

Discrete maximal regularity of an implicit Euler–Maruyama scheme with non-uniform time discretisation for a class of stochastic partial differential equations

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Abstract

An implicit Euler–Maruyama method with non-uniform step-size applied to a class of stochastic partial differential equations is studied. A spectral method is used for the spatial discretization and the truncation of the Wiener process. A discrete analogue of maximal L^2 -regularity of the scheme and the discretised stochastic convolution is established, which has the same form as their continuous counterpart.

Keywords: multiplicative noise; non-uniform time discretisation; implicit Euler–Maruyama scheme.

AMS MSC 2010: 60H15; 60H35.

Submitted to ECP on October 20, 2017, final version accepted on April 4, 2018.

1 Introduction

Our interest in this paper lies in a discrete analogue of maximal regularity for a class of stochastic partial differential equations (SPDEs) of parabolic type. In more detail, with a positive self-adjoint generator $-A$ with compact inverse densely defined on a separable Hilbert space H , we consider the equation

$$\begin{cases} dX(t) = AX(t)dt + B(t, X(t))dW(t), & \text{for } t \in (0, 1] \\ X(0) = \xi, \end{cases} \quad (1.1)$$

where the mild solution X takes values in H . The assumption on B and the Q -Wiener process W will be discussed later. The aim of this paper is to show a property of a prototypical discretisation to simulate the solution of such equations: we show a discrete analogue of an estimate called maximal regularity (Corollary 4.6).

Maximal regularity is a fundamental concept in the theory of deterministic partial differential equations (see, for example [2, 18, 21] and references therein). Similarly, in the study of stochastic partial differential equations, the maximal regularity is an important analysis tool [9, 8] as well as an active research area [27, 26, 5, 28]. In our setting, the above equation (1.1) can be shown to satisfy the maximal regularity estimate of the form

$$\int_0^1 \mathbb{E}[\|X(s)\|_{D(A^{\iota+\frac{1}{2}})}^2] ds \leq \|\xi\|_{D(A^{\iota+\frac{1}{2}})}^2 + \int_0^1 \mathbb{E}\|B(r, X(r))\|_{\mathcal{L}_2(H_0, D(A^{\iota}))}^2 dr, \quad (1.2)$$

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where $\iota \geq 0$ is a suitable parameter depending on the operator B , $D(A^{\iota+\frac{1}{2}})$ is the domain of the fractional power $A^{\iota+\frac{1}{2}}$ of A in H , and $\mathcal{L}_2(H_0, D(A^\iota))$ is the space of Hilbert–Schmidt operator from H_0 , the Cameron–Martin space associated with Q , to $D(A^\iota)$. More details will be discussed later.

In recent years, the study of discrete analogues of the maximal regularity has been attracting attention for deterministic partial differential equations [1, 4, 7, 13, 14, 15, 20]; to the best of the author’s knowledge, corresponding properties of numerical methods for stochastic PDEs have not been addressed in the literature.

Maximal regularity of stochastic and deterministic equations are different in nature. As we will see in (1.2), given a suitable smoothness of the initial data, the solution is “one-half spatially smoother” than the range of the diffusion operator $B(t, x)$. This estimate is optimal, in that the solution cannot be spatially smoother in general (see [17, Example 5.3]). To put it another way, as described in [9, Chapter 6], the regularity one can obtain is the half of the corresponding regularity for the deterministic case.

We focus on the case where the operator A and the covariance operator Q share the same eigensystems. This prototypical setting is partly motivated by applications in environmental modelling and astrophysics, where covariance operators—of the random fields [6, 22], and of the Wiener process for the stochastic heat equations [19, 3], for example—the eigenspaces of which are the same as those of the Laplace operators play important roles. In simulations, it is desirable that discretisations users employ inherit properties of the solution of the model considered. Our results show the method we consider preserves a spatial regularity—maximal regularity—in a suitable sense.

As a spatial discretisation we consider the spectral-Galerkin method. The Wiener process, which is assumed to admit a series representation, takes its value in an infinite-dimensional space. In practice, we can simulate only finitely many of them. We approximate the Wiener process by truncation, i.e., we use a type of truncated Karhunen–Loève approximation.

Temporally, we consider the implicit Euler–Maruyama method with a non-uniform time discretisation. The aforementioned approximation of the Wiener process introduces one-dimensional Wiener processes multiplied by varying scalars—the eigenvalues of the covariance operator. Motivated by this observation, we allow the discretisation of each Wiener process to depend on these scalars. The algorithm we consider is first proposed by Müller-Gronbach and Ritter [24, 23], for the stochastic heat equation on the unit cube. In [24, 23], the resulting non-uniform scheme was shown to achieve an asymptotic optimality under a suitable step size, which in general cannot be achieved by schemes with uniform step-size.

The results we establish show that the non-uniform discretisation still preserves maximal regularity in a suitable sense. The algorithm we consider includes the implicit Euler–Maruyama method with the uniform time discretisation as a special case—the case where one uses the same step size for all one-dimensional Wiener processes—even though we, in general, lose the aforementioned optimality. As a consequence, we obtain a discrete analogue of maximal regularity for the standard implicit Euler–Maruyama method: the discretisation with the uniform step size.

The structure of this paper is as follows. Section 2 recalls some definitions and basic results needed in this paper. Section 3 introduces the discretised scheme we consider. Then, in Section 4 we show a discrete maximal regularity. Then, we conclude this paper in Section 5.

2 Setting

By H we denote a separable \mathbb{R} -Hilbert space $(H, \langle \cdot, \cdot \rangle, \|\cdot\|)$. Let $-A : D(A) \subset H \rightarrow H$ be a self-adjoint, positive definite linear operator that is densely defined on H , with compact inverse. Then, A is the generator of the C_0 -semigroup $(S(t))_{t \geq 0} := (e^{At})_{t \geq 0}$ acting on H that is analytic. Further, there exists a complete orthonormal system $\{h_j\}$ for H such that $-Ah_j = \lambda_j h_j$, each eigenspace is of finite dimensional, and

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_j < \dots,$$

and $\lambda_j \rightarrow \infty$ as $j \rightarrow \infty$ unless the compact inverse $-A^{-1}$ is finite rank. For simplicity, we assume the dimension of each eigenspace is 1. Then, we have the spectral representation

$$S(t)x = \sum_{j=1}^{\infty} e^{-\lambda_j t} \langle x, h_j \rangle h_j \in H, \quad \text{for } x \in H.$$

For $r \in \mathbb{R}$, let us define the domain $D(A^r)$ of the fractional power A^r of A by

$$D(A^r) := \left\{ x \in H \mid \|x\|_{D(A^r)}^2 = \sum_{j=1}^{\infty} \lambda_j^{2r} \langle x, h_j \rangle^2 < \infty \right\}.$$

We obtain a separable Hilbert space $(D(A^r), \langle \cdot, \cdot \rangle_{D(A^r)}, \|\cdot\|_{D(A^r)})$ by setting $\langle \cdot, \cdot \rangle_{D(A^r)} := \langle A^r \cdot, A^r \cdot \rangle$.

