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The Time Evolution of a Class of Meta-stable States

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Abstract: The non-relativistic quantum mechanical description of meta-stable states which arise by perturbation of embedded eigenvalues is considered. The model given by the Hamiltonian

$$H = \begin{pmatrix} -\Delta & \lambda u \\ \lambda u & -\Delta + v \end{pmatrix}$$

is studied for small λ. If $-\Delta + v$ has a positive eigenvalue then, when $\lambda = 0$, H has an embedded eigenvalue. The corresponding eigenstate, Φ, is a meta-stable state for $\lambda \neq 0$. The time evolution of Φ under $H, e^{-itH}\Phi$, is estimated uniformly in t.

1. Introduction

One of the more striking effects described by quantum mechanics is the decay of an unstable state and the observed transformation of matter which accompanies the decay. Unstable states which remain close enough to their initial conditions for a long enough time to be observed are sometimes said to be meta-stable. The physics of quantum mechanical meta-stability is fairly well understood and formal schemes for approximating the relevent quantities have been developed ([GW, LL, M]). It is expected that the probability that a meta-stable state remains in its initial condition decreases exponentially in time, behaving like $e^{-\frac{t}{\tau}}$. The constant τ is said to be the lifetime of the state. Further, when a meta-stable state finally decays, the distribution of energies of the final state is found to be peaked about the energy of the initial state, the distribution having the Lorentzian shape, $\frac{1}{E^2+\gamma^2}$. Here E is the difference between the energies of the meta-stable state and the final state. The constant γ is called the width. It is found that (in units in which $\hbar = 1$) $\gamma = \frac{1}{\tau}$.

The problem of making these formal considerations precise has a long history (see [S1, S2 and RS]: it is well known that exponential decay cannot persist as $t \to \infty$). One approach which has had some success is the technique of dilation analyticity ([AC, BC, RS]). The dilation of a wave function $\phi \in L^2(\mathbf{R}^N)$ is given by $\phi(x) \mapsto a^{\frac{n}{2}}\phi(ax)$ for some $a > 0$. Such dilations are implemented by a unitary transformation. In its original form, the idea of dilation analyticity is to write $a = e^{\imath\theta}$ and then to analytically continue from imaginary θ to real θ. More generally, if H is the Hamiltonian describing a given system, and if $U(\theta)$ is a unitary operator for pure imaginary θ, then one considers analytic continuations in θ of $H_\theta = U(\theta)HU(\theta)^{-1}$. For real θ the operator H_θ is generally not self-adjoint. For appropriately chosen U it is shown that real eigenvalues of H_θ are eigenvalues of H. Complex eigenvalues are defined to be resonances. This technique has provided proofs in some examples that embedded eigenvalues are unstable under perturbation, by showing that they become complex eigenvalues of H_θ ([FW, RS, OY, Si, S1, S2]). This shows, in these examples, that states corresponding to embedded eigenvalues become unstable when perturbed in the sense that they are not eigenfunctions of the time evolution operator e^{-itH}.

To do better, the action of the time evolution operator e^{-itH} on a meta-stable state must be controlled. One would like to track the time evolution of a meta-stable state to see in what sense there is exponential decay and to find the state which describes its decay products. As Simon [S2] has pointed out, the leading order expression for the absolute value of the imaginary part of a resonance obtained from the dilation analyticity technique is equal to the inverse of the lifetime calculated by the physicists' methods. This suggests that dilation analyticity might be used to control time evolution.

There are some results on exponential decay ([D, FW, H, Ki, Sk]). The greatest generality is obtained by Hunziker [H] who considers a class of models amenable to dilation analyticity techniques. The Hamiltonians in these models are of the form $H = H_0 + \lambda V$, where λ is a small parameter. He studied the matrix element $\langle\phi, e^{-itH}\phi\rangle$, where ϕ is an eigenfunction of H_0 which perturbs to a resonant state in the sense of dilation analyticity. It was found (we quote his result to leading order only and in the simplest case where the eigenvalue of H_0 corresponding to ϕ is simple) that

$$\langle\phi, e^{-itH}\phi\rangle = e^{-it\varepsilon} + \mathcal{O}(\lambda^2),$$

where ε is the resonance corresponding to ϕ. Further, $\operatorname{Im}\varepsilon \sim -\lambda^2$. Since $\|e^{-itH}\| = 1$ one can see that $e^{-itH}\phi \sim \phi$ for times small compared to $|\operatorname{Im}\varepsilon|^{-1}$. This justifies the interpretation of $|\operatorname{Im}\varepsilon|^{-1}$ as the lifetime of ϕ. Since $|\operatorname{Im}\varepsilon|$ is small one concludes that ϕ is meta-stable under the action of e^{-itH}.

In this note we extend the ideas of Hunziker [H] to estimate the full time evolution of a meta-stable state. In particular we obtain asymptotics for the state describing the decay products. We consider a model which, though admittedly unphysical, is nonetheless nontrivial and captures much of the problem. Explicitly, we consider the model given by the Hamiltonian

$$H = \begin{pmatrix} -\Delta & \lambda u \\ \lambda u & -\Delta + v \end{pmatrix}$$

acting in $\mathbf{C}^2 \otimes L^2(\mathbf{R}^n)$; restrictions on u and v will be given in the next section. If $-\Delta + v$ has an isolated positive eigenvalue, ε_0, with corresponding eigenfunction ϕ, then one expects the state $\Phi = \binom{0}{\phi}$ to be unstable, the instability arising through the coupling λu to the continuous spectrum of $-\Delta$. In such models we are able to estimate $\Phi_t = e^{-itH}\Phi$ to leading order in λ uniformly in t. The approximations anticipated in the physics literature are obtained.

An example is given by taking $n = 1$, $v(x) = x^2$, $u(x) = e^{-x^2}$ and $\varepsilon_0 = 1$. The eigenfunction corresponding to $\varepsilon_0 = 1$ is $\phi(x) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}x^2}$. It is found that there is an $\tilde{\varepsilon} \in \mathbf{C}$, with $|1 - \tilde{\varepsilon}| = \mathcal{O}(\lambda^2)$ and

$$\operatorname{Im} \tilde{\varepsilon} = -\frac{\pi^{\frac{1}{2}}}{3e^{\frac{1}{3}}} \lambda^2 + \mathcal{O}(\lambda^4),$$

such that, using the notation $\hat{f}$ for the Fourier transform of f,

$$\hat{\Phi}_t(p) = e^{-i\tilde{\varepsilon}t} \begin{pmatrix} 0 \\ \hat{\phi}(p) \end{pmatrix} + \lambda \left(\frac{4\pi}{9} \right)^{\frac{1}{4}} \frac{e^{-\frac{1}{6}p^2}(e^{-i|p|^2 t} - e^{-i\tilde{\varepsilon}t})}{|p|^2 - \tilde{\varepsilon}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}(\lambda^{\frac{1}{2}}).$$

This estimate is uniform in t.

