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The remainder in the renewal theorem

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Abstract

If the step distribution in a renewal process has finite mean and regularly varying tail with index $-\alpha$, $1 < \alpha < 2$, the first two terms in the asymptotic expansion of the renewal function have been known for many years. Here we show that, without making any additional assumptions, it is possible to give, in all cases except for $\alpha = 3/2$, the exact asymptotic behaviour of the next term. In the case $\alpha = 3/2$ the result is exact to within a slowly varying correction. Similar results are shown to hold in the random walk case.

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1 Introduction and results

We consider a renewal process $(S_n, n \geq 0)$, i.e. a random walk with non-negative, i.i.d. increments $X_1, X_2, \cdots$ with a distribution F whose tail $\overline{F} \in RV(-\alpha)$ (i.e. is regularly varying at infinity with index $-\alpha$) where $\alpha \in (1,2)$. We write $EX_1 = m$ and define a distribution Φ via its density function

$$\phi(y) = \frac{P(X_1 > y)}{m} := m^{-1}\overline{F}(y),\ y \geq 0, \text{and write } \overline{\Phi}(x) = \int_x^\infty \phi(y)dy. \tag{1.1}$$

The object of our study is the renewal function $U(x) := U([0,x])$, where the renewal measure is defined by

$$U(dx) := \sum_0^\infty P(S_n \in dx), \tag{1.2}$$

with $S_0 \equiv 0$. Since Φ is the limiting and stationary distribution in the process of overshoots in S, its importance is well-known, and the following result dates from the 70s: see Mohan, [3], who improves earlier results in [6].

$$U(x) - m^{-1}x - m^{-1}\int_0^x \overline{\Phi}(y)dy = o(\int_0^x \overline{\Phi}(y)dy) \text{ as } x \to \infty. \tag{1.3}$$

Later Sgibnev showed, in [5], that (1.3) actually holds whenever m is finite and $EX_1^2 = \infty$, so that the assumption of a regularly varying tail is redundant. This in turn suggests that if we do make this assumption we should be able to improve on (1.3). Under our

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assumptions $\overline{\Phi} \in RV(-\beta)$, where $\beta = \alpha - 1$, so any statement that the LHS of (1.3) is $O(x^\gamma)$ with $\gamma < 1 - \beta$ would be an improvement. In fact we can be much more precise than this.

We write ϕ_2 for the convolution $\phi * \phi$ and define real-valued functions g and $\overline{G}$ on $[0, \infty)$ by

$$\begin{aligned} g(y) &= 2\phi(y) - \phi_2(y), & (1.4) \\ \overline{G}(x) &= \int_x^\infty g(z)dz, \text{ so that } \overline{G}(0) = \int_0^\infty g(z)dz = 1. & (1.5) \end{aligned}$$

To state our result, we set

$$U(x) - m^{-1}x - m^{-1} \int_0^x \overline{\Phi}(y)dy = m^{-1}V(x), \tag{1.6}$$

so that the known result (1.3) says that $V(x) = o(\overline{\overline{\Phi}}(x))$, where $\overline{\overline{\Phi}}(x) := \int_0^x \overline{\Phi}(y)dy \in RV(1-\beta)$.

Theorem 1.1 (Renewal processes)**.** *Recall that* $\alpha \in (1, 2)$ *and* $\beta = \alpha - 1$.

(i) Define a constant by

$$c_\alpha = (1 - 2\beta) \int_0^1 \frac{dw}{w^\beta (1-w)^\beta} = \frac{(1-2\beta)\Gamma(1-\beta)^2}{\Gamma(2-2\beta)}.$$

Then

$$\lim_{x \to \infty} \frac{\overline{G}(x)}{\overline{\Phi}(x)^2} = c_\alpha.$$

(ii) $V \in RV(1 - 2\beta)$ *unless* $\beta = 1/2$. *More precisely, as* $x \to \infty$,

$$\begin{aligned} V(x) &\backsim \frac{|c_\alpha| x \overline{\Phi}(x)^2}{|2\beta - 1|} \text{ if } \beta \neq 1/2, & (1.7) \\ V(x) &\to \int_0^\infty \overline{G}(y)dy \text{ if } \beta = 1/2 \text{ and } \int_0^\infty \overline{\Phi}(y)^2 dy < \infty, & (1.8) \\ V(x) &= o\left(\int_0^x \overline{\Phi}(y)^2 dy\right) \text{ if } \beta = 1/2 \text{ and } \int_0^\infty \overline{\Phi}(y)^2 dy = \infty. & (1.9) \end{aligned}$$

