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Absolute continuity of the martingale limit in branching processes in random environment

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Abstract

We consider a supercritical branching process Z_n in a stationary and ergodic random environment $\xi = (\xi_n)_{n\geq 0}$. Due to the martingale convergence theorem, it is known that the normalized population size $W_n = Z_n/(\mathbb{E}[Z_n|\xi])$ converges almost surely to a random variable W. We prove that if W is not concentrated at 0 or 1 then for almost every environment ξ the law of W conditioned on the environment ξ is absolutely continuous with a possible atom at 0. The result generalizes considerably the main result of [10], and of course it covers the well-known case of the martingale limit of a Galton-Watson process. Our proof combines analytical arguments with the recursive description of W.

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1 Introduction and statement of the main result

We consider a supercritical branching process Z_n in a stationary and ergodic random environment $\xi = (\xi_n)_{n\geq 0}$, defined as follows. Let Δ be the space of probability measures on $\mathbb{N}_0 = \{0, 1, 2, ...\}$ - the set of possible offspring distributions. Let $\xi = (\xi_n)_{n\geq 0}$ be a stationary and ergodic process taking values in Δ. The sequence $(\xi_n)_{n\geq 0}$ is called a “random environment” or “environment sequence”. All our random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The process $(Z_n : n \geq 0)$ with values in $\mathbb{N}_0$ is called a branching process in random environment ξ if Z_0 is independent of ξ and it satisfies

$$\mathcal{L}(Z_n|\xi, Z_0, \dots Z_{n-1}) = \xi_{n-1}^{*Z_{n-1}} \quad \text{a.s.} \tag{1.1}$$

where ξ_{n-1}^{*k} is the k fold convolution. Conditioned on the past and on the environment sequence, Z_n may be viewed as the sum of Z_{n-1} independent and identically distributed random variables $Y_{n-1,i}$, each having law ξ_{n-1}. The process $\{Z_n\}_{n=0}^{\infty}$ conditioned on the

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environment ξ is called a branching process in varying environment. For an environment sequence ξ we denote

$$f_n(s) = \sum_{k=0}^{\infty} s^k \xi_n(\{k\}), \quad s \in \mathbb{C},\ |s| \le 1,$$

the sequence of probability generating functions associated with ξ and

$$m_n = m(\xi_n) = f_n'(1) = \sum_{k=0}^{\infty} k\xi_n(\{k\}),$$

the sequence of the means. Let

$$M_n = \mathbb{E}_\xi[Z_n] = m_0 \cdot \ldots \cdot m_{n-1}$$

and

$$W_n = \frac{Z_n}{M_n}.$$

By $\mathbb{P}_\xi$ we denote the measure $\mathbb{P}$ conditioned on the environment ξ. The corresponding mean and variance are denoted by $\mathbb{E}_\xi$ and Var_ξ i.e. for any random variable X we have $\mathbb{E}_\xi[X] = \mathbb{E}[X|\xi]$ and $\mathrm{Var}_\xi(X) = \mathbb{E}\big[(X - \mathbb{E}_\xi X)^2|\xi\big]$. Finally, for any random variable X we also introduce the conditional law $\mathcal{L}_\xi(X)$ by $\mathcal{L}_\xi(X)(A) = \mathbb{P}(X \in A|\xi)$, where the equality is valid for any measurable set A. Then $(W_n)_{n\ge0}$ is a nonnegative martingale under $\mathbb{P}_\xi$. Therefore,

$$\lim_{n\to\infty} W_n = W$$

exists $\mathbb{P}_\xi$–almost surely.

There has been a lot of interest in asymptotic properties of W, convergence rates of $W - W_n$ as well as limit theorems for Z_n and large deviations principles. Positive and negative, annealed and quenched, moments of W were investigated. Most of that was done for i.i.d. environments, because then properties of the so-called "associated random walks" could be applied, but some results hold also in a stationary and ergodic environment. For a sample of results see [3, 5, 4, 8, 9] and references therein.

However, except of [10] the local regularity of the law of W has not been studied. Due to the recursive equation (1.2) satisfied by W, see below, it is closely related to the local regularity for stationary solutions to affine type equations, see (1.3) below. This motivated us to study absolute continuity of the law of W. More precisely, the following recursive formula will be crucial for our proof. The definition of the process Z_n yields that W satisfies the relation

$$W = \frac{1}{m_0} \sum_{j=1}^{Z_1} W_j, \tag{1.2}$$

where under $\mathbb{P}_\xi$, the random variables W_j are independent of each other and independent of Z_1 with distribution $\mathbb{P}_\xi(W_j \in \cdot) = \mathbb{P}_{T\xi}(W \in \cdot)$. Here, T is the translation operator defined by $T(\xi_0, \xi_1, \dots) = (\xi_1, \xi_2, \dots)$.

The question about local regularity of $\mathcal{L}_\xi(W)$ fits very well into a number of similar problems being investigated recently [6, 7, 12, 13, 18, 22]. For the Galton Watson process the W_j's have the same law as W and so then (1.2) is an example of the so-called smoothing equation. By the latter we mean

$$Y = \sum_{j\ge1} A_j Y_j + C, \tag{1.3}$$

where the equality is meant in law, $(C, A_1, A_2, \dots)$ is a given sequence of real or complex random variables and $Y_1, Y_2, \dots$ are independent copies of the variable Y and independent of $(C, A_1, A_2, \dots)$. Let N be a random number of A_j's that are not zero. As long as $\mathbb{E}N > 1$ the transform

$$S(\mu) = \text{Law of } (\sum_{j\geq 1} A_j Y_j + C),$$

where μ is the law of Y_1, improves local regularity of the measure, and so it is expected that the fixed points of S are absolutely continuous even when the A_j's and C are discrete. This is indeed the case, see [7], [12], [13] and references therein.