Φ has, to $\mathcal{O}(\lambda^{\frac{1}{2}})$, the behavior expected of a meta-stable state. It exhibits approximately exponential decay

$$\langle \Phi, \Phi_t \rangle = e^{-\frac{1}{3}\pi^{\frac{1}{2}} e^{-\frac{1}{3}} \lambda^2 t} + \mathcal{O}(\lambda^{\frac{1}{2}}).$$

Further, the transition probability from Φ to a state of momentum p, given by ([LL, GW])

$$\lim_{t \to \infty} |\hat{\Phi}_t(p)|^2 = \lambda^2 \left(\frac{4\pi}{9} \right)^{\frac{1}{2}} \frac{e^{-\frac{1}{3}p^2}}{(|p|^2 - 1)^2 + \frac{1}{9}\pi e^{-\frac{2}{3}} \lambda^4} + \mathcal{O}(\lambda^{\frac{1}{2}}),$$

is peaked about $|p| = 1$ with the expected Lorentzian form and with width, $\frac{1}{3}\pi^{\frac{1}{2}} e^{-\frac{1}{3}} \lambda^2$, equal to the inverse of the lifetime. Note that the decay products all lie in the $\binom{1}{0}$ sector and, in the long time limit, evolve according to the time evolution generated by $-\Delta$.

The paper is organized as follows. In Sect. 2 the full definition of the model is given and the main result is stated. In Sect. 3 it is shown that H has a dilation analytic resonance which is a perturbation of ε_0 in the sense that it approaches ε_0 as $\lambda \to 0$. In Sect. 4 it is shown how the results of Sect. 3 can be used to control the time development of the corresponding meta-stable state.

2. Description of the Model and Statement of Results

As mentioned in Sect. 1, we consider the model given by the Hamiltonian

$$H = \begin{pmatrix} -\Delta & \lambda u \\ \lambda u & -\Delta + v \end{pmatrix}$$

acting in $\mathbf{C}^2 \otimes L^2(\mathbf{R}^n)$. Here u and v are multiplication operators corresponding to the functions $u(x)$ and $v(x)$. Enough restrictions will be put on u and v so that the calculations can be carried through with a minimum of technical difficulty. We assume the following (in all that follows $\sigma(A)$ denotes the spectrum of the operator A).

Assumptions. *There is an* $\varepsilon_0 \in (\frac{1}{2}, \infty)$ *such that for all* θ *with* $\operatorname{Re}\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$ *the following is true.*

i) $v_\theta(x) = v(e^{i\theta}x)$ *is an analytic function of* θ *which is real valued when* $\theta = 0$ *such that* $-e^{-2i\theta}\Delta + v_\theta$ *is an analytic family of operators in the sense of Kato.*

ii) $u_\theta(x) = u(e^{i\theta}x)$ *and* $(x \cdot \nabla u)_\theta(x) = (x \cdot \nabla u)(e^{i\theta}x)$ *are analytic* $L^\infty(\mathbf{R}^n)$ *valued functions of* θ *which are real valued when* $\theta = 0$. *Further,* $e^{ie^{i\theta}p \cdot x}u_\theta(x)$ *is an analytic* $L^2(\mathbf{R}^n)$ *valued function of* θ.

iii) ε_0 *is a simple eigenvalue of* $-\Delta + v$. *Let* $\phi \in L^2(\mathbf{R}^n)$ *be its eigenfunction.*

iv) ε_0 *is an isolated point of* $\sigma(-e^{-2i\theta}\Delta + v_\theta)$.

v) $\langle e^{i\sqrt{\varepsilon_0}\,\hat{p} \cdot x}u, \phi\rangle \neq 0$ *as a function of* $\hat{p}$ *on* $S_{n-1} = \{p \in \mathbf{R}^n \,|\, |p| = 1\}$.

vi) *Both* $x_j u\phi$ *and* $x_j x_k u\phi$ *are in* $L^1(\mathbf{R}^n)$ *for all* $j,k \in \{1,2,3,\dots,n\}$.

Assumptions (i) and (ii) are made to insure that we can apply the complex dilation technique to H. Insisting that $e^{ie^{i\theta}p \cdot x}u_\theta(x)$ be analytic in $L^2(\mathbf{R}^n)$ is not necessary, but simplifies the exposition. Assumption (iii) is also a simplifying assumption. Allowing ε_0 to be finitely degenerate introduces no fundamental new difficulties into the problem. Assumption (iv) insures that ε_0 is in the pure point spectrum of $-\Delta + v$ and that $-\Delta + v$ has no resonances in the sense of dilation analyticity near ε_0. Assumption (v) is a sufficient condition for ε_0 to perturb to a resonance of H. It insures that the coupling of ϕ to the continuous spectrum of $-\Delta$ occurs at leading order. Finally, assumption (vi) insures that $\langle e^{ip \cdot x}u, \phi\rangle$ is twice differentiable in p. This simplifies certain parts of the proof.

We now state the main result that is obtained. A discussion of the result follows below.

Main Result. *Let*

$$r = \frac{1}{2}\min\left\{\operatorname{dist}(\sigma(-\Delta + v)\backslash\{\varepsilon_0\}, \varepsilon_0), \frac{1}{2}\right\},$$

let

$$\Phi(x) = \begin{pmatrix} 0 \\ \phi(x) \end{pmatrix},$$

and let $\widehat{u\phi}(p) = (2\pi)^{-\frac{n}{2}}\langle e^{ip \cdot x}u, \phi\rangle$ *denote the Fourier transform of the function* $u\phi$. *If* λ *is small enough there is an* $\tilde{\varepsilon} \in \mathbf{C}$ *with,*

$$|\operatorname{Re}\tilde{\varepsilon} - \varepsilon_0| \leqq \text{const } \lambda^2,$$

and

$$\left|\operatorname{Im}\tilde{\varepsilon} + \lambda^2\frac{\pi}{2}(\sqrt{\varepsilon_0})^{n-2}\int_{S_{n-1}} |\widehat{u\phi}(\sqrt{\varepsilon_0}\hat{p})|^2\, d\Omega(\hat{p})\right| \leqq \text{const } \lambda^4,$$

where $S_{n-1} = \{p \in \mathbf{R}^n \,|\, |p| = 1\}$ *and* $d\Omega$ *is the surface element on* S_{n-1}, *such that if*

$$\Phi_t^{(0)}(x) = e^{-i\tilde{\varepsilon}t}\Phi(x) + \lambda \int_{|p|^2 \in (\varepsilon_0 - \frac{r}{4},\, \varepsilon_0 + \frac{r}{4})} \frac{\widehat{u\phi}(p)}{|p|^2 - \tilde{\varepsilon}}\left(e^{-i|p|^2 t} - e^{-i\tilde{\varepsilon}t}\right)\begin{pmatrix} e^{ip \cdot x} \\ 0 \end{pmatrix} d^n p,$$

then

$$\|e^{-itH}\Phi - \Phi_t^{(0)}\| \leqq \text{const } \lambda^{\frac{1}{2}}.$$

Assumption (v) is a necessary and sufficient condition for

$$\int_{S_{n-1}} |\langle e^{i\sqrt{\varepsilon_0}p \cdot x}u, \phi\rangle|^2 d\Omega(\hat{p}) > 0,$$

so that $\operatorname{Im} \tilde{\varepsilon} \sim -\lambda^2$. In particular, assumption (v) is a sufficient condition for $\operatorname{Im} \tilde{\varepsilon} < 0$. We will see that $\operatorname{Im} \tilde{\varepsilon} = 0$ implies that $\tilde{\varepsilon}$ is an embedded eigenvalue for H. One does not expect H to have embedded eigenvalues. Rather one expects that $\operatorname{Im} \tilde{\varepsilon} < 0$ with some generality. If so, then, as mentioned above, $\tilde{\varepsilon}$ is generally referred to as a resonance for H in the sense of dilation analyticity. If it happens that $\langle e^{i\sqrt{\varepsilon_0}p \cdot x}u, \phi\rangle = 0$ for all $\hat{p} \in S_{n-1}$, then it is presumably sufficient for $\langle e^{ip \cdot x}u, \phi\rangle \neq 0$ as a function of p to show that $\operatorname{Im} \tilde{\varepsilon} < 0$. We have not investigated this possibility.

