Electron. Commun. Probab. **25** (2020), no. 2, 1–7.
`https://doi.org/10.1214/19-ECP281`
ISSN: `1083-589X`

**ELECTRONIC
COMMUNICATIONS
in PROBABILITY**

Vanishing of the anchored isoperimetric profile in bond percolation at p_c*

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Abstract

We consider the anchored isoperimetric profile of the infinite open cluster, defined for $p > p_c$, whose existence has been recently proved in [3]. We extend adequately the definition for $p = p_c$, in finite boxes. We prove a partial result which implies that, if the limit defining the anchored isoperimetric profile at p_c exists, it has to vanish.

Keywords: critical percolation; isoperimetric constant.
AMS MSC 2010: Primary 60K35, Secondary 82B43.
Submitted to ECP on March 18, 2019, final version accepted on December 8, 2019.
Supersedes `arXiv:1903.08065`.
Supersedes `HAL:hal-02071030`.

1 Introduction

The most well–known open question in percolation theory is to prove that the percolation probability vanishes at p_c in dimension three. In fact, the interesting quantities associated to the model are very difficult to study at the critical point or in its vicinity. We study here a very modest intermediate question. We consider the anchored isoperimetric profile of the infinite open cluster, defined for $p > p_c$, whose existence has been recently proved in [3]. We extend adequately the definition for $p = p_c$, in finite boxes. We prove a partial result which implies that, if the limit defining the anchored isoperimetric profile at p_c exists, it has to vanish.

The Cheeger constant For a graph $\mathcal{G}$ with vertex set V and edge set E, we define the edge boundary $\partial_{\mathcal{G}} A$ of a subset A of V as

$$\partial_{\mathcal{G}} A = \Big\{ e = \langle x, y \rangle \in E : x \in A, y \notin A \Big\} .$$

We denote by $|B|$ the cardinal of the finite set B. The Cheeger constant of the graph $\mathcal{G}$ is defined as

$$\varphi_{\mathcal{G}} = \min \left\{ \frac{|\partial_{\mathcal{G}} A|}{|A|} \, : \, A \subset V, 0 < |A| \leq \frac{|V|}{2} \right\} .$$

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[*] The second author is laureate of the Séphora Berrebi Scholarship in Mathematics in 2019. The second author would like to thank Association Séphora Berrebi for their support.

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This constant was introduced by Cheeger in his thesis [2] in order to obtain a lower bound for the smallest eigenvalue of the Laplacian.

The anchored isoperimetric profile $\varphi_n(p)$ Let $d \geq 2$. We consider an i.i.d. supercritical bond percolation on $\mathbb{Z}^d$, every edge is open with a probability $p > p_c(d)$, where $p_c(d)$ denotes the critical parameter for this percolation. We know that there exists almost surely a unique infinite open cluster $\mathcal{C}_\infty$ [5]. We say that H is a valid subgraph of $\mathcal{C}_\infty$ if H is connected and $0 \in H \subset \mathcal{C}_\infty$. We define the anchored isoperimetric profile $\varphi_n(p)$ of $\mathcal{C}_\infty$ as follows. We condition on the event $\{0 \in \mathcal{C}_\infty\}$ and we set

$$\varphi_n(p) = \min \left\{ \frac{|\partial_{\mathcal{C}_\infty} H|}{|H|} : \ H \text{ valid subgraph of } \mathcal{C}_\infty,\, 0 < |H| \leq n^d \right\} .$$

The following theorem from [3] asserts the existence of the limit of $n\varphi_n(p)$ when $p > p_c(d)$.

Theorem 1.1. *Let $d \geq 2$ and $p > p_c(d)$. There exists a positive real number $\varphi(p)$ such that, conditionally on $\{0 \in \mathcal{C}_\infty\}$,*

$$\lim_{n\to\infty} n\varphi_n(p) \ = \ \varphi(p) \text{ almost surely.}$$

We wish to study how this limit behaves when p is getting closer to p_c. To do so, we need to extend the definition of the anchored isoperimetric profile so that it is well defined at $p_c(d)$. We say that H is a valid subgraph of $\mathcal{C}(0)$, the open cluster of 0, if H is connected and $0 \in H \subset \mathcal{C}(0)$. We define $\widehat{\varphi}_n(p)$ for every $p \in [0,1]$ as

$$\widehat{\varphi}_n(p) = \min \left\{ \frac{|\partial_{\mathcal{C}(0)} H|}{|H|} : \ H \text{ valid subgraph of } \mathcal{C}(0),\, 0 < |H| \leq n^d \right\} .$$

