

Classical and Quantum Integrable Systems in $\tilde{\mathfrak{gl}}(2)^{+*}$ and Separation of Variables

J. Harnad^{1,2} P. Winternitz²

¹ Department of Mathematics and Statistics, Concordia University, 7141, Sherbrooke W., Montréal Québec, Canada H4B 1R6

e-mail address: harnad@alcor.concordia.ca, harnad@mathen.umontreal.ca

² Centre de recherches mathématiques, Université de Montréal, C.P. 6128-A, Montréal, Québec, Canada H3C 3J7 e-mail address: wintern@ere.umontreal.ca

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Abstract: Classical integrable Hamiltonian systems generated by elements of the Poisson commuting ring of spectral invariants on rational coadjoint orbits of the loop algebra $\tilde{\mathfrak{gl}}^{+*}(2, \mathbb{R})$ are integrated by separation of variables in the Hamilton–Jacobi equation in hyperellipsoidal coordinates. The canonically quantized systems are then shown to also be completely integrable and separable within the same coordinates. Pairs of second class constraints defining reduced phase spaces are implemented in the quantized systems by choosing one constraint as an invariant, and interpreting the other as determining a quotient (i.e. by treating one as a first class constraint and the other as a gauge condition). Completely integrable, separable systems on spheres and ellipsoids result, but those on ellipsoids require a further modification of order $\mathcal{O}(\hbar^2)$ in the commuting invariants in order to assure self-adjointness and to recover the Laplacian for the case of free motion. For each case – in the ambient space \mathbb{R}^n , the sphere and the ellipsoid – the Schrödinger equations are completely separated in hyperellipsoidal coordinates, giving equations of generalized Lamé type.

Introduction

A general method for realizing integrable Hamiltonian systems as isospectral flows in rational coadjoint orbits of loop algebras was developed in [AHP, AHH1–AHH4]. This approach begins with a moment map embedding of certain Hamiltonian quotients of symplectic vector spaces into finite dimensional Poisson subspaces of the dual $\tilde{\mathfrak{gl}}(r)^{+*}$ of the positive frequency part of the loop algebra $\tilde{\mathfrak{gl}}(r)$ (or certain subalgebras thereof). The Adler–Kostant–Symes (AKS) theorem [A, K, S] then implies that the spectral invariants provide commuting integrals inducing isospectral flows determined by matrix Lax equations. The level sets of these commuting invariants are shown to determine Lagrangian foliations on the rational coadjoint orbits, and hence completely integrable systems. Finally, a special

set of canonical coordinates, the *spectral Darboux coordinates* are introduced, in which the Liouville generating function, which determines the linearizing canonical transformation, is expressed in completely separated form as an abelian integral on the associated invariant spectral curve. The resulting linearizing map is essentially the Abel map to the Jacobi variety of the spectral curve, thus providing a link, through purely Hamiltonian methods, with the algebro-geometric linearization methods of [Du, KN, AvM]. This approach has been applied to the study of a large number of integrable classical Hamiltonian systems, as well as the determination of finite dimensional quasi-periodic solutions of integrable systems of PDE's [H, HW, AHH3, AHH4, W, TW].

In the present work we focus on the case $\mathfrak{gl}(2)^{+*}$, taking an equivalent approach to integrability based, first of all, on separation of variables in the Hamilton–Jacobi equation. The relevant “spectral Darboux coordinates” in this case simply reduce to hyperellipsoidal coordinates. The purpose of this reformulation is to prepare the passage to the corresponding quantum systems and the study of integrability and separation of variables in the associated Schrödinger equation. As it turns out, each such classical integrable system has an integrable quantum analogue, for which the Schrödinger equation is completely separable in the same coordinates. One case of separation of variables in such systems, the quantized Neumann oscillator (an anisotropic harmonic oscillator constrained to the surface of a sphere), was studied in [BT], and the results extended to the quantum Rosochatius system in [Mc]. Other special cases, involving quantized free motion in various symmetric spaces and reductions thereof, were studied in [K, KM, KMW, BKW1, BKW2, ORW, Ku, To]. All these systems may be placed in a loop algebra setting using the moment map embedding of [AHP], and canonically quantized. The resulting formulation is equivalent to a Gaudin spin chain [G, Ku], and the separation of variables interpreted as a functional Bethe ansatz [Sk1, Sk2].

In [Mo], the algebraic geometry of a number of classical integrable systems constrained to quadrics in \mathbf{R}^n was examined. The integration of these and related systems was given a loop algebra formulation based on $\mathfrak{gl}^+(2, \mathbf{C})$ and reductions thereof in [AHP, AHH4]. In the present work, such systems will be reexamined in terms of separation of variables in the Hamilton–Jacobi equation. Their quantum analogues will then be studied through constrained canonical quantization, making use of the loop algebra formulation to identify the commuting invariants in terms of “quantum determinants.” The corresponding Schrödinger equations will be shown to separate within the same coordinates as the classical systems. Constraints leading to dynamics on spheres and ellipsoids will be shown to lead to integrable quantum systems, also separable in the same coordinates as the classical ones.

In Sect. 1, the appropriate loop algebra formulation of the systems in question is given. In each case, the coadjoint orbit is identified with the quotient $\mathbf{R}^{2n}/(\mathbf{Z}_2)^n$ and integrable isospectral flows are examined both in this space and on constrained submanifolds identified with the cotangent bundle of a sphere $S^{n-1} \subset \mathbf{R}^n$, or an ellipsoid $\mathcal{E}^{n-1} \subset \mathbf{R}^n$. The key step consists of using a Lagrange interpolation formula to express the invariant spectral polynomial in terms of its values at the associated spectral divisor points, and noting that these values coincide with the squared canonical momentum components. The separation of variables follows from identification of residues in the interpolation formula.

In Sect. 2, the corresponding quantum systems are obtained by canonical quantization in the ambient phase space \mathbf{R}^{2n} before quotienting. The resulting

Schrödinger equation is again expressed in terms of hyperellipsoidal coordinates through Lagrange interpolation, and the completely separated form is deduced, again by identification of residues, giving various types of generalized Lamé equations. The associated one dimensional Schrödinger operators are seen to give the quantized form of the invariant spectral curves (cf. [SK3]). In the case of the constrained systems on the sphere S^{n-1} the same formulation, together with second class constraints, leads without difficulty to completely separable integrable systems. In the case of the ellipsoid \mathcal{E}^{n-1} , however, a new problem arises, since the separation of variables, while holding for the Schrödinger equation, does not hold for the volume element, giving rise at first to non-self-adjoint operators. This is easily rectified by noting that the resulting Schrödinger operators are nevertheless self-adjoint with respect to a scalar product determined by a different measure than the one associated to the induced volume form. Conjugating the operators by the map relating the two scalar products, self-adjoint operators are obtained with respect to the standard measure. However, to recover the Laplacian in the case of free motion, a further scalar term of order $\mathcal{O}(\hbar^2)$ must be added to the quantum Hamiltonians. Since this correction gives the same semi-classical limit and does not destroy the integrability (or separability), it may be viewed as a satisfactory quantized version of the associated classical integrable systems.

1. Classical Systems in $\widetilde{\mathfrak{gl}}(2)^{+*}$

1a. Ambient Space. Following the general approach of [AHP, AHH2, H], we define a Poisson map

$$\widetilde{J}_A: \mathbf{R}^{2n} \rightarrow \widetilde{\mathfrak{gl}}(2)^{+*}, \tag{1.1a}$$

$$\widetilde{J}_A: (\mathbf{x}, \mathbf{y}) \mapsto \mathcal{N}_0(\lambda), \tag{1.1b}$$

where

$$\mathcal{N}_0(\lambda) := \frac{1}{2} \begin{pmatrix} -\sum_{i=1}^n \frac{x_i y_i - \mu_i}{\lambda - \alpha_i} - \sum_{i=1}^n \frac{y_i^2 - \frac{\mu_i^2}{x_i^2}}{\lambda - \alpha_i} \\ \sum_{i=1}^n \frac{x_i^2}{\lambda - \alpha_i} \quad \sum_{i=1}^n \frac{x_i y_i + \mu_i}{\lambda - \alpha_i} \end{pmatrix}, \tag{1.2}$$

$\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ have components $(x_i, y_i)_{i=1, \dots, n}$ and $\{\mu_i, \alpha_i\}_{i=1, \dots, n}$ is a set of $2n$ arbitrary real constants, the α_i 's being chosen as distinct. Here, the loop algebra $\widetilde{\mathfrak{gl}}(2)$ consists of smooth maps $X: S^1 \rightarrow \mathfrak{gl}(2)$ from a fixed circle S^1 , centered at the origin of the complex λ -plane, to $\mathfrak{gl}(2)$, the subalgebra $\widetilde{\mathfrak{gl}}(2)^+$ consists of elements $X \in \widetilde{\mathfrak{gl}}(2)$ admitting a holomorphic extension to the interior of S^1 , and the (smooth) dual space $\widetilde{\mathfrak{gl}}(2)^{+*}$ is identified with the subalgebra $\widetilde{\mathfrak{gl}}(2)^-$ of elements X admitting a holomorphic extension outside S^1 , with $X(\infty) = 0$. The dual pairing \langle, \rangle is defined by integration:

$$\begin{aligned} \langle X, Y \rangle &:= \frac{1}{2\pi i} \oint_{S^1} \text{tr}(X(\lambda) Y(\lambda)) d\lambda, \\ X \in \widetilde{\mathfrak{gl}}^{+*}(2) \sim \widetilde{\mathfrak{gl}}(2)^-, \quad Y \in \widetilde{\mathfrak{gl}}^+(2). \end{aligned} \tag{1.3}$$

To assure that $\mathcal{N}_0 \in \widetilde{\mathfrak{gl}}(2)^{+*}$, we must choose the constants $\{\alpha_i\}$ entering in definition (1.2) of the Poisson map \tilde{J}_A to be in the interior of S^1 . The image of the map is a coadjoint orbit in $\widetilde{\mathfrak{gl}}(2)^{+*}$, identified with the quotient $\mathbf{R}^{2n}/(\mathbf{Z}_2)^n$ of the phase space \mathbf{R}^{2n} by the symplectic action of the group $(\mathbf{Z}_2)^n$ of reflections in the coordinate hyperplanes, and $\{x_i, y_i\}_{i=1, \dots, n}$ are canonical coordinates defining the standard symplectic form

