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Practical criteria for R-positive recurrence of unbounded semigroups*

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Abstract

The goal of this note is to show how recent results on the theory of quasi-stationary distributions allow us to deduce general criteria for the geometric convergence of normalized unbounded semigroups.

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1 Introduction

Let E be a measurable space and $(P_n, n \in \mathbb{Z}_+)$ be a positive semigroup of operators on the space $L^\infty(\psi_1)$ to itself, where $\psi_1 : E \to (0, +\infty)$ is measurable and $L^\infty(\psi_1)$ is the set of measurable $f : E \to \mathbb{R}$ such that $|f|/\psi_1$ is bounded, endowed with the norm $\|f\|_{\psi_1} = \||f|/\psi_1\|_\infty$. We define the dual action of $(P_n, n \in \mathbb{Z}_+)$ on non-negative measures μ on E such that $\mu(\psi_1) < +\infty$ as

$$\mu P_n f = \int_E P_n f(x)\mu(\mathrm{d}x). \tag{1.1}$$

Our aim is to provide sufficient conditions for the existence of $\theta_0 > 0$ such that $(\theta_0^{-n} P_n)_{n\in\mathbb{N}}$ converges geometrically toward a non-trivial limit.

In this setting, given c such that $P_1\psi_1 \leq c\psi_1$, the operators $Q_n = \frac{P_n(\cdot\psi_1)}{c^n\psi_1}$ defines a sub-Markov semigroup corresponding to a stochastic process with killing. The asymptotic behavior of such semigroups is the subject of the theory of quasi-stationary distributions based on various tools, including the theory of R-recurrent Markov chains [31, 29, 28, 17], spectral theoretic results (e.g. Krein-Rutman theorem [13], spectral theory of symetric operators [8, 24], or other general criteria of convergence of normalized semigroups such as the work of Birkhoff [7] and its extensions) and Doeblin's conditions and Foster-Lyapunov criteria [9, 10]. In this note, we apply the results of [10] to the semigroup $(Q_n, n \in \mathbb{Z}_+)$ to give a necessary and sufficient condition for the existence of a non-negative eigenfunction η of P_1 with eigenvalue θ_0 and the geometric convergence of

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$\theta_0^{-n}P_n$. We also extend these results to continuous-time semigroups. In particular, our results provide practical criteria for the general theory of R-positive recurrence of unbounded semigroups as developed in [29, Section 6.2] and [28]. The notion of R-positive recurrence has strong implications for the study of penalized Markov processes [14, 15], of the long time behaviour of Markov branching processes (see for instance [20, 21, 22, 6, 23, 11, 5, 3, 4]), of non-conservative PDEs (see e.g. [1, 2] and references therein) and of measure-valued Pólya processes and reinforced processes [25].

The recent article [2] proposes similar criteria for R-positive recurrence of continous-time semigroups with nice applications to growth-fragmentation equations. The extent of our results and approaches sensibly differ. Concerning the results, our criteria apply to a larger class of semigroups including non-irreducible ones (see Remark 2.5 below). Concerning the approaches, the authors of [2] make use of tools developed in the proofs of [10] adapted to the semigroup setting. We show here how these R-positivity criteria can be directly derived as corollaries of the results of [10], applied to the sub-Markov semigroup $(Q_n, n\in\mathbb{Z}_+)$. This approach also has the advantage to allow one to deduce with little extra effort sufficient criteria for the convergence of unbounded semigroups from the abundant theory of sub-Markov processes (cf. e.g. [13, 12, 32, 18, 24, 19]). Note that a similar approach has been used in [5] to describe the asymptotic behaviour of the growth-fragmentation equation using Krein-Rutman theorem and other criteria for R-positivity. Finally, the authors of [2] also establish a counterpart assuming the existence of a positive eigenfunction of the semigroup and using the approach of [9]. In Theorem 2.3, we extend this counterpart by allowing the eigenfunction to vanish and exhibit the link with the classical theory of V-ergodicity [27, 16].

Section 2 is devoted to the statement and the proof of our main results. In Section 3, we provide two applications of these general results to penalized semigroups associated to perturbed (discrete-time) dynamical systems (Subsection 3.1) and diffusion processes (Subsection 3.2).

2 Main result

We first introduce the assumptions on which our results are based. We state them following the same structure as Assumption (E) in [10] to emphasize their similarity.

Condition (G). There exist positive real constants $\theta_1,\theta_2,c_1,c_2,c_3$, an integer $n_1\geq 1$, two functions $\psi_1:E\to(0,+\infty)$, $\psi_2:E\to\mathbb{R}_+$ and a probability measure ν on a measurable subset K of E such that

(G1) *(Local Dobrushin coefficient).* $\forall x\in K$ and all measurable $A\subset K$,

$$P_{n_1}(\psi_1\mathbb{1}_A)(x)\geq c_1\nu(A)\psi_1(x).$$

(G2) *(Global Lyapunov criterion).* We have $\theta_1<\theta_2$ and

$$\begin{aligned}&\inf_{x\in K}\psi_2(x)/\psi_1(x)>0,\ \sup_{x\in E}\psi_2(x)/\psi_1(x)\leq 1,\\ &P_1\psi_1(x)\leq\theta_1\psi_1(x)+c_2\mathbb{1}_K(x)\psi_1(x),\ \forall x\in E,\\ &P_1\psi_2(x)\geq\theta_2\psi_2(x),\ \forall x\in E.\end{aligned}$$

(G3) *(Local Harnack inequality).* We have

$$\sup_{n\in\mathbb{Z}_+}\frac{\sup_{y\in K}P_n\psi_1(y)/\psi_1(y)}{\inf_{y\in K}P_n\psi_1(y)/\psi_1(y)}\leq c_3.$$