However, in the case of a random environment, the equation (1.2) is not exactly of the form in (1.3) and so a different approach had to be elaborated.

If $N = 1$ a.s., (1.3) becomes

$$Y = AY + C \tag{1.4}$$

and absolute continuity of the solution is much harder to prove if (A, C) does not possess a priori any regularity, as for instance in the case of Bernoulli convolutions A is concentrated at λ, for some $0 < \lambda < 1$ and C is a Bernoulli random variable, i.e. C takes the values $+1, -1$ each with probability $1/2$. If $0 < \lambda < 1/2$ then the law ν_λ of $Y = \lambda Y + C$ is continuous but singular with respect to Lebesgue measure and if $\lambda = 1/2$ then ν_λ is the uniform distribution on $[-2, 2]$. However, when $1/2 < \lambda < 1$, ν_λ is absolutely continuous for almost every such λ or even better: it is absolutely continuous outside of a subset of $\lambda \in (1/2, 1)$ of Hausdorff dimension 0. Moreover, if particular λ's are considered, absolute continuity of ν_λ depends on delicate algebraic properties of λ, see [22] for an overview of the recent developments on Bernoulli convolutions.

When we go beyond Bernoulli convolutions there is no general theory about regularity of ν. Further examples of singular (A, C) that give rise to absolutely continuous solutions as well as to singular ones are available, see [6], [14], [18].

Let

$$q(\xi) = \mathbb{P}_\xi\Big(\lim_{n\to\infty} Z_n = 0 \Big| Z_0 = 1\Big)$$

be the extinction probability of the process Z_n. Since ξ is ergodic, $\mathbb{P}(q(\xi) < 1)$ equals 0 or 1 a.s.. We assume that the random variable $\log m_0$ is integrable. If $\mathbb{E}\log m_0 \leq 0$ then it is easy to see that $\mathbb{P}(q(\xi) = 1) = 1$, see also [19], unless $\xi_0 = \delta_1$ a.s. Therefore, we will assume

$$0 < \mu := \mathbb{E}\log m_0 < \infty \tag{1.5}$$

The question whether $\mathbb{P}(q(\xi) < 1)$ is 0 or 1 is well understood (c.f. [16, 17, 2] and [11]):

Proposition 1.1. *Suppose that*

$$\mathbb{E}(\log m_0)^- < \mathbb{E}(\log m_0)^+ \leq \infty, \tag{1.6}$$

and

$$\mathbb{E}|\log(1 - f_0(0))| < \infty, \tag{1.7}$$

holds. Then $\mathbb{P}(q(\xi) < 1) = 1$.

Conversely, if $\mathbb{E}(\log m_0)^+ < \infty$, $\mathbb{P}(q(\xi) < 1) = 1$ *and* $(\xi_n)_{n\geq 0}$ *forms a sequence of i.i.d. random variables then* (1.7) *holds.*

Our main result is the following description of the law of W under $\mathbb{P}_\xi$.

Theorem 1.2. *Suppose that the environment sequence* ξ *is stationary and ergodic and* (1.5) *holds. Let* $\mathcal{L}_\xi$ *be the law of* W *under* $\mathbb{P}_\xi$. *Then exactly one of the following three cases occurs:*

(i) $\mathcal{L}_\xi = \delta_0$ *a.s.*

(ii) $\mathcal{L}_\xi = \delta_1$ *a.s.*

(iii) $q(\xi) < 1$ *and* $\mathcal{L}_\xi(W) = q(\xi)\delta_0 + \nu_\xi$*, a.s where* ν_ξ *is absolutely continuous with respect to the Lebesgue measure.*

Remark 1.3. Note that the statement in case (i) of the Theorem 1.2 is equivalent to $W \equiv 0$. The question when W is not identically zero is well-studied. Below, we provide a sufficient condition for W not to be identically 0, see Theorem 2.3. On the other hand, the statement in case (ii) of Theorem 1.2 clearly holds when the probability measure ξ_0 is a Dirac measure almost surely. In this case, the process $(Z_n : n \geq 0)$ is deterministic under $\mathbb{P}_\xi$. For instance, ξ_0 could have values in $\{\delta_2, \delta_3\}$. We show below, see Theorem 2.1, that if one excludes the case where $W \equiv 0$ and the case where ξ_0 is a Dirac measure, almost surely, then indeed case (iii) in Theorem 1.2 occurs.

In order to prove Theorem 1.2, we will need some additional statements provided in the next section.

