

Stein type characterization for G -normal distributions

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Abstract

In this article, we provide a Stein type characterization for G -normal distributions: Let $\mathcal{N}[\varphi] = \sup_{\mu \in \Theta} \mu[\varphi]$, $\varphi \in C_{b,Lip}(\mathbb{R})$, be a sublinear expectation. \mathcal{N} is G -normal if and only if for any $\varphi \in C_b^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \left[\frac{x}{2} \varphi'(x) - G(\varphi''(x)) \right] \mu^\varphi(dx) = 0,$$

where μ^φ is a realization of φ associated with \mathcal{N} , i.e., $\mu^\varphi \in \Theta$ and $\mu^\varphi[\varphi] = \mathcal{N}[\varphi]$.

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1 Introduction

Peng (2007) introduced the notion of G -normal distribution via the viscosity solutions of the G -heat equation below

$$\begin{aligned} \partial_t u - G(\partial_x^2 u) &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \\ u(0, x) &= \varphi(x), \end{aligned}$$

where $G(a) = \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$, $a \in \mathbb{R}$ with $0 \leq \underline{\sigma} \leq \bar{\sigma} < \infty$, and $\varphi \in C_{b,Lip}(\mathbb{R})$, the collection of bounded Lipschitz functions on \mathbb{R} .

Then the one-dimensional G -normal distribution is defined by

$$\mathcal{N}_G[\varphi] = u^\varphi(1, 0),$$

where u^φ is the viscosity solution to the G -heat equation with the initial value φ .

The above G -heat equation has a unique viscosity solution. We refer to [2] for the definition, existence, uniqueness and comparison theorem of this type of parabolic PDEs (see also [10] for this specific situation). In this article, we consider only the non-degenerate G , i.e., $\underline{\sigma} > 0$. Then the above G -heat equation has a unique $C^{1,2}$ -solution (see, e.g., [6]). More precisely, there exists $\alpha \in (0, 1)$ such that for any $0 < a < b < \infty$,

$$\|u\|_{C^{1+\frac{\alpha}{2}, 2+\alpha}([a,b] \times \mathbb{R})} < \infty.$$

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By the comparison theorem of the G -heat equation, it can be easily checked that \mathcal{N}_G is a sublinear expectation on $C_{b,Lip}(\mathbb{R})$, i.e., a functional on $C_{b,Lip}(\mathbb{R})$ satisfies

- E1.** $\mathcal{N}_G[\varphi] \geq \mathcal{N}_G[\phi]$, for $\varphi \geq \phi$;
- E2.** $\mathcal{N}_G[\lambda\varphi] = \lambda\mathcal{N}_G[\varphi]$, for $\lambda \geq 0$;
- E3.** $\mathcal{N}_G[\varphi + c] = \mathcal{N}_G[\varphi] + c$, for $c \in \mathbb{R}$;
- E4.** $\mathcal{N}_G[\varphi + \phi] \leq \mathcal{N}_G[\varphi] + \mathcal{N}_G[\phi]$.

Moreover, \mathcal{N}_G is continuous from above: for $\varphi_n \in C_{b,Lip}(\mathbb{R})$, $\varphi_n \downarrow 0$, we have $\mathcal{N}_G[\varphi_n] \downarrow 0$. A sublinear expectation with this property is called *regular*, which happens if and only if it can be represented as the supremum expectation of a *tight* family of probability measures Θ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (see [3]).

Throughout this article, we shall only consider sublinear expectations which are regular.

As a regular sublinear expectation, the G -normal distribution can be represented as

$$\mathcal{N}_G[\varphi] = \sup_{\mu \in \Theta_G} \mu[\varphi], \text{ for all } \varphi \in C_{b,Lip}(\mathbb{R}),$$

where Θ_G is a tight family of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For $\varphi \in C_{b,Lip}(\mathbb{R})$, we call $\mu \in \Theta_G$ a *realization* of φ associated with \mathcal{N}_G if $\mathcal{N}_G[\varphi] = \mu[\varphi]$. To ensure that each $\varphi \in C_{b,Lip}(\mathbb{R})$ has a realization, Θ_G will always be chosen as *weakly compact*.

As is well known, the fact that $\mu = \mathcal{N}(0, \sigma^2)$ if and only if

$$\int_{\mathbb{R}} [x\varphi'(x) - \sigma^2\varphi''(x)]\mu(dx) = 0, \text{ for all } \varphi \in C_b^2(\mathbb{R}). \tag{1.1}$$

This is the characterization of the normal distribution presented in Stein (1972), which is the basis of Stein's method for normal approximation (see Chen, Goldstein and Shao (2011) and the references therein for more details).

What is the proper counterpart of (1.1) for G -normal distributions? An immediate conjecture should be

$$\mathcal{N}_G[\mathcal{L}_G\varphi] = 0, \text{ for all } \varphi \in C_b^2(\mathbb{R}),$$

where $\mathcal{L}_G\varphi(x) = \frac{x}{2}\varphi'(x) - G(\varphi''(x))$. However, the above equality does not hold generally as was pointed out in Hu et al. (2015) by a counterexample.

By calculating some examples, we try to find the proper generalization of (1.1) for G -normal distributions.

Example 1.1. Set $\beta = \frac{\bar{\sigma}}{\sigma}$ and $\sigma = \frac{\bar{\sigma} + \sigma}{2}$. Song (2015) defined a periodic function ϕ_β as a variant of the trigonometric function $\cos x$ (see Figure 1).

$$\phi_\beta(x) = \begin{cases} \frac{2}{1+\beta} \cos(\frac{1+\beta}{2}x) & \text{for } x \in [-\frac{\pi}{1+\beta}, \frac{\pi}{1+\beta}); \\ \frac{2\beta}{1+\beta} \cos(\frac{1+\beta}{2\beta}x + \frac{\beta-1}{2\beta}\pi) & \text{for } x \in [\frac{\pi}{1+\beta}, \frac{(2\beta+1)\pi}{1+\beta}). \end{cases} \tag{1.2}$$

It was proved that

$$G(\phi_\beta''(x)) = -\frac{\sigma^2}{2}\phi_\beta(x)$$

and that $u(t, x) := e^{-\frac{1}{2}\sigma^2 t}\phi_\beta(x)$ is a solution to the G -heat equation. Therefore

$$u(t, x) = \mathcal{N}_G[\phi_\beta(x + \sqrt{t}\cdot)] = \mu^{t,x}[\phi_\beta(x + \sqrt{t}\cdot)], \tag{1.3}$$

