

Two observations on the capacity of the range of simple random walks on \mathbb{Z}^3 and \mathbb{Z}^4

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Abstract

We prove a weak law of large numbers for the capacity of the range of simple random walks on \mathbb{Z}^4 . On \mathbb{Z}^3 , we show that the capacity, properly scaled, converges in distribution towards the corresponding quantity for three dimensional Brownian motion. The paper answers two of the three open questions raised by Asselah, Schapira and Sousi in [2, Section 6].

Keywords: capacity; range of simple random walks.

AMS MSC 2010: 60J10.

Submitted to ECP on December 6, 2016, final version accepted on April 26, 2017.

Supersedes arXiv:1611.08654.

1 Introduction

Let $(X_n)_n$ be a simple random walk on \mathbb{Z}^d with $d = 3$ or 4 . We are interested in the scaling limit of the capacity of the random set

$$X[0, n] \stackrel{\text{def}}{=} \{X_0, \dots, X_n\},$$

where the capacity $\text{Cap}(F)$ of a set F of vertices on \mathbb{Z}^d is defined as the sum of escape probabilities:

$$\text{Cap}(F) \stackrel{\text{def}}{=} \sum_{x \in F} \mathbb{E}^x [\forall n \geq 1, X_n \notin F]. \quad (1.1)$$

The capacity of the range of random walks is closely related with the intersection probability of two independent random walks. In fact, many estimates on $\text{Cap}(X[0, n])$ were deduced from that. We refer the reader to the Lawler's classical book [7] and the reference there.

In the present paper, we show that when $d = 4$, the second moment of $\text{Cap}(X[0, n])$ is asymptotically equivalent to the square of the first moment, which implies a weak law of large numbers:

Theorem 1.1. *For a SRW $(X_n)_n$ on \mathbb{Z}^4 ,*

$$\lim_{n \rightarrow \infty} \text{Var}(\text{Cap}(X[0, n])) / \mathbb{E}[\text{Cap}(X[0, n])]^2 = 0, \quad (1.2)$$

which implies that

$$\text{Cap}(X[0, n]) / \mathbb{E}[\text{Cap}(X[0, n])] \stackrel{\text{Probability}}{\rightarrow} 1, \quad n \rightarrow \infty. \quad (1.3)$$

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Such a weak law of large numbers was conjectured by Asselah, Schapira and Sousi [2, Section 6]. Besides, on \mathbb{Z}^3 , they also expect a random scaling limit for $\text{Cap}(X[0, n])/\mathbb{E}[\text{Cap}(X[0, n])]$ and there is no such kind of weak law of large numbers. We affirm this as a corollary (see Remark 4.1) of our second main result, which states that as $n \rightarrow \infty$, $\text{Cap}(X[0, n])/\sqrt{n}$ has a random limit in distribution, which is the corresponding quantity for three dimensional Brownian motion. To be more precise, let $(M_t)_{t \geq 0}$ be the standard Brownian motion on \mathbb{R}^3 . Recall the Green function for Brownian motions on \mathbb{R}^3 , see e.g. [9, Theorem 3.33]:

$$G(x, y) = \int_0^\infty (2\pi t)^{-3/2} e^{-\|x-y\|_2^2/2t} dt = \frac{1}{2\pi} \|x - y\|_2^{-1}.$$

The corresponding (Brownian motion) capacity of a Borel set F is given by

$$\text{Cap}_{BM}(F)^{-1} = \inf \left\{ \int \int G(x, y) \mu(dx) \mu(dy) : \mu \text{ is a probability measure on } F \right\}, \quad (1.4)$$

see e.g. [9, Definition 8.18]. We have the following result on the fluctuation of $\text{Cap}(X[0, n])$ on \mathbb{Z}^3 .

Theorem 1.2. *For a SRW $(X_n)_n$ on \mathbb{Z}^3 , as $n \rightarrow \infty$, $\text{Cap}(X[0, n])/\sqrt{n}$ converges to $\frac{1}{3\sqrt{3}} \text{Cap}_{BM}(M[0, 1])$ in distribution.*

The law of large numbers for $(\text{Cap}(X[0, n]))_n$ had already been obtained in dimension 5 and larger by Jain and Orey [5]. In [2], Asselah, Schapira and Sousi established a central limit theorem in dimension larger than or equal to 6. To understand the model of random interlacement invented by Sznitman [12], Ráth and Sapozhnikov [10, 11] established moments and deviation bounds for the capacity of the union of ranges of paths. During the study of the simple random walk loop percolation on \mathbb{Z}^d [4], together with Sapozhnikov, when $d = 4$, we improved the upper bounds for the second moment of $\text{Cap}(X[0, n])$ by showing that it is comparable with the square of the first moment. Theorem 1.1 sharpens our result in [4], which implies a weak law of large numbers for $\text{Cap}(X[0, n])_n$ on \mathbb{Z}^4 . Soon after this and very recently, Asselah, Schapira and Sousi [1] greatly improved the result in \mathbb{Z}^4 by proving the strong law of large numbers and the central limit theorem. In another paper [3] by the same authors, the strong law of large numbers was established for the Wiener sausage, which is the continuous counterpart of the discrete simple random walk. We refer the reader to [3] for more references and historical remarks on the Wiener sausages.

