsilon \pi_i e^{t_i}} \right] \\ &\leq \lim_{\epsilon \downarrow 0} E_{\mu^{\epsilon x \delta_x}} [\mathcal{O}_{xy}^\pi] \leq \min \left\{ \frac{1}{\pi_x}, \frac{1}{\pi_y} \right\}. \end{aligned} \tag{3.11}$$

In particular all the displayed limits exist.

The proof will be a consequence of a more general result, where the family ϵ is replaced by another family ϵa and the random variable $\mathcal{O}_{xy}^\pi 1_{\{R(T)=\{x\}\}}$ is multiplied by an additional factor χ .

Take $a = (a_i)_{i \in V}$ with $a_i \geq 0$ for all i and $a_x > 0$ and an additional density function $\chi : \mathbb{R}^V \rightarrow (0, 1]$. We will study the expectation

$$E_{\mu^{\epsilon a}} [\chi(t + \log \epsilon) \mathcal{O}_{xy}^\pi 1_{\{R(T)=\{x\}\}}], \tag{3.12}$$

where we abbreviate $t + \log \epsilon := (t_i + \log \epsilon)_{i \in V}$. The main contribution to the expectation coming from values of the t_i 's close to $-\log \epsilon$ motivates us to shift the t_i 's by $\log \epsilon$. For this purpose, we introduce new variables:

$$t'_i := t_i + \log \epsilon, \quad s'_i := \epsilon^{-1}(s_i - s_x) \quad \text{for all } i \in V. \tag{3.13}$$

With this definition, $s'_x = 0$. Therefore, we will use as new integration variables $(t'_i)_{i \in V}$, s_x , and $(s'_i)_{i \in V \setminus \{x\}}$. For any fixed configuration t', s' of these new variables, we consider an auxiliary random variable S' on some probability space, taking values s'_i with probabilities

$$P_{a, t', s'}(S' = s'_i) := \frac{a_i e^{t'_i}}{z_{a, t'}}, \quad i \in V, \tag{3.14}$$

where

$$z_{a, t'} := \sum_{i \in V} a_i e^{t'_i} \tag{3.15}$$

is the normalizing constant. We denote by $E_{a, t', s'}$ and $\text{Var}_{a, t', s'}$ the corresponding expectation and variance operators, respectively. In order to have a compact notation, we abbreviate in the following

$$\begin{aligned} \kappa(dt' ds') &:= \prod_{j \in V} \frac{dt'_j e^{-t'_j}}{2\pi} \prod_{j \in V \setminus \{x\}} ds'_j \prod_{(i \sim j) \in E} e^{-\beta_{ij}(B_{ij}(t', s') - 1)} \\ &\cdot \sum_{T \in \mathcal{T}} \prod_{(i \sim j) \in F(T)} \beta_{ij} e^{t'_i + t'_j} e^{t'_x} \frac{e^{t'_x + t'_y}}{\pi_x e^{t'_x} + \pi_y e^{t'_y}} 1_{\{R(T)=\{x\} \text{ and } x \overset{T}{\leftrightarrow} y\}}. \end{aligned} \tag{3.16}$$

Note that κ depends also on the fixed quantities π , $(\beta_{ij})_{(i \sim j) \in E}$, and on the vertices x, y , although this is not displayed. We remind that $s'_x = 0$ by construction.

Lemma 3.3. *With all the definitions above, we have*

$$\begin{aligned}
 & e^{-\epsilon \sum_{i \in V} a_i} E_{\mu^{\epsilon a}} [\chi(t + \log \epsilon) \mathcal{O}_{xy}^\pi 1_{\{R(T)=\{x\}\}}] \\
 &= \int_{\mathbb{R}^V \times \mathbb{R}^{V \setminus \{x\}}} \kappa(dt' ds') \sqrt{\frac{2\pi}{z_{a,t'}}} e^{-\frac{z_{a,t'}}{2} \epsilon^2 \text{Var}_{a,t',s'}(S')} \prod_{i \in V} e^{-\frac{1}{2} a_i (\epsilon^{t'_i} + \epsilon^2 e^{-t'_i})} \cdot a_x \chi(t') \\
 & \uparrow_{\epsilon \downarrow 0} \int_{\mathbb{R}^V \times \mathbb{R}^{V \setminus \{x\}}} \kappa(dt' ds') \sqrt{\frac{2\pi}{z_{a,t'}}} \prod_{i \in V} e^{-\frac{1}{2} a_i \epsilon^{t'_i}} \cdot a_x \chi(t') > 0. \tag{3.17}
 \end{aligned}$$