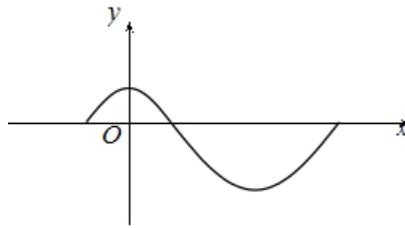


Figure 1: $\phi_\beta(x)$

where $\mu^{t,x}$ is a realization of $\phi_\beta(x + \sqrt{t}\cdot)$. Since $\mu^{s,x}[\phi_\beta(x + \sqrt{t}\cdot)]$ considered as a function of s attains its maximum at $s = t$, we get

$$\partial_t u(t, x) = \int_{\mathbb{R}} \phi'_\beta(x + \sqrt{t}y) \frac{y}{2\sqrt{t}} \mu^{t,x}(dy) = \frac{1}{t} \int_{\mathbb{R}} \partial_y \phi_\beta(x + \sqrt{t}y) \frac{y}{2} \mu^{t,x}(dy)$$

by taking the formal derivation on the equality (1.3) with respect to t . On the other hand, noting $u(t, x) = e^{-\frac{1}{2}\sigma^2 t} \phi_\beta(x)$, we have

$$\begin{aligned} \partial_t u(t, x) &= -\frac{1}{2}\sigma^2 u(t, x) \\ &= -\frac{1}{2}\sigma^2 \int_{\mathbb{R}} \phi_\beta(x + \sqrt{t}y) \mu^{t,x}(dy) \\ &= \int_{\mathbb{R}} G(\phi''_\beta(x + \sqrt{t}y)) \mu^{t,x}(dy) \\ &= \frac{1}{t} \int_{\mathbb{R}} G(\partial_y^2 \phi_\beta(x + \sqrt{t}y)) \mu^{t,x}(dy). \end{aligned}$$

Combining the above arguments, we get the following equality

$$\int_{\mathbb{R}} \left[\frac{y}{2} \partial_y \phi_\beta(x + \sqrt{t}y) - G(\partial_y^2 \phi_\beta(x + \sqrt{t}y)) \right] \mu^{t,x}(dy) = 0.$$

Inspired by this example, we predict the following result generally holds.

Proposition 1.2. Let $\varphi \in C_b^2(\mathbb{R})$. If μ^φ is a realization of φ associated with the G -normal distribution \mathcal{N}_G , we have

$$\int_{\mathbb{R}} \left[\frac{x}{2} \varphi'(x) - G(\varphi''(x)) \right] \mu^\varphi(dx) = 0.$$

To convince ourselves, let us calculate another simple example.

Example 1.3. Let $\phi \in C^2(\mathbb{R})$ satisfy, for some $\rho \geq 0$,

$$\frac{x}{2} \phi'(x) + G(\phi''(x)) = \rho \phi(x).$$

It is easy to check that $u(t, x) = (1 + t)^\rho \phi(\frac{x}{\sqrt{1+t}})$ is a solution to the G -heat equation. Therefore

$$u(t, x) = \mathcal{N}_G[\phi(x + \sqrt{t}\cdot)] = \mu^{t,x}[\phi(x + \sqrt{t}\cdot)] = (1 + t)^\rho \phi\left(\frac{x}{\sqrt{1+t}}\right), \tag{1.4}$$

where $\mu^{t,x}$ is a realization of $\phi(x + \sqrt{t}\cdot)$. By taking the formal derivation on (1.4) with respect to t , we get

$$\partial_t u(t, x) = \int_{\mathbb{R}} \phi'(x + \sqrt{t}y) \frac{y}{2\sqrt{t}} \mu^{t,x}(dy) \tag{1.5}$$

$$= \rho(1 + t)^{\rho-1} \phi\left(\frac{x}{\sqrt{1+t}}\right) - \frac{x}{2}(1 + t)^{\rho-\frac{3}{2}} \phi'\left(\frac{x}{\sqrt{1+t}}\right). \tag{1.6}$$

Similarly, by taking the formal derivation on (1.4) with respect to x , we get

$$\partial_x u(t, x) = \int_{\mathbb{R}} \phi'(x + \sqrt{t}y) \mu^{t,x}(dy) = (1+t)^{\rho-\frac{1}{2}} \phi'\left(\frac{x}{\sqrt{1+t}}\right). \quad (1.7)$$

Note that (1.6) \times (1+t) + (1.7) \times $\frac{x}{2}$ - (1.4) \times $\rho = 0$, which implies

$$\int_{\mathbb{R}} \left[\frac{y}{2\sqrt{t}} \phi'(x + \sqrt{t}y) - G(\phi''(x + \sqrt{t}y)) \right] \mu^{t,x}(dy) = 0.$$

More precisely, we have

$$\int_{\mathbb{R}} \left[\frac{y}{2} \partial_y \phi(x + \sqrt{t}y) - G(\partial_y^2 \phi(x + \sqrt{t}y)) \right] \mu^{t,x}(dy) = 0,$$

which is exactly the conclusion of Proposition 1.2.

Returning to the linear case, the closed linear span of the family of functions considered in either of the previous two examples is the space of continuous functions, which increases our confidence that the conclusion of Proposition 1.2 is correct.

Just like Stein's characterization of (classical) normal distributions, we are also concerned about the converse problem:

(Q) Let $\mathcal{N}[\varphi] = \sup_{\mu \in \Theta} \mu[\varphi]$, $\varphi \in C_{b,Lip}(\mathbb{R})$, be a sublinear expectation. Assuming \mathcal{N} satisfies the Stein type formula (SH) below, does it follow that $\mathcal{N} = \mathcal{N}_G$?

(SH) For $\varphi \in C_b^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}} [G(\varphi''(x)) - \frac{x}{2} \varphi'(x)] \mu^\varphi(dx) = 0,$$

where μ^φ is a realization of φ associated with \mathcal{N} , i.e., $\mu^\varphi \in \Theta$ and $\mu^\varphi[\varphi] = \mathcal{N}[\varphi]$.