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i. \tag{1.4}$$

Fixing an element

$$Y = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in sl(2, \mathbf{R}), \tag{1.5}$$

we let

$$\begin{aligned} \mathcal{N}(\lambda) &:= Y + \mathcal{N}_0(\lambda) \\ &= \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -\sum_{i=1}^n \frac{x_i y_i - \mu_i}{\lambda - \alpha_i} - \sum_{i=1}^n \frac{y_i^2 - \frac{\mu_i^2}{x_i^2}}{\lambda - \alpha_i} \\ \sum_{i=1}^n \frac{x_i^2}{\lambda - \alpha_i} & \sum_{i=1}^n \frac{x_i y_i + \mu_i}{\lambda - \alpha_i} \end{pmatrix} \in \widetilde{\mathfrak{gl}}(2). \end{aligned} \tag{1.6}$$

The ring \mathcal{S}_{AKS}^Y of commuting invariants is chosen, according to the Adler–Kostant–Symes (AKS) theorem, by restricting the ring $\mathcal{S}(\widetilde{\mathfrak{gl}}^*(2))$ of Ad^* invariants on the dual space $\widetilde{\mathfrak{gl}}^*(2)$ of the full loop algebra to the translate of the coadjoint orbit $\mathcal{C}_{\mathcal{N}_0} \subset \widetilde{\mathfrak{gl}}(2)^{+*} \subset \widetilde{\mathfrak{gl}}(2)^*$ by the fixed element Y . Picking any element $\phi \in \mathcal{S}_{AKS}^Y$ as Hamiltonian, the AKS theorem implies that the equations of motion take the Lax form

$$\frac{\partial \mathcal{N}}{\partial t} = [(d\phi)(\mathcal{N})_+, \mathcal{N}], \tag{1.7}$$

and hence determine an isospectral flow. Since \mathcal{S}_{AKS}^Y is just the ring of spectral invariants, fixing simultaneous level sets of its elements amounts to fixing the coefficients of the characteristic polynomial

$$\mathcal{P}(\lambda, \zeta) := \det(\mathcal{N}(\lambda) - \zeta \mathbf{I}_2) = \zeta^2 - \zeta \sum_{i=1}^n \frac{\mu_i}{\lambda - \alpha_i} + \frac{P(\lambda)}{a(\lambda)}, \tag{1.8}$$

where

$$\det(\mathcal{N}(\lambda)) := \frac{P(\lambda)}{a(\lambda)} = \frac{1}{2} \sum_{i=1}^n \frac{I_i}{\lambda - \alpha_i} - (a^2 + bc), \tag{1.9}$$

$$a(\lambda) := \prod_{i=1}^n (\lambda - \alpha_i), \tag{1.10}$$

and

$$I_i := \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^n \frac{(x_i y_j - x_j y_i)^2 - \mu_i^2 \frac{x_j^2}{x_i^2} - \mu_j^2 \frac{x_i^2}{x_j^2} + 2\mu_i \mu_j}{\alpha_i - \alpha_j} + 2ax_i y_i - bx_i^2 + c \left(y_i^2 - \frac{\mu_i^2}{x_i^2} \right) \tag{1.11}$$

are the (generalized) Devaney–Uhlenbeck invariants (cf. [Mo]). The leading term of the polynomial

$$P(\lambda) = \sum_{i=0}^n P_i \lambda^i \tag{1.12}$$

has constant coefficient

$$P_n = -(a^2 + bc), \tag{1.13}$$

while the remaining coefficients $\{P_0, \dots, P_{n-1}\}$ are independent generators of the ring of commuting invariants.

Letting

$$\frac{1}{2} \sum_{i=1}^n \frac{\mu_i}{\lambda - \alpha_i} := \frac{K(\lambda)}{a(\lambda)}, \tag{1.14a}$$

$$z := a(\lambda) \left(\zeta - \frac{1}{2} \sum_{i=1}^n \frac{\mu_i}{\lambda - \alpha_i} \right), \tag{1.14b}$$

the invariant spectral curve \mathcal{C} is defined by

$$z^2 = K^2(\lambda) - a(\lambda) P(\lambda), \tag{1.15}$$

and thus is hyperelliptic. Since $P(\lambda)$ has leading terms of the form

$$P(\lambda) = -(a^2 + bc)\lambda^n + \left(\frac{1}{2} \sum_{i=1}^n I_i + (a^2 + bc) \sum_{i=1}^n \alpha_i \right) \lambda^{n-1} + \mathcal{C}(\lambda^{n-2}), \tag{1.16}$$

\mathcal{C} generically has genus $g = n - 1$ if $a^2 + bc \neq 0$ or $\sum_{i=1}^n I_i \neq 0$ and $g = n - 2$ if $a^2 + bc = 0$ and $\sum_{i=1}^n I_i = 0$.

In the following, instead of considering individual Hamiltonians with the ring of spectral invariants, it will be convenient (as in [BT]) to treat the invariant polynomial $P(\lambda)$ as a one parameter linear family of Hamiltonians, all commuting amongst themselves. It will be necessary to distinguish two cases, depending on whether $c = 0$ or $c \neq 0$.

*1b. Case (i) $c = 0$. Restriction to T^*S^{n-1} .* Following the general procedure of [AHH4, H] the spectral Darboux coordinates $\{q, p, \lambda_\mu, \zeta_\mu\}_{\mu=1, \dots, n-1}$ are defined by

$$\sum_{i=1}^n \frac{x_i^2}{\lambda - \alpha_i} = \frac{e^q Q(\lambda)}{a(\lambda)}, \tag{1.17a}$$

$$Q(\lambda) := \prod_{\mu=1}^{n-1} (\lambda - \lambda_\mu), \tag{1.17d}$$

$$e^q := \sum_{i=1}^n x_i^2, \tag{1.17c}$$

and

$$\zeta_\mu := \frac{1}{2} \sum_{i=1}^n \frac{x_i y_i}{\lambda_\mu - \alpha_i}, \tag{1.17b}$$

$$p := \frac{1}{2} \sum_{i=1}^n x_i y_i. \tag{1.17e}$$

Here $\{q, \lambda_\mu\}_{\mu=1, \dots, n-1}$ are, essentially, hyperellipsoidal coordinates on \mathbf{R}^n , and $\{p, \zeta_\mu\}_{\mu=1, \dots, n-1}$ are the canonically conjugate momenta. To make such an identification, we must also assume that the α_i 's are all real and positive, and choose an ordering such that, e.g.,

$$\alpha_n < \lambda_{n-1} < \alpha_{n-1} < \lambda_{n-2} < \dots < \lambda_1 < \alpha_1. \tag{1.18}$$

The canonical 1-form on \mathbf{R}^{2n} is

$$\theta = \sum_{i=1}^n y_i dx_i = pdq + \sum_{\mu=1}^{n-1} \zeta_\mu d\lambda_\mu. \tag{1.19}$$

The Jacobian matrix of the coordinate change is defined by the partial derivatives

$$\frac{\partial x_i}{\partial \lambda_\mu} = \frac{1}{2} \frac{x_i}{\lambda_\mu - \alpha_i}, \quad \frac{\partial x_i}{\partial q} = \frac{x_i}{2}, \tag{1.20}$$

and its inverse by

$$\frac{\partial \lambda_\mu}{\partial x_i} = -\frac{2a(\lambda_\mu)x_i}{Q'(\lambda_\mu)(\lambda_\mu - \alpha_i)}, \quad \frac{\partial q}{\partial x_i} = 2x_i e^{-q}, \tag{1.21}$$

The coordinate frame fields are thus

$$\frac{\partial}{\partial \lambda_\mu} = \frac{1}{2} \sum_{i=1}^n \frac{x_i}{\lambda_\mu - \alpha_i} \frac{\partial}{\partial x_i}, \quad \mu = 1, \dots, n-1, \tag{1.22a}$$

$$\frac{\partial}{\partial q} = \frac{1}{2} \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}. \tag{1.22b}$$

The \mathbf{R}^n euclidean metric in these coordinates is

$$\sum_{i=1}^n dx_i^2 = \frac{1}{4} e^q dq^2 - \frac{e^q}{4} \sum_{\mu=1}^{n-1} \frac{Q'(\lambda_\mu)}{a(\lambda_\mu)} d\lambda_\mu^2, \tag{1.23}$$

and the volume form is

$$dV = \frac{e^{\frac{nq}{2}} \prod_{\nu < \mu}^{n-1} (\lambda_\mu - \lambda_\nu)}{2^n \left| \prod_{\mu=1}^{n-1} a(\lambda_\mu) \right|^{\frac{1}{2}}} dq \wedge d\lambda_1 \wedge \dots \wedge d\lambda_{n-1}. \tag{1.24}$$

In order to express the linear family of Hamiltonians generated by $P(\lambda)$ in terms of the hyperellipsoidal coordinates, we note first that it follows from (1.6) and (1.8) that the values of $P(\lambda_\mu)$ are given by

$$\frac{P(\lambda_\mu)}{a(\lambda_\mu)} = -(\zeta_\mu - a)^2 + \frac{1}{4} \left(\sum_{i=1}^n \frac{\mu_i}{\lambda_\mu - \alpha_i} \right)^2 \tag{1.25a}$$

$$= -(\zeta_\mu - a)^2 + \frac{K(\lambda_\mu)^2}{a(\lambda_\mu)^2}. \tag{1.25b}$$

