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Short proofs in extrema of spectrally one sided Lévy processes*

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Abstract

We provide short and simple proofs of the continuous time ballot theorem for processes with cyclically interchangeable increments and Kendall's identity for spectrally positive Lévy processes. We obtain the later result as a direct consequence of the former. The ballot theorem is extended to processes having possible negative jumps. Then we prove through straightforward arguments based on the law of bridges and Kendall's identity, Theorem 2.4 in [20] which gives an expression for the law of the supremum of spectrally positive Lévy processes. An analogous formula is obtained for the supremum of spectrally negative Lévy processes.

Keywords: cyclically interchangeable process; spectrally one sided Lévy process; Ballot theorem; Kendall's identity; past supremum; bridge.
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1 Introduction

The series of notes from J. Bertrand [5], É. Barbier [2] and D. André [1] which appeared in 1887 has inspired an extensive literature on the famous ballot theorem for discrete and continuous time processes. In the same year, the initial question raised by J. Bertrand was related by himself to the ruin problem. Using modern formalism, it can be stated in terms of the simple random walk $(S_n)_{n\geq 0}$ as follows:

$$\mathbb{P}(T_k = n \,|\, S_n = -k) = \frac{k}{n}, \quad k, n \geq 1, \tag{1.1}$$

where $T_k = \inf\{j : S_j = -k\}$. The first substantial extension was obtained in 1962 by L. Takács [26] who proved that identity (1.1) is actually satisfied if $(S_n)_{n\geq 0}$ is any downward skip free sequence with interchangeable increments such that $S_0 = 0$. Then the same author considered this question in continuous time and proved the identity

$$\mathbb{P}(T_x = t \,|\, X_t = -x) = \frac{x}{t}\,, \quad x, t > 0, \tag{1.2}$$

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where $(X_s, 0 \le s \le t) = (Y_s - s, 0 \le s \le t)$ and $(Y_s, 0 \le s \le t)$ is an increasing continuous time stochastic process with cyclically interchangeable increments and $T_x = \inf\{s : X_s = -x\}$, see [24]. The first step of this note is to provide a short and elementary proof of a more general result than identity (1.2) which also applies to processes with negative jumps.

Identity (1.2) cannot be extended to all continuous time processes with cyclically interchangeable increments. A problem appears when the process has unbounded variation. In particular, if $(X_s, 0 \le s \le t)$ is a spectrally positive Lévy process with unbounded variation, then we can check that $\mathbb{P}(T_x = t \,|\, X_t = -x) = 0$. However, by considering the process on the whole half line, it is still possible to compare the measures $\mathbb{P}(T_x \in \mathrm{d}t)\,\mathrm{d}x$ and $\mathbb{P}(-X_t \in \mathrm{d}x)\,\mathrm{d}t$ on $(0,\infty)^2$ in order to obtain the following analogous result:

$$\mathbb{P}(T_x \in dt)\,\mathrm{d}x = \frac{x}{t}\mathbb{P}(-X_t \in \mathrm{d}x)\,\mathrm{d}t\,. \tag{1.3}$$

Identity (1.3) was first obtained in the particular case of compound Poisson processes by D. Kendall in [15] where the problem of the first emptiness of a reservoir is solved. It has later been extended by J. Keilson in [14] and A.A. Borovkov in [7] to all spectrally positive Lévy processes. Since then several proofs have been given using fluctuation identities, chap. VII of J. Bertoin's book [4] or martingale identities and change of measures, K. Borovkov and Z. Burq [8]. We shall see in the next section that identity (1.3) can actually be obtained as a direct consequence of (1.2) for Lévy processes with bounded variation and extended to the general case in a direct way.

These results on first passage times will naturally lead us in Section 3 to the law of the past supremum $\overline{X}_t$ of spectrally positive Lévy processes. In a recent work, Z. Michna, Z. Palmowski and M. Pistorius [20] obtained the identity

$$\mathbb{P}(\overline{X}_t > x, X_t \in \mathrm{d}z) = \int_0^t \frac{x-z}{s} p_s(z-x) p_{t-s}(x)\,\mathrm{d}s\,\mathrm{d}z\,, \quad x > z\,, \tag{1.4}$$

where $p_t(x)$ is the density of X_t. As in [20], our proof of identity (1.4) is based on an application of Kendall's identity. However, we show in Theorem 3.1 that a quite simple computation involving the law of the bridge of the Lévy process allows us to provide a very short proof of (1.4). It is first obtained for the dual process $-X$ and then derived for X from the time reversal property of Lévy processes. As a consequence of this result, we obtain in Corollary 3.5 an integro-differential equation characterizing the entrance law of the excursion measure of the Lévy process X reflected at its infimum.