(G4) *(Aperiodicity).* For all $x\in K$, there exists $n_4(x)$ such that for all $n\geq n_4(x)$,

$$P_n(\mathbb{1}_K\psi_1)>0.$$

Theorem 2.1. *Assume that Condition (G) holds true. Then there exist a positive measure ν_P on E such that $\nu_P(\psi_1) = 1$ and $\nu_P(\psi_2) > 0$, and two constants $C < +\infty$ and $\alpha \in (0,1)$ such that, for all measurable functions $f : E \to \mathbb{R}$ satisfying $|f| \leq \psi_1$ and all positive measures μ on E such that $\mu(\psi_1) < +\infty$ and $\mu(\psi_2) > 0$,*

$$\left|\frac{\mu P_n f}{\mu P_n \psi_1} - \nu_P(f)\right| \leq C\alpha^n \frac{\mu(\psi_1)}{\mu(\psi_2)}, \quad \forall n \in \mathbb{Z}_+. \tag{2.1}$$

In addition, there exist $\theta_0 > 0$ such that $\nu_P P_n = \theta_0^n \nu_P$ and a function $\eta : E \to \mathbb{R}_+$ such that $\theta_0^{-n} P_n \psi_1$ converges uniformly and geometrically toward η in $L^\infty(\psi_1)$ and such that $P_1\eta = \theta_0\eta$ and $\nu_P(\eta) = \nu_P(\psi_1) = 1$. Moreover, there exist two constants $C' > 0$ and $\beta \in (0,1)$ such that, for all measurable functions $f : E \to \mathbb{R}$ satisfying $|f| \leq \psi_1$ and all positive measures μ on E such that $\mu(\psi_1) < +\infty$,

$$\left|\theta_0^{-n}\mu P_n f - \mu(\eta)\nu_P(f)\right| \leq C'\beta^n \mu(\psi_1). \tag{2.2}$$

Remark 2.2. Note that (G2) implies that $P_n\psi_1 \leq cP_n\psi_2$ on K for all $n \geq 0$ and some constant $c > 0$ (see [10, Lemma 9.6]). Hence we have, for all $x \in K$,

$$P_n\psi_1(x)/\psi_1(x) \leq c\, P_n\psi_2(x)/\psi_1(x) \leq c\, P_n\psi_2(x)/\psi_2(x)$$

and

$$P_n\psi_2(x)/\psi_2(x) \leq P_n\psi_1(x)/\psi_2(x) \leq \sup_K \frac{\psi_1}{\psi_2}\, P_n\psi_1(x)/\psi_1(x).$$

Therefore, replacing ψ_1 by ψ_2 in (G1) and/or (G3) give equivalent versions of Condition (G).

Proof. Assumption (G2) implies that $P_1\psi_1 \leq (\theta_1 + c_2)\psi_1$, so that $Q_1 f := \frac{P_1(f\psi_1)}{(\theta_1+c_2)\psi_1}$ defines a submarkovian kernel generating the semigroup $(Q_n)_{n\in\mathbb{N}}$ defined by

$$Q_n(f) = \frac{P_n(f\,\psi_1)}{(\theta_1 + c_2)^n\psi_1},\ \forall n \geq 0,\ \|f\|_\infty \leq 1.$$

It is straightforward to check that this semigroup satisfies conditions (E1–E4) in [10] with $\varphi_1 = 1$ and $\varphi_2 = \psi_2/\psi_1$, using $\theta_1/(\theta_1 + c_2)$ in place of θ_1, $\theta_2/(\theta_1 + c_2)$ in place of θ_2 and $c_1/(\theta_1 + c_2)^{n_1}$ in place of c_1. Using Theorem 2.1 in this reference applied to Q_n, we deduce that there exist constants $C > 0, \alpha \in (0,1)$ and a probability measure ν_{QSD} on E such that, for all bounded measurable functions $g : E \to \mathbb{R}$ and all probability measures υ such that $\upsilon(\varphi_2) > 0$,

$$\left|\frac{\upsilon Q_n g}{\upsilon Q_n \mathbb{1}} - \nu_{QSD}(g)\right| \leq C\alpha^n \frac{\|g\|_\infty}{\upsilon(\varphi_2)}.$$

Setting $\nu_P(dx) = \frac{1}{\psi_1(x)}\nu_{QSD}(dx)$, $\mu(dx) = \frac{1}{\psi_1(x)}\upsilon(dx)$ and $f = g\,\psi_1$, one obtains (2.1). Similarly, Theorem 2.5 of [10] for Q_n states that there exist $\theta_Q > 0$ such that $\nu_{QSD}Q_n = \theta_Q^n\nu_{QSD}$ and a function $\eta_Q : E \to \mathbb{R}_+$ such that $\theta_Q^{-n}Q_n\mathbb{1}$ converges uniformly and geometrically toward η_Q in L^∞ and such that $Q_1\eta_Q = \theta_Q\eta_Q$. Setting $\eta = \eta_Q\psi_1$ and $\theta_0 = \theta_Q(\theta_1 + c_2)$ gives the result on geometric convergence of $\theta_0^{-n}P_n\psi_1$ to η in $L^\infty(\psi_1)$.

It remains to prove (2.2). Note that it is sufficient to prove it for any $\mu = \delta_x$. If $\eta(x) = 0$, this is implied by the above geometric convergence. If $\eta(x) > 0$, then $\eta_Q(x) > 0$ and the convergence of [10, Theorem 2.7] applied to Q_n implies that there exists $C' < +\infty$ and $\tilde\alpha \in (0,1)$ such that, for all measurable $g : E \to \mathbb{R}$ satisfying $|g| \leq 1/\eta_Q$,

$$\left|\theta_Q^{-n}\frac{Q_n(g\eta_Q)(x)}{\eta_Q(x)} - \nu_{QSD}(g\eta_Q)\right| \leq C'\tilde\alpha^n \frac{1}{\eta_Q(x)}.$$

Multiplying both sides by $\eta_Q(x)\psi_1(x)$ and setting $f = g\eta_Q\psi_1$ ends the proof of (2.2). □

Whether Assumption (G) is necessary for (2.1) is still an open problem up to our knowldge. However, if one assumes that there exists a positive eigenfunction η such that (2.2) holds true, then one recovers easily Assumption (G) by applying the classical counterpart of Forster-Lyapunov criteria for conservative semigroups. Here, the conservative semigroup is the one associated to the h-tranform of P_n defined by $R_n f := \frac{\theta_0^{-n}}{\eta} P_n(\eta f)$ (which is called Q-process in the sub-Markovian case, cf. e.g. [26]). The only difficulty in the proof of the following theorem is that η may vanish on some subset of E.