In particular, if 0 is not connected to $\partial[-n/2, n/2]^d$ by a p-open path, then $|\mathcal{C}(0)| < n^d$ and taking $H = \mathcal{C}(0)$, we see that $\widehat{\varphi}_n(p)$ is equal to 0. Thanks to Theorem 1.1, we have

$$\forall p > p_c \qquad \lim_{n\to\infty} n\widehat{\varphi}_n(p) \ = \ \theta(p)\delta_{\varphi(p)} + (1-\theta(p))\delta_0 \,,$$

where $\theta(p)$ is the probability that 0 belongs to an infinite open cluster. The techniques of [3] to prove the existence of this limit rely on coarse–graining estimates which can be employed only in the supercritical regime. Therefore we are not able so far to extend the above convergence at the critical point p_c. Naturally, we expect that $n\widehat{\varphi}_n(p_c)$ converges towards 0 as n goes to infinity, unfortunately we are only able to prove a weaker statement.

Theorem 1.2. *With probability one, we have*

$$\liminf_{n\to\infty} n\widehat{\varphi}_n(p_c) \ = \ 0 \,.$$

We shall prove this theorem by contradiction. We first define an exploration process of the cluster of 0 that remains inside the box $[-n,n]^d$. If the statement of the theorem does not hold, then the cluster of 0 satisfies a d-dimensional anchored isoperimetric inequality. It follows that the number of sites that are revealed in the exploration of the cluster of 0 will grow fast enough of order n^{d-1}. Then, we can prove that the intersection of the cluster that we have explored with the boundary of the box $[-n,n]^d$ is of order n^{d-1}. Using the fact that there is no percolation in a half-space, we obtain a contradiction. Before starting the precise proof, we recall some results from [3] on the meaning of the limiting value $\varphi(p)$.

The Wulff theorem We denote by $\mathcal{L}^d$ the d-dimensional Lebesgue measure and by $\mathcal{H}^{d-1}$ denotes the $(d-1)$–Hausdorff measure in dimension d. Given a norm τ on $\mathbb{R}^d$ and a subset E of $\mathbb{R}^d$ having a regular enough boundary, we define $\mathcal{I}_\tau(E)$, the surface tension of E for the norm τ, as

$$\mathcal{I}_\tau(E) = \int_{\partial E} \tau(n_E(x))\mathcal{H}^{d-1}(dx)\,.$$

We consider the anisotropic isoperimetric problem associated with the norm τ:

$$\text{minimize } \frac{\mathcal{I}_\tau(E)}{\mathcal{L}^d(E)} \text{ subject to } \mathcal{L}^d(E) \leq 1\,. \tag{1.1}$$

The famous Wulff construction provides a minimizer for this anisotropic isoperimetric problem. We define the set $\widehat{W}_\tau$ as

$$\widehat{W}_\tau = \bigcap_{v\in\mathbb{S}^{d-1}} \left\{x \in \mathbb{R}^d : \, x\cdot v \leq \tau(v)\right\}\,,$$

where $\cdot$ denotes the standard scalar product and $\mathbb{S}^{d-1}$ is the unit sphere of $\mathbb{R}^d$. Up to translation and Lebesgue negligible sets, the set

$$\frac{1}{\mathcal{L}^d(\widehat{W}_\tau)^{1/d}}\widehat{W}_\tau$$

is the unique solution to the problem (1.1).

Representation of $\varphi(p)$ In [3], we build an appropriate norm β_p for our problem that is directly related to the open edge boundary. We define the Wulff crystal W_p as the dilate of $\widehat{W}_{\beta_p}$ such that $\mathcal{L}^d(W_p) = 1/\theta(p)$, where $\theta(p) = \mathbb{P}(0 \in \mathcal{C}_\infty)$. We denote by $\mathcal{I}_p$ the surface tension associated with the norm β_p. In [3], we prove that

$$\forall p > p_c(d) \qquad \varphi(p) = \mathcal{I}_p(W_p)\,.$$

2 Proofs

We prove next the following lemma, which is based on two important results due to Zhang [9] and Rossignol and Théret [6]. To alleviate the notation, the critical point $p_c(d)$ is denoted simply by p_c.