2 Further results

In general, for a supercritical BPRE, W may vanish almost surely and conditions for that to happen are well known. Recall that $\mathcal{L}_\xi$ is the law of W under $\mathbb{P}_\xi$. Notice that due to (1.2), the sets $\{\xi : \mathcal{L}_\xi = \delta_0\}$ and $\{\xi : \mathcal{L}_{T\xi} = \delta_0\}$ coincide. Therefore, by ergodicity, $\mathbb{P}(\xi : \mathcal{L}_\xi = \delta_0) \in \{0, 1\}$. If $\mathbb{P}(\xi : \mathcal{L}_\xi = \delta_0) = 0$, i.e. if W is not identically zero, let $z(\xi) = \mathbb{P}_\xi(W = 0) < 1$. In fact, as explained below, it is known, that if $z(\xi) < 1$ then $z(\xi) = q(\xi)$ but we will not need this information for our proof of Theorem 2.1. We say that a measure is *degenerate* if it is concentrated at a point.

Theorem 2.1. *Suppose that the environment sequence* ξ *is stationary and ergodic,* (1.5) *holds,* $\mathbb{P}(\xi : \mathcal{L}_\xi = \delta_0) = 0$ *and* $\mathbb{P}(\xi : \xi_0$ *not degenerate*$) > 0$*. Then*

$$\mathcal{L}_\xi = z(\xi)\delta_0 + \nu_\xi \text{ a.s.,}$$

where ν_ξ *is absolutely continuous with respect to Lebesgue measure.*

Remark 2.2. Theorem 1.2 follows directly from Theorem 2.1. Indeed, if $\mathbb{P}(\xi : \mathcal{L}_\xi = \delta_0) = 1$ then (i) in Theorem 1.2 holds. If $\mu > 0$ (recall (1.5)) and $\mathbb{P}(\xi : \xi_0$ degenerate$) = 1$ then W_n is concentrated at 1 for every n, hence the same is true for W. Moreover, if W is not identically zero then $z(\xi) = q(\xi)$, see [20] and [21]. Let us provide a short argument for the latter statement. If (1.5) holds, then by [20, Theorem 1] there exists a sequence of random variables $c_n(\xi)$ and a nonnegative random variable U such that

$$\lim_{n\to\infty} c_n^{-1} Z_n = U \text{ a.s.}$$

and

$$\mathbb{P}_\xi(U = 0) = q(\xi), \quad \mathbb{P}_\xi(U < \infty) = 1.$$

Since

$$\frac{Z_n}{M_n} = \frac{Z_n}{c_n}\frac{c_n}{M_n},$$

and

$$\mathbb{P}_\xi(U = 0) \leq \mathbb{P}_\xi(W = 0) < 1, \tag{2.1}$$

we conclude that

$$\lim_{n\to\infty} \frac{c_n}{M_n} = L(\xi) < \infty \; a.s.$$

On the other hand, $L(\xi)$ is constant under $\mathbb{P}_\xi$, and therefore, $L(\xi) = 0$ would imply $\mathbb{P}_\xi(W = 0) = 1$ which is a contradiction. Hence $W = L(\xi)U$ and $\mathbb{P}_\xi(W = 0) = \mathbb{P}_\xi(U = 0) = q(\xi)$.

The question when W is not identically zero is well-studied. For a stationary and ergodic environment a sufficient condition was given in [1]:

Theorem 2.3. *(see [1]) Let $Z_0 = 1$. Suppose that* (1.5) *is satisfied and*

$$\mathbb{E}[m_0^{-1} Z_1 \log^+ Z_1] < \infty. \tag{2.2}$$

Then

$$W = \lim_{n\to\infty} \frac{Z_n}{M_n} \text{ is not identically zero.} \tag{2.3}$$

Furthermore,

$$\mathbb{P}_\xi(W = 0) = q(\xi) \quad \text{and} \quad \mathbb{E}_\xi W = 1 \quad a.s. \tag{2.4}$$

Moreover, it was proved in [21] that if (ξ_n) is an i.i.d. sequence then condition (2.2) is in fact equivalent to (2.3). Another proof for i.i.d. environments (ξ_n) is contained in [11]. For i.i.d. environments, assuming (1.5), (2.2) and $\mathbb{E}_\xi W = 1$ a.s. are equivalent. In general, when the sequence (ξ_n) is assumed to be only stationary and ergodic (2.2) is not necessary for W to be not identically zero [21]. In this case the necessary condition is $\sum_{n=0}^\infty m_n^{-1}\Big(\sum_{k\geq M_{n+1}} k\xi_n(k)\Big) < \infty$ $a.s.$ The sufficient condition is only a little bit stronger (see Theorem 1, [21]). Under this sufficient condition, (2.4) holds.

We write

$$\psi(t,\xi) = \mathbb{E}_\xi[e^{itW}]$$

for the conditional characteristic function of W. We now derive a second recursive formula which is crucial for our proofs. Define

$$F_n(s,\xi) = \mathbb{E}_\xi[s^{Z_n} | Z_0, \dots, Z_{n-1}] = f_{n-1}(s)^{Z_{n-1}} \quad \text{a.s.} \tag{2.5}$$

Then by the recursive relation (1.2) we obtain

$$\begin{aligned} \psi(t,\xi) &= f_0(\psi(t/m_0^{-1}, T\xi)) \\ &= f_0 \circ \dots \circ f_{n-1}(\psi(t/M_n, T^n\xi)) = F_n(\psi(t/M_n, T^n\xi)), \end{aligned} \tag{2.6}$$

where F_n is the probability generating function of Z_n given by (2.5) and T the translation operator defined above.

In order to prove Theorem 2.1 we use the following analytical result.