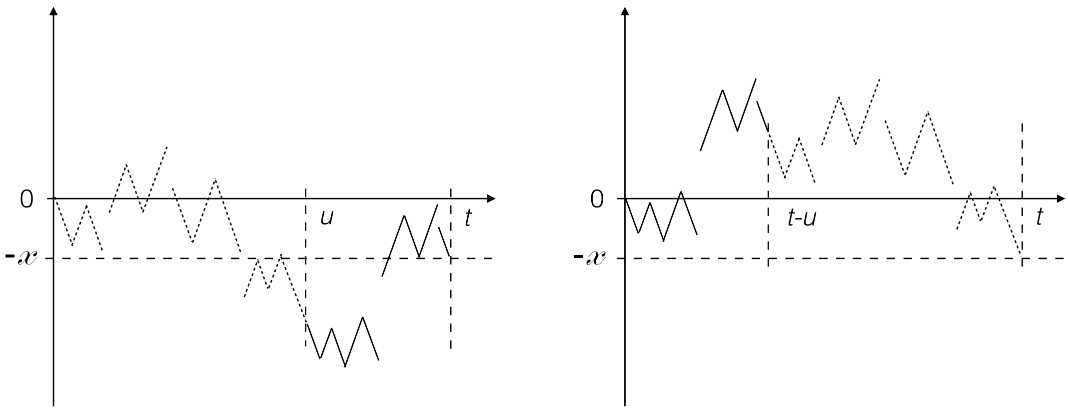
2 Continuous ballot theorem and Kendall's identity

Let $\mathcal{D} = \mathcal{D}([0,\infty))$ and for $t > 0$ let $\mathcal{D}_t = \mathcal{D}([0,t])$ be the spaces of càdlàg functions defined on $[0,\infty)$ and $[0,t]$, respectively. Denote by X the canonical process of the coordinates, i.e. for all $\omega \in \mathcal{D}$ and $s \ge 0$ or for all $\omega \in \mathcal{D}_t$ and $s \in [0,t]$, $X_s(\omega) = \omega(s)$. The spaces $\mathcal{D}$ and $\mathcal{D}_t$ are endowed with their Borel sigma fields $\mathcal{F}$ and $\mathcal{F}_t$, respectively. For any $u \in [0,t]$, we define the family of transformations $\theta_u : \mathcal{D}_t \to \mathcal{D}_t$, as follows:

$$\theta_u(\omega)_s = \begin{cases} \omega(0) + \omega(s+u) - \omega(u), & \text{if } s < t-u \\ \omega(s-(t-u)) + \omega(t) - \omega(u) & \text{if } t-u \le s \le t\,. \end{cases} \tag{2.1}$$

The transformation θ_u consists in inverting the paths $\{\omega(s),\, 0 \le s \le u\}$ and $\{\omega(s),\, u \le s \le t\}$ in such a way that the new path $\theta_u(\omega)$ has the same values as ω at times 0 and 1, i.e. $\theta_u(\omega)(0) = \omega(0)$ and $\theta_u(\omega)(t) = \omega(t)$. We call θ_u the *shift* at time u over the interval

[0,t], see the picture below.



A path ω of $\mathcal{D}_t$ on the left and the shifted path $\theta_u(\omega)$ on the right.

We say that the process $X = (X_s,\, 0 \le s \le t)$ has cyclically interchangeable increments under some probability measure $\mathbb{P}$ on $(\mathcal{D}_t, \mathcal{F}_t)$ if

$$\theta_u(X) \stackrel{(d)}{=} X\,, \quad \text{for all } u \in [0,t]. \tag{2.2}$$

The process $(X, \mathbb{P})$ will be called a *CEI process on* $[0,t]$. Let us note that Lévy processes are CEI processes on $[0,t]$, for all > 0. We define the past supremum and the past infimum of X before time $s \ge 0$ by

$$\overline{X}_s = \sup_{u \le s} X_u \quad \text{and} \quad \underline{X}_s = \inf_{u \le s} X_u\,,$$

this definition being valid for all $s \in [0,t]$ on $\mathcal{D}_t$ and for all $s \ge 0$ on $\mathcal{D}$. For a stochastic process Z defined on $\mathcal{D}$ or $\mathcal{D}_t$, and $x > 0$, we define the first passage time at $-x$ by Z,

$$T_x(Z) = \inf\{s : Z_s = -x\}\,,$$

with the convention that $\inf \emptyset = \infty$. For the canonical process, we will often simplify this notation by setting $T_x := T_x(X)$.

Here is an extension of Theorem 3 in [24], which is known as the continuous time Ballot theorem.

Theorem 2.1. *Let $t > 0$ and $(X, \mathbb{P})$ be a CEI process on $[0,t]$ such that $X_0 = 0$ and $X_t = -x < 0$, a.s., then*

$$\mathbb{P}(T_x = t) = \frac{1}{t}\mathbb{E}(\lambda(E_{t,x}))\,, \tag{2.3}$$

where λ is the Lebesgue measure on $\mathbb{R}$ and $E_{t,x}$ is the random set

$$E_{t,x} = \{s \in [0,t] : X_s = \underline{X}_s \text{ and } X_s \in [\underline{X}_t, \underline{X}_t + x)\}\,.$$

In particular if X is of the form $X_s = Y_s - cs$, where Y is a pure jump, non-decreasing CEI process and c is some positive constant, then

$$\mathbb{P}(T_x = t) = \frac{x}{ct}\,. \tag{2.4}$$

Proof. First observe that for all $u \in [0,t]$,

$$T_x(\theta_u(X)) = t \text{ if and only if } X_u = \underline{X}_u \text{ and } X_u \in [\underline{X}_t, \underline{X}_t + x)\,. \tag{2.5}$$

This fact is readily seen on the graph of X, see for instance the picture above. Then let U be a uniformly distributed random variable on $[0,t]$ which is independent of X under $\mathbb{P}$. The CEI property immediately yields that under $\mathbb{P}$,

$$\theta_U(X) \stackrel{(d)}{=} X\,. \tag{2.6}$$

From (2.5), we obtain $\{T_x(\theta_U(X)) = t\} = \{U \in E_{t,x}\}$ and from (2.6), we derive the equalities,

$$\begin{aligned}\mathbb{P}(T_x(X) = t) &= \mathbb{P}(T_x(\theta_U(X)) = t)\\ &= \mathbb{P}(U \in E_{t,x})\\ &= \frac{1}{t}\mathbb{E}(\lambda(E_{t,x}))\,.\end{aligned}$$