Remark 1.2. Since $\overline{\Phi}(x)^2 \in RV(-1)$ when $\beta = 1/2$ we see that in (1.9) $\int_0^x \overline{\Phi}(y)^2 dy$ is slowly varying. Also in (1.8) $\int_0^\infty \overline{G}(y)dy = 0$ iff

$$\frac{1 - \hat{\phi}(\lambda)}{\sqrt{\lambda}} = \frac{\int_0^\infty (1 - e^{-\lambda x}) \overline{F}(x)dx}{m\sqrt{\lambda}} \to 0 \text{ as } \lambda \downarrow 0. \tag{1.10}$$

Remark 1.3. If we consider the case that $\alpha = 2$ and $\int_0^\infty y^2 dF(y) = \infty$, we cannot give the exact behaviour of V, but it is not difficult to show that $V(x) = o(x^{\varepsilon - 1})$ for any fixed $\varepsilon > 0$.

2 Proofs

(i) Recall that $\phi(x) = m^{-1}\overline{F}(x)$ is decreasing, bounded and is in $RV(-\alpha)$. Then write

$$\begin{aligned} \overline{G}(x) &= \int_x^\infty (2\phi(y) - \int_0^y \phi(y-w)\phi(w)dw)dy \\ &= \int_x^\infty (2\phi(y) \int_0^{y/2} \phi(w)dw - 2\int_0^{y/2} \phi(y-w)\phi(w)dw)dy + 2\int_x^\infty \phi(y)\overline{\Phi}(y/2)dy. \\ : &= I_1 + I_2. \end{aligned}$$

Since $\overline{\Phi}(y/2) \backsim 2^\beta\overline{\Phi}(y)$, we see that $I_2 \backsim 2^\beta\overline{\Phi}(x)^2$. Also

$$\begin{aligned}
-I_1 &= 2\int_x^\infty dy\int_0^{y/2}(\phi(y-w)-\phi(y))\phi(w)dw \\
&= 2\int_0^\infty \phi(w)dw\int_{2w\vee x}^\infty(\phi(y-w)-\phi(y))dy \\
&= 2\int_0^{x/2}\phi(w)(\overline{\Phi}(x-w)-\overline{\Phi}(x))dw + 2\int_{x/2}^\infty \phi(w)(\overline{\Phi}(w)-\overline{\Phi}(2w))dw.
\end{aligned}$$

As $\overline{\Phi}(w)-\overline{\Phi}(2w) \backsim (1-2^{-\beta})\overline{\Phi}(w)$ we see that the second term is asymptotic to $(1-2^{-\beta})\overline{\Phi}(x/2)^2$, or equivalently $2^\beta(2^\beta-1)\overline{\Phi}(x)^2$. Also we can write the first term as

$$2\int_0^{x/2}\phi(w)dw\int_{x-w}^x\phi(y)dy = 2(x\phi(x))^2\int_0^{1/2}\frac{\phi(xw)}{\phi(x)}dw\int_{1-w}^1\frac{\phi(xy)}{\phi(x)}dy. \tag{2.1}$$

Now take a fixed $\varepsilon > 0$ such that $\alpha+\varepsilon < 2$, and use Potter's bounds (see [1], p. 25) to see that the integrand on the RHS here is bounded above by a multiple of

$$w^{-(\alpha+\varepsilon)}\{(1-w)^{-(\beta+\varepsilon)}-1) \text{ for } wx > 1.$$

Since

$$\int_0^{1/x}\frac{w\phi(xw)}{\phi(x)}dw \le \int_0^{1/x}\frac{wdw}{\phi(x)} \to 0$$

we conclude that the double integral converges to

$$\begin{aligned}
J_\alpha &= \int_0^{1/2}w^{-\alpha}dw\int_{1-w}^1 y^{-\alpha}dy = \beta^{-1}\int_0^{1/2}w^{-\alpha}\{(1-w)^{-\beta}-1\}dw \\
&= -\beta^{-2}2^\beta(2^\beta-1)+\beta^{-1}\int_0^{1/2}w^{-\beta}(1-w)^{-\alpha}dw.
\end{aligned}$$