Lemma 2.4. *Let ν be a probability measure on $(\mathbb{R}, \mathcal{B})$ with finite first moment and let ψ be its characteristic function. If $|\psi'|$ is integrable then $\nu = c\delta_0 + \nu_{\rm abs}$ where $\nu_{\rm abs}$ is absolutely continuous with respect to the Lebesgue measure.*

Proof. $\partial_t \psi(t)\, dt$ defines a tempered distribution, see [15], part 2. Moreover, its Fourier inverse satisfies

$$\mathcal{F}^{-1}(\partial_t\psi(t)\, dt) = \mathcal{F}^{-1}(\partial_t\psi(t))\, dx =: f(x)\, dx,$$

where f is a complex valued function vanishing at infinity. In the above formula the first $\mathcal{F}^{-1}$ means the inverse Fourier transform of a tempered distribution and the second $\mathcal{F}^{-1}$ the inverse Fourier transform of an integrable function. On the other hand

$$\mathcal{F}^{-1}(\partial_t\psi(t)\, dt) = -ix\mathcal{F}^{-1}(\psi(t)\, dt) = -ix\nu,$$

as tempered distributions. Hence

$$-ix\nu = f(x)\, dx.$$

This shows that $\nu \mathbb{1}_{\mathbb{R}\setminus\{0\}}$ has density given by $ix^{-1}f(x)$ and the conclusion follows. □

Remark 2.5. Theorem 2.1 generalizes considerably Theorem 1 in [10] but, what is more important, Kaplan's proof contains essential gaps that concern the integrability of $|\psi'(\cdot,\xi)|$. We don't think that they are easily reparable within his approach and instead we suggest our proof which is contained in Theorem 2.6 below. However, the idea to show the integrability of $|\psi'(\cdot,\xi)|$ is borrowed from [10].

The key step in the proof of Theorem 2.1 is the following theorem.

Theorem 2.6. *Suppose that ξ is stationary and ergodic,* (1.5) *holds and for a.e. ξ,*

$$\rho(\xi) := \sup_{|t|\geq 1} |\psi(t,\xi)| < 1. \tag{2.7}$$

Then for a.e. ξ, $\int_{\mathbb{R}} |\psi'(t,\xi)|\,dt < \infty$.

It turns out that (2.7) can be quite easily guaranteed.

Theorem 2.7. *Assume that the environment sequence ξ is stationary and ergodic such that* (1.5) *holds. If W is not identically zero and $\mathbb{P}(\xi_0$ not degenerate$) > 0$, then*

$$\limsup_{|t|\to\infty} |\psi(t,\xi)| < 1.$$

Proof of Theorem 2.1. Suppose that W is not degenerate and (1.5) is satisfied. Then it follows from Theorems 2.7 and 2.6 that for almost every ξ, $\int_{\mathbb{R}} |\psi'(t,\xi)|\,dt < \infty$. Hence by Lemma 2.4, (iii) in Theorem 2.1 holds. Moreover, $z(\xi) < 1$ a.s.

If W is degenerate then it follows from Lemma 3.1 below that $\mathbb{P}(\xi_0$ is degenerate$) = 1$. □

3 Proof of Theorem 2.7

We first need some auxiliary results.

Lemma 3.1. *Suppose that W is not identically zero and W is degenerate, i.e. $\mathrm{Var}_\xi W = 0$. Then $\mathbb{P}(\xi_0$ is degenerate$) = 1$.*

Proof. Taking conditional expectation of both sides of (1.2), we see that $\mathbb{E}_\xi W = \mathbb{E}_{T\xi} W$ and so by ergodicity, $\mathbb{E}_\xi W$ is a strictly positive constant, call it γ. Moreover, due to (1.2),

$$\mathrm{Var}_\xi W = \frac{1}{m_0}\mathrm{Var}_{T\xi} W + \frac{\gamma^2}{m_0^2}\mathrm{Var}_\xi Z_1, \tag{3.1}$$

(which holds also in the case when one of the terms is infinite). Suppose that $\mathrm{Var}_\xi W = 0$. Then iterating (3.1), we have that $\mathrm{Var}_{T^i\xi} Z_1 = 0$ for all $i \in \mathbb{N}$, which is not possible. Indeed, if $\mathbb{P}(\xi_0$ is not degenerate$) > 0$ then by Birkhoff's ergodic theorem for a.e. ξ there is i such that $(T^i\xi)_0 = \xi_i$ is not degenerate. □

Lemma 3.2. *Assume that W is not identically zero and that $\mathbb{P}(\xi_0$ not degenerate$) > 0$ and* (1.5) *holds. Then there is a measurable function $\xi \mapsto (N(\xi), c(\xi)) \in \mathbb{N} \times [0,1]$ such that for a.e. ξ, $c(\xi) > 0$ and*

$$|\psi(t,\xi)| \leq 1 - c(\xi)t^2, \quad \textit{for } 0 \leq t \leq \tfrac{1}{2N(\xi)}$$

Proof. Let W' be a random variable such that under $\mathbb{P}_\xi$, W and W' are i.i.d. Then for almost all ξ we have

$$\lim_{t\to 0}(1 - |\psi(t,\xi)|^2)t^{-2} = \lim_{t\to 0}\big(1 - \mathbb{E}_\xi\big[e^{it(W-W')}\big]\big)t^{-2} = \mathrm{Var}_\xi W \in (0,\infty]. \tag{3.2}$$