Since $\operatorname{Im} \tilde{\varepsilon} \sim -\lambda^2$ is small $\Phi_t^{(0)}$ has precisely the behavior expected of a meta-stable state with lifetime $|\operatorname{Im} \tilde{\varepsilon}|^{-1}$. For short times, $t \sim \mathcal{O}(1), \Phi_t^{(0)} = \Phi + \mathcal{O}(\lambda)$. As t increases the factor $e^{-i\tilde{\varepsilon}t}$ gives the expected exponential decrease of the component of $\Phi_t^{(0)}$ in the subspace spanned by the initial condition, Φ. As $t \to \infty$ the state $\Phi_t^{(0)}$ describes, to leading order in λ, the decay products of the unstable state Φ. For long times, $t \gg \frac{1}{\lambda^2}$, we have

$$\Phi_t^{(0)}(x) = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} \int_{|p|^2 \in (\varepsilon_0 - \frac{r}{4}, \varepsilon_0 + \frac{r}{4})} \frac{\widehat{u\phi}(p)}{|p|^2 - \tilde{\varepsilon}} e^{-i|p|^2 t + ip \cdot x} d^n p + \mathcal{O}(e^{\operatorname{Im} \tilde{\varepsilon} t}) .$$

This is the approximation for the state describing the decay products of Φ obtained by the formal arguments found in the physics literature ([GW, LL, M]). The transmission probability

$$\lim_{t \to \infty} |\hat{\Phi}_t^{(0)}(p)|^2 = \lambda^2 \frac{|\widehat{u\phi}(p)|^2}{(|p|^2 - \operatorname{Re} \tilde{\varepsilon})^2 + (\operatorname{Im} \tilde{\varepsilon})^2}$$

has the Lorentzian shape characteristic of decay phenomena and, indeed, the width is the inverse of the lifetime of the state.

3. The Dilation Analytic Resonance for H

In this section we show that there is a neighborhood of ε_0 in which H has no eigenvalues but has a resonance, $\tilde{\varepsilon}$, in the sense of dilation analyticity. Basic to the technique of dilation analyticity are the definitions

$$\psi_\theta(x) = e^{i\frac{n}{2}\theta} \psi(e^{i\theta} x) \tag{3.1}$$

for $\psi \in L^2(\mathbf{R}^n)$ and

$$H_\theta = \begin{pmatrix} -e^{-2i\theta} \Delta & \lambda u_\theta \\ \lambda u_\theta & -e^{-2i\theta} \Delta + v_\theta \end{pmatrix} . \tag{3.2}$$

If $\operatorname{Re} \theta = 0$ then the transformations (3.1) and (3.2) are unitary. As mentioned in Sect. 1, the dilation analyticity technique involves considering $\operatorname{Re} \theta > 0$. Eigenvalues of H_θ with non-zero imaginary parts are called dilation analytic resonances of H. We begin by deriving an expression for the resolvent of H_θ which we will be able to use to establish the analyticity of H_θ in θ and then afterwards to study the eigenvalue problem for H_θ.

Lemma 3.1. *Let*

$$h_\theta(\eta) = -e^{-2i\theta} \Delta + v_\theta - \lambda^2 u_\theta (-e^{-2i\theta} \Delta - \eta)^{-1} u_\theta \tag{3.3}$$

in $L^2(\mathbf{R}^n)$ *and let* $\eta \in \mathbf{C} \backslash e^{-2i\theta}[0, \infty)$.

a) *η is in the resolvent set of H_θ iff $h_\theta(\eta)-\eta$ is boundedly invertible. If so then*

$$(H_\theta-\eta)^{-1}=\begin{pmatrix} A_\theta(\eta) & -\lambda B_\theta(\eta) \\ -\lambda C_\theta(\eta) & D_\theta(\eta)\end{pmatrix}, \tag{3.4}$$

where

$$\begin{aligned}
A_\theta(\eta)&=(-e^{-2i\theta}\Delta-\eta)^{-1}+\lambda^2(-e^{-2i\theta}\Delta-\eta)^{-1}u_\theta(h_\theta(\eta)-\eta)^{-1}u_\theta(-e^{-2i\theta}\Delta-\eta)^{-1},\\
B_\theta(\eta)&=(-e^{-2i\theta}\Delta-\eta)^{-1}u_\theta(h_\theta(\eta)-\eta)^{-1},\\
C_\theta(\eta)&=(h_\theta(\eta)-\eta)^{-1}u_\theta(-e^{-2i\theta}\Delta-\eta)^{-1},\\
D_\theta(\eta)&=(h_\theta(\eta)-\eta)^{-1}.
\end{aligned} \tag{3.5}$$

b) *If η is in the resolvent set of H_θ then $(h_\theta(\eta)-\eta)^{-1}$ is an analytic bounded operator valued function of θ. It follows that H_θ is an analytic family in the sense of Kato.*

Proof. Since $\eta\notin e^{-2i\theta}[0,\infty)$ the resolvent $(-e^{-2i\theta}\Delta-\eta)^{-1}$ exists and is bounded. If, in addition, $h_\theta(\eta)-\eta$ is boundedly invertible then the operators in (3.5) are well defined and bounded. Equation (3.4) then follows from elementary algebra and is clearly bounded. Thus η is in the resolvent set of H_θ.

Conversely, if η is in the resolvent set of H_θ then for all $\Phi\in\mathbf{C}^2\otimes L^2(\mathbf{R}^n)$ there is a unique Ψ in the domain of H_θ with $\Phi=(H_\theta-\eta)\Psi$. If we write $\Phi=\binom{\phi_1}{\phi_2}$ and $\Psi=\binom{f_1}{f_2}$, where $\phi_j\in L^2(\mathbf{R}^n)$, f_1 is in the domain of $-\Delta$ and f_2 is in the domain of $-e^{-2i\theta}\Delta+v_\theta$, then

$$\begin{aligned}
\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}&=\begin{pmatrix} -e^{-2i\theta}\Delta-\eta & \lambda u_\theta \\ \lambda u_\theta & -e^{-2i\theta}\Delta+v_\theta-\eta\end{pmatrix}\begin{pmatrix} f_1\\ f_2\end{pmatrix}\\
&=\begin{pmatrix}(-e^{-2i\theta}\Delta-\eta)f_1+\lambda u_\theta f_2\\ \lambda u_\theta f_1+(-e^{-2i\theta}\Delta+v_\theta-\eta)f_2\end{pmatrix}.
\end{aligned}$$

But since $\eta\notin e^{-2i\theta}[0,\infty)$ we have

$$f_1=(-e^{-2i\theta}\Delta-\eta)^{-1}\phi_1-\lambda(-e^{-2i\theta}\Delta-\eta)^{-1}u_\theta f_2,$$

so that

$$\phi_2-\lambda u_\theta(-e^{-2i\theta}\Delta-\eta)^{-1}\phi_1=(h_\theta(\eta)-\eta)f_2.$$

Since the domains of $-e^{-2i\theta}\Delta+v_\theta$ and $h_\theta(\eta)$ are equal it follows that for all $\phi\in L^2(\mathbf{R}^3)$ there is an f in the domain of $h_\theta(\eta)$ such that $\phi=(h_\theta(\eta)-\eta)f$. Thus $h_\theta(\eta)-\eta$ is boundedly invertible. This finishes the proof of part (a).