Throughout this article, we suppose the following additional properties:

(H1) Θ is weakly compact;

(H2) $\lim_{N \rightarrow \infty} \mathcal{N}[|x|1_{[|x|>N]}] = 0$.

Clearly, Θ and $\bar{\Theta}^w$ generate the same sublinear expectation on $C_{b,Lip}(\mathbb{R})$. Here, we emphasize by (H1) that Θ is weakly compact, which ensures that there exists a realization μ_φ for any $\varphi \in C_{b,Lip}(\mathbb{R})$. (H2) is a condition (strictly) stronger than $\mathcal{N}[|x|] < \infty$, but weaker than $\mathcal{N}[|x|^\alpha] < \infty$ for some $\alpha > 1$, which is employed to ensure that the functions generated by \mathcal{N} have better analytic properties.

Actually, we also find evidence for the converse statement from some simple examples.

Example 1.4. Assume that \mathcal{N} is a sublinear expectation on $C_{b,Lip}(\mathbb{R})$ satisfying the Stein type formula (SH). Set $u(t, x) := \mathcal{N}[\phi_\beta(x + \sqrt{t}\cdot)]$. We shall "prove" that u is the solution to the G -heat equation. Actually, noting that $u(t, x) = \mathcal{N}[\phi_\beta(x + \sqrt{t}\cdot)] = \mu^{t,x}[\phi_\beta(x + \sqrt{t}\cdot)]$ with $\mu^{t,x}$ a realization of $\phi_\beta(x + \sqrt{t}\cdot)$, we get

$$\partial_t u(t, x) = \int_{\mathbb{R}} \phi'_\beta(x + \sqrt{t}y) \frac{y}{2\sqrt{t}} \mu^{t,x}(dy).$$

So, from Hypothesis (SH), we get

$$\partial_t u(t, x) = \int_{\mathbb{R}} G(\phi''_\beta(x + \sqrt{t}y)) \mu^{t,x}(dy) = -\frac{\sigma^2}{2} \int_{\mathbb{R}} \phi_\beta(x + \sqrt{t}y) \mu^{t,x}(dy) = -\frac{\sigma^2}{2} u(t, x).$$

Then $u(t, x) = e^{-\frac{\sigma^2}{2}t} \phi_\beta(x)$, which is the solution to the G -heat equation with $u(0, x) = \phi_\beta(x)$.

Our purpose is to prove the Stein type formula for G -normal distributions (Proposition 1.2) and its converse problem (Q). In order to do so, we first prove a weaker version of the Stein type characterization below.

Theorem 1.5. *Let $\mathcal{N}[\varphi] = \sup_{\mu \in \Theta} \mu[\varphi]$, $\varphi \in C_{b,Lip}(\mathbb{R})$, be a sublinear expectation. \mathcal{N} is G -normal if and only if for any $\varphi \in C_b^2(\mathbb{R})$, we have*

$$\sup_{\mu \in \Theta_\varphi} \int_{\mathbb{R}} [G(\varphi''(x)) - \frac{x}{2}\varphi'(x)]\mu(dx) = 0, \tag{SHw}$$

where $\Theta_\varphi = \{\mu \in \Theta : \mu[\varphi] = \mathcal{N}[\varphi]\}$.

Since (SH) implies (SHw), the necessity part of Theorem 1.5 follows from Proposition 1.2. At the same time, the converse argument (Q) follows from the sufficiency part of Theorem 1.5.

In Section 2, we provide several lemmas to show how the differentiation penetrates the sublinear expectations, which makes sense the "formal derivation" in the above examples. In Section 3, we give a proof to Theorem 1.5. We shall prove Proposition 1.2 in Section 5 based on the G -expectation theory, and as a preparation we list some basic definitions and notations concerning G -expectation in Section 4.

2 Some useful lemmas

Let $\mathcal{N}[\varphi] = \sup_{\mu \in \Theta} \mu[\varphi]$ be a sublinear expectation on $C_{b,Lip}(\mathbb{R})$.

Define $\xi : \mathbb{R} \rightarrow \mathbb{R}$ by $\xi(x) = x$. Sometimes, we write $\mathcal{N}[\varphi]$, $\mu[\varphi]$ by $\mathbb{E}[\varphi(\xi)]$, $E_\mu[\varphi(\xi)]$, respectively. For $\varphi \in C_{b,Lip}(\mathbb{R})$, set $\Theta_\varphi = \{\mu \in \Theta : E_\mu[\varphi(\xi)] = \mathbb{E}[\varphi(\xi)]\}$.

Let $\psi : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a bounded function satisfying $\psi(t, \cdot) \in C_{b,Lip}(\mathbb{R})$ for each $t \in [a, b]$ and for each $n \in \mathbb{N}$, there exists $L_n > 0$ such that, for $s, t \in [a, b]$ and $|x| \leq n$,

$$|\psi(t, x) - \psi(s, x)| \leq L_n|t - s|.$$

We denote by $C_{b,loc}([a, b] \times \mathbb{R})$ the totality of such functions. For $\psi \in C_{b,loc}([a, b] \times \mathbb{R})$, we sometimes employ the following assumption: there exists a continuous function $\dot{\psi}_{t_0}(x)$ such that at point $t_0 \in [a, b]$ the properties below hold.

(A1) $\mathbb{E}[\mathfrak{R}_{\delta;t_0,\xi}^\psi] = o(\delta)$ as $\delta \rightarrow 0$, where $\mathfrak{R}_{\delta;t_0,\xi}^\psi = \psi(t_0 + \delta, \xi) - \psi(t_0, \xi) - \dot{\psi}_{t_0}(\xi)\delta$;

(A2) $\lim_{N \rightarrow \infty} \mathcal{N}[\dot{\psi}_{t_0}(x)1_{|x|>N}] = 0$.

Remark 2.1. For most cases, the function $\dot{\psi}_{t_0}(x)$ would be chosen as $\partial_t \psi(t_0, x)$.

For a function $\alpha : \mathbb{R} \rightarrow \mathbb{R}$, define $\partial_t^+ \alpha(t) = \lim_{\delta \downarrow 0} \frac{\alpha(t+\delta) - \alpha(t)}{\delta}$ (respectively, $\partial_t^- \alpha(t) = \lim_{\delta \downarrow 0} \frac{\alpha(t-\delta) - \alpha(t)}{-\delta}$) if the corresponding limits exist.

