Commun. Math. Phys. 171, 279 – 322 (1995)

Communications in Mathematical Physics

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A Large k Asymptotics of Witten's Invariant of Seifert Manifolds

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Received: 15 April 1993 / in revised form: 28 November 1994

Abstract: We calculate a large k asymptotic expansion of the exact surgery formula for Witten's $SU(2)$ invariant of some Seifert manifolds. The contributions of all flat connections are identified. An agreement with the 1-loop formula is checked. A contribution of the irreducible connections appears to contain only a finite number of terms in the asymptotic series. A 2-loop correction to the contribution of the trivial connection is found to be proportional to Casson's invariant.

Contents

[1] Work supported by NSF Grant 9009850 and R.A. Welch Foundation.

1. Introduction

A Chern–Simons action is an "almost" gauge invariant function of a gauge connection on a 3-dimensional manifold $\mathscr{M}$:

$$S_{CS} = \frac{1}{4\pi}\varepsilon^{\mu\nu\rho}\mathrm{Tr}\int_{\mathscr{M}} \left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) d^3x \tag{1.1}$$

(a trace is taken in the fundamental representation of the gauge group G). A quantum field theory built upon this action is topological. This means that a partition function presented by a path integral over the gauge equivalence classes of connections

$$Z(\mathscr{M},k) = \int \mathscr{D}A_\mu e^{ikS_{CS}[A_\mu]} \tag{1.2}$$

does not depend on the metric of the manifold $\mathscr{M}$ and is therefore its topological invariant. An exact calculation of this invariant was carried out by E. Witten in his paper [1] on the Jones polynomial. The calculation requires a construction of $\mathscr{M}$ by a surgery on a link in S^3 (or in some other simple manifold, say $S^1 \times S^2$). Reshetikhin and Turaev proved in [2] that Witten's procedure really leads to a topological invariant. Their proof does not use path integral (1.2) which is not a rigorously defined object for mathematicians yet.

A Chern–Simons action enters the exponential of the path integral (1.2) with an arbitrary integer factor k. Its inverse k^{-1} (or rather $2\pi k^{-1}$) plays a role of the Planck constant $\hbar$, which appears in quantum theories and sets a scale of quantum effects. A stationary phase approximation for the integral (1.2) in the limit of $k \to \infty$ expresses a partition function $Z(\mathscr{M},k)$ as an asymptotic series in k^{-1}. Physicists call this series a "loop expansion," because the terms of order k^{1-n} come from the n-loop Feynman diagrams.

The loop expansion of $Z(\mathscr{M},k)$ has been studied in [3,4 and 5], as well as in papers [6 and 7] which were aimed at producing Vassiliev's knot invariants. Feynman rules were formulated, however the actual calculation of loop corrections for particular manifolds went only up to the 1-loop order. The 1-loop correction was found in [1,8,9] to contain such invariants of the manifold $\mathscr{M}$ as the Reidemeister–Ray–Singer torsion, spectral flow and dimensions of cohomologies.

Thus there are two different methods of calculating $Z(\mathscr{M},k)$: a "surgery calculus" of Witten–Reshetikhin–Turaev and a loop expansion which is a standard method of quantum field theory. Both methods should give the same value of $Z(\mathscr{M},k)$ if the path integral (1.2) has the properties that physicists expect it to have. D. Freed and R. Gompf suggested to check this by computing an exact value of $Z(\mathscr{M},k)$ for large values of k through the surgery calculus and then comparing it to the quantum field theory 1-loop approximation. They carried out their program in [8] for some lens spaces and Brieskorn spheres. A computer calculation showed a close correspondence between the values of exact and 1-loop partition functions. In a subsequent paper [9], L. Jeffrey used a Poisson resummation trick to derive analytically a large k expansion of an exact surgery formula for lens spaces and mapping tori. She also observed a correspondence between the surgical and 1-loop expressions (at least up to some minor factors, which we will discuss in the next section). Similar results were obtained in [10].

In this paper we carry out a large k expansion of an exact surgery formula for Seifert manifold $SU(2)$ invariants. We identify the contributions of all flat connections and show that they correspond to the slightly modified 1-loop approximation formula of [8 and 9]. In contrast to the lens spaces, Seifert manifolds have irreducible flat connections. We find rather surprisingly that although the reducible connections contribute to all orders in loop expansion, the contribution of irreducible connections appears to be finite loop exact. We also find that a 2-loop correction to the contribution of the trivial connection is proportional to Casson's invariant.

