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On stochastic heat equation with measure initial data

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Abstract

The aim of this short note is to obtain the existence, uniqueness and moment upper bounds of the solution to a stochastic heat equation with measure initial data, without using the iteration method in [1, 2, 3].

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1 Introduction

Consider the stochastic heat equation

$$\frac{\partial u}{\partial t} = \mathcal{L}u + b(u) + \sigma(u)\dot{W} \tag{1.1}$$

for $(t,x) \in (0,\infty) \times \mathbb{R}^d (d \geqslant 1)$ where $\mathcal{L}$ is the generator of a Lévy process $X = \{X_t\}_{t \geqslant 0}$. $\dot{W}$ is a centered Gaussian noise with covariance formally given by

$$\mathrm{E}\big(\dot{W}(t,x)\dot{W}(s,y)\big) = \delta(s-t)f(x-y)\,,$$

where f is some nonnegative and nonnegative definite function whose Fourier transform is denoted by

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\xi}dx$$

in distributional sense, and δ denotes the Dirac delta function at 0. For some technical reasons, we will assume that f is either lower semicoutinuous (see Lemma 2.1 below), or $f = \delta$, which corresponds to the space time white noise.

Let Φ be the Lévy exponent of X_t, we will assume that

$$\exp(-\mathrm{Re}\Phi) \in L^t(\mathbb{R}^d) \text{ for all } t > 0\,. \tag{1.2}$$

Thus according to Proposition 2.1 in [5], X_t has a transition function $p_t(x)$ and we can (and will) find a version of $p_t(x)$ which is continuous on $(0,\infty)\times\mathbb{R}^d$ and uniformly continuous for all $(t,x) \in [\eta,\infty)\times\mathbb{R}^d$ for every $\eta > 0$, and that p_t vanishes at infinity for all $t > 0$.

The initial condition $u(0,\cdot)$ is assumed to be a (positive) measure $\mu(\cdot)$ such that

$$\int_{\mathbb{R}^d} p_t(x-y)\mu(dy) < \infty \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d\,. \tag{1.3}$$

To avoid trivialities, we assume that $\mu(\cdot) \not\equiv 0$.

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Using iteration method, the existence, uniqueness and some moment bounds of the solution have been obtained in [1, 2, 3] for the case $b \equiv 0$ and for some specific choice of $\mathcal{L}$. However, these approaches rely on the structure (or asymptotic structure) of $p_t(x)$. In this article, we will study the equation (1.1) with also a Lipschitz drift term b and establish the existence, uniqueness and p-th moment upper bound, without using the iteration method in [1, 2, 3], also, our criteria only need some integrability of the Lévy exponent.

To state the result, let us recall that by a solution u to (1.1) we mean a mild solution. That is, (i) u is a predictable random field on a complete probability space $\{\Omega, \mathcal{F}, P\}$, with respect to the Brownian filtration generated by the cylindrical Brownian motion defined by $B_t(\phi) := \int_{[0,t]\times\mathbb{R}^d} \phi(y)W(ds,dy)$, for all $t \geqslant 0$ and measurable $\phi : \mathbb{R}^d \to \mathbb{R}$ such that $\int_{\mathbb{R}^d\times\mathbb{R}^d} \phi(y)\phi(z)f(y-z)dydz < \infty$; and (ii) for any $(t,x) \in (0,\infty)\times\mathbb{R}^d$, we have that $\mathrm{E}[u(t,x)^2] < \infty$ and the following equation holds a.s.

$$u(t,x) = \int_{\mathbb{R}^d} p_t(x-y)\mu(dy) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)b(u(s,y))dyds + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy)\,. \tag{1.4}$$

where $p_t(x)$ is the transition function for X_t and the stochastic integral above is in the sense of Walsh [6]. The following theorem is the main result of this paper.

