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A rotor configuration with maximum escape rate*

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Abstract

The rotor walk is a deterministic analogue of the simple random walk. For any given graph, we construct a rotor configuration for which the escape rate of the corresponding rotor walk is equal to the escape rate of the simple random walk, and thus answer a question of Florescu, Ganguly, Levine, and Peres (2014).

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1 Introduction

Let $G := (V, E)$ be a graph that is connected, simple (i.e., no loops and multiple edges) and locally finite (i.e., every vertex has finitely many neighbors). In a rotor walk [WLB96, PDDK96], each vertex has a *rotor*, which is an outgoing edge of the vertex. All of the rotors together constitute a *rotor configuration*, which is encoded by a function ρ that maps every vertex of G to one of its outgoing edges. To each vertex x we assign a fixed *rotor mechanism*, which is a cyclic ordering on the set of outgoing edges $\mathcal{E}_x$ of x, and is encoded by a bijection $m_x : \mathcal{E}_x \to \mathcal{E}_x$ that has only one orbit.

The rotor walk evolves in the following manner. A particle is initially located at a fixed vertex o. At each time step, the rotor at the particle's current location is first incremented to the next edge in the cyclic order, and the particle moves to the target vertex of the new rotor.

Propp [Pro03] proposed the rotor walk as a derandomized version of the simple random walk, and this naturally invited a comparison between the two walks. One such comparison is given by the following experiment. Start with an initial rotor configuration ρ, and with n particles initially located at o. At each time step, each of these n particles will take turns in performing one step of the rotor walk, and the particle is removed if it ever returns to o. Denote by $I(\rho, n)$ the number of particles that never return to o.

Schramm [HP10, FGLP14] showed that the *escape rate* of the rotor walk is always bounded above by the escape rate of the simple random walk. That is to say, for any rotor configuration ρ:

$$\limsup_{n\to\infty} \frac{I(\rho, n)}{n} \leq \alpha_G, \tag{1.1}$$

where α_G is the probability for the simple random walk starting at o to never return to o.

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The result of Schramm inspired Florescu, Ganguly, Levine, and Peres [FGLP14] to ask if there is always a rotor configuration with escape rate equal to α_G. Such a configuration has been constructed for certain choices of G, such as for the binary tree [LL09]; for transient trees [AH11]; for $\mathbb{Z}^d$ with $d \geq 3$ [He14]; and for transient vertex-transitive graphs [Cha19].

In this paper, we resolve the question of Florescu et al. by constructing a rotor configuration with maximum escape rate for any given graph. We focus on the case when G is a transient graph, as any rotor configuration on a recurrent graph has escape rate equal to 0 by (1.1).

Let $\mathcal{G} : V \to \mathbb{R}_{\geq 0}$ be the *Green function* of G, which maps $x \in V$ to the expected number of visits to x by the simple random walk on G starting at o. We define the *weight* of a directed edge (x, y) of G to be

$$w(x,y) := \frac{-1}{\deg(x)} \sum_{i=0}^{\deg(x)-1} i \frac{\mathcal{G}(y_{i+1})}{\deg(y_{i+1})}, \tag{1.2}$$

where $(x, y_i) := m_x^i(x, y)$ is the edge obtained by incrementing the edge (x, y) for i consecutive times by using the rotor mechanism at x.

Theorem 1.1. *Let G be a transient graph that is connected, simple, and locally finite. If $\rho_{\min}$ is a rotor configuration such that, for any vertex x and any outgoing edge (x, y) of x,*

$$w(\rho_{\min}(x)) \leq w(x,y), \tag{1.3}$$

then

$$\lim_{n\to\infty} \frac{I(\rho_{\min}, n)}{n} = \alpha_G.$$

Theorem 1.1 is proved by constructing an invariant of the rotor walk that balances between the Green function of the location of the particles and the weight of the edges in the rotor configuration at any given time.

Note that one can always construct a rotor configuration ρ satisfying (1.3), by defining $\rho(x)$ for any $x \in V$ to be the edge for which its weight is the minimum among all outgoing edges of x. Also note that (1.3) is not a necessary condition, as almost all configurations with maximum escape rate from other works (mentioned above) do not satisfy (1.3).