For more details for the set up above, see for example [12, 21, 25, 29].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration satisfying the usual conditions. By $W : [0, 1] \times \Omega \rightarrow H$ we denote the Q -Wiener process with a covariance operator Q of the trace class. We assume that the Wiener process W is adapted to the filtration. Further, we assume that the eigenfunctions h_ℓ of A is also eigenfunctions of Q with

$$Qh_\ell = q_\ell h_\ell,$$

such that $\text{Tr}(Q) = \sum_{\ell=1}^{\infty} \langle Qh_\ell, h_\ell \rangle = \sum_{\ell=1}^{\infty} q_\ell < \infty$. It is well-known that W taking values in H can be characterised as

$$W(t) = \sum_{\ell=1}^{\infty} \sqrt{q_\ell} \beta_\ell(t) h_\ell \quad \text{a.s.,}$$

where β_ℓ are independent one-dimensional standard Brownian motions with the zero initial condition realised on $(\Omega, \mathcal{F}, \mathbb{P})$ that are adapted to the underlying filtration, and that the series converges in the Bochner space $L^2(\Omega; C([0, 1]; H))$. The Q -Wiener process takes values in H by construction. Here, since A and Q are assumed to share the same eigenfunctions, we can provide finer characterisations of the regularity.

Remark 2.1. Let $r \geq 0$ and $t \in (0, 1]$. Then, $\sum_{\ell=1}^{\infty} \lambda_\ell^{2r} q_\ell < \infty$ if and only if $W(t) \in D(A^r)$, a.s. Indeed, we have $\mathbb{E}[\|W(t)\|_{D(A^r)}^2] = t \sum_{\ell=1}^{\infty} \lambda_\ell^{2r} q_\ell$.

We introduce the Hilbert space $H_0 = Q^{1/2}(H)$ equipped with the inner product

$$\langle h_1, h_2 \rangle_0 = \langle Q^{-1/2} h_1, Q^{-1/2} h_2 \rangle \quad \text{for } h_1, h_2 \in H,$$

where $Q^{-1/2} := (Q^{1/2}|_{(\ker(Q^{1/2}))^\perp})^{-1} : H_0 \rightarrow (\ker(Q^{1/2}))^\perp$ is the pseudo-inverse of $Q^{1/2}$.

In the following, $a \preceq b$ means that a can be bounded by some constant times b uniformly with respect to any parameters on which a and b may depend. Throughout this paper, we assume the following.

Assumption 2.2. We assume $B: [0, 1] \times H \rightarrow \mathcal{L}_2(H_0, H)$ is $\mathcal{B}([0, 1]) \otimes \mathcal{B}(H) / \mathcal{B}(\mathcal{L}_2(H_0, H))$ -measurable, where for a given normed space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ the Borel σ -algebra associated with the norm topology is denoted by $\mathcal{B}(\mathcal{X})$. Further, let B satisfy

$$\|B(t, u) - B(t, v)\|_{\mathcal{L}_2(H_0, H)} \preceq \|u - v\|, \quad \text{for } t \in [0, 1], u, v \in H. \quad (2.1)$$

Moreover, let $\iota \in [0, 1/2]$ be given. We assume for any $t \in [0, 1]$, $u \in D(A^\iota)$ we have $B(t, u) \in \mathcal{L}_2(H_0, D(A^\iota))$ and

$$\|B(t, u)\|_{\mathcal{L}_2(H_0, D(A^\iota))} \preceq 1 + \|u\|_{D(A^\iota)}. \quad (2.2)$$

The condition (2.2) implies $\sup_{t \in [0, 1]} \|B(t, 0)\|_{\mathcal{L}_2(H_0, D(A^\iota))} \preceq 1$. Thus, together with (2.1) we see that Assumption 2.2 implies

$$\|B(t, u)\|_{\mathcal{L}_2(H_0, H)} \preceq C_\iota(1 + \|u\|) < \infty, \quad (2.3)$$

for $t \in [0, 1]$, $u \in H$, with a constant $C_\iota > 0$.

We recall the following existence result, which can be found in, for example, [9].

Theorem 2.3. Suppose that the mapping B satisfies Assumption 2.2 with some $\iota \geq 0$. Then, for $\xi \in H$ there exists an H -valued continuous process $(X(t))_{t \in [0, 1]}$ adapted to the underlying filtration satisfying the usual conditions such that

$$X(t) = S(t)\xi + \int_0^t S(t-s)B(s, X(s))dW(s), \quad t \in [0, 1] \quad \text{a.s.} \quad (2.4)$$

Moreover, this process is uniquely determined a.s., and it is called the mild solution of (1.1). Further, for any $p \geq 2$ we have

$$\sup_{t \in [0, 1]} \mathbb{E}\|X(t)\|^p < \infty. \quad (2.5)$$

For more details, see for example [9, Sec. 7.1]. For the mild solution X , let

$$X(t) = \sum_{j=1}^{\infty} X_j(t)h_j, \quad X_j(t) = \langle X(t), h_j \rangle.$$

Then, the processes $X_j = (X_j(t))_{t \in [0, 1]}$ satisfy the following bi-infinite system of stochastic differential equations:

$$\begin{cases} dX_j(t) &= -\lambda_j X_j(t)dt + \sum_{\ell=1}^{\infty} \sqrt{q_\ell} \langle B(t, X(t))h_\ell, h_j \rangle d\beta_\ell(t) \\ X_j(0) &= \langle \xi, h_j \rangle, \quad \text{for } j \in \mathbb{N}. \end{cases}$$

Each process X_j is given as

$$\begin{aligned} X_j(t) &= e^{-\lambda_j t} \langle \xi, h_j \rangle \\ &\quad + \sum_{\ell=1}^{\infty} \sqrt{q_\ell} \int_0^t e^{-\lambda_j(t-s)} \langle B(s, X(s))h_\ell, h_j \rangle d\beta_\ell(s), \end{aligned}$$

where the series in the second term is convergent in $L^2(\Omega)$, due to (2.5) and Assumption 2.2.

We have the following spatial regularity result.

Proposition 2.4. Suppose that Assumption 2.2 is satisfied with some $\iota \in [0, 1/2]$, and that the initial condition satisfies $\xi \in D(A^\iota)$. Then, we have the estimate

$$\int_0^1 \mathbb{E}\|X(s)\|_{D(A^{\iota+1/2})}^2 ds \leq \|\xi\|_{D(A^\iota)}^2 + \int_0^1 \mathbb{E}\|B(r, X(r))\|_{\mathcal{L}_2(H_0, D(A^\iota))}^2 dr. \quad (2.6)$$

Proof. Itô’s isometry yields

$$\begin{aligned} \lambda_j^{2\iota+1} \mathbb{E}(X_j(s))^2 &= \exp(-2\lambda_j s) \lambda_j^{2\iota+1} \langle \xi, h_j \rangle^2 \\ &\quad + \int_0^s \exp(-2\lambda_j(s-r)) \lambda_j \mathbb{E} \|B^*(r, X(r)) \lambda_j^{\iota+\frac{1}{2}} h_j\|_{H_0}^2 dr, \end{aligned}$$

where $B^*(r, X(r))$ denotes the adjoint operator of $B(r, X(r))$. Therefore, it holds that $\int_0^1 \mathbb{E}[\lambda_j^{2\iota+1} |X_j(s)|^2] ds \leq \lambda_j^{2\iota} \langle \xi, h_j \rangle^2 + \int_0^1 \mathbb{E} \|B^*(r, X(r)) \lambda_j^{2\iota} h_j\|_{H_0}^2 dr$, and thus summing over $j \geq 1$ yields the desired result.