Since $x\phi(x) \backsim \beta\overline{\Phi}(x)$, we see that the the RHS of (2.1) is asymptotic to $2(\beta\overline{\Phi}(x))^2J_\alpha$, which establishes the result, and gives

$$c_\alpha = 2^{2\beta}-2\beta I_\alpha, \text{ where } I_\alpha = \int_0^{1/2}\frac{dw}{(1-w)\{w(1-w)\}^\beta}. \tag{2.2}$$

But

$$\begin{aligned}
I_\alpha &= \int_0^{1/2}\frac{w+(1-w)dw}{(1-w)\{w(1-w)\}^\beta} = \int_0^{1/2}\frac{w^{1-\beta}dw}{(1-w)^{1+\beta}}+\int_0^{1/2}\frac{dw}{w^\beta(1-w)^\beta} \\
&= \beta^{-1}2^{2\beta-1}+\beta^{-1}(\beta-1)\int_0^{1/2}\frac{w^{-\beta}dw}{(1-w)^\beta}+\int_0^{1/2}\frac{dw}{w^\beta(1-w)^\beta} \\
&= \beta^{-1}2^{2\beta-1}+(1-(2\beta)^{-1})\int_0^1\frac{dw}{w^\beta(1-w)^\beta},
\end{aligned}$$

so $c_\alpha = (1-2\beta)B(1-\beta,1-\beta)$, as required.

(ii) We start by noting that the stationarity of ϕ gives $\int_0^x\phi(x-y)U(y)dy = m^{-1}x$, and then

$$\begin{aligned}
\int_0^x\phi_2(x-y)U(y)dy &= \int_0^x\int_0^{x-y}\phi(x-y-z)\phi(z)dzU(y)dy \\
&= \int_0^x\phi(z)dz\int_0^{x-z}\phi(x-y-z)U(y)dy \\
&= m^{-1}\int_0^x(x-z)\phi(z)dz = m^{-1}(x-\int_0^x\overline{\Phi}(y)dy).
\end{aligned}$$

Thus

$$\int_0^x g(x-y)U(y)dy = m^{-1}(x + \int_0^x \overline{\Phi}(y)dy),$$

and

$$m^{-1}V(x) = U(x) - m^{-1}(x + \int_0^x \overline{\Phi}(y)dy) = U(x) - \int_0^x g(x-y)U(y)dy, \tag{2.3}$$

so integration by parts gives

$$V(x) = m\int_{[0,x)} \overline{G}(x-y)U(dy). \tag{2.4}$$

Although statement (1.7) unifies the cases $\beta \in (0, 1/2)$ and $\beta \in (1/2, 1)$ their proofs differ. In the first case $\int_0^x \overline{\Phi}^2(y)dy \to \infty$, and we can use (2.4) in conjunction with the following, which is Theorem 4 in [5], and shows that Theorem 2.1 in [3] holds without assuming asymptotic stability.

Lemma 2.1 (Sgibnev)**.** *Let Q be a non-negative, non-increasing bounded function and put $A(x) = \int_0^x Q(y)dy$. Then if $A(\infty) = \infty$.*

$$\int_0^x Q(x-y)dU(y) \backsim m^{-1}A(x) \text{ as } x \to \infty. \tag{2.5}$$

If $\beta \in (0, 1/2)$ we have $c_\alpha > 0$, so given $\varepsilon > 0$ $\exists x_0$ such that for all $x > x_0$

$$(c_\alpha - \varepsilon)Q(x) \le \overline{G}(x) \le (c_\alpha + \varepsilon)Q(x), \tag{2.6}$$

where $Q(x) = \overline{\Phi}^2(x)$ satisfies the conditions of Lemma 2.1. Since the contribution to the integral in (2.4) from $[0, x_0]$ is $O(\overline{G}(x))$, which is neglible, it follows that

$$m^{-1}V(x) \sim m^{-1}\int_0^x Q(y)dy, \text{ so } V(x) \sim \frac{c_\alpha x\overline{\Phi}^2(x)}{1-2\beta},$$