If $X_s = Y_s - cs$, for a pure jump non-decreasing process Y and a constant $c > 0$, then X has bounded variation and for all $t \ge 0$,

$$\begin{aligned}\underline{X}_t &= \int_0^t \mathbb{1}_{\{X_u = \underline{X}_u\}}\,\mathrm{d}X_u\\ &= \sum_{u \le t} \mathbb{1}_{\{X_u = \underline{X}_u\}}(Y_u - Y_{u-}) - c\int_0^t \mathbb{1}_{\{X_u = \underline{X}_u\}}\,\mathrm{d}u.\end{aligned}$$

But since X is càdlàg with no negative jumps, if u is such that $X_u = \underline{X}_u$, then $Y_u = Y_{u-}$. Therefore $\underline{X}_t = -c\int_0^t \mathbb{1}_{\{X_u = \underline{X}_u\}}\,\mathrm{d}u$, so that on the set $\{s \in [0,t] : X_s = \underline{X}_s\}$, the Lebesgue measure satisfies $\lambda(\mathrm{d}s) = \mathrm{d}s = -c^{-1}\,\mathrm{d}\underline{X}_s$, and in particular $\lambda(E_{t,x}) = x/c$, a.s. □

Note that in [24], Theorem 2.1 has been proved for *separable* processes of the form $X_s = Y_s - s$, where Y is a pure jump non decreasing CEI process. Separability implies that the past infimum of the process is measurable and this property can be considered as the minimal assumption for a CEI process to satisfy the ballot theorem. Our proof would still apply, up to slight changes, under this more general assumption. However, since this paper is mainly concerned with Lévy processes, we have chosen the more classical framework of càdlàg processes in which they are usually defined.

Let us now focus on three applications of identity (2.3):

1. Let $x > 0$ and P be some probability measure on $(\mathcal{D}_t, \mathcal{F}_t)$ such that $P(X_t = -x) = 1$. Let U be a random variable on $[0,t]$ which is uniformly distributed and independent of the canonical process X under P. Denote by $\mathbb{P}$ the law of $\theta_U(X)$ under P. Then one easily proves that $(X, \mathbb{P})$ is a CEI process on $[0,t]$. It is also straightforward to show that for all $u \in [0,t]$,

$$E_{t,x} \circ \theta_u = E_{t,x} + t - u, \text{ mod}\,(t)\,, \tag{2.7}$$

and (2.7) implies that $\lambda(E_{t,x}) = \lambda(E_{t,x} \circ \theta_u)$. It follows from these two observations and (2.3) that

$$P(T_x = t) = \mathbb{P}(T_x = t)\,,$$

which allows us to provide many examples of CEI processes $(X, \mathbb{P})$ such that $\mathbb{P}(T_x = t)$ is explicit. Suppose for instance that under P, the canonical process is almost surely equal to the deterministic function

$$\omega(s) = \begin{cases} s^2 & \text{if } 0 \le s < \frac{t}{4} \\ -s^3 - x & \text{if } \frac{t}{4} \le s < \frac{t}{2} \\ -(t-s)^3 - x & \text{if } \frac{t}{2} \le s \le t \end{cases},$$

and set $\underline{\omega}(t) = \inf_{s \le t} \omega(s)$, then

$$\lambda(\{s \in [0,t] : \omega_s = \underline{\omega}_s \text{ and } \omega_s \in [\underline{\omega}_t, \underline{\omega}_t + x)\}) = \frac{t}{4},$$

so that from (2.3),

$$\mathbb{P}(T_x = t) = \frac{1}{4}.$$

2. For our second application, we assume that $(X, \mathbb{P})$ is the bridge with length t of a Lévy process from 0 to $-x < 0$ and we set $\widehat{X} = -X$. Then the process $(\widehat{X}, \mathbb{P})$ is the bridge of the dual Lévy process from 0 to x. By the time reversal property of Lévy processes,

$$(\widehat{X}, \mathbb{P}) = ((x + X_{(t-s)-}, \, 0 \le s \le t), \mathbb{P}),$$

where we set $X_{0-} = X_0$. Hence $\mathbb{P}(T_x = t) = \mathbb{P}(\inf_{0 \le s \le t} \widehat{X}_s \ge 0)$ and from (2.3),

$$\mathbb{P}(\inf_{0 \le s \le t} \widehat{X}_s \ge 0) = \mathbb{P}(\sup_{0 \le s \le t} X_s \le 0) = \frac{1}{t}\mathbb{E}(\lambda(E_{t,x})).$$

Integrating this equality over x with respect to the law $\mathbb{P}(X_t \in \mathrm{d}x)$, this shows that, for the Lévy process, $\sup_{0 \le s \le t} X_s \le 0$ with positive probability if and only if the set $\{s : \underline{X}_s = X_s\}$ has positive Lebesgue measure. Recall that the downward ladder time process is the inverse of the local time defined on this set. Then we have recovered the well-know fact that for a Lévy process, 0 is not regular for $(0, \infty)$ if and only if the downward ladder time process has positive drift, see [11]. Note that when X has no negative jumps, this is also equivalent to the fact that it has bounded variation.