We note that for $\iota \in [0, 1/2]$ the right hand side of (2.6) is finite. To see this, we first note that (2.3) together with (2.5) implies

$$\int_0^1 \mathbb{E} \|X(s)\|_{D(A^{1/2})}^2 ds \leq \|\xi\|^2 + \int_0^1 \mathbb{E} \|B(r, X(r))\|_{\mathcal{L}_2(H_0, H)}^2 dr < \infty.$$

Thus, from (2.2) we have

$$\begin{aligned} \int_0^1 \mathbb{E} \|B(r, X(r))\|_{\mathcal{L}_2(H_0, D(A^\iota))}^2 dr &\leq 1 + \int_0^1 \mathbb{E} \|X(r)\|_{D(A^\iota)}^2 dr \\ &\leq c_\iota \left(1 + \int_0^1 \mathbb{E} \|X(r)\|_{D(A^{1/2})}^2 dr \right) < \infty, \end{aligned}$$

for some constant $c_\iota > 0$. □

Remark 2.5. We note that the solution is spatially one half smoother than the range of $B(t, x)$. This is in general optimal, in that the solution cannot be spatially smoother in general ([17, Example 5.3]). For more details, see [17, 16] and references therein. For recent developments of maximal regularity theory, see [27, 26].

3 Discretisation

This section introduces the scheme proposed by Müller-Gronbach and Ritter [24, 23]. In this regard, let us first discretise the interval $[0, 1]$ with a uniform partition, i.e., we partition the interval with $t_i = i/n$, for $i = 0, 1, 2, \dots, n$. For integers $J, L \in \mathbb{N}$, an Itô–Galerkin approximation $\bar{X}(t_i)$ to (2.4) with the temporal discretisation being the implicit Euler–Maruyama scheme with a uniform time discretisation is given by

$$\bar{X}^{J,L}(t_i) = \sum_{j=1}^J \bar{X}_j^{J,L}(t_i) h_j, \quad \text{for } i = 0, \dots, N, \tag{3.1}$$

with coefficients $\langle \bar{X}(t_i), h_j \rangle$ defined by $\bar{X}_j^{J,L}(0) = \langle \xi, h_j \rangle$, and

$$\begin{aligned} \bar{X}_j^{J,L}(t_i) &= \left(1 + \frac{\lambda_j}{n} \right)^{-1} \left(\bar{X}_j^{J,L}(t_{i-1}) \right. \\ &\quad \left. + \sum_{\ell=1}^L \sqrt{q_\ell} \left\langle B(t_{i-1}, \bar{X}^{J,L}(t_{i-1})) h_\ell, h_j \right\rangle (\beta_\ell(t_i) - \beta_\ell(t_{i-1})) \right). \end{aligned}$$

Müller-Gronbach and Ritter [24, 23] noted that the projected Q -Wiener processes $\sqrt{q_\ell} \beta_\ell = \sqrt{\langle Q h_\ell, h_\ell \rangle} \beta_\ell = \langle W(t), h_\ell \rangle$ have varying variances depending on the index ℓ . This observation motivated them to use different step-sizes depending on ℓ . Following them, we evaluate the standard one-dimensional Wiener process β_ℓ at each level $\ell = 1, \dots, L$ at the corresponding $n_\ell \in \mathbb{N}$ nodes

$$0 < t_{1,\ell} < \dots < t_{n_\ell,\ell} = 1, \quad \text{where } t_{i,\ell} = \frac{i}{n_\ell} \quad \text{for } i = 0, \dots, n_\ell.$$

Then, the discretisation of the truncated Q -Wiener process $\sum_{\ell=1}^L \sqrt{q_\ell} \beta_\ell h_\ell$ in general results in a non-uniform time discretisation:

$$0 =: \tau_0 < \dots < \tau_N := 1, \quad \text{where} \quad \{\tau_0, \dots, \tau_N\} := \bigcup_{\ell=1}^L \{t_{0,\ell}, \dots, t_{n_\ell,\ell}\},$$

and $t_{0,\ell} = \tau_0 = 0$ for all $\ell \in \mathbb{N}$. To write our scheme in the recursive form, we introduce the following notations. Let

$$\mathcal{K}_\eta := \{\ell \in \{0, 1, \dots, L\} \mid \tau_\eta \in \{t_{0,\ell}, \dots, t_{n_\ell,\ell}\}\},$$

for $\eta = 0, \dots, N$ and we define $s_{\eta,\ell}$ for $\eta = 1, \dots, N$ and $\ell = 1, \dots, L$ by

$$s_{\eta,\ell} := \max \{ \{t_{0,\ell}, \dots, t_{n_\ell,\ell}\} \cap [0, \tau_\eta] \}.$$

We further introduce the following notation for the product of eigenvalues of the operator $(I - \frac{1}{\tau_\nu - \tau_{\nu-1}} A)^{-1}$, which we use for the approximation of the semigroup generated by A . For any $\tau_{\eta_1} \leq \tau_{\eta_2}$, we let

$$\mathfrak{R}_j(\tau_{\eta_1}, \tau_{\eta_2}) := \prod_{\nu=\eta_1+1}^{\eta_2} \frac{1}{1 + \lambda_j(\tau_\nu - \tau_{\nu-1})}, \tag{3.2}$$

with the convention $\prod_\emptyset = 1$. Note that $s_{\eta,\ell}, t_{i-1,\ell} \in \{\tau_1, \dots, \tau_N\}$. Then, for $\eta = 1, \dots, N$, the drift-implicit Euler-Maruyama scheme in the recursive form is given by,

$$\begin{aligned} \widehat{X}_j^{J,L}(\tau_\eta) = \mathfrak{R}_j(\tau_{\eta-1}, \tau_\eta) & \left(\widehat{X}_j^{J,L}(\tau_{\eta-1}) + \sum_{\ell \in \mathcal{K}_\eta} \sqrt{q_\ell} \left\langle B(s_{\eta,\ell}, \widehat{X}^{J,L}(s_{\eta,\ell})) h_\ell, h_j \right\rangle \right. \\ & \left. \times \mathfrak{R}_j(s_{\eta,\ell}, \tau_{\eta-1}) (\beta_\ell(\tau_\eta) - \beta_\ell(s_{\eta,\ell})) \right). \end{aligned} \tag{3.3}$$

Equivalently, the above can be written in the convolution form

$$\begin{aligned} \widehat{X}_j^{J,L}(\tau_\eta) = \mathfrak{R}_j(\tau_0, \tau_\eta) \langle \xi, h_j \rangle + \sum_{\ell=1}^L \sum_{\tau_1 \leq t_{i,\ell} \leq \tau_\eta} \sqrt{q_\ell} \left\langle B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell})) h_\ell, h_j \right\rangle \\ \times \mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta) (\beta_\ell(t_{i,\ell}) - \beta_\ell(t_{i-1,\ell})). \end{aligned} \tag{3.4}$$

Then, we use

$$\widehat{X}^{J,L}(\tau_\eta) = \sum_{j=1}^J \widehat{X}_j^{J,L}(\tau_\eta) h_j \tag{3.5}$$

for $\eta = 1, \dots, N$ as our approximate solution.