2 Proof of Theorem 1.1

We now give a formal definition to the experiment in Section 1. Let ρ be the initial rotor configuration, and let n be the number of particles. The location of the particles $X_t^{(0)}, X_t^{(1)}, \ldots, X_t^{(n-1)}$ and the rotor configuration ρ_t at the t-th step of the experiment $(t \geq 0)$ are given by the following recurrence:

- Initially, $X_0^{(i)} = o$ for $i \in \{0, 1, \ldots, n-1\}$ and $\rho_0 = \rho$;
- Write $i_t := t + 1 \bmod n$. If the i_t-th particle has returned to o (i.e. $X_t^{(i_t)} = o$ and $X_s^{(i_t)} \neq o$ for some $s < t$), then

$$\rho_{t+1} = \rho_t, \quad \text{and} \quad X_{t+1}^{(i)} = X_t^{(i)} \text{ for } i \in \{0, \ldots, n-1\}.$$

- If the i_t-th particle has not returned to o, then

$$\rho_{t+1}(x) = \begin{cases} m_x(\rho_t(x)) & \text{if } x = X_t^{(i_t)}; \\ \rho_t(x) & \text{otherwise.} \end{cases}$$

$$X_{t+1}^{(i)} = \begin{cases} \text{target vertex of } \rho_{t+1}(X_t^{(i)}) & \text{if } i = i_t; \\ X_t^{(i)} & \text{otherwise.} \end{cases}$$

That is, at time t, the i_t-th particle performs one step of a rotor walk if it has not returned to o, and does nothing if it has returned to o.

We denote by $R_t := R_t(\rho, n)$ the *range* of the experiment at time t,

$$R_t := \{X_s^{(i)} \mid i \in \{0, 1, \dots, n-1\} \text{ and } s \leq t\}.$$

We now define an invariant of the rotor walk that is a a special case of the invariant introduced in [HP10, Proposition 13]; a related invariant has been used in [HS11] and [HS12] to study the rotor-router aggregation of comb lattices. Let $M_t := M_t(\rho, n)$ $(t \geq 0)$ be given by:

$$M_t := \sum_{i=0}^{n-1} \frac{\mathcal{G}(X_t^{(i)})}{\deg(X_t^{(i)})} + \frac{\min\{t, n\}}{\deg(o)} + \sum_{x \in R_t} \left(w(\rho_t(x)) - w(\rho(x)) \right). \tag{2.1}$$

Proposition 2.1. *For any initial rotor configuration ρ, any $n \geq 1$, and any $t \geq 0$, we have*

$$M_t = n \frac{\mathcal{G}(o)}{\deg(o)}.$$

We will use the fact that the Green function is a voltage function when a unit current enters G through o [LP16, Proposition 2.1]. That is, for any $x \in V$,

$$\frac{1}{\deg(x)} \sum_{y \sim x} \frac{\mathcal{G}(y)}{\deg(y)} = \frac{\mathcal{G}(x)}{\deg(x)} - \frac{\mathbb{1}\{x = o\}}{\deg(o)}, \tag{2.2}$$

where $y \sim x$ means that y is a neighbor of x in G.

Proof of Proposition 2.1. It follows directly from the definition that $M_0 = n \frac{\mathcal{G}(o)}{\deg(o)}$. Therefore it suffices to show that, for any $t \geq 0$:

$$M_{t+1} - M_t = 0.$$

Recall that $i_t := t + 1 \bmod n$. Write $\alpha_t := X_t^{(i_t)}$ and $\beta_t := X_{t+1}^{(i_t)}$. If the i_t-th particle has returned to o by time t, then no action is performed at time t, and $M_{t+1} = M_t$. If the i_t-th particle has not returned to o by time t, then it follows from the definition of M_t and M_{t+1} in (2.1) that

$$\begin{aligned} M_{t+1} - M_t = & \frac{\mathcal{G}(\beta_t)}{\deg(\beta_t)} - \frac{\mathcal{G}(\alpha_t)}{\deg(\alpha_t)} + \frac{\mathbb{1}\{t \leq n-1\}}{\deg(o)} \\ & + w(\rho_{t+1}(\alpha_t)) - w(\rho_t(\alpha_t)). \end{aligned} \tag{2.3}$$

On the other hand, we have from the definition of w in (1.2) that

$$\begin{aligned} w(\rho_{t+1}(\alpha_t)) - w(\rho_t(\alpha_t)) = & \frac{-1}{\deg(\alpha_t)} \left(\deg(\alpha_t) \frac{\mathcal{G}(\beta_t)}{\deg(\beta_t)} - \sum_{y \sim \alpha_t} \frac{\mathcal{G}(y)}{\deg(y)} \right) \\ = & - \frac{\mathcal{G}(\beta_t)}{\deg(\beta_t)} + \frac{1}{\deg(\alpha_t)} \sum_{y \sim \alpha_t} \frac{\mathcal{G}(y)}{\deg(y)}. \end{aligned}$$

