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Inverse Scattering Problem for the Schrödinger Equation with Magnetic Potential at a Fixed Energy

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Abstract: In this article we consider the Schrödinger operator in $R^n, n \geqq 3$, with electric and magnetic potentials which decay exponentially as $|x| \to \infty$. We show that the scattering amplitude at fixed positive energy determines the electric potential and the magnetic field.

1. Introduction

Consider the Schrödinger equation in R^n, $n \geqq 3$, with magnetic potential $A(x) = (A_1(x), \dots, A_n(x))$ and electric potential $V(x)$:

$$-\sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} + iA_j(x) \right)^2 u + V(x)u = k^2 u , \tag{1}$$

$k > 0$, or equivalently

$$-\Delta u - 2i \sum_{j=1}^{n} A_j(x) \frac{\partial u}{\partial x_j} + q(x)u = k^2 u , \tag{1'}$$

where

$$q(x) = \sum_{j=1}^{n} \left(A_j^2(x) - i\frac{\partial A_j}{\partial x_j} \right) + V(x) . \tag{2}$$

We will assume that the potentials A and V are real-valued and exponentially decreasing, i.e.

$$\left| \frac{\partial^\alpha V(x)}{\partial x^\alpha} \right| \leqq C_\alpha e^{-\delta|x|}, \qquad \left| \frac{\partial^\beta A_j}{\partial x^\beta} \right| \leqq C_\beta e^{-\delta|x|}, \quad j = 1, \dots, n , \tag{3}$$

for $0 \leqq |\alpha| \leqq P, 0 \leqq |\beta| \leqq P + 1$, where $P = n + 4$. We consider the solutions of (1) of the form

$$u = e^{ik\omega \cdot x} + v(x, \omega, k) , \tag{4}$$

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where v is the outgoing solution of

$$-\Delta v - 2i\sum_{j=1}^{n} A_j(x)\frac{\partial v}{\partial x_j} + (q(x) - k^2)v = e^{ik\omega\cdot x}\left(-2k\sum_{j=1}^{n}\omega_j A_j(x) - q(x)\right) \tag{5}$$

obtained by the limiting absorption method. By this argument v exists and is unique whenever k^2 is not an embedded eigenvalue, and, combining Sect. 5 of Hörmander [4] with the proof of Theorem 3.3 of Agmon [1], one sees that (3) implies there are no embedded eigenvalues. Representing v in terms of the outgoing fundamental solution of $\Delta + k^2$, it follows that as $|x| \to \infty$,

$$v(x,\omega,k) = \frac{e^{ik|x|}}{|x|^{\frac{n-1}{2}}}\left(a\left(\frac{x}{|x|},\omega,k\right) + O\left(\frac{1}{|x|}\right)\right), \tag{6}$$

where $a(\theta,\omega,k)$ is defined to be the scattering amplitude. Our objective is to prove

Theorem 1. *Fix $k > 0$. Then one can recover $V(x)$ and the magnetic field $B =$ curl A from the scattering amplitude $a(\theta,\omega,k), (\theta,\omega) \in S^{n-1} \times S^{n-1}$.*

Note that, if A and A' satisfy (3) and *curl* A = *curl* A', then $A' - A$ is the gradient of function φ satisfying

$$\left|\frac{\partial^p \varphi}{\partial x^p}\right| \leqq C_p e^{-\delta|x|}, \quad 0 \leqq |p| \leqq P. \tag{7}$$

To see that changing A to $A' = A + \frac{\partial\varphi}{\partial x}$ does not change the scattering amplitude note that, if one replaces $u(x)$ by $w(x) = u(x)e^{-i\varphi(x)}$, then $w(x)$ will satisfy

$$-\left(\frac{\partial}{\partial x} + iA(x) + i\frac{\partial\varphi}{\partial x}\right)^2 w + V(x)w = k^2 w.$$

However, this does not change the scattering amplitude, since

$$w = u(x)e^{-i\varphi(x)} = e^{-i\varphi(x)}\left(e^{ik\omega\cdot x} + a\left(\frac{x}{|x|},\omega,k\right)\frac{e^{ik|x|}}{|x|^{\frac{n-1}{2}}} + O\left(\frac{1}{|x|^{\frac{n+1}{2}}}\right)\right)$$
$$= e^{ik\omega\cdot x} + a\left(\frac{x}{|x|},\omega,k\right)\frac{e^{ik|x|}}{|x|^{\frac{n-1}{2}}} + O\left(\frac{1}{|x|^{\frac{n+1}{2}}}\right).$$

In this article as in [2] we will use $h(\xi,k\omega,k)$, the Fourier transform of $-(\Delta + k^2)v$, to study the scattering amplitude. Since v is obtained by limiting absorption,

$$v(x,\omega,k) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{h(\xi,k\omega,k)e^{ix\cdot\xi}}{|\xi|^2 - k^2 - i0}d\xi, \tag{8}$$

and, taking the asymptotics of (8) when $\theta = x/|x|$ is fixed and $|x| \to \infty$, one obtains

$$a(\theta,\omega,k) = C_{n,k}h(k\theta,k\omega,k), C_{n,k} = \frac{1}{4\pi}\left(\left(\frac{k}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{i\pi}{4}}\right)^{n-3}. \tag{9}$$

From (5) one sees that h satisfies

$$h(\xi,\zeta,k)+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{q_0(\xi-\eta,\eta)h(\eta,\zeta,k)}{|\eta|^2-k^2-i0}d\eta=-q_0(\xi-\zeta,\zeta)\,, \tag{10}$$

where

$$q_0(\xi,\zeta)=2\sum_{j=1}^{n}\hat{A}_j(\xi)\zeta_j+\hat{q}(\xi)\,. \tag{11}$$

Note that (3) implies that $q_0(\xi-\zeta,\zeta)$ is analytic in (ξ,ζ) for $|\mathrm{Im}\ \xi|<\delta/2$, $|\mathrm{Im}\ \zeta|<\delta/2$. For fixed λ, the integral operator

$$T_\lambda w=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{q_0(\xi-\eta,\eta)w(\eta)}{|\eta|^2-\lambda-i0}d\eta \tag{12}$$

is compact in the space $H_{\alpha,N}, 0<\alpha<1, n-1<N<n+4$. Here $H_{\alpha,N}$ is the weighted Hölder space used in [2]: let $\|f\|_{\alpha,N}=\|(1+|\xi|^2)^{N/2}f\|_\alpha$, where $\|\ \|_\alpha$. is the standard Hölder norm, and define $H_{\alpha,N}$ as the completion of $C_0^\infty(R^n)$ is $\|\ \|_{\alpha,N}$. Moreover, T_λ depends analytically on λ for Im $\lambda>0$ and extends continuously to the positive real axis, $\lambda>0$. In the same way that Theorem 5.2 of [4] showed that the homogeneous equation corresponding to (5) had no nontrivial square-integrable solutions, it can be used here to show the $I+T_{k^2}$ has no nontrivial solutions in $H_{\alpha,N}(R^n)$. Hence we see that the Fredholm operator $I+T_{k^2}$ is invertible on $H_{\alpha,N}$ for $k>0$. This will be useful in what follows.

In the case that the magnetic field B is small uniqueness results at fixed energy have been obtained previously by Henkin and Novikov [6] and by Sun [9]. Recently Nakamura, Sun and Uhlmann [5] obtained the uniqueness result analogous to Theorem 1 for the Dirichlet to Neumann map. This implies Theorem 1 for magnetic and electric potentials of compact support. In fact, when the magnetic and electric potentials have compact support, as in [9], uniqueness for inverse scattering at fixed energy and uniqueness for the Dirichlet-to-Neumann map inverse problem at fixed energy are equivalent.

For potentials without compact support the previous work which influenced us considerably was by Novikov [8]. He proved Theorem 1 in the case of zero magnetic potential, and the methods of [8] could be used to give a different proof of some of the results in Sect. 2.

Finally, we are deeply indebted to Adrian Nachman for calling our attention to a serious error in the first version of Sect. 2.

2. Faddeev-Type Scattering Amplitudes

Following Faddeev [3] and Novikov–Khenkin [6], we introduce a new scattering amplitude which will contain a large parameter. The later will be helpful in solving the inverse scattering problem.

Let v be an arbitrary unit vector, $|v|=1$, and $E_{v,\sigma}(x)$ be the following fundamental solution to the equation $(-\Delta-k^2)u=f$:

$$E_{v,\sigma}(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{e^{ix\cdot\eta}d\eta}{\eta\cdot\eta-k^2+i0(\eta_v-\sigma)}\,, \tag{13}$$

where $\eta_\nu = \eta \cdot \nu$ and $-k < \sigma < k$. Comparing $E_{\nu,\sigma}(x)$ with the fundamental solution

$$E_0(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{ix \cdot \eta}}{\eta \cdot \eta - k^2 - i0} d\eta , \tag{14}$$

we have

$$E_{\nu,\sigma}(x) = E_0(x) - \frac{i\pi k^{n-2}}{(2\pi)^n} \int_{k\omega\cdot\nu>\sigma} e^{ix \cdot k\omega} d\omega , \tag{15}$$

where $d\omega$ is the area element of the unit sphere in R^n. Analogously to (10) consider the following integral equation

$$h_{\nu,\sigma}(\xi,\zeta,k) + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{q_0(\xi-\eta,\eta) h_{\nu,\sigma}(\eta,\zeta,k)}{\eta \cdot \eta - k^2 + i0(\eta_\nu - \sigma)} d\eta = -q_0(\xi-\zeta,\zeta) . \tag{16}$$

Set

$$v_{\nu,\sigma}(x,\zeta,k) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{h_{\nu,\sigma}(\xi,\zeta,k) e^{ix \cdot \xi}}{\xi \cdot \xi - k^2 + i0(\xi_\nu - \sigma)} d\xi , \tag{17}$$

assuming that $h_{\nu,\sigma}(\xi,\zeta,k)$ is the solution of (16). Then $v_{\nu,\sigma}(x,\zeta,k)$ is a solution of the differential equation (5) for $\zeta = k\omega$ with asymptotics at infinity that can be obtained by applying the stationary phase method to (17).

Now we shall find the relation between $h_{\nu,\sigma}(\xi,\zeta,k)$ and $h(\xi,\zeta,k)$. Analogously to (15) we have

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{q_0(\xi-\eta,\eta) h_{\nu,\sigma}(\eta,\zeta,k)}{\eta \cdot \eta - k^2 + i0(\eta_\nu - \sigma)} d\eta = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{q_0(\xi-\eta,\eta) h_{\nu,\sigma}(\eta,\zeta,k)}{\eta \cdot \eta + k^2 - i0} d\eta$$
$$- \frac{i\pi k^{n-2}}{(2\pi)^n} \int_{k\omega \cdot \nu>\sigma} q_0(\xi - k\omega, k\omega) h_{\nu,\sigma}(k\omega,\zeta,k) d\omega . \tag{18}$$

It follows from (16) and (18) that

$$h_{\nu,\sigma}(\xi,\zeta,k) + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{q_0(\xi-\eta,\eta) h_{\nu,\sigma}(\eta,\zeta,k)}{\eta \cdot \eta - k^2 - i0} d\eta$$
$$= -\frac{i\pi k^{n-2}}{(2\pi)^n} \int_{k\omega \cdot \nu>\sigma} q_0(\xi - k\omega, k\omega) h_{\nu,\sigma}(k\omega,\zeta,k) d\omega - q_0(\xi-\zeta,\zeta) . \tag{19}$$

Set

$$A(q_0)w = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{q_0(\xi-\eta,\eta) w(\eta)}{\eta \cdot \eta - k^2 - i0} d\eta , \tag{20}$$

and

$$A(h)w = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{h(\xi,\eta,k) w(\eta)}{\eta \cdot \eta - k^2 - i0} d\eta . \tag{21}$$