Theorem 2.3. *Assume that there exist a positive function $\psi : E \to (0, +\infty)$ and a non-negative eigenfunction $\eta \in L^\infty(\psi)$ of P_1 for the eigenvalue $\theta_0 > 0$, such that*

$$\left|\theta_0^{-n} P_n f(x) - \eta(x)\nu_P(f)\right| \leq \zeta_n \psi(x) \tag{2.3}$$

is satisfied for all $x \in E$ and all measurable functions $f : E \to \mathbb{R}$ such that $|f| \leq \psi$, where $(\zeta_n)_{n\geq 0}$ is some positive sequence converging to 0. Then Assumption (G) is satisfied with $\psi_2 = \eta$ and with some function $\psi_1 \in L^\infty(\psi)$ such that $\psi \in L^\infty(\psi_1)$.

Proof. We define $E' = \{x \in E,\ \eta(x) > 0\}$ and introduce the conservative semigroup R on functions $g : E' \to \mathbb{R}$ such that $|g(x)| \leq \psi(x)/\eta(x)$ defined by

$$R_n g(x) = \frac{\theta_0^{-n}}{\eta(x)} P_n(\eta g)(x),\ \forall x \in E' \text{ and } n \geq 0.$$

Applying (2.3) to $f = g\eta$ and setting $\nu_R(dx) = \eta(x)\nu_P(dx)$, we deduce that, for all $x \in E'$ and all measurable function $g : E' \to \mathbb{R}$ such that $|g| \leq \psi/\eta$

$$|R_n g(x) - \nu_R(g)| \leq \zeta_n \frac{\psi(x)}{\eta(x)}.$$

This is the classical V-uniform ergodicity condition (with $V = \psi/\eta$), for which necessary and sufficient conditions are well-known. First, it implies V-uniform geometric ergodicity, i.e. one can replace ζ_n by $C\beta^n$ for some $C > 0, \beta \in (0,1)$ in the above equation (see for instance Proposition 15.2.3 in [16]). Second, we deduce using for example Theorem 15.2.4(b) in [16] that, for any integer m such that $C^{1/m}\beta < 1$ and any λ, ρ such that $C^{1/m}\beta \leq \lambda < \rho < 1$, there exist $d, C_R < +\infty$ such that

$$R_1 V_0(x) \leq \rho V_0(x) + C_R \mathbb{1}_K(x), \quad \forall x \in E', \tag{2.4}$$

with

$$V_0 = \sum_{k=0}^{m-1} \lambda^{-k} R_k\left(\frac{\psi}{\eta}\right)$$

and $K := \{\psi/\eta \leq d\} \cap E'$ is an accessible small set for R. This last property means that there exists a probability measure ν_R on E' and a constant $c_R > 0$ such that, for all $A \subset K$ measurable,

$$R_{n_1'} \mathbb{1}_A(x) \geq c_R \nu_R(A), \quad \forall x \in K,$$

for some constant integer $n_1' \geq 1$. Since K is accessible, there exists $n_1'' \geq 0$ such that $a := \nu_R R_{n_1''} \mathbb{1}_K > 0$. Setting $n_1 = n_1' + n_1''$, it then follows that

$$P_{n_1}(\psi \mathbb{1}_A)(x) \geq c_R \theta_0^{n_1} \eta(x)\, \nu_R R_{n_1''}\left(\mathbb{1}_K \mathbb{1}_A \frac{\psi}{\eta}\right), \quad \forall x \in K.$$

Due to the definition of K, we deduce that (G1) holds true with $c_1 = a c_R \theta_0^{n_1}/d$ and the probability measure $\nu(dx) = \frac{\psi(x)}{a\eta(x)} \mathbb{1}_K(x)(\nu_R R_{n_1''})(dx)$.

Defining $\psi_1 = \eta V_0$, we also deduce from (2.4) that,

$$P_1\psi_1(x) \leq \theta_0\rho\psi_1(x) + C_R \mathbb{1}_K(x)\eta(x) \leq \theta_0\rho\psi_1(x) + \frac{C_R}{\sup_E |\eta|/\psi_1}\mathbb{1}_K(x)\psi_1(x), \quad \forall x \in E'.$$

In view of the definition of $V_0(x)$ for all $x \in E'$, we have

$$\psi_1(x) = \sum_{k=0}^{m-1} (\lambda\theta_0)^{-k} P_k\psi(x),$$

which also makes sense for $x \in E \setminus E'$. For such an x, we deduce from (2.3) that $P_n\psi(x) \leq \zeta_n\theta_0^n\psi(x)$. Without loss of generality, increasing m, λ and ρ if necessary, we can assume that $\zeta_m^{1/m} \leq \lambda < \rho < 1$. Then,

$$P_1\psi_1(x) = \lambda\theta_0\psi_1(x) - \lambda\theta_0\psi(x) + (\lambda\theta_0)^{1-m}P_m\psi \leq \lambda\theta_0\psi_1(x), \quad \forall x \in E \setminus E'.$$

Hence, we have checked that $P_1\psi_1 \leq \theta_0\rho\psi_1 + c_2\mathbb{1}_K\psi_1$ on E for some constants $\rho < 1$ and $c_2 < +\infty$. Since $P_1\eta = \theta_0\eta$, the proof of (G2) is completed. Note also that $\psi \leq \psi_1$ and the fact that $\psi_1 \in L^\infty(\psi)$ follows from the inequality $P_n\psi_1 \leq A_n\psi_1$ for some constant A_n, which is an immediate consequence of (2.3) and the fact that $\eta \in L^\infty(\psi_1)$.