3. The third application is concerned with d-dimensional subordinators, that is d-dimensional Lévy processes whose coordinates are non decreasing. The problem originates from a multidimensional extension of the first emptiness problem of reservoirs raised by D. Kendall [15]. It is considered for instance in Section 3 of [16] where a rough proof is given and in Proposition 6.1 of [22] where Corollary 2.2 is stated with a long proof, sometimes difficult to follow.

Let $\mathrm{X}_t = (X_t^{(1)}, ..., X_t^{(d)})$, $t \ge 0$ be a d-dimensional subordinator under some probability measure P, and assume that for each $t \ge 0$, the law of X_t is absolutely continuous, with density $p_t^{\mathrm{X}}(\mathrm{x})$, $\mathrm{x} \in [0, \infty)^d$. We denote by $\mathrm{x} \cdot \mathrm{y}$ the usual scalar product of x and y. Let φ_{X} be the Laplace exponent of X, that is

$$E(e^{-\mathrm{z} \cdot \mathrm{X}_t}) = e^{-t\varphi_X(\mathrm{z})}, \quad \mathrm{z} \in [0, \infty)^d.$$

Fix $\mathrm{r} = (r_1, ..., r_d)$ with $r_i \ge 0$ and define the one-dimensional process Z by $Z_u = u - \mathrm{r} \cdot \mathrm{X}_u$, $u \ge 0$. Then (Z, P) is clearly a Lévy process with bounded variation and no positive jumps. Assume that $\sum_{i=1}^d r_i E(X_1^{(i)}) \le 1$, so that Z does not drift to $-\infty$. Then let us define the first passage time process of Z by $\tau_t = \inf\{u : Z_u = t\}$, $t \ge 0$ and let $\mathrm{Y}_t = (Y_t^{(1)}, \dots, Y_t^{(d)})$ be the d-dimensional process whose coordinates are defined by

$$Y_t^{(i)} = X^{(i)}(\tau_t), \quad t \ge 0.$$

It is readily seen that Y is a d-dimensional subordinator. Its distribution is described as follows.

Corollary 2.2. *The law of the subordinator* Y *is absolutely continuous and its density function is given by*

$$p_t^{\mathrm{Y}}(\mathrm{y}) = \frac{t}{t+\mathrm{r}\cdot\mathrm{y}} p_{t+\mathrm{r}\cdot\mathrm{y}}^{\mathrm{X}}(\mathrm{y}), \quad t>0,\, \mathrm{y}\in[0,\infty)^d. \tag{2.8}$$

Moreover, the Laplace exponent $\varphi_{\mathrm{Y}}(\mathrm{z})$ *of the process* Y *satisfies*

$$\varphi_{\mathrm{Y}}(\mathrm{z}) = \varphi_{\mathrm{X}}(\mathrm{z}+\varphi_{\mathrm{Y}}(\mathrm{z})\mathrm{r}), \quad \mathrm{z}\in[0,\infty)^d. \tag{2.9}$$

Proof. Note that from the definition of Z, for all y such that $s-\mathrm{r}\cdot\mathrm{y}=t$, conditionally on $\mathrm{X}_s=\mathrm{y}$, the process $(Z_u, 0\le u\le s)$ has interchangeable increments and satisfies $Z_s=t$, so that from (2.4), we have

$$P(\tau_t = s\,|\,\mathrm{X}_s=\mathrm{y}) = \frac{t}{s}. \tag{2.10}$$

Now let us write

$$P(\mathrm{Y}_t\in \mathrm{dy}) = \int_0^\infty P(\mathrm{X}_s\in\mathrm{dy},\, \tau_t\in \mathrm{d}s), \tag{2.11}$$

and note that by definition,

$$Z(\tau_t) = t = \tau_t - \mathrm{r}\cdot\mathrm{X}(\tau_t),$$

which shows that the measure $P(\mathrm{X}_s\in\mathrm{dy},\, \tau_t\in\mathrm{d}s)$ is carried out by the set

$$\{s\ge 0, \mathrm{y}\in[0,\infty)^d : s-\mathrm{r}\cdot\mathrm{y}=t\}.$$

Then we can write

$$\begin{aligned} P(\mathrm{X}_s\in\mathrm{dy},\, \tau_t\in\mathrm{d}s) &= P(\mathrm{X}_s\in\mathrm{dy},\, \tau_t=s)\delta_{\{t+\mathrm{r}\cdot\mathrm{y}\}}(\mathrm{d}s) \\ &= P(\mathrm{X}_s\in\mathrm{dy})\mathbb{P}(\tau_t=s\,|\,\mathrm{X}_s=\mathrm{y})\delta_{\{t+\mathrm{r}\cdot\mathrm{y}\}}(\mathrm{d}s) \\ &= p_s^{\mathrm{X}}(\mathrm{y})\frac{t}{s}\delta_{\{t+\mathrm{r}\cdot\mathrm{y}\}}(\mathrm{d}s)\mathrm{dy}, \end{aligned}$$

where the last equality follows from (2.10). Then we derive (2.8) by integrating over s and using (2.11).