and (1.7) holds. If $\beta = 1/2$ we have $c_a = 0$ but (2.6) still holds and provided $\int_0^\infty Q(y)dy = \infty$ the conditions of Lemma 2.1 are satisfied and the proof of (1.9) follows. In the remaining cases it is clear that $\overline{G}$ is Directly Riemann Integrable, so the Key Renewal Theorem applies to (2.4) to give $V(x) \to \int_0^\infty \overline{G}(y)dy$, and we need only show when this is 0. From (1.4) we see that the ordinary Laplace transforms of ϕ and g are related by

$$1 - \hat{g}(\lambda) = (1 - \hat{\phi}(\lambda))^2 \sim \lambda^{2\beta}L(\lambda) \text{ as } \lambda \to 0,$$

where L is slowly varying at zero, so we have $(1 - \hat{g}(\lambda))/\lambda \to 0$ as $\lambda \to 0$ iff $\beta > 1/2$ or $\beta = 1/2$ and (1.10) holds. But since g is bounded in absolute value by the integrable function $2\phi + \phi_2$, we can interchange orders of integration to see that

$$(1 - \hat{g}(\lambda))/\lambda = \int_0^\infty e^{-\lambda x}\overline{G}(x)dx,$$

and the conclusion follows by letting λ go to 0.

For the case $\beta \in (1/2, 1)$ we write $g^*, \overline{G^*}$ for $-g, -\overline{G}$, and we claim first that $\overline{G^*}$ is eventually positive and monotone, which follows from the fact

$$\liminf_{x\to\infty} \frac{g^*(x)}{2x\phi(x)^2} \ge \frac{-c_\alpha}{\beta} > 0. \tag{2.7}$$

To see that (2.7) holds, write

$$\begin{aligned} g^*(x) &= 2\left(\int_0^{x/2}\phi(w)\{\phi(x-w)-\phi(x)\}dw-\phi(x)\overline{\Phi}(x/2)\right)\\ &= 2x\phi(x)^2\left(\int_0^{1/2}\frac{\phi(xw)}{\phi(x)}\{\frac{\phi(x-xw)}{\phi(x)}-1\}dw-\frac{\overline{\Phi}(x/2)}{x\phi(x)}\right). \end{aligned}$$

Since the integrand converges pointwise to $w^{-\alpha}\{(1-w)^{-\alpha}-1\}$ it follows from Fatou's Lemma that

$$\begin{aligned} \liminf_{x\to\infty}\frac{g^*(x)}{2x\phi(x)^2} &\geq \int_0^{1/2} w^{-\alpha}\{(1-w)^{-\alpha}-1\}dw-\beta^{-1}2^{\beta}\\ &= I_\alpha+\beta J_\alpha-\beta^{-1}2^{\beta}=\frac{-c_\alpha}{\beta}, \end{aligned}$$

as claimed. So we can fix x_0 with $g^*(x)>0$ for $x>x_0$, and then, as in the above referenced proof in [5], given any $\varepsilon>0$ we can find $x_1>x_0$ with

$$\begin{aligned} \int_{x_1}^{x}\overline{G^*}(x-y)dU(y) &\leq \frac{1+\varepsilon}{m}\int_{x_1}^{x}\overline{G^*}(x-y)dy\\ &= \frac{1+\varepsilon}{m}\int_{x-x_1}^{\infty}\overline{G}(z)dz \backsim \frac{1+\varepsilon}{m}\frac{c_\alpha x\overline{\Phi}(x)^2}{2\beta-1}, \end{aligned}$$

where we have used $\int_0^\infty \overline{G}(z)dz=0$, and $\int_{x-x_1}^{x}\overline{G}(z)dz=O(\overline{\Phi}(x)^2)$. Using a corresponding lower bound and the fact that $\int_{[0,x_1)}\overline{G^*}(x-y)dU(y)=O(\overline{\Phi}(x)^2)$, (1.7) follows.

3 The random walk case

If the variables $X_1, X_2, \cdots$ can take positive and negative values, we will still define the renewal measure by (1.2), and study $U(x)=U([0,x])$ as $x\to\infty$. (For a different interpretation of the renewal function see [4].) In this case it is also shown in [5] that (1.3) holds only assuming $m=EX_1\in(0,\infty)$ and $E(X_1^+)^2=\infty$. The idea of that proof is to express U in terms of $U^\uparrow$ and $U^\downarrow$, the renewal measures for the process of increasing and decreasing ladder heights, and then use (1.3) for $U^\uparrow$. We will use a similar argument to give an extension of (ii) of our Theorem 1.1 to the random walk case.