Finally, we briefly outline the proof. The argument for Theorem 1.1 is a refinement of that in [4]. We consider two independent simple random walks $(X_i^{(0)})_{i=0, \dots, n}$ starting from 0 and $(X_i^{(1)})_{i \geq 0}$ starting from “infinity”. In order to estimate $\text{Cap}(X^0[0, n])$, we estimate the intersection probability of $(X_i^{(0)})_{i=0, \dots, n}$ and $(X_i^{(1)})_{i \geq 0}$. We use the strong Markov property at time τ_1 , where τ_1 is the first time that $(X_i^{(0)})_{i=0, \dots, n}$ meets the trajectory of $(X_i^{(1)})_{i \geq 0}$. A key observation is that $\mathbb{P}[\tau_1/n \in \cdot | \tau_1 \leq n]$ converges to the uniform distribution on $[0, 1]$ when $n \rightarrow \infty$. Together with the sharp estimate

$$\mathbb{E}[\text{Cap}(X[0, n])] \stackrel{n \rightarrow \infty}{\sim} \frac{\pi^2}{8} \frac{n}{\log n}$$

from [2], we conclude the desired result. Theorem 1.2 is proved via a coupling between SRW paths and Brownian motion paths. We crucially use the fact that two independent SRW (Brownian motion) paths are very likely to intersect for $d = 3$, by the result of Lawler [6, Lemmas 2.4, 2.6].

Organization of the paper We introduce necessary notation in Section 2. Then, we prove Theorem 1.1 and 1.2 in separate sections.

2 Notation

We collect notation in the following.

- ℓ_1 -balls on \mathbb{Z}^d : $B_{\mathbb{Z}^d}(x, r) = \{z \in \mathbb{Z}^d : |x - z|_1 \leq r\}$ for $x \in \mathbb{Z}^d$ and $r \geq 0$.
- ℓ_1 -balls on \mathbb{R}^d : $B_{\mathbb{R}^d}(x, r) = \{z \in \mathbb{R}^d : |x - z|_1 \leq r\}$ for $x \in \mathbb{R}^d$ and $r \geq 0$.
- Simple random walk: $(X_n)_{n \geq 0}$.
- Brownian motion: $(M_t)_{t \geq 0}$.
- Range of a SRW: $X[0, n] = \{X_0, \dots, X_n\}$.
- Range of a Brownian motion: $M[0, t] = \cup_{s \in [0, t]} \{M_s\}$.
- First entrance time in a set F : $\tau(F) = \inf\{n \geq 0 : X_n \in F\}$.
- Hitting time of a set F : $\tau^+(F) = \inf\{n \geq 1 : X_n \in F\}$.
- Green function for SRWs: $G(x, y) = \mathbb{E}^x[\sum_{n \geq 0} \mathbb{1}_{\{X_n = y\}}]$.
- Green function for Brownian motions on \mathbb{R}^d ($d \geq 3$):

$$G(x, y) = \int_0^\infty (2\pi t)^{-d/2} e^{-\|x-y\|_2^2/2t} dt = \frac{\Gamma(d/2)}{(d-2)\pi^{d/2}} \|x-y\|_2^{2-d}.$$

- SRW capacity of a set F : $\text{Cap}(F) = \sum_{x \in F} \mathbb{P}^x[\tau^+(F) = \infty]$.
- Brownian motion capacity of a Borel set F :

$$\text{Cap}_{BM}(F)^{-1} = \inf \left\{ \int \int G(x, y) \mu(dx) \mu(dy) : \mu \text{ is a probability measure on } F \right\}.$$

3 Four dimension: concentration of $\text{Cap}(X[0, n])$ around its mean

We prove Theorem 1.1 in this section. Before that, we need to state three auxiliary lemmas.