In Sect. 2 we review the basic features of loop expansion and surgery calculus. Section 3 describes an application of the Poisson resummation to the surgery formula for Seifert manifolds with 3 fibers. In Sect. 4 the asymptotic expansion of the surgery formula for those manifolds is compared to the 1-loop formula. In Sect. 5 a Poisson resummation is applied to a general n-fibered Seifert manifold and the contributions of both irreducible and reducible flat connections are calculated.

Summary of the Results. For reader's convenience we summarize briefly the main results of the calculations in Sect. 3. We present the $SU(2)$ Witten's invariant of a 3-fibered Seifert manifold $X(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$ as a sum over flat connections in the spirit of Eqs. (2.3) and (2.4). The structure of the flat connections is described in Sect. 4.

An irreducible flat connection is labelled by three integer numbers n_1, n_2, n_3 and a number $\lambda = 0, \frac{1}{2}$. Its holonomies around the fibers and along the central element of the fundamental group are given by Eqs. (4.2)–(4.4). The contribution of the irreducible connection is 2-loop exact:

$$Z_{\mathrm{st}}^{(n_1,n_2,n_3;\lambda)} = \frac{1}{2}\exp\left[2\pi i K \sum_{i=1}^{3}\left(\frac{r_i}{p_i}\tilde{n}_i^2 - q_i s_i \lambda^2\right)\right] e^{2\pi i\lambda + i\frac{3\pi}{4}\mathrm{sign}(\frac{H}{P})}\mathrm{sign}(P)$$
$$\times\left[\prod_{i=1}^{3}\frac{1}{\sqrt{|p_i|}} 2i\sin 2\pi\left(\frac{r_i}{p_i}\tilde{n}_i + s_i\lambda\right)\right] e^{-\frac{i\pi}{2K}\phi}\,. \qquad (1.3)$$

Here

$$P = p_1p_2p_3, \qquad H = p_1p_2q_3 + p_1q_2p_3 + q_1p_2p_3, \qquad \lambda = 0, \tfrac{1}{2}\,,$$
$$\tilde{n}_i = n_i + q_i\lambda, \qquad n_i \in \mathbf{Z}, \qquad p_is_i - q_ir_i = 1, \qquad p_i, q_i, s_i, r_i \in \mathbf{Z}\,, \qquad (1.4)$$

$s(q,p)$ is a Dedekind sum. For more details see Subsect. 4.1. The phase

$$\phi = 3\mathrm{sign}\left(\frac{H}{P}\right) + \sum_{i=1}^{3}\left(12s(q_i,p_i) - \frac{q_i}{p_i}\right) \qquad (1.5)$$

is the 2-loop correction. As we will see, this phase appears in the contributions of all flat connections.

A reducible connection is labelled by three integer numbers n_1, n_2, n_3. Its holonomies are given by Eqs. (4.15), (4.16). The contribution of this connection contains an asymptotic series of loop corrections:

$$Z_{\mathrm{cst}}^{(n_1,n_2,n_3)} = -\exp 2\pi i K\left[\sum_{i=1}^{3}\frac{r_i}{p_i}n_i^2 + \frac{H}{P}c_0^2\right]\frac{e^{i\frac{\pi}{2}\mathrm{sign}(\frac{H}{P})}}{\sqrt{2K|H|}}\mathrm{sign}(P)e^{-\frac{i\pi}{2K}\phi}$$
$$\times\sum_{j=0}^{\infty}\frac{1}{j!}(8\pi i K)^{-j}\left(\frac{P}{H}\right)^j\left[\partial_c^{(2j)}\frac{\prod_{i=1}^{3}2i\sin\left(2\pi\frac{r_in_i+c}{p_i}\right)}{2i\sin 2\pi c}\right]\Bigg|_{c=c_0}, \qquad (1.6)$$

here $c_0 = \frac{P}{H}\sum_{i=1}^{3}\frac{n_i}{p_i}$, for notations see Subsect. 4.2. A logarithm of this series has to be calculated in order to put (1.6) into the form (2.4).

A reducible flat connection for which $c_0 = 0, 1/2$ (see "point on a face" in Subsect. 4.3), contributes