Using Lagrange interpolation, $P(\lambda)$ may then be expressed as

$$P(\lambda) = Q(\lambda) \left(\left(-\lambda + \sum_{\mu=1}^{n-1} \lambda_\mu + \sum_{i=1}^n \alpha_i \right) a^2 + \frac{1}{2} \sum_{i=1}^n I_i \right) - \sum_{\mu=1}^{n-1} \frac{Q(\lambda) a(\lambda_\mu)}{(\lambda - \lambda_\mu) Q'(\lambda_\mu)} \left((\zeta_\mu - a)^2 - \frac{K(\lambda_\mu)^2}{a(\lambda_\mu)^2} \right), \tag{1.26}$$

where

$$\frac{1}{2} \sum_{i=1}^n I_i = 2ap - \frac{b}{2} e^q. \tag{1.27}$$

To write the Hamilton–Jacobi equation, we must reinterpret the coefficients of the invariant polynomial $P(\lambda)$ in (1.26), not as functions on the phase space, but rather as integration constants and replace the canonical momentum components $\{\zeta_\mu, p\}_{\mu=1, \dots, n-1}$ by the partial derivatives $\left\{ \frac{\partial S}{\partial \lambda_\mu}, \frac{\partial S}{\partial q} \right\}_{\mu=1, \dots, n-1}$ of the Hamilton characteristic function. The resulting form for the time independent Hamilton–Jacobi equation is then

$$\begin{aligned} & - \sum_{\mu=1}^{n-1} \frac{Q(\lambda) a(\lambda_\mu)}{(\lambda - \lambda_\mu) Q'(\lambda_\mu)} \left(\left(\frac{\partial S}{\partial \lambda_\mu} - a \right)^2 - \frac{K(\lambda_\mu)^2}{a(\lambda_\mu)^2} \right) \\ & + Q(\lambda) \left(-a^2 \lambda + 2a \frac{\partial S}{\partial q} - \frac{b}{2} e^q + a^2 \left(\sum_{i=1}^n \alpha_i - \sum_{\mu=1}^{n-1} \lambda_\mu \right) \right) = P(\lambda), \end{aligned} \tag{1.28}$$

where the leading coefficient P_n of $P(\lambda)$ is given in (1.13), and the remaining n coefficients $\{P_0, \dots, P_{n-1}\}$ are interpreted as integration constants determining the “energies” for the parametric family of Hamiltonians defined in (1.26).

The integration of (1.28) then proceeds by separation of variables. Expressing S in the separated form

$$S(\lambda_1, \dots, \lambda_{n-1}, q) = s_0(q) + \sum_{\mu=1}^{n-1} s_\mu(\lambda_\mu), \tag{1.29}$$

dividing both sides of (1.28) by $Q(\lambda)$ and equating the leading terms in λ at ∞ , as well as the residues at $\{\lambda = \lambda_\mu\}_{\mu=1, \dots, n-1}$ in the resulting equation gives

$$\left(\frac{\partial s_\mu}{\partial \lambda_\mu} - a \right)^2 = \frac{K(\lambda_\mu)^2 - a(\lambda_\mu) P(\lambda_\mu)}{a(\lambda_\mu)^2}, \tag{1.30a}$$

$$2a \frac{\partial s_0}{\partial q} = \frac{b}{2} e^q - a^2 \sum_{i=1}^n \alpha_i + P_{n-1}. \tag{1.30b}$$

If $a \neq 0$, this may be integrated to give the completely separated solution

$$\begin{aligned} S(\lambda_1, \dots, \lambda_{n-1}, q) &= \frac{b}{4a} e^q + \frac{q}{2a} \left(P_{n-1} - a^2 \sum_{i=1}^n \alpha_i \right) + a \sum_{\mu=1}^{n-1} \lambda_\mu + a \sum_{\mu=1}^{n-1} \\ &+ \sum_{\mu=1}^n \int_0^{\lambda_\mu} \sqrt{\frac{K^2(\lambda) - a(\lambda) P(\lambda)}{a^2(\lambda)}} d\lambda. \end{aligned} \tag{1.31}$$

The linearizing coordinates are then

$$Q_i := \frac{\partial S}{\partial P_i} = \frac{1}{2} \sum_{\mu=1}^{n-1} \int_0^{\lambda_\mu} \frac{\lambda^i}{\sqrt{K^2(\lambda) - a(\lambda) P(\lambda)}} d\lambda, \quad i = 0, \dots, n-2, \tag{1.32a}$$

$$Q_{n-1} := \frac{\partial S}{\partial P_{n-1}} = \frac{q}{2a} + \frac{1}{2} \sum_{\mu=1}^{n-1} \int_0^{\lambda_\mu} \frac{\lambda^i}{\sqrt{K^2(\lambda) - a(\lambda) P(\lambda)}} d\lambda. \tag{1.32b}$$

The first $n - 1$ of these, defined by Eq. (1.32a), involve abelian integrals of the first kind and essentially define the Abel map to the Jacobi variety $\mathcal{J}(\mathcal{C})$. The last one, defined by (1.32b), is an abelian integral of the third kind, the integrand having a pair of simple poles over $\lambda = \infty$. The linear flow induced by any Hamiltonian $\phi = \phi(P_0, \dots, P_{n-1})$ in the ring \mathcal{I}_{AKS}^Y of spectral invariants is then given by

$$Q = Q_{i0} + \frac{\partial \phi}{\partial P_i} t, \quad i = 0, \dots, n - 1. \tag{1.33}$$

By standard Jacobi inversion techniques (cf. [Du, GH, AHH4]), any function of the coordinates $\{\lambda_\mu, q\}_{\mu=1, \dots, n-1}$ that is symmetric in the λ_μ 's, can be given an explicit form in terms of the Riemann theta functions associated to the curve \mathcal{C} .

If $a = 0$ and $b = 0$, both q and p are conserved quantities, and Hamilton's equations may be integrated on the invariant symplectic submanifold given by fixing a level set of q and p . By Eqs. (1.17c), (1.17e), this defines the cotangent bundle $T^*S^{n-1} \subset \mathbf{R}^{2n}$ to the sphere $S^{n-1} \subset \mathbf{R}^n$. The Hamilton–Jacobi equation (1.28), may then be interpreted on S^{n-1} , by choosing $S = S(\lambda_1, \dots, \lambda_{n-1})$. The separated form is again given by (1.29), (1.30a), with $s_0 = 0$ and (1.30b) omitted. Both the leading and next to leading terms in Eqs. (1.26) vanish, so $P(\lambda)$ is of degree $n - 2$. The completely separated solution is

$$S(\lambda_1, \dots, \lambda_{n-1}, q) = \sum_{\mu=1}^n \int_0^{\lambda_\mu} \sqrt{\frac{K^2(\lambda) - a(\lambda)P(\lambda)}{a^2(\lambda)}} d\lambda, \tag{1.34}$$

and the linearizing equations are given by (1.32a) and (1.33), for $i = 0, \dots, n - 1$. Since the genus of \mathcal{C} is $g = n - 2$, the $i = n - 2$ integral in (1.32a) becomes singular, the integrand having simple poles over $\lambda = \infty$.

If $a = 0$ and $b \neq 0$, q is still a conserved quantity, but p is not. Since the linear family of Hamiltonians $P(\lambda)$ now has no dependence on p , to apply the Hamilton–Jacobi theory, the rôles of p and $-q$ must be interchanged, and the term e^q in (1.28) replaced by $e^{\frac{is}{p}}$. The Hamilton characteristic function S is now a function of $\{\lambda_\mu; p\}_{\mu=1, \dots, n-1}$, and the solution is obtained in completely separated form as

$$S(\lambda_1, \dots, \lambda_{n-1}, p) = p \ln\left(\frac{-2P_{n-1}}{b}\right) + \sum_{\mu=1}^n \int_0^{\lambda_\mu} \sqrt{\frac{K^2(\lambda) - a(\lambda)P(\lambda)}{a^2(\lambda)}} d\lambda. \tag{1.35}$$

The linear flow equations (1.32a) and (1.33) remain the same for $i = 1, \dots, n - 2$, while (1.32b) is replaced by

$$Q_{n-1} := \frac{\partial S}{\partial P_{n-1}} = \frac{p}{P_{n-1}} + \frac{1}{2} \sum_{\mu=1}^{n-1} \int_0^{\lambda_\mu} \frac{\lambda^{n-1}}{\sqrt{K^2(\lambda) - a(\lambda)P(\lambda)}} d\lambda, \quad i = 0, \dots, n - 2. \tag{1.36}$$

Even if $a \neq 0$, and q and p are not individually conserved quantities, we may still impose the second class constraints

$$\sum_{i=1}^n x_i^2 = 1, \tag{1.37a}$$

$$\sum_{i=1}^n x_i y_i = 0, \tag{1.37b}$$

or, equivalently,

$$q = 0, \tag{1.38a}$$

$$p = 0, \tag{1.38b}$$

which define the symplectic submanifold $T^*S^{n-1} \subset \mathbf{R}^{2n}$ as phase space. The term P_{n-1} in $P(\lambda)$ is no longer viewed as an independent dynamic variable, or an integration constant, but rather the fixed constant defined by

$$P_{n-1} := a^2 \sum_{i=1}^n x_i - \frac{b}{2}. \tag{1.39}$$

The hyperellipsoidal coordinates $\{\lambda_\mu\}_{\mu=1, \dots, n-1}$, given by (1.17a), (1.17b) are now interpreted as defined on $S^{n-1} \subset \mathbf{R}^n$ and, together with the conjugate momenta $\{\xi_\mu\}_{\mu=1, \dots, n-1}$, these provide a canonical system on T^*S^{n-1} . The invariant coefficients $\{P_0, \dots, P_{n-2}\}$ of $P(\lambda)$ still form a Poisson commutative set when constrained to T^*S^{n-1} , even though q and p are not individually conserved quantities. (This follows from the fact that in the ambient space, the combination (1.27) commutes with all the P_i 's.)

For later use in Sect. 2, we note that the induced metric on S^{n-1} in the hyperellipsoidal coordinates $\{\lambda_1, \dots, \lambda_{n-1}\}$ is

$$\sum_{i=1}^n dx_i^2|_{S^{n-1}} = \frac{1}{4} \sum_{\mu=1}^{n-1} \frac{Q'(\lambda_\mu)}{a(\lambda_\mu)} d\lambda_\mu^2, \tag{1.40}$$

the volume form on S^{n-1} is

$$dV_{S^{n-1}} = \frac{1}{2^n} \frac{\prod_{\nu < \mu}^{n-1} (\lambda_\mu - \lambda_\nu)}{|\prod_{\mu} a(\lambda_\mu)|^{\frac{1}{2}}} d\lambda_1 \wedge \dots \wedge d\lambda_{n-1}, \tag{1.41}$$

and the scalar Laplacian is

$$\Delta_{S^{n-1}} = 4 \sum_{\mu=1}^{n-1} \frac{a(\lambda_\mu)}{Q'(\lambda_\mu)} \left(\frac{\partial^2}{\partial \lambda_\mu^2} + \frac{1}{2} \sum_{j=1}^n \frac{1}{\lambda_\mu - x_j} \frac{\partial}{\partial \lambda_\mu} \right). \tag{1.42}$$

The Hamilton–Jacobi equation is the same as in the unconstrained case (1.28), but with the $\frac{\partial S}{\partial q}$ term omitted. The solution in completely separated form is again just

$$S(\lambda_1, \dots, \lambda_{n-1}) = \sum_{\mu=1}^n \int_0^{\lambda_\mu} \sqrt{\frac{K^2(\lambda) - a(\lambda)(\lambda)}{a^2(\lambda)}}, \tag{1.43}$$

and the linearizing variables and flow are again defined by Eqs. (1.32a), (1.33).