Let η be in the resolvent set of H_θ. By assumption $(-e^{-2i\theta}\Delta+v_\theta-\eta)^{-1}$ and u_θ are analytic bounded operator valued functions of θ. Since $\eta\notin e^{-2i\theta}[0,\infty)$ the resolvent $(-e^{-2i\theta}\Delta-\eta)^{-1}$ is also an analytic bounded operator valued function of θ. It follows immediately that $(h_\theta(\eta)-\eta)^{-1}$ is a bounded operator valued analytic function of θ.

Finally, since $(-e^{-2i\theta}\Delta-\eta)^{-1}$ is a bounded operator valued analytic function of θ, it now follows easily from part (a) and the assumptions that $(H_\theta-\eta)^{-1}$ is a bounded operator valued analytic function of θ. Thus H_θ is an analytic family in the sense of Kato. This finishes the proof of part (b). $\square$

We now show that H_θ has an eigenvalue which approaches ε_0 as $\lambda \to 0$. The imaginary part of this eigenvalue is estimated and is found to be negative. This establishes the existence, and estimates the position, of a dilation analytic resonance of H near ε_0.

Proposition 3.2. *Let*

$$r = \frac{1}{2}\min\left\{\operatorname{dist}(\sigma(-\Delta+v)\backslash\{\varepsilon_0\},\varepsilon_0), \frac{1}{2}\right\}.$$

Fix $\theta \in (\frac{\pi}{12}, \frac{\pi}{4})$, let H_θ be as in (3.2) *and let $h_\theta(\eta)$ be as in* (3.3). *There is a $\lambda_0 > 0$ such that if $\lambda < \lambda_0$ then the following is true.*

a) *There is an $\tilde{\varepsilon} \in \mathbf{C}$ such that $(h_\theta(\eta)-\eta)^{-1}$ is analytic in η for $\eta \in D_r(\varepsilon_0)\backslash\{\tilde{\varepsilon}\}$ and has a simple pole at $\eta = \tilde{\varepsilon}$.*

b) *If $D_r(\varepsilon_0) = \{\varepsilon \in \mathbf{C} \mid |\varepsilon - \varepsilon_0| < r\}$ then*

$$\sigma(H_\theta) \cap D_r(\varepsilon_0) = \{\tilde{\varepsilon}\}.$$

Further, $\tilde{\varepsilon}$ is a simple eigenvalue of H_θ. $\operatorname{Re}\tilde{\varepsilon}$ *satisfies*

$$|\operatorname{Re}\tilde{\varepsilon} - \varepsilon_0| \leqq \text{const } \lambda^2$$

and $\operatorname{Im}\tilde{\varepsilon} < 0$ *and is estimated by*

$$\left|\operatorname{Im}\tilde{\varepsilon} + \lambda^2 \frac{(\sqrt{\varepsilon_0})^{n-2}}{4(2\pi)^{n-1}} \int_{S_{n-1}} |\langle e^{i\sqrt{\varepsilon_0}\hat{p}\cdot x} u, \phi\rangle|^2 \, d\Omega(\hat{p})\right| \leqq \text{const } \lambda^4,$$

where $S_{n-1} = \{p \in \mathbf{R}^n \mid |p| = 1\}$ and $d\Omega$ is the surface element on S_{n-1}.

Proof. First note that $(h_\theta(\eta)-\eta)^{-1}$ is a bounded operator valued analytic function of η for η in the resolvent set of H_θ and $\eta \notin e^{-2i\theta}[0,\infty)$. If $|\eta - \varepsilon_0| = r$ then by assumption (iii) the resolvent $(-e^{-2i\theta}\Delta + v_\theta - \eta)^{-1}$ is bounded so that

$$\|(-e^{-2i\theta}\Delta + v_\theta - \eta)^{-1} u_\theta (-e^{-2i\theta}\Delta - \eta)^{-1} u_\theta\| \leqq \text{const}.$$

It follows that for λ small enough

$$\begin{aligned}(h_\theta(\eta)-\eta)^{-1} =& (-e^{-2i\theta}\Delta + v_\theta - \eta)^{-1} \\ &\times (1 - \lambda^2 u_\theta(-e^{-2i\theta}\Delta - \eta)^{-1} u_\theta (-e^{-2i\theta}\Delta + v_\theta - \eta)^{-1})^{-1}\end{aligned}$$

exists and is bounded and satisfies

$$\|(h_\theta(\eta)-\eta)^{-1} - (-e^{-2i\theta}\Delta + v_\theta - \eta)^{-1}\| \leqq \text{const } \lambda^2. \tag{3.6}$$

That $(h_\theta(\eta)-\eta)^{-1}$ has a simple pole, $\tilde{\varepsilon}$, in $D_r(\varepsilon_0)$ with $|\tilde{\varepsilon} - \varepsilon_0| \leqq \text{const } \lambda^2$ follows from the general theory of analytic perturbations [K]. Standard arguments show that

$$|\tilde{\varepsilon} + \lambda^2 \langle \phi_{-\theta}, u_\theta(-e^{-2i\theta}\Delta - \varepsilon_0)^{-1} u_\theta \phi_\theta\rangle| \leqq \text{const } \lambda^4. \tag{3.7}$$

This establishes part (a).

That $\sigma(H_\theta)\cap D_r(\varepsilon_0)=\{\tilde{\varepsilon}\}$ and that $\tilde{\varepsilon}$ is a simple eigenvalue of H_θ follows from part (a) of Lemma 3.1. To establish the estimates for $\tilde{\varepsilon}$ we note first that (3.7) implies that both the real and imaginary parts of $\tilde{\varepsilon}$ are bounded by a constant times λ^2. To estimate the imaginary part of $\tilde{\varepsilon}$ note that ϕ_θ is an eigenfunction of an analytic family of operators and thus is an $L^2(\mathbf{R}^n)$ valued analytic function of θ. It follows that the inner product $\langle\phi_{-\theta},u_\theta(-e^{-2i\theta}\Delta-\eta)^{-1}u_\theta\phi_\theta\rangle$ is analytic in θ. Further, this inner product is independent of $\operatorname{Im}\theta$, and is therefore independent of θ, in the strip $0<\operatorname{Re}\theta<\frac{\pi}{4}$. Representing the integral kernel for $(-e^{-2i\theta}\Delta-\eta)^{-1}$ using the Fourier transform we have

$$
\begin{aligned}
&\langle\phi_{-\theta},u_\theta(-e^{-2i\theta}\Delta-\eta)^{-1}u_\theta\phi_\theta\rangle\\
&=\lim_{\theta\downarrow 0}\langle\phi_{-\theta},u_\theta(-e^{-2i\theta}\Delta-\varepsilon_0)^{-1}u_\theta\phi_\theta\rangle\\
&=\frac{1}{(2\pi)^n}\lim_{\theta\downarrow 0}\int_0^\infty\int_{S_{n-1}}\langle\phi_{-\theta},u_\theta e^{i\zeta p\,\cdot\,x}\rangle\langle e^{i\zeta p\,\cdot\,x}u_{-\theta},\phi_\theta\rangle\,d\Omega(\hat{p})\frac{1}{e^{-2i\theta}\zeta^2-\varepsilon_0}\zeta^{n-1}d\zeta\\
&=\frac{1}{(2\pi)^n}\lim_{\xi\downarrow 0}\left(\int_0^{\sqrt{\varepsilon_0}-\xi}\int_{S_{n-1}}|\langle e^{i\zeta p\,\cdot\,x}u,\phi\rangle|^2\,d\Omega(\hat{p})\frac{1}{\zeta^2-\varepsilon_0}\zeta^{n-1}d\zeta\right.\\
&\qquad\left.+\int_{\sqrt{\varepsilon_0}+\xi}^\infty\int_{S_{n-1}}|\langle e^{i\zeta p\,\cdot\,x}u,\phi\rangle|^2\,d\Omega(\hat{p})\frac{1}{\zeta^2-\varepsilon_0}\zeta^{n-1}d\zeta\right)\\
&\quad+i\frac{(\sqrt{\varepsilon_0})^{n-2}}{4(2\pi)^{n-1}}\int_{S_{n-1}}|\langle e^{i\sqrt{\varepsilon_0}p\,\cdot\,x}u,\phi\rangle|^2\,d\Omega(\hat{p})\,,
\end{aligned}
$$