That (10) has a unique solution is equivalent (cf. [2]) to the equality

$$(I + A(q_0))(I + A(h)) = I . \tag{22}$$

Since $I + A(q_0)$ has an inverse, it follows from (22) that

$$(I + A(h))(I + A(q_0)) = I \tag{23}$$

or equivalently

$$h(\xi,\zeta,k) + q_0(\xi-\zeta,\zeta) + \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{h(\xi,\eta,k)q_0(\eta-\zeta,\zeta)}{\eta\cdot\eta-k^2-i0}d\eta = 0\,. \tag{23$'$}$$

Applying $I + A(h)$ to (19) and using (23) and (23$'$), we obtain (cf. [3] and [6], formula (1.7)):

$$h_{\nu,\sigma}(\xi,\zeta,k) = h(\xi,\zeta,k) - \frac{i\pi k^{n-2}}{(2\pi)^n}\int_{k\omega\,\cdot\,\nu>\sigma} h(\xi,k\omega,k)h_{\nu,\sigma}(k\omega,\zeta,k)d\omega\,. \tag{24}$$

Since $I + A(q_0)$ is invertible, Eq. (24) has a unique solution for any $h(\xi,\zeta,k)$ if and only if Eq. (16) has a unique solution. Indeed, if $\varphi(\xi)$ is a solution of the homogeneous equation corresponding to (16), i.e.

$$\varphi(\xi) + (2\pi)^{-n}\int_{\mathbb{R}^n}\frac{q_0(\xi-\eta,\eta)\varphi(\eta)}{\eta\cdot\eta-k^2+i0(\eta_\nu-\sigma)}d\eta = 0\,, \tag{25}$$

then from (25) and (18) with h_ν replaced with φ we conclude that

$$\varphi(\xi) + (2\pi)^{-n}\int_{\mathbb{R}^n}\frac{q_0(\xi-\eta,\eta)\varphi(\eta)d\eta}{\eta\cdot\eta-k^2-i0} = \frac{i\pi k^{n-2}}{(2\pi)^n}\int_{k\omega\,\cdot\,\nu>\sigma} q_0(\xi-k\omega,k\omega)\varphi(k\omega)d\omega\,.$$

Applying $(I + A(h))$ to both sides of this, we have

$$0 = \varphi(\xi) + \frac{i\pi k^{n-2}}{(2\pi)^n}\int_{k\omega\,\cdot\,\nu>\sigma} h(\xi,k\omega,k)\varphi(k\omega)d\omega\,, \tag{26}$$

i.e. φ restricted to $|\xi| = k$ solves the homogeneous equation corresponding to (24). Conversely, suppose $\varphi(\xi)$ is a nonzero solution of the preceding equation (26) on the sphere of radius k. Then (26) extends φ to R^n, since $h(\xi,k\omega,k)$ is defined for $\xi \in R^n$. Applying $I + A(q_0)$ to both sides of (26), we see that φ satisfies (25).

Denote by $E_\nu(x,z)$ the following function:

$$E_\nu(x,z) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{e^{ix\,\cdot\,\eta}d\eta}{(\eta+z\nu)\cdot(\eta+z\nu)-k^2},\qquad \operatorname{Im} z > 0\,.$$

Note that $E_\nu(x,z)$ is a fundamental solution for $(-i\frac{\partial}{\partial x}+z\nu)\cdot(-i\frac{\partial}{\partial x}+z\nu)-k^2$, i.e.

$$\left[\left(-i\frac{\partial}{\partial x}+z\nu\right)\cdot\left(-i\frac{\partial}{\partial x}+z\nu\right)-k^2\right]E_\nu(x,z) = \delta(x)\,.$$

Note that the distribution $[(\eta+z\nu)\cdot(\eta+z\nu)-k^2]^{-1}$ is not analytically dependent on z for $\operatorname{Im} z > 0$. This gives rise to the $\bar{\partial}$-equation in inverse scattering (see, for example [6]).

Denote by $h_\nu(\xi,\zeta,k,z)$ the solution of the following integral equation:

$$h_\nu(\xi,\zeta,k,z)+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{q_0(\xi-\eta,\eta+z\nu)h_\nu(\nu,\zeta,k,z)}{(\eta+z\nu)\cdot(\eta+z\nu)-k^2}d\eta$$
$$=-q_0(\xi-\zeta,\zeta+z\nu),\quad z=i\tau,\quad \tau>0\,. \tag{27}$$

Let $T_{i\tau}^{(1)}$ denote the operator

$$[T_{i\tau}^{(1)}f](\xi)=(2\pi)^{-n}\int_{\mathbb{R}^n}\frac{q_0(\xi-\eta,\eta+i\tau\nu)f(\eta)d\eta}{(\eta+i\tau\nu)\cdot(\eta+i\tau\nu)-k^2}\,. \tag{28}$$

Then (27) can be written

$$[(I+T_{i\tau}^{(1)})h_\nu](\xi)=-q_0(\xi-\zeta,\zeta+i\tau\nu)$$

and

$$h_\nu(\xi,\zeta,k,i\tau)=-[(I+T_{i\tau})^{-1}q_0(\,\cdot\,-\zeta,\zeta+i\tau\nu)](\xi)\,,$$

provided $(I+T_{i\tau}^{(1)})^{-1}$ exists. The analyticity of h_ν in τ will be important for us. Thus we need to study the analyticity of $T_{i\tau}^{(1)}f$ in τ when $f(\eta)$ is analytic in a strip $|\mathrm{Im}\ \eta|<\varepsilon$. We will use coordinates $\eta_\nu=\eta\cdot\nu$, $\eta'=\eta-\eta_\nu\nu$, $r=|\eta'|$ and $\omega'=\eta'/|\eta'|$. For η real and $\tau=\mu+i\sigma$,

$$\mathrm{Im}((\eta+i\tau\nu)\cdot(\eta+i\tau\nu)-k^2)=2\mu\eta_\nu-2\mu\sigma\,.$$

Hence, for $|\eta_\nu|>\varepsilon_1$, Re $\tau>0$ and $|\mathrm{Im}\,\tau|<\varepsilon_1/2$ the denominator in the integral defining $T_{i\tau}^{(1)}$ does not vanish. Thus, choosing $\chi\in C_0^\infty(R)$ such that $\chi(s)$ is supported in $|s|<2\varepsilon_1$ and $1-\chi(s)$ is supported in $|s|>\varepsilon_1$, we have

$$[T_{i\tau}^{(1)}f](\xi)=(2\pi)^{-n}\int_{\mathbb{R}^n}\frac{\chi(\eta_\nu)q_0(\xi-\eta,\eta+i\tau\nu)f(\eta)d\eta}{(\eta+i\tau\nu)\cdot(\eta+i\tau\nu)-k^2}$$
$$+(2\pi)^{-n}\int_{\mathbb{R}^n}\frac{(1-\chi(\eta_\nu))q_0(\xi-\eta,\eta+i\tau\nu)f(\eta)d\eta}{(\eta+i\tau\nu)\cdot(\eta+i\tau\nu)-k^2}$$
$$\equiv[V_\tau^{(1)}f](\xi)+[V_\tau^{(2)}f](\xi)\,,$$

where $[V_\tau^{(2)}f](\xi)$ is analytic in (ξ,τ) in the set $|\mathrm{Im}\,\xi|<\delta$, Re $\tau>0$ and $|\mathrm{Im}\,\tau|<\varepsilon_1/2$.

In our coordinates we have

$$(\eta+i\tau\nu)\cdot(\eta+i\tau\nu)-k^2=(r-\sqrt{B})(r+\sqrt{B})\,,$$

where $B=k^2+(\tau-i\eta_\nu)^2$. Using $\tau=\mu+i\sigma$ again, we have Re $B=k^2+\mu^2-\sigma^2+2\sigma\eta_\nu-\eta_\nu^2$, and Im $B=2\mu\sigma-2\mu\eta_\nu$. Hence for $k^2>8\varepsilon_1^2$, Re $B>k^2/8$ for $|\eta_\nu|<2\varepsilon_1$ and $|\mathrm{Im}\ \tau|<\varepsilon_1/2$, and we fix $\sqrt{B}$ as the square root in the right half plane. We wish to define $V_\tau^{(1)}$, and hence $T_{i\tau}^{(1)}$, by analytic continuation from $\tau>0$. When $\tau>0$, i.e. when $\mu>0$ and $\sigma=0$, $r-\sqrt{B}\neq 0$ for $\eta_\nu\neq 0$, and we have sgn (Im B) $=-$sgn η_ν. Therefore, we will deform the integration in r in

$$[V_\tau^{(1)}f](\xi)=\int_{S^{n-2}}d\omega'\int_{\mathbb{R}}d\eta_\nu\left(\int_0^\infty\frac{\chi(\eta_\nu)q_0(\xi-\eta,\eta+i\tau\nu)f(\eta)r^{n-2}}{(r-\sqrt{B})(r+\sqrt{B})}dr\right)$$

into the upper half plane for $\eta_\nu > 0$ and into the lower half plane for $\eta_\nu < 0$. We need to deform $[0,\infty)$ far enough that $r - \sqrt{B}$ will not vanish on the new contour for τ in a complex neighborhood of $[0,\tau_0]$. Note that for $\tau = \mu + i\sigma$,

$$\sqrt{B} = \sqrt{\mu^2 + k^2 + 2i(\sigma - \eta_\nu)\mu - (\sigma - \eta_\nu)^2}$$

$$= \sqrt{\mu^2 + k^2} + i(\sigma - \eta_\nu)\frac{\mu}{\sqrt{\mu^2 + k^2}} + O((\sigma - \eta_\nu)^2)\,.$$

Hence, for $|\sigma| < \varepsilon_1/2$ and $|\eta_\nu| < 2\varepsilon_1$, we have $|\mathrm{Re}(\sqrt{B} - \sqrt{\mu^2 + k^2})| < C\varepsilon_1^2$ and $|\mathrm{Im}\,\sqrt{B}| < 5\varepsilon_1/2 + C\varepsilon_1^2$. We now fix $\varepsilon_1 > 0$ such that $C\varepsilon_1^2 < k/3, 5\varepsilon_1/2 + C\varepsilon_1^2 < \varepsilon/2$ and $8\varepsilon_1^2 < k^2$. Then we may deform the r integration in $V_\tau^{(1)}f$ to the piecewise linear curve Γ from 0 to $k/2$ to $k/2 + i\varepsilon/2$ sgn η_ν to $\sqrt{k^2 + \tau_0^2} + k/2 + i\varepsilon/2$ sgn η_ν to $\sqrt{k^2 + \tau_0} + k/2$ to ∞. With this choice of $\Gamma, r - \sqrt{B}$ will not vanish on Γ for $|\eta_\nu| < 2\varepsilon_1, |\sigma| < \varepsilon_1/2$ and $0 \leqq \mu \leqq \tau_0$. Thus we have proven:

Lemma 1. *If $f(\eta)$ is analytic in $|\mathrm{Im}\,\eta| < \varepsilon$, satisfying $|f(\eta)| \leqq C(1 + |\eta|)^{-n-1}$ for $|\mathrm{Im}\,\eta| < \varepsilon$, then $[T_{i\tau}^{(1)}f](\xi)$ has an analytic extension from $\tau > 0$ to the half strip $\{(\xi,\tau) : |\mathrm{Im}\,\xi| < \delta - \varepsilon,\ \mathrm{Re}\,\tau > 0,\ |\mathrm{Im}\,\tau| < \varepsilon_1/2\}$.*

Let $A_{N,r}$ denote the space of functions $f(\eta)$, analytic on $S_r = \{\eta \in C^n : |\mathrm{Im}\,\eta| < r\}$ and continuous on $\overline{S}_r$, which satisfy

$$|f(\eta)| \leqq C(1 + |\eta|)^{-N}$$

on S_r. $A_{N,r}$ is a Banach space in the norm

$$\|f\|_{N,r} = \sup_{S_r}(1 + |\eta|)^N |f(\eta)|\,.$$

Proposition 1. *For ε_1 sufficiently small $T_{i\tau}^{(1)}$ is a family of compact operators on $A_{n+1,\delta/3}$, depending continuously on τ in the closed half strip $D = \{\tau = \mu + i\sigma : \mu \geqq 0,\ |\sigma| \leqq \varepsilon_1/2\}$ and analytically on τ in $\mathring{D}$, the interior of D.*

Remark 1. The choice $N = n + 1$ is made simply to make the Banach spaces used here compatible with those used in Sect. 3. The δ here is from (3).