Thanks to Remark 2.2, it is sufficient to check (G3) with $\psi_2 = \eta$ instead of ψ_1. Since η is an eigenfunction of P_1, (G3) is trivial.

Since $K \subset E'$, it follows from (2.3) that, for all $x \in K$, $\theta_0^{-n}P_n(\mathbb{1}_K\psi_1)(x)$ converges as $n \to +\infty$ to $\eta(x)\nu_P(\mathbb{1}_K\psi_1) > 0$. Hence (G4) is clear. □

For continuous time semigroups $(P_t)_{t\in[0,+\infty)}$, the conclusions of Theorem 2.1 can be easily deduced from properties on the discrete skeleton $(P_{nt_0})_{n\in\mathbb{N}}$ (similar properties where already observed in Theorem 5 of [31] and in [10]). In the following result, the function η and the positive measure ν_P are the one of Theorem 2.1 applied to the discrete skeleton $(P_{nt_0})_{n\in\mathbb{N}}$.

Corollary 2.4. *Let $(P_t)_{t\in[0,+\infty)}$ be a continuous time semigroup. Assume that there exists $t_0 > 0$ such that $(P_{nt_0})_{n\in\mathbb{N}}$ satisfies Assumption (G), $\left(\frac{P_t\psi_1}{\psi_1}\right)_{t\in[0,t_0]}$ is upper bounded by a constant $\bar{c} > 0$ and $\left(\frac{P_t\psi_2}{\psi_2}\right)_{t\in[0,t_0]}$ is lower bounded by a constant $\underline{c} > 0$. Then there exist some constants $C'' > 0$ and $\gamma > 0$ such that, for all measurable functions $f : E \to \mathbb{R}$ satisfying $|f| \leq \psi_1$ and all positive measure μ on E such that $\mu(\psi_1) < +\infty$ and $\mu(\psi_2) > 0$,*

$$\left|\frac{\mu P_t f}{\mu P_t\psi_1} - \nu_P(f)\right| \leq C'' e^{-\gamma t}\frac{\mu(\psi_1)}{\mu(\psi_2)}, \quad \forall t \in [0, +\infty), \tag{2.5}$$

In addition, there exists $\lambda_0 \in \mathbb{R}$ such that $\nu_P P_t = e^{\lambda_0 t}\nu_P$ for all $t \geq 0$, and $e^{-\lambda_0 t}P_t\psi_1$ converges uniformly and exponentially toward η in $L^\infty(\psi_1)$ when $t \to +\infty$. Moreover, there exist some constants $C''' > 0$ and $\gamma' > 0$ such that, for all measurable functions $f : E \to \mathbb{R}$ satisfying $|f| \leq \psi_1$ and all positive measures μ on E such that $\mu(\psi_1) < +\infty$,

$$\left|e^{-\lambda_0 t}\mu P_t f - \mu(\eta)\nu_P(f)\right| \leq C''' e^{-\gamma' t}\mu(\psi_1), \quad \forall t \in [0, +\infty). \tag{2.6}$$

Remark 2.5. In [2], a similar result is obtained, but with the additional assumptions that $\psi_2 > 0$ on E and $n_1 = 1$. In this restricted case, one can check using Remark 2.2 that their assumptions are equivalent to ours. The fact that ψ_2 can vanish allows to deal with non-irreducible semigroups (see [10, Section 6]).

Remark 2.6. The adaptation of the counterpart of Theorem 2.3 to the countinuous-time setting is straightforward. A similar result was proven in [2], where the authors assume in addition that ζ_n is geometrically decreasing and that η is positive.

Proof. Assuming without loss of generality that $t_0 = 1$ and applying (2.1) to μP_t, where $t \in [0,1]$, and f such that $\mu(\psi_1) < +\infty$ and $|f| \leq \psi_1$, one deduces that

$$\left|\frac{\mu P_{t+n} f}{\mu P_{t+n}\psi_1} - \nu_P(f)\right| \leq C\alpha^n \frac{\mu P_t \psi_1}{\mu P_t \psi_2} \leq \frac{C\bar{c}}{\alpha \underline{c}} \alpha^{n+t} \frac{\mu(\psi_1)}{\mu(\psi_2)},$$

which implies (2.5). Then, applying this inequality to $\mu = \nu_P$ and letting n go to infinity shows that $\nu_P P_t f / \nu_P P_t \psi_1 = \nu_P f$ for all $t \geq 0$. Choosing $f = P_s \psi_1$ entails $\nu_P P_{t+s}\psi_1 = \nu_P P_t \psi_1 \, \nu_P P_s \psi_1$ for all $s, t \geq 0$, and hence $\nu_P P_t \psi_1 = e^{\lambda_0 t} \nu_P \psi_1$ for all $t \geq 0$ for some constant $\lambda_0 \in \mathbb{R}$ (note that $\theta_0 = e^{\lambda_0}$).