Lemma 2.2. For $\psi \in C_{b,loc}([a, b] \times \mathbb{R})$ and a sublinear expectation \mathcal{N} on $C_{b,Lip}(\mathbb{R})$, set

$$\alpha(t) := \mathbb{E}[\psi(t, \xi)], \Theta_t := \Theta_{\psi(t, \cdot)} \text{ for } t \in [a, b].$$

1) For any $t_0 \in [a, b]$, $\{t_n\}_{n \geq 1} \subset [a, b]$ and $\mu_n \in \Theta_{t_n}$, $n \geq 1$, such that $t_n \rightarrow t_0$ and $\mu_n \xrightarrow{\text{weakly}} \mu$ as n goes to infinity, we have $\mu \in \Theta_{t_0}$.

Denote by $\bar{\Theta}_{t_0}$ the totality of $\mu \in \Theta_{t_0}$ defined above corresponding to $t_n \downarrow t_0$ and by $\underline{\Theta}_{t_0}$ corresponding to $t_n \uparrow t_0$.

2) Furthermore, if **(A1)**, **(A2)** hold at t_0 , we have

$$\partial_t^+ \alpha(t_0) = \sup_{\mu \in \Theta_{t_0}} E_\mu[\dot{\psi}_{t_0}(\xi)] = E_{\bar{\mu}}[\dot{\psi}_{t_0}(\xi)], \text{ for any } \bar{\mu} \in \bar{\Theta}_{t_0}, \tag{2.1}$$

$$\partial_t^- \alpha(t_0) = \inf_{\mu \in \Theta_{t_0}} E_\mu[\dot{\psi}_{t_0}(\xi)] = E_{\underline{\mu}}[\dot{\psi}_{t_0}(\xi)], \text{ for any } \underline{\mu} \in \underline{\Theta}_{t_0}. \tag{2.2}$$

Proof. Proof to 1).

Since ψ belongs to $C_{b,loc}([a, b] \times \mathbb{R})$, it is easy to prove that $\{\alpha(t)\}_{t \in [a, b]}$ is continuous based on the assumption **(H2)** on \mathcal{N} . By similar arguments, we can show that, as n goes to infinity,

$$|E_{\mu_n}[\psi(t_n, \xi)] - E_{\mu_n}[\psi(t_0, \xi)]| \leq \mathbb{E}[|\psi(t_n, \xi) - \psi(t_0, \xi)|] \rightarrow 0.$$

Therefore, noting that $\mu_n \xrightarrow{\text{weakly}} \mu$, we have

$$E_{\mu}[\psi(t_0, \xi)] = \lim_{n \rightarrow \infty} E_{\mu_n}[\psi(t_0, \xi)] = \lim_{n \rightarrow \infty} E_{\mu_n}[\psi(t_n, \xi)] = \lim_{n \rightarrow \infty} \alpha(t_n) = \alpha(t_0),$$

which means $\mu \in \Theta_{t_0}$.

Proof to 2).

By the definition of the function α we have, for any $\mu^\delta \in \Theta_{t_0+\delta}$,

$$\frac{\alpha(t_0 + \delta) - \alpha(t_0)}{\delta} = \frac{1}{\delta} \mathbb{E}[\psi(t_0 + \delta, \xi)] - \frac{1}{\delta} \mathbb{E}[\psi(t_0, \xi)] \quad (2.3)$$

$$= \frac{1}{\delta} E_{\mu^\delta}[\psi(t_0 + \delta, \xi)] - \frac{1}{\delta} \mathbb{E}[\psi(t_0, \xi)] \quad (2.4)$$

$$\leq \frac{1}{\delta} E_{\mu^\delta}[\psi(t_0 + \delta, \xi)] - \frac{1}{\delta} E_{\mu^\delta}[\psi(t_0, \xi)] \quad (2.5)$$

$$= E_{\mu^\delta}[\dot{\psi}_{t_0}(\xi)] + o(1). \quad (2.6)$$

The last equality follows from Assumption (A1). Let $\delta_n \downarrow 0$ be a sequence such that

$$\limsup_{\delta \downarrow 0} \frac{\alpha(t_0 + \delta) - \alpha(t_0)}{\delta} = \lim_{n \rightarrow \infty} \frac{\alpha(t_0 + \delta_n) - \alpha(t_0)}{\delta_n}. \quad (2.7)$$

Since Θ is weakly compact, there exists a subsequence, also denoted by δ_n , such that

$$\mu^{\delta_n} \xrightarrow{\text{weakly}} \bar{\mu} \in \Theta.$$

From 1) of this lemma, we know $\bar{\mu} \in \bar{\Theta}_{t_0}$. By (2.6) and Assumption (2), we have

$$\limsup_{\delta \downarrow 0} \frac{\alpha(t_0 + \delta) - \alpha(t_0)}{\delta} \leq \lim_{n \rightarrow \infty} E_{\mu^{\delta_n}}[\dot{\psi}_{t_0}(\xi)] = E_{\bar{\mu}}[\dot{\psi}_{t_0}(\xi)].$$

On the other hand, for any $\mu \in \Theta_{t_0}$ we get

$$\frac{\alpha(t_0 + \delta) - \alpha(t_0)}{\delta} = \frac{1}{\delta} \mathbb{E}[\psi(t_0 + \delta, \xi)] - \frac{1}{\delta} E_{\mu}[\psi(t_0, \xi)] \quad (2.8)$$

$$\geq \frac{1}{\delta} E_{\mu}[\psi(t_0 + \delta, \xi)] - \frac{1}{\delta} E_{\mu}[\psi(t_0, \xi)] \quad (2.9)$$

$$= E_{\mu}[\dot{\psi}_{t_0}(\xi)] + o(1). \quad (2.10)$$

The last equality follows from Assumption (A1).