Theorem 1.1. *Assume that the initial condition satisfies* (1.3) *and assume that*

$$\Upsilon(\beta) := \sup_{t>0} \int_0^t \int_{\mathbb{R}^d} \exp\left[-2s\mathrm{Re}\Phi\left((1-\frac{s}{t})\xi\right) - 2(t-s)\mathrm{Re}\Phi\left(\frac{s}{t}\xi\right)\right] e^{-2\beta(t-s)}\hat{f}(\xi)d\xi ds < \infty \tag{1.5}$$

and

$$\tilde{\Upsilon}(\beta) := \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)d\xi}{\beta + \mathrm{Re}\Phi(\xi)} < \infty \tag{1.6}$$

for any $\beta > 0$. And assume that σ and b are Lipschitz functions with Lipschitz coefficients $L_\sigma, L_b > 0$ respectively. Then there exists a unique mild solution to equation (1.1). *Moreover, define*

$$\bar{\gamma}(p) := \limsup_{t\to\infty} \frac{1}{t} \sup_{x\in\mathbb{R}^d} \log \left\| \frac{u(t,x)}{\tau + p_t * \mu(x)} \right\|_{L^p(\Omega)}, \tag{1.7}$$

where

$$\tau = \max\left\{\frac{|b(0)|}{L_b}, \frac{|\sigma(0)|}{L_\sigma}\right\}. \tag{1.8}$$

Then,

$$\bar{\gamma}(p) \leqslant \inf\{\beta > 0 : B(\beta,p) < 1\} \quad \textit{for all integers } p \geqslant 2\,, \tag{1.9}$$

where

$$B(\beta,p) := \frac{L_b}{\beta} + \frac{z_p L_\sigma}{(2\pi)^{d/2}} \left(\sqrt{\frac{\tilde{\Upsilon}(\beta)}{2}} + \sqrt{\Upsilon(\beta)}\right), \tag{1.10}$$

and z_p denotes the largest positive zero of the Hermite polynomial, actually, one may bound z_p above by $2\sqrt{p}$.

Remark 1.2. If we choose $\mathcal{L}$ to be the generator of an a-stable Lévy process D^a_θ for $1 < a < 2$, where θ is the skewness and $|\theta| < 2 - a$(see [2]), or the Laplacian $\frac{1}{2}\Delta$ ($a = 2$), then the classical Dalang's condition

$$\int_{\mathbb{R}^d} \frac{\hat{f}(\xi)d\xi}{1+|\xi|^a} < \infty \tag{1.11}$$

implies condition (1.5), since in this case $\mathrm{Re}\Phi(\xi) = C|\xi|^a$ for some $C > 0$. Also, in the case $d = 1$ and $\dot{W}$ is a space-time white noise, that is, $\hat{f}(\xi) \equiv 1$, condition (1.6) clearly guarantees that (1.2) holds.

2 Proof of Theorem 1.1

In the proof of Theorem 1.1 we will need some results about taking Fourier transforms, which we now state.

Lemma 2.1 (Corollary 3.4 in [5]). *Assume that f is lower semicontinuous, then for all Borel probability measures ν on $\mathbb{R}^d$,*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\nu(dx)\nu(dy) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{f}(\xi)|\hat{\nu}(\xi)|^2 d\xi\,.$$

Lemma 2.2. *If f is lower semicontinuous, then*

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{t-s}(x-y_1)p_s*\mu(y_1)p_{t-s}(x-y_2)p_s*\mu(y_2)f(y_1-y_2)dy_1dy_2\\&\quad\leqslant \frac{[p_t*\mu(x)]^2}{(2\pi)^d}\int_{\mathbb{R}^d}\exp\left[-2s\mathrm{Re}\Phi\left((1-\frac{s}{t})\xi\right)-2(t-s)\mathrm{Re}\Phi\left(\frac{s}{t}\xi\right)\right]\hat{f}(\xi)d\xi\,.\end{aligned}$$

Proof. We begin by noting that

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{t-s}(x-y_1)p_s*\mu(y_1)p_{t-s}(x-y_2)p_s*\mu(y_2)f(y_1-y_2)dy_1dy_2\\&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{p_{t-s}(x-y_1)p_s(y_1-z_1)}{p_t(x-z_1)}\frac{p_{t-s}(x-y_2)p_s(y_2-z_2)}{p_t(x-z_2)}f(y_1-y_2)dy_1dy_2\\&\qquad\times p_t(x-z_1)p_t(x-z_2)\mu(dz_1)\mu(dz_2)\,,\end{aligned}$$

and as a function of y, the quotient $\frac{p_{t-s}(x-y)p_s(y-z)}{p_t(x-z)}$ is the probability density of the Lévy bridge $\tilde{X}_{z,x,t} = \{\tilde{X}_{z,x,t}(s)\}_{0\leqslant s\leqslant t}$ which is at z when $s = 0$ and at x when $s = t$. Actually, $\tilde{X}_{z,x,t}(s)$ can be written as