Indeed, if $\mathrm{Var}_\xi W < \infty$ then (3.2) follows by applying twice L'Hospital's rule and the fact that $\mathbb{E}_\xi[(W-W')^2] = 2\mathrm{Var}_\xi W$. On the other hand, if $\mathrm{Var}_\xi W = \infty$ then

$$\begin{aligned}\liminf_{t\to 0}\big(1-\mathbb{E}_\xi\big[e^{it(W-W')}\big]\big)t^{-2} &= \liminf_{t\to 0}\mathbb{E}_\xi[1-\cos(t(W-W'))]t^{-2}\\ &\geq \limsup_{t\to 0}\mathbb{E}_\xi\Big[\big(1-\cos(t(W-W'))\big)\mathbb{1}_{[|W-W'|<|t|^{-1/2}]}\Big]t^{-2}.\end{aligned}$$

Next, since for $0 \leq x \leq 1$ the function $x \mapsto x^{-2}(1-\cos x)$ is decreasing, we conclude that for $|t(W-W')| < |t|^{1/2} \leq 1$ it holds

$$\big(1-\cos(t(W-W'))\big)t^{-2} \geq (W-W')^2 \cdot \frac{1-\cos(|t|^{1/2})}{|t|}.$$

Hence

$$\lim_{t\to 0}\big(1-\mathbb{E}_\xi\big[e^{it(W-W')}\big]\big)t^{-2} \geq \limsup_{t\to 0}\frac{1}{2}\mathbb{E}_\xi\Big[(W-W')^2\mathbb{1}_{[|W-W'|<|t|^{-1/2}]}\Big] = \infty,$$

by the monotone convergence theorem. It follows that, for $c(\xi) := \frac{1}{8}\min(1, \mathrm{Var}_\xi W)$ we have

$$|\psi(t,\xi)| \leq \sqrt{1-2c(\xi)t^2} \leq 1 - c(\xi)t^2,$$

on some neighbourhood of 0. In particular,

$$\tau(\xi) := \inf\{s : |\psi(s,\xi)| \geq 1 - c(\xi)s^2\}| > 0,$$

and since $\xi \to \psi(t,\xi)$ is measurable, τ is measurable as well. The lemma now holds with $N(\xi) := \lceil \tau(\xi)^{-1}\rceil$. □

Lemma 3.3. *Assume that the environment sequence ξ is stationary and ergodic such that* (1.5) *holds. If W is not degenerate then for any $0 < \beta < 1$ there are constants $c > 0$ and $t_0 \leq 1$ such that for a.e. ξ there is a sequence of natural numbers n_i such that*

$$(1-\beta)i \leq n_i \leq i, \quad \text{for } i \geq i_0 \tag{3.3}$$

and

$$|\psi(t, T^{n_i}\xi)| \leq 1 - ct^2, \quad \mathit{for}\ 0 \leq t \leq t_0.$$

Proof. Given $\beta \in (0,1)$ there are $c > 0$, $N \in \mathbb{N}$ such that probability of the set

$$S = \{\xi : c(\xi) \geq c,\ N(\xi) \leq N\}$$

is larger than $1-\beta$. By the ergodic theorem, we have for sufficiently large n

$$\sum_{j=1}^{n}\mathbb{1}_S(T^j\xi) \geq (1-\beta)n.$$

Therefore, for every large enough i we can find $(1-\beta)i \leq n_i \leq i$ such that $T^{n_i}\xi \in S$. In view of Lemma 3.2, for $t \leq t_0 := \frac{1}{2N}$ we have $|\psi(t, T^{n_i}\xi)| \leq 1 - ct^2$. □

Proof of Theorem 2.7. We write, using (2.6),

$$\begin{aligned}\psi(t,\xi) &= \mathbb{E}_\xi\left[\psi\Big(\frac{t}{M_n}, T^n\xi\Big)^{Z_n}\right]\\ &= \mathbb{E}_\xi\left[\psi\Big(\frac{t}{M_n}, T^n\xi\Big)^{Z_n}\mathbb{1}_{[W=0]}\right] + \mathbb{E}_\xi\left[\psi\Big(\frac{t}{M_n}, T^n\xi\Big)^{Z_n}\mathbb{1}_{[W>0]}\right]\end{aligned}$$

and the absolute value of the first term above is bounded by $\mathbb{P}_\xi(W=0)<1$. It remains to show that the second term converges to zero as $|t|\to\infty$ ($n=n(t)$ will be adjusted to t).