Lemma 3.1 ([8, Proposition 4.6.4]). *For a transient graph and a subset F of vertices, by last passage time decomposition,*

$$\mathbb{P}^x[\tau(F) < \infty] = \sum_{z \in F} G(x, z) \mathbb{P}^z[\tau^+(F) = \infty].$$

Lemma 3.2 ([8, Theorem 4.3.1]). *For a simple random walk on \mathbb{Z}^d , $d \geq 3$, there exist $0 < c(d) \leq C(d) < \infty$ such that*

$$c(d)(1 + \|x - y\|_\infty)^{2-d} \leq G(x, y) \leq C(d)(1 + \|x - y\|_\infty)^{2-d}.$$

More precisely, $G(x, y) = \frac{d\Gamma(d/2)}{(d-2)\pi^{d/2}} (\|x-y\|_2 + 1)^{2-d} + O((\|x-y\|_2 + 1)^{-d})$ as $\|x-y\|_2 \rightarrow \infty$.

Lemma 3.3 ([2, Corollary 1.4]). *For a SRW $(X_n)_n$ on \mathbb{Z}^4 ,*

$$\lim_{n \rightarrow \infty} \frac{\log n}{n} \mathbb{E}[\text{Cap}(X[0, n])] = \frac{\pi^2}{8}. \tag{3.1}$$

It is known that the capacity of a set is closely related to the hitting probability of that set, see Lemma 3.1. We will prove Theorem 1.1 by refining the argument in [4, Lemma 2.4].

Proof of Theorem 1.1. Let $(X_n^0)_{n \geq 0}, (X_n^1)_{n \geq 0}, (X_n^2)_{n \geq 0}$ be three independent simple random walks. Denote by $\mathbb{E}_{(i)}^x$ the expectation corresponding to the random walk X^i starting from x . Similarly, we define $(\mathbb{E}_{(i),(j)}^{x,y})_{i \neq j}$. For simplicity of notation, we denote by $\mathbb{E}^{x,y,z}$ (or $\mathbb{P}^{x,y,z}$) the expectation (or probability) corresponding to X^0, X^1 and X^2 with initial

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points x, y and z , respectively. Recall that $X^0[0, n]$ is the range of X^0 up to time n . Similarly, we define $X^1[0, \infty)$ and $X^2[0, \infty)$.

Let $x_0 = (Kn, 0, 0, 0)$, where K will be sent to infinity in the end. By Lemmas 3.1 and 3.2, we have that

$$\text{Cap}(X^0[0, n]) = \mathbb{P}_{(1)}^{x_0} [X^0[0, n] \cap X^1[0, \infty) \neq \emptyset] \cdot s(4)K^2n^2(1 + O(K^{-1})),$$

where $s(4) = \frac{(d-2)\pi^{d/2}}{d\Gamma(d/2)} \Big|_{d=4} = \pi^2/2$. Hence,

$$\begin{aligned} \mathbb{E}[\text{Cap}(X^0[0, n])] &= s(4)K^2n^2(1 + O(K^{-1})) \\ &\quad \times \mathbb{P}_{(0),(1)}^{0,x_0} [X^0[0, n] \cap X^1[0, \infty) \neq \emptyset], \\ \mathbb{E}[(\text{Cap}(X^0[0, n]))^2] &= s(4)^2K^4n^4(1 + O(K^{-1})) \\ &\quad \times \mathbb{P}^{0,x_0,x_0} [X^0[0, n] \cap X^1[0, \infty) \neq \emptyset, X^0[0, n] \cap X^2[0, \infty) \neq \emptyset]. \end{aligned}$$

For $i = 1$ and 2 , define $\tau_i = \inf\{j \geq 0 : X_j^0 \in X^i[0, \infty)\}$. By symmetry,

$$\mathbb{P}^{0,x_0,x_0} [X^0[0, n] \cap X^1[0, \infty) \neq \emptyset, X^0[0, n] \cap X^2[0, \infty) \neq \emptyset] \leq 2\mathbb{P}^{0,x_0,x_0} [\tau_1 \leq \tau_2 \leq n].$$

By conditioning on X^1 and X^2 and then applying the strong Markov property for X^0 at time τ_1 ,

$$\begin{aligned} \mathbb{P}^{0,x_0,x_0} [\tau_1 \leq \tau_2 \leq n] &= \mathbb{E}^{0,x_0,x_0} \left[\mathbb{1}_{\{\tau_1 \leq \tau_2\}} \cdot \mathbb{1}_{\{\tau_1 \leq n\}} \cdot \mathbb{E}_{(0)}^{X_{\tau_1}^0} [\mathbb{1}_{\{X^0[0, n-\tau_1] \cap X^2[0, \infty) \neq \emptyset\}}] \right] \\ &\leq \mathbb{E}^{0,x_0,x_0} \left[\mathbb{1}_{\{\tau_1 \leq n\}} \cdot \mathbb{E}_{(0)}^{X_{\tau_1}^0} [\mathbb{1}_{\{X^0[0, n-\tau_1] \cap X^2[0, \infty) \neq \emptyset\}}] \right]. \end{aligned}$$