Applying (2.2) to the equation above then gives us

$$w(\rho_{t+1}(\alpha_t)) - w(\rho_t(\alpha_t)) = - \frac{\mathcal{G}(\beta_t)}{\deg(\beta_t)} + \frac{\mathcal{G}(\alpha_t)}{\deg(\alpha_t)} - \frac{\mathbb{1}\{\alpha_t = o\}}{\deg(o)}. \tag{2.4}$$

Combining (2.3) and (2.4), we then get

$$M_{t+1} - M_t = \frac{\mathbb{1}\{t \leq n-1\}}{\deg(o)} - \frac{\mathbb{1}\{\alpha_t = o\}}{\deg(o)}.$$

Since the i_t-th particle has not returned to o yet by time t, this means that the $\alpha_t = o$ if and only if $t \leq n-1$ (i.e., the i_t-th particle has not left o yet). This then implies $M_{t+1} - M_t = 0$ by the equation above, and the proof is complete. ☐

Let $I_t(\rho, n)$ be the number of particles that have not returned to o by time t.

Proposition 2.2. *If $\rho_{\min}$ is a configuration that satisfies* (1.3)*, then for any $n \geq 1$ and any $t \geq n$,*

$$\frac{I_t(\rho_{\min}, n)}{n} \geq \alpha_G.$$

Proof. Let $S_t \subseteq \{0, 1, \ldots, n-1\}$ be the set of particles that has returned to o by time t. Since the Green function is a nonnegative function, we have:

$$\sum_{i=0}^{n-1} \frac{\mathcal{G}(X_t^{(i)})}{\deg(X_t^{(i)})} \geq \sum_{i \in S_t} \frac{\mathcal{G}(X_t^{(i)})}{\deg(X_t^{(i)})} = (n - I_t(\rho_{\min}, n)) \frac{\mathcal{G}(o)}{\deg(o)}. \tag{2.5}$$

Since $\rho_{\min}$ satisfies (1.3), we also have

$$\sum_{x \in R_t} \left(w(\rho_t(x)) - w(\rho_{\min}(x)) \right) \geq 0. \tag{2.6}$$

Plugging (2.5) and (2.6) into the definition of M_t in (2.1), we have that, for $t \geq n$,

$$M_t \geq (n - I_t(\rho_{\min}, n)) \frac{\mathcal{G}(o)}{\deg(o)} + \frac{n}{\deg(o)},$$

which is equivalent to

$$\frac{I_t(\rho_{\min}, n)}{n} \geq 1 + \frac{1}{\mathcal{G}(o)} - \frac{\deg(o)}{n\, \mathcal{G}(o)} M_t.$$

Plugging in the value of M_t from Proposition 2.1, we then have:

$$\frac{I_t(\rho_{\min}, n)}{n} \geq \frac{1}{\mathcal{G}(o)},$$

and the conclusion now follows by noting that $\alpha_G = 1/\,\mathcal{G}(o)$. ☐

We now present the proof of Theorem 1.1.

Proof of Theorem 1.1. We have for any $n \geq 1$,

$$\frac{I(\rho_{\min}, n)}{n} = \lim_{t \to \infty} \frac{I_t(\rho_{\min}, n)}{n} \geq \alpha_G,$$

where the inequality is due to Proposition 2.2. The theorem now follows by combining the inequality above with (1.1). ☐

3 Open problems

(i) Classify all $c \geq 0$ for which there is a rotor configuration ρ such that its escape rate is equal to c, i.e., $\lim_{n \to \infty} I(\rho, n)/n = c$.

To date the only known result of this kind is due to Landau and Levine [LL09], which shows that, for the complete binary tree, the constant c can range from 0 to α_G.

(ii) Consider the random rotor configuration ρ where $(\rho(x))_{x \in V}$ are independent and uniformly distributed among the outgoing edges of x. What is the probability that ρ has escape rate equal to α_G?

Angel and Holroyd [AH11] showed that this probability is 1 if G is a complete b-ary tree. The author [Cha19] also showed the same result if G is a transient vertex-transitive graph and the configuration is sampled from the oriented wired spanning forest measure.

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