$$Z_{\mathrm{cst}}^{(n_1,n_2,n_3)} = -\exp 2\pi i K\left[\sum_{i=1}^{3}\frac{r_i}{p_i}n_i^2 + \frac{H}{P}c_0^2\right]\frac{e^{i\frac{\pi}{2}\mathrm{sign}(\frac{H}{P})}}{\sqrt{2K|H|}}e^{2\pi i c_0}\mathrm{sign}(P)e^{-\frac{i\pi}{2K}\phi}$$
$$\times\left\{e^{i\frac{\pi}{4}\mathrm{sign}(\frac{H}{P})}\sqrt{\frac{K}{8}\left|\frac{H}{P}\right|}\prod_{i=1}^{3}2i\sin\left(2\pi\frac{r_i n_i + c_0}{p_i}\right)\right.$$
$$+\sum_{j=0}^{\infty}\frac{1}{j!}(8\pi i K)^{-j}\left(\frac{P}{H}\right)^{j}\partial_\varepsilon^{(2j)}$$
$$\left.\times\left[\frac{\prod_{i=1}^{3}2i\sin\left(2\pi\frac{r_i n_i + c_0 + \varepsilon}{p_i}\right)}{2i\sin 2\pi\varepsilon} - \frac{\prod_{i=1}^{3}2i\sin\left(2\pi\frac{r_i n_i + c_0}{p_i}\right)}{4\pi i\varepsilon}\right]\Bigg|_{\varepsilon=0}\right\}. \tag{1.7}$$

The contribution of the trivial connection is

$$Z_{\mathrm{cst}}^{(\mathrm{triv})} = -\frac{e^{i\frac{\pi}{2}\mathrm{sign}(\frac{H}{P})}}{\sqrt{8K|H|}}\mathrm{sign}(P)e^{-\frac{i\pi}{2K}\phi}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\pi}{2iK}\frac{P}{H}\right)^{j}\left[\partial_\varepsilon^{(2j)}\frac{\prod_{i=1}^{3}2i\sin\frac{\varepsilon}{p_i}}{2i\sin\varepsilon}\right]\Bigg|_{\varepsilon=0}. \tag{1.8}$$

A remarkable feature of this formula is that its full 2-loop term is proportional to Casson's invariant (4.14).

The formulas analogous to Eqs. (1.6), (1.7) and (1.8) for the n-fibered Seifert manifold are Eqs. (5.49), (5.53) and (5.55).

2. Calculation of Witten's Invariant

2.1. Loop Expansion. We start with a brief description of a stationary phase approximation to the path integral (1.2). The stationary phase points are the extrema of the action (1.1). Since

$$\frac{\delta S}{\delta A_\mu} = \frac{1}{2\pi}\varepsilon^{\mu\nu\rho}F_{\nu\rho}, \tag{2.1}$$

these extrema are flat connections, i.e. connections with $F_{\mu\nu} = 0$. The gauge equivalence classes of flat connections are in one-to-one correspondence with the homomorphisms

$$\pi_1(\mathcal{M}) \xrightarrow{A} G, \qquad A : x \mapsto g(x) \in G \tag{2.2}$$

(G is a gauge group) up to a conjugacy, that is, the homomorphisms $g(x)$ and $h^{-1}g(x)h$ are considered equivalent.

Each stationary phase point $A^{(i)}$ contributes a classical exponential $\exp[ikS_i]$ times an asymptotic series in k:

$$Z(\mathcal{M},k) = \sum_i Z^{(i)}(\mathcal{M},k), \qquad Z^{(i)}(\mathcal{M},k) = e^{ikS_i}\left(\sum_{n=0}^{\infty}k^{-n}\Delta_n^{(i)}\right). \tag{2.3}$$

Another form of presenting the same expansion is

$$Z^{(i)}(\mathcal{M},k)=\sum_i \Delta_0^{(i)} \exp ik \left[S_i+\sum_{n=2}^{\infty} k^{-n} S_n^{(i)}\right] . \tag{2.4}$$

Here S_i are Chern–Simons invariants of the flat connections $A_\mu^{(i)}$, and $S_n^{(i)}$ are the n-loop quantum corrections coming from the n-loop 1-particle irreducible Feynman diagrams. A set of Feynman rules for their calculation has been developed in [3], however the actual calculations have been carried out only up to the 1-loop order.

Generally in quantum field theory a 1-loop factor is an inverse square root of a determinant of the second order variations of the classical action taken at the stationary phase point. However a gauge invariance of the action (1.1) requires a gauge fixing and an introduction of the Faddeev–Popov ghost determinant (see [1] for details). So for the Chern–Simons theory

$$\Delta_0=\frac{\left|\det\left(\frac{k}{4\pi^2}\Delta\right)\right|}{\left[\det\left(-i\frac{k}{4\pi^2}L_-\right)\right]^{1/2}} . \tag{2.5}$$

Here Δ is a covariant Laplacian

$$\Delta=D_\mu D^\mu, \qquad D_\mu=\partial_\mu+A_\mu \tag{2.6}$$

acting on the Lie algebra valued functions, while L_- is

$$L_-=\begin{bmatrix} *d_A & -d_{A^*} \\ d_{A^*} & 0 \end{bmatrix} \tag{2.7}$$

acting on the Lie algebra valued 1-forms and 3-forms. A differential d_A is built upon a covariant derivative $D_\mu, d_A^2=0$ for flat connections.