where $S_{n-1}=\{p\in\mathbf{R}^n\,\big|\,|p|=1\}$ and $d\Omega$ is the surface element on S_{n-1}. The estimate for $\operatorname{Im}\tilde{\varepsilon}$ now follows from

$$
\operatorname{Im}\langle\phi_{-\theta},u_\theta(-e^{-2i\theta}\Delta-\varepsilon_0)^{-1}u_\theta\phi_\theta\rangle=\frac{(\sqrt{\varepsilon_0})^{n-2}}{4(2\pi)^{n-1}}\int_{S_{n-1}}|\langle e^{i\sqrt{\varepsilon_0}p\,\cdot\,x}u,\phi\rangle|^2\,d\Omega(\hat{p})\,,
$$

which is clearly non-negative. That it is also non-zero follows from assumption (v). It follows that $\operatorname{Im}\tilde{\varepsilon}<0$ if λ is small enough. □

Using Lemma 3.1, Proposition 3.2 and $\operatorname{Im}\tilde{\varepsilon}<0$ it can be shown, as in [RS] volume IV theorem XIII.36, that H has no point spectrum in $(\varepsilon_0-r,\varepsilon_0+r)$, since real eigenvalues of H_θ in $D_r(\varepsilon_0)$ are eigenvalues of H and conversely. This is enough to conclude that the state $\binom{0}{\phi}$ is unstable since it is not close to any eigenfunctions of H. In the next section its time evolution is estimated.

4. The Propogation Estimate

If $\lambda=0$ then ε_0 is an eigenvalue of H. Let Φ be its eigenvector. In Sect. 3 we showed that, for $\lambda\neq 0$, H has no eigenvalues near ε_0. One expects Φ to represent a meta-stable state under the time evolution generated by H, e^{-itH}. In this section we use the results of Sect. 3 to estimate $e^{-itH}\Phi$.

Proposition 4.1. *Let $\phi \in L^2(\mathbf{R}^n)$ be the normalized eigenfunction of $-\Delta + v$ corresponding to the eigenvalue ε_0, let $r = \frac{1}{2}\min\{\operatorname{dist}(\sigma(-\Delta+v)\backslash\{\varepsilon_0\}, \varepsilon_0), \frac{1}{2}\}$, let $p^2 \in (\varepsilon_0 - \frac{r}{4}, \varepsilon_0 + \frac{r}{4})$, let $\xi \in C_0^\infty(\mathbf{R})$ satisfy*

$$\xi(\varepsilon) = \begin{cases} 1 & \text{if } \varepsilon \in [\varepsilon_0 - \frac{r}{2}, \varepsilon_0 + \frac{r}{2}] \\ 0 & \text{if } \varepsilon \in (-\infty, \varepsilon_0 - r] \cup [\varepsilon_0 + r, \infty) \end{cases} \tag{4.1}$$

and let $\tilde{\varepsilon}$ be the resonance found in Proposition 3.2. *Let*

$$\Phi(x) = \begin{pmatrix} 0 \\ \phi(x) \end{pmatrix}$$

and

$$\Psi_p(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \begin{pmatrix} e^{ip \cdot x} \\ 0 \end{pmatrix}.$$

If λ is small enough then for each $N \in \{1,2,3,\dots\}$ there is a constant $c(N)$ and for each $j \in \{1,2\}$ a function $a_j(p)$ which satisfies

$$|a_j(p) - (2\pi)^{-\frac{n}{2}}\langle e^{ip\cdot x}u, \phi\rangle| \leqq \text{ const } \lambda^2$$

such that

$$|\langle \Psi_p, e^{-itH}\xi(H)\Phi\rangle - \frac{\lambda}{p^2 - \tilde{\varepsilon}}(a_1(p)e^{-ip^2t} - a_2(p)e^{-i\tilde{\varepsilon}t})| \leqq c(N)\lambda^3(1+t)^{-N}.$$

Proof. Using Stone's formula we have

$$\langle \Psi_p, e^{-itH}\xi(H)\Phi\rangle = \frac{1}{2\pi i}\lim_{\zeta\downarrow 0}\int_{-\infty}^{\infty}\xi(\varepsilon)e^{-i\varepsilon t}\langle \Psi_p, ((H - \varepsilon + i\zeta)^{-1} - (H - \varepsilon - i\zeta)^{-1})\Phi\rangle\, d\varepsilon.$$

For $\eta \notin \mathbf{R}$ the operator $H - \eta$ is invertible. As in (3.4) we have

$$(H - \eta)^{-1} = \begin{pmatrix} A(\eta) & -\lambda B(\eta) \\ -\lambda C(\eta) & D(\eta) \end{pmatrix},$$

where as a general convention we drop the subscript θ when $\theta = 0$. It follows that

$$\langle \Psi_p, e^{-itH}\xi(H)\Phi\rangle = \frac{i\lambda}{(2\pi)^{\frac{n}{2}+1}}\lim_{\zeta\downarrow 0}\int_{-\infty}^{\infty}\xi(\varepsilon)e^{-i\varepsilon t}(\langle e^{ip\cdot x}, B(\varepsilon - i\zeta)\phi\rangle - \langle e^{ip\cdot x}, B(\varepsilon + i\zeta)\phi\rangle)\, d\varepsilon.$$

Explicitly we have

$$\begin{aligned}
&\langle e^{ip\cdot x}, B(\varepsilon \pm i\zeta)\phi\rangle \\
&= \langle e^{ip\cdot x}, (-\Delta - \varepsilon \pm i\zeta)^{-1}u(-\Delta + v - \varepsilon \pm i\zeta - \lambda^2 u(-\Delta - \varepsilon \pm i\zeta)^{-1}u)^{-1}\phi\rangle \\
&= \frac{1}{p^2 - \varepsilon \pm i\zeta}\langle e^{ip\cdot x}u, \left(-\Delta + v - \varepsilon \pm i\zeta - \lambda^2 u(-\Delta - \varepsilon \pm i\zeta)^{-1}u\right)^{-1}\phi\rangle.
\end{aligned}$$

Let

$$\begin{aligned} f_\theta(\eta) &= \langle e^{ie^{-i\theta}p\,\cdot\,x}u_{-\theta}, (-e^{-2i\theta}\Delta + v_\theta - \eta - \lambda^2 u_\theta(-e^{-2i\theta}\Delta - \eta)^{-1}u_\theta)^{-1}\phi_\theta\rangle \\ &= \langle e^{ie^{-i\theta}p\,\cdot\,x}u_{-\theta}, (h_\theta(\eta) - \eta)^{-1}\phi_\theta\rangle\,. \end{aligned} \tag{4.2}$$