We note that this scheme generalises the aforementioned approximation $\overline{X}^{J,L}$ with the uniform time step as in (3.1): $\overline{X}^{J,L}$ is nothing but $\widehat{X}^{J,L}$ with $n_\ell = N$ for $\ell = 1, \dots, L$.

4 Discrete regularity estimate

First, let $\mathcal{P}_{Jx} := \sum_{j=1}^J \langle x, h_j \rangle h_j$ for $x \in H$. Further, by writing $\prod_\emptyset = I$ we let

$$R(\tau_{\eta_1}, \tau_{\eta_2}; A) := \prod_{\nu=\eta_1+1}^{\eta_2} \left(I - \frac{1}{\tau_\nu - \tau_{\nu-1}} A \right)^{-1}, \tag{4.1}$$

where the meaning of the product symbol is unambiguous due to the commutativity of $(I - \frac{1}{\tau_\nu - \tau_{\nu-1}} A)^{-1} \mathbf{s}$.

For $j \in \{1, \dots, J\}$ and $\eta \in \{1, \dots, N\}$, define

$$\begin{aligned}
 & [R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]^L(\tau_\eta) \\
 & := \sum_{\ell=1}^L \sum_{\tau_1 \leq t_{i,\ell} \leq \tau_\eta} \mathcal{P}_J R(t_{i-1,\ell}, \tau_\eta; A) B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell})) \sqrt{q_\ell} h_\ell (\beta_\ell(t_{i,\ell}) - \beta_\ell(t_{i-1,\ell})).
 \end{aligned} \tag{4.2}$$

For $\xi = 0$ and $B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell})) = B(t_{i-1,\ell})$ the equation (4.2) is a discrete analogue of the stochastic convolution. The Fourier coefficients of (4.2) are given by

$$\begin{aligned}
 & [R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]^L_j(\tau_\eta) := \left\langle [R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]^L(\tau_\eta), h_j \right\rangle \\
 & = \sum_{\ell=1}^L \sum_{\tau_1 \leq t_{i,\ell} \leq \tau_\eta} \sqrt{q_\ell} \mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta) \left\langle B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell})) h_\ell, h_j \right\rangle (\beta_\ell(t_{i,\ell}) - \beta_\ell(t_{i-1,\ell})),
 \end{aligned}$$

for $j \in \{1, \dots, J\}$ and $\eta \in \{1, \dots, N\}$. Then, noting that by the assumptions on A we have $((I - \lambda A)^{-1})^* = (I - \lambda A)^{-1}$ for $\lambda \in (0, \infty)$, the Fourier coefficients of the discretised solution are given by

$$\widehat{X}_j^{J,L}(\tau_\eta) = \mathfrak{R}_j(\tau_0, \tau_\eta) \langle \xi, h_j \rangle + [R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]^L_j(\tau_\eta).$$

For any $r \geq 0$ we have

$$\mathbb{E} \|\widehat{X}^{J,L}(\tau_\eta)\|_{D(A^r)}^2 = \sum_{j=1}^J \lambda_j^{2r} |\mathfrak{R}_j(\tau_0, \tau_\eta) \langle \xi, h_j \rangle|^2 + \sum_{j=1}^J \lambda_j^{2r} \mathbb{E} |[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]^L_j(\tau_\eta)|^2. \tag{4.3}$$

Our first goal is to estimate the second term in the right hand side of (4.3). We see this term as the stochastic integral of a representation of an elementary process.

Let $\mathcal{P}_\ell x := \langle x, h_\ell \rangle h_\ell$ for $\ell \geq 1$, and let $\iota \geq 0$ be the index from Assumption 2.2. For $\nu \in \{1, \dots, \eta\}$, we define an $\mathcal{L}_2(H_0, H)$ -valued random variable $(\phi_\ell^{J,(\eta)})_{\nu-1}$ by

$$(\phi_\ell^{J,(\eta)})_{\nu-1} := \begin{cases} \mathcal{P}_J R(s_{\nu,\ell}, \tau_\eta; A) B(s_{\nu,\ell}, \widehat{X}^{J,L}(s_{\nu,\ell})) \mathcal{P}_\ell & \text{if } \ell \in \Xi_\nu \\ 0_{H_0 \rightarrow H} & \text{if } \ell \notin \Xi_\nu, \end{cases} \tag{4.4a}$$

$$\tag{4.4b}$$

where

$$\Xi_\nu := \{\ell \in \{1, \dots, L\} \mid \ell \in \mathcal{K}_\mu \text{ for some } \mu \in \{\nu, \dots, \eta\}\}. \tag{4.5}$$

We elaborate on the notation. First, note the following: for $\ell \notin \mathcal{K}_\nu$, $\nu \in \{0, \dots, \eta\}$ if the index $i' \in \{1, \dots, n_\ell\}$ is such that $s_{\nu,\ell} = t_{i'-1,\ell}$, then we have $\tau_\nu < t_{i',\ell}$. The separate treatment (4.4b) corresponds to the construction of the algorithm: suppose $\ell \in \{1, \dots, L\}$ and $i^* \in \{1, \dots, n_\ell\}$ satisfy $s_{\eta,\ell} = t_{i^*-1,\ell}$ and $\tau_\eta < t_{i^*,\ell}$, then the evaluation $\beta_\ell(t_{i^*,\ell})$ of the Brownian motion β_ℓ at $t_{i^*,\ell}$ is not used to obtain $\widehat{X}_j^{J,L}(\tau_\eta)$; only up to $\beta_\ell(t_{0,\ell}), \dots, \beta_\ell(t_{i^*-1,\ell})$ are used.

Let us define the elementary process $\Phi_\ell^{J,(\eta)} : \Omega \times [0, \tau_\eta] \rightarrow \mathcal{L}_2(H_0, H)$ by

$$\Phi_\ell^{J,(\eta)}(\omega, t) := \sum_{\nu=1}^{\eta} (\phi_\ell^{J,(\eta)})_{\nu-1}(\omega) \mathbb{I}_{(\tau_{\nu-1}, \tau_\nu]}(t). \tag{4.6}$$

Then, we have the following.