An example of such a constrained integrable system on T^*S^{n-1} is generated by the invariant

$$\begin{aligned} \phi_R &:= -2P_{n-2} \\ &= \frac{1}{2} \left(\sum_{i=1}^n x_i^2 \right) \left(\sum_{j=1}^n y_j^2 \right) - \frac{1}{2} \left(\sum_{i=1}^n x_i y_i \right)^2 - \frac{1}{2} \left(\sum_{i=1}^n x_i^2 \right) \left(\sum_{j=1}^n \frac{\mu_j^2}{x_j^2} \right) \\ &\quad - 2a \sum_{i=1}^n \alpha_i x_i y_i + b \sum_{i=1}^n \alpha_i x_i^2, \end{aligned} \tag{1.44}$$

which, on the constrained manifold defined by (1.37a), (1.37b) becomes

$$\phi_R = \frac{1}{2} \sum_{i=1}^n y_i^2 - \frac{1}{2} \frac{\mu_j^2}{x_j^2} - 2a \sum_{i=1}^n \alpha_i x_i y_i + b \sum_{i=1}^n \alpha_i x_i^2. \tag{1.45}$$

For $a = 0$, this gives the Rosochatius system [Mo, GHHW, AHP]. If all the μ_i 's also vanish, it reduces to the Neumann oscillator system [Mo, H].

1c. Case (ii) $c \neq 0$. Restriction to $T^\mathcal{E}^{n-1}$.* In this case, the spectral Darboux coordinates $\{\lambda_\mu, \zeta_\mu\}_{\mu=1, \dots, n}$ are defined by the relations

$$\sum_{i=1}^n \frac{x_i^2}{\lambda - \alpha_i} + 2c = 2c \frac{Q(\lambda)}{a(\lambda)}, \tag{1.46a}$$

$$Q(\lambda) := \prod_{\mu=1}^n (\lambda - \lambda_\mu), \tag{1.46b}$$

$$\zeta_\mu := \frac{1}{2} \sum_{i=1}^n \frac{x_i y_i}{\lambda_\mu - \alpha_i}. \tag{1.46c}$$

The Jacobian matrix is thus again given by

$$\frac{\partial x_i}{\partial \lambda_\mu} = \frac{1}{2} \frac{x_i}{\lambda_\mu - \alpha_i}, \tag{1.47}$$

and its inverse by

$$\frac{\partial \lambda_\mu}{\partial x_i} = -\frac{1}{c} \frac{a(\lambda_\mu)}{Q'(\lambda_\mu)} \frac{x_i}{\lambda_\mu - \alpha_i}. \tag{1.48}$$

The coordinate frame fields are therefore

$$\frac{\partial}{\partial \lambda_\mu} = \frac{1}{2} \sum_{i=1}^n \frac{x_i}{\lambda_\mu - \alpha_i} \frac{\partial}{\partial x_i}, \quad \mu = 1, \dots, n, \tag{1.49}$$

and the canonical 1-form is

$$\theta = \sum_{i=1}^n y_i dx_i = \sum_{\mu=1}^n \zeta_\mu d\lambda_\mu. \tag{1.50}$$

To identify $\{\lambda_\mu\}_{\mu=1, \dots, n}$ as hyperellipsoidal coordinates, the constants $\{\alpha_i\}_{i=1, \dots, n}$ must again be chosen as real and positive, and an ordering fixed, e.g., by

$$\lambda_n < \alpha_n < \lambda_{n-1} < \dots < \lambda_1 < \alpha_1. \tag{1.51}$$

The \mathbf{R}^n euclidean metric in these coordinates is

$$\sum_{i=1}^n dx_i^2 = -\frac{c}{2} \sum_{\mu=1}^n \frac{Q'(\lambda_\mu)}{a(\lambda_\mu)} d\lambda_\mu^2, \tag{1.52}$$

and the volume form is

$$dV = \left(\frac{c}{2}\right)^{\frac{n}{2}} \frac{\prod_{\nu < \mu}^n (\lambda_\mu - \lambda_\nu)}{[\prod_{\mu}^n a(\lambda_\mu)]^{\frac{1}{2}}} d\lambda_1 \wedge \dots \wedge d\lambda_n. \tag{1.53}$$

For reference in Sect. 2, we note that the scalar Laplacian is

$$\Delta = \frac{2}{c} \sum_{\mu=1}^n \frac{a(\lambda_\mu)}{Q'(\lambda_\mu)} \left(\frac{\partial^2}{\partial \lambda_\mu^2} + \frac{1}{2} \sum_{j=1}^n \frac{1}{\lambda_\mu - \alpha_j} \frac{\partial}{\partial \lambda_\mu} \right). \tag{1.54}$$

The values $\{P(\lambda_\mu)\}_{\mu=1, \dots, n}$ are again given by

$$\frac{P(\lambda_\mu)}{a(\lambda_\mu)} = -(\zeta_\mu - a)^2 + \frac{K(\lambda_\mu)^2}{a(\lambda_\mu)^2}, \tag{1.55}$$

and, using Lagrange interpolation, $P(\lambda)$ may be expressed as

$$P(\lambda) = - \sum_{\mu=1}^n \frac{Q(\lambda)a(\lambda_\mu)}{(\lambda - \lambda_\mu)Q'(\lambda_\mu)} \left((\zeta_\mu - a)^2 - \frac{K(\lambda_\mu)^2}{a(\lambda_\mu)^2} \right) - Q(\lambda)(a^2 + bc). \tag{1.56}$$

To obtain the Hamilton–Jacobi equation, we again reinterpret the coefficients of the invariant polynomial $P(\lambda)$ in (1.56) as integration constants and replace the canonical momentum components ζ_μ by the partial derivatives $\frac{\partial S}{\partial \lambda_\mu}$, giving

$$- \sum_{\mu=1}^n \frac{Q(\lambda)a(\lambda_\mu)}{(\lambda - \lambda_\mu)Q'(\lambda_\mu)} \left(\left(\frac{\partial S}{\partial \lambda_\mu} - a \right)^2 - \frac{K^2(\lambda_\mu)}{a^2(\lambda_\mu)} \right) - Q(\lambda)(a^2 + bc) = P(\lambda), \tag{1.57}$$

where the leading term of $P(\lambda)$ is $-(a^2 + bc)\lambda^n$ and the remaining terms are independent integration constants.

The integration of (1.57) again proceeds by separation of variables. Expressing S in the separated form,

$$S(\lambda_1, \dots, \lambda_n) = \sum_{\mu=1}^n s_\mu(\lambda_\mu), \tag{1.58}$$

dividing both sides of (1.57) by $Q(\lambda)$, and equating the residues at $\{\lambda = \lambda_\mu\}_{\mu=1, \dots, n}$ gives

$$\left(\frac{\partial s_\mu}{\partial \lambda_\mu} - a \right)^2 = \frac{K(\lambda_\mu)^2 - a(\lambda_\mu)P(\lambda_\mu)}{a(\lambda_\mu)^2}. \tag{1.59}$$

This may be integrated to give

$$S(\lambda_1, \dots, \lambda_{n-1}) = a \sum_{\mu=1}^n \lambda_\mu + \sum_{\mu=1}^n \int_0^{\lambda_\mu} \sqrt{\frac{K^2(\lambda) - a(\lambda)P(\lambda)}{a^2(\lambda)}} d\lambda. \tag{1.60}$$

The linearizing coordinates are thus

$$Q_i := \frac{\partial S}{\partial P_i} = \frac{1}{2} \sum_{\mu=1}^n \int_0^{\lambda_\mu} \frac{\lambda^i}{\sqrt{K^2(\lambda) - a(\lambda)P(\lambda)}} d\lambda, \quad i = 0, \dots, n - 1. \tag{1.61}$$

The first $n - 1$ abelian integrals in (1.61), with $i = 0, \dots, n - 2$, are all of the first kind, again defining the Abel map to the Jacobi variety $\mathcal{J}(\mathcal{C})$, while the remaining one, giving Q_{n-1} , is again singular, the integrand having a pair of simple poles over $\lambda = \infty$. The linear flow induced by any Hamiltonian $\phi = \phi(P_0, \dots, P_{n-1})$ in the ring of spectral invariants \mathcal{S}_{AKS}^Y is, as before, given by

$$Q_i = Q_{i0} + \frac{\partial \phi}{\partial P_i} t, \quad i = 0, \dots, n - 1. \tag{1.62}$$