Proof. For $\tau \in \mathring{D}$, $T_{i\tau}^{(1)}f = V_\tau^{(1)}f + V_\tau^{(2)}f$ by definition. Since $r^2 + (\eta_\nu + i\tau)^2 - k^2$ does not vanish for $r \in \Gamma$ and $\tau \in D$, the operator $V_\tau^{(1)}$ satisfies

$$|[V_\tau^{(1)}f](\xi)| \leqq C_\tau \int_{S^{n-2}} d\omega' \int_{\mathbb{R}} d\eta_\nu \int_\Gamma \frac{|q_0(\xi - \eta, \eta + i\tau)||f(\eta)||r^{n-2}||dr|}{(1 + |\eta|)^2}\,, \tag{29}$$

where the constant C_τ is uniformly bounded on compact subsets of D. By hypothesis (3) for any $\delta' < \delta$,

$$|q_0(\xi - \eta, \eta + i\tau\nu)| \leqq C_{\tau,\delta'}(1 + |\xi - \eta|)^{-n-4}(1 + |\eta|) \tag{30}$$

for $\xi \in S_{\delta'-\varepsilon}$ and $\eta \in S_\varepsilon$, where again $C_{\tau,\delta'}$ is uniformly bounded on compact subsets of D. Since $|f(\eta)| \leqq (1+|\eta|)^{-n-1}\|f\|_{n+1,\varepsilon}$ on S_ε, the integrand in (29) is bounded by

$$C_{\tau,\delta'}\frac{|r^{n-2}|}{(1+|\xi-\eta|)^{n+2}(1+|\eta|)^{n+2}}.$$

Since for any $p>0$,

$$(1+|\xi|)^p(1+|\xi-\eta|)^{-p}(1+|\eta|)^{-p} \leqq C((1+|\xi-\eta|)^{-p}+(1+|\eta|)^{-p}),$$

we conclude

$$(1+|\xi|)^{n+2}|[V_\tau^{(1)}f](\xi)| \leqq C\|f\|_{n+1,\varepsilon}. \tag{31}$$

Taking $\varepsilon=\delta/3$ and $\delta'=5\delta/6$, we have $[V_\tau^{(1)}f](\xi)$ analytic in $S_{\delta/2}$. Thus for $\tau\in D$, $V_\tau^{(1)}$ maps $A_{n+1,\delta/3}$ into $A_{n+2,\delta/2}$ with norm uniformly bounded on compact subsets of D. Hence $V_\tau^{(1)}$ is compact for $\tau \in D$.

In proving Lemma 1 we showed that for $f\in A_{n+1,\delta/3}$, $[V_\tau^{(1)}f](\xi)$ was analytic in (ξ,τ) for $\tau\in D$ and $\xi\in S_{\delta/2}$. Since the norm of $V_\tau^{(1)}$ as an operator on $A_{n+1,\delta/3}$ is uniformly bounded on compact subsets it follows by Cauchy's formula that $V_\tau^{(1)}$ is an analytic family of operators for $\tau\in D$.

For $\tau\in \mathring{D}$ the preceding arguments apply equally well to $V_\tau^{(2)}$, and we may conclude that $T_{\iota\tau}^{(1)}$ is an analytic family of compact operators in $\mathring{D}$. However, since

$$\begin{aligned}[V_{\mu+i\sigma}^{(2)}f](\xi) &= (2\pi)^{-n}\int_{\mathbb{R}^n}\frac{(1-\chi(\eta_\nu))q_0(\xi-\eta,\eta-\sigma\nu+i\mu\nu)f(\eta)}{|\eta-\sigma\nu|^2-k^2-\mu^2+2i\mu(\eta_\nu-\sigma)}d\eta\\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\frac{(1-\chi(\eta_\nu+\sigma))q_0(\xi-\eta-\sigma\nu,\eta+i\mu\nu)f(\eta+\sigma\nu)}{|\eta|^2-k^2-\mu^2+2i\mu\eta_\nu}d\eta,\end{aligned}$$

we need to show that $V_{\mu+i\sigma}^{(2)}$ extends continuously to $\mu=0$ from $\mu>0$. Since η_ν does not vanish on the support of $(1-\chi(\eta_\nu+\sigma))$ for $|\sigma|<\varepsilon_1/2$, we can again deform the integration in r into $\operatorname{Im} r>0$ for $\eta_\nu>0$ and into $\operatorname{Im} r<0$ for $\eta_\nu<0$, using the piecewise linear contour Γ' connecting 0 to $\varepsilon/2+i\varepsilon/2 \operatorname{sgn}\eta_\nu$ to $3k/2+i\varepsilon/2 \operatorname{sgn}\eta_\nu$ to $3k/2$ to ∞. Then for $r\in\Gamma'$ and $0\leqq\mu\leqq\varepsilon_{1/2}$,

$$\begin{aligned}|\eta\cdot\eta-k^2-\mu^2+2i\mu\eta_\nu|^{-1} &= |r^2+\eta_\nu^2-k^2-\mu^2+2i\mu\eta_\nu|^{-1}\\ &\leqq C_{k,\varepsilon/2}(|r|^2+|\eta_\nu-(\operatorname{sgn}\eta_\nu)k|)^{-1},\end{aligned}$$

because $r=(1+i\operatorname{sgn}\eta_\nu)t$ on the first segment of Γ' and $r^2=2i(\operatorname{sgn}\eta_\nu)t^2$. Since $(|r|^2+|\eta_\nu-(\operatorname{sgn}\eta_\nu)k|)^{-1}$ is locally integrable with respect to $|r|^{n-2}d|r|d\eta_\nu$, we may argue as follows. Removing small disks about $(r,\eta_\nu)=(0,\pm k)$ in the integral defining $V_{\mu+i\sigma}^{(2)}f$, we get an operator to which our previous arguments apply. Since this operator differs in norm from $V_{\mu+\iota\sigma}^{(2)}$ by an amount which goes to zero with the radius of disks, uniformly for $0\leqq\mu\leqq\varepsilon_1/2$, we conclude that $V_{\mu+i\sigma}^{(2)}$ extends continuously to a compact operator on $\mu=0$. $\square$

In Sect. 3 we will show that $I + T_{i\tau}^{(1)}$ is invertible on $H_{0,n+1}$ for $\tau \gg 0$. This implies immediately that it is invertible on $A_{n+1,\delta/3}$, since the null space of $I + T_{i\tau}^{(1)}$ on $A_{n+1,\delta/3}$ is a subspace of its nullspace on $H_{0,n+1}$. Therefore, by Proposition 1 the set Z where $I + T_{i\tau}^{(1)}$ is not invertible is discrete in $\overset{\circ}{D}$ and closed of measure zero in $D \cap \{\mathrm{Re}\ \tau = 0\}$. In particular, there is an open interval $I = (\sigma_1, \sigma_2) \subset (-\varepsilon_1/2, \varepsilon_1/2)$ such that $I + T_{i\tau}^{(1)}$ is invertible for $\tau = -i\sigma, \sigma \in I$. Hence

$$h_\nu(\xi, \zeta, k, i\tau) = [(I + T_{i\tau}^{(1)})^{-1} q_0(\,\cdot\, - \zeta, \zeta + i\tau\nu)](\xi)$$

exists for $\tau \in D \backslash Z$ and is analytic in (ξ, ζ, τ) on $S_{\delta/2} \times S_{\delta/2} \times \overset{\circ}{D} \backslash Z$.

Our goal is to recover $h_\nu(\xi, \zeta, k, i\tau)$ from the scattering data. To make the connection with scattering data we will need to use $\tau = -i\sigma$ and identify h_ν with a translate of $h_{\nu,\sigma}$. Since denominator $(\eta + i\tau\nu) \cdot (\eta + i\tau\nu) - k^2$ with $\tau = \mu - i\sigma$ goes to $\eta \cdot \eta + 2\sigma\eta_\nu + \sigma^2 - k^2$ as $\mu \downarrow 0$, we can remove the contour deformation in the definition of $V_\tau^{(1)} f$. However, since the integration in r is deformed into the upper half-plane when $\eta_\nu > 0$ and the lower half-plane when $\eta_\nu < 0$, we have

$$[T_\sigma^{(1)} f](\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta + \sigma\nu) f(\eta)}{\eta \cdot \eta + 2\sigma\eta_\nu + \sigma^2 - k^2 + i0\eta_\nu} d\eta\,,$$

and for $\sigma \in I, h_\nu(\xi, \zeta, k, \sigma)$ is the unique solution in $A_{n+1,\delta/3}$ to

$$f(\xi, \zeta) + (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta + \sigma\nu) f(\eta, \zeta)}{\eta \cdot \eta + 2\sigma\eta_\nu + \sigma^2 - k^2 + i0\eta_\nu} d\eta = -q_0(\xi - \zeta, \zeta + \sigma\nu)\,. \quad (32)$$

Since the changes of variables $\eta \to \eta - \sigma\nu$, $\xi \to \xi - \sigma\nu$ and $\zeta \to \zeta - \sigma\nu$, transform Eq. (32) to (16), we conclude that $h_\nu(\xi - \sigma\nu, \zeta - \sigma\nu, k, \sigma)$ is the unique solution of (16) in $A_{n+1,\delta/3}$ and hence for $\sigma \in I$,

$$h_\nu(\xi - \sigma\nu,\ \zeta - \sigma\nu, k, \sigma) = h_{\nu,\sigma}(\xi, \zeta, k)\,. \quad (33)$$

Therefore, assuming the results of Sect. 3, we have proven the following theorem:

Theorem 2. *The solution $h_\nu(\xi, \zeta, k, i\tau)$ of* (27) *exists for $\tau \in D \backslash Z$ and is analytic in (ξ, ζ, τ) on $S_{\delta/3} \times S_{\delta/3} \times (\overset{\circ}{D} \backslash Z)$. The limiting values of $h_\nu(\xi, \zeta, k, i\tau)$ when $\tau \to -i\sigma$ satisfy* (33), *where $h_{\nu,\sigma}(\xi, \zeta, k)$ is the solution of* (16).

Since the unique solvability of (16) in $A_{n+1,\delta/3}$ implies the unique solvability of (24) in $C(S^{n-1})$, we know that (24) has a unique solution for $\sigma \in I$. Hence, knowing the scattering amplitude $h(\xi, \zeta, k)$ for $|\xi|^2 = |\zeta|^2 = k^2$, we can find $h_{\nu,\sigma}(\xi, \zeta, k)$ for $|\xi|^2 = |\zeta|^2 = k^2$ and $\sigma \in I$, which translates (by (33)) to knowing $h_\nu(\xi, \zeta, k, \sigma)$ for $|\xi + \sigma\nu|^2 = |\zeta + \sigma\nu|^2 = k^2$, for $\sigma \in I$. Since $h_\nu(\xi, \zeta, k, i\tau)$ is analytic for $(\xi, \zeta, \tau) \in S_{\delta/3} \times S_{\delta/3} \times (\overset{\circ}{D} \backslash Z)$ with a continuous extension to $S_{\delta/3} \times S_{\delta/3} \times (-iI)$, we can determine it on the variety

$$(\xi + i\tau\nu) \cdot (\xi + i\tau\nu) = (\zeta + i\tau\nu) \cdot (\zeta + i\tau\nu) = k^2$$

for $(\xi, \zeta, \tau) \in S_{\delta/3} \times S_{\delta/3} \times (\overset{\circ}{D} \backslash Z)$ by analytic continuation.