To clarify, for $n\geq 1$ we write τ_n for the n^{th} strict increasing ladder epoch (i.e. the time at which the n^{th} strict maximum occurs) and σ_n for the n^{th} weak decreasing ladder epoch, and put $\tau_0=\sigma_0=0$. The corresponding ladder height processes are defined by

$$H_n^\uparrow = S_{\tau_n} \text{ and } H_n^\downarrow = |S_{\sigma_n}|,$$

and we denote their renewal measures by

$$U^\uparrow(dx)=\sum_0^\infty P(H_n^\uparrow\in dx) \text{ and } U^\downarrow(dx)=\sum_0^\infty P(H_n^\downarrow\in dx).$$

Since $m>0$ we know that S drifts to ∞ : thus $H_1^\downarrow$ is improper and $U^\downarrow$ is a finite measure, whereas $H_1^\uparrow$ is proper. Everything depends on the following simple observation:

Lemma 3.1. *We have*

$$U(dx)=\int_0^\infty U^\downarrow(dy)U^\uparrow(y+dx),\ x>0. \tag{3.1}$$

Proof. Since the Fourier transforms of the measures $U, U^\uparrow$ and $U^\downarrow$ are $(1-E(e^{i\theta S_1}))^{-1}$, $(1-E(e^{i\theta H_1^\uparrow}))^{-1}$, and $(1-E(e^{i\theta H_1^\downarrow}))^{-1}$, this is immediate from the Wiener-Hopf factorisation, which states that

$$1-E(e^{i\theta S_1}) = (1-E(e^{i\theta H_1^\uparrow}))(1-E(e^{i\theta H_1^\downarrow})): \tag{3.2}$$

see e.g. [2] chapter XVIII, p. 605. □

Remark 3.2. This paraphrases the Lemma on p. 790 of [5].

If we divide (3.2) by θ and let $\theta \downarrow 0$ we see immediately that $P(H_1^\downarrow < \infty) = m/m^\uparrow$, where $m^\uparrow = EH_1^\uparrow$, and this yields

$$\int_0^\infty U^\downarrow(dy) = C, \text{ where } C = \frac{m^\uparrow}{m}.$$

Moreover it follows from the duality lemma (see [2] chapter XII, p. 395) that

$$P(H_1^\uparrow > w) = \int_0^\infty U^\downarrow(dy) P(S_1 > w+y),$$

so since $P(S_1 > w+y) \backsim P(S_1 > w)$ for each fixed y we see that $P(H_1^\uparrow > w) \backsim CP(S_1 > w)$ as $w \to \infty$. Thus if we set

$$\overline{\Phi^\uparrow}(z) = \frac{1}{m^\uparrow}\int_z^\infty P(H_1^\uparrow > w)dw$$

we get

$$\overline{\Phi^\uparrow}(z) \backsim \frac{C\int_z^\infty P(S_1 > w)dw}{m^\uparrow} = \overline{\Phi}(z) \text{ as } z \to \infty. \tag{3.3}$$

Remark 3.3. Actually what is shown in [5] is that

$$U(x) - m^{-1}x \backsim m^{-1}\int_0^x \overline{\Phi^\uparrow}(y)dy, \tag{3.4}$$

and then a version of (3.3) is used to obtain (1.3). But in examining the remainder it is important that we use the exact expression (3.4).

Notice in the following extension of Theorem 1.1 there is no restriction on the left-hand tail of F, other than that imposed by the existence of the mean. So we are not assuming that S is asymptotically stable.

Theorem 3.4 (Random walks)**.** *Assume that $ES_1 = m \in (0,\infty)$ and the right-hand tail $\overline{F} \in RV(-\alpha)$ with $\alpha \in (1,2)$. Write $\overline{\Phi^\uparrow}$ and $\overline{G^\uparrow}$ for the functions $\overline{\Phi}$ and $\overline{G}$ evaluated for the renewal process $(H_n^\uparrow, n \ge 0)$, and set*

$$\begin{aligned}\Psi(x) &= \frac{1}{m^\uparrow}\int_0^\infty U^\downarrow(dy)\int_y^{x+y}\overline{\Phi^\uparrow}(z)dz - K, \text{ where}\\ K &= 0 \text{ if } \int_0^\infty \overline{\Phi}(y)^2 dy = \infty,\ K = \int_0^\infty U^\downarrow(dy)V^\uparrow(y) \text{ if } \int_0^\infty \overline{\Phi}(y)^2 dy < \infty.\end{aligned}$$

Then if we put

$$m^{-1}\tilde{V}(x) = U(x) - \frac{x}{m} - \Psi(x),$$

we have that the statements (1.7), (1.8) and (1.9) of Theorem 1.1 hold with V replaced by $\tilde{V}$.