We may also impose the constraints

$$\sum_{i=1}^n \frac{X_i^2}{\alpha_i} = 2c, \tag{1.63a}$$

$$\sum_{i=1}^n \frac{X_i Y_i}{\alpha_i} = 0, \tag{1.63b}$$

defining the cotangent bundle $T^*\mathcal{E}^{n-1}$ to the ellipsoid $\mathcal{E}^{n-1} \subset \mathbf{R}^n$ defined by (1.63a). (Here, the constant c must be chosen as positive; more generally, arbitrary signs may be allowed for the α_i 's and c , thereby defining various hyperboloids.) In terms of the canonical coordinates $\{\lambda_\mu, \zeta_\mu\}_{\mu=1, \dots, n}$ the two constraints are equivalent to

$$\lambda_n = 0, \tag{1.64a}$$

$$\zeta_n = 0. \tag{1.64b}$$

For reference in Sect. 2, we note that the induced metric on \mathcal{E}^{n-1} is

$$\sum_{i=1}^n dx_i^2|_{\mathcal{E}^{n-1}} = -\frac{c}{2} \sum_{\mu=1}^{n-1} \frac{\lambda_\mu Q'(\lambda_\mu)}{a(\lambda_\mu)} d\lambda_\mu^2, \tag{1.65}$$

the corresponding volume form is

$$dV_{\mathcal{E}^{n-1}} = \left(\frac{c}{2}\right)^{\frac{n-1}{2}} \prod_{\nu < \mu}^{n-1} (\lambda_\mu - \lambda_\nu) \prod_{\mu=1}^{n-1} \frac{\lambda_\mu^{\frac{1}{2}}}{|a(\lambda_\mu)|^{\frac{3}{2}}} d\lambda_1 \wedge \dots \wedge d\lambda_{n-1}, \tag{1.66}$$

and the scalar Laplacian is

$$\Delta_{\mathcal{E}^{n-1}} = \frac{2}{c} \sum_{\mu=1}^{n-1} \frac{a(\lambda_\mu)}{\lambda_\mu Q_0'(\lambda_\mu)} \left(\frac{\partial^2}{\partial \lambda_\mu^2} + \frac{1}{2} \left[\sum_{j=1}^n \frac{1}{\lambda_\mu - \alpha_j} - \frac{1}{\lambda_\mu} \right] \frac{\partial}{\partial \lambda_\mu} \right). \tag{1.67}$$

Equations (1.64a), (1.64b) determine the cotangent bundle $T^*\mathcal{E}^{n-1} \subset \mathbf{R}^{2n}$ as a symplectic submanifold; i.e., they are purely second class, and the restriction of the remaining coordinates $\{\lambda_\mu, \zeta_\mu\}_{\mu=1, \dots, n-1}$ provide canonical coordinates on this constrained manifold. The restriction of the canonical 1-form defines the canonical 1-form on $T^*\mathcal{E}^{n-1}$,

$$\theta|_{T^*\mathcal{E}^{n-1}} = \sum_{\mu=1}^{n-1} \zeta_\mu d\lambda_\mu. \tag{1.68}$$

Although neither of the constraints (1.63a), (1.63b) is individually invariant under the AKS flows generated by the invariants P_0, \dots, P_{n-1} , they are equivalent to the pair

$$Q(0) = 0, \tag{1.69a}$$

$$\frac{P(0)}{a(0)} = \frac{P_0}{a(0)} = -a^2 + \frac{1}{4} \left(\sum_{i=1}^n \frac{\mu_i}{\alpha_i} \right)^2, \tag{1.69b}$$

and the second of these is invariant. It follows that the restrictions of the remaining invariants $\{P_1, \dots, P_{n-1}\}$ to the constrained manifold $T^*\mathcal{E}^{n-1} \subset \mathbf{R}^{2n}$ also Poisson commute, generating completely integrable systems. On $T^*\mathcal{E}^{n-1}$, we may write

$$Q(\lambda) = \lambda Q_0(\lambda), \tag{1.70}$$

where

$$Q_0(\lambda) := \prod_{\mu=1}^{n-1} (\lambda - \lambda_\mu). \tag{1.71}$$

Proceeding again by Lagrange interpolation, we have

$$P(\lambda) = \sum_{\mu=1}^{n-1} \frac{\lambda a(\lambda_\mu)}{(\lambda - \lambda_\mu) \lambda_\mu Q'_0(\lambda_\mu)} \left(-(\zeta_\mu - a)^2 + \frac{K^2(\lambda_\mu)}{a^2(\lambda_\mu)} \right) + Q_0(\lambda) \left(-(a^2 + bc)\lambda + \frac{a(0)}{Q_0(0)} \left(-a^2 + \frac{1}{4} \left(\sum_{i=1}^n \frac{\mu_i^2}{x_i} \right)^2 \right) \right), \tag{1.72}$$

which with suitable reinterpretation of $P(\lambda)$ in terms of integration constants, gives the Hamilton–Jacobi equation on $T^* \mathcal{E}^{n-1}$ as

$$\sum_{\mu=1}^{n-1} \frac{\lambda a(\lambda_\mu)}{(\lambda - \lambda_\mu) \lambda_\mu Q'_0(\lambda_\mu)} \left(- \left(\frac{\partial S}{\partial \lambda_\mu} - a \right)^2 + \frac{K^2(\lambda_\mu)}{a^2(\lambda_\mu)} \right) + Q_0(\lambda) \left(-(a^2 + bc)\lambda + \frac{a(0)}{Q_0(0)} \left(-a^2 + \frac{1}{4} \left(\sum_{i=1}^n \frac{\mu_i^2}{x_i} \right)^2 \right) \right) = P(\lambda), \tag{1.73}$$

with $P(0)$ fixed to satisfy (1.69b). Expressing $S(\lambda_1, \dots, \lambda_{n-1})$ in separated form as

$$S(\lambda_1, \dots, \lambda_{n-1}) = \sum_{\mu=1}^{n-1} s_\mu(\lambda_\mu), \tag{1.74}$$

and proceeding as in the unconstrained case again gives (1.59), for $\mu = 1, \dots, n - 1$, which upon integration gives

$$S(\lambda_1, \dots, \lambda_{n-1}) = a \sum_{\mu=1}^{n-1} \lambda_\mu + \sum_{\mu=1}^{n-1} \int_0^{\lambda_\mu} \sqrt{\frac{K^2(\lambda) - a(\lambda)P(\lambda)}{a^2(\lambda)}} d\lambda. \tag{1.75}$$

The linearizing coordinates are therefore

$$Q_i := \frac{\partial S}{\partial P_i} = \frac{1}{2} \sum_{\mu=1}^{n-1} \int_0^{\lambda_\mu} \frac{\lambda^i}{\sqrt{K^2(\lambda) - a(\lambda)P(\lambda)}} d\lambda, \quad i = 1, \dots, n - 1, \tag{1.76}$$

and the linear flow induced by any Hamiltonian $\phi = \phi(P_1, \dots, P_{n-1})$ in the ring \mathcal{S}_{AKS}^Y of spectral invariants is again given by

$$Q_i = Q_{i0} + \frac{\partial \phi}{\partial P_i} t, \quad i = 1, \dots, n - 1. \tag{1.77}$$

In general, the abelian integrals in (1.76) are all of the first kind, so the linearizing map is again the Abel map to the Jacobi variety $\mathcal{J}(\mathcal{C})$.

As an example of such a constrained system within the commuting family of systems generated by \mathcal{S}_{AKS}^Y , consider the Hamiltonian

$$\begin{aligned} \phi_J &:= 2P_{n-1} = \sum_{i=1}^n I_i + (a^2 + bc) \sum_{i=1}^n \alpha_i \\ &= c \sum_{i=1}^n \left(y_i^2 - \frac{\mu_i^2}{x_i^2} \right) + 2a \sum_{i=1}^n x_i y_i - b \sum_{i=1}^n x_i^2. \end{aligned} \tag{1.78}$$

Taking $c = \frac{1}{2}$, $a = b = 0$, $\{\mu_i = 0\}_{i=1, \dots, n}$, this reduces to the Jacobi system, determining geodesic motion on an ellipsoid [Mo]. More generally, if $b \neq 0$, $\mu_i \neq 0$, we have motion on the ellipsoid in the presence of both a harmonic force, directed towards the origin, and forces derived from the $\sum_{i=1}^n \frac{\mu_i^2}{x_i^2}$ potential, as in the Rosochatius system.

2. Quantum Integrable Systems in $\tilde{\mathfrak{gl}}(2)^{+*}$

2a. *Ambient Space.* We quantize the systems considered above first in the ambient space \mathbf{R}^n , using canonical quantization within the Schrödinger representation. The wave functions will thus be taken as smooth functions $\Psi(x_1, \dots, x_n)$ on \mathbf{R}^n , square integrable with respect to the standard measure. Using the canonical quantization rule (with $\hbar \equiv 1$)

$$x_j \rightarrow \hat{x} := x_j \quad y_j \rightarrow \hat{y}_j := -i \frac{\partial}{\partial x_j} := -i \hat{\partial}_j, \tag{2.1}$$

we obtain a quantum analogue of the loop algebra element $\mathcal{A}(\lambda)$,

$$\mathcal{A}(\lambda) \rightarrow \hat{\mathcal{A}}(\lambda) := \begin{pmatrix} \hat{h}(\lambda) & \hat{e}(\lambda) \\ \hat{f}(\lambda) & \hat{g}(\lambda) \end{pmatrix}, \tag{2.2}$$

where

$$\hat{h}(\lambda) := \frac{1}{2} \sum_{j=1}^n \frac{i x_j \hat{\partial}_j + \mu_j}{\lambda - \alpha_j} + a, \tag{2.3a}$$

$$\hat{e}(\lambda) := \frac{1}{2} \sum_{j=1}^n \frac{\hat{\partial}_j^2 + \frac{\mu_j^2}{x_j^2}}{\lambda - \alpha_j} + b, \tag{2.3b}$$

$$\hat{f}(\lambda) := \frac{1}{2} \sum_{j=1}^n \frac{x_j^2}{\lambda - \alpha_j} + c, \tag{2.3c}$$

$$\hat{g}(\lambda) := \frac{1}{2} \sum_{j=1}^n \frac{-i x_j \hat{\partial}_j + \mu_j}{\lambda - \alpha_j} - a. \tag{2.3d}$$

Note that we have not chosen to order these operators so as to assure self-adjointness; they will only enter at an intermediate state in subsequent calculations and need not themselves be viewed as quantized dynamic variables. We also define an operator that plays essentially the rôle of $\det \mathcal{A}(\lambda)$ in the proceeding calculations:

$$\hat{\mathcal{D}}(\lambda) := \hat{h}(\lambda) \hat{g}(\lambda) - \hat{f}(\lambda) \hat{e}(\lambda). \tag{2.4}$$

Again, we are not concerned with whether this is an appropriately ordered “quantum determinant,” since it will only appear as a calculational convenience in what follows; the choice of ordering in (2.4) is the most convenient in the subsequent Lagrange interpolation.