We show identity (2.9) by noticing that

$$\tau_t = t+\tilde\tau_{\mathrm{r}\cdot\mathrm{X}_t} \quad \text{and} \quad \mathrm{X}_{\tau_t} = \mathrm{X}_t + \tilde{\mathrm{X}}_{\tilde\tau_{\mathrm{r}\cdot\mathrm{X}_t}},$$

where $\tilde{\mathrm{X}}_s = \mathrm{X}_{t+s}-\mathrm{X}_t$, $s\ge 0$ and $\tilde\tau_s = \inf\{u : u-\mathrm{r}\cdot\tilde{\mathrm{X}}_u = s\}$. Then we can write from the independence between X_t and $\tilde{\mathrm{X}}$, and the fact that $\tilde{\mathrm{X}}$ has the same law as X,

$$\begin{aligned} \varphi_{\mathrm{Y}}(\mathrm{z}) &= E(e^{-\mathrm{z}\cdot(\mathrm{X}_t+\tilde{\mathrm{X}}_{\tilde\tau_{\mathrm{r}\cdot\mathrm{X}_t}})}) \\ &= \int_{[0,\infty)^d} e^{-\mathrm{z}\cdot\mathrm{x}}E(e^{-\mathrm{z}\cdot\mathrm{X}_{\tau_{\mathrm{r}\cdot\mathrm{x}}}})p_t^{\mathrm{X}}(\mathrm{x})\,\mathrm{dx} \\ &= \varphi_{\mathrm{X}}(\mathrm{z}+\varphi_{\mathrm{Y}}(\mathrm{z})\mathrm{r})\,, \end{aligned}$$

which ends the proof. □

From now on we will consider stochastic processes defined on the whole positive half line. In particular, X is now the canonical process of $\mathcal{D}$. We shall see in the proof of the following theorem that Kendall's identity is a direct consequence of the Ballot theorem.

Theorem 2.3. *Let* $(X,\mathbb{P})$ *be a spectrally positive Lévy process such that* $\mathbb{P}(X_0=0)=1$. *If* $(X,\mathbb{P})$ *is not a subordinator, then the following identity between measures:*

$$\mathbb{P}(T_x\in\mathrm{d}t)\,\mathrm{d}x = \frac{x}{t}\mathbb{P}(-X_t\in\mathrm{d}x)\,\mathrm{d}t \tag{2.12}$$

holds on $(0,\infty)^2$.

Proof. Assume first that X has bounded variation, that is $X_t = Y_t - ct$, where Y is a subordinator with no drift and $c > 0$ is a constant. Let f and g be any two Borel positive functions defined on $\mathbb{R}$. It follows directly from (2.4) by conditioning on X_t that $\mathbb{E}(\mathbb{1}_{\{X_t=\underline{X}_t\}}f(X_t)) = -\mathbb{E}\left(\frac{X_t}{ct}f(X_t)\mathbb{1}_{\{X_t\le 0\}}\right)$, so that

$$\mathbb{E}\left(\int_0^\infty g(t)\mathbb{1}_{\{X_t=\underline{X}_t\}}f(X_t)\,\mathrm{d}t\right) = -\int_0^\infty g(t)\mathbb{E}\left(X_t f(X_t)\mathbb{1}_{\{X_t\le 0\}}\right)\frac{\mathrm{d}t}{ct}. \tag{2.13}$$

Recall from the end of the proof of Theorem 2.1 (applied to processes defined on $[0,\infty)$) that $\mathrm{d}t = -c^{-1}\,\mathrm{d}\underline{X}_t$ on the set $\{t : X_t = \underline{X}_t\}$, so that from the change of variables $t = T_x$,

$$\begin{aligned}\mathbb{E}\left(\int_0^\infty g(t)\mathbb{1}_{\{X_t=\underline{X}_t\}}f(X_t)\,\mathrm{d}t\right) &= -\mathbb{E}\left(\int_0^\infty g(t)f(X_t)\,c^{-1}\,\mathrm{d}\underline{X}_t\right)\\ &= \int_0^\infty \mathbb{E}\left(g(T_x)f(-x)\mathbb{1}_{\{T_x<\infty\}}\right)\frac{\mathrm{d}x}{c}. \end{aligned}\tag{2.14}$$

Then (2.12) follows by comparing the right hand sides of (2.13) and (2.14).

Now if X has unbounded variation and Laplace exponent

$$\varphi(\lambda) := \log\mathbb{E}(e^{-\lambda X_1}) = -a\lambda + \frac{\sigma^2\lambda^2}{2} + \int_{(0,\infty)}(e^{-\lambda x} - 1 + \lambda x\mathbb{1}_{\{x<1\}})\,\pi(\mathrm{d}x),\quad \lambda > 0,$$

then the spectrally positive Lévy process $X^{(n)}$ with Laplace exponent

$$\varphi_n(\lambda) := \log\mathbb{E}(e^{-\lambda X_1^{(n)}}) = -a\lambda + \sigma^2(\lambda\sqrt{n} + n(e^{-\lambda/\sqrt{n}} - 1)) + \int_{(1/n,\infty)}(e^{-\lambda x} - 1 + \lambda x\mathbb{1}_{\{x<1\}})\,\pi(\mathrm{d}x)$$

has bounded variation and the sequence $X_t^{(n)}$, $n \ge 1$ converges weakly toward X_t, for all t. Recall that φ and φ_n are strictly convex functions. Then let ρ and ρ_n be the largest roots of $\varphi(s) = 0$ and $\varphi_n(s) = 0$, respectively. Since X and $X^{(n)}$ are not subordinators, ρ and ρ_n are finite and ρ_n tends to ρ as $n \to \infty$. The first passage time $T_x^{(n)}$ by $X^{(n)}$ at $-x$ has Laplace exponent φ_n^{-1}, where φ_n^{-1} is the inverse of φ_n, on $[\rho_n, \infty)$, see chap. VII in [4]. From these arguments, φ_n^{-1} converges toward the Laplace exponent $\varphi^{-1}$of T_x, so that $T_x^{(n)}$ converges weakly toward T_x, for all $x > 0$. Since $X^{(n)}$ satisfies identity (2.12) for each $n \ge 1$, so does X.