Lemma 4.1. Let $[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]_j^L(\cdot)$ be defined by (4.2) and let Assumption 2.2 hold with $\iota \geq 0$. Then, for $j = 1, \dots, J$ we have

$$[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]_j^L(\tau_\eta) = \left\langle \int_0^{\tau_\eta} \sum_{\ell=1}^L \Phi_\ell^{J,(\eta)}(s) dW(s), h_j \right\rangle.$$

Proof. Fix $\eta \in \{1, \dots, N\}$. Let $\mathcal{S}_\mu := \mathcal{K}_{\eta-\mu} \setminus (\cup_{\mu' \in \{0, \dots, \mu-1\}} \mathcal{K}_{\eta-\mu'})$ for $\mu, \eta \in \{1, \dots, N\}$ with $\mu \leq \eta$, and let $\mathcal{S}_0 := \mathcal{K}_\eta$. Then, we have

$$\begin{aligned} & [R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]_j^L(\tau_\eta) \\ &= \sum_{\nu=1}^{\eta} \sum_{\ell \in \mathcal{K}_\eta} \sqrt{q_\ell} \mathfrak{R}_j(s_{\nu,\ell}, \tau_\eta) \left\langle B(s_{\nu,\ell}, \widehat{X}^{J,L}(s_{\nu,\ell})) h_\ell, h_j \right\rangle (\beta_\ell(\tau_\nu) - \beta_\ell(\tau_{\nu-1})) \\ &+ \sum_{\nu=1}^{\eta-1} \sum_{\ell \in \mathcal{S}_1} \sqrt{q_\ell} \mathfrak{R}_j(s_{\nu,\ell}, \tau_\eta) \left\langle B(s_{\nu,\ell}, \widehat{X}^{J,L}(s_{\nu,\ell})) h_\ell, h_j \right\rangle (\beta_\ell(\tau_\nu) - \beta_\ell(\tau_{\nu-1})) \\ &\vdots \\ &+ \sum_{\nu=1}^{\eta-\mu} \sum_{\ell \in \mathcal{S}_\mu} \sqrt{q_\ell} \mathfrak{R}_j(s_{\nu,\ell}, \tau_\eta) \left\langle B(s_{\nu,\ell}, \widehat{X}^{J,L}(s_{\nu,\ell})) h_\ell, h_j \right\rangle (\beta_\ell(\tau_\nu) - \beta_\ell(\tau_{\nu-1})) \\ &\vdots \\ &+ \sum_{\ell \in \mathcal{S}_{\eta-1}} \sqrt{q_\ell} \mathfrak{R}_j(s_{1,\ell}, \tau_\eta) \left\langle B(s_{1,\ell}, \widehat{X}^{J,L}(s_{1,\ell})) h_\ell, h_j \right\rangle (\beta_\ell(\tau_1) - \beta_\ell(\tau_0)). \end{aligned}$$

Further, we can rewrite the above as

$$\begin{aligned} & [R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]_j^L(\tau_\eta) \\ &= \sum_{\mu=0}^{\eta-1} \sum_{\nu=1}^{\eta-\mu} \sum_{\ell \in \mathcal{S}_\mu} \left\langle B(s_{\nu,\ell}, \widehat{X}^{J,L}(s_{\nu,\ell})) \mathcal{P}_\ell(W(\tau_\nu) - W(\tau_{\nu-1})), R(s_{\nu,\ell}, \tau_\eta; A) \mathcal{P}_J h_j \right\rangle. \end{aligned}$$

By the assumptions on A we have $((I - \lambda A)^{-1})^* = (I - \lambda A)^{-1}$ for $\lambda \in (0, \infty)$, and thus

$$[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]_j^L(\tau_\eta) = \left\langle \sum_{\nu=1}^{\eta} \left(\sum_{\ell=1}^L (\phi_\ell^{J,(\eta)})_{\nu-1} \right) (W(\tau_\nu) - W(\tau_{\nu-1})), h_j \right\rangle.$$

By definition of the stochastic integral of elementary processes the statement follows. \square

Using the previous result, we obtain the following estimate.

Proposition 4.2. Let Assumption 2.2 hold. Let $\eta \in \{1, \dots, N\}$. For $p \geq 1$, suppose that the process defined by (4.4a)–(4.4b) satisfies

$$\mathbb{E} \left[\sum_{\nu=1}^{\eta} \left\| \sum_{\ell=1}^L (\phi_\ell^{J,(\eta)})_{\nu-1} \right\|_{\mathcal{L}_2(H_0, D(A^\iota))}^2 (\tau_\nu - \tau_{\nu-1}) \right] < \infty. \tag{4.7}$$

Then, we have

$$\mathbb{E} \left[\left\| [R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]_j^L(\tau_\eta) \right\|_{D(A^\iota)}^2 \right] \leq \mathbb{E} \left[\sum_{\nu=1}^{\eta} \left\| \sum_{\ell=1}^L (\phi_\ell^{J,(\eta)})_{\nu-1} \right\|_{\mathcal{L}_2(H_0, D(A^\iota))}^2 (\tau_\nu - \tau_{\nu-1}) \right]. \tag{4.8}$$

Proof. For any $\eta \in \{1, \dots, N\}$, from Lemma 4.1 we have

$$\mathbb{E} \left[\left\| [R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]_j^L(\tau_\eta) \right\|_{D(A^\iota)}^2 \right] = \mathbb{E} \left[\sum_{j=1}^J \lambda_j^{2r} \left| \left\langle \int_0^{\tau_\eta} \sum_{\ell=1}^L \Phi_\ell^{J,(\eta)}(s) dW(s), h_j \right\rangle \right|^2 \right].$$

It follows that

$$\begin{aligned} \mathbb{E} \left[\sum_{j=1}^J \left| \left\langle \int_0^{\tau_\eta} \sum_{\ell=1}^L \Phi_\ell^{J,(\eta)}(s) dW(s), \lambda_j^r h_j \right\rangle \right|^2 \right] &\leq \mathbb{E} \left[\left\| \int_0^{\tau_\eta} \sum_{\ell=1}^L \Phi_\ell^{J,(\eta)}(s) dW(s) \right\|_{D(A^t)}^2 \right] \\ &= \mathbb{E} \left[\int_0^{\tau_\eta} \left\| \sum_{\ell=1}^L \Phi_\ell^{J,(\eta)}(s) \right\|_{\mathcal{L}_2(H_0, D(A^t))}^2 ds \right] \\ &= \mathbb{E} \left[\sum_{\nu=1}^{\eta} \left\| \sum_{\ell=1}^L (\phi_\ell^{J,(\eta)})_{\nu-1} \right\|_{\mathcal{L}_2(H_0, D(A^t))}^2 (\tau_\nu - \tau_{\nu-1}) \right] < \infty, \end{aligned}$$

where in the first equality Itô’s isometry, and in the last inequality the condition (4.7) is used. Thus, the statement follows. \square

We need the following estimate for the process $(\phi_\ell^{J,(\eta)})_{\nu-1}$ as in (4.4a) and (4.4b) in terms of the Hilbert–Schmidt norm.