$$\begin{aligned}\tilde{X}_{z,x,t}(s) &= X_s - \frac{s}{t}X_t + z + \frac{s}{t}(x-z)\\&=(1-\frac{s}{t})X_s - \frac{s}{t}(X_t - X_s) + z + \frac{s}{t}(x-z)\,,\end{aligned}$$

hence by the independence of increment of Lévy process, we have

$$\mathrm{E}e^{i\xi\tilde{X}_{z,x,t}(s)} = \exp\left(-s\Phi\left((1-\frac{s}{t})\xi\right)-(t-s)\Phi\left(-\frac{s}{t}\xi\right)\right)e^{i(z+\frac{s}{t}(x-z))}\,.$$

Thus, an application of Lemma 2.1 to $\nu_j(dy) = \frac{p_{t-s}(x-y)p_s(y-z_j)}{p_t(x-z_j)}dy$, $j = 1, 2$, yields

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{p_{t-s}(x-y_1)p_s(y_1-z_1)}{p_t(x-z_1)}\frac{p_{t-s}(x-y_2)p_s(y_2-z_2)}{p_t(x-z_2)}f(y_1-y_2)dy_1dy_2\\&\quad=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{E}e^{i\xi\tilde{X}_{z_1,x,t}(s)}\overline{\mathrm{E}e^{i\xi\tilde{X}_{z_2,x,t}(s)}}\hat{f}(\xi)d\xi\\&\quad\leqslant\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\exp\left[-2s\mathrm{Re}\Phi\left((1-\frac{s}{t})\xi\right)-2(t-s)\mathrm{Re}\Phi\left(-\frac{s}{t}\xi\right)\right]\hat{f}(\xi)d\xi\,,\end{aligned}$$

which proves the lemma. □

In the case that $f = \delta$, which corresponds to space-time white noise case, Lemma 2.2 is replaced by

Lemma 2.3. *We have that*

$$\int_{\mathbb{R}^d} (p_{t-s}(x-y)p_s * \mu(y))^2 \, dy$$
$$\leqslant \frac{[p_t * \mu(x)]^2}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\left[-2s\mathrm{Re}\Phi\left((1-\frac{s}{t})\xi\right) - 2(t-s)\mathrm{Re}\Phi\left(\frac{s}{t}\xi\right)\right] d\xi \, .$$

The proof of Lemma 2.3 goes in the same way as that of Lemma 2.2, except that instead of using Lemma 2.1, we use Plancherel's identity. We omit the details of the proof.

To prove Theorem 1.1, we first define a norm for all $\beta, p > 0$ and all predictable random fields $v := v(t,x)$,

$$\|v\|_{\beta,p} = \sup_{t>0} e^{-\beta t} \sup_{x\in\mathbb{R}^d} \|v(t,x)\|_{L^p(\Omega)} \, . \tag{2.1}$$

Let $\mathcal{B}_{\beta,p}$ denote the collection of all predictable random fields $v := \{v(t,x)\}_{t\geqslant 0, x\in\mathbb{R}^d}$ such that $\|v\|_{\beta,p} < \infty$. We note that after the usual identification of a process with its modifications, $\mathcal{B}_{\beta,p}$ is a Banach space (see Section 5 in [5]).

Proof of Theorem 1.1. We will only do the case that f is a lower semicontinuous function, the case that $f = \delta$ is proved in the same way. We use Picard iteration. Set

$$u^0(t,x) := p_t * \mu(x) \, ,$$
$$u^{n+1}(t,x) := p_t * \mu(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) b(u^n(s,y)) dy ds$$
$$+ \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\sigma(u^n(s,y)) W(ds,dy) \, .$$

We first show that whenever β is chosen such that $B(\beta,p) < 1$, where $B(\beta,p)$ is defined in (1.10), then, for any $n \geqslant 1$,

$$\left\| \frac{\tau + |u^n|}{\tau + p * \mu} \right\|_{\beta,p} < \infty \, . \tag{2.2}$$

Note that by the dominated convergence theorem, the condition $B(\beta,p) < 1$ can be achieved if β is sufficiently large.