Thus, by (2.10), we get

$$\liminf_{\delta \downarrow 0} \frac{\alpha(t_0 + \delta) - \alpha(t_0)}{\delta} \geq \sup_{\mu \in \Theta_{t_0}} E_{\mu}[\dot{\psi}_{t_0}(\xi)].$$

Combining the above arguments, we have

$$\lim_{\delta \downarrow 0} \frac{\alpha(t_0 + \delta) - \alpha(t_0)}{\delta} = \sup_{\mu \in \Theta_{t_0}} E_{\mu}[\dot{\psi}_{t_0}(\xi)] = E_{\bar{\mu}}[\dot{\psi}_{t_0}(\xi)].$$

Since $\lim_{\delta \downarrow 0} \frac{\alpha(t_0 + \delta, x) - \alpha(t_0, x)}{\delta}$ exists, the equality (2.7) holds for any sequence $\delta_n \downarrow 0$. Hence, the equality (2.1) holds for any $\bar{\mu} \in \bar{\Theta}_{t_0}$. (2.2) can be proved similarly. \square

Applying Lemma 2.2, we shall present the derivative formulas for two types of functions in the remainder of this section.

For a sublinear expectation \mathcal{N} on $C_{b,Lip}(\mathbb{R})$ and $\varphi \in C_{b,Lip}(\mathbb{R})$, set

$$u(t, x) := \mathcal{N}[\varphi(x + \sqrt{t}\cdot)] = \mathbb{E}[\varphi(x + \sqrt{t}\xi)], \quad (t, x) \in [0, \infty) \times \mathbb{R}.$$

Lemma 2.3. For a sublinear expectation \mathcal{N} on $C_{b,Lip}(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R})$, we have, for $t > 0$,

$$\partial_t^+ u(t, x) = \sup_{\mu \in \Theta_{t,x}} E_\mu \left[\frac{\xi}{2\sqrt{t}} \varphi'(x + \sqrt{t}\xi) \right], \tag{2.11}$$

$$\partial_t^- u(t, x) = \inf_{\mu \in \Theta_{t,x}} E_\mu \left[\frac{\xi}{2\sqrt{t}} \varphi'(x + \sqrt{t}\xi) \right],$$

$$\partial_x^+ u(t, x) = \sup_{\mu \in \Theta_{t,x}} E_\mu [\varphi'(x + \sqrt{t}\xi)], \tag{2.12}$$

$$\partial_x^- u(t, x) = \inf_{\mu \in \Theta_{t,x}} E_\mu [\varphi'(x + \sqrt{t}\xi)],$$

where $\Theta_{t,x} = \Theta_{\varphi(x+\sqrt{t}\cdot)}$. Furthermore, we have

$$\partial_t^+ u(t, x) = \lim_{\delta \downarrow 0} \partial_t^+ u(t + \delta, x) = \lim_{\delta \downarrow 0} \partial_t^- u(t + \delta, x), \tag{2.13}$$

$$\partial_t^- u(t, x) = \lim_{\delta \downarrow 0} \partial_t^+ u(t - \delta, x) = \lim_{\delta \downarrow 0} \partial_t^- u(t - \delta, x). \tag{2.14}$$

Similar relations hold for $\partial_x^+ u(t, x), \partial_x^- u(t, x)$.

Proof. We shall only give proof to (2.11) and (2.13). The other conclusions can be proved similarly.

Step 1. Proof to (2.11).

Set $\psi(t, x) = \varphi(x_0 + \sqrt{t}x)$. For $t_0 > 0$, choose $b > t_0 > a > 0$. Clearly, $\psi(t, x)$ belongs to $C_{b,loc}([a, b] \times \mathbb{R})$. It suffices to show that ψ satisfies (A1), (A2) at point t_0 with $\psi_{t_0}(x) = \partial_t \psi(t_0, x)$.

Step 1.1 ψ satisfies (A1) at point t_0 .

$\mathfrak{R}_{\delta;t_0,\xi}^\psi = \psi(t_0 + \delta, \xi) - \psi(t_0, \xi) - \partial_t \psi(t_0, \xi)\delta = \delta \int_0^1 [\partial_t \psi(t_0 + s\delta, \xi) - \partial_t \psi(t_0, \xi)] ds$. Now, we should prove that $\lim_{\delta \rightarrow 0} \mathbb{E} | \int_0^1 [\partial_t \psi(t_0 + s\delta, \xi) - \partial_t \psi(t_0, \xi)] ds | = 0$. Note that

$$\partial_t \psi(t_0 + s\delta, \xi) - \partial_t \psi(t_0, \xi) = \xi \left[\frac{\varphi'(x_0 + \sqrt{t_0 + s\delta}\xi)}{2\sqrt{t_0 + s\delta}} - \frac{\varphi'(x_0 + \sqrt{t_0}\xi)}{2\sqrt{t_0}} \right]$$

and $\frac{\varphi'(x_0 + \sqrt{t}x)}{2\sqrt{t}}$ belongs to $C_{b,loc}([a, b] \times \mathbb{R})$. For any $\varepsilon > 0$, by (H2), there exists $N > 0$ such that

$$\mathbb{E} \left| \left[\frac{\varphi'(x_0 + \sqrt{t_0 + s\delta}\xi)}{2\sqrt{t_0 + s\delta}} - \frac{\varphi'(x_0 + \sqrt{t_0}\xi)}{2\sqrt{t_0}} \right] \right| \mathbb{1}_{\{|\xi| > N\}} < \frac{\varepsilon}{2}.$$

For $|\xi| \leq N$ there exists $\delta_0 > 0$ such that for any $\delta < \delta_0$ we have

$$\left| \frac{\varphi'(x_0 + \sqrt{t_0 + s\delta}\xi)}{2\sqrt{t_0 + s\delta}} - \frac{\varphi'(x_0 + \sqrt{t_0}\xi)}{2\sqrt{t_0}} \right| |\xi| \mathbb{1}_{\{|\xi| \leq N\}} < \frac{\varepsilon}{2}.$$

Thus, we have

$$\mathbb{E} \left| \int_0^1 [\partial_t \psi(t_0 + s\delta, \xi) - \partial_t \psi(t_0, \xi)] ds \right| < \varepsilon.$$

Step 1.2 ψ satisfies (A2) at point t_0 .

Note that $\partial_t \psi(t_0, x) = \frac{\varphi'(x_0 + \sqrt{t_0}x)}{2\sqrt{t_0}}x$. By (H2) we know that ψ satisfies (A2) at point t_0 .

Step 2. Proof to (2.13).