Fix $l \in R^n, \mu \in R^n, n \geqq 3$, such that

$$l \cdot \nu = 0, \qquad \mu \cdot \nu = 0, \qquad l \cdot \mu = 0, \qquad \mu \cdot \mu = 1, \tag{34}$$

and put

$$\begin{aligned} \xi(s) &= \frac{1}{2} l + s\mu, \\ \zeta(s) &= \frac{-1}{2} l + s\mu, \\ z(s) &= i\tau(s) = i\sqrt{s^2 + \frac{1}{4} l \cdot l - k^2}, \end{aligned} \tag{35}$$

$s \geqq s_0$, s_0 large. We have that $h_\nu(\xi(s), \zeta(s), k, z(s))$ is analytic in s for $s > s_0$ and

$$(\xi(s) + i\tau(s)\nu) \cdot (\xi(s) + i\tau(s)\nu) = (\zeta(s) + i\tau(s)\nu) \cdot (\zeta(s) + i\tau(s)\nu) = k^2 .$$

Hence $h_\nu(\xi(s), \zeta(s), k, z(s))$ is known for $s > s_0$.

Remark 1. In the case $A(x) = 0$ the operator $T_{i\tau}^{(1)}$ has a small norm in $H_{\zeta, n+1}$ (see Proposition 4) when $\tau > 0$ is large. Substituting $\xi = \xi(s), \zeta = \zeta(s), z = z(s) = i\tau(s)$ in (27) and passing to the limit when $s \to +\infty$, we obtain that the integral in (27) tends to zero, and we can recover

$$\hat{V}(l) = \lim_{s \to \infty} h_\nu(\xi(s), \zeta(s), k, z(s)) .$$

Thus we obtain an alternate proof of R. Novikov's result [8].

3. Solution of an Integral Equation

In this section we set $z = i\tau$ and only consider τ real and positive.

In order to solve the integral equation (27) when τ is large and positive we will pass to an equivalent differential equation. Let

$$v_\nu(x, \zeta, k, z) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{h_\nu(\eta, \xi, k, z) e^{ix \cdot \eta}}{(\eta + z\nu) \cdot (\eta + z\nu) - k^2} d\eta, \quad z = i\tau, \ \tau > 0 . \tag{36}$$

Then v_ν satisfies the differential equation

$$\begin{aligned} &[(-i\partial/\partial x + z\nu)^2 - k^2 + 2A(x) \cdot (-i\partial/\partial x + z\nu) + q(x)] v_\nu \\ &= -2(\zeta + z\nu) \cdot A(x) e^{ix \cdot \zeta} - q(x) e^{ix \cdot \zeta} . \end{aligned} \tag{37}$$

Our strategy will be to construct solutions of the equation

$$[(-i\partial/\partial x + z\nu)^2 - k^2 + 2A(x) \cdot (-i\partial/\partial x + z\nu) + q(x)] v = f \tag{37$'$}$$

for all f in the Banach space $H_{0,n+1}(R^n)$, where $H_{0,N}(R^n)$ is defined as the closure of $C_0^\infty(R^n)$ in the norm, $\|f\|_{0,N} = \sup_\xi (1 + |\xi|)^N |\hat{f}(\xi)|$, i.e. $H_{0,N}$ is the Fourier transform of $H_{0,N}$. Then

$$h(\xi) = \int_{\mathbb{R}^n} ((-i\partial/\partial x + z\nu)^2 - k^2) v(x) e^{-ix \cdot \xi} dx$$

will be a solution of (27) with the inhomogeneous term replaced by $\hat{f}(\xi)$, i.e.

$$h(\xi) + (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta + zv)h(\eta)}{(\eta + zv) \cdot (\eta + zv) - k^2} d\eta = \hat{f}(\xi), \tag{38}$$

and we will show that $h \in H_{0,n+1}$. Thus we can conclude that $I + T_{i\tau}^{(1)}$ (see (28)) maps $H_{0,n+1}$ *onto* $H_{0,n+1}$ for $\tau \gg 0$. Since $T_{i\tau}^{(1)}$ is also compact on $H_{0,n+1}$ for $\tau > 0$, it follows that $I + T_{i\tau}^{(1)}$ is invertible on $H_{0,n+1}$ for $\tau \gg 0$, and (27) is uniquely solvable in $H_{0,n+1}$, when τ is sufficiently large positive.

We will look for a solution of (37′) in the form

$$v(x, \zeta, z) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{c(x, \eta, z)\tilde{g}(\eta, \zeta, z)e^{ix \cdot \eta}}{(\eta + zv) \cdot (\eta + zv) - k^2} d\eta, \tag{39}$$

where $z = i\tau, \tau > 0$. Here $g(x, \zeta, z)$ is the new unknown and $\tilde{g}(\eta, \zeta, z)$ is its Fourier transform in the first variable. The factor $c(x, \eta, z)$ will be chosen so that the analogue of Eq. (27) for $\tilde{g}$ will not have the unbounded terms in $q_0(\xi - \eta, \eta + zv)$. For this reason we choose $c(x, \eta, z)$ as a solution of the transport equation

$$-2i\frac{\partial c}{\partial x} \cdot (\eta + zv) + 2A(x) \cdot (\eta + zv)\chi_1(\eta, z)c = 0 \tag{40}$$

of the form $c = \exp(-i\chi_1\varphi)$. Thus φ must satisfy

$$(\eta + zv) \cdot \frac{\partial \varphi}{\partial x} = A(x) \cdot (\eta + zv), \tag{40′}$$

and we choose

$$\varphi = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{\hat{A}(\xi) \cdot (\eta + zv)e^{ix \cdot \xi}}{i\xi \cdot (\eta + zv)} d\xi. \tag{41}$$

The function $\chi_1(\eta, z)$ is (40) is a cutoff to a neighborhood of $(\eta + zv) \cdot (\eta + zv) = k^2$. The cancellation of unbounded terms is not needed outside this neighborhood, and it is convenient to have $c \equiv 1$ there. We choose $\chi(t) \in C_0^\infty(R)$ such that $\chi(t) \geqq 0, \chi(t) = 1$ on $|t| < \varepsilon/2$ and $\chi(t) = 0$ on $|t| > \varepsilon$, and define

$$\chi_1(\eta, z) = \chi\left(\frac{|(\eta + zv) \cdot (\eta + zv) - k^2|}{|\eta|^2 + \tau^2 + k^2}\right).$$

Since, setting $\eta_v = \eta \cdot v$,

$$|(\eta + zv) \cdot (\eta + zv) - k^2| = ((|\eta|^2 - \tau^2 - k^2)^2 + 4\tau^2\eta_v^2)^{1/2}, \tag{42}$$

it follows that on the support of χ_1

$$\varepsilon(|\eta|^2 + \tau^2 + k^2) \geqq ||\eta|^2 - (\tau^2 + k^2)|,$$

and hence

$$\left(\frac{1 - \varepsilon}{1 + \varepsilon}\right)|\eta|^2 < \tau^2 + k^2 < \left(\frac{1 + \varepsilon}{1 - \varepsilon}\right)|\eta|^2. \tag{43}$$

Setting $\eta' = \eta - (\eta \cdot \nu)\nu$, (42) also implies that on the support of χ_1,

$$2\varepsilon(|\eta'|^2 + \eta_\nu^2 + \tau^2 + k^2) \geqq ||\eta'|^2 + \eta_\nu^2 - \tau^2 - k^2| + 2\tau|\eta_\nu|,$$

and hence, using (43),

$$\begin{aligned}(1+2\varepsilon)|\eta'|^2 &\geqq (1-2\varepsilon)(\tau^2+k^2) - (1+2\varepsilon)\eta_\nu^2 + 2\tau|\eta_\nu| \\ &\geqq (1-2\varepsilon)(\tau^2+k^2) + \left(2\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{1/2}\frac{\tau}{(\tau^2+k^2)^{1/2}} - (1+2\varepsilon)\right)\eta_\nu^2 .\end{aligned}$$

Thus, choosing ε sufficiently small and τ_0 sufficiently large, we have for $\tau \geqq \tau_0$,

$$(\tau^2 + k^2) + \eta_\nu^2 \leqq C_\varepsilon |\eta'|^2 \tag{44}$$

on support χ_1.

We will need some detailed estimates on φ. The behavior of φ in the x-variables is strongly dependent on η. We introduce $\mu = \eta'/|\eta'|$, and use the orthogonal expansion $x = x_1\nu + x_2\mu + x_\perp$, where $x_\perp$ is the projection of x on the orthogonal complement of span $\{\nu, \eta\}$.

Proposition 2. *Assume that $B(x)$ is a vector-valued function satisfying* (3) *and define*

$$\psi(x, \eta + z\nu) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{\hat{B}(\xi) \cdot (\eta + z\nu)}{\xi \cdot (\eta + z\nu)} e^{ix \cdot \xi} d\xi .$$

Then for $(\eta, z) \in \operatorname{supp} \chi_1, \tau \geqq \tau_0$ and $|\alpha| + |\beta| \leqq P$ in (3′) *one has*

$$\left| \frac{\partial^{|\alpha|+|\beta|}\psi}{\partial x^\alpha \partial \eta^\beta} \right| \leqq C_{\alpha\beta}\tau^{-|\beta|} e^{-\frac{\delta}{2}|x_\perp|} . \tag{45}$$

Proof. By contour integration one computes

$$(2\pi)^{-2} \int_{\mathbb{R}^2} \frac{e^{i(x_1\xi_1 + x_2\xi_2)}}{\xi \cdot (\eta + z\nu)} d\xi_1\, d\xi_2 = \frac{1}{2\pi} \frac{1}{|\eta'|x_1 - (\eta_\nu + z)x_2} .$$

Thus

$$\psi(x, \eta + z\nu) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{B(x - y_1\nu - y_2\mu) \cdot (\eta + z\nu)}{|\eta'|y_1 - (\eta_\nu + z)y_2} dy , \tag{46}$$

and, using (3′), for $|\alpha| \leqq P$,

$$\left| \frac{\partial^{|\alpha|}\psi}{\partial x^\alpha}(x, \eta + z\nu) \right| \leqq \int_{\mathbb{R}^2} \frac{Ce^{-\delta|(x_1-y_1)\nu + (x_2-y_2)\mu + x_\perp|}|\eta + z\nu|}{||\eta'|y_1 - (\eta_\nu + z)y_2|} dy . \tag{47}$$

Since (43) and (44) imply that

$$\begin{aligned}||\eta'|y_1 - (\eta_\nu + z)y_2| &= ((|\eta'|y_1 - \eta_\nu y_2)^2 + \tau^2 y_2^2)^{1/2} \\ &\geqq C\tau(y_1^2 + y_2^2)^{1/2} = C\tau|y| ,\end{aligned} \tag{48}$$

it follows from (43) and (47) that

$$\left|\frac{\partial^{|\alpha|}\psi}{\partial x^\alpha}(x,\eta+zv)\right| \leqq C_\alpha e^{-\frac{\delta}{2}|x_\perp|}$$

for $|\alpha| \leqq P$, where C_α is independent of η and z.