Lemma 2.1. *We have*

$$\lim_{\substack{p\to p_c\\ p>p_c}} \Big(\theta(p)\delta_{\mathcal{I}_p(W_p)} + (1-\theta(p))\delta_0\Big) \;=\; \delta_0\,.$$

Proof. If $\lim_{p\to p_c}\theta(p) = 0$, then the result is clear. Otherwise, let us assume that

$$\lim_{\substack{p\to p_c\\ p>p_c}} \theta(p) = \delta > 0\,.$$

Let B be a subset of $\mathbb{R}^d$ having a regular boundary and such that $\mathcal{L}^d(B) = 1/\delta$. As the map $p \mapsto \theta(p)$ is non-decreasing and $\mathcal{L}^d(W_p) = 1/\theta(p)$, we have

$$\forall p > p_c \qquad \mathcal{L}^d(W_p) \leq \mathcal{L}^d(B)\,.$$

Moreover as W_p is the dilate of the minimizer associated to the isoperimetric problem (1.1), we have

$$\forall p > p_c \qquad \mathcal{I}_p(W_p) \leq \mathcal{I}_p(B)\,.$$

In [9], Zhang proved that $\beta_{p_c}=0$. In [6], Rossignol and Théret proved the continuity of the flow constant. Combining these two results, we get that

$$\lim_{\substack{p\to p_c\\ p>p_c}} \beta_p \,=\, \beta_{p_c} \,=\, 0 \quad \text{and so} \quad \lim_{\substack{p\to p_c\\ p>p_c}} \mathcal{I}_p(B) \,=\, 0\,.$$

Finally, we obtain

$$\lim_{\substack{p\to p_c\\ p>p_c}} \mathcal{I}_p(W_p) \,=\, 0\,.$$

This yields the result. □

Proof of Theorem 1.2. We assume by contradiction that

$$\mathbb{P}\left(\liminf_{n\to\infty}\; n\widehat{\varphi}_n(p_c)=0\right)<1\,.$$

Therefore there exist positive constants c and δ such that

$$\mathbb{P}\left(\liminf_{n\to\infty}\; n\widehat{\varphi}_n(p_c)>c\right)=\lim_{n\to\infty}\mathbb{P}\left(\inf_{k\geq n}\; k\widehat{\varphi}_k(p_c)>c\right)=\delta\,. \tag{2.1}$$

Therefore, there exists a positive integer n_0 such that

$$\mathbb{P}\left(\inf_{k\geq n_0}\; k\widehat{\varphi}_k(p_c)>c\right)\geq\frac{\delta}{2}\,. \tag{2.2}$$

In what follows, we condition on the event

$$\left\{\inf_{k\geq n_0}\; k\widehat{\varphi}_k(p_c)>c\right\}.$$

Note that on this event, 0 is connected to infinity by a p_c-open path. For H a subgraph of $\mathbb{Z}^d$, we define

$$\partial^o H=\left\{\,e\in\partial H,\; e \text{ is open}\,\right\}.$$

Note that if $H\subset\mathcal{C}_\infty$, then $\partial_{\mathcal{C}_\infty}H=\partial^o H$. Moreover, if H is equal to $\mathcal{C}(0)$, the open cluster of 0, then $\partial_{\mathcal{C}(0)}H=\partial^o H=\varnothing$. We define next an exploration process of the cluster of 0. We set $\mathcal{C}_0=\{0\}$, $\mathcal{A}_0=\emptyset$. Let us assume that $\mathcal{C}_0,\dots,\mathcal{C}_l$ and $\mathcal{A}_0,\dots,\mathcal{A}_l$ are already constructed. We define

$$\mathcal{A}_{l+1}=\left\{x\in\mathbb{Z}^d\,:\,\exists y\in\mathcal{C}_l\quad\langle x,y\rangle\in\partial^o\mathcal{C}_l\right\}$$