A direct computation shows that $\hat{\mathcal{G}}(\lambda)$ may be expressed as

$$\begin{aligned} \hat{\mathcal{G}}(\lambda) = & \frac{1}{2} \sum_{i=1}^n \frac{\hat{I}_i}{\lambda - \alpha_i} - \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n \frac{x_i \hat{c}_i}{(\lambda - \alpha_i)(\lambda - \alpha_j)} + \frac{1}{2} \sum_{i=1}^n \frac{x_i \hat{c}_i}{(\lambda - \alpha_i)^2} \\ & + \frac{ia}{2} \sum_{i=1}^n \frac{1}{\lambda - \alpha_i} - (a^2 + bc), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \hat{I}_i := & \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^n \frac{-(x_i \hat{c}_j - x_j \hat{c}_i)^2 - \mu_i^2 \frac{x_j^2}{x_i^2} - \mu_j^2 \frac{x_i^2}{x_j^2} + 2\mu_i \mu_j}{\alpha_i - \alpha_j} - 2ia \left(x_i \hat{c}_i + \frac{1}{2} \right) \\ & - bx_i^2 - c \left(\hat{c}_i^2 + \frac{\mu_i^2}{x_i^2} \right). \end{aligned} \tag{2.6}$$

Here the \hat{I}_i 's are self-adjoint, and represent the quantized version of the Poisson commuting invariants I_i of the preceding section. Moreover, a direct computation shows that these \hat{I}_i 's also commute amongst themselves

$$[\hat{I}_i, \hat{I}_j] = 0 \quad \forall i, j = 1, \dots, n. \tag{2.7}$$

Using either definition (1.17a)–(1.17b) or (1.46a)–(1.46b) for the coordinates λ_μ , and the fact that $\hat{f}(\lambda_\mu) = 0$, it follows from Eq. (1.22a) or (1.49) that,

$$\hat{\mathcal{G}}(\lambda) = \frac{\partial^2}{\partial \lambda_\mu^2} - 2ia \frac{\partial}{\partial \lambda_\mu} + \frac{1}{2} \sum_{j=1}^n \frac{x_j \hat{c}_j}{(\lambda_\mu - \alpha_j)^2} + \frac{K^2(\lambda_\mu)}{a^2(\lambda_\mu)} - a^2, \tag{2.8}$$

where it is understood that the factor $(\lambda_\mu - \alpha_j)^2$ in the denominator precedes the derivation $x_j \hat{c}_j$. Define the differential operator valued polynomial $\hat{P}(\lambda)$ by

$$\frac{\hat{P}(\lambda)}{a(\lambda)} := \frac{1}{2} \sum_{i=1}^n \frac{\hat{I}_i}{\lambda - \alpha_i}. \tag{2.9}$$

Equating the expressions for $\hat{\mathcal{G}}(\lambda_\mu)$ given by Eqs. (2.5) and (2.8), gives

$$\frac{\hat{P}(\lambda_\mu)}{a(\lambda_\mu)} = \left(\frac{\partial}{\partial \lambda_\mu} - ia \right)^2 + \frac{1}{2} \sum_{j=1}^n \frac{1}{\lambda_\mu - \alpha_j} \left(\frac{\partial}{\partial \lambda_\mu} - ia \right) + \frac{K^2(\lambda_\mu)}{a^2(\lambda_\mu)} + a^2 + bc, \tag{2.10}$$

where again, the ordering on the LHS of (2.10) is understood to place $a(\lambda_\mu)$ term in the denominator *before* the differential operator in the numerator. Note also, that in the definition (2.9) of $\hat{P}(\lambda)$, the leading term is

$$\hat{P}(\lambda) = \frac{1}{2} \sum_{i=1}^n \hat{I}_i \lambda^{n-1} + \mathcal{O}(\lambda_{n-2}), \tag{2.11}$$

where

$$\hat{P}_{n-1} := \frac{1}{2} \sum_{i=1}^n \hat{I}_i = -\frac{1}{2} \sum_{i=1}^n \left(2ia(x_i \hat{c}_i + \frac{1}{2}) + bx_i^2 + c \left(\hat{c}_i^2 + \frac{\mu_i^2}{x_i^2} \right) \right). \tag{2.12}$$

Thus, contrary to the definition of $P(\lambda)$ in (1.9), $\hat{P}(\lambda)$ here does not contain the constant leading term $-(a^2 + bc)\lambda^n$, and is an operator-valued polynomial of degree $n - 1$.

In the following sections, we again must distinguish between the cases $c = 0$ and $c \neq 0$.

2b. Case (i) $c = 0$. Restriction to S^{n-1} . Taking (1.17a)–(1.17c) as the definition of our coordinates $\{\lambda_\mu, q\}_{\mu=1, \dots, n-1}$ and of $Q(\lambda)$, and using Eqs. (2.9), (2.11), (2.12), Lagrange interpolation gives

$$\hat{P}(\lambda) = \sum_{\mu=1}^n \frac{Q(\lambda)a(\lambda_\mu)}{(\lambda - \lambda_\mu)Q'(\lambda_\mu)} \left(\left(\frac{\hat{c}}{\partial \lambda_\mu} - ia \right)^2 + \frac{1}{2} \sum_{j=1}^n \frac{1}{\lambda_\mu - \alpha_j} \left(\frac{\hat{c}}{\partial \lambda_\mu} - ia \right) + \frac{K^2(\lambda_\mu)}{a^2(\lambda_\mu)} + a^2 \right) - Q(\lambda) \left(2ia \frac{\hat{c}}{\partial q} + \frac{nia}{2} + \frac{b}{2} e^q \right) \tag{2.13}$$

as the quantum analogue of (1.26). Here we have used

$$\frac{1}{2} \sum_{j=1}^n \hat{I}_j = -2ia \frac{\hat{c}}{\partial q} - \frac{nia}{2} - \frac{b}{2} e^q. \tag{2.14}$$

Letting

$$\frac{P^E(\lambda)}{a(\lambda)} := \frac{1}{2} \sum_{i=1}^n \frac{I_i^E}{\lambda - \alpha_i}, \tag{2.15}$$

where $\{I_i^E\}_{i=1, \dots, n}$ is a set of simultaneous eigenvalues of the operators $\{\hat{I}_i^E\}_{i=1, \dots, n}$, the time independent Schrödinger equation corresponding to the 1-parameter linear family of commuting Hamiltonians $\hat{P}(\lambda)$ is thus

$$\sum_{\mu=1}^{n-1} \frac{Q(\lambda)a(\lambda_\mu)}{(\lambda - \lambda_\mu)Q'(\lambda_\mu)} \left(\left(\frac{\hat{c}}{\partial \lambda_\mu} - ia \right)^2 + \frac{1}{2} \sum_{j=1}^n \frac{1}{\lambda_\mu - \alpha_j} \left(\frac{\hat{c}}{\partial \lambda_\mu} - ia \right) + \frac{K^2(\lambda_\mu)}{a^2(\lambda_\mu)} + a^2 \right) \Psi + Q(\lambda) \left(-2ia \frac{\hat{c}}{\partial q} - \frac{nia}{2} - \frac{b}{2} \sum_{j=1}^n x_j^2 \right) \Psi = P^E(\lambda) \Psi, \tag{2.16}$$

where $\Psi = \Psi(\lambda_1, \dots, \lambda_{n-1}, q)$.

Choosing Ψ in the completely separated form,

$$\Psi(\lambda_1, \dots, \lambda_{n-1}, q) = \phi(q) \prod_{\mu=1}^{n-1} \psi_\mu(\lambda_\mu) e^{ia \sum_{\mu=1}^{n-1} \lambda_\mu}, \tag{2.17}$$

dividing (2.16) by $Q(\lambda)$ and equating the leading terms in λ and the residues at $\lambda = \lambda_\mu$, we find that each $\psi_\mu(\lambda_\mu)$ satisfies the same separated equation

$$\left(\frac{\partial^2}{\partial \lambda^2} + \frac{1}{2} \sum_{j=1}^n \frac{1}{\lambda - \alpha_j} \frac{\partial}{\partial \lambda} + \frac{K^2(\lambda)}{a^2(\lambda)} - \frac{1}{2} \sum_{j=1}^n \frac{I_j^E}{\lambda - \alpha_j} + a^2 \right) \psi(\lambda) = 0, \tag{2.18a}$$

while $\phi(q)$ satisfies

$$\left(-2ia \frac{\partial}{\partial q} - \frac{nia}{2} - \frac{b}{2} e^q \right) \phi(q) = \frac{1}{2} \sum_{j=1}^n I_j^E \phi(q). \tag{2.18b}$$

Equation (2.18a) is a fuchsian differential equation of the generalized Lamé type already encountered in previously studied examples of such systems [BT, Mc]. Note also that, if λ and ζ are viewed as canonically conjugate variables, (2.18a) may be interpreted as the quantized form of the characteristic equation (1.15) defining the classical spectral curve \mathcal{C} . Thus, the completely separated form of the Schrödinger equation may be viewed as the “quantized spectral curve” (cf. [Sk3]).

If $a = 0$, q is a conserved quantity and if $b \neq 0$, Eq. (2.18b) should be interpreted as fixing the constant value of q in terms of the invariant $\sum_{i=1}^n I_j^E$. Then Eq. (2.18b) implies that, rather than choosing ϕ to be an L^2 function, it should be chosen as a delta function within the coordinate representation in which q is fixed by the eigenvalues I_j^E . If $a = 0$ and $b = 0$, this choice may still be made, but the eigenvalues I_j^E are not all independent, since they must sum to 0.

$$\sum_{i=1}^n I_j^E \equiv 0, \tag{2.19}$$

but the invariant q may be set equal to any constant value independently.