To estimate η derivatives of ψ we first observe that (48) implies

$$\left|\frac{\partial}{\partial\eta_j}\left(\frac{1}{|\eta'|y_1-(\eta_v+z)y_2}\right)\right| = \left|\frac{\frac{\partial|\eta'|}{\partial\eta_j}y_1 - \frac{\partial\eta_v}{\partial\eta_j}y_2}{(|\eta'|y_1-(\eta_v+z)y_2)^2}\right| \leqq \frac{C}{\tau^2|y|}\,.$$

Thus, differentiating (46),

$$\begin{aligned}\left|\frac{\partial\psi}{\partial\eta_j}\right| &\leqq \frac{C}{\tau}\int_{\mathbb{R}^2}\frac{|B(x-y_1v-y_2\mu)|dy}{|y|} + \frac{C}{\tau}\int_{\mathbb{R}^2}\left|\frac{\partial B}{\partial x}(x-y_1v-y_2\mu)\right|dy \\ &\leqq \frac{C}{\tau}e^{-\frac{\delta}{2}|x_\perp|}\,.\end{aligned}$$

Repeating the same argument and noting that $\partial^{|\gamma|}/\partial\eta^\gamma(|\eta'|y_1-(\eta_v+z)y_2)^{-1}$ is homogeneous of degree -1 in y for any γ, one concludes

$$\left|\frac{\partial^{|\alpha|+|\beta|}\psi}{\partial x^\alpha\partial\eta^\beta}\right| \leqq \frac{C_{\alpha\beta}}{\tau^{|\beta|}}e^{-\frac{\delta}{2}|x_\perp|} \tag{49}$$

for $|\alpha|+|\beta| \leqq P$ and $\tau \geqq \tau_0$ on the support of χ_1. □

To study φ in (41) we will use Proposition 2. We introduce

$$w = x_1-(\eta_v+z)|\eta'|^{-1}x_2 \quad\text{and}\quad w' = y_1-(\eta_v+z)|\eta'|^{-1}y_2$$

and observe that

$$\frac{1}{w-w'} = \frac{1}{w(1-\frac{w'}{w})} = \sum_{k=0}^{N}\frac{(w')^k}{w^{k+1}} + \frac{(w')^{N+1}}{w^{N+1}(w-w')}\,. \tag{50}$$

Then we can write (46) with B replaced by A/i in the form

$$\begin{aligned}\varphi(x,\eta+zv) &= \frac{1}{2\pi i}\int_{\mathbb{R}^2}\frac{A(y_1v+y_2\mu+x_\perp)\cdot(\eta+zv)}{|\eta'|(x_1-y_1)-(\eta_v+z)(x_2-y_2)}dy \\ &= \frac{1}{2\pi|\eta'|i}\int_{\mathbb{R}^2}\frac{A(y_1v+y_2\mu+x_\perp)\cdot(\eta+zv)}{w-w'}dy\,. \end{aligned}\tag{51}$$

Using (50) to expand (51), the remainder term in (50) contributes a term to φ of the form

$$\frac{1}{2\pi i}\frac{1}{w^{N+1}}\int_{\mathbb{R}^2}\frac{B_N(x-y_1v-y_2\mu,\eta,z)\cdot(\eta+zv)}{|\eta'|y_1-(\eta_v+z)y_2}dy\,,$$

where $B_N(x,\eta,z) = (x_1-(\eta_v+z)|\eta'|^{-1}x_2)^{N+1}A(x)$ satisfies (3) uniformly in (η,z) on the support of χ_1 for $\tau \geqq \tau_0$. The other terms in (50) contribute terms to φ of the form

$$\frac{1}{2\pi i}\frac{1}{w^{k+1}}\int_{\mathbb{R}^2}|\eta'|^{-1}A(x^\perp+y_1v+y_2\mu)\cdot(\eta+zv)(w')^k\,dy\,.$$

Thus we see that for any $N \geqq 0$, when (η, z) is in the support of χ_1 and $\tau \geqq \tau_0$,

$$\varphi = \sum_{k=1}^{N-1} w^{-k} b_k(x_\perp, \eta, z) + w^{-N} b_N , \tag{52}$$

where $\psi = b_N$ satisfies (45) and $b_k(x_\perp, \eta, z)$ is exponentially decreasing in $x_\perp$ together with its derivatives up to order P uniformly in (η, z).

Substituting (39) into (37′) and using (40), we obtain

$$C(x, D, z)g + T_1 g + T_2 g + T_3 g = f , \tag{53}$$

where

$$[T_1 g](x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{(-2iA \cdot \frac{\partial c}{\partial x} + qc)\hat{g}(\eta)e^{ix \cdot \eta}}{(\eta + zv) \cdot (\eta + zv) - k^2} d\eta ,$$

$$[T_2 g](x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{(-\Delta c)\hat{g}(\eta)e^{ix \cdot \eta} d\eta}{(\eta + zv) \cdot (\eta + zv) - k^2} ,$$

$$[T_3 g](x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{2(1 - \chi_1)A \cdot (\eta + zv)c\hat{g}(\eta)e^{ix \cdot \eta}}{(\eta + zv) \cdot (\eta + zv) - k^2} d\eta ,$$

and $C(x, D, z)$ is a pseudo-differential operator with symbol $c(x, \eta, z)$.

In Sects. 4 and 5 we will need uniform estimates on the norms of the operators $e^{-ix \cdot \zeta} T_j e^{ix \cdot \zeta}, j = 1, 2, 3$, and $e^{-ix \cdot \zeta} C e^{ix \cdot \zeta}$. Since multiplication by $e^{ix \cdot \zeta}$ is not bounded on $H_{0,N}$ (for $N > 0$) and $\zeta \to \infty$, these estimates do not follow from estimates on the norms of the $T_j, j = 1, 2, 3$ and C on $H_{0,N}$. To prove what we will use later efficiently we are going to equip $H_{0,N}$ with a family of norms, $\| \ \|_{\zeta,N}$ so that estimates in these norms *uniform in* ζ will imply the needed estimates for Sects. 4 and 5. We will refer to $H_{0,N}$ with the norm $\| \ \|_{\zeta,N}$ as "$H_{\zeta,N}$."

Proposition 3. *Let $H_{\zeta,N}(R^n)$ be the closure of $C_0^\infty(R^n)$ in the norm $\|f\|_{\zeta,N} = \sup_{\mathbb{R}^n}(1 + |\xi - \zeta|)^N |\hat{f}(\xi)|$. Then $C(x, D, z)$ is invertible as an operator on $H_{\zeta,n+1}$ (R^n) for τ sufficiently large.*

Proof. Our approach here will be to show that $C(x, D)$ and the operator $C^{(-1)}(x, D)$ with the reciprocal symbol $e^{i\chi_1 \varphi}$ are bounded on $H_{\zeta,n+1}$. Then the composition formula for pseudo-differential operators and Proposition 2 will be used to show

$$C^{(-1)} C = I + T , \tag{54}$$

where the norm of T on $H_{\zeta,n+1}$ goes to zero as $\tau \to \infty$ uniformly in ζ.

The proof that C and $C^{(-1)}$ are uniformly bounded on $H_{\zeta,n+1}$ uses only (52). Expanding $c(x, \eta, z) = \exp(-i\varphi\chi_1)$ in a Taylor series in $\varphi\chi_1$, it is clear that $c - 1$ also has an expansion of the form (52) for $\tau \geqq \tau_0$. A linear transformation of R^n takes w in (52) to the standard complex variable $z = s + it$. Hence analytic functions of w are annihilated by the pull-back of $\partial/\partial\bar{z}$ under this transformation which is

$\frac{\partial}{\partial \overline{w}} = \frac{1}{2}(\frac{\partial}{\partial x_2} + (\eta_v + z)|\eta'|^{-1}\frac{\partial}{\partial x_1})$. From (52) we have $\|(\partial^{|\alpha|}/\partial x^\alpha)\partial c/\partial \overline{w}\|_{L^1(\mathbb{R}^n)} \leqq C$ for $|\alpha| < P$ uniformly on support χ_1 for $\tau > \tau_0$. Thus setting $v_0 = \partial c/\partial \overline{w}$,

$$|\hat{v}_0(\xi, \eta, z)| \leqq C(1 + |\xi|)^{-P+1}. \tag{55}$$

Thus, since $P \geqq n+2$, the inverse Fourier transform of $v_0(\xi)(\xi_2 + (\eta_v + z)|\eta'|^{-1}\xi_1)^{-1}$ is continuous, tending to zero as $|x| \to 0$. Since c is bounded, we conclude (by Liouville's theorem)

$$c(x, \eta, z) = 1 + (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{2\hat{v}_0(\xi)e^{ix\cdot\xi}}{i(\xi_2 + (\eta_v + z)|\eta'|^{-1}\xi_1)}\, d\xi$$

$$= 1 + (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{2\hat{v}_0(\xi)|\eta'|e^{ix\cdot\xi}}{i\xi \cdot (\eta + zv)}\, d\xi. \tag{56}$$

Using (55) and (56), given $C(x, D, z)g = h$, we have, setting $c_1 = c - 1$,

$$\hat{h}(\xi) = \hat{g}(\xi) + \int_{\mathbb{R}^n} \tilde{c}_1(\xi - \eta, \eta, z)\hat{g}(\eta)\, d\eta,$$

where $\tilde{c}_1(\xi, \eta, \zeta)$ has support in the support of χ_1 and satisfies

$$|\tilde{c}_1(\xi, \eta, z)| \leqq C|\eta'|(1 + |\xi|)^{-n-1}|\xi \cdot (\eta + zv)|^{-1}. \tag{57}$$

Hence

$$\sup_{\xi}(1 + |\xi - \zeta|)^{n+1}|\hat{h}(\xi)| \leqq (1 + \sup_{\xi,\zeta} \int_{\mathbb{R}^n} (1 + |\xi - \zeta|)^{n+1}|\hat{c}_1(\xi - \eta, \eta, z)|$$
$$(1 + |\eta - \zeta|)^{-n-1}\, d\eta) \sup_{\xi}(1 + |\xi - \zeta|)^{n+1}|\hat{g}(\xi)|,$$

and the boundness of $C(x, D, z)$ on $H_{\zeta,n+1}(R^n)$ uniformly in (ζ, z) for $\tau \geqq \tau_0$ follows from (57) and the estimate

$$(1 + |\xi - \zeta|)^{n+1}(1 + |\xi - \eta|)^{-n-1}(1 + |\eta - \zeta|)^{-n-1}$$
$$\leqq C((1 + |\xi - \eta|)^{-n-1} + (1 + |\eta - \zeta|)^{-n-1}). \tag{58}$$

To see that C is *invertible* on $H_{\zeta,n+1}$ when τ is large, we recall that the integral remainder formula for Taylor series implies that the symbol of $C^{(-1)}(x, D, z)C(x, D, z) - I$ is given by

$$r(x, \eta, z) = \sum_{|\alpha|=1} (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\int_0^1 e^{ix\cdot\zeta} \frac{\partial c^{-1}}{\partial \eta^\alpha}(x, \eta + t\zeta)\zeta^\alpha\, dt\right) \tilde{c}_1(\zeta, \eta)\, d\zeta.$$

The analogue of (57) for $\partial c^{-1}/\partial \eta^\alpha$, $|\alpha| = 1$, is

$$\left|\frac{\partial \tilde{c}^{-1}}{\partial \eta^\alpha}(\xi, \eta, z)\right| \leqq C\frac{|\eta'|}{\tau}(1 + |\xi|)^{-n-1}|\xi \cdot (\eta + zv)|^{-1}.$$

We can now apply the argument, used above to show that $C(x, D, z)$ is bounded on $H_{\zeta,n+1}$, to $R(x, D, z)$. The superpositions in ζ and τ produce no new difficulties and the factor of $1/\tau$ in the estimate for $\partial c^{-1}/\partial\eta^\alpha$ above makes $\|R(x,D)\|$ go to zero as $\tau \to \infty$. Thus C is invertible for τ sufficiently large. $\square$

Proposition 4. *The norms of the operators $T_1(\tau), T_2(\tau)$ and $T_3(\tau)$ on $H_{\zeta,n+1}(R^n)$ tend to zero as $\tau \to \infty$ uniformly in ζ.*

Proof. Let $\tilde{T}_k(\xi-\eta,\eta,z)$ be the kernel of the Fourier transform of T_k, $k=1,2,3$, i.e.

$$\widehat{T_k g}(\xi) = \int_{\mathbb{R}^n} \tilde{T}_k(\xi-\eta,\eta,z)\hat{g}(\eta)\,d\eta\,.$$

In order to show that the norm of T_k on $H_{\zeta,n+1}(R^n)$, is arbitrarily small for τ large uniformly in ζ, it suffices to prove that

$$\sup_{\xi,\zeta} \int_{\mathbb{R}^n} (1+|\xi-\zeta|)^{n+1}|\tilde{T}_k(\xi-\eta,\eta,z)|(1+|\eta-\zeta|)^{-n-1}\,d\eta \leqq \frac{C}{\tau}\log\tau\,. \tag{59}$$

On the support of $1-\chi_1$ we have $|(\eta+zv)\cdot(\eta+zv)-k^2| \geqq \frac{\varepsilon}{2}(|\eta|^2+\tau^2+k^2)$. Hence

$$|\tilde{T}_3(\xi-\eta,\eta,z)| \leqq C(1+|\xi-\eta|)^{-n-1}\frac{|\eta+zv|}{|\eta|^2+\tau^2+k^2} \leqq \frac{C}{\tau}(1+|\xi-\eta|)^{-n-1}\,,$$

and (59) for $k=3$ follows from (58).