Proof. Note that on the event $\{R(T) = \{x\}\}$, the root contribution in (2.5) is given by $\epsilon a_x e^{t_x}$. Using (2.6), we get

$$\begin{aligned}
 & E_{\mu^{\epsilon a}} [\chi(t + \log \epsilon) \mathcal{O}_{xy}^\pi 1_{\{R(T)=\{x\}\}}] \\
 &= \sum_{T \in \mathcal{T}} \int_{(\mathbb{R}^V)^2} \prod_{j \in V} \frac{dt'_j ds'_j e^{-t'_j}}{2\pi} \prod_{(i \sim j) \in E} e^{-\beta_{ij} (B_{ij}(t,s) - 1)} \prod_{i \in V} e^{-\epsilon a_i (B_{i\rho}(t,s) - 1)} \prod_{(i \sim j) \in F(T)} \beta_{ij} e^{t_i + t_j} \\
 & \cdot \epsilon a_x e^{t_x} \chi(t + \log \epsilon) \frac{e^{t_x + t_y}}{\pi_x e^{t_x} + \pi_y e^{t_y}} 1_{\{R(T)=\{x\} \text{ and } x \overset{T}{\leftrightarrow} y\}}. \tag{3.18}
 \end{aligned}$$

Changing variables according to (3.13), we get $(s_i - s_j)^2 e^{t_i + t_j} = (s'_i - s'_j)^2 e^{t'_i + t'_j}$ and

$$\begin{aligned}
 & E_{\mu^{\epsilon a}} [\chi(t + \log \epsilon) \mathcal{O}_{xy}^\pi 1_{\{R(T)=\{x\}\}}] \\
 &= \sum_{T \in \mathcal{T}} \int_{\mathbb{R}^V} \prod_{j \in V} \frac{dt'_j \epsilon e^{-t'_j}}{2\pi} \int_{\mathbb{R}} ds_x \int_{\mathbb{R}^{V \setminus \{x\}}} \prod_{j \in V \setminus \{x\}} \epsilon ds'_j \prod_{(i \sim j) \in E} e^{-\beta_{ij} (B_{ij}(t',s') - 1)} \\
 & \cdot \prod_{i \in V} e^{-\frac{1}{2} a_i (\epsilon^{t'_i} + \epsilon^2 e^{-t'_i} - 2\epsilon + (\epsilon s'_i + s_x)^2 e^{t'_i})} \prod_{(i \sim j) \in F(T)} \beta_{ij} \epsilon^{-2} e^{t'_i + t'_j} \cdot a_x e^{t'_x} \\
 & \cdot \chi(t') \frac{\epsilon^{-1} e^{t'_x + t'_y}}{\pi_x e^{t'_x} + \pi_y e^{t'_y}} 1_{\{R(T)=\{x\} \text{ and } x \overset{T}{\leftrightarrow} y\}} \\
 &= e^{\epsilon \sum_{i \in V} a_i} \int_{\mathbb{R}} ds_x \int_{\mathbb{R}^V \times \mathbb{R}^{V \setminus \{x\}}} \kappa(dt' ds') \prod_{i \in V} e^{-\frac{1}{2} a_i (\epsilon^{t'_i} + \epsilon^2 e^{-t'_i} + (\epsilon s'_i + s_x)^2 e^{t'_i})} \cdot a_x \chi(t'). \tag{3.19}
 \end{aligned}$$

For counting powers of ϵ in the last equality, we have used that $F(T)$ is a spanning tree of \mathcal{G} for $|R(T)| = 1$ and consequently $|F(T)| = |V| - 1$. Next we integrate out s_x . In terms of the auxiliary random variable S' as specified in (3.14), the part of the exponent in (3.19) containing s_x can be rewritten as follows:

$$\begin{aligned}
 \sum_{i \in V} a_i e^{t'_i} (\epsilon s'_i + s_x)^2 &= z_{a,t'} E_{a,t',s'} [(\epsilon S' + s_x)^2] \\
 &= z_{a,t'} \text{Var}_{a,t',s'}(\epsilon S' + s_x) + z_{a,t'} E_{a,t',s'} [\epsilon S' + s_x]^2 \\
 &= z_{a,t'} \epsilon^2 \text{Var}_{a,t',s'}(S') + z_{a,t'} (\epsilon E_{a,t',s'}[S'] + s_x)^2 \tag{3.20}
 \end{aligned}$$

This yields

$$\begin{aligned}
 & \int_{\mathbb{R}} ds_x \exp \left[-\frac{1}{2} \sum_{i \in V} a_i e^{t'_i} (\epsilon s'_i + s_x)^2 \right] \\
 &= \exp \left[-\frac{z_{a,t'}}{2} \epsilon^2 \text{Var}_{a,t',s'}(S') \right] \int_{\mathbb{R}} ds_x \exp \left[-\frac{z_{a,t'}}{2} (\epsilon E_{a,t',s'}[S'] + s_x)^2 \right] \\
 &= \sqrt{\frac{2\pi}{z_{a,t'}}} \exp \left[-\frac{z_{a,t'}}{2} \epsilon^2 \text{Var}_{a,t',s'}(S') \right]. \tag{3.21}
 \end{aligned}$$

Inserting this into (3.19), we obtain the equality claimed in (3.17).