Whether $a = 0$ or not, we may impose the second class constraints (1.38a), (1.38b) in the quantum problem by requiring the wave function to satisfy

$$\left(\frac{\hat{c}}{\hat{c}q} + \frac{n}{4}\right) \Psi|_{q=0} = 0, \tag{2.20}$$

which is the quantum analogue of the constraint (1.38b), and setting $q = 0$ in the expression (2.13) defining $\hat{P}(\lambda)$. Restricting to the subspace of Ψ 's satisfying (2.20), the coefficients of $\hat{P}(\lambda)$ generate an $n - 1$ dimension linear space of commuting operators on the sphere S^{n-1} . This is spanned, e.g., by the restrictions of the operators \hat{I}_j , which, because of (2.20) satisfy

$$\sum_{j=1}^n \hat{I}_j = -b. \tag{2.21}$$

Applying the constraints, (2.16) reduces to

$$\sum_{\mu=1}^n \frac{Q(\lambda)a(\lambda_\mu)}{(\lambda - \lambda_\mu)Q'(\lambda_\mu)} \left(\left(\frac{\hat{c}}{\hat{c}\lambda_\mu} - ia\right)^2 + \frac{1}{2} \sum_{j=1}^n \frac{1}{\lambda_\mu - \lambda_j} \left(\frac{\hat{c}}{\hat{c}\lambda_\mu} - ia\right) + \frac{K^2(\lambda_\mu)}{a^2(\lambda_\mu)} + a^2 \right) \Psi_0 - bQ(\lambda) \Psi_0 = P^E(\lambda) \Psi_0, \tag{2.22}$$

where

$$\Psi(\lambda_1, \dots, \lambda_{n-1}, q) := e^{-\frac{nq}{4}} \Psi_0(\lambda_1, \dots, \lambda_{n-1}), \tag{2.23}$$

and the eigenvalues I_j^E must satisfy

$$\sum_{i=1}^n I_j^E = -b. \tag{2.24}$$

The volume form on S^{n-1} in these coordinates is given by (1.41) and the polynomial family of operators appearing on the LHS of (2.22) are all self-adjoint with respect to the corresponding scalar product. The separation of variables for the resulting systems on the constrained space S^{n-1} is again obtained by setting

$$\Psi_0(\lambda_1, \dots, \lambda_{n-1}) = \prod_{i=1}^{n-1} \psi_\mu(\lambda_\mu) e^{ia \sum_{\mu=1}^{n-1} \lambda_\mu} \tag{2.25}$$

in (2.22), dividing by $Q(\lambda)$ and equating residues at $\lambda = \lambda_\mu$. The resulting separated equations for the $\psi_\mu(\lambda_\mu)$ are all again given by (2.18a). The particular case of this family of systems corresponding to choosing all $\mu_i = 0$ and $a = 0$, as in the classical

case, gives the quantum Neumann oscillator system (cf. [BT]), while if the μ_i 's are non-zero, we obtain the quantum Rosochatius system (cf. [Mc]).

2c. Case (ii) $c \neq 0$. Restriction to \mathcal{E}^{n-1} . We begin again with the unconstrained system in the ambient space \mathbf{R}^n . Taking (1.46a), (1.46b) as the definition of the coordinates $\{\lambda_\mu\}_{\mu=1, \dots, n}$, and of $Q(\lambda)$, and again using Eqs. (2.9)–(2.12), Lagrange interpolation gives

$$\hat{P}(\lambda) = \sum_{\mu=1}^n \frac{Q(\lambda)a(\lambda_\mu)}{(\lambda - \lambda_\mu)Q'(\lambda_\mu)} \left(\left(\frac{\partial}{\partial \lambda_\mu} - ia \right)^2 + \frac{1}{2} \sum_{j=1}^n \frac{1}{\lambda_\mu - \alpha_j} \left(\frac{\partial}{\partial \lambda_\mu} - ia \right) + \frac{K^2(\lambda_\mu)}{a^2(\lambda_\mu)} + a^2 + bc \right) \quad (2.26)$$

as the quantum analogue of (1.56). Defining the polynomial family of eigenvalues $P^E(\lambda)$ again by Eq. (2.15) in terms of the set $\{I_i^E\}_{i=1, \dots, n}$ of simultaneous eigenvalues of the operators $\{I_i\}_{i=1, \dots, n}$, the time independent Schrödinger equation corresponding to the 1-parameter linear family of commuting Hamiltonians $\hat{P}(\lambda)$ is now

$$\sum_{\mu=1}^n \frac{Q(\lambda)a(\lambda_\mu)}{(\lambda - \lambda_\mu)Q'(\lambda_\mu)} \left(\left(\frac{\partial}{\partial \lambda_\mu} - ia \right)^2 + \frac{1}{2} \sum_{j=1}^n \frac{1}{\lambda_\mu - \alpha_j} \left(\frac{\partial}{\partial \lambda_\mu} - ia \right) + \frac{K^2(\lambda_\mu)}{a^2(\lambda_\mu)} + a^2 + bc \right) \Psi = P^E(\lambda) \Psi, \quad (2.27)$$

where $\Psi = \Psi(\lambda_1, \dots, \lambda_n)$. Choosing Ψ in the completely separated form,

$$\Psi(\lambda_1, \dots, \lambda_n) = e^{ia \sum_{\mu=1}^{n-1} \lambda_\mu} \prod_{\mu=1}^n \psi_\mu(\lambda_\mu), \quad (2.28)$$

dividing (2.16) by $Q(\lambda)$ and equating the residues at $\lambda = \lambda_\mu$, we find again that each $\psi_\mu(\lambda_\mu)$ satisfies the same separated equation

$$\left(\frac{\partial^2}{\partial \lambda^2} + \frac{1}{2} \sum_{j=1}^n \frac{1}{\lambda - \alpha_j} \frac{\partial}{\partial \lambda} + \frac{K^2(\lambda)}{a^2(\lambda)} - \frac{1}{2} \sum_{j=1}^n \frac{I_j^E}{\lambda - \alpha_j} + a^2 + bc \right) \psi(\lambda) = 0. \quad (2.29)$$

In the quantum problem corresponding to the system with second class constraints (1.64a) (1.64b), it is preferable to impose the invariant constraint (cf. Eq. (1.69b))

$$\hat{P}(0) \Psi = P^E(0) \Psi, \quad (2.30)$$

where

$$\frac{P^E(0)}{a(0)} = -\frac{1}{2} \sum_{i=1}^n \frac{I_i^E}{\alpha_i} = \frac{K^2(0)}{a^2(0)} + bc. \quad (2.31)$$

Letting

$$A_\mu := \left(\frac{\partial}{\partial \lambda_\mu} - ia \right)^2 + \frac{1}{2} \sum_{j=1}^n \frac{1}{\lambda_\mu - \alpha_j} \left(\frac{\partial}{\partial \lambda_\mu} - ia \right) + \frac{K^2(\lambda_\mu)}{a^2(\lambda_\mu)} + a^2 + bc, \quad (2.32)$$

we have

$$\begin{aligned} \hat{P}(\lambda) &= \frac{Q_0(\lambda)a(\lambda_n)}{Q_0(\lambda_n)} \Delta_n + \sum_{\mu=1}^{n-1} \frac{(\lambda - \lambda_n)Q_0(\lambda)a(\lambda_\mu)}{(\lambda - \lambda_\mu)(\lambda_\mu - \lambda_n)Q'_0(\lambda_\mu)} \Delta_\mu \\ &= Q_0(\lambda) \left[\hat{R}(\lambda) - \hat{R}(\lambda_n) + \frac{a(\lambda_n)}{Q_0(\lambda_n)} \Delta_n \right], \end{aligned} \quad (2.33)$$

where

$$\hat{R}(\lambda) := \sum_{\mu=1}^{n-1} \frac{a(\lambda_\mu)}{(\lambda - \lambda_\mu)Q'_0(\lambda_\mu)} \Delta_\mu. \quad (2.34)$$

It follows that, for Ψ satisfying the constraint (2.30), we have

$$\hat{P}(\lambda)\Psi = \hat{P}_0(\lambda)\Psi, \quad (2.35)$$

where

$$\begin{aligned} \hat{P}_0(\lambda) &:= Q_0(\lambda) \left(\hat{R}(\lambda) - \hat{R}(0) + \frac{P^E(0)}{Q_0(0)} \right) \\ &= Q_0(\lambda) \left[\sum_{\mu=1}^{n-1} \frac{\lambda a(\lambda_\mu)}{(\lambda - \lambda_\mu)\lambda_\mu Q'_0(\lambda_\mu)} \Delta_\mu + \frac{P^E(0)}{Q_0(0)} \right]. \end{aligned} \quad (2.36)$$

Since the coordinate λ_n does not enter in Eq. (2.35), it may be treated as a parameter, and set equal to 0 in Ψ . This may be viewed as the gauge condition associated to the first class constraint (2.30). Since the constraint commutes with the family of operators $\hat{P}(\lambda)$, the reduced operators $\hat{P}_0(\lambda)$ defined by (2.36) still commute amongst themselves. The wave function Ψ , with $\lambda_n = 0$, is defined on the ellipsoid \mathcal{E}^{n-1} , and satisfies the polynomial family of Schrödinger equations,

$$Q_0(\lambda) \left[\sum_{\mu=1}^{n-1} \frac{\lambda a(\lambda_\mu)}{(\lambda - \lambda_\mu)\lambda_\mu Q'_0(\lambda_\mu)} \Delta_\mu + \frac{P^E(0)}{Q_0(0)} \right] \Psi = P^E(\lambda) \Psi, \quad (2.37)$$

where $P^E(0)$ is given by (2.31). The coefficients of $\hat{P}_0(\lambda)$ thus generate an $n - 1$ parameter family of commuting invariants, which may be simultaneously diagonalized. The simultaneous eigenfunction Ψ is independent of the parameter λ , so dividing (2.37) by $Q_0(\lambda)$ and taking the limit $\lambda \rightarrow \infty$ gives

$$\hat{R}(0)\Psi = \left(\frac{P^E(0)}{Q_0(0)} - \frac{1}{2} \sum_{i=1}^n I_i^E \right) \Psi, \quad (2.38)$$

and hence

$$\hat{P}_0(\lambda)\Psi = Q_0(\lambda) \left(\hat{R}(\lambda) + \sum_{i=1}^n I_i^E \right) \Psi. \quad (2.39)$$

The Schrödinger equation (2.37) may therefore equivalently be written

$$Q_0(\lambda) \left[\sum_{\mu=1}^{n-1} \frac{a(\lambda_\mu)}{(\lambda - \lambda_\mu)\lambda_\mu Q'_0(\lambda_\mu)} \Delta_\mu + \sum_{i=1}^n I_i^E \right] \Psi = P^E(\lambda) \Psi. \quad (2.40)$$

Either way, $\Psi(\lambda_1, \dots, \lambda_{n-1})$ may be chosen in completely separated form

$$\Psi(\lambda_1, \dots, \lambda_{n-1}) = e^{ia \sum_{\mu=1}^{n-1} \lambda_\mu} \prod_{\mu=1}^n \psi_\mu(\lambda_\mu), \tag{2.41}$$

where each $\psi_\mu(\lambda_\mu)$ still satisfies (2.29).