To estimate $\tilde{T}_1$ we note that (42) implies that for all (η,z),

$$\begin{aligned}|\eta+zv)\cdot(\eta+zv)-k^2| &\geqq \frac{1}{2}(||\eta|^2-(\tau^2+k^2)|+2\tau|\eta_v|)\\ &= \frac{1}{2}(||\eta|-(\tau^2+k^2)^{1/2}||\,|\eta|+(\tau^2+k^2)^{1/2}|+2\tau|\eta_v|)\\ &\geqq \frac{\tau}{2}(||\eta|-(\tau^2+k^2)^{1/2}|+|\eta_v|)\,.\end{aligned} \tag{60}$$

Since $c-1$ has an expansion of the form (52), qc and $A\cdot\frac{\partial c}{\partial x}$ satisfy (3) with constants uniform in (η,z) for $\tau>\tau_0$. Thus, from (58) and (60),

$$\begin{aligned}&\sup_{\xi,\zeta}\int_{\mathbb{R}^n}(1+|\xi-\zeta|)^{n+1}|\tilde{T}_1(\xi-\eta,\eta,z)|(1+|\eta-\zeta|)^{-n-1}\,d\eta\\ &\leqq \frac{C}{\tau}\sup_{\xi,\zeta}\int_{\mathbb{R}^n}\frac{(1+|\xi-\eta|)^{-n-1}+(1+|\eta-\zeta|)^{-n-1}}{||\eta|-(\tau^2+k^2)^{1/2}|+|\eta_v|}\,d\eta\\ &\leqq \frac{2C}{\tau}\sup_{\xi}\int_{\mathbb{R}^n}\frac{(1+|\xi-\eta|)^{-n-1}}{||\eta|-(\tau^2+k^2)^{1/2}|+|\eta_v|}\,d\eta\,.\end{aligned} \tag{61}$$

Setting $R = (\tau^2 + k^2)^{1/2}$, $\eta = R\zeta$ and $l(\zeta) = ((|\zeta| - 1)^2 + \zeta_\nu^2)^{1/2}$ in the last line of (61), this gives

$$\sup_{\xi,\zeta} \int_{\mathbb{R}^n} (1 + |\xi - \zeta|)^{n+1} |\tilde{T}_1(\xi - \eta, \eta, z)| (1 + |\eta - \zeta|)^{-n-1} \, d\eta$$

$$\leqq \frac{C}{\tau} \sup_{\xi} \int_{\mathbb{R}^n} (1 + |\xi - R\zeta|)^{-n-1} (l(\zeta))^{-1} R^{n-1} \, d\zeta$$

$$\leqq \frac{C}{\tau} \Bigg[\sup_{\xi} \int_{\mathbb{R}^n} (1 + |\xi - R\zeta|)^{-n-1} R^{n-1} \, d\zeta$$

$$+ \sup_{\xi} \int_{l(\zeta)<\varepsilon_0} (1 + |\xi - R\zeta|)^{-n-1} (l(\zeta))^{-1} R^{n-1} \, d\zeta \Bigg] .$$

$$\leqq \frac{C}{\tau} \left[\frac{1}{R} + R^{n-1} \sup_{\xi} \int_{l(\zeta)<\varepsilon_0} (1 + |\xi - R\zeta|^{-n-1} (l(\zeta))^{-1} \, d\zeta \right] .$$

Here ε_0 is any fixed constant, and we assume $\varepsilon_0 \ll 1$. Since $\tau \sim R$ for $\tau > \tau_0$, it suffices to show

$$\tau^{n-1} \sup_{\xi} \int_{l(\zeta)<\varepsilon_0} (1 + \tau|\xi - \zeta|)^{-n-1} (l(\zeta))^{-1} \, d\zeta < C \tag{62}$$

for $\tau > \tau_0$ to conclude that (59) holds for $k = 1$.

To prove (62) we note first that when $|\xi'| < \frac{1}{2}$,

$$\int_{l(\zeta)<\varepsilon_0} (1 + \tau|\xi - \zeta|)^{-n-1} (l(\zeta))^{-1} \, d\zeta \leqq \int_{l(\zeta)<\varepsilon_0} (1 + c_0\tau)^{-n-1} (l(\zeta))^{-1} \, d\zeta ,$$

where $c_0 = \min_{l(\zeta)<\varepsilon_0} |\xi - \zeta| > 0$, and (62) holds.

To establish (62) for $|\xi'| > \frac{1}{2}$ we will use spherical coordinates in the hyperplane $\zeta \cdot \nu = 0$ with $r = |\zeta'|$ and polar angle $\theta = \cos^{-1}(\frac{\zeta'}{|\zeta'|} \cdot \frac{\xi'}{|\xi'|})$. Then we have $d\zeta = r^{n-2} dr d\omega d\zeta_\nu$, where $d\omega$ is the volume form on S^{n-2}, and we also have

$$|\zeta - \xi| = (r^2 - 2|\xi'| r \cos\theta + |\xi'|^2 + (\zeta_\nu - \xi_\nu)^2)^{1/2}$$

$$\geqq \frac{1}{2} (((r - |\xi'| \cos\theta)^2 + (\zeta_\nu - \xi_\nu)^2)^{1/2} + |\xi'| |\sin\theta|) . \tag{63}$$

Likewise, there is $c > 0$ such that

$$l(\zeta) \geqq c((r - 1)^2 + \zeta_\nu^2)^{1/2} . \tag{64}$$

Now we consider $v = (r - 1, \zeta_\nu)$ and $v_0 = (|\xi'| \cos\theta - 1, \xi_\nu)$ as vectors in R^2 and use $\| \ \|$ to denote the norm on R^2. From (63) and (64) we have

$$\int_{l(\zeta)<\varepsilon_0} (1 + \tau|\xi - \zeta|)^{-n-1} (l(\zeta))^{-1} \, d\zeta$$

$$\leqq C \int_{\mathbb{R}^2 \times S^{n-2}} \frac{(1 + \tau(\|v - v_0\| + |\sin\theta|))^{-n-1}}{\|v\|} \, dr d\zeta_\nu d\omega .$$

We split the integral over $R^2 \times S^{n-2}$ into an integral over $\{\zeta : \|v\| \geqq \|v - v_0\|\}$ in which we replace $\|v\|$ by $\|v - v_0\|$ and an integral over $\{\zeta : \|v\| < \|v - v_0\|\}$ in which we replace $\|v - v_0\|$ by $\|v\|$. Since the two integrands that are produced this way differ only by a translation in the (r, ζ_v)-plane, we have the estimate

$$\begin{aligned}
&\int\limits_{l(\zeta)<\varepsilon_0} (1 + \tau|\xi - \zeta|)^{-n-1}(l(\zeta))^{-1}\, d\zeta \\
&\leqq C \int\limits_{\mathbb{R}^2 \times S^{n-2}} \frac{(1 + \tau((s^2 + t^2)^{1/2} + |\sin\theta|))^{-n-1}}{(s^2 + t^2)^{1/2}}\, ds\, dt\, d\omega \\
&\leqq C \int\limits_0^\infty \int\limits_{S^{n-2}} (1 + \tau(u + |\sin\theta|))^{-n-1} du\, d\omega \\
&\leqq C \int\limits_0^\infty \int\limits_0^{\pi/2} (1 + \tau(u + \theta))^{-n-1}\theta^{n-3} du\, d\theta
\end{aligned}$$

and, setting $\tau u = r, \tau\theta = s$, we have

$$\int\limits_{l(\zeta)<\varepsilon_0} (1 + \tau|\xi - \zeta|)^{-n-1}(l(\zeta))^{-1}\, d\zeta \leqq \tau^{-n+1} C \int\limits_0^\infty \int\limits_0^\infty (1 + r + s)^{-n-1} s^{n-3} dr\, ds\,.$$

Thus, since the integral is finite, we have (62), and (59) holds for $k = 1$, in the stronger form

$$\sup_{\xi,\zeta} \int\limits_{\mathbb{R}^n} (1 + |\xi - \zeta|)^{n+1} |\tilde{T}_1(\xi - \eta, \eta, z)| (1 + |\eta - \zeta|)^{-n-1}\, d\eta \leqq \frac{C}{\tau}\,. \tag{64'}$$

From (56) one sees that

$$|\widetilde{\Delta c}(\xi - \eta, \eta)| \leqq C(1 + |\xi - \eta|)^{-P+3} |\eta'| |(\xi - \eta) \cdot (\eta + zv)|^{-1}\,,$$

and hence

$$|\tilde{T}_2(\xi - \eta, \eta, z)| \leqq \frac{C(1 + |\xi - \eta|)^{-P+3}|\eta'|}{|(\xi - \eta) \cdot (\eta + zv)||(\eta + zv) \cdot (\eta + zv) - k^2|}\,,$$

and by the reasoning that leads to (61), we have (note $P \geqq n + 4$ is needed):

$$\begin{aligned}
&\sup_{\xi,\zeta} \int\limits_{\mathbb{R}^n} (1 + |\xi - \zeta|)^{n+1} |\tilde{T}_2(\xi - \eta, \eta, z)| (1 + |\eta - \zeta|)^{-n-1}\, d\eta \\
&\leqq \frac{C}{\tau} \sup_{\xi} \int\limits_{\mathbb{R}^n} \frac{(1 + |\xi - \eta|)^{-n-1} |\eta'|\, d\eta}{|(\xi - \eta) \cdot (\eta + i\tau v)|(||\eta| - (\tau^2 + k^2)^{1/2}| + |\eta_v|)}\,.
\end{aligned} \tag{65}$$

Setting $R = (\tau^2 + k^2)^{1/2}$, $\beta = \tau(\tau^2 + k^2)^{-1/2}$, $\eta = R\zeta$ and $l(\zeta) = ((|\zeta| - 1)^2 + \zeta_v^2)^{1/2}$, (65) becomes

$$\begin{aligned}
&\sup_{\xi,\zeta} \int\limits_{\mathbb{R}^n} (1 + |\xi - \zeta|)^{n+1} |\tilde{T}_2(\xi - \eta, \eta, z)| (1 + |\eta - \zeta|)^{-n-1}\, d\eta \\
&\leqq \frac{C}{\tau} R^{n-1} \sup_{\xi} \int\limits_{\mathbb{R}^n} \frac{(1 + |\xi - R\zeta|)^{-n-1}|\zeta'| d\zeta}{|(\xi - R\zeta) \cdot (\zeta + i\beta v)| l(\zeta)} \\
&= \frac{C}{\tau} R^{n-2} \sup_{\xi} \int\limits_{\mathbb{R}^n} \frac{(1 + R|\xi - \zeta|)^{-n-1}|\zeta'|\, d\zeta}{(((\xi - \zeta) \cdot \zeta)^2 + \beta^2(\xi_v - \zeta_v)^2)^{1/2} l(\zeta)}\,.
\end{aligned}$$

Since $\beta \to 1$ as $\tau \to \infty$ and $\beta R = \tau$, to show $\|T_2\| \to 0$ as $\tau \to \infty$, it suffices to show for $\tau > \tau_0$ that

$$\tau^{n-2} \sup_{\xi} \int_{\mathbb{R}^n} \frac{(1+\tau|\xi-\zeta|)^{-n-1}|\zeta'|\,d\zeta}{(((\xi-\zeta)\cdot\zeta)^2+(\xi_\nu-\zeta_\nu)^2)^{1/2}\mathrm{l}(\zeta)} \leqq C\log\tau\,. \tag{66}$$