and

$$\mathcal{C}_{l+1}=\mathcal{C}_l\cup\mathcal{A}_{l+1}\,.$$

We have

$$\partial^o\mathcal{C}_l\subset\{\langle x,y\rangle\in\mathbb{E}^d\,:\,x\in\mathcal{A}_{l+1}\}$$

so that $|\partial^o\mathcal{C}_l|\leq 2d|\mathcal{A}_{l+1}|$. Since $\mathcal{A}_{l+1}$ and $\mathcal{C}_l$ are disjoint, we have

$$|\mathcal{C}_{l+1}|=|\mathcal{C}_l|+|\mathcal{A}_{l+1}|\geq|\mathcal{C}_l|+\frac{|\partial^o\mathcal{C}_l|}{2d}\,. \tag{2.3}$$

Let us set $\alpha=1/n_0^d$ so that $|\mathcal{C}_0|=\alpha n_0^d$. Let k be the smallest integer greater than $2^{d+1}d/c$. We recall that c and n_0 were defined in (2.1) and (2.2). Let us prove by induction on n that

$$\forall n\geq n_0\qquad|\mathcal{C}_{(n-n_0)k}|\geq\alpha n^d\,. \tag{2.4}$$

This is true for $n=n_0$. Let us assume that this inequality is true for some integer $n\geq n_0$. If $|\mathcal{C}_{(n+1-n_0)k}|\geq n^d$, then we are done. Suppose that $|\mathcal{C}_{(n+1-n_0)k}|<n^d$. In this case, for

any integer $l \leq k$, we have also $|\mathcal{C}_{(n-n_0)k+l}| < n^d$, and since $\mathcal{C}_{(n-n_0)k+l}$ is a valid subgraph of $\mathcal{C}(0)$ and $\widehat{\varphi}_n(p_c) > c/n$, we conclude that

$$\frac{|\partial^o \mathcal{C}_{(n-n_0)k+l}|}{|\mathcal{C}_{(n-n_0)k+l}|} \geq \frac{c}{n}$$

and so $|\partial^o \mathcal{C}_{(n-n_0)k+l}| \geq \alpha c n^{d-1}$. Thanks to inequality (2.3) applied k times, we have

$$|\mathcal{C}_{(n+1-n_0)k}| \geq \alpha \left(n^d + \frac{ck}{2d} n^{d-1} \right) .$$

As $k \geq 2^{d+1} d/c$, we get

$$|\mathcal{C}_{(n+1-n_0)k}| \geq \alpha (n^d + 2^d n^{d-1}) \geq \alpha (n+1)^d .$$

This concludes the induction.

Let $\eta > 0$ be a constant that we will choose later. In [1], Barsky, Grimmett and Newman proved that there is no percolation in a half-space at criticality. An important consequence of the result of Grimmett and Marstrand [4] is that the critical value for bond percolation in a half-space equals to the critical parameter $p_c(d)$ of bond percolation in the whole space, *i.e.*, we have

$$\mathbb{P}(0 \text{ is connected to infinity by a } p_c\text{-open path in } \mathbb{N} \times \mathbb{Z}^{d-1}) = 0 ,$$

so that for n large enough,

$$\mathbb{P}(\exists \gamma \text{ a } p_c\text{-open path starting from } 0 \text{ in } \mathbb{N} \times \mathbb{Z}^{d-1} \text{ such that } |\gamma| \geq n) \leq \eta .$$

In what follows, we will consider an integer n such that the above inequality holds. By construction the set $\mathcal{C}_n$ is inside the box $[-n,n]^d$. Starting from this cluster, we are going to resume our exploration but with the constraint that we do not explore anything outside the box $[-n,n]^d$. We set $\mathcal{C}'_0 = \mathcal{C}_n$ and $\mathcal{A}'_0 = \emptyset$. Let us assume $\mathcal{C}'_0, \dots, \mathcal{C}'_l$ and $\mathcal{A}'_0, \dots, \mathcal{A}'_l$ are already constructed. We define

$$\mathcal{A}'_{l+1} = \left\{ x \in [-n,n]^d \, : \, \exists y \in \mathcal{C}'_l \quad \langle x, y \rangle \in \partial^o \mathcal{C}'_l \right\}$$

and

$$\mathcal{C}'_{l+1} = \mathcal{C}'_l \cup \mathcal{A}'_{l+1} .$$

We stop the process when $\mathcal{A}'_{l+1} = \emptyset$. As the number of vertices in the box $[-n,n]^d$ is finite, this process of exploration will eventually stop for some integer l. We have that $|\mathcal{C}'_l| \leq n^d$ and $n\hat{\varphi}_k(p_c) > c$ so that

$$|\partial^o \mathcal{C}'_l| \, \geq \, \frac{c}{n} |\mathcal{C}'_l| \, \geq \, \frac{c}{n} |\mathcal{C}_n| .$$