Lemma 4.3. *Suppose that Assumption 2.2 is satisfied. Fix an arbitrary integer $\eta \in \{1, \dots, N\}$. Then, for any $\nu \in \{0, \dots, \eta\}$, we have*

$$\begin{aligned} \left\| \sum_{\ell=1}^L (\phi_\ell^{J,(\eta)})_{\nu-1} \right\|_{\mathcal{L}_2(H_0, D(A^t))} &= \left\| \sum_{\ell \in \Xi_\nu} (\phi_\ell^{J,(\eta)})_{\nu-1} \right\|_{\mathcal{L}_2(H_0, D(A^t))} \\ &\leq \left(\sum_{\ell \in \Xi_\nu} \sum_{j=1}^J \lambda_j^{2t} |\mathfrak{R}_j(s_{\nu, \ell}, \tau_\eta)|^2 \left| \left\langle B(s_{\nu, \ell}, \widehat{X}^{J,L}(s_{\nu, \ell})) \sqrt{q_\ell} h_\ell, h_j \right\rangle \right|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where Ξ_ν is defined by (4.5).

Proof. Note that if $\ell \notin \Xi_\nu$, then $\left\| (\phi_\ell^{J,(\eta)})_{\nu-1} \sqrt{q_\ell} h_\ell \right\|_{D(A^t)}^2 = 0$. Thus, noting that $\mathcal{P}_\ell h_{\ell'} = 0$ unless $\ell = \ell'$, from the definition of $(\phi_\ell^{J,(\eta)})_{\nu-1}$ we have

$$\begin{aligned} \left\| \sum_{\ell=1}^L (\phi_\ell^{J,(\eta)})_{\nu-1} \right\|_{\mathcal{L}_2(H_0, D(A^t))}^2 &= \sum_{\ell'=1}^L \left\| (\phi_{\ell'}^{J,(\eta)})_{\nu-1} \sqrt{q_{\ell'}} h_{\ell'} \right\|_{D(A^t)}^2 \\ &= \sum_{\ell' \in \Xi_\nu} \left\| (\phi_{\ell'}^{J,(\eta)})_{\nu-1} \sqrt{q_{\ell'}} h_{\ell'} \right\|_{D(A^t)}^2. \end{aligned}$$

Fix $\ell \in \Xi_\nu$. For any $\eta \in \{1, \dots, N\}$ and $\nu \in \{1, \dots, \eta\}$ we have

$$\begin{aligned} &\left\| (\phi_\ell^{J,(\eta)})_{\nu-1} \sqrt{q_\ell} h_\ell \right\|_{D(A^t)}^2 \\ &= \sum_{j=1}^{\infty} \lambda_j^{2t} \left| \left\langle \mathcal{P}_J R(s_{\nu, \ell}, \tau_\eta; A) B(s_{\nu, \ell}, \widehat{X}^{J,L}(s_{\nu, \ell})) \sqrt{q_\ell} h_\ell, h_j \right\rangle \right|^2 \\ &\leq \sum_{j=1}^J \lambda_j^{2t} |\mathfrak{R}_j(s_{\nu, \ell}, \tau_\eta)|^2 \left| \left\langle B(s_{\nu, \ell}, \widehat{X}^{J,L}(s_{\nu, \ell})) \sqrt{q_\ell} h_\ell, h_j \right\rangle \right|^2. \end{aligned}$$

Hence, the statement follows. \square

The following lemma is important to show the maximal regularity estimate of the same form as the continuous counterpart (2.6), studied in [9, Proposition 6.18] and [8].

Lemma 4.4. For any $j \geq 1$, $\ell \geq 1$, and $i \in \{1, \dots, n_\ell\}$, we have

$$\sum_{t_{i,\ell} \leq \tau_\eta \leq \tau_N} |\mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta)|^2 (\tau_\eta - \tau_{\eta-1}) \leq \frac{2}{\lambda_j},$$

where $\mathfrak{R}_j(\cdot, \cdot)$ is defined by (3.2).

Proof. For $\tau_{\eta_0} \in \{\tau_0, \dots, \tau_N\}$ define a continuous interpolation $\mathcal{S}_j(\tau_{\eta_0}, \cdot): [0, 1] \rightarrow \mathbb{R}$ of $\mathfrak{R}_j(\tau_{\eta_0}, \tau_\eta)$ by

$$\mathcal{S}_j(\tau_{\eta_0}, t) := \prod_{\nu=\eta_0+1}^N \frac{1}{1 + \lambda_j(t \wedge \tau_\nu - t \wedge \tau_{\nu-1})}, \quad t \in [0, 1]. \tag{4.9}$$

Then, for $t \in (\tau_{\eta-1}, \tau_\eta]$, $\eta \in \{1, \dots, N\}$, we have

$$\mathcal{S}_j(\tau_{\eta_0}, t) \mathbb{I}_{\{(\tau_{\eta-1}, \tau_\eta]\}}(t) = \mathcal{S}_j(\tau_{\eta_0}, t) \geq \mathfrak{R}_j(\tau_{\eta_0}, \tau_\eta) \mathbb{I}_{\{(\tau_{\eta-1}, \tau_\eta]\}}(t).$$

Further, for $\ell = 1, \dots, L$ and $i = 1, \dots, n_\ell$, let

$$\tau_{\eta^*} := \tau_{\eta^*(i,\ell)} := t_{i,\ell}.$$

Then, we have

$$\begin{aligned} & \sum_{t_{i,\ell} \leq \tau_\eta \leq \tau_N} |\mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta)|^2 (\tau_\eta - \tau_{\eta-1}) \\ & \leq \int_{\tau_{\eta^*-1}}^1 \sum_{\eta=\eta^*(i,\ell)}^N |\mathcal{S}_j(t_{i-1,\ell}, s)|^2 \mathbb{I}_{\{(\tau_{\eta-1}, \tau_\eta]\}}(s) \, ds \\ & = \int_{\tau_{\eta^*-1}}^1 |\mathcal{S}_j(t_{i-1,\ell}, s)|^2 \, ds \leq \int_{t_{i-1,\ell}}^1 |\mathcal{S}_j(t_{i-1,\ell}, s)|^2 \, ds. \end{aligned}$$

For $t \in [t_{\kappa-1,\ell}, t_{\kappa-1,\ell}]$ with $\kappa \geq i$, the elementary inequality $\frac{1}{1+(b-a)} \frac{1}{1+(c-b)} \leq \frac{1}{1+(c-a)}$ ($0 \leq a \leq b \leq c$) implies

$$\mathcal{S}_j(t_{i-1,\ell}, t) \leq \frac{1}{(1 + \lambda_j \frac{1}{n_\ell})^{\kappa-i}} \cdot \frac{1}{1 + \lambda_j(t - t_{\kappa-1,\ell})},$$

and therefore

$$\begin{aligned} & \int_{t_{i-1,\ell}}^1 |\mathcal{S}_j(t_{i-1,\ell}, s)|^2 \, ds = \sum_{\kappa=i}^{n_\ell} \int_{t_{\kappa-1,\ell}}^{t_{\kappa,\ell}} |\mathcal{S}_j(t_{i-1,\ell}, s)|^2 \, ds \\ & \leq \sum_{\kappa=i}^{n_\ell} \frac{1}{(1 + \frac{\lambda_j}{n_\ell})^{2\kappa-2i}} \int_{t_{\kappa-1,\ell}}^{t_{\kappa,\ell}} \frac{1}{(1 + \lambda_j(s - t_{\kappa-1,\ell}))^2} \, ds \\ & = \sum_{\kappa=i}^{n_\ell} \frac{1}{(1 + \frac{\lambda_j}{n_\ell})^{2\kappa-2i}} \frac{1}{\lambda_j + 1/(t_{\kappa,\ell} - t_{\kappa-1,\ell})} \leq \frac{1}{\lambda_j + n_\ell} \sum_{\kappa=i}^{n_\ell} \frac{1}{(1 + \frac{\lambda_j}{n_\ell})^{2\kappa-2i}}. \end{aligned}$$