Similarly, inequality (2.2) applied to $\mu = \delta_x P_t$ and $f = \psi_1$ on the one hand and to $\mu = \delta_x$ and $f = P_t \psi_1$ on the other hand implies that $P_t \eta(x) = \eta(x) \nu_P(P_t \psi_1) = e^{\lambda_0 t} \eta(x)$ for all $t \geq 0$. Applying again (2.2) to $\mu = \delta_x P_t$ entails that

$$\left|\theta_0^{-n} P_{t+n} f(x) - P_t \eta(x) \nu_P(f)\right| \leq C' \beta^n P_t \psi_1(x) \leq \frac{C'\bar{c}}{\beta} \beta^{n+t} \psi_1(x).$$

In particular, for all $t \geq 0$,

$$\left|e^{-\lambda_0 t} P_t f(x) - \eta(x) \nu_P(f)\right| \leq \frac{C'\bar{c}}{\beta} \beta^t \psi_1(x)$$

and $e^{-\lambda_0 t} P_t \psi_1$ converges geometrically to η in $L^\infty(\psi_1)$. This concludes the proof of Corollary 2.4 □

3 Some applications

Given a positive semigroup P acting on measurable functions on E, one can try to directly check Assumption (G) by finding appropriate functions ψ_1 and ψ_2. Another natural and equivalent strategy is to find a function ψ such that the semigroup defined by $Q_n f = \frac{P_n(\psi f)}{c^n \psi}$ is sub-Markovian and check that it satisfies Assumption (E) of [10]. The main advantage of this last approach is that Q has a probabilistic interpretation as the semigroup of a sub-Markov process. As such, one can apply all the criteria developed in the above mentioned reference and, more generally, use the intuitions and toolboxes of the theory of stochastic processes. Since both approaches are equivalent, this is rather a question of taste.

In Subsection 3.1, we consider the case of a penalized perturbed dynamical system and check Assumption (G) directly. In subsection 3.2, we consider the case of a penalized diffusion processes and check Assumption (E).

3.1 Perturbed dynamical systems

Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be a locally bounded measurable function and consider the perturbed dynamical system $X_{n+1} = F(X_n) + \xi_n$ with $(\xi_i)_{i \in \mathbb{Z}_+}$ i.i.d. non-degenerate Gaussian random variables. We are interested in the asymptotic behaviour of the associated Feynman-Kac semigroup

$$P_n f(x) = \mathbb{E}_x \left(\prod_{k=1}^{n} G(X_k) \mathbb{1}_{X_k \in E} f(X_n) \right),$$

where E is a measurable subset of $\mathbb{R}^d$ with positive Lebesgue measure and $G : E \to (0, +\infty)$ is a measurable function.

Proposition 3.1. *Assume that $1/G$ is locally bounded, $G(x) \leq C \exp(|x|)$ for all $x \in E$ and some constant $C > 0$ and there exists $p > 1$ such that $|x| - p|F(x)| \to +\infty$ when $|x| \to +\infty$, then the semigroup $(P_n)_{n \in \mathbb{N}}$ satisfies Assumption (G).*

Proof. One easily checks that $\psi_1(x)=\exp(a|x|)$, where $a>0$ is such that $1/a<p-1$, satisfies

$$P_1\psi_1(x)\le C\mathbb{E}\left(e^{(1+a)|F(x)+\xi_1|}\right)\le C'\,\psi_1(x)\,\exp\left(-a\left(|x|-p|F(x)|\right)\right), \tag{3.1}$$

where $C'=C\mathbb{E}e^{(1+a)|\xi_1|}$. Now, assume without loss of generality that $B(0,1)\cap E$ has positive Lebesgue measure and set $\theta_2:=\inf_{x\in B(0,1)\cap E}P_1\mathbb{1}_{B(0,1)\cap E}(x)/2$, which is clearly positive. It then follows from Markov's property that

$$\theta_2^{-n}\inf_{x\in B(0,1)\cap E}P_n\mathbb{1}_{B(0,1)\cap E}(x)\ge\theta_2^{-n}\inf_{x\in B(0,1)\cap E}\mathbb{E}_x\left[\prod_{k=1}^n G(X_k)\mathbb{1}_{B(0,1)\cap E}(X_k)\right]\ge 2^n\to+\infty,$$

when $n\to+\infty$. One easily deduces that, for all $R\ge 1$, $\theta_2^{-n}\inf_{x\in B(0,R)\cap E}P_n\mathbb{1}_{B(0,1)\cap E}(x)\to+\infty$, and hence that $\theta_2^{-n}\inf_{x\in B(0,R)\cap E}P_n\mathbb{1}_{B(0,R)\cap E}(x)\to+\infty$ when $n\to+\infty$.

We set $\theta_1=\theta_2/2$ and fix $R\ge 1$ large enough so that $C'e^{-a(|x|-p|F(x)|)}\le\theta_1$ for all $|x|\ge R$. It then follows from (3.1) that $P_1\psi_1\le\theta_1\psi_1+c_2\mathbb{1}_K\psi_1$, where $K:=B(0,R)\cap E$. Setting $\psi_2(x)=\sum_{k=0}^{n_0}\theta_2^{-k}P_k\mathbb{1}_K(x)$, we deduce that, for all $x\in E$,

$$P_1\psi_2(x)=\sum_{k=0}^{n_0}\theta_2^{-k}P_{k+1}\mathbb{1}_K(x)=\theta_2\psi_2(x)+\theta_2\left[\theta_2^{-(n_0+1)}P_{n_0+1}\mathbb{1}_K(x)-\mathbb{1}_K(x)\right]\ge\theta_2\psi_2(x)$$

for n_0 chosen large enough. Since in addition $P_k\mathbb{1}_K\le P_k\psi_1\le(\theta_1+c_2)^k\psi_1$, $\psi_2\in L^\infty(\psi_1)$ and, for all $x\in K$, $\psi_2(x)\ge 1\ge e^{-aR}\psi_1(x)$. Hence, dividing ψ_2 by $\|\psi_2/\psi_1\|_\infty$ ends the proof of (G2).

In order to prove (G1), (G3) and (G4), we follow similar arguments as for [10, Prop. 7.2]. Since the adaptation of these arguments is a bit tricky because the function ψ_1 needs to be taken into account appropriately, we give the details below.