When $\mathrm{l}(\zeta) > \varepsilon_0$, the integrand in (66) is essentially the same as the one we considered for T_1: note that $(\xi-\zeta)\cdot\zeta = |\zeta-\xi/2|^2 - |\xi/2|^2$. Thus we again assume that $\mathrm{l}(\zeta) < \varepsilon_0 \ll 1$. We have

$$\begin{aligned}
(((\xi-\zeta)\cdot\zeta)^2+(\xi_\nu-\zeta_\nu)^2)^{1/2} &> \frac{1}{2}(|(\xi-\zeta)\cdot\zeta| + |\xi_\nu-\zeta_\nu|)\\
&= \frac{1}{2}(||\zeta'-\xi'/2|^2 - |\xi'/2|^2 + \zeta_\nu(\zeta_\nu-\xi_\nu)| + |\xi_\nu-\zeta_\nu|)\\
&\geqq \frac{1}{2}(||\zeta'-\xi'/2|^2 - |\xi'/2|^2| + (1-\varepsilon_0)|\xi_\nu-\zeta_\nu|)\,.
\end{aligned}$$

Again using the coordinates $r = |\zeta'|, \theta = \cos^{-1}(\zeta'/|\zeta'|\cdot\xi'/|\xi'|)$, we have

$$|\zeta'-\xi'/2|^2 - |\xi'/2|^2 = r^2 - r|\xi'|\cos\theta$$

and

$$(((\xi-\zeta)\cdot\zeta)^2+(\xi_\nu-\zeta_\nu)^2)^{1/2} \geqq c((r-|\xi'|\cos\theta)^2+(\xi_\nu-\zeta_\nu)^2)^{1/2} = c\|v-v_0\|\,,$$

in the notation used earlier. Thus, using (64), for $|\xi'| \leqq 1/2$,

$$\int_{l(\zeta)<\varepsilon_0} \frac{(1+\tau|\xi-\zeta|)^{-n-1}|\zeta'|\,d\zeta}{(((\xi-\zeta)\cdot\zeta)^2+(\xi_\nu-\zeta_\nu)^2)^{1/2}l(\zeta)} \leqq C\int_{l(\zeta)<\varepsilon_0} \frac{(1+c_0\tau)^{-n-1}dr\,d\zeta_\nu\,d\omega}{\|v-v_0\|\;\|v\|}\,,$$

and, since $|\xi'| \leqq 1/2$ implies $\|v_0\| \geqq \frac{1}{2}$, this is bounded by $C\tau^{-n-1}$. Hence we may assume that $|\xi'| > 1/2$, and in this case (63) implies

$$\begin{aligned}
&\int_{l(\zeta)<\varepsilon_0} \frac{(1+\tau|\xi-\zeta|)^{-n-1}|\zeta'|\,d\zeta}{(((\xi-\zeta)\cdot\zeta)^2+(\xi_\nu-\zeta_\nu)^2)^{1/2}l(\zeta)}\\
&\quad\leqq C\int_{l(\zeta)<\varepsilon_0} \frac{(1+\tau(\|v-v_0\|+|\sin\theta|))^{-n-1}dr d\zeta_\nu d\omega}{\|v-v_0\|\;\|v\|} \equiv I_1\,.
\end{aligned}$$

Since (64) implies $\|v\| < \varepsilon_0$ when $l(\zeta) < \varepsilon_0$, we see that contribution to I_1 from integration over $\{\theta : \|v_0(\theta\| \geqq \frac{1}{2}\}$ is bounded by $C\tau^{-n-1}$. Thus we may replace the domain of integration in I_1 by $\{l(\zeta) < \varepsilon_0\} \cap \{\|v_0\| < \frac{1}{2}\}$.

At this point the argument used for T_1 leads to divergent integrals, and we need to use the fact that the factors in the denominator only vanish simultaneously when

$|\xi'|\cos\theta = 1$. To bound I_1, we set $z = (\|v_0(\theta)\|^{-1})v$. Then

$$\begin{aligned}
I_1 &\leqq C \int_{S^{n-2}\times\{\|z\|\leqq\|v_0\|^{-1}\})\cap\{\|v_0\|<1/2\}} \frac{(1+\tau|\sin\theta|)^{-n-1}}{\|z - v_0/\|v_0\|\ \|\ \|z\|} dz d\omega \\
&\leqq C \int_{S^{n-2}\cap\{\|v_0\|<1/2\}} (1+\tau|\sin\theta|)^{-n-1} \log(\|v_0(\theta)\|^{-1})\, d\omega \\
&\leqq C \int_0^{\pi/2} (1+\tau\theta)^{-n-1} \max\{\log 2, -\log\|v_0(\theta)\|\}\theta^{n-3} d\theta \\
&\leqq C\tau^{2-n} \int_0^{\pi\tau/2} (1+\beta)^{-n-1}\beta^{n-3} \max\left\{\log 2, -\log\left\|v_0\left(\frac{\beta}{\tau}\right)\right\|\right\} d\beta
\end{aligned}$$

$$I_1 \leqq c\tau^{2-n} \int_0^{\pi\tau/2} (1+\beta)^{-n-1}\beta^{n-3} \max\left\{\log 2, -\log\left|1 - |\xi'|\cos\frac{\beta}{\tau}\right|\right\} d\beta\,. \tag{67}$$

If $1/2 \leqq |\xi'| \leqq 1$, then $|1 - |\xi'|\cos\beta/\tau| \geqq c_0^2\beta^2\tau^{-2}$ with c_0 independent of $|\xi'|$. Hence, in this case $I_1 \leqq c\tau^{2-n}\log\tau$ for τ large. If $|\xi'| > 1$, then $1 - |\xi'|\cos\theta = 0$ has a unique solution θ_0 in the interval $[0, \pi/2]$ and we have

$$|1 - |\xi'|\cos\theta| \geqq c_0^2(\theta - \theta_0)^2$$

with $0 < c_0 < 1$ and c_0 independent of $|\xi'|$. Thus

$$|1 - |\xi'|\cos\beta/\tau| \geqq \frac{c_0^2}{\tau^2}(\beta - \beta_0)^2\,,$$

where $\beta_0 = \tau\theta_0$. Thus for $\tau > 1$.

$$\begin{aligned}
&\max\{\log 2, -\log|1 - |\xi'|\cos\frac{\beta}{\tau}|\} \\
&\qquad \leqq \log 2 + 2\log\tau - 2\log c_0 + 2(-\log|\beta - \beta_0|)_+\,.
\end{aligned} \tag{68}$$

Combining (68) with (67) we see that $I_1 \leqq C\tau^{2-n}\log\tau$ for τ large in this case also. Thus (66) holds and the proof of Proposition 4 is complete. □

It follows from Propositions 3 and 4 that for $\tau \gg 0$ there exists a unique solution g in $H_{0,n+1}$ of the integral equation (53), given by

$$g = (I + (I+T)^{-1}C^{(-1)}(T_1 + T_2 + T_3))^{-1}(I+T)^{-1}C^{(-1)}f\,, \tag{69}$$

where T is the operator in (54). Thus v, given by (39) with this choice of g, is a solution of (37′). Thus to complete the proof that (27) has a unique solution in $H_{0,n+1}(R^n)$ when $\tau \gg 0$, we need only show that $\check{h}$ given by

$$\check{h}(x) = ((-i\partial/\partial x + zv)^2 - k^2)v$$

is in $H_{0,n+1}$. From (39) we see that

$$\check{h} = Cg + T_2 g + Sg\,,$$

where T_2 is the operator in (53) and

$$Sg = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{-2i\frac{\partial c}{\partial x} \cdot (\eta + zv)\hat{g}(\eta)e^{ix\cdot\eta}}{(\eta + zv) \cdot (\eta + zv) - k^2} d\eta$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{-2A(x) \cdot (\eta + zv)\chi_1(\eta, z)c\hat{g}(\eta)e^{ix\cdot\eta}}{(\eta + zv) \cdot (\eta + zv) - k^2} d\eta \tag{70}$$

by (40). From (70) one sees that S is an operator of the same type as T_1 in (53) with an additional factor of $\eta + zv$ in the numerator. However, since we showed that the norm of T_1 on $H_{\zeta,n+1}$ was $O(\tau^{-1})$ uniformly in ζ for $\tau \to \infty$, and $|\eta + zv| \leqq c\tau$ on support χ_1 (see (43)), it follows that S is bounded on $H_{\zeta,n+1}$, uniformly in (ζ, τ) for $\tau > \tau_0$. This completes the verification that $h_v(\xi, \zeta, k, i\tau) \in H_{\zeta,n+1}$.

4. Recovering the Magnetic Field

Proposition 5. *Let $h_v(\xi, \zeta, k, z)$ be the unique solution of (27) in $H_{0,n+1}$ for $\tau \gg 0$, and let $g_v(x, \zeta, k, z)$ be the unique solution in $H_{0,n+1}$ of (53) with $f = -(q(x) + 2(\zeta + zv).\ A(x))exp(ix \cdot \zeta)$ for $\tau \gg 0$. Then*

$$h_v(\xi, \zeta, k, z) = \tilde{g}_v(\xi, \zeta, k, z) \tag{71}$$

when $(\xi + zv) \cdot (\xi + zv) - k^2 = 0$.

Proof. We have

$$v_v(x, \zeta, k, z) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{h_v(\eta, \zeta, k, z)e^{ix\cdot\eta}d\eta}{(\eta + zv) \cdot (\eta + zv) - k^2}$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{c(x, \eta, z)\tilde{g}_v(\eta, \zeta, k, z)e^{ix\cdot\eta}d\eta}{(\eta + zv) \cdot (\eta + zv) - k^2}. \tag{72}$$

As we observed earlier $c_1 = c(x, \eta, z) - 1$ has an expansion of the form (52) for $\tau > \tau_0$. Thus, as in the proof of the bound on T_2 in Proposition 4, we see that

$$f(\xi, \zeta, k, z) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{\tilde{c}_1(\xi - \eta, \eta, z)\tilde{g}_v(\eta, \zeta, k, z)d\eta}{(\eta + zv) \cdot (\eta + zv) - k^2}$$

belongs to $H_{0,n+1}$ as a function of ξ, and hence is continuous in ξ. Since the Fourier transform of (72) gives (a.e. in ξ)

$$\frac{h_v(\xi, \zeta, z)}{(\xi + zv) \cdot (\xi + zv) - k^2} = \frac{\tilde{g}_v(\xi, \zeta, k, z)}{(\xi + zv) \cdot (\xi + zv) - k^2} + f(\xi, \zeta, k, z),$$

where h_v and $\tilde{g}_v$ are also continuous in ξ, (71) follows immediately. □

By Proposition 1 and the discussion following it we can recover $h_\nu(\xi(s),\zeta(s),k, z(s))$ from the scattering amplitude $h(k\Theta,k\omega,k)$. Recall (see (34), (35)) that given the orthogonal frame $\{\nu,\mu,l\}$ with $|\mu|=|\nu|=1$,

$$\xi(s)=\frac{1}{2}l+s\mu\,,$$

$$\zeta(s)=-\frac{1}{2}l+s\mu\,,$$

$$z(s)=i\tau(s)=i\sqrt{s^2+|l|^2/4-k^2} \tag{73}$$

for $s>s_0$. Since $(\xi(s)+z(s)\nu)\cdot(\xi(s)+z(s)\nu)-k^2=0$, it follows from Proposition 5 that $h(k\theta,k\omega,k)$ determines $\tilde{g}_\nu(\xi(s),\zeta(s),k,z(s))$ for $s>s_0$.