If $\operatorname{Im}\eta > 0$ then, by analyticity arguments as in [RS] volume IV theorem XIII.36, η is in the resolvent set of H_θ for all θ with $0 \leqq \operatorname{Re}\theta < \frac{\pi}{4}$. Thus, by part (b) of Lemma 3.1 and assumptions (i) and (ii), $f_\theta(\eta)$ is analytic in θ in the strip $0 < \operatorname{Re}\theta < \frac{\pi}{4}$. Further, $f_\theta(\eta)$ is independent of $\operatorname{Im}\theta$. It follows that $f_\theta(\varepsilon + i\zeta)$ is independent of θ in this strip. Similarly, since $h_\theta(\eta)^* = h_{-\theta}(\bar\eta)$, $f_{-\theta}(\varepsilon - i\zeta)$ is independent of θ in this same strip. Thus

$$\begin{aligned} f_{\pm\theta}(\varepsilon \pm i\zeta) &= \lim_{\operatorname{Re}\theta\downarrow 0} f_{\pm\theta}(\varepsilon \pm i\zeta) \\ &= \langle e^{ip\,\cdot\,x}u, (-\Delta + v - \varepsilon \mp i\zeta - \lambda^2 u(-\Delta - \varepsilon \mp i\zeta)^{-1}u)^{-1}\phi\rangle\,, \end{aligned}$$

so that we can write

$$\langle \Psi_p, e^{-itH}\xi(H)\Phi\rangle = \frac{i\lambda}{(2\pi)^{\frac{n}{2}+1}} \lim_{\zeta\downarrow 0} \int_{-\infty}^{\infty} \xi(\varepsilon)e^{-i\varepsilon t}\left(\frac{f_{-\theta}(\varepsilon - i\zeta)}{p^2 - \varepsilon + i\zeta} - \frac{f_\theta(\varepsilon + i\zeta)}{p^2 - \varepsilon - i\zeta}\right) d\varepsilon\,. \tag{4.3}$$

Fix $\theta \in (\frac{\pi}{12}, \frac{\pi}{4})$. Let $\tilde\xi \in C_0^\infty(\mathbf{R})$ be a non-negative cutoff function with

$$\tilde\xi(\varepsilon) = \begin{cases} \frac{r}{4} & \text{if } \varepsilon \in \left[\varepsilon_0 - \frac{r}{4}, \varepsilon_0 + \frac{r}{4}\right] \\ 0 & \text{if } \varepsilon \in \left(-\infty, \varepsilon_0 - \frac{r}{2}\right] \cup \left[\varepsilon_0 + \frac{r}{2}, \infty\right) \end{cases}$$

and let $\Gamma \subset \mathbf{C}$ be the curve parameterized by $\Gamma(\varepsilon) = \varepsilon - i\tilde\xi(\varepsilon)$. In the interior of the closed curve $\mathbf{R} \cup \Gamma$ the function $f_\theta(\eta)$ is meromorphic in η with a simple pole at $\tilde\varepsilon$ while $f_{-\theta}(\eta)$ is analytic. Since $p^2 \in (\varepsilon_0 - \frac{r}{4}, \varepsilon_0 + \frac{r}{4})$ we have

$$\begin{aligned} \langle \Psi_p, e^{-itH}\xi(H)\Phi\rangle =& \frac{i\lambda}{(2\pi)^{\frac{n}{2}+1}} \int_\Gamma \xi(\operatorname{Re}\eta)\frac{e^{-i\eta t}}{p^2 - \eta}(f_{-\theta}(\eta) - f_\theta(\eta))\,d\eta \\ & - \frac{\lambda}{(2\pi)^{\frac{n}{2}}} e^{-ip^2 t} f_\theta(p^2) + \frac{\lambda}{(2\pi)^{\frac{n}{2}}}\frac{1}{p^2 - \tilde\varepsilon} e^{-i\tilde\varepsilon t}\operatorname{Res}(f_\theta(\varepsilon), \tilde\varepsilon)\,, \end{aligned}$$

where $\operatorname{Res}(f(z), w)$ is the residue of f at w.

Consider first the integral over Γ. If we let

$$g_\theta(\eta) = f_\theta(\eta) - \langle e^{ip\,\cdot\,x}u, \phi\rangle \frac{1}{\varepsilon_0 - \eta}$$

for $\eta \in \Gamma$ then, recalling that $(-e^{-2i\theta}\Delta + v_\theta)\phi_\theta = \varepsilon_0\phi_\theta$ and noting that $\langle e^{ie^{-i\theta}p\,\cdot\,x}u_\theta, \phi_\theta\rangle$ is independent of θ, it follows that

$$\begin{aligned} g_\theta(\eta) =& \frac{1}{\varepsilon_0 - \eta}\sum_{m=1}^{\infty} \lambda^{2m}\langle e^{ie^{-i\theta}p\,\cdot\,x}u_{-\theta}, \\ & \times ((-e^{-2i\theta}\Delta + v_\theta - \eta)^{-1}u_\theta(-e^{-2i\theta}\Delta - \eta)^{-1}u_\theta)^m\phi_\theta\rangle \end{aligned}$$

converges uniformly on $\{\eta \in \Gamma | \xi(\operatorname{Re}\eta) \neq 0\}$. Using this expansion it is straightforward to show that for each $N \in \{0, 1, 2, \ldots\}$ there is a constant $c_1(N)$ such that

$$\left|\frac{d^N g_\theta(\eta)}{d\eta^N}\right| \leqq c_1(N)\,\lambda^2 .$$

It follows that there is a constant $c_2(N)$ such that

$$\begin{aligned}\int_\Gamma \xi(\operatorname{Re}\eta)\frac{e^{-i\eta t}}{p^2-\eta}(f_{-\theta}(\eta)-f_\theta(\eta))\,d\eta &= \int_\Gamma \xi(\operatorname{Re}\eta)\frac{e^{-i\eta t}}{p^2-\eta}(g_{-\theta}(\eta)-g_\theta(\eta))\,d\eta \\ &\leqq c_2(N)\,\lambda^2(1+t)^{-N} .\end{aligned}$$

Now consider the second of the two residues. We have, using (3.6),

$$\begin{aligned}\operatorname{Res}(f_\theta(\varepsilon), \tilde{\varepsilon}) &= \frac{1}{2\pi i}\int_{|\eta-\varepsilon_0|=r} f_\theta(\eta)\,d\eta \\ &= \frac{1}{2\pi i}\int_{|\eta-\varepsilon_0|=r} \langle e^{ie^{-i\theta}p\,\cdot\,x}u_{-\theta}, (h_\theta(\eta)-\eta)^{-1}\phi_\theta\rangle d\eta \\ &= -\langle e^{ie^{-i\theta}p\,\cdot\,x}u_{-\theta}, \phi_\theta\rangle + \mathcal{O}(\lambda^2) \\ &= -\langle e^{ip\,\cdot\,x}u, \phi\rangle + \mathcal{O}(\lambda^2)\end{aligned}$$

so that

$$\frac{\lambda}{p^2-\tilde{\varepsilon}}e^{-i\tilde{\varepsilon}t}\operatorname{Res}(f_\theta(\varepsilon),\tilde{\varepsilon}) = \frac{\lambda}{p^2-\tilde{\varepsilon}}\varepsilon^{-i\tilde{\varepsilon}t}(-\langle e^{ip\,\cdot\,x}u,\phi\rangle + \mathcal{O}(\lambda^2)) .$$