The first step consists in proving that there exists a constant $c>0$ such that, for all measurable $A\subset K$, for all $x\in E$ and all $y\in K$,

$$\frac{P_1(\psi_1\mathbb{1}_A)(x)}{\psi_1(x)}\le c\frac{P_1(\psi_1\mathbb{1}_A)(y)}{\psi_1(y)}. \tag{3.2}$$

This implies easily (G1) for $n_1=1$ and (G4) then follows directly from (G1) (since $n_1=1$).

To prove (3.2), we observe that (recall that $A\subset K=E\cap B(0,R)$)

$$\frac{P_1(\psi_1\mathbb{1}_A)(x)}{\psi_1(x)}\le P_1(\psi_1\mathbb{1}_A)(x)\le\sup_{|z|\le R}\left[G(z)\psi_1(z)\right]\,\mathbb{P}(F(x)+\xi_1\in E\cap A\cap B(0,R)).$$

Because ξ_1 is a non-degenerate gaussian random variable, it is not hard to check that there exists a constant C_R depending only on R (and not on $x\in E$ and $y\in K$) such that $\mathbb{P}(F(x)+\xi_1\in E\cap A\cap B(0,R))\le C_R\mathbb{P}(F(y)+\xi_1\in E\cap A\cap B(0,R))$. Therefore,

$$\frac{P_1(\psi_1\mathbb{1}_A)(x)}{\psi_1(x)}\le C_R\frac{\sup_{|z|\le R}G(z)\psi_1(z)}{\inf_{|z|\le R}G(z)}\mathbb{E}_y\left[G(X_1)\psi_1(X_1)\mathbb{1}_{X_1\in E\cap A}\right]\le c\frac{P_1(\psi_1\mathbb{1}_A)(y)}{\psi_1(y)},$$

where $c=C_Re^{aR}\sup_{|z|\le R}G(z)\psi_1(z)/\inf_{|z|\le R}G(z)$. Hence (3.2) is proved.

Next, we observe that the Markov property combined with (G2) implies that, for all $x\in E$ and all $n\ge 1$,

$$\mathbb{E}_x\left[\prod_{k=1}^n G(X_k)\mathbb{1}_{X_k\in E\setminus K}\psi_1(X_n)\right]\le(\theta_1+c_2)\theta_1^{n-1}\psi_1(x). \tag{3.3}$$

We also have the property that there exists a constant $c' > 0$ such that, for all $y \in K$ and all $0 \le k \le n$,

$$\frac{P_n\psi_1(y)}{\psi_1(y)} \ge c'\theta_2^k \frac{P_{n-k}\psi_1(y)}{\psi_1(y)}. \tag{3.4}$$

As observed in Remark 2.2, since we already proved (G2), the last property is equivalent to the same one with ψ_2 instead of ψ_1. Since $P_1\psi_2 \ge \theta_2\psi_2$ on K (3.4) is then clear.

The proof of (G3) can then be done by combining the last inequalities. We first decompose $P_n\psi_1$ depending on the value of the first return time in K: for all $x \in E$,

$$\begin{aligned}
P_n\psi_1(x) &= \mathbb{E}_x\left[\prod_{k=1}^n G(X_k)\mathbb{1}_{X_k\in E\setminus K}\psi_1(X_n)\right] \\
&\qquad\qquad + \sum_{\ell=1}^n \mathbb{E}_x\left[\prod_{k=1}^{\ell-1} G(X_k)\mathbb{1}_{X_k\in E\setminus K} G(X_\ell)\mathbb{1}_{X_\ell\in K} P_{n-\ell}\psi_1(X_\ell)\right] \\
&\le (\theta_1+c_2)\theta_1^{n-1}\psi_1(x) \\
&\qquad + \sum_{\ell=1}^n \mathbb{E}_x\left[\prod_{k=1}^{\ell-1} G(X_k)\mathbb{1}_{X_k\in E\setminus K}\mathbb{E}_{X_{\ell-1}}\left[G(X_1)\mathbb{1}_{X_1\in K}P_{n-\ell}\psi_1(X_1)\right]\right],
\end{aligned}$$

where we used (3.3) and Markov's property in the second line. We then proceed by using (3.2) for some fixed $y \in K$ first, (3.3) next, and finally (3.4) twice:

$$\begin{aligned}
\frac{P_n\psi_1(x)}{\psi_1(x)} &\le (\theta_1+c_2)\theta_1^{n-1} + \frac{c}{\psi_1(x)}\sum_{\ell=1}^n \mathbb{E}_x\left[\prod_{k=1}^{\ell-1} G(X_k)\mathbb{1}_{X_k\in E\setminus K}\psi_1(X_{\ell-1})\right] \\
&\qquad\qquad\qquad\qquad \times \frac{\mathbb{E}_y\left[G(X_1)\mathbb{1}_{X_1\in K}P_{n-\ell}\psi_1(X_1)\right]}{\psi_1(y)} \\
&\le \frac{\theta_1+c_2}{\theta_1}\theta_1^n + \frac{c(\theta_1+c_2)}{\theta_1}\sum_{\ell=1}^n \theta_1^{\ell-1}\frac{P_{n-\ell+1}\psi_1(y)}{\psi_1(y)} \\
&\le \left[\frac{\theta_1+c_2}{c'\theta_1}\left(\frac{\theta_1}{\theta_2}\right)^n + \frac{c(\theta_1+c_2)}{c'\theta_1}\sum_{\ell=1}^n\left(\frac{\theta_1}{\theta_2}\right)^{\ell-1}\right]\frac{P_n\psi_1(y)}{\psi_1(y)}.
\end{aligned}$$