By Step 1 and Lemma 2.2, there exists $\mu^\delta \in \Theta_{t+\delta, x}$ such that $\partial_t^+ u(t + \delta, x) = E_{\mu^\delta}[\frac{\xi}{2\sqrt{t+\delta}}\varphi'(x + \sqrt{t + \delta}\xi)]$. Noting that

$$E_{\mu^\delta}[\frac{\xi}{2\sqrt{t+\delta}}\varphi'(x + \sqrt{t + \delta}\xi)] - E_{\mu^\delta}[\frac{\xi}{2\sqrt{t}}\varphi'(x + \sqrt{t}\xi)] \rightarrow 0,$$

it suffices to prove that $E_{\mu^\delta}[\frac{\xi}{2\sqrt{t}}\varphi'(x + \sqrt{t}\xi)] \rightarrow \partial_t^+ u(t, x)$. Actually, for any subsequence of (μ^δ) , there is a sub-subsequence $(\mu^{\delta'})$ such that $\mu^{\delta'} \xrightarrow{weakly} \mu^* \in \Theta$. By Lemma 2.2, we have $\mu^* \in \Theta_{t, x}$ and $\partial_t^+ u(t, x) = E_{\mu^*}[\frac{\xi}{2\sqrt{t}}\varphi'(x + \sqrt{t}\xi)]$. Thus, we conclude that $E_{\mu^{\delta'}}[\frac{\xi}{2\sqrt{t}}\varphi'(x + \sqrt{t}\xi)] \rightarrow \partial_t^+ u(t, x)$. \square

For any $\varphi \in C_{b, Lip}(\mathbb{R})$, let $v(t, x)$ be the solution to the G -heat equation with an initial value φ . For a sublinear expectation \mathcal{N} on $C_{b, Lip}(\mathbb{R})$, set $w_{\mathcal{N}}(s) := \mathbb{E}[v(s, \sqrt{1-s}\xi)]$, $s \in [0, 1]$.

Lemma 2.4. For a sublinear expectation \mathcal{N} on $C_{b, Lip}(\mathbb{R})$, we have, for $t \in (0, 1)$

$$\partial_t^+ w_{\mathcal{N}}(t) = \sup_{\mu \in \Theta_{v(t, \sqrt{1-t}\cdot)}} E_{\mu}[\partial_t v(t, \sqrt{1-t}\xi) - \partial_x v(t, \sqrt{1-t}\xi)\frac{\xi}{2\sqrt{1-t}}], \tag{2.15}$$

$$\partial_t^- w_{\mathcal{N}}(t) = \inf_{\mu \in \Theta_{v(t, \sqrt{1-t}\cdot)}} E_{\mu}[\partial_t v(t, \sqrt{1-t}\xi) - \partial_x v(t, \sqrt{1-t}\xi)\frac{\xi}{2\sqrt{1-t}}]. \tag{2.16}$$

Proof. Set $\psi(t, x) = v(t, \sqrt{1-t}x)$. For $1 > t_0 > 0$, choose $1 > b > t_0 > a > 0$. By the regularity property of the G -heat equation (see [6]), we know that $v \in C_b^{1,2}([\kappa, 1])$ for any $\kappa > 0$. So it is easy to show that $\psi(t, x)$ belongs to $C_{b, loc}([a, b] \times \mathbb{R})$, and that ψ satisfies (A1), (A2) at point t_0 with

$$\dot{\psi}_{t_0}(x) = \partial_t \psi(t_0, x) = \partial_t v(t_0, \sqrt{1-t_0}x) - \partial_x v(t_0, \sqrt{1-t_0}x)\frac{x}{2\sqrt{1-t_0}}$$

by similar arguments as those in Lemma 2.3. Then by Lemma 2.2 we get the desired results. \square

3 Proof to Theorem 1.5

We shall prove Theorem 1.5 based mainly on the lemmas introduced in Section 2.

Proof. Necessity.

Assume that \mathcal{N} is G -normal. Then, for $\varphi \in C_b^2(\mathbb{R})$, $u(t, x) := \mathcal{N}[\varphi(x + \sqrt{t}\cdot)] = \mathbb{E}[\varphi(x + \sqrt{t}\xi)]$ is the solution to G -heat equation with initial value φ .

Step 1. For $\mu \in \Theta_\varphi$,

$$\partial_t u(1, 0) = E_{\mu}[\frac{\xi}{2}\varphi'(\xi)].$$

Actually, by Lemma 2.3, we have

$$\sup_{\mu \in \Theta_\varphi} E_{\mu}[\frac{\xi}{2}\varphi'(\xi)] = \partial_t^+ u(1, 0) = \partial_t u(1, 0) = \partial_t^- u(1, 0) = \inf_{\mu \in \Theta_\varphi} E_{\mu}[\frac{\xi}{2}\varphi'(\xi)].$$

Step 2. $\partial_t u(1, 0) = \sup_{\mu \in \Theta_\varphi} E_{\mu}[G(\varphi''(\xi))]$.

Note that $u(1 + \delta, 0) = \mathbb{E}[u(\delta, \xi)]$ and $u(\delta, x) = \mathbb{E}[\varphi(x + \sqrt{\delta}\xi)] = \varphi(x) + \delta G(\varphi''(x)) + o(\delta)$ uniformly with respect to x . Set $\psi(t, x) = u(t, x)$ and $\alpha(t) = \mathbb{E}[u(t, \xi)]$. Then ψ belongs to $C_{b, loc}([0, 1] \times \mathbb{R})$ and satisfies Assumptions (A1), (A2) at $t_0 = 0$ with $\psi_0(x) = G(\varphi''(x))$.

So, by Lemma 2.2, we have

$$\lim_{\delta \downarrow 0} \frac{u(1 + \delta, 0) - u(1, 0)}{\delta} = \partial_t^+ \alpha(0) = \sup_{\mu \in \Theta_\varphi} E_{\mu}[G(\varphi''(\xi))].$$

Sufficiency. Assume \mathcal{N} is a sublinear expectation on $C_{b,Lip}(\mathbb{R})$ satisfying Hypothesis (SHw). For any $\varphi \in C_{b,Lip}(\mathbb{R})$, let $v(t, x)$ be the solution to the G -heat equation with initial value φ . For $s \in [0, 1]$, set $w(s) := \mathbb{E}[v(s, \sqrt{1-s}\xi)]$. To prove the theorem, it suffices to show that $w(0) = w(1)$.