The ϵ -dependent integrand in (3.17) increases as $\epsilon \downarrow 0$. Hence, by the monotone convergence theorem, we get the claimed limit. \square

Proof of Lemma 3.2. To prove (3.10), we compare two special cases of formula (3.17) in Lemma 3.3, namely $\chi = 1$ with first $a = \pi$ and second $a = \pi_x \delta_x$. For $a = \pi_x \delta_x$ we have $\text{Var}_{\pi_x \delta_x, t', s'}(S') = 0$, hence

$$\begin{aligned} & \sqrt{\frac{2\pi}{z_{\pi, t'}}} e^{-\frac{z_{\pi, t'}}{2} \epsilon^2 \text{Var}_{\pi, t', s'}(S')} \prod_{i \in V} e^{-\frac{1}{2} \pi_i (e^{t'_i} + \epsilon^2 e^{-t'_i})} \leq \sqrt{\frac{2\pi}{\pi_x e^{t'_x}}} e^{-\frac{1}{2} \pi_x (e^{t'_x} + \epsilon^2 e^{-t'_x})} \\ & = \sqrt{\frac{2\pi}{z_{\pi_x \delta_x, t'}}} e^{-\frac{z_{\pi_x \delta_x, t'}}{2} \epsilon^2 \text{Var}_{\pi_x \delta_x, t', s'}(S')} e^{-\frac{1}{2} \pi_x (e^{t'_x} + \epsilon^2 e^{-t'_x})}. \end{aligned} \tag{3.22}$$

Inserting this in the equality in Lemma 3.3 yields claim (3.10) as follows

$$e^{-\sum_{i \in V} \epsilon_i} E_{\mu^\epsilon} [\mathcal{O}_{xy}^\pi 1_{\{R(T)=\{x\}\}}] \leq e^{-\epsilon_x} E_{\mu^{\epsilon_x \delta_x}} [\mathcal{O}_{xy}^\pi 1_{\{R(T)=\{x\}\}}] = e^{-\epsilon_x} E_{\mu^{\epsilon_x \delta_x}} [\mathcal{O}_{xy}^\pi]. \tag{3.23}$$

Note that the event $\{R(T) = \{x\}\}$ holds $\mu^{\epsilon_x \delta_x}$ -almost surely.

To prove the remaining claim (3.11), we compare three special cases of formula (3.17) in Lemma 3.3.

Case 1: $a = \pi, \chi = 1$;

$$\text{Case 2: } a = \pi_x \delta_x, \chi(t') = \sqrt{\frac{\pi_x e^{t'_x}}{\sum_{i \in V} \pi_i e^{t'_i}}} \prod_{i \in V \setminus \{x\}} e^{-\frac{1}{2} \pi_i e^{t'_i}} = \sqrt{\frac{\pi_x e^{t'_x}}{\sum_{i \in V} \pi_i e^{t'_i}}} \prod_{i \in V \setminus \{x\}} e^{-\frac{1}{2} \epsilon \pi_i e^{t'_i}};$$

Case 3: $a = \pi_x \delta_x, \chi = 1$.

Note that

$$\sqrt{\frac{2\pi}{z_{\pi, t'}}} = \sqrt{\frac{2\pi}{z_{\pi_x \delta_x, t'}}} \sqrt{\frac{\pi_x e^{t'_x}}{\sum_{i \in V} \pi_i e^{t'_i}}} \leq \sqrt{\frac{2\pi}{z_{\pi_x \delta_x, t'}}}. \tag{3.24}$$

Consequently, the limits in (3.17) for the first two cases coincide, while the limit in the third case yields an upper bound for the other two cases. Hence,

$$\begin{aligned} & 0 < \lim_{\epsilon \downarrow 0} E_{\mu^\epsilon} [\mathcal{O}_{xy}^\pi 1_{\{R(T)=\{x\}\}}] \\ & = \lim_{\epsilon \downarrow 0} E_{\mu^{\epsilon_x \delta_x}} \left[\mathcal{O}_{xy}^\pi \sqrt{\frac{\pi_x e^{t'_x}}{\sum_{i \in V} \pi_i e^{t'_i}}} \prod_{i \in V \setminus \{x\}} e^{-\frac{1}{2} \epsilon \pi_i e^{t'_i}} 1_{\{R(T)=\{x\}\}} \right] \\ & \leq \lim_{\epsilon \downarrow 0} E_{\mu^{\epsilon_x \delta_x}} [\mathcal{O}_{xy}^\pi 1_{\{R(T)=\{x\}\}}]. \end{aligned} \tag{3.25}$$

Recall that the event $\{R(T) = \{x\}\}$ holds $\mu^{\epsilon_x \delta_x}$ -almost surely. Consequently, we can drop the indicator function in the last two expectations.