Finally we consider $f_\theta(p^2)$. Separating out the singular part of $f_\theta(\eta)$ near $\eta = \tilde{\varepsilon}$ we have

$$f_\theta(p^2) = \frac{1}{p^2-\tilde{\varepsilon}}\operatorname{Res}(f_\theta(\varepsilon),\tilde{\varepsilon}) + R(p^2) ,$$

where

$$R(p^2) = \frac{1}{2\pi i}\int_{|\eta-\varepsilon_0|=r} (\eta-p^2)^{-1} f_\theta(\eta)\,d\eta$$

is the regular part of f_θ at p^2. Using (3.6) and recalling that $p^2 \in (\varepsilon_0 - \frac{r}{4}, \varepsilon_0 + \frac{r}{4})$ we have

$$\left|R(p^2) - \frac{1}{2\pi i}\int_{|\eta-\varepsilon_0|=r}\frac{1}{\eta-p^2}\frac{\langle e^{ip\,\cdot\,x}u,\phi\rangle}{\varepsilon_0-\eta}d\eta\right| \leqq \text{ const } \lambda^2 .$$

But

$$\int_{|\eta-\varepsilon_0|=r}\frac{1}{(n-p^2)(\varepsilon_0-\eta)}d\eta = 0$$

so that

$$|R(p^2)| \leqq \text{ const } \lambda^2 .$$

It follows that

$$f_\theta(p^2) = \frac{1}{p^2-\tilde{\varepsilon}}(-\langle e^{ip\,\cdot\,x}u,\phi\rangle + \mathcal{O}(\lambda^2)) .$$

Thus we arrive at the estimate

$$\langle \Psi_p, e^{-itH}\xi(H)\Phi\rangle = \frac{\lambda}{(2\pi)^{\frac{n}{2}}}\frac{1}{p^2-\tilde{\varepsilon}}(\langle e^{ip\cdot x}u,\phi\rangle + \mathcal{O}(\lambda^2))\left(e^{-ip^2t}-e^{-i\tilde{\varepsilon}t}\right)$$
$$+\mathcal{O}(\lambda^3(1+t)^{-N}).$$

This proves the proposition. □

We end this section by using Proposition 4.1 to prove the main result quoted in Sect. 1. Let

$$H_0 = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta + v \end{pmatrix}$$

and let $P = P_{(\varepsilon_0-\frac{r}{4},\varepsilon_0+\frac{r}{4})}$ be the spectral projection for H_0 onto the subspace of $\mathbf{C}^2 \otimes L^2(\mathbf{R}^n)$ in which H_0 has spectrum $(\varepsilon_0 - \frac{r}{4}, \varepsilon_0 + \frac{r}{4})$. By assumption, P has the integral kernel

$$P(x,y) = \Phi(x)\otimes\Phi^*(y) + \frac{1}{(2\pi)^n}\int_{|p|^2\in(\varepsilon_0-\frac{r}{4},\varepsilon_0+\frac{r}{4})}\Psi_p(x)\otimes\Psi_p^*(y)\,d^np\,, \tag{4.4}$$

where Φ and Ψ_p are as in Proposition 4.1 and $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}^* = (\bar{\phi}_1\ \bar{\phi}_2)$. Recall the cut-off function ξ introduced in Proposition 4.1. The content of Proposition 4.1 is an estimate for the continuum matrix elements $\langle\Psi_p, e^{-itH}\xi(H)\Phi\rangle$. The matrix element $\langle\Phi, e^{-itH}\xi(H)\Phi\rangle$ is estimated by Hunziker in Theorem 1 of [H]. It is found that there is an a, depending on λ and satisfying $|a-1| \leqq \text{const}\,\lambda^2$, and, for each $N\in\{1,2,3,\ldots\}$, a constant $b(N)$ such that

$$|\langle\Phi, e^{-itH}\xi(H)\Phi\rangle - ae^{-i\tilde{\varepsilon}t}| \leqq b(N)\,\lambda^2(1+t)^{-N}\,, \tag{4.5}$$

uniformly for $t\in[0,\infty)$. Using (4.5) and Proposition 4.1 we can estimate $Pe^{-itH}\xi(H)\Phi$. We now show that this is enough to estimate $e^{-itH}\Phi$ itself.

Proposition 4.2. *Let*

$$\Phi_t^{(0)}(x) = e^{-i\tilde{\varepsilon}t}\Phi(x) + \lambda\int_{|p|^2\in(\varepsilon_0-\frac{r}{4},\varepsilon_0+\frac{r}{4})}\frac{\widehat{u\phi}(p)}{|p|^2-\tilde{\varepsilon}}(e^{-i|p|^2t}-e^{-i\tilde{\varepsilon}t})\begin{pmatrix}e^{ip\cdot x}\\ 0\end{pmatrix}d^np\,,$$

where $\Phi(x) = \begin{pmatrix}0\\ \phi\end{pmatrix}$ *is as in Proposition* 4.1 *and* $\widehat{u\phi}(p) = (2\pi)^{-\frac{n}{2}}\langle e^{ip\cdot x}u,\phi\rangle$ *is the Fourier transform of the function* $u\phi$. *If* $\operatorname{Im}\tilde{\varepsilon}\neq 0$ *then*

$$\|e^{-itH}\Phi - \Phi_t^{(0)}\| \leqq \text{const}\,\lambda^{\frac{1}{2}}$$

uniformly in t.

Proof. We must bound

$$\|e^{-itH}\Phi - \Phi_t^{(0)}\|^2 = 1 + \|\Phi_t^{(0)}\|^2 - 2\operatorname{Re}\langle e^{-itH}\Phi, \Phi_t^{(0)}\rangle\,. \tag{4.6}$$

We begin by computing the inner product $\langle e^{-itH}\Phi, \Phi_t^{(0)}\rangle$. Note that $P\Phi_t^{(0)} = \Phi_t^{(0)}$ so that $\langle e^{-itH}\Phi, \Phi_t^{(0)}\rangle = \langle Pe^{-itH}\Phi, \Phi_t^{(0)}\rangle$. Recalling the cutoff function ξ introduced in Proposition 4.1 we have

$$\langle e^{-itH}\Phi, \Phi_t^{(0)}\rangle = \|\Phi_t^{(0)}\|^2 + \langle Pe^{-itH}\xi(H)\Phi - \Phi_t^{(0)}, \Phi_t^{(0)}\rangle + \langle Pe^{-itH}(1-\xi(H))\Phi, \Phi_t^{(0)}\rangle .$$

As in [H] we can use (4.5) to bound $\|(1-\xi(H))\Phi\|$. The only condition on ξ is that it be in $C_0^\infty(\mathbf{R})$ and satisfy (4.1). Thus (4.5) remains true if ξ is replaced by ξ^2. We have

$$\|(1-\xi(H))\Phi\|^2 = 1 - 2\langle\Phi, \xi(H)\Phi\rangle + \langle\Phi, \xi(H)^2\Phi\rangle .$$

Setting $t = 0$ in (4.5) we find that $\langle\Phi, \xi(H)\Phi\rangle = 1 + \mathcal{O}(\lambda^2)$ and $\langle\Phi, \xi(H)^2\Phi\rangle = 1 + \mathcal{O}(\lambda^2)$ so that

$$\|(1-\xi(H))\Phi\| \leqq \text{const}\,\lambda .$$

It follows that

$$|\langle Pe^{-itH}(1-\xi(H))\Phi, \Phi_t^{(0)}\rangle| \leqq \text{const}\,\lambda\|\Phi_t^{(0)}\| .$$