Then, we take the expectation with respect to X^2 and get that

$$\begin{aligned} &\mathbb{P}^{0,x_0,x_0} [\tau_1 \leq \tau_2 \leq n] \\ &\leq \mathbb{E}_{(0),(1)}^{0,x_0} \left[\mathbb{1}_{\{X^0[0, n] \cap X^1[0, \infty) \neq \emptyset\}} \cdot \sup_{y \in B(0, n)} \mathbb{P}_{(0),(2)}^{y,x_0} [X^0[0, n - \tau_1] \cap X^2[0, \infty) \neq \emptyset] \right]. \quad (3.2) \end{aligned}$$

By last passage time decomposition (Lemma 3.1) and translation invariance, we get that for $y \in B(0, n)$,

$$\begin{aligned} &\mathbb{P}_{(0),(2)}^{y,x_0} [X^0[0, n - \tau_1] \cap X^2[0, \infty) \neq \emptyset] \\ &= \mathbb{E}_{(0)}^y \left[\sum_{z \in \partial X^0[0, n-\tau_1]} G(x_0, z) \mathbb{P}^z[\tau^+(X^0[0, n - \tau_1]) = \infty] \right] \\ &= \mathbb{E}_{(0)}^0 \left[\sum_{z \in \partial X^0[0, n-\tau_1]} G(x_0, y + z) \mathbb{P}^z[\tau^+(X^0[0, n - \tau_1]) = \infty] \right], \end{aligned}$$

and

$$\begin{aligned} &\mathbb{P}_{(0),(2)}^{0,x_0} [X^0[0, n - \tau_1] \cap X^2[0, \infty) \neq \emptyset] \\ &= \mathbb{E}_{(0)}^0 \left[\sum_{z \in \partial X^0[0, n-\tau_1]} G(x_0, z) \mathbb{P}^z[\tau^+(X^0[0, n - \tau_1]) = \infty] \right], \end{aligned}$$

Since $y + \partial X^0[0, n - \tau_1] \subset B(0, 2n)$ and $x_0 = (Kn, 0, \dots, 0)$, we have that

$$\|x_0 - y - z\|_2 = (1 + O(K^{-1}))\|x_0 - z\|_2.$$

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By the Green function estimate (Lemma 3.2) (or by gradient estimate for harmonic functions [8, Theorem 6.3.8]), we have that $G(x_0, y + z) = (1 + O(K^{-1}))G(x_0, z)$. Hence,

$$\begin{aligned} \sum_{z \in \partial X^0[0, n - \tau_1]} G(x_0, y + z) \mathbb{P}^z[\tau^+(X^0[0, n - \tau_1]) = \infty] \\ = (1 + O(K^{-1})) \sum_{z \in \partial X^0[0, n - \tau_1]} G(x_0, z) \mathbb{P}^z[\tau^+(X^0[0, n - \tau_1]) = \infty] \end{aligned}$$

and we have that

$$\begin{aligned} \sup_{y \in B(0, n)} \mathbb{P}_{(0), (2)}^{y, x_0}[X^0[0, n - \tau_1] \cap X^2[0, \infty) \neq \emptyset] \\ = (1 + O(K^{-1})) \cdot \mathbb{P}_{(0), (2)}^{0, x_0}[X^0[0, n - \tau_1] \cap X^2[0, \infty) \neq \emptyset]. \end{aligned} \quad (3.3)$$

Hence, by (3.2) and (3.3), we obtain that

$$\begin{aligned} \mathbb{P}^{0, x_0, x_0}[\tau_1 \leq \tau_2 \leq n] \\ \leq (1 + O(K^{-1})) \mathbb{E}_{(0), (1)}^{0, x_0} \left[\mathbb{1}_{\{X^0[0, n] \cap X^1[0, \infty) \neq \emptyset\}} \cdot \mathbb{P}_{(0), (2)}^{0, x_0}[X^0[0, n - \tau_1] \cap X^2[0, \infty) \neq \emptyset] \right]. \end{aligned} \quad (3.4)$$