Next, we argue that the last limit is finite. Clearly, from (2.14), we have

$$\mathcal{O}_{xy}^\pi \leq \min \left\{ \frac{e^{t_x}}{\pi_y}, \frac{e^{t_y}}{\pi_x} \right\}. \tag{3.26}$$

By formula (A.7) in the appendix, we conclude

$$\begin{aligned} E_{\mu^{\epsilon_x \delta_x}} [\mathcal{O}_{xy}^\pi] & \leq E_{\mu^{\epsilon_x \delta_x}} \left[\min \left\{ \frac{e^{t_x}}{\pi_y}, \frac{e^{t_y}}{\pi_x} \right\} \right] \\ & \leq \min \left\{ \frac{1}{\pi_y} E_{\mu^{\epsilon_x \delta_x}} [e^{t_x}], \frac{1}{\pi_x} E_{\mu^{\epsilon_x \delta_x}} [e^{t_y}] \right\} = \min \left\{ \frac{1}{\pi_x}, \frac{1}{\pi_y} \right\}. \end{aligned} \tag{3.27}$$

Since the upper bound is independent of ϵ , we have the same bound for the limit: $\lim_{\epsilon \downarrow 0} E_{\mu^{\epsilon_x \delta_x}} [\mathcal{O}_{xy}^\pi] \leq \min\{\pi_x^{-1}, \pi_y^{-1}\}$. \square

The next lemma deals with the lower order corrections coming from forests with at least two roots.

Lemma 3.4 (Contribution of at least two roots).

$$\lim_{\epsilon \downarrow 0} E_{\mu^\epsilon} [\mathcal{O}_{xy}^\pi 1_{\{|R(T)| > 1\}}] = 0. \tag{3.28}$$

Proof. Let S be a fixed spanning tree of \mathcal{G} . We drop the interaction terms $\beta_{ij}(B_{ij} - 1) \geq 0$ along the edges $(i \sim j) \notin S \cup \{x \sim \rho\}$. This yields

$$\begin{aligned} & E_{\mu^\epsilon} [\mathcal{O}_{xy}^\pi 1_{\{|R(T)| > 1\}}] \\ &= \sum_{T \in \mathcal{T}} 1_{\{|R(T)| > 1\}} \int_{(\mathbb{R}^V)^2} \prod_{j \in V} \frac{dt_j ds_j e^{-t_j}}{2\pi} \prod_{(i \sim j) \in E} e^{-\beta_{ij}(B_{ij}(t,s)-1)} \prod_{i \in V} e^{-\epsilon_i(B_{i\rho}(t,s)-1)} \\ &\quad \cdot \prod_{(i \sim j) \in F(T)} \beta_{ij} e^{t_i+t_j} \prod_{i \in R(T)} \epsilon_i e^{t_i} \cdot \frac{e^{t_x+t_y}}{\pi_x e^{t_x} + \pi_y e^{t_y}} 1_{\{(x \sim \rho) \in T \text{ and } x \leftrightarrow y\}} \\ &\leq \sum_{T \in \mathcal{T}} 1_{\{|R(T)| > 1\}} \int_{(\mathbb{R}^V)^2} \prod_{j \in V} \frac{dt_j ds_j e^{-t_j}}{2\pi} \prod_{(i \sim j) \in S} e^{-\beta_{ij}(B_{ij}(t,s)-1)} \cdot e^{-\epsilon_x(B_{x\rho}(t,s)-1)} \\ &\quad \cdot \prod_{(i \sim j) \in F(T)} \beta_{ij} e^{t_i+t_j} \prod_{i \in R(T)} \epsilon_i e^{t_i} \cdot \frac{e^{t_x}}{\pi_y}. \end{aligned} \tag{3.29}$$

In the following, we first change variables to s_x and gradient variables $s_{ij} := s_i - s_j$, $(i \sim j) \in S$, along the spanning tree S , where the edges in S are oriented in a fixed, but arbitrary way. Since S is a spanning tree, this is a well defined coordinate change. Then we integrate the new variables out.