To recover the magnetic field we can begin with representation for g_ν given by (69) with $f=-(q(x)+2(\zeta+z\nu)\cdot A(x))\exp(ix\cdot\zeta)$, take the Fourier transform in x, evaluate at $\xi=\xi(s)$, $\zeta=\zeta(s)$, $z=z(s)$ as in (73), divide by $z(s)$ and take the limit as $s\to\infty$. Since the norms of T,T_1,T_2 and T_3 on $H_{\zeta(s),n+1}$ go to zero and $\frac{1}{|z(s)|}\|f\|_{\zeta(s),n+1}$ is bounded as $s\to\infty$, it follows that $h(k\theta,k\omega,k)$ determines

$$\lim_{s\to\infty}(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(-2)(\zeta(s)+z(s)\nu)}{z(s)}\cdot\hat{A}(\eta-\zeta(s))$$

$$\times\, e^{-ix\cdot(\xi(s)-\eta)+i\chi_1(\eta,z(s))\varphi(x,\eta+z(s)\nu)}d\eta dx\,. \tag{74}$$

Replacing $\eta-\zeta(s)$ by η, (74) becomes

$$\lim_{s\to\infty}(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(-2)(\zeta(s)+z(s)\nu)}{z(s)}\cdot\hat{A}(\eta)$$

$$\times\, e^{ix\cdot\eta-ix\cdot(\xi(s)-\zeta(s))+i\chi_1(\eta+\zeta(s),z(s))\varphi(x,\eta+\zeta(s)+z(s)\nu)}d\eta dx\,. \tag{75}$$

By (73) $\xi(s)-\zeta(s)=l$ and $\lim_{s\to\infty}(\zeta(s)+z(s)\nu)/z(s)=\nu-i\mu$. Also (see definition of χ_1 before (42))

$$\lim_{s\to\infty}\chi_1(\eta+\zeta(s),z(s))=\chi(0)=1\,.$$

Finally

$$\lim_{s\to\infty}\varphi(x,\eta+\zeta(s)+z(s)\nu)=\lim_{s\to\infty}(2\pi)^{-n}\int_{\mathbb{R}^n}\frac{\hat{A}(\xi)\cdot(\eta+\zeta(s)+z(s)\nu)e^{ix\cdot\xi}}{i\xi\cdot(\eta+\zeta(s)+z(s)\nu)}d\xi$$

$$=(2\pi)^{-n}\int_{\mathbb{R}^n}\frac{\hat{A}(\xi)\cdot(\mu+i\nu)}{i\xi\cdot(\mu+i\nu)}e^{ix\cdot\xi}d\xi\equiv\varphi(x,\mu+i\nu)\,. \tag{76}$$

Hence the limit in (75) equals

$$I \equiv -2(2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot l + i\varphi(x,\mu+i\nu) + ix \cdot \eta}(\nu - i\mu) \cdot \hat{A}(\eta)d\eta dx$$

$$= 2i \int_{\mathbb{R}^n} e^{-ix \cdot l + i\varphi(x,\mu+i\nu)}(\mu + i\nu) \cdot A(x)dx . \tag{77}$$

Comparing (76) with (40′), we see that

$$(\mu + i\nu) \cdot \frac{\partial\varphi}{\partial x} = (\mu + i\nu) \cdot A(x) ,$$

and hence, using the coordinates $(x_1,x_2,x^\perp)$ introduced before Proposition 2, we have

$$I = 2 \int_{\mathbb{R}^{n-2}} e^{-il \cdot x^\perp} \left(\int_{\mathbb{R}^2} \left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_1} \right) e^{i\varphi} dx_1 dx_2 \right) dx^\perp .$$

We have

$$\int_{\mathbb{R}^2} \left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_1} \right) e^{i\varphi} dx_1 dx_2 = \lim_{R\to\infty} \int_{x_1^2+x_2^2 \le R^2} \left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_1} \right) e^{i\varphi} dx_1 dx_2$$

$$= \lim_{R\to\infty} \int_0^{2\pi} e^{i\varphi(R\cos\theta, R\sin\theta, x^\perp, \mu+i\nu)}(\sin\theta + i\cos\theta)R\mathrm{d}\theta ,$$

by Green's theorem with $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. Returning to the expansion (52) for φ, we have

$$\varphi = \frac{1}{2\pi i}\frac{1}{x_1 - ix_2} \int_{\mathbb{R}^2} A(y_1\nu + y_2\mu + x^\perp) \cdot (\mu + i\nu)dy_1dy_2 + O((x_1 - ix_2)^{-2}) .$$

Thus

$$\lim_{R\to\infty} \int_0^{2\pi} e^{i\varphi(R\cos\theta, R\sin\theta, x^\perp, \mu+i\nu)}(\sin\theta + i\cos\theta)R\mathrm{d}\theta$$

$$= i\int_{\mathbb{R}^2} A(y_1\nu + y_2\mu + x^\perp) \cdot (\mu + i\nu)dy_1dy_2 ,$$

and

$$I = 2i \int_{\mathbb{R}^{n-2}} e^{-il\cdot x^\perp} \left(\int_{\mathbb{R}^2} A(y_1\nu + y_2\mu + x^\perp) \cdot (\mu + i\nu)dy_1dy_2 \right) dx^\perp$$

$$= 2i\hat{A}(l) \cdot (\mu + i\nu) .$$

Since μ and ν are a general orthonormal pair perpendicular to l, we conclude that for all $l \in R^n$, I determines $\hat{A}(l) - (\hat{A}(l) \cdot l)l/|l|^2$. In other words I determines A modulo the gradient of

$$\rho(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{il \cdot x} i\hat{A}(l) \cdot l/|l|^2 dl = -\Delta^{-1}(\nabla \cdot A) , \tag{78}$$

and hence I determines *curl* A.

5. Recovering the Electric Potential

To recover $V(x)$ we need to compute the next term in the asymptotic expansion of (69) which yielded (74) as the leading term. We have determined $A(x)$ modulo the gradient of a function of the form (78). Hence, we may assume that we know the scattering data for the problem with the $A(x)$ here and $q = q' \equiv A \cdot A - i\nabla \cdot A$, since the scattering data only depends on the magnetic field $B = \operatorname{curl} A$. This scattering data determines the Fourier transform of the solution g_0 of (53) with $f = f_0 \equiv -(q' + 2(\zeta + zv) \cdot A(x))\exp(ix \cdot \zeta)$ on the set $(\xi, \zeta, z) = (\xi(s), \zeta(s), z(s))$ given by (73). Among the operators in (69) only T_1 is changed when we replaced g by g_0, and we denote the new operator by $T_{1,0}$. Thus, subtracting the representation (69) for g_0 from the representation (69) for g, we may assume that we know the Fourier transform on the curve $(\xi(s), \zeta(s), z(s))$ of

$$
\begin{aligned}
&(I + (I+T)^{-1}C^{(-1)}(T_1 + T_2 + T_3))^{-1}(I+T)^{-1}C^{(-1)}(f - f_0) \\
&\quad - (I + (I+T)^{-1}C^{(-1)}(T_1 + T_2 + T_3))^{-1}(I+T)^{-1}C^{(-1)}(T_1 - T_{1,0})) \\
&\quad\quad \cdot (I + (I+T)^{-1}C^{(-1)}(T_{1,0} + T_2 + T_3))^{-1}(I+T)^{-1}C^{(-1)}f_0 \,. \qquad (79)
\end{aligned}
$$

Taking the limit in the Fourier transform of (79) at $(\xi(s), \zeta(s), z(s))$ as $s \to \infty$, we recover

$$
\begin{aligned}
&\lim_{s\to\infty} (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} -\hat{V}(\eta - \zeta(s)) e^{-ix\cdot(\xi(s)-\eta)+i\chi_1(\eta,z(s))\varphi(x,\eta+z(s)v)} d\eta dx \\
&\quad - \lim_{s\to\infty} \mathscr{F}(C^{(-1)}(T_1 - T_{1,0})C^{(-1)}f_0)(\xi(s), \zeta(s), z(s)) \equiv J_1 - J_2 \,.
\end{aligned}
$$

By the same computation that derived (77) from (75), we have

$$
J_1 = -\int_{\mathbb{R}^n} e^{-ix \cdot l + i\varphi(x,\mu+iv)} V(x) dx \,. \qquad (80)
$$

To compute J_2 we argue as follows. $T_1 - T_{1,0} = VCL$, where L multiplies the Fourier transform by $((\eta + zv) \cdot (\eta + zv) - k^2)^{-1}$. Since $[V, C]$ goes to zero and $C^{(-1)}C$ goes to the identity as $s \to \infty$, we can conclude that

$$
\begin{aligned}
J_2 = \lim_{s\to\infty} (2\pi)^{-2n} &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\hat{V}(\xi(s) - \eta)}{(\eta + z(s)v) \cdot (\eta + z(s)v) - k^2} \\
&\times (-2(\zeta(s) + z(s)v) \cdot \hat{A}(\delta - \zeta(s))) \\
&\times e^{ix \cdot (\delta - \eta) + i\chi_1(\delta, z(s))\varphi(x, \delta + z(s)v)} d\delta dx d\eta \,.
\end{aligned}
$$

Replacing δ by $\delta+\zeta(s)$ and η by $\eta+\zeta(s)$, and arguing as before (recall $(\zeta(s)+z(s)v)\cdot(\zeta(s)+z(s)v)=k^2$), we have

$$J_2 =(2\pi)^{-2n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\hat{V}(l-\eta)}{2\eta\cdot(\mu+iv)}(-2(\mu+iv)\cdot\hat{A}(\delta))$$

$$\times e^{ix\cdot(\delta-\eta)+i\varphi(x,\mu+iv)}d\delta dx d\eta$$

$$=(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} -\frac{\hat{V}(\ell-\eta)(\mu+iv)\cdot A(x)e^{-ix\cdot\eta+i\varphi(x,\mu+iv)}}{(\mu+iv)\cdot\eta}dx d\eta\,.$$

Proceeding as before with $x_1=x\cdot v$ and $x_2=x\cdot\mu$,

$$\int_{\mathbb{R}^n} e^{-ix\cdot\eta+i\varphi(x,\mu+iv)}(\mu+iv)\cdot A(x)dx$$

$$=\int_{\mathbb{R}^{n-2}} e^{-ix^\perp\cdot\eta^\perp}dx^\perp\int_{\mathbb{R}^2} e^{-i(x_1\eta_1+x_2\eta_2)}(-i)\left(\frac{\partial}{\partial x_2}+i\frac{\partial}{\partial x_1}\right)(\overline{c}(x,\mu+iv)-1)dx_1dx_2\,,$$

and by Green's theorem

$$\int_{\mathbb{R}^2} e^{-i(x_1\eta_1+x_2\eta_2)}(-i)\left(\frac{\partial}{\partial x_2}+i\frac{\partial}{\partial x_1}\right)(\overline{c}-1)dx_1dx_2$$

$$=\lim_{R\to\infty}\Bigg[\int_{x_1^2+x_2^2\leqq R^2} e^{-i(x_1\eta_2+x_2\eta_2)}(\eta_2+i\eta_1)(\overline{c}-1)dx_1dx_2$$

$$+\int_0^{2\pi} e^{iR(\eta_2\cos\theta+\eta_1\sin\theta)}R(\sin\theta+i\cos\theta)$$

$$\times(\overline{c}(R\cos\theta,\ R\sin\theta,x^\perp,\mu+iv)-1)d\theta\Bigg]\,. \tag{81}$$

Since

$$\overline{c}(R\cos\theta,R\sin\theta,x^\perp,\mu+iv)-1=\frac{1}{2\pi R}\cdot\frac{f(x^\perp)}{\cos\theta-i\sin\theta}+O\left(\frac{1}{R^2}\right),$$

the second integral in the limit in (81) goes to zero as R goes to infinity when $(\eta_1,\eta_2)\neq 0$. The first integral just goes to the Fourier transform of $\overline{c}-1$ in (x_1,x_2) multiplied by $(\eta_2+i\eta_1)=(\mu+iv)\cdot\eta$. Thus

$$J_2=-\int_{\mathbb{R}^n} e^{-iy\cdot l}V(y)(e^{i\varphi(y,\mu+iv)}-1)dy\,.$$

Thus $J_1-J_2=-\int_{\mathbb{R}^n}e^{-iy\cdot l}V(y)dy$. Since l is arbitrary, we have determined the Fourier transform of V and the proof is complete.

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