We denote by $k(n)$ the averaged capacity $\mathbb{E}[\text{Cap}(X^0[0, n])]$. Then,

$$\begin{aligned} \mathbb{E}[\text{Cap}(X^0[0, n])^2] &\leq (1 + O(K^{-1})) s(4) K^2 n^2 \mathbb{E}_{(0), (1)}^{0, x_0} [2k(n - \tau_1) \cdot \mathbb{1}_{\{\tau_1 \leq n\}}] \\ &= (1 + O(K^{-1})) s(4) K^2 n^2 \mathbb{E}_{(0), (1)}^{0, x_0} [2k(n - \tau_1) | \tau_1 \leq n] \mathbb{P}_{(0), (1)}^{0, x_0} [\tau_1 \leq n] \\ &= (1 + O(K^{-1})) k(n) \mathbb{E}_{(0), (1)}^{0, x_0} [2k(n - \tau_1) | \tau_1 \leq n]. \end{aligned} \quad (3.5)$$

Since

$$\begin{aligned} \mathbb{P}[\tau_1 \leq m] &= \mathbb{P}_{(0), (1)}^{0, x_0} [X^0[0, m] \cap X^1[0, \infty) \neq \emptyset] \\ &= (1 + O(K^{-1})) (s(4))^{-1} K^{-2} n^{-2} \mathbb{E}[\text{Cap}(X^0[0, m])], \end{aligned}$$

we have that

$$\mathbb{P}[\tau_1 \leq m | \tau_1 \leq n] = (1 + O(K^{-1})) k(m) / k(n).$$

Hence, by (3.1), we see that

$$\mathbb{P}[\tau_1 / n \in \cdot | \tau_1 \leq n] \rightarrow \text{Uniform distribution on } [0, 1], \text{ as } K, n \rightarrow \infty. \quad (3.6)$$

By (3.1) and (3.6), we have that

$$\lim_{K, n \rightarrow \infty} \mathbb{E}_{(0), (1)}^{0, x_0} [2k(n - \tau_1) | \tau_1 < n] / k(n) = 1$$

and consequently,

$$\lim_{n \rightarrow \infty} \text{Var}(X^0[0, n]) / k(n)^2 = 0. \quad \square$$

4 Three dimension: $(\text{Cap}(X[0, n]) / \sqrt{n})_n$ has a random limit

In this section, for $d = 3$, we will show that $\text{Cap}(X[0, n]) / \sqrt{n}$ converges to the corresponding quantity of a Brownian motion. The reason is that two 3D Brownian motion paths (or simple random walk paths) are very likely to intersect when they get close, see [6, Lemmas 2.4 and 2.6]. As an application of Skorokhod embedding theorem, one could closely couple SRWs and Brownian motions. For these two reasons, with a high probability, a SRW path is as hittable as a Brownian motion path. Accordingly, by last passage time decomposition and the scaling invariance of Brownian motion, we prove Theorem 1.2, which affirms the conjecture in [2, Section 6], see the remark below.

Remark 4.1. It was conjectured that $\text{Cap}(X[0, n])/\mathbb{E}[\text{Cap}(X[0, n])]$ has a random limit as $n \rightarrow \infty$ for $d = 3$, see [2, Section 6]. By Theorem 1.2, we confirm this conjecture. Indeed, by considering a ball containing $X[0, n]$, it was proved that there exists $C < \infty$ such that $\mathbb{E}[(\text{Cap}(X[0, n]))^2] \leq Cn$, see the proof of [10, Lemma 5]. Hence, $(\text{Cap}(X[0, n])/\sqrt{n})_n$ is uniformly integrable and $\lim_{n \rightarrow \infty} \mathbb{E}[\text{Cap}(X[0, n])]/\sqrt{n} = \mathbb{E}[\frac{1}{3\sqrt{3}} \text{Cap}_{BM}(M[0, 1])]$, which implies that $\text{Cap}(X[0, n])/\mathbb{E}[\text{Cap}(X[0, n])]$ converges in distribution towards $\text{Cap}_{BM}(M[0, 1])/\mathbb{E}[\text{Cap}_{BM}(M[0, 1])]$ as $n \rightarrow \infty$. Note that for all $\varepsilon > 0$, $\mathbb{P}[|M_t| \leq \varepsilon, \forall t \in [0, 1]] > 0$, which, by monotonicity of capacities, implies that $\mathbb{P}[\text{Cap}_{BM}(M[0, 1]) < \varepsilon] > 0, \forall \varepsilon > 0$. Also, note that $\mathbb{E}[\text{Cap}_{BM}(M[0, 1])] > 0$. Hence, $\text{Var}(\text{Cap}_{BM}(M[0, 1])) > 0$.

Proof of Theorem 1.2. By Skorokhod embedding, there exists a coupling between a pair of independent SRWs and a pair of independent Brownian motions. To be more precise, there exists a probability space such that the following holds, see [6, Lemma 3.1].