$$\begin{aligned} & \text{r.h.s. in (3.29)} \\ &= \sum_{T \in \mathcal{T}} 1_{\{|R(T)| > 1\}} \int_{\mathbb{R}^V} \prod_{j \in V} \frac{dt_j e^{-t_j}}{2\pi} \int_{\mathbb{R}} ds_x e^{-\epsilon_x(\cosh t_x - 1 + \frac{1}{2}s_x^2 e^{t_x})} \\ &\quad \cdot \int_{\mathbb{R}^S} \prod_{(i \sim j) \in S} ds_{ij} \prod_{(i \sim j) \in S} e^{-\beta_{ij}(\cosh(t_i-t_j)-1 + \frac{1}{2}s_{ij}^2 e^{t_i+t_j})} \prod_{(i \sim j) \in F(T)} \beta_{ij} e^{t_i+t_j} \prod_{i \in R(T)} \epsilon_i e^{t_i} \cdot \frac{e^{t_x}}{\pi_y} \\ &= \sum_{T \in \mathcal{T}} 1_{\{|R(T)| > 1\}} \int_{\mathbb{R}^V} \prod_{j \in V} \frac{dt_j e^{-t_j}}{\sqrt{2\pi}} \cdot e^{-\epsilon_x(\cosh t_x - 1)} \epsilon_x^{-\frac{1}{2}} e^{-\frac{1}{2}t_x} \\ &\quad \cdot \prod_{(i \sim j) \in S} e^{-\beta_{ij}(\cosh(t_i-t_j)-1)} \beta_{ij}^{-\frac{1}{2}} e^{-\frac{1}{2}(t_i+t_j)} \prod_{(i \sim j) \in F(T)} \beta_{ij} e^{t_i+t_j} \prod_{i \in R(T)} \epsilon_i e^{t_i} \cdot \frac{e^{t_x}}{\pi_y}. \end{aligned} \tag{3.30}$$

Next, we set

$$t'_x := t_x - \log \epsilon, \quad \tau_i := t_i - t'_x - \log \epsilon \tag{3.31}$$

for $i \in V$. In particular, $\tau_x = 0$; thus, we use t'_x and τ_i , $i \in V \setminus \{x\}$, as new integration variables. Note that this substitution is different from the one in the proof of Lemma 3.2. Heuristically speaking, the reason is that in the case of $|R(T)| > 1$ most of the mass of the t_x -integral is located near $t_x \approx +\log \epsilon$, while in the case of one root $R(T) = \{x\}$ the mass is essentially located near $t_x \approx -\log \epsilon$. To do the power counting for ϵ and $e^{t'_x}$ in the following calculation, we use

$$|F(T)| + |R(T)| = |V_\rho| - 1 = |V| = |S| + 1. \tag{3.32}$$

We obtain

$$\begin{aligned}
 (3.30) &= \sum_{T \in \mathcal{T}} 1_{\{|R(T)| > 1\}} \int_{\mathbb{R}} \frac{dt'_x e^{-t'_x}}{\epsilon \sqrt{2\pi}} e^{-\epsilon \pi_x [\frac{1}{2}(\epsilon e^{t'_x} + \epsilon^{-1} e^{-t'_x}) - 1]} (\epsilon \pi_x)^{-\frac{1}{2}} \epsilon^{-\frac{1}{2}} e^{-\frac{1}{2} t'_x} \\
 &\cdot \int_{\mathbb{R}^{V \setminus \{x\}}} \prod_{j \in V \setminus \{x\}} \frac{d\tau_j e^{-\tau_j}}{\epsilon \sqrt{2\pi}} \prod_{(i \sim j) \in S} [e^{-\beta_{ij}(\cosh(\tau_i - \tau_j) - 1)} \beta_{ij}^{-\frac{1}{2}} \epsilon^{-1} e^{-t'_x} e^{-\frac{1}{2}(\tau_i + \tau_j)}] \\
 &\prod_{(i \sim j) \in F(T)} \beta_{ij} \epsilon^2 e^{2t'_x + \tau_i + \tau_j} \prod_{i \in R(T)} \epsilon^2 \pi_i e^{t'_x + \tau_i} \cdot \frac{\epsilon e^{t'_x}}{\pi_y} \\
 &= \epsilon e^{\epsilon \pi_x} \sum_{T \in \mathcal{T}} 1_{\{|R(T)| > 1\}} \int_{\mathbb{R}} \frac{dt'_x}{\sqrt{2\pi}} e^{(\frac{3}{2} - |R(T)|)t'_x} e^{-\frac{\pi_x}{2}(\epsilon^2 e^{t'_x} + e^{-t'_x})} \pi_x^{-\frac{1}{2}} \\
 &\cdot \int_{\mathbb{R}^{V \setminus \{x\}}} \prod_{j \in V \setminus \{x\}} \frac{d\tau_j e^{-\tau_j}}{\sqrt{2\pi}} \prod_{(i \sim j) \in S} [e^{-\beta_{ij}(\cosh(\tau_i - \tau_j) - 1)} \beta_{ij}^{-\frac{1}{2}} e^{-\frac{1}{2}(\tau_i + \tau_j)}] \\
 &\prod_{(i \sim j) \in F(T)} \beta_{ij} e^{\tau_i + \tau_j} \prod_{i \in R(T)} \pi_i e^{\tau_i} \cdot \frac{1}{\pi_y}. \tag{3.33}
 \end{aligned}$$