Since the last factor is bounded in n, this ends the proof of Proposition 3.1. □

3.2 Diffusion processes

Let $(X_t)_{t\in[0,+\infty)}$ be solution to the SDE

$$\mathrm{d}X_t = \mathrm{d}B_t + b(X_t)\,\mathrm{d}t, \quad X_0 \in (0,+\infty)^d, \tag{3.5}$$

where $B = (B^{(1)},\dots,B^{(d)})$ is a standard d-dimensional Brownian motion and $b : \mathbb{R}^d \to \mathbb{R}^d$ is locally Hölder. Let $r : (0,+\infty)^d \to \mathbb{R}$ be locally bounded and consider the semigroup $(P_t)_{t\in[0,+\infty)}$ defined by

$$P_t f(x) = \mathbb{E}_x\left(e^{\int_0^t r(X_u)\,\mathrm{d}u} f(X_t)\,\mathbb{1}_{X_s\in(0,+\infty)^d,\ \forall s\in[0,t]}\right). \tag{3.6}$$

The term $\mathbb{1}_{X_s\in(0,+\infty)^d,\ \forall s\in[0,t]}$ above corresponds to a killing at the boundary of the domain $(0,+\infty)^d$. Note that the solution to (3.5) may explode in finite time if b does not satisfy the linear growth condition. However, we assume by convention that $X_t \notin (0,+\infty)^d$ after the explosion time, so that (3.6) makes sense. We refer to [10, Sections 4.1 and 12.1] for the precise construction of the process.

One motivation for the study of this semigroup comes from the Feynam-Kac formula. Indeed, when the coefficients are smooth enough (see for instance [30, Section 1.3.3]), this semigroup is solution to the Cauchy linear parabolic partial differential equation

$$\begin{aligned} &rv - \frac{\partial v}{\partial t} + \mathcal{L}v = 0, \text{ on } [0,+\infty)\times(0,+\infty)^d \\ &v(0,\cdot) = f, \text{ on } (0,+\infty)^d, \end{aligned}$$

where $\mathcal{L}$ is the differential operator of second order

$$\mathcal{L}\varphi(x) = \frac{1}{2}\Delta\varphi(x) + b(x)\cdot\nabla\varphi(x), \quad \forall\varphi\in C^2(\mathbb{R}^d),$$

with Dirichlet boundary conditions.

Proposition 3.2. *Assume that*

$$r(x) + \sum_{i=1}^d b_i(x) \xrightarrow[|x|\to\infty,\ x\in(0,\infty)^d]{} -\infty. \tag{3.7}$$

Then the semigroup $(P_t)_{t\in[0,+\infty)}$ satisfies the assumptions of Corollary 2.4.

Proof. We first observe that, setting $\psi(x) = \exp\left(\sum_{i=1}^d x_i\right)$ and $a := d/2 + \sup_{x\in(0,\infty)^d} r(x) + \sum_{i=1}^d b_i(x)$, we have, for all $x\in(0,+\infty)$,

$$\begin{aligned} Q_t f(x) &:= e^{-at}\frac{P_t(f\psi)(x)}{\psi(x)} \\ &= \mathbb{E}_x\left(e^{-\frac{d}{2}t+\sum_{i=1}^d B_t^{(i)}}\, e^{\int_0^t\left(r(X_u)+\sum_{i=1}^d b_i(X_u)-a+\frac{d}{2}\right)\mathrm{d}u}\, f(X_t)\,\mathbb{1}_{X_s\in(0,+\infty)^d,\ \forall s\in[0,t]}\right). \end{aligned}$$

Using Girsanov's theorem, we deduce that

$$Q_t f(x) = \mathbb{E}_x\left(e^{-\int_0^t \kappa(\bar X_u)\,\mathrm{d}u}\, f(\bar X_t)\,\mathbb{1}_{\bar X_s\in(0,+\infty)^d,\ \forall s\in[0,t]}\right),$$

where $\kappa(y) = a - r(y) - \frac{d}{2} - \sum_{i=1}^d b_i(y) \geq 0$ and $\bar X = (\bar X^{(1)},\dots,\bar X^{(d)})$ is solution to the SDE $\mathrm{d}\bar X_t^{(i)} = \mathrm{d}B_t^{(i)} + (1+b_i(\bar X_t))\,\mathrm{d}t$ with $\bar X_0^{(i)} = x_i$.

Assumption (3.7) thus implies that the conditions of [10, Theorem 4.5] are satified[1] and hence that Q satisfies Assumption (F) therein, which implies that Assumption (E) is satisfied by the semigroup Q_{nt_0} for some $t_0 > 0$ and some Lyapunov functions φ_1 and φ_2, that $\left(\frac{Q_t\varphi_1}{\varphi_1}\right)_{t\in[0,t_0]}$ is uniformly bounded, and that there exist a positive function $\eta_Q \in L^\infty(\varphi_1)$ and a constant $\lambda_0 > 0$ such that $Q_t\eta_Q = e^{-\lambda_0 t}\eta_Q$ for all $t\in[0,+\infty)$.

To conclude, it remains to observe that the same procedure as the one used in the proof of Theorem 2.1 above allows to deduce from these properties that $(P_{nt_0})_{n\geq 0}$ satisfies Assumption (G) with $\psi_1 = \psi\varphi_1$ and $\psi_2 = \psi\eta_Q$. Observing also that ψ_2 is the function η of Theorem 2.1, we deduce that $(P_t)_{t\in[0,+\infty)}$ satisfies the assumptions of Corollary 2.4. □

References

[1] Vincent Bansaye, Bertrand Cloez, and Pierre Gabriel, *Ergodic behavior of non-conservative semigroups via generalized doeblin's conditions*, Acta Applicandae Mathematica (2019), To appear.

[1] To prove (4.12) therein, one can use the same argument as the one used in Corollary 4.3 of this reference.

[2] Vincent Bansaye, Bertrand Cloez, Pierre Gabriel, and Aline Marguet, *A non-conservative Harris' ergodic theorem*, arXiv e-prints (2019), arXiv:1903.03946.

[3] Jean Bertoin, *On a Feynman-Kac approach to growth-fragmentation semigroups and their asymptotic behaviors*, arXiv e-prints (2018), arXiv:1804.04905.