By (2.15) in Lemma 2.4 and Hypothesis (SHw), we get $\partial_s^+ w(s) = 0, s \in (0, 1)$. Noting that w is continuous on $[0, 1]$ and locally Lipschitz continuous on $(0, 1)$, we get $w(0) = w(1)$. \square

Corollary 3.1. Let $\mathcal{N}[\varphi] = \sup_{\mu \in \Theta} \mu[\varphi], \varphi \in C_{b,Lip}(\mathbb{R})$, be a sublinear expectation. Then \mathcal{N} is G -normal if for any $\varphi \in C_b^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}} [G(\varphi''(x)) - \frac{x}{2}\varphi'(x)]\mu^\varphi(dx) = 0, \tag{SH}$$

where μ^φ is a realization of φ associated with \mathcal{N} , i.e., $\mu^\varphi \in \Theta$ and $\mu^\varphi[\varphi] = \mathcal{N}[\varphi]$.

4 Some definitions and notations about G -expectation

We review some basic notions and definitions of the related spaces under G -expectation. The readers may refer to [9], [10], [11], [12] and [14] for more details.

Let $\Omega_T = C_0([0, T]; \mathbb{R}^d)$ be the space of all \mathbb{R}^d -valued continuous paths $\omega = (\omega(t))_{t \in [0, T]}$ with $\omega(0) = 0$ and let $B_t(\omega) = \omega(t)$ be the canonical process.

Let us recall the definitions of G -Brownian motion and its corresponding G -expectation introduced in [10]. Set

$$Lip(\Omega_T) := \{\varphi(\omega(t_1), \dots, \omega(t_n)) : t_1, \dots, t_n \in [0, T], \varphi \in C_{b,Lip}((\mathbb{R}^d)^n), n \in \mathbb{N}\},$$

where $C_{b,Lip}(\mathbb{R}^d)$ is the collection of bounded Lipschitz functions on \mathbb{R}^d .

We are given a function

$$G : \mathbb{S}_d \mapsto \mathbb{R}$$

satisfying the following monotonicity, sublinearity and positive homogeneity:

- A1.** $G(a) \geq G(b)$, if $a, b \in \mathbb{S}_d$ and $a \geq b$;
- A2.** $G(a + b) \leq G(a) + G(b)$, for each $a, b \in \mathbb{S}_d$;
- A3.** $G(\lambda a) = \lambda G(a)$ for $a \in \mathbb{S}_d$ and $\lambda \geq 0$.

Remark 4.1. When $d = 1$, we have $G(a) := \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$, for $0 \leq \underline{\sigma}^2 \leq \bar{\sigma}^2$.

For each $\xi(\omega) \in Lip(\Omega_T)$ of the form

$$\xi(\omega) = \varphi(\omega(t_1), \omega(t_2), \dots, \omega(t_n)), \quad 0 = t_0 < t_1 < \dots < t_n = T,$$

we define the following conditional G -expectation

$$\mathbb{E}_t[\xi] := u_k(t, \omega(t); \omega(t_1), \dots, \omega(t_{k-1}))$$

for each $t \in [t_{k-1}, t_k], k = 1, \dots, n$. Here, for each $k = 1, \dots, n, u_k = u_k(t, x; x_1, \dots, x_{k-1})$ is a function of (t, x) parameterized by $(x_1, \dots, x_{k-1}) \in (\mathbb{R}^d)^{k-1}$, which is the solution of the following PDE (G -heat equation) defined on $[t_{k-1}, t_k] \times \mathbb{R}^d$:

$$\partial_t u_k + G(\partial_x^2 u_k) = 0$$

with terminal conditions

$$u_k(t_k, x; x_1, \dots, x_{k-1}) = u_{k+1}(t_k, x; x_1, \dots, x_{k-1}, x), \text{ for } k < n$$

and $u_n(t_n, x; x_1, \dots, x_{n-1}) = \varphi(x_1, \dots, x_{n-1}, x)$.

The G -expectation of $\xi(\omega)$ is defined by $\mathbb{E}[\xi] = \mathbb{E}_0[\xi]$. From this construction we obtain a natural norm $\|\xi\|_{L_G^p} := \mathbb{E}[|\xi|^p]^{1/p}$, $p \geq 1$. The completion of $L_{ip}(\Omega_T)$ under $\|\cdot\|_{L_G^p}$ is a Banach space, denoted by $L_G^p(\Omega_T)$. The canonical process $B_t(\omega) := \omega(t)$, $t \geq 0$, is called a G -Brownian motion in this sublinear expectation space $(\Omega, L_G^1(\Omega), \mathbb{E})$.

Definition 4.2. A process $\{M_t\}$ with values in $L_G^1(\Omega_T)$ is called a G -martingale if $\mathbb{E}_s(M_t) = M_s$ for any $s \leq t$. If $\{M_t\}$ and $\{-M_t\}$ are both G -martingales, we call $\{M_t\}$ a symmetric G -martingale.

Theorem 4.3. ([3]) There exists a weakly compact subset $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$, the set of probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that

$$\mathbb{E}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \text{ for all } \xi \in L_{ip}(\Omega_T).$$

\mathcal{P} is called a set that represents \mathbb{E} .

Definition 4.4. A function $\eta(t, \omega) : [0, T] \times \Omega_T \rightarrow \mathbb{R}$ is called a step process if there exists a time partition $\{t_i\}_{i=0}^n$ with $0 = t_0 < t_1 < \dots < t_n = T$, such that for each $k = 0, 1, \dots, n - 1$ and $t \in (t_k, t_{k+1}]$

$$\eta(t, \omega) = \xi_{t_k} \in L_{ip}(\Omega_{t_k}).$$

We denote by $M^0(0, T)$ the collection of all step processes.

For a step process $\eta \in M^0(0, T)$, we set the norm

$$\|\eta\|_{H_G^p}^p := \mathbb{E}[\{\int_0^T |\eta_s|^2 ds\}^{p/2}], p \geq 1$$

and denote by $H_G^p(0, T)$ the completion of $M^0(0, T)$ with respect to the norms $\|\cdot\|_{H_G^p}$.