□

Note that if $X = (X_s,\, s \ge 0)$ is a stochastic process such that $X = (X_s,\, 0 \le s \le t)$ is a CEI process for all $t > 0$, then it has actually interchangeable increments, that is for any $t > 0$, $n \ge 1$ and for any permutation σ of the set $\{1, \dots, n\}$,

$$(X_{kt/n} - X_{(k-1)t/n}, k = 1, \dots, n) \stackrel{(d)}{=} (X_{\sigma(k)t/n} - X_{(\sigma(k)-1)t/n}, k = 1, \dots, n)\,.$$

A canonical representation for these processes has been given in Theorem 3.1 of [13]. In particular, conditionally on the tail σ-field $\mathcal{G} = \cap_{t\ge 0}\{X_s : s \ge t\}$, the process X is a Lévy process. By performing again the proof of Theorem 2.3 under the conditional probability $\mathbb{P}(\,\cdot\,|\,\mathcal{G})$ we show that (2.12) is actually valid for all processes with interchangeable increments and no negative jumps.

3 The law of the extrema of spectrally one sided Lévy processes

Throughout this section we are assuming that,

(i) the process $(X, \mathbb{P})$ is a spectrally positive Lévy process which is not a subordinator and such that $\mathbb{P}(X_0 = 0) = 1$.

(ii) For all $t > 0$, the law $p_t(\mathrm{d}x)$ of X_t is absolutely continuous with respect to the Lebesgue measure. We shall denote by $p_t(x)$ its density.

We recall that under assumption (i), 0 is always regular for $(-\infty, 0)$ and that 0 is regular for $(0, \infty)$ if and only if X has unbounded variation, see Corollary 5 in Chap. VII of [4]. Let us also mention that condition (ii) is satisfied for instance if the Lévy measure π of $(X, \mathbb{P})$ is absolutely continuous and satisfies $\pi(0, \infty) = \infty$, see Theorem 27.7 in [21].

Now we briefly recall the definition of bridges of Lévy processes. The law $\mathbb{P}_y^t$ of the bridge from 0 to $y \in \mathbb{R}$, with length $t > 0$ of the Lévy process $(X, \mathbb{P})$ is a regular version of the conditional law of $(X_s,\, 0 \le s \le t)$ given $X_t = y$, under $\mathbb{P}$. It satisfies $\mathbb{P}_y^t(X_0 = 0, X_t = y) = 1$ and for all $s < t$, this law is absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{F}_s$, with density $p_{t-s}(y - X_s)/p_t(y)$, i.e.

$$\mathbb{P}_y^t(\Lambda) = \mathbb{E}\left(\mathbb{1}_\Lambda \frac{p_{t-s}(y - X_s)}{p_t(y)}\right), \quad \text{for all } \Lambda \in \mathcal{F}_s\,. \tag{3.1}$$

Note that from Theorem (3.3) in [23], $p_t(y) > 0$, for all $t > 0$ and $y \in \mathbb{R}$ if and only if for all $c \ge 0$, the process $(|X_t - ct|,\, t \ge 0)$ is not a subordinator. But from assumptions (i) and (ii), the later condition is always satisfied in our framework.

Formula (3.3) below was proved in Theorem 2.4 of [20], see also [19] and Theorem 12 of [17] for the stable case. Here we first prove an analogous formula for the dual process in (3.2) from which (3.3) is immediately derived.

Theorem 3.1. *The laws of $(\overline{X}_t, X_t)$ and $(\underline{X}_t, X_t)$ admit the following expressions,*

$$\mathbb{P}(\underline{X}_t < -x, X_t \in \mathrm{d}z) = \int_0^t \frac{x}{s} p_s(-x) p_{t-s}(x+z)\,\mathrm{d}s\,\mathrm{d}z, \quad -x \le z, \quad x > 0, \tag{3.2}$$

$$\mathbb{P}(\overline{X}_t > x, X_t \in \mathrm{d}z) = \int_0^t \frac{x-z}{s} p_s(z-x) p_{t-s}(x)\,\mathrm{d}s\,\mathrm{d}z, \quad x > z, \quad x \ge 0. \tag{3.3}$$

The process $(X, \mathbb{P})$ has bounded variation if and only if for all $t \ge 0$, $\mathbb{P}(\overline{X}_t = 0) > 0$ (and equivalently $\mathbb{P}(\underline{X}_t = X_t) > 0$). In this case, the expressions (3.2) and (3.3) can be completed by the following one,

$$\mathbb{P}(\overline{X}_t = 0,\, X_t \in \mathrm{d}z) = \mathbb{P}(\underline{X}_t = X_t \in \mathrm{d}z) = -\frac{z}{ct} p_t(z)\,\mathrm{d}z, \quad z < 0\,, \tag{3.4}$$

where $-c$ is the drift of X.