- $(X_n^0)_n$ and $(X_n^1)_n$ are both SRWs on \mathbb{Z}^3 starting from 0.
- $(M_t^0)_t$ and $(M_t^1)_t$ are both Brownian motions on \mathbb{R}^3 starting from 0.
- (X^0, M^0) is independent of (X^1, M^1) .
- For all $\varepsilon > 0$, there exist $\gamma > 0$ and $C < \infty$ such that for all $n \geq 1$,

$$\mathbb{P} \left[\sum_i \max_{s \leq n} |X_{[3s]}^i - M_s^i|_1 > n^{1/4+\varepsilon} \right] \leq C \cdot e^{-n^\gamma}. \tag{4.1}$$

We take $\varepsilon \leq \frac{1}{1000}$. We define several events as follows:

- $E_1 \stackrel{\text{def}}{=} \{ \max_{s \leq n} |X_{[3s]}^0 - M_s^0|_1 \leq n^{\frac{1}{4}+\varepsilon} \}$.
- $E_2 \stackrel{\text{def}}{=} \{ X^0[0, n] \subset B_{\mathbb{Z}^3}(0, n^{\frac{1}{2}+\frac{\varepsilon}{4}}) \}$.
- $E_{3,SRW} \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \forall z \in \text{Nbd}(X^0[0, n], n^{\frac{1}{2}-\varepsilon}) \cap \mathbb{Z}^3, \text{ we have that} \\ \mathbb{P}[X^0[0, n] \cap (z + X^1[0, \infty)) = \emptyset | X^0[0, n]] < n^{-\delta} \end{array} \right\}$, where

$$\text{Nbd}(A, s) = \cup_{x \in A} B_{\mathbb{R}^3}(x, s), \quad \text{for } A \subset \mathbb{R}^3 \text{ and } s \geq 0.$$

Similarly, we define

$$E_{3,BM} \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \forall z \in \text{Nbd}(M^0[0, n/3], n^{\frac{1}{2}-\varepsilon}), \text{ we have that} \\ \mathbb{P}[M^0[0, n/3] \cap (z + M^1[0, \infty)) = \emptyset | M^0[0, n/3]] < n^{-\delta} \end{array} \right\}.$$

We define $E_3 = E_{3,SRW} \cup E_{3,BM}$.

As we mentioned above, by Skorokhod approximation, $\mathbb{P}[E_1^c] \leq C \cdot e^{-n^{\gamma(\varepsilon)}}$ where $C < \infty$ does not depend on n . By union bounds and Hoeffding's inequality, $\mathbb{P}[E_2^c] \leq Cne^{-n^{\varepsilon/2}/C}$ where $C < \infty$ does not depend on n . By [6, Lemmas 2.4, 2.6], for all $N \geq 1, \exists \delta > 0$ (in the definition of E_3) and $C < \infty$ such that for all $n \geq 1, \mathbb{P}[E_3^c] \leq C \cdot n^{-N}$. (Use union bounds and the fact that the diameter of a SRW (or Brownian motion) path up to time n is of order \sqrt{n} with a two-sided stretched exponential tail distribution.) We define $E = E_1 \cap E_2 \cap E_3$, take $N = 2$ and choose δ accordingly such that

$$\exists C < \infty, \mathbb{P}[E^c] \leq C \cdot n^{-N}. \tag{4.2}$$

Take $y_n \in \mathbb{Z}^d$ such that $\|y_n\|_2 = \lfloor n^{\frac{1}{2}+\varepsilon} \rfloor$. By the independence between (X^0, M^0) and (X^1, M^1) , the last passage time decomposition and the Green function estimate, we have that

$$\begin{aligned} & \text{Cap}(X^0[0, n])\mathbb{1}_E \\ &= \frac{2\pi}{3} (1 + o(1))\mathbb{1}_E n^{\frac{1}{2}+\varepsilon} \mathbb{P}[X^0[0, n] \cap (y_n + X^1[0, \infty)) \neq \emptyset | X^0[0, n], M^0[0, n/3]], \end{aligned} \tag{4.3}$$

and similarly, by [9, Theorem 8.8, Theorem 8.27 and Definition 8.18],

$$\begin{aligned} & \text{Cap}(M^0[0, n/3])\mathbb{1}_E \\ &= 2\pi(1 + o(1))\mathbb{1}_E n^{\frac{1}{2} + \varepsilon} \mathbb{P} [M^0[0, n/3] \cap (y_n + M^1[0, \infty)) \neq \emptyset | X^0[0, n], M^0[0, n/3]]. \end{aligned} \quad (4.4)$$