Theorem 4.5. ([15]) For $\xi \in L_G^\beta(\Omega_T)$ with some $\beta > 1$, $X_t = \mathbb{E}_t(\xi)$, $t \in [0, T]$ has the following decomposition:

$$X_t = X_0 + \int_0^t Z_s dB_s + K_t, \text{ q.s.},$$

where $\{Z_t\} \in H_G^1(0, T)$ and $\{K_t\}$ is a continuous non-increasing G -martingale. Furthermore, the above decomposition is unique and $\{Z_t\} \in H_G^\alpha(0, T)$, $K_T \in L_G^\alpha(\Omega_T)$ for any $1 \leq \alpha < \beta$.

5 Proof to Proposition 1.2

Let \mathcal{P} be a weakly compact set that represents \mathbb{E} . Then, the corresponding G -normal distribution can be represented as

$$\mathcal{N}_G[\varphi] = \sup_{P \in \mathcal{P}} E_P[\varphi(B_1)], \text{ for all } \varphi \in C_{b, Lip}(\mathbb{R}).$$

Clearly, \mathcal{N}_G satisfies condition (H2) and $\Theta := \{P \circ B_1^{-1} \mid P \in \mathcal{P}\}$ is weakly compact. Also, Proposition 1.2 can be restated in the following form.

Proposition 5.1. Let $\varphi \in C_b^2(\mathbb{R})$. For $P \in \mathcal{P}$ such that $E_P[\varphi(B_1)] = \mathbb{E}[\varphi(B_1)]$, we have

$$E_P\left[\frac{B_1}{2}\varphi'(B_1) - G(\varphi''(B_1))\right] = 0.$$

Proof. For $\varphi \in C_b^2(\mathbb{R})$, set $u(t, x) = \mathbb{E}[\varphi(x + B_t)]$. As a solution to the G -heat equation, we know $u \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R})$. Particularly, we have

$$\lim_{\delta \downarrow 0} \frac{u(1 + \delta, 0) - u(1, 0)}{\delta} = \lim_{\delta \downarrow 0} \frac{u(1 - \delta, 0) - u(1, 0)}{-\delta} = \partial_t u(1, 0).$$

Set $\mathcal{P}_\varphi = \{P \in \mathcal{P} : E_P[\varphi(B_1)] = \mathbb{E}[\varphi(B_1)]\}$. In the proof to Theorem 1.5, we have already proved that for $P \in \mathcal{P}_\varphi$, $\partial_t u(1, 0) = E_P[\frac{B_1}{2}\varphi'(B_1)]$ and that $\partial_t u(1, 0) = \sup_{P \in \mathcal{P}_\varphi} \times E_P[G(\varphi''(B_1))]$. We shall only prove $\partial_t u(1, 0) = \inf_{P \in \mathcal{P}_\varphi} E_P[G(\varphi''(B_1))]$.

By the G -martingale representation theorem, we have

$$\begin{aligned} u(1 - \delta, 0) &= \varphi(B_{1-\delta}) - \int_0^{1-\delta} z_s^\delta dB_s - K_{1-\delta}^\delta, \\ u(1, 0) &= \varphi(B_1) - \int_0^1 z_s dB_s - K_1, \end{aligned}$$

where $\{K_t^\delta\}, \{K_t\}$ are non-increasing G -martingales with $K_0^\delta = K_0 = 0$. Thus

$$\begin{aligned} u(1 - \delta, 0) - u(1, 0) &= \mathbb{E}[\varphi(B_{1-\delta}) - \varphi(B_1) + K_1] \\ &= \mathbb{E}[-\frac{1}{2}\varphi''(B_{1-\delta})(B_1 - B_{1-\delta})^2 + K_1] + o(\delta). \end{aligned}$$

For each $P \in \mathcal{P}_\varphi$,

$$\begin{aligned} \frac{u(1-\delta,0)-u(1,0)}{-\delta} &= \frac{1}{-2\delta} \mathbb{E}[-\varphi''(B_{1-\delta})(B_1 - B_{1-\delta})^2 + K_1] + o(1) \\ &\leq \frac{1}{2\delta} E_P[\varphi''(B_{1-\delta})(B_1 - B_{1-\delta})^2] + o(1) \\ &\leq E_P[G(\varphi''(B_{1-\delta}))] + o(1). \end{aligned}$$

Thus

$$\sup_{P \in \mathcal{P}_\varphi} E_P[G(\varphi''(B_1))] = \partial_t u(1, 0) \leq \inf_{P \in \mathcal{P}_\varphi} E_P[G(\varphi''(B_1))].$$

Consequently, for $P \in \mathcal{P}_\varphi$, we have $\partial_t u(1, 0) = E_P[G(\varphi''(B_1))]$. □

Remark 5.2. In [4], the authors used a similar idea to obtain the variation equation for the cost functional associated with the stochastic recursive optimal control problem.

Corollary 5.3. Let $H \in C^2(\mathbb{R})$ with polynomial growth satisfy, for some $\rho > 0$,

$$\frac{x}{2}H'(x) - G(H''(x)) = \rho H(x).$$

Then we have

$$\mathbb{E}[H(B_1)] = 0.$$

The proof is immediate from Proposition 1.2. Actually, for $P \in \mathcal{P}$ such that $E_P[H(B_1)] = \mathbb{E}[H(B_1)]$, we have

$$\rho \mathbb{E}[H(B_1)] = \rho E_P[H(B_1)] = E_P[\frac{B_1}{2}H'(B_1) - G(H''(B_1))] = 0.$$

Below we give a direct proof.

Proof. Let $X_t^x = e^{-\frac{t}{2}}x + \int_0^t e^{-\frac{1}{2}(t-s)}dB_s$. Applying Itô's formula to $e^{\rho t}H(X_t^x)$, we have

$$\begin{aligned} e^{\rho t}H(X_t^x) &= H(x) + \int_0^t e^{\rho s}(\rho H(X_s^x) - \frac{1}{2}X_s^x H'(X_s^x) + G(H''(X_s^x)))ds \\ &\quad + \int_0^t e^{\rho s}H'(X_s^x)dB_s + \frac{1}{2} \int_0^t e^{\rho s}H''(X_s^x)d\langle B \rangle_s - \int_0^t e^{\rho s}G(H''(X_s^x))ds. \end{aligned}$$

So $\mathbb{E}[H(X_t^x)] = e^{-\rho t}H(x)$ and

$$\mathbb{E}[H(B_1)] = \lim_{t \rightarrow \infty} \mathbb{E}[H(X_t^x)] = 0. \quad \square$$

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