Proof. From (3.1) applied at the stopping time $T_x = \inf\{s : X_s = -x\}$, we obtain

$$\begin{aligned}
\mathbb{P}_z^t(\underline{X}_t < -x) &= \mathbb{P}_z^t(T_x < t)\\
&= \mathbb{E}\left(\mathbb{1}_{\{T_x < t\}} \frac{p_{t-T_x}(z - X_{T_x})}{p_t(z)}\right)\\
&= \mathbb{E}\left(\mathbb{1}_{\{T_x < t\}} \frac{p_{t-T_x}(x + z)}{p_t(z)}\right),
\end{aligned}$$

where in the third equality we used the fact that X has no negative jumps. Recalling the definition of the law $\mathbb{P}_z^t$, we derive from the above equality that

$$\begin{aligned}
\mathbb{P}(\underline{X}_t < -x,\, X_t \in \mathrm{d}z) &= \mathbb{E}(\mathbb{1}_{\{T_x < t\}} p_{t-T_x}(x+z))\,\mathrm{d}z\\
&= \int_0^t \mathbb{P}(T_x \in \mathrm{d}s) p_{t-s}(x+z)\,\mathrm{d}z\,.
\end{aligned}$$

Then (3.2) is obtained by plunging Kendall's identity (2.12) in the right hand side of the above equality.

Identity (3.3) follows by replacing x by $x-z$ in (3.2) and by applying the time reversal property of Lévy processes, that is under $\mathbb{P}$,

$$(X_s, 0 \le s < t) \stackrel{(d)}{=} (X_t - X_{(t-s)-}, 0 \le s < t). \tag{3.5}$$

If $(X, \mathbb{P})$ has bounded variation, then 0 is not regular for the half line $(0, \infty)$, so that for all $t \ge 0$, $\mathbb{P}(\overline{X}_t = 0) > 0$ and $\mathbb{P}(\underline{X}_t = X_t) > 0$, where the second inequality follows from the time reversal property (3.5). Then (3.4) follows directly from (2.13). $\square$

We can derive from Theorem 3.1 a series of immediate corollaries. First we obtain the distribution functions of $\overline{X}_t$ and $\underline{X}_t$ by integrating identities (3.2), (3.3) and (3.4) over z.

Corollary 3.2. *For all $t \ge 0$ and $x > 0$,*

$$\begin{aligned} \mathbb{P}(\underline{X}_t < -x) &= \int_0^t \mathbb{P}(X_{t-s} > 0) p_s(-x) \frac{\mathrm{d}s}{s} + \mathbb{P}(X_t < -x), \qquad (3.6) \\ \mathbb{P}(\overline{X}_t > x) &= \int_0^t \mathbb{E}(X_s^-) p_{t-s}(x) \frac{\mathrm{d}s}{s} + \mathbb{P}(X_t > x). \qquad (3.7) \end{aligned}$$

If X has bounded variation with drift $-c$, then for all $t > 0$,

$$\mathbb{P}(\overline{X}_t = 0) = -\frac{\mathbb{E}(X_t \mathbb{1}_{\{X_t \le 0\}})}{ct}. \tag{3.8}$$

Note that we can derive from (2.12) the following simpler expression for the distribution function of $\underline{X}_t$.

$$\mathbb{P}(\underline{X}_t < -x) = \int_0^t x p_s(-x) \frac{\mathrm{d}s}{s}. \tag{3.9}$$

There exists a huge literature on the law of the extrema of spectrally one sided Lévy processes. First explicit results were obtained for processes with bounded variation in [25]. Then the stable case has received particular attention. In [6] it is proved that $\underline{X}_t$ has a Mittag-Leffler distribution. Then the law of $\overline{X}_t$ was first characterized in [3] and was followed by more explicit forms in [12], [19] and Theorem 12 in [17]. In the general case, one is tempted to derive expressions for the density of the extrema by differentiating (3.6), (3.7) and (3.9) but proving conditions allowing us to do so is an open problem. Only some estimates of these densities have been given in [10] and [18].

Multiplying each side of identities (3.6) and (3.7) by $e^{-\lambda x}$ or x^n and integrating we obtain the following other immediate consequence of Theorem 3.1.

Corollary 3.3. *The Laplace transform of $\overline{X}_t$ and $\underline{X}_t$ are given for $\lambda \ge 0$ by*

$$\mathbb{E}(e^{\lambda \underline{X}_t}) = -\lambda \int_0^t \mathbb{P}(X_{t-s} > 0) \mathbb{E}(e^{\lambda X_s} \mathbb{1}_{\{X_s \le 0\}}) \frac{\mathrm{d}s}{s} + \mathbb{E}(e^{\lambda X_t} \mathbb{1}_{\{X_t \le 0\}}) + \mathbb{P}(X_t > 0),$$

$$\mathbb{E}(e^{-\lambda \overline{X}_t}) = -\lambda \int_0^t \mathbb{E}(X_s^-) \mathbb{E}(e^{-\lambda X_{t-s}} \mathbb{1}_{\{X_{t-s} > 0\}}) \frac{\mathrm{d}s}{s} + \mathbb{E}(e^{-\lambda X_t} \mathbb{1}_{\{X_t > 0\}}) + \mathbb{P}(X_t \le 0).$$