Next, we drop the term $e^{-\frac{\pi_x}{2} \epsilon^2 e^{t'_x}} \leq 1$. For any $T \in \mathcal{T}$ with $|R(T)| \geq 2$, we obtain

$$\begin{aligned}
 \int_{\mathbb{R}} \frac{dt'_x}{\sqrt{2\pi}} e^{(\frac{3}{2} - |R(T)|)t'_x} e^{-\frac{\pi_x}{2}(\epsilon^2 e^{t'_x} + e^{-t'_x})} &\leq \int_{\mathbb{R}} \frac{dt'_x}{\sqrt{2\pi}} e^{(\frac{3}{2} - |R(T)|)t'_x} e^{-\frac{\pi_x}{2} e^{-t'_x}} \\
 &=: c_4(\pi, |R(T)|) < \infty. \tag{3.34}
 \end{aligned}$$

Note that in this integral, the integrand decays superexponentially for t'_x near $-\infty$ and exponentially for t'_x near $+\infty$. Thus, we get

$$\begin{aligned}
 (3.33) &\leq \epsilon e^{\epsilon \pi_x} \sum_{T \in \mathcal{T}} 1_{\{|R(T)| > 1\}} c_4(\pi, |R(T)|) \pi_x^{-\frac{1}{2}} \int_{\mathbb{R}^{V \setminus \{x\}}} \prod_{j \in V \setminus \{x\}} \frac{d\tau_j e^{-\tau_j}}{\sqrt{2\pi}} \\
 &\prod_{(i \sim j) \in S} [e^{-\beta_{ij}(\cosh(\tau_i - \tau_j) - 1)} \beta_{ij}^{-\frac{1}{2}} e^{-\frac{1}{2}(\tau_i + \tau_j)}] \prod_{(i \sim j) \in F(T)} \beta_{ij} e^{\tau_i + \tau_j} \prod_{i \in R(T)} \pi_i e^{\tau_i} \cdot \frac{1}{\pi_y} \\
 &=: \epsilon e^{\epsilon \pi_x} c_5(\pi, \beta, \mathcal{G}). \tag{3.35}
 \end{aligned}$$

Note that $c_5(\pi, \beta, \mathcal{G}) < \infty$ because the product over $e^{-\beta_{ij}[\cosh(\tau_i - \tau_j) - 1]}$ decays superexponentially fast (recall that $\tau_x = 0$). Summarizing, we get

$$0 \leq E_{\mu^\epsilon} [\mathcal{O}_{xy}^\pi 1_{\{|R(T)| > 1\}}] \leq \epsilon e^{\epsilon \pi_x} c_5(\pi, \beta, \mathcal{G}) \xrightarrow{\epsilon \downarrow 0} 0. \tag{3.36}$$

□

The main theorem 2.2 is now proved by a combination of Lemmas 3.2 and 3.4:

Proof of Theorem 2.2. From (3.9), we get

$$E_{\mu^\epsilon} [\mathcal{O}_{xy}^\pi] = E_{\mu^\epsilon} [\mathcal{O}_{xy}^\pi 1_{\{R(T) = \{x\}\}}] + E_{\mu^\epsilon} [\mathcal{O}_{xy}^\pi 1_{\{|R(T)| > 1\}}]. \tag{3.37}$$

Combining this with Lemma 3.2 and Lemma 3.4 yields

$$\begin{aligned}
 0 &< \lim_{\epsilon \downarrow 0} E_{\mu^\epsilon} [\mathcal{O}_{xy}^\pi] = \lim_{\epsilon \downarrow 0} E_{\mu^{\epsilon x \delta_x}} \left[\mathcal{O}_{xy}^\pi \sqrt{\frac{\pi_x e^{t_x}}{\sum_{i \in V} \pi_i e^{t_i}}} \prod_{i \in V \setminus \{x\}} e^{-\frac{1}{2} \epsilon \pi_i e^{t_i}} \right] \\
 &\leq \lim_{\epsilon \downarrow 0} E_{\mu^{\epsilon x \delta_x}} [\mathcal{O}_{xy}^\pi] < \infty. \tag{3.38}
 \end{aligned}$$

Using (2.15), we obtain

$$\epsilon E_{\mu^\epsilon} [G_{xy}^\epsilon] = E_{\mu^\epsilon} [\mathcal{O}_{xy}^\pi + \mathcal{O}_{yx}^\pi]. \tag{3.39}$$

Applying (3.38) twice, as it is and with x and y interchanged, the claim follows. \square

Finally, specializing the theorem down to ladder graphs, we transfer our results from [3] concerning exponential decay of weights in the case of pinning at one point to the case of uniform pinning (or more general pinning):

Proof of Corollary 2.3. Recall that $c_3 = \min\{\pi_x, \pi_y\}^{-1}$. We estimate

$$\mathcal{O}_{xy}^\pi \leq \frac{e^{t_x+t_y}}{\pi_x e^{t_x} + \pi_y e^{t_y}} \leq c_3 \min\{e^{t_x}, e^{t_y}\} \leq c_3 e^{t_x} e^{\frac{1}{4}(t_y-t_x)}. \tag{3.40}$$