Moreover, for $n \geq k n_0$, we have, thanks to inequality (2.4),

$$|\mathcal{C}_n| \, \geq \, \left| \mathcal{C}_{\lfloor \frac{n}{k} \rfloor k} \right| \, \geq \, \left| \mathcal{C}_{(\lfloor \frac{n}{k} \rfloor - n_0)k} \right| \, \geq \, \alpha \Big(\Big\lfloor \frac{n}{k} \Big\rfloor \Big)^d .$$

We suppose that n is large enough so that $n \geq k n_0$ and $\lfloor \frac{n}{k} \rfloor \geq n/2k$. Combining the two previous display inequalities, we conclude that

$$|\partial^o \mathcal{C}'_l| \, \geq \, \frac{c\alpha}{2^d k^d} n^{d-1} .$$

Therefore, for n large enough, there exists one face of $[-n,n]^d$ such that there are at least $c\alpha n^{d-1}/(2^d k^d 2d)$ vertices that are connected to 0 by a p_c-open path that remains inside the box $[-n,n]^d$ and so

$$\mathbb{P}\left(\begin{array}{c}\text{there exists one face of } [-n,n]^d \text{ with at least}\\ c\alpha n^{d-1}/(2^d k^d 2d) \text{ vertices that are connected to } 0 \text{ by a}\\ p_c\text{-open path that remains inside the box } [-n,n]^d\end{array}\right) \geq \frac{\delta}{2}\,. \tag{2.5}$$

Let us denote by X_n the number of vertices in the face $\{-n\}\times[-n,n]^{d-1}$ that are connected to 0 by a p_c-open path inside the box $[-n,n]^d$. We have

$$\begin{aligned}\mathbb{E}(X_n) &\leq \left|(\{-n\}\times[-n,n]^{d-1})\cap\mathbb{Z}^d\right|\,\mathbb{P}\left(\begin{array}{c}\exists\gamma \text{ a } p_c\text{-open path starting}\\ \text{from } 0 \text{ in } \mathbb{N}\times\mathbb{Z}^{d-1} \text{ such that}\\ |\gamma|\geq n\end{array}\right)\\ &\leq (2n+1)^{d-1}\eta\,. \end{aligned}\tag{2.6}$$

Moreover, we have

$$\mathbb{E}(X_n)\;\geq\;\frac{c\alpha}{2d2^dk^d}n^{d-1}\;\mathbb{P}\left(X_n>\frac{c\alpha}{2d2^dk^d}n^{d-1}\right)\,. \tag{2.7}$$

Finally, combining inequalities (2.6) and (2.7), we get

$$\mathbb{P}\left(X_n>\frac{c\alpha}{2d2^dk^d}n^{d-1}\right)\leq\frac{2d\eta 3^{d-1}2^dk^d}{c\alpha}\,.$$

Therefore, we can choose η small enough such that

$$\mathbb{P}\left(X_n>\frac{c\alpha}{2d2^dk^d}n^{d-1}\right)\leq\frac{\delta}{10d}$$

and so using the symmetry of the lattice

$$\begin{aligned}&\mathbb{P}\left(\begin{array}{c}\text{there exists one face of } [-n,n]^d \text{ such there are at least}\\ c\alpha n^{d-1}/(2^d k^d 2d) \text{ vertices that are connected to } 0 \text{ by a } p_c\text{-open}\\ \text{path that remains inside the box } [-n,n]^d\end{array}\right)\\ &\qquad\qquad\leq 2d\,\mathbb{P}\left(X_n>\frac{c\alpha}{2d2^dk^d}n^{d-1}\right)\leq\frac{\delta}{5}\,.\end{aligned}$$

This contradicts inequality (2.5) and yields the result. □

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