Notice, however, that the reduced operator $\hat{P}_0(\lambda)$ obtained in this way is not self-adjoint with respect to the scalar product

$$\langle \Phi, \Psi \rangle := \int_{\mathcal{E}^{n-1}} \Phi \Psi dV_{\mathcal{E}^{n-1}}, \tag{2.42}$$

with $dV_{\mathcal{E}^{n-1}}$ the standard volume form (1.66) on \mathcal{E}^{n-1} . It is, however, with respect to the scalar product

$$\langle \Phi, \Psi \rangle := \int_{\mathcal{E}^{n-1}} \Phi \Psi d\tilde{V}_{\mathcal{E}^{n-1}}, \tag{2.43}$$

where $dV_{\mathcal{E}^{n-1}}$ is the modified volume form defined by

$$dV_{\mathcal{E}^{n-1}} = \left(\frac{c}{2}\right)^{\frac{n-1}{2}} \prod_{\nu < \mu}^{n-1} (\lambda_\mu - \lambda_\nu) \prod_{\mu=1}^{n-1} \frac{\lambda_\mu}{|a(\lambda_\mu)|^{\frac{1}{2}}} d\lambda_1 \wedge \dots \wedge d\lambda_{n-1}. \tag{2.44}$$

Making the transformation

$$\begin{aligned} \Psi &\rightarrow \tilde{\Psi} := \prod_{\mu=1}^{n-1} \lambda_\mu^{\frac{1}{2}} \Psi, \\ \hat{P}_0 &\rightarrow \tilde{P}_0 := \prod_{\mu=1}^{n-1} \lambda_\mu^{\frac{1}{2}} \circ \hat{P}_0 \circ \prod_{\mu=1}^{n-1} \lambda_\mu^{-\frac{1}{2}} \end{aligned} \tag{2.45}$$

gives the equivalent Schrödinger equation

$$\tilde{P}_0(\lambda) \tilde{\Psi} = P^E(\lambda) \tilde{\Psi}, \tag{2.46}$$

where the polynomial family of operators $\tilde{P}_0(\lambda)$, given by

$$\begin{aligned} \tilde{P}_0(\lambda) := & \sum_{\mu=1}^{n-1} \frac{\lambda Q(\lambda) a(\lambda_\mu)}{(\lambda - \lambda_\mu) \lambda_\mu Q'(\lambda_\mu)} \left[\left(\frac{\hat{c}}{\partial \lambda_\mu} - ia \right)^2 \right. \\ & \left. + \frac{1}{2} \left(\sum_{j=1}^n \frac{1}{\lambda_\mu - \alpha_j} - \frac{1}{\lambda_\mu} \right) \left(\frac{\hat{c}}{\partial \lambda_\mu} - ia \right) + \frac{K^2(\lambda_\mu)}{a^2(\lambda_\mu)} + a^2 + bc + \mathcal{U}(\lambda_\mu) \right], \end{aligned} \tag{2.47}$$

with

$$\mathcal{U}(\lambda_\mu) := \frac{5}{16 \lambda_\mu^2} - \frac{1}{\lambda_\mu} \sum_{j=1}^n \frac{1}{\lambda_\mu - \alpha_j}, \tag{2.48}$$

is now self adjoint with respect to (2.42).

The terms $\mathcal{U}(\lambda_\mu)$ in (2.47) are of order $\ell(\hbar^2)$, and hence disappear in the semi-classical limit. If we subtract these, we obtain the polynomial family of self adjoint operators

$$\begin{aligned} \hat{P}_c(\lambda) := & \sum_{\mu=1}^{n-1} \frac{\lambda Q(\lambda) a(\lambda_\mu)}{(\lambda - \lambda_\mu) \lambda_\mu Q'(\lambda_\mu)} \left[\left(\frac{\hat{c}}{\partial \lambda_\mu} - ia \right)^2 \right. \\ & \left. + \frac{1}{2} \left(\sum_{j=1}^n \frac{1}{\lambda_\mu - \alpha_j} - \frac{1}{\lambda_\mu} \right) \left(\frac{\hat{c}}{\partial \lambda_\mu} - ia \right) + \frac{K^2(\lambda_\mu)}{a^2(\lambda_\mu)} + a^2 + bc \right], \end{aligned} \tag{2.49}$$

which have the same semi-classical limit as (2.47). The coefficients of $\hat{P}_c(\lambda)$ may again be simultaneously diagonalized by separation of variables in the corresponding Schrödinger equation

$$\hat{P}_c(\lambda) \Psi_c = P_c^E(\lambda) \Psi_c, \tag{2.50}$$

where the eigenvalues $I_i^{E,c}$, defined by

$$\frac{P_c^E(\lambda)}{a(\lambda)} := \sum_{i=1}^n \frac{I_i^{E,c}}{\lambda - \alpha_i}, \tag{2.51}$$

are again constrained to satisfy

$$\frac{P_c^E(0)}{a(0)} = -\frac{1}{2} \sum_{i=1}^n \frac{I_i^{E,c}}{\alpha_i} = \frac{K^2(0)}{a^2(0)} + bc. \tag{2.52}$$

Expressing Ψ_c in the completely separated form

$$\Psi(\lambda_1, \dots, \lambda_n) = e^{ia \sum_{\mu=1}^{n-1} \lambda_\mu} \prod_{\mu=1}^{n-1} \psi_{c,\mu}(\lambda_\mu), \tag{2.53}$$

each $\psi_{c,\mu}(\lambda_\mu)$ must satisfy

$$\left(\frac{\partial^2}{\partial \lambda^2} + \frac{1}{2} \left(\sum_{j=1}^n \frac{1}{\lambda - \alpha_j} - \frac{1}{\lambda} \right) \frac{\partial}{\partial \lambda} + \frac{K^2(\lambda)}{a^2(\lambda)} - \frac{1}{2} \sum_{j=1}^n \frac{I_j^E}{\lambda - \alpha_j} + a^2 + bc \right) \psi_c(\lambda) = 0. \tag{2.54}$$

Since the self-adjoint operators $\tilde{P}_0(\lambda)$ and $\hat{P}_c(\lambda)$ have the same semi-classical limits, and both are completely integrable through separation of variables, either may be regarded as a valid quantization of the corresponding constrained classical system on \mathcal{E}^{n-1} . Ambiguities of order $\mathcal{O}(\hbar^2)$ are well known to occur in the constrained quantization procedure [D], and can only be resolved by appealing to some further physical principle. In the case $a = b = \mu_i = 0$, the degree $n - 1$ coefficient in $\hat{P}_c(\lambda)$, which in the classical system generates free motion on \mathcal{E}^{n-1} (cf. Eq. (1.78)), gives the Laplacian on \mathcal{E}^{n-1} , whereas that in $\tilde{P}_0(\lambda)$ contains additional $\mathcal{O}(\hbar^2)$ potential terms due to the $\mathcal{W}(\lambda_\mu)$ factors in (2.47). This may be a valid reason to regard $\hat{P}_c(\lambda)$ as the correct quantization of this family of integrable systems.

3. Discussion

We have seen that in the ambient space \mathbf{R}^n and the sphere, the quantization of the integrable systems of Sect. 1 proceeds straightforwardly, whereas on the ellipsoid, constrained quantization leads to $\mathcal{O}(\hbar^2)$ ambiguities. The origin of the problem may be understood by noting that, whereas in the coordinates of Sects. 1b, 2b, which are adapted to the sphere, not only does separation with respect to the transversal coordinate q hold for the ambient space Schrödinger equation, but also for the volume form (1.24), giving rise to the standard sphere volume form (1.41) upon factorization. In the case of the ellipsoid, however, although the Schrödinger equation again separates in the adapted coordinates of Sects. 1c, 2c, the volume form (1.53) does not admit such a factorization with respect to the transversal coordinate λ_n . If one just imposes the constraint $\lambda_n = 0$ defining the ellipsoid, and eliminates the $d\lambda_n$ factor in (1.53), the volume form $d\tilde{V}_{\mathcal{E}^{n-1}}$ of Eq. (2.44) results, not

the standard one (1.66) determined by the induced metric. More generally, when the constrained manifold is obtained as an orbit in configuration space under a large symmetry group, and the commuting operators of the system are expressible, as for the sphere, in terms of invariant operators in the enveloping algebra, the constraining procedure may compatibly be deduced through separation of the remaining “radial” variable. This is the case, for example, in the case of free motion on symmetric spaces [BKW1, BKW2, K, KM, ORW]. For the ellipsoid, or more general constrained systems in which no such transitive symmetry group is present, further considerations regarding self-adjointness invariably enter, and $\mathcal{L}(\hbar^2)$ ambiguities of the type encountered here may arise.

The systems studied here (prior to the constraints) may, in view of the form of the operators $\hat{\mathcal{H}}(\lambda)$ of Eq. (2.2), and the choice of commuting invariants given by its “quantum determinant” be regarded equivalently as defining a noncompact version of the Gaudin spin lattice [G, Sk1-Sk3, Ku] (with $su(2)$ replaced by $sl(2, \mathbf{R})$). From this viewpoint, the separation of variables may be interpreted as a “functional Bethe ansatz” (cf. [Sk1-Sk3]).

Two types of generalization of the systems studied here naturally suggest themselves, both at the classical and quantum levels. The first is to allow degeneration of the parameters $\{\alpha_1, \dots, \alpha_n\}$ defining the image of the Poisson map (1.1b), (1.2), including the possibility of higher order poles. This is known (cf. [K, Ku]) to lead to separable coordinates other than the generic, hyperellipsoidal ones appearing here, adapted to symmetry groups larger than the finite group $(\mathbf{Z}_2)^n$ encountered in Sect. 1. The second generalization consists of extending the present considerations to higher rank algebras, such as $\mathfrak{gl}(r)$, and reductions thereof. Whereas the classical systems so arising are known to be integrable and separable under generic assumptions regarding the rational coadjoint orbits and initial data [AHH4], very little is known about the corresponding quantum systems. These questions merit further study, and will be addressed in future work.

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