By Lemma A.1, with respect to $\mu_L^{\epsilon_x \delta_x}$, the random variables e^{t_x} and $e^{\frac{1}{4}(t_y-t_x)}$ are stochastically independent and the distribution of $e^{\frac{1}{4}(t_y-t_x)}$ is independent of ϵ_x . Furthermore, $E_{\mu_L^{\epsilon_x \delta_x}} [e^{t_x}] = 1$. Thus, for every $\epsilon > 0$, we have

$$\begin{aligned} E_{\mu_L^{\epsilon_x \delta_x}} [\mathcal{O}_{xy}^\pi] &\leq c_3 E_{\mu_L^{\epsilon_x \delta_x}} [e^{t_x} e^{\frac{1}{4}(t_y-t_x)}] = c_3 E_{\mu_L^{\epsilon_x \delta_x}} [e^{t_x}] E_{\mu_L^{\epsilon_x \delta_x}} [e^{\frac{1}{4}(t_y-t_x)}] \\ &= c_3 E_{\mu_L^{\epsilon_x \delta_x}} [e^{\frac{1}{4}(t_y-t_x)}] = c_3 E_{\mu_L^{\delta_x}} [e^{\frac{1}{4}(t_y-t_x)}]; \end{aligned} \tag{3.41}$$

in the last expectation we replaced ϵ_x by 1.

Let z denote the copy of x at the level of y , i.e. if $x = (n, v)$ and $y = (m, w)$, then $z := (m, v)$. Using the Cauchy-Schwarz inequality, we obtain

$$E_{\mu_L^{\delta_x}} [e^{\frac{1}{4}(t_y-t_x)}] = E_{\mu_L^{\delta_x}} [e^{\frac{1}{4}(t_y-t_z)} e^{\frac{1}{4}(t_z-t_x)}] \leq E_{\mu_L^{\delta_x}} [e^{\frac{1}{2}(t_y-t_z)}]^{\frac{1}{2}} E_{\mu_L^{\delta_x}} [e^{\frac{1}{2}(t_z-t_x)}]^{\frac{1}{2}}. \tag{3.42}$$

By Theorem 2.1 in [3], there exist constants $c_6, c_7 > 0$ depending only on Γ_0 and β such that

$$E_{\mu_L^{\delta_x}} [e^{\frac{1}{2}(t_z-t_x)}] \leq c_6 e^{-c_7|z-x|} = c_6 e^{-c_7|y-x|}. \tag{3.43}$$

For the points y and z on the same level, estimate (7.6) from [3] states

$$E_{\mu_L^{\delta_x}} [e^{\frac{1}{2}(t_y-t_z)}] \leq c_8 \tag{3.44}$$

with a constant c_8 depending only on Γ_0 and β . Summarizing, (3.41)–(3.44) yield

$$E_{\mu_L^{\epsilon_x \delta_x}} [\mathcal{O}_{xy}^\pi] \leq c_3 (c_6 c_8)^{\frac{1}{2}} e^{-\frac{1}{2}c_7|y-x|} =: \frac{c_1 c_3}{2} e^{-c_2|y-x|} \tag{3.45}$$

with constants $c_1(\Gamma_0, \beta), c_2(\Gamma_0, \beta) > 0$ uniformly in $\epsilon > 0$. This shows

$$\lim_{\epsilon \downarrow 0} E_{\mu_L^{\epsilon_x \delta_x}} [\mathcal{O}_{xy}^\pi] \leq \frac{c_1 c_3}{2} e^{-c_2|y-x|}. \tag{3.46}$$

Interchanging the roles of x and y , we get the same upper bound for $\lim_{\epsilon \downarrow 0} E_{\mu_L^{\epsilon_y \delta_y}} [\mathcal{O}_{yx}^\pi]$. An application of Theorem 2.2 yields the claim. \square

A Appendix: Product structure of the model with single pinning

When transforming the model $H^{2|2}$ with pinning at one point to gradient variables, it exhibits a certain product structure coming from scaling symmetry. This is made precise in the following lemma.

Lemma A.1. *With respect to $\mu^{\varepsilon_x \delta_x}$, the random pair (t_x, s_x) is independent of the random vector consisting of the (rescaled) gradient variables*

$$(t'_i := t_i - t_x, s'_i := (s_i - s_x)e^{t_x})_{i \in V \setminus \{x\}}. \tag{A.1}$$

The joint distribution of (t_x, s_x) with respect to $\mu^{\varepsilon_x \delta_x}$ has the density

$$\frac{\varepsilon_x}{2\pi} \exp \left[-\varepsilon_x \left(\cosh t_x - 1 + \frac{1}{2} s_x^2 e^{t_x} \right) \right], \tag{A.2}$$

independently of the graph \mathcal{G} . In particular,

$$E_{\mu^{\varepsilon_x \delta_x}} [e^{t_x}] = 1. \tag{A.3}$$

The joint distribution of $(t'_i, s'_i)_{i \in V \setminus \{x\}}$ does not depend on ε_x .