Assume moreover that X admits a moment of order $n \ge 1$. Then $\overline{X}_t$ and $\underline{X}_t$ admit a moment of order n and the later are given by,

$$\mathbb{E}((-\underline{X}_t)^n) = n \int_0^t \mathbb{P}(X_{t-s} > 0) \mathbb{E}((-X_s)^{n-1} \mathbb{1}_{\{X_s < 0\}}) \frac{\mathrm{d}s}{s} + \mathbb{E}((X_t^-)^n),$$

$$\mathbb{E}(\overline{X}_t^n) = n \int_0^t \mathbb{E}(X_s^-) \mathbb{E}(X_{t-s}^{n-1} \mathbb{1}_{\{X_{t-s} \ge 0\}}) \frac{\mathrm{d}s}{s} + \mathbb{E}((X_t^+)^n).$$

Then for $\lambda \geq 0$ and $z < 0$, define the Laplace transform of the function $t \mapsto t^{-1} p_t(z)$ by

$$\varphi(\lambda, z) = \int_0^\infty e^{-\lambda t} t^{-1} p_t(z) \, \mathrm{d}t \, .$$

Corollary 3.4. *The Laplace transform $\varphi(\lambda, z)$ satisfies the equation*

$$\varphi(\lambda, z) = \varphi(0, z) + e^{-z\Phi(\lambda)}\,, \quad \lambda \geq 0\,, \; z < 0\,, \tag{3.10}$$

where $\Phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) t^{-1} p_t(0) \, \mathrm{d}t$.

Proof. Letting $x = 0$ in identity (3.3), we obtain for $z < 0$, $p_t(z) = \int_0^t \frac{-z}{s} p_s(z) p_{t-s}(0) \, \mathrm{d}s$. Taking the Laplace transform of each side of this identity gives

$$\frac{\partial}{\partial \lambda} \varphi(\lambda, z) = -z \varphi(\lambda, z) \int_0^\infty e^{-\lambda t} p_t(0) \, \mathrm{d}t \, ,$$

whose solution is given by (3.10).

□

Recall from [9] and [10] the definition of the entrance laws $q_t(\mathrm{d}x)$ (resp. $q_t^*(\mathrm{d}x)$) of the excursions reflected at the supremum (resp. at the infimum) of X. Both reflected processes $\overline{X} - X$ and $X - \underline{X}$ are homogeneous Markov processes. We denote by n and n^* the characteristic measures of the corresponding Poisson point processes of excursions away from 0, see [9]. Then $q_t(\mathrm{d}x)$ and $q_t^*(\mathrm{d}x)$ are defined by

$$n(f(X_t), \, t < \zeta) = \int_{[0,\infty)} f(x) q_t(\mathrm{d}x) \quad \text{and} \quad n^*(f(X_t), \, t < \zeta) = \int_{[0,\infty)} f(x) q_t^*(\mathrm{d}x)\,,$$

where ζ denotes the life time of the excursions and f is any positive Borel function. We also recall that if $p_t(\mathrm{d}x)$ is absolutely continuous, then so are $q_t(\mathrm{d}x)$ and $q_t^*(\mathrm{d}x)$, see part (3) of Lemma 1, p. 1208 in [9]. We will denote the corresponding densities by $q_t(x)$ and $q_t^*(x)$. Thanks to the absence of negative jumps, the entrance law $q_t(\mathrm{d}x)$ can be related to the law of X_t through the relation,

$$q_t(x) = \frac{x}{t} p_t(-x)\,, \tag{3.11}$$

which is valid for all $t > 0$ and $x \geq 0$, see (5.10), p.1208 in [9]. We now use this fact and Theorem 3.1, in order to describe the entrance law $q_t^*(\mathrm{d}x)$.

Corollary 3.5. *The entrance law $q_t^*(x)$ satisfies the equation,*

$$\int_0^t \frac{x - z}{t - s} p_{t-s}(z - x) q_s^*(x) \, \mathrm{d}s = -\frac{\mathrm{d}}{\mathrm{d}x} \int_0^t \frac{x - z}{t - s} p_{t-s}(z - x) p_s(x) \, \mathrm{d}s\,, \tag{3.12}$$

for all $t > 0$, $x > 0$ and $z < x$.

Proof. Let us recall that from Theorem 6 in [9], the law of the couple $(\overline{X}_t, X_t)$ is given in terms of q_t and q_t^* as follows,

$$\mathbb{P}(\overline{X}_t \in \mathrm{d}x, \, X_t \in \mathrm{d}z) = \int_0^t q_s^*(x) q_{t-s}(x - z) \, \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}z\,, \tag{3.13}$$

for $x > 0$ and $z < x$. Then plunging (3.11) into (3.13) and comparing this expression with (3.3) where we performed the time change $s \to t - s$ and we differentiated in $x > 0$, we obtain (3.12). □

Let us finally point out that actually Theorem 6 in [9] gives the following disintegrated version of (3.13),

$$\mathbb{P}(g_t \in \mathrm{d}s, \overline{X}_t \in \mathrm{d}x,\, X_t \in \mathrm{d}z) = q_s^*(x) q_{t-s}(x-z) \mathbb{1}_{[0,t]}(s)\, \mathrm{d}x\, \mathrm{d}z\,, \tag{3.14}$$

on $(0,\infty)^2 \times \mathbb{R}$, where g_t is the unique time at which the past supremum of $(X, \mathbb{P})$ occurs on $[0,t]$. This result suggests a possibility of disintegrating also (3.3) according to the law of g_t. Then comparing this disintegrated form with (3.14) would provide a means to obtain an expression for the density $q_t^*(x)$ in terms of $p_t(x)$. However, this problem remains open.

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