If $\frac{\lambda_j}{n_\ell} \geq 1$, then $\frac{1}{\lambda_j + n_\ell} \sum_{\kappa=i}^{n_\ell} \frac{1}{(1 + \lambda_j/n_\ell)^{2\kappa-2i}} \leq \frac{2}{\lambda_j}$, and otherwise $(1 + \frac{\lambda_j}{n_\ell})^2 \leq 4$ and thus

$$\frac{1}{\lambda_j + n_\ell} \sum_{\kappa=i}^{n_\ell} \frac{1}{(1 + \frac{\lambda_j}{n_\ell})^{2\kappa-2i}} \leq \frac{1}{n_\ell} \frac{1}{1 - 1/(1 + \frac{\lambda_j}{n_\ell})^2} \leq \frac{4}{2\lambda_j + \lambda_j^2/n_\ell} \leq \frac{2}{\lambda_j}.$$

Hence, we have $\sum_{t_{i,\ell} \leq \tau_\eta \leq \tau_N} |\mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta)|^2 (\tau_\eta - \tau_{\eta-1}) \leq \frac{2}{\lambda_j}$, as claimed. □

We are ready to state our main result.

Theorem 4.5. *Suppose Assumption 2.2 is satisfied with some $\iota \in [0, 1/2]$. Then, we have*

$$\begin{aligned} & \sum_{\eta=1}^N \mathbb{E} \left[\left\| [R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]^L(\tau_\eta) \right\|_{D(A^{\iota+1/2})}^2 \right] (\tau_\eta - \tau_{\eta-1}) \\ & \leq 2 \mathbb{E} \left[\sum_{\ell=1}^\infty \sum_{i=1}^{n_\ell} \left\| \mathcal{P}_J B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell})) \mathcal{P}_L \sqrt{q_\ell} h_\ell \right\|_{D(A^\iota)}^2 (t_{i,\ell} - t_{i-1,\ell}) \right]. \end{aligned}$$

In particular, $\overline{X}^{J,L}$ defined as in (3.1) satisfies

$$\begin{aligned} & \sum_{i=1}^N \mathbb{E} \left[\left\| [R^J \diamond B(\cdot, \overline{X}^{J,L}(\cdot))]^L(t_i) \right\|_{D(A^{\iota+1/2})}^2 \right] (t_i - t_{i-1}) \\ & \leq 2 \sum_{i=1}^N \mathbb{E} \left[\left\| \mathcal{P}_J B(t_{i-1}, \overline{X}^{J,L}(t_{i-1})) \mathcal{P}_L \right\|_{\mathcal{L}_2(H_0, D(A^\iota))}^2 \right] (t_i - t_{i-1}). \end{aligned}$$

Proof. We first show that for $\eta = 1, \dots, N$, we have

$$\mathbb{E} \left[\sum_{\nu=1}^\eta \left\| \sum_{\ell=1}^L (\phi_\ell^{J,(\eta)})_{\nu-1} \right\|_{\mathcal{L}_2(H_0, D(A^{\iota+1/2}))}^2 (\tau_\nu - \tau_{\nu-1}) \right] < \infty. \tag{4.10}$$

In view of Lemma 4.3, we have

$$\begin{aligned} & \sum_{\eta=1}^N \mathbb{E} \left[\sum_{\nu=1}^\eta \left\| \sum_{\ell=1}^L (\phi_\ell^{J,(\eta)})_{\nu-1} \right\|_{\mathcal{L}_2(H_0, D(A^{\iota+1/2}))}^2 (\tau_\nu - \tau_{\nu-1}) \right] (\tau_\eta - \tau_{\eta-1}) \\ & \leq \sum_{\eta=1}^N \mathbb{E} \left[\sum_{j=1}^J \sum_{\nu=1}^\eta \sum_{\ell \in \Xi_\nu} \lambda_j^{2\iota+1} |\mathfrak{R}_j(s_{\nu,\ell}, \tau_\eta)|^2 \right. \\ & \quad \times \left. \left| \left\langle B(s_{\nu,\ell}, \widehat{X}^{J,L}(s_{\nu,\ell})) \sqrt{q_\ell} h_\ell, h_j \right\rangle \right|^2 (\tau_\nu - \tau_{\nu-1}) \right] (\tau_\eta - \tau_{\eta-1}) \\ & = \mathbb{E} \left[\sum_{j=1}^J \sum_{\ell=1}^L \sum_{\eta=1}^N \sum_{\tau_1 \leq t_{i,\ell} \leq \tau_\eta} \lambda_j^{2\iota+1} |\mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta)|^2 \right. \\ & \quad \times \left. \left| \left\langle B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell})) \sqrt{q_\ell} h_\ell, h_j \right\rangle \right|^2 (t_{i,\ell} - t_{i-1,\ell}) (\tau_\eta - \tau_{\eta-1}) \right]. \tag{4.11} \end{aligned}$$

Since $\bigcup_{\eta=1}^N \bigcup_{\tau_1 \leq t_{i,\ell} \leq \tau_\eta} \{\tau_\eta, t_{i,\ell}\} = \bigcup_{i=1}^{m_\ell} \bigcup_{t_{i,\ell} \leq \tau_\eta \leq \tau_N} \{\tau_\eta, t_{i,\ell}\}$, the right hand side of (4.11) can be rewritten as

$$\begin{aligned} & \mathbb{E} \left[\sum_{j=1}^J \sum_{\ell=1}^L \sum_{i=1}^{n_\ell} \sum_{t_{i,\ell} \leq \tau_\eta \leq \tau_N} \lambda_j^{2\iota+1} |\mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta)|^2 \right. \\ & \quad \times \left. \left| \left\langle B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell})) \sqrt{q_\ell} h_\ell, h_j \right\rangle \right|^2 (t_{i,\ell} - t_{i-1,\ell}) (\tau_\eta - \tau_{\eta-1}) \right] \\ & = \mathbb{E} \left[\sum_{j=1}^J \sum_{\ell=1}^L \sum_{i=1}^{n_\ell} \lambda_j^{2\iota+1} \left| \left\langle B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell})) \sqrt{q_\ell} h_\ell, h_j \right\rangle \right|^2 \right. \\ & \quad \times \left. (t_{i,\ell} - t_{i-1,\ell}) \sum_{t_{i,\ell} \leq \tau_\eta \leq \tau_N} |\mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta)|^2 (\tau_\eta - \tau_{\eta-1}) \right]. \tag{4.12} \end{aligned}$$

From Lemma 4.4, (4.11) and (4.12), due to Assumption 2.2 we have (4.10).