Proof. Recall the definition of μ^ε given in (2.6). In the special case $\varepsilon = \varepsilon_x \delta_x$, the random tree T contains $\mu^{\varepsilon_x \delta_x}$ -almost surely the edge $x \sim \rho$, but no other edge of the type $i \sim \rho$, $i \neq x$. Hence, we get

$$\begin{aligned} & \mu^{\varepsilon_x \delta_x} (dt ds dT) \\ &= \prod_{j \in V} \frac{dt_j ds_j e^{-t_j}}{2\pi} dT e^{-\varepsilon_x (B_{x\rho}(t,s)-1)} \varepsilon_x e^{t_x} \prod_{(i \sim j) \in E} e^{-\beta_{ij} (B_{ij}(t,s)-1)} \prod_{(i \sim j) \in F(T)} \beta_{ij} e^{t_i + t_j}. \end{aligned} \tag{A.4}$$

Let $\nu^{\varepsilon_x \delta_x}$ denote the joint distribution of $(t_x, s_x, (t'_i, s'_i)_{i \neq x})$. We set $t'_x := 0$ and $s'_x := 0$. Note that $(s_i - s_j)^2 e^{t_i + t_j} = (s'_i - s'_j)^2 e^{t'_i + t'_j}$. Changing variables accordingly and denoting the set of spanning trees of the graph \mathcal{G} by $\mathcal{T}_{\mathcal{G}}$, we obtain

$$\begin{aligned} \nu^{\varepsilon_x \delta_x} (dt_x ds_x dt' ds') &= \frac{dt_x ds_x e^{-t_x}}{2\pi} \prod_{j \in V \setminus \{x\}} \frac{dt'_j ds'_j e^{-2t_x - t'_j}}{2\pi} \\ & e^{-\varepsilon_x (\cosh(t_x) - 1 + \frac{1}{2} s_x^2 e^{t_x})} \varepsilon_x e^{t_x} \prod_{(i \sim j) \in E} e^{-\beta_{ij} [B_{ij}(t', s') - 1]} \sum_{T \in \mathcal{T}_{\mathcal{G}}} \prod_{(i \sim j) \in T} \beta_{ij} e^{2t_x + t'_i + t'_j} \\ &= \frac{dt_x ds_x \varepsilon_x}{2\pi} e^{-\varepsilon_x (\cosh(t_x) - 1 + \frac{1}{2} s_x^2 e^{t_x})} \\ & \cdot \prod_{j \in V \setminus \{x\}} \frac{dt'_j ds'_j e^{-t'_j}}{2\pi} \prod_{(i \sim j) \in E} e^{-\beta_{ij} [B_{ij}(t', s') - 1]} \sum_{T \in \mathcal{T}_{\mathcal{G}}} \prod_{(i \sim j) \in T} \beta_{ij} e^{t'_i + t'_j}. \end{aligned} \tag{A.5}$$

In the special case of the graph \mathcal{G} consisting of only one point x , i.e. $V = \{x\}$ and $E = \emptyset$, the measure $\nu^{\varepsilon_x \delta_x}$ has the density given in (A.2). Since $\nu^{\varepsilon_x \delta_x}$ is a probability measure, the density in (A.2) is normalized to have total mass one. Consequently, given the product structure in (A.5), for a general graph \mathcal{G} , the random vectors (t_x, s_x) and (t', s') are independent with the claimed first marginal and the second marginal not depending on ε_x . Finally, we calculate

$$\begin{aligned} E_{\mu^{\varepsilon_x \delta_x}} [e^{t_x}] &= \frac{\varepsilon_x}{2\pi} \int_{\mathbb{R}^2} e^{t_x} \exp[-\varepsilon_x (\cosh t_x - 1 + \frac{1}{2} s_x^2 e^{t_x})] ds_x dt_x \\ &= \sqrt{\frac{\varepsilon_x}{2\pi}} \int_{\mathbb{R}} e^{\frac{t_x}{2}} \exp[-\varepsilon_x (\cosh t_x - 1)] dt_x \\ &\stackrel{\text{(by symmetry)}}{=} \sqrt{\frac{\varepsilon_x}{2\pi}} \int_{\mathbb{R}} e^{-\frac{t_x}{2}} \exp[-\varepsilon_x (\cosh t_x - 1)] dt_x \\ &= E_{\mu^{\varepsilon_x \delta_x}} [1] = 1. \end{aligned} \tag{A.6}$$

□

Using supersymmetry, identity (A.3) can be generalized as follows.

Lemma A.2 (Formula (B.3) in [6]). *For any $y \in V$ and any choice of ε we have*

$$E_{\mu^\varepsilon}[e^{t y}] = 1. \quad (\text{A.7})$$

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