The inner product $\langle Pe^{-itH}\xi(H)\Phi - \Phi_t^{(0)}, \Phi_t^{(0)}\rangle$ is estimated by noting that

$$(Pe^{-itH}\xi(H)\Phi)(x) = \langle\Phi, e^{-itH}\xi(H)\Phi\rangle\Phi(x) + \frac{1}{(2\pi)^n}\int_{|p|^2\in(\varepsilon_0-\frac{r}{4},\,\varepsilon_0+\frac{r}{4})} \langle\Psi_p, e^{-itH}\xi(H)\Phi\rangle\,\Psi_p(x)\,d^np$$

and using Proposition 4.1 and (4.5). One finds that

$$|\langle Pe^{-itH}\xi(H)\Phi - \Phi_t^{(0)}, \Phi_t^{(0)}\rangle| \leqq \text{const}\,\lambda^2\,\|\Phi_t^{(0)}\| .$$

Thus

$$|\langle e^{-itH}\Phi, \Phi_t^{(0)}\rangle - \|\Phi_t^{(0)}\|^2| \leqq \text{const}\,\lambda\,\|\Phi_t^{(0)}\| . \tag{4.7}$$

All that remains is to estimate the norm of $\Phi_t^{(0)}$. We have

$$\|\Phi_t^{(0)}\|^2 = |e^{-i\tilde\varepsilon t}|^2 + \frac{\lambda^2}{(2\pi)^n}\int_{|p|^2\in(\varepsilon_0-\frac{r}{4},\,\varepsilon_0+\frac{r}{4})}\left|\frac{\langle e^{ip\cdot x}u, \phi\rangle}{|p|^2-\tilde\varepsilon}(e^{-i|p|^2t} - e^{-i\tilde\varepsilon t})\right|^2 d^np$$
$$= e^{2\operatorname{Im}\tilde\varepsilon t} + \lambda^2\int_{\sqrt{\varepsilon_0-\frac{r}{4}}}^{\sqrt{\varepsilon_0+\frac{r}{4}}}\frac{1-2e^{\operatorname{Im}\tilde\varepsilon t}\cos((\zeta^2-\operatorname{Re}\tilde\varepsilon)t) + e^{2\operatorname{Im}\tilde\varepsilon t}}{(\zeta^2-\operatorname{Re}\tilde\varepsilon)^2 + (\operatorname{Im}\tilde\varepsilon)^2}\,g(\zeta)\,\zeta d\zeta ,$$

where

$$g(\zeta) = (2\pi)^{-n}\zeta^{n-2}\int_{S_{n-1}}|\langle e^{i\zeta\hat p\cdot x}u, \phi\rangle|^2\,d\Omega(\hat p) .$$

It follows from assumption (vi) that $g'(\zeta)$ and $g''(\zeta)$ are bounded on $[\sqrt{\varepsilon_0 - \frac{r}{4}}, \sqrt{\varepsilon_0 + \frac{r}{4}}]$. Using Taylor's theorem we can estimate the integral

$$\int_{\sqrt{\varepsilon_0-\frac{r}{4}}}^{\sqrt{\varepsilon_0+\frac{r}{4}}} \frac{1 - 2e^{\operatorname{Im}\tilde{\varepsilon}t}\cos((\zeta^2 - \operatorname{Re}\tilde{\varepsilon})t) + e^{2\operatorname{Im}\tilde{\varepsilon}t}}{(\zeta^2 - \operatorname{Re}\tilde{\varepsilon})^2 + (\operatorname{Im}\tilde{\varepsilon})^2} g(\zeta)\,\zeta d\zeta$$

$$= \frac{1}{2}\int_{\varepsilon_0-\operatorname{Re}\tilde{\varepsilon}-\frac{r}{4}}^{\varepsilon_0-\operatorname{Re}\tilde{\varepsilon}+\frac{r}{4}} \frac{1 - 2e^{\operatorname{Im}\tilde{\varepsilon}t}\cos(xt) + e^{2\operatorname{Im}\tilde{\varepsilon}t}}{x^2 + (\operatorname{Im}\tilde{\varepsilon})^2} g(\sqrt{\operatorname{Re}\tilde{\varepsilon} + x})\,dx$$

$$= \frac{1}{2} g(\sqrt{\operatorname{Re}\tilde{\varepsilon}}) \int_{\varepsilon_0-\operatorname{Re}\tilde{\varepsilon}-\frac{r}{4}}^{\varepsilon_0-\operatorname{Re}\tilde{\varepsilon}+\frac{r}{4}} \frac{1 - 2e^{\operatorname{Im}\tilde{\varepsilon}t}\cos(xt) + e^{2\operatorname{Im}\tilde{\varepsilon}t}}{x^2 + (\operatorname{Im}\tilde{\varepsilon})^2} dx + \mathcal{O}(1)$$

$$= \frac{1}{\lambda^2}(1 - e^{2\operatorname{Im}\tilde{\varepsilon}t}) + \mathcal{O}(1),$$

where we have noted that $\operatorname{Im}\tilde{\varepsilon} = \frac{\pi}{2} g(\sqrt{\varepsilon_0})\lambda^2 + \mathcal{O}(\lambda^4)$. It follows that

$$\|\Phi_t^{(0)}\| = 1 + \mathcal{O}(\lambda^2). \tag{4.8}$$

Substituting (4.7) and (4.8) in (4.6) the proposition follows. □

As a final comment we point out that Proposition 4.1 and (4.5) allow for a much sharper estimate for the propogation of the restricted state $\xi(H)\Phi$. Explicitly, if we let

$$\Phi_t^{(1)} = ae^{-i\tilde{\varepsilon}t}\Phi + \lambda\int \frac{1}{|p|^2 - \tilde{\varepsilon}}(a_1(p)e^{-i|p|^2t} - a_2(p)e^{-i\tilde{\varepsilon}t}) \begin{pmatrix} e^{ip\cdot x} \\ 0 \end{pmatrix} d^n p,$$

where the functions $a_j(p)$ are those introduced in Proposition 4.1 for $|p|^2 \in (\varepsilon_0 - \frac{r}{4}, \varepsilon_0 + \frac{r}{4})$ and are zero otherwise and a is the constant introduced in (4.5), one has for every $N \in \{1, 2, 3, \ldots\}$ a constant $K(N)$ such that

$$\|e^{-itH}\xi(H)\Phi - \Phi_t^{(1)}\| \leqq K(N)\,\lambda(1+t)^{-\frac{N}{2}}.$$

Not only does $\Phi_t^{(1)}$ approximate $e^{-itH}\xi(H)\Phi$ to within order λ rather than $\lambda^{\frac{1}{2}}$ but the error goes to zero as $t \to \infty$ as well. Simply replacing $\Phi_t^{(0)}$ by $\Phi_t^{(1)}$ in Proposition 4.2 will not help. A factor of $\lambda^{\frac{1}{2}}$ and the decrease of the error with increasing t would both be lost in bounding $\|(1 - \xi(H))\Phi\|$, that is, in replacing $e^{-itH}\xi(H)\Phi$ with $e^{-itH}\Phi$. In fact one does not expect the error $\|e^{-itH}\Phi - \Phi_t^{(0)}\|$ to decrease as t increases without further assumptions on H. Indeed, if H has other resonances whose real parts lie somewhere in the support of $1 - \xi$ then one expects these resonances to become excited as $e^{-itH}\Phi$ evolves, producing meta-stable states which themselves decay. Thus, higher order terms in λ should contain terms which behave like $\Phi_t^{(0)}$, but with $\tilde{\varepsilon}$ replaced by these other resonances. These terms would not decrease with increasing t.

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