Proof. From (3.1) we have

$$U(x) = \int_0^\infty U^{\downarrow}(dy)\left(U^{\uparrow}(y+x) - U^{\uparrow}(y)\right),$$

so that if we substitute (1.6) for $U^{\uparrow}$ we get

$$\begin{aligned} U(x) &= \frac{1}{m^{\uparrow}}\int_0^\infty U^{\downarrow}(dy)\left(x + \int_y^{x+y}\overline{\Phi^{\uparrow}}(z)dz + V^{\uparrow}(x+y) - V^{\uparrow}(y)\right) \\ &= \frac{Cx}{m^{\uparrow}} + \Psi(x) + \frac{1}{m^{\uparrow}}\int_0^\infty U^{\downarrow}(dy)(V^{\uparrow}(x+y) - V^{\uparrow}(y)) + \frac{K}{m^{\uparrow}} \\ : &= \frac{x}{m} + \Psi(x) + \frac{I(x)}{m^{\uparrow}}, \end{aligned}$$

and we need to examine the behaviour of $I(x)$. Note that for $\beta > 1/2$ we have $K = \int_0^\infty U^{\downarrow}(dy)V^{\uparrow}(y) \in (0,\infty)$, and $\int_0^\infty U^{\downarrow}(dy)V^{\uparrow}(x+y) \backsim CV^{\uparrow}(x)$. For $\beta < 1/2$ we have $K = 0, V^{\uparrow}(x) \to \infty$ and

$$\frac{V^{\uparrow}(x+y) - V^{\uparrow}(y)}{V^{\uparrow}(x)} \to 1,$$

and we can modify the argument in [5] to show that dominated convergence applies to give the result. Similar arguments deal with the case $\beta = 1/2$. □

4 Concluding remarks

It is easy to see that in the renewal case we can expand $\int_0^\infty e^{-\lambda x}U(x)dx$ in powers of $1 - \hat{\phi}(\lambda)$ as follows:

$$\hat{U}(\lambda) = \frac{1}{\lambda} + \frac{1}{m\lambda^2}\left(1 + \sum_1^\infty (1 - \hat{\phi}(\lambda))^r\right)$$

Now

$$\begin{aligned} \frac{(1-\hat{\phi}(\lambda))}{m\lambda^2} &= m^{-1}\int_0^\infty e^{-\lambda x}\int_0^x \overline{\Phi}(y)dy, \\ \frac{(1-\hat{\phi}(\lambda))^2}{m\lambda^2} &= m^{-1}\int_0^\infty e^{-\lambda x}\int_0^x \overline{G}(y)dy, \end{aligned}$$

and in fact for any $r \geq 2$

$$\frac{(1-\hat{\phi}(\lambda))^r}{m\lambda^2} = m^{-1}\int_0^\infty e^{-\lambda x}\int_0^x \overline{G_r}(y)dy,$$

where $\overline{G_r}(y) = \int_y^\infty g_r(z)dz$ and the sequence of functions g_r are defined by

$$g_2 = g = 2\phi - \phi * \phi \text{ and } g_{r+1} = \phi + g_r - \phi * g_r,\ r \geq 2.$$

If one could justify inverting the transform, writing $\overline{G_1}$ for $\overline{\Phi}$, this would yield a complete asymptotic expansion

$$U(x) = 1 + \frac{x}{m} + \frac{1}{m}\sum_1^\infty \int_0^x \overline{G_r}(y)dy,$$

and our results involve only the first two terms in the sum. The crux of our result is the justification of the relation $\overline{G_2}(y) \backsim c_\alpha \overline{\Phi}(x)^2$, so a natural question is whether one can show that $\overline{G_r}(y) \backsim c\overline{\Phi}(x)^r$. This seems to be impossible without making extra assumptions, but it seems that the not unnatural assumption that F has a monotone density would permit verification of this when $r = 3$. This would then give an extra term in our result when $\beta < 1/2$.

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