Fix $0<\beta<\frac{1}{25}$. Then, by the ergodic theorem we get that for almost every ξ

$$e^{(1-\beta)i\mu} \le M_i \le e^{(1+\beta)i\mu} \tag{3.4}$$

for sufficiently large i. In view of (3.4) and (3.3), for large i we have

$$\frac{M_{n_{i+1}}}{M_{n_i}} \le e^{(1+\beta)(n_{i+1})\mu-(1-\beta)n_i\mu} \le e^{((1+\beta)(i+1)-(1-\beta)^2 i)\mu} \le e^{3\beta\mu i} =: \alpha_i. \tag{3.5}$$

For large enough $i_0=i_0(\xi)$ the intervals $[\alpha_i^{-1}t_0M_{n_{i+1}}, t_0M_{n_{i+1}}]$, for $i\ge i_0$, cover $[t_0M_{n_{i_0}},\infty)$. Indeed, given $x\ge M_{n_{i_0}}t_0$, let $M_{n_i}=\max\{M_{n_k}: t_0M_{n_k}\le x, k\ge i_0\}$. Moreover, we may assume that i is maximal with that property. Then $x<t_0M_{n_{i+1}}$ and $x\in[\alpha_i^{-1}t_0M_{n_{i+1}}, t_0M_{n_{i+1}}]$. Further, for $\alpha_i^{-1}t_0M_{n_{i+1}}\le|t|\le t_0M_{n_{i+1}}$ we have

$$\begin{aligned}\left|\mathbb{E}_\xi\left[\psi\Big(\frac{t}{M_{n_{i+1}}},T^{n_{i+1}}\xi\Big)^{Z_{n_{i+1}}}\mathbb{1}_{[W>0]}\right]\right| &\le \mathbb{E}_\xi\left[\left|\psi\Big(\frac{t}{M_{n_{i+1}}},T^{n_{i+1}}\xi\Big)\right|^{Z_{n_{i+1}}}\mathbb{1}_{[W>0]}\right]\\ &\le \mathbb{E}_\xi\left[\big(1-c(|t|M_{n_{i+1}}^{-1})^2\big)^{Z_{n_{i+1}}}\mathbb{1}_{[W>0]}\right]\\ &\le \mathbb{E}_\xi\left[\big(1-ct_0^2\alpha_i^{-2}\big)^{Z_{n_{i+1}}}\mathbb{1}_{[W>0]}\right]\end{aligned}$$

Then since $W_{n_{i+1}}=\frac{Z_{n_{i+1}}}{M_{n_{i+1}}}$ by applying the inequality $1-x\le e^{-x}$, valid for $x\ge 0$, we get

$$\begin{aligned}\left|\mathbb{E}_\xi\left[\psi\Big(\frac{t}{M_{n_{i+1}}},T^{n_{i+1}}\xi\Big)^{Z_{n_{i+1}}}\mathbb{1}_{[W>0]}\right]\right| &\le \mathbb{E}_\xi\left[\exp(-ct_0^2\alpha_i^{-2}Z_{n_{i+1}})\mathbb{1}_{[W>0]}\right]\\ &= \mathbb{E}_\xi\left[\exp(-ct_0^2e^{-6\beta\mu i}M_{n_{i+1}}W_{n_{i+1}})\mathbb{1}_{[W>0]}\right].\end{aligned}$$

Since, for large enough i, we have $M_{n_{i+1}}\ge e^{(1-\beta)^2(i+1)\mu}$ and by the choice of β, $(1-\beta)^2>6\beta$, the dominated convergence theorem gives

$$\limsup_{i\to\infty}\mathbb{E}_\xi\left[\exp(-ct_0^2e^{-6\beta\mu i}M_{n_{i+1}}W_{n_{i+1}})\mathbb{1}_{[W>0]}\right]=0. \qquad \square$$

4 Integrability of ψ'

In this section we prove Theorem 2.6. To this end, we need the following auxiliary result.

Lemma 4.1. *Fix ξ_0 such that $\xi_0(0)<1$. Let $f=f_0$ and*

$$h(r)=\frac{1-r}{1-f(r)}f'(r), \quad 0\le r<1.$$

Then

$$h(r)\le\frac{1}{1+f'(1)^{-1}\sum_{k=2}^\infty\xi_0(k)(1-r)^{k-1}}\le 1.$$

Remark 4.2. The idea to consider the function h is borrowed from [5].

Proof. First, let us observe that for any $0<r<1$ we have

$$1=f(1)\ge\sum_{k=0}^\infty\frac{f^{(k)}(r)}{k!}(1-r)^k.$$

Indeed, by applying Taylor's theorem at r we get that for any natural number N

$$1 = f(1) = \sum_{k=0}^{N} \frac{f^{(k)}(r)}{k!}(1-r)^k + \frac{f^{(N+1)}(s)}{(N+1)!}(1-r)^{N+1},$$

for some $1-r < s < 1$ and since all the derivatives are positive we can take the limit for $N \to \infty$ and obtain the desired inequality. Next, we conclude that

$$h(r) \le \frac{(1-r)f'(r)}{f'(r)(1-r) + R_1} \le 1,$$

where the reminder R_1 is given by

$$R_1 = \sum_{k=2}^{\infty} \frac{f^{(k)}(r)}{k!}(1-r)^k.$$

From the fact that $\xi_0(0) < 1$ a.s. we infer

$$f'(r) = \sum_{m=1}^{\infty} m\xi_0(m) r^{m-1} > 0, \quad a.s.$$

and

$$h(r) \le \frac{1}{1 + f'(r)^{-1}(1-r)^{-1}R_1}.$$

Since all derivatives of f are nonnegative and so nondecreasing, we conclude

$$f'(r) \le f'(1)$$

and

$$\xi_0(k)k! = f^{(k)}(0) \le f^{(k)}(r).$$

In particular, we can estimate the reminder from below

$$R_1 \ge \sum_{k=2}^{\infty} \xi_0(k)(1-r)^k$$

and for $0 \le r \le 1$ we have

$$h(r) \le \frac{1}{1 + f'(1)^{-1}\sum_{k=2}^{\infty} \xi_0(k)(1-r)^{k-1}}. \qquad \square$$