We will show that (4.3) = $\frac{1}{3}(1 + o(1)) \cdot (4.4)$ which is equivalent to

$$\begin{aligned} & \mathbb{1}_E \mathbb{P} [X^0[0, n] \cap (y_n + X^1[0, \infty)) \neq \emptyset | X^0[0, n], M^0[0, n/3]] \\ &= (1 + o(1))\mathbb{1}_E \mathbb{P} [M^0[0, n/3] \cap (y_n + M^1[0, \infty)) \neq \emptyset | X^0[0, n], M^0[0, n/3]]. \end{aligned} \quad (4.5)$$

We first find several quantities, which are asymptotically equivalent to the left hand side of (4.5). By the definition of E_3 and the strong Markov property of X^1 , we get that

$$\begin{aligned} & \mathbb{1}_E \mathbb{P} [X^0[0, n] \cap (y_n + X^1[0, \infty)) \neq \emptyset | X^0[0, n], M^0[0, n/3]] \\ &= (1 + o(1))\mathbb{1}_E \mathbb{P} [\text{Nbd}(X^0[0, n], n^{\frac{1}{2} - \varepsilon}) \cap (y_n + X^1[0, \infty)) \neq \emptyset | X^0[0, n], M^0[0, n/3]] \\ &= (1 + o(1))\mathbb{1}_E \mathbb{P} [\text{Nbd}(X^0[0, n], n^{\frac{1}{2} - 4\varepsilon}) \cap (y_n + X^1[0, \infty)) \neq \emptyset | X^0[0, n], M^0[0, n/3]]. \end{aligned} \quad (4.6)$$

Next, we will show that X^1 could be replaced by M^1 in the following sense:

$$\begin{aligned} (4.6) &= (1 + o(1)) \cdot \mathbb{1}_E \\ &\cdot \mathbb{P} [\text{Nbd}(X^0[0, n], n^{\frac{1}{2} - 2\varepsilon}) \cap (y_n + M^1[0, \infty)) \neq \emptyset | X^0[0, n], M^0[0, n/3]], \end{aligned} \quad (4.7)$$

which would follow from Skorokhod approximation (4.1) up to time $n^{1+8\varepsilon}$ and the following three equations:

$$\begin{aligned} & \mathbb{1}_E \mathbb{P} [\text{Nbd}(X^0[0, n], n^{\frac{1}{2} - 4\varepsilon}) \cap (y_n + X^1[0, \infty)) \neq \emptyset | X^0[0, n], M^0[0, n/3]] \\ &= (1 + o(1))\mathbb{1}_E \mathbb{P} [\text{Nbd}(X^0[0, n], n^{\frac{1}{2} - 4\varepsilon}) \cap (y_n + X^1[0, n^{1+8\varepsilon})) \neq \emptyset | X^0[0, n], M^0[0, n/3]], \end{aligned} \quad (4.8)$$

$$\begin{aligned} & \mathbb{1}_E \mathbb{P} [\text{Nbd}(X^0[0, n], n^{\frac{1}{2} - \varepsilon}) \cap (y_n + X^1[0, \infty)) \neq \emptyset | X^0[0, n], M^0[0, n/3]] \\ &= (1 + o(1))\mathbb{1}_E \mathbb{P} [\text{Nbd}(X^0[0, n], n^{\frac{1}{2} - \varepsilon}) \cap (y_n + X^1[0, n^{1+8\varepsilon})) \neq \emptyset | X^0[0, n], M^0[0, n/3]], \end{aligned} \quad (4.9)$$

and that

$$\begin{aligned} & \mathbb{1}_E \mathbb{P} [\text{Nbd}(X^0[0, n], n^{\frac{1}{2} - 2\varepsilon}) \cap (y_n + M^1[0, \infty)) \neq \emptyset | X^0[0, n], M^0[0, n/3]] \\ &= (1 + o(1))\mathbb{1}_E \mathbb{P} [\text{Nbd}(X^0[0, n], n^{\frac{1}{2} - 2\varepsilon}) \cap (y_n + M^1[0, \frac{n^{1+8\varepsilon}}{3}]) \neq \emptyset | X^0[0, n], M^0[0, n/3]] \end{aligned} \quad (4.10)$$

Indeed, by union bounds, Markov property, the last passage time decomposition and the Green function estimate, there exists $c > 0$ such that

$$\begin{aligned} & \mathbb{1}_E \mathbb{P} [\text{Nbd}(X^0[0, n], n^{\frac{1}{2} - 4\varepsilon}) \cap (y_n + X^1[n^{1+8\varepsilon}, \infty)) \neq \emptyset | X^0[0, n], M^0[0, n/3]] \\ &\leq \mathbb{1}_E \mathbb{P} [y_n + X_{n^{1+8\varepsilon}}^1 \leq n^{\frac{1}{2} + 2\varepsilon}] + \mathbb{1}_E \cdot c \cdot n^{-\frac{1}{2} - 2\varepsilon} \text{Cap}(\text{Nbd}(X^0[0, n], n^{\frac{1}{2} - 4\varepsilon})) \\ &\leq \mathbb{1}_E \cdot c \cdot (n^{-6\varepsilon} + n^{-\frac{1}{2} - 2\varepsilon} \text{Cap}(\text{Nbd}(X^0[0, n], n^{\frac{1}{2} - 4\varepsilon}))). \end{aligned} \quad (4.11)$$