Recall that τ is defined in (1.8). We start with the inequality

$$\frac{\tau + |u^{n+1}(t,x)|}{\tau + p_t * \mu(x)} \leqslant 1 + \left| \int_0^t \int_{\mathbb{R}^d} \frac{p_{t-s}(x-y)[\tau + p_s * \mu(y)]}{\tau + p_t * \mu(x)} \frac{b(u^n(s,y))}{\tau + p_s * \mu(y)} dy ds \right|$$
$$+ \left| \int_0^t \int_{\mathbb{R}^d} \frac{p_{t-s}(x-y)(\tau + p_s * \mu(y))}{\tau + p_t * \mu(x)} \frac{\sigma(u^n(s,y))}{\tau + p_s * \mu(y)} W(ds,dy) \right| \, .$$

(2.2) is clearly true for $n = 0$. Using induction, assume (2.2) is true for some n, using Burkholder inequality (see [4]) and the assumption on σ and b, we obtain

$$\left\| \frac{\tau + |u^{n+1}(t,x)|}{\tau + p_t * \mu(x)} \right\|_{L^p(\Omega)}$$
$$\leqslant 1 + L_b \int_0^t \int_{\mathbb{R}^d} \frac{p_{t-s}(x-y)[\tau + p_s * \mu(y)]}{\tau + p_t * \mu(x)} \left\| \frac{\tau + |u^n(s,y)|}{\tau + p_s * \mu(y)} \right\|_{L^p(\Omega)} dy ds$$
$$+ z_p L_\sigma \Bigg(\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{p_{t-s}(x-y_1)(\tau + p_s * \mu(y_1))}{\tau + p_t * \mu(x)} \frac{p_{t-s}(x-y_1)(\tau + p_s * \mu(y_1))}{\tau + p_t * \mu(x)}$$
$$\times \left\| \frac{\tau + |u^n(s,y_1)|}{\tau + p_s * \mu(y_1)} \right\|_{L^p(\Omega)} \left\| \frac{\tau + |u^n(s,y_2)|}{\tau + p_s * \mu(y_2)} \right\|_{L^p(\Omega)} f(y_1 - y_2) dy_1 dy_2 ds \Bigg)^{1/2} ,$$

multiplying both sides by $e^{-\beta t}$ and applying Minkowski's inequality to the third summand above we obtain

$$
\begin{aligned}
&e^{-\beta t}\left\|\frac{\tau+|u^{n+1}(t,x)|}{\tau+p_t*\mu(x)}\right\|_{L^p(\Omega)} \\
&\leqslant 1+L_b\left\|\frac{\tau+|u^n|}{\tau+p*\mu}\right\|_{\beta,p}\int_0^t\int_{\mathbb{R}^d}e^{-\beta(t-s)}\frac{p_{t-s}(x-y)[\tau+p_s*\mu(y)]}{\tau+p_t*\mu(x)}dyds \\
&\quad+z_pL_\sigma\left\|\frac{\tau+|u^n|}{\tau+p*\mu}\right\|_{\beta,p} \\
&\qquad\times\left(\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{-2\beta(t-s)}p_{t-s}(x-y_1)p_{t-s}(x-y_2)f(y_1-y_2)dy_1dy_2ds\right)^{1/2} \\
&\quad+z_pL_\sigma\left\|\frac{\tau+|u^n|}{\tau+p*\mu}\right\|_{\beta,p} \\
&\qquad\times\left(\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{-2\beta(t-s)}\frac{p_{t-s}(x-y_1)p_s*\mu(y_1)}{p_t*\mu(x)}\frac{p_{t-s}(x-y_2)p_s*\mu(y_2)}{p_t*\mu(x)}\right. \\
&\qquad\quad\left.\times f(y_1-y_2)dy_1dy_2ds\vphantom{\int_0^t}\right)^{1/2} \\
&:=1+I_1+I_2+I_3\,,
\end{aligned}
$$

where in obtaining I_2 and I_3 above, we have used the bound

$$
\frac{p_{t-s}(x-y)\tau}{\tau+p_t*\mu(x)}\leqslant p_{t-s}(x-y)\quad\text{and}\quad\frac{p_{t-s}(x-y)p_s*\mu(y)}{\tau+p_t*\mu(x)}\leqslant\frac{p_{t-s}(x-y)p_s*\mu(y)}{p_t*\mu(x)}\,. \tag{2.3}
$$

We will estimate I_1, I_2, I_3 separately. For I_1, the semigroup property of $p_t(x)$ yields

$$
I_1\leqslant\frac{L_b}{\beta}\left\|\frac{\tau+|u^n|}{\tau+p*\mu}\right\|_{\beta,p}\,.
$$