From (4.10), we note that Proposition 4.2 implies

$$\begin{aligned} & \sum_{\eta=1}^N \mathbb{E} \left[\left\| [R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]^L(\tau_\eta) \right\|_{D(A^{\iota+1/2})}^2 \right] (\tau_\eta - \tau_{\eta-1}) \\ & \leq \sum_{\eta=1}^N \mathbb{E} \left[\sum_{\nu=1}^{\eta} \left\| \sum_{\ell=1}^L (\phi_\ell^{J,(\eta)})_{\nu-1} \right\|_{\mathcal{L}_2(H_0, D(A^{\iota+1/2}))}^2 (\tau_\nu - \tau_{\nu-1}) \right] (\tau_\eta - \tau_{\eta-1}). \end{aligned}$$

Therefore, again from Lemma 4.4 together with (4.11) and (4.12) we obtain

$$\begin{aligned} & \sum_{\eta=1}^N \mathbb{E} \left[\left\| [R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]^L(\tau_\eta) \right\|_{D(A^{\iota+1/2})}^2 \right] (\tau_\eta - \tau_{\eta-1}) \\ & \leq 2\mathbb{E} \left[\sum_{j=1}^J \sum_{\ell=1}^L \sum_{i=1}^{n_\ell} \lambda_j^{2\iota} \left| \left\langle B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell})) \sqrt{q_\ell} h_\ell, h_j \right\rangle \right|^2 (t_{i,\ell} - t_{i-1,\ell}) \right] \\ & = 2\mathbb{E} \left[\sum_{\ell=1}^\infty \sum_{i=1}^{n_\ell} \left\| \mathcal{P}_J B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell})) \mathcal{P}_L \sqrt{q_\ell} h_\ell \right\|_{D(A^\iota)}^2 (t_{i,\ell} - t_{i-1,\ell}) \right]. \end{aligned}$$

When $n_\ell = N$ for all $\ell \in \{1, \dots, L\}$, we have $t_{i,\ell} - t_{i-1,\ell} = t_i - t_{i-1}$ ($i = 1, \dots, N$). Thus, repeating the same argument as above completes the proof. \square

As a consequence of the previous result, given a suitable regularity of the initial condition, the approximate solution has the spatial regularity “one-half smoother”—the same as the continuous counterpart [9]—than the range of the operator $B(t, x)$.

Corollary 4.6. *Suppose Assumption 2.2 is satisfied with some $\iota \in [0, 1/2]$, and let $\xi \in D(A^\iota)$. Then, we have*

$$\begin{aligned} & \left(\sum_{\eta=1}^N \mathbb{E} [\|\widehat{X}^{J,L}(\tau_\eta)\|_{D(A^{\iota+1/2})}^2] (\tau_\eta - \tau_{\eta-1}) \right)^{\frac{1}{2}} \leq \|\mathcal{P}_J \xi\|_{D(A^\iota)} \\ & + \left(\mathbb{E} \left[\sum_{\ell=1}^\infty \sum_{i=1}^{n_\ell} \left\| \mathcal{P}_J B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell})) \mathcal{P}_L \sqrt{q_\ell} h_\ell \right\|_{D(A^\iota)}^2 (t_{i,\ell} - t_{i-1,\ell}) \right] \right)^{\frac{1}{2}}. \end{aligned}$$

In particular, $\overline{X}^{J,L}$ defined as in (3.1) satisfies

$$\begin{aligned} & \left(\sum_{i=1}^N \mathbb{E} [\|\overline{X}^{J,L}(t_i)\|_{D(A^{\iota+1/2})}^2] (t_i - t_{i-1}) \right)^{\frac{1}{2}} \\ & \leq \|\mathcal{P}_J \xi\|_{D(A^\iota)} + \left(\sum_{i=1}^N \mathbb{E} \left[\left\| \mathcal{P}_J B(t_{i-1}, \overline{X}^{J,L}(t_{i-1})) \mathcal{P}_L \right\|_{\mathcal{L}_2(H_0, D(A^\iota))}^2 \right] (t_i - t_{i-1}) \right)^{\frac{1}{2}}. \end{aligned}$$

Proof. From Lemma 4.4 we have

$$\begin{aligned} & \sum_{\eta=1}^N \mathbb{E} \left[\|R(\tau_0, \tau_\eta; A) \mathcal{P}_J \xi\|_{D(A^{\iota+1/2})}^2 \right] (\tau_\eta - \tau_{\eta-1}) \\ & = \mathbb{E} \left[\sum_{j=1}^J \lambda_j^{2\iota+1} |\langle \xi, h_j \rangle|^2 \sum_{\eta=1}^N |\mathfrak{R}_j(\tau_0, \tau_\eta)|^2 (\tau_\eta - \tau_{\eta-1}) \right] \\ & \leq 2\mathbb{E} \left[\sum_{j=1}^J \lambda_j^{2\iota} |\langle \xi, h_j \rangle|^2 \right]. \end{aligned}$$

Then, from (4.3) and Theorem 4.5 the first statement follows. Letting $n_\ell = N$ for $\ell = 1, \dots, L$ establishes the second statement. \square

Remark 4.7. The results in this section can be generalised to non-uniform grids on each level. Let $0 < t_{1,\ell} < \dots < t_{n_\ell,\ell} = 1$ be the temporal grids that satisfies the following: Letting $\delta_\ell^{\max} := \max_{i=1,\dots,n_\ell} \{t_{i,\ell} - t_{i-1,\ell}\}$, $\delta_\ell^{\min} := \min_{i=1,\dots,n_\ell} \{t_{i,\ell} - t_{i-1,\ell}\}$, we have a constant $c_{\text{disc}} \geq 1$ such that $\delta_\ell^{\max}/\delta_\ell^{\min} \leq c_{\text{disc}}$ holds. Then, the statement of Lemma 4.4 can be replaced by

$$\sum_{t_{i,\ell} \leq \tau_\eta \leq \tau_N} |\mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta)|^2 (\tau_\eta - \tau_{\eta-1}) \leq \frac{2c_{\text{disc}}}{\lambda_j},$$

and that of Theorem 4.5 by

$$\begin{aligned} & \sum_{\eta=1}^N \mathbb{E} \left[\left\| [R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))]^L(\tau_\eta) \right\|_{D(A^{\iota+1/2})}^2 \right] (\tau_\eta - \tau_{\eta-1}) \\ & \leq 2c_{\text{disc}} \mathbb{E} \left[\sum_{\ell=1}^{\infty} \sum_{i=1}^{n_\ell} \left\| \mathcal{P}_J B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell})) \mathcal{P}_L \sqrt{q_\ell} h_\ell \right\|_{D(A^\iota)}^2 (t_{i,\ell} - t_{i-1,\ell}) \right]. \end{aligned}$$

5 Conclusion

In this paper, we considered an implicit Euler–Maruyama scheme for a class of stochastic partial differential equations with a non-uniform time discretisation. For this scheme, we showed that a discrete analogue of the maximal L^2 -regularity holds, which has the same form as the maximal regularity of the original problem.

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Acknowledgments. I would like to thank an anonymous reviewer for their helpful comments.