By the last passage time decomposition, the Green function estimate, monotonicity and translation invariance of the capacity and the estimate for the capacity of a ball, there exists $c > 0$ such that

$$\begin{aligned} \mathbb{1}_E \mathbb{P} \left[\text{Nbd}(X^0[0, n], n^{\frac{1}{2}-4\epsilon}) \cap (y_n + X^1[0, \infty)) \neq \emptyset \mid X^0[0, n], M^0[0, n/3] \right] \\ \geq \mathbb{1}_E \cdot c \cdot n^{-\frac{1}{2}-\epsilon} \text{Cap}(\text{Nbd}(X^0[0, n], n^{\frac{1}{2}-4\epsilon})) \\ \geq \mathbb{1}_E \cdot c \cdot n^{-\frac{1}{2}-\epsilon} \text{Cap}(B(0, n^{\frac{1}{2}-4\epsilon})) \geq \mathbb{1}_E \cdot c^2 \cdot n^{-5\epsilon}. \end{aligned} \quad (4.12)$$

Comparing (4.11) with (4.12), we see that (4.8) holds. And (4.9) and (4.10) could be derived in a similar way. (Alternatively, one could use strong Markov property at the hitting time $\tau(\partial B(0, n^{\frac{1}{2}+3\epsilon}))$ with the bound

$$\mathbb{P}[\tau(\partial B(0, n^{\frac{1}{2}+3\epsilon})) > n^{1+8\epsilon}] \leq e^{-cn^{2\epsilon}},$$

for some $c = c(\epsilon) > 0$ independent of n .)

Next, we derive several quantities which are equivalent to the right hand side of (4.5). Similarly to (4.6), we obtain that

$$\begin{aligned} \mathbb{1}_E \mathbb{P} \left[\text{Nbd}(M^0[0, n/3], n^{\frac{1}{2}-\epsilon}) \cap (y_n + M^1[0, \infty)) \neq \emptyset \mid X^0[0, n], M^0[0, n/3] \right] \\ = (1 + o(1)) \mathbb{1}_E \mathbb{P} \left[M^0[0, n/3] \cap (y_n + M^1[0, \infty)) \neq \emptyset \mid X^0[0, n], M^0[0, n] \right]. \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} \mathbb{1}_E \mathbb{P} \left[\text{Nbd}(M^0[0, n/3], n^{\frac{1}{2}-4\epsilon}) \cap (y_n + M^1[0, \infty)) \neq \emptyset \mid X^0[0, n], M^0[0, n/3] \right] \\ = (1 + o(1)) \mathbb{1}_E \mathbb{P} \left[M^0[0, n/3] \cap (y_n + M^1[0, \infty)) \neq \emptyset \mid X^0[0, n], M^0[0, n] \right]. \end{aligned} \quad (4.14)$$

By the definition of E_1 (the Skorokhod approximation), for n sufficient large, we have that

$$\text{Nbd}(M^0[0, n/3], n^{\frac{1}{2}-4\epsilon}) \subset \text{Nbd}(X^0[0, n], n^{\frac{1}{2}-2\epsilon}) \subset \text{Nbd}(M^0[0, n/3], n^{\frac{1}{2}-\epsilon})$$

and hence, (4.14) \leq (4.7) \leq (4.13). Therefore, (4.5) holds and equivalently,

$$\text{Cap}(X^0[0, n]) \mathbb{1}_E = \frac{1}{3} (1 + o(1)) \text{Cap}(M^0[0, n/3]) \mathbb{1}_E. \quad (4.15)$$

By Brownian scaling, $\text{Cap}(M^0[0, n/3])$ has the same distribution as $\frac{\sqrt{n}}{\sqrt{3}} \text{Cap}(M^0[0, 1])$. Hence, together with (4.2), we get that $\text{Cap}(X^0[0, n])/\sqrt{n}$ converges in distribution towards $\frac{1}{3\sqrt{3}} \text{Cap}(M^0[0, 1])$ as $n \rightarrow \infty$. \square

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Acknowledgments. The author thanks Sapozhnikov for valuable advices and useful comments. The author thanks the reviewer for useful suggestions and careful reading.

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