For I_2, an application of Lemma 2.1 to $\nu(dy)=p_{t-s}(x-y)dy$ yields

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{-2\beta(t-s)}p_{t-s}(x-y_1)p_{t-s}(x-y_2)f(y_1-y_2)dy_1dy_2ds \\
&\quad=\frac{1}{(2\pi)^d}\int_0^t\int_{\mathbb{R}^d}e^{-2(t-s)\mathrm{Re}\Phi(\xi)}\hat{f}(\xi)d\xi e^{-2\beta(t-s)}ds\leqslant\frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat{f}(\xi)d\xi}{\beta+\mathrm{Re}\Phi(\xi)}\,,
\end{aligned}
$$

thus we obtain

$$
I_2\leqslant z_pL_\sigma\left(\frac{1}{2(2\pi)^d}\tilde{\Upsilon}(\beta)\right)^{1/2}\left\|\frac{\tau+|u^n|}{\tau+p*\mu}\right\|_{\beta,p}\,.
$$

Finally, an application of Lemma 2.2 yields

$$
I_3\leqslant z_pL_\sigma\left\|\frac{\tau+|u^n|}{\tau+p*\mu}\right\|_{\beta,p}\left(\frac{1}{(2\pi)^d}\Upsilon(\beta)\right)^{1/2}\,.
$$

Combining the estimates for I_1, I_2, I_3, we arrive at

$$
\left\|\frac{\tau+|u^{n+1}|}{\tau+p*\mu}\right\|_{\beta,p}\leqslant 1+B(\beta,p)\left\|\frac{\tau+|u^n|}{\tau+p*\mu}\right\|_{\beta,p}\,,
$$

where $B(\beta,p)$ is defined in (1.10). Using the iteration, we see that (2.2) holds for all $n\geqslant 1$ if $B(\beta,p)<1$.

Here we note that the stochastic integral

$$\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) \frac{\sigma(u^n(s,y))}{\tau + p_t * \mu(x)} W(ds,dy)$$
$$= \int_0^t \int_{\mathbb{R}^d} \frac{p_{t-s}(x-y)[\tau + p_s * \mu(y)]}{\tau + p_t * \mu(x)} \frac{\sigma(u^n(s,y))}{\tau + p_s * \mu(y)} W(ds,dy)$$

maps bounded predictable process $\frac{u^n}{\tau + p*\mu}$ in $\mathcal{B}_{\beta,p}$ to a predictable process, thus the u^{n+1} in the iteration is predictable. See also the discussion after Definition 5.1 in [5].

The same technique applied to $\frac{u^{n+1}(t,x) - u^n(t,x)}{\tau + p_t * \mu(x)}$ yields that

$$\left\| \frac{u^{n+1} - u^n}{\tau + p * \mu} \right\|_{\beta,p} \leqslant B(\beta,p) \left\| \frac{u^n - u^{n-1}}{\tau + p * \mu} \right\|_{\beta,p},$$

and if β is chosen such that $B(\beta,p) < 1$, we will obtain that

$$\sum_{n=1}^{\infty} \left\| \frac{u^n - u^{n-1}}{\tau + p * \mu} \right\|_{\beta,p} < \infty .$$

Therefore, we can find a predictable random field $u^\infty \in \mathcal{B}_{\beta,p}$ such that $\lim_{n\to\infty} u^n = u^\infty$ in $\mathcal{B}_{\beta,p}$. It is easy to see that this u^∞ is a solution to equation (1.4), and uniqueness is checked by a standard argument.

To prove (1.9), we note that since $u \in \mathcal{B}_{\beta,p}$ for those β such that $B(\beta,p) < 1$,

$$\sup_{x\in\mathbb{R}^d} \left\| \frac{u(t,x)}{\tau + p_t * \mu(x)} \right\|_{L^p(\Omega)} \leqslant \sup_{x\in\mathbb{R}^d} \frac{\tau}{\tau + p_t * \mu(x)} + Ce^{\beta t}$$

for some $C > 0$ which does not depend on t, thus (1.9) is proved. The condition $u(t,x) \in L^2(\Omega)$ for each $(t,x) \in (0,\infty) \times \mathbb{R}^d$ follows easily from the case $p = 2$. The proof of Theorem 1.1 is complete. □

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