Now we are ready to prove Theorem 2.6, but first let us sketch the idea. Similarly to the proof of Theorem 2.7 the intervals $[M_{n_i}, M_{n_i}\alpha_i]$ cover some half line $[y,\infty)$ and therefore the integrability of ψ' will follow once we prove the finiteness of $\sum_i \int_{|t|\in I_i} |\psi'(t)|dt$, where $I_i = [M_{n_i}, M_{n_i}\alpha_i]$. On each such interval I_i we can use the relation (2.6) and then apply the chain rule. By doing so we get a product of derivatives of functions f_i' which in general is not easy to handle. However, replacing f_i' by h_i, which are bounded by 1 and counting those that are bounded away from 1 leads to exponential decay of $|\psi'(t)|$, uniformly for $M_{n_i} \le |t| \le M_{n_i}\alpha_i$. Finally, as $\alpha_i = e^{3\beta i}$ with arbitrary small β we conclude the integrability of ψ'.

Proof of Theorem 2.6. First notice that due to $\mu > 0$,

$$\mathbb{P}(\xi_0(0) + \xi_0(1) < 1) > 0. \tag{4.1}$$

Moreover, our assumptions imply that

$$\mathbb{P}(\xi_0(0) < 1) = 1. \tag{4.2}$$

Indeed, let $\tilde{S} = \{\xi : \xi_0(0) = 1\}$ and $\mathbb{P}(\tilde{S}) > 0$. Then, by the Poincaré recurrence theorem, for a.e. ξ there is n such that $T^n\xi \in \tilde{S}$ i.e. $\xi_0(0) = 1$ and so $Z_{n+1} = 0$ hence $W = 0$ a.s., which contradicts (2.7). Let us introduce

$$b(\xi) = \frac{1}{1 + f_0'(1)^{-1}\sum_{k=2}^{\infty}\xi_0(k)(1-\rho(T\xi))^{k-1}}. \tag{4.3}$$

In view of (4.1), there is $0 < \eta < 1$ such that for $S = \{\xi : b(\xi) < \eta\}$ we have $\mathbb{P}(S) > \frac{1}{2}$. Take $0 < \beta < \frac{1}{4}$ small enough such that $|\log\eta| > 24\beta\mu$, and then choose $0 < d(\beta) < 1$ such that for $S_1 = \{\xi : \rho(\xi) < d\}$ we have $\mathbb{P}(S_1) > (1-\beta)$. By the ergodic theorem we conclude that for a.e. ξ and sufficiently large n

$$\sum_{j=1}^{n}\mathbb{1}_{S_1}(T^j\xi) > (1-\beta)n.$$

Therefore, we may choose a sequence $n_i \to \infty$ such that

$$(1-\beta)i < n_i \le i \tag{4.4}$$

and

$$\rho(T^{n_i}\xi) < d(\beta)$$

for sufficiently large i. Then $M_{n_i} \to \infty$ and for sufficiently large i

$$\frac{M_{n_i}}{M_j} \ge 1, \quad \text{provided } \frac{n_i}{8} \le j \le \frac{n_i}{2}. \tag{4.5}$$

Indeed, in view of (3.4),

$$M_{n_i}M_j^{-1} \ge e^{\mu(1-\beta)n_i}e^{-\mu(1+\beta)j} = e^{\mu(1-\beta)n_i}e^{-\mu(1+\beta)\frac{n_i}{2}} \ge 1,$$

by the choice of β. As before, in view of (4.5) there is $i(\xi)$ such that for $i \ge i(\xi)$ the intervals $[M_{n_i}, M_{n_i}\alpha_i]$ cover $[M_{n_{i(\xi)}}, \infty)$. Therefore,

$$\begin{aligned}\int_{|t|\ge M_{n_{i(\xi)}}}|\psi'(t,\xi)|\,dt &\le \sum_{i\ge i(\xi)}\int_{M_{n_i}\le|t|\le\alpha_i M_{n_i}}|\psi'(t,\xi)|\,dt\\ &= \sum_{i\ge i(\xi)}\int_{M_{n_i}\le|t|\le\alpha_i M_{n_i}}|F_{n_i}'(\psi(t/M_{n_i},T^{n_i}\xi)\psi'(t/M_{n_i},T^{n_i}\xi)|M_{n_i}^{-1}\,dt\\ &= \sum_{i\ge i(\xi)}\int_{1\le|y|\le\alpha_i}|F_{n_i}'(\psi(y,T^{n_i}\xi))||\psi'(y,T^{n_i}\xi)|\,dy\end{aligned}$$

since $\psi(t,\xi) = F_{n_i}(\psi(t/M_{n_i},T^{n_i}\xi))$. Moreover,

$$|\psi'(y,T^{n_i}\xi)| \le \mathbb{E}_{T\xi}W \le 1.$$

For any n and any complex number z in the unit disk we have:

$$\begin{aligned}|F_n'(z,\xi)| &= \prod_{j=0}^{n-1}|f_j'(f_{j+1}\circ\ldots\circ f_{n-1}(z)| \le \prod_{j=0}^{n-1}f_j'(|f_{j+1}\circ\ldots\circ f_{n-1}(z)|)\\ &= \prod_{j=0}^{n-1}\frac{1-|f_{j+1}\circ\ldots\circ f_{n-1}(z)|}{1-|f_j\circ\ldots\circ f_{n-1}(z)|}f_j'(|f_{j+1}\circ\ldots\circ f_{n-1}(z)|)\times\frac{1-|f_0\circ\ldots\circ f_{n-1}(z)|}{1-|z|}\\ &\le \prod_{j=0}^{n-1}\frac{1-|f_{j+1}\circ\ldots\circ f_{n-1}(z)|}{1-f_j(|f_{j+1}\circ\ldots\circ f_{n-1}(z)|)}f_j'(|f_{j+1}\circ\ldots\circ f_{n-1}(z)|)\times\frac{1-|f_1\circ\ldots\circ f_{n-1}(z)|}{1-|z|},\end{aligned}$$

since $|f_i^{(k)}(z)| \le f_i^{(k)}(|z|)$ for any $k,\ i \ge 0$ and any complex $|z| \le 1$. We intend to prove that for sufficiently large i

$$\prod_{j=0}^{n_i-1} \frac{1-|f_{j+1}\circ\ldots\circ f_{n_i-1}(z)|}{1-f_j(|f_{j+1}\circ\ldots\circ f_{n_i-1}(z)|)} f_j'(|f_{j+1}\circ\ldots\circ f_{n_i-1}(z)|) \le \eta^{\frac{n_i}{8}-2}. \tag{4.6}$$

uniformly for $s=\psi(y,T^{n_i}\xi)$ and $|y|\ge 1$. Hence,

$$\int_{1\le|y|\le\alpha_{n_i}} |F_{n_i}'(\psi(y,T^{n_i}\xi))||\psi'(y,T^{n_i}\xi)|\,dt \le (1-d(\beta))^{-1}\eta^{\frac{n_i}{8}-2}\alpha_i \le (1-d(\beta))^{-1}\eta^{\frac{i}{8}-2}e^{3\beta\mu i}.$$

Then, by the choice of β, the sequence $\eta^{\frac{i}{8}}e^{3\beta\mu i}$ decays exponentially fast and so $\psi'(\cdot,\xi)$ is integrable.

We return now to show that the inequality (4.6) holds. To this end, we first prove that for almost every ξ, sufficiently large i, $\frac{n_i}{8}\le j\le\frac{n_i}{2}$ and $|y|\ge1$, we have

$$|f_{j+1}\circ\ldots\circ f_{n_i-1}(\psi(y,T^{n_i}\xi))| \le \rho(T^{j+1}\xi).$$

Indeed,

$$f_j\circ\ldots\circ f_{n_i-1}(\psi(y,T^{n_i}\xi)) = \psi(M_{n_i}M_j^{-1}y, T^j\xi).$$

So by (4.5), for $|y|\ge1$

$$|\psi(M_{n_i}M_j^{-1}y,T^j\xi)| \le \sup_{|y|\ge1}|\psi(y,T^j\xi)| = \rho(T^j\xi)|.$$

For $r\in[0,1]$ consider

$$h_j(r) = \frac{1-r}{1-f_j(r)}f_j'(r).$$

By Lemma 4.1, for $0\le r\le1$

$$h_j(r) \le \frac{1}{1+f_j'(1)^{-1}\sum_{k=2}^\infty \xi_j(k)(1-r)^{k-1}} \le 1.$$

Let $r=|f_{j+1}\circ\ldots\circ f_{n_i-1}(\psi(y,T^{n_i}\xi))|$. For $n_i/8\le j\le n_i/2$ and i sufficiently large we have

$$\begin{aligned} h_j(r) &\le \frac{1}{1+f_j'(1)^{-1}\sum_{k=2}^\infty \xi_j(k)(1-r)^{k-1}} \\ &\le \frac{1}{1+f_j'(1)^{-1}\sum_{k=2}^\infty \xi_j(k)(1-\rho(T^{j+1}\xi))^{k-1}} \\ &=: b(T^j\xi). \end{aligned}$$

Hence

$$\prod_{j=0}^{n_i-1} \frac{1-|f_{j+1}\circ\ldots\circ f_{n_i-1}(z)|}{1-f_j(|f_{j+1}\circ\ldots\circ f_{n_i-1}(z)|)} f_j'(|f_{j+1}\circ\ldots\circ f_{n_i-1}(z)|) \le \prod_{j=\lceil n_i/8\rceil}^{\lfloor n_i/2\rfloor} b(T^j\xi)$$

for a.e. ξ and sufficiently large i. Since $n_i\to\infty$

$$\frac{1}{\lfloor n_i/2\rfloor}\sum_{j=0}^{\lfloor n_i/2\rfloor}\mathbb{1}_S(T^j\xi) \to \mathbb{P}(S) > \frac12, \quad \text{a.s..} \tag{4.7}$$

So there is $N(\xi)$ such that for $n_i\ge N(\xi)$, $T^j\xi\in S$ at least $\frac12\lfloor n_i/2\rfloor \ge n_i/4-1$ times, that is $b(T^j\xi)<\eta$ at least $n_i/8-2$ times for $j>n_i/8$. Finally, for large enough i,

$$\prod_{j=1}^{n_i} h_j(r) \le \eta^{n_i/8-2}. \qquad \square$$

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