[4] Jean Bertoin and Alexander Watson, *The strong Malthusian behavior of growth-fragmentation processes*, arXiv e-prints (2019), arXiv:1901.07251.

[5] Jean Bertoin and Alexander Watson, *A probabilistic approach to spectral analysis of growth-fragmentation equations*, Journal of Functional Analysis **274** (2018), no. 8, 2163–2204.

[6] John D. Biggins and Andreas Kyprianou, *Measure change in multitype branching*, Adv. in Appl. Probab. **36** (2004), no. 2, 544–581. MR-2058149

[7] Garrett Birkhoff, *Extensions of Jentzsch's theorem*, Trans. Amer. Math. Soc. **85** (1957), 219–227. MR-0087058

[8] Patrick Cattiaux, Pierre Collet, Amaury Lambert, Servet Martínez, Sylvie Méléard, and Jaime San Martín, *Quasi-stationary distributions and diffusion models in population dynamics*, Ann. Probab. **37** (2009), no. 5, 1926–1969. MR-2561437

[9] Nicolas Champagnat and Denis Villemonais, *Exponential convergence to quasi-stationary distribution and Q-process*, Probab. Theory Related Fields **164** (2016), no. 1, 243–283.

[10] Nicolas Champagnat and Denis Villemonais, *General criteria for the study of quasi-stationarity*, arXiv e-prints (2017), arXiv:1712.08092.

[11] Bertrand Cloez, *Limit theorems for some branching measure-valued processes*, Advances in Applied Probability **49** (2017), no. 2, 549–580.

[12] Pierre Collet, Servet Martínez, Sylvie Méléard, and Jaime San Martín, *Quasi-stationary distributions for structured birth and death processes with mutations*, Probability Theory and Related Fields **151** (2011), 191–231, 10.1007/s00440-010-0297-4.

[13] Pierre Collet, Servet Martínez, and Jaime San Martín, *Quasi-stationary distributions*, Probability and its Applications (New York), Springer, Heidelberg, 2013, Markov chains, diffusions and dynamical systems. MR-2986807

[14] Pierre Del Moral, *Feynman-Kac formulae*, Probability and its Applications (New York), Springer-Verlag, New York, 2004, Genealogical and interacting particle systems with applications. MR-2044973

[15] Pierre Del Moral, *Mean field simulation for Monte Carlo integration*, Monographs on Statistics and Applied Probability, vol. 126, CRC Press, Boca Raton, FL, 2013. MR-3060209

[16] Randal Douc, Eric Moulines, Pierre Priouret, and Philippe Soulier, *Markov chains*, Springer, 2018.

[17] P. A. Ferrari, H. Kesten, and S. Martínez, *R-positivity, quasi-stationary distributions and ratio limit theorems for a class of probabilistic automata*, Ann. Appl. Probab. **6** (1996), no. 2, 577–616. MR-1398060

[18] Grégoire Ferré, Mathias Rousset, and Gabriel Stoltz, *More on the long time stability of Feynman-Kac semigroups*, arXiv e-prints (2018), arXiv:1807.00390.

[19] Alexandru Hening and Martin Kolb, *Quasistationary distributions for one-dimensional diffusions with singular boundary points*, Stochastic Processes and their Applications **129** (2019), no. 5, 1659–1696.

[20] Nobuyuki Ikeda, Masao Nagasawa, and Shinzo Watanabe, *Branching Markov processes. I*, J. Math. Kyoto Univ. **8** (1968), 233–278. MR-0232439

[21] Nobuyuki Ikeda, Masao Nagasawa, and Shinzo Watanabe, *Branching Markov processes. II*, J. Math. Kyoto Univ. **8** (1968), 365–410. MR-0238401

[22] Nobuyuki Ikeda, Masao Nagasawa, and Shinzo Watanabe, *Branching Markov processes. III*, J. Math. Kyoto Univ. **9** (1969), 95–160. MR-0246376

[23] Peter Jagers, *General branching processes as Markov fields*, Stochastic Process. Appl. **32** (1989), no. 2, 183–212. MR-1014449

[24] Martin Kolb and David Steinsaltz, *Quasilimiting behavior for one-dimensional diffusions with killing*, Ann. Probab. **40** (2012), no. 1, 162–212. MR-2917771

[25] Cécile Mailler and Denis Villemonais, *Stochastic approximation on non-compact measure spaces and application to measure-valued Pólya processes*, Ann. Appl. Probab. (2020), To appear.

[26] Sylvie Méléard and Denis Villemonais, *Quasi-stationary distributions and population processes*, Probab. Surv. **9** (2012), 340–410. MR-2994898

[27] Sean Meyn and Richard L. Tweedie, *Markov chains and stochastic stability*, second ed., Cambridge University Press, Cambridge, 2009, With a prologue by Peter W. Glynn. MR-2509253

[28] Seppo Niemi and Esa Nummelin, *On nonsingular renewal kernels with an application to a semigroup of transition kernels*, Stochastic Process. Appl. **22** (1986), no. 2, 177–202. MR-860932

[29] Esa Nummelin, *General irreducible Markov chains and nonnegative operators*, Cambridge Tracts in Mathematics, vol. 83, Cambridge University Press, Cambridge, 1984. MR-776608

[30] Huyên Pham, *Continuous-time stochastic control and optimization with financial applications*, Stochastic Modelling and Applied Probability, vol. 61, Springer-Verlag, Berlin, 2009. MR-2533355

[31] Pekka Tuominen and Richard L. Tweedie, *Exponential decay and ergodicity of general Markov processes and their discrete skeletons*, Adv. in Appl. Probab. **11** (1979), no. 4, 784–803. MR-544195

[32] Aurélien Velleret, *Unique Quasi-Stationary Distribution, with a possibly stabilizing extinction*, arXiv e-prints (2018), arXiv:1802.02409.