According to [1], the absolute value of the ratio (2.5) is a square root of the Reidemeister–Ray–Singer torsion $\tau_R(A)$. A detailed expression for the phase of that ratio has been worked out in [8]. The 1-loop formula for $Z(\mathcal{M},k)$ presented there is a sum (2.3) over the flat connections $A^{(i)}$ in which:

$$Z^{(i)}(\mathcal{M},k)=e^{-i\frac{\pi}{4}(\dim G)(1+b^1)}e^{2\pi iKS_{CS}^{(i)}}\tau_R^{\frac{1}{2}}(A^{(i)})e^{-i\frac{\pi}{2}I_i}C_0C_1 , \tag{2.8}$$

here $K=k+c_v$, c_v is a dual Coxeter number or, equivalently, a quadratic Casimir invariant of the adjoint representation, b^1 is the first Betti number and I_i is a spectral flow. The factors C_0 and C_1 reflect the presence of the 0-form (i.e. 3-form) and 1-form zero modes in the operators Δ and L_- of Eq. (2.5). These factors have to be slightly modified from their original values in [8].

The zero modes are related to the elements of the cohomology spaces $H^0(\mathcal{M},d_{A^{(i)}})$ and $H^1(\mathcal{M},d_{A^{(i)}})$. For each element of H^0 there is a zero mode of Δ and a zero mode of L_-. For each element of H^1 there is another zero mode of L_-. It is also known that H^0 can be identified with a tangent space of the symmetry group H_i of the connection $A_\mu^{(i)}$. The group H_i consists of the gauge transformations that do not change $A_\mu^{(i)}$. Equivalently, H_i is a subgroup of G whose elements commute with the image of the homomorphism (2.2). As for H^1, its elements represent infinitesimal deformations of the connection $A_\mu^{(i)}$ which do not violate the flatness condition. This picture is reminiscent of the string theory. There the zero modes of the ghosts c and b were identified with the elements of tangent spaces of the symmetry group and moduli of the complex structure. However in our case generally not all the elements of H^1 can be extended to finite deformations of flat connections. In other words, $\dim H^1 \geqq \dim X_i(\mathscr{M})$, where X_i is a connected component of the moduli space of flat connections.

Let us first assume that $\dim H^1 = \dim X_i$. If operators Δ, L_- have zero modes, then the Reidemeister torsion can still be obtained from Eq. (2.5) if the zero modes and zero eigenvalues are removed from there. L. Jeffrey noted in [9] that $\tau_R^{1/2}$ thus defined is an element of $\Lambda^{\max}H^0 \otimes (\Lambda^{\max}H^1)^*$. She suggested to take a canonical element $v \in (\Lambda^{\max}H^0)^*$ derived from the basic inner product on H^0 which is a Lie algebra of H_i. A pairing of v and $\tau_R^{1/2}$ produces a volume form on the moduli space X_i. A sum over the flat connections in Eq. (2.8) then includes a natural integration over X_i. However, according to [9], this procedure does not quite agree with the leading term in the $1/k$ expansion of $Z(S^3,k)$.

We propose a slightly different prescription. We take any element $v \in (\Lambda^{\max}H^0)^*$ and balance the integral over X_i, defined by pairing of v and $\tau_R^{1/2}$, with a factor of $1/\mathrm{Vol}(H_i)$, volume of H_i being defined by the same element v.[2] We show in the Appendix why this factor should appear after the removal of the zero modes from Eq. (2.5) by considering a simple finite dimensional version of a gauge invariant path integral. We also demonstrate in the end of Subsect. 2.3 how our prescription fits the value of $Z(S^3,k)$.

There is another consequence of dropping the zero modes from the determinants in Eq. (2.5). Each non-zero mode of the operator Δ carries a factor of $k/4\pi^2$ in Eq. (2.5) and each non-zero mode of L_- carries there a factor[3] of $(-ik/4\pi^2)^{-1/2}$. By dropping the modes, we loose these factors. Therefore dropping an element of H^0 produces an extra factor $(ik/4\pi^2)^{-1/2}$ while dropping an element of H^1 creates a factor $(-ik/4\pi^2)^{1/2}$. Thus

$$C_0 = \frac{1}{\mathrm{Vol}(H_i)}\left(\frac{ik}{4\pi^2}\right)^{-(\dim H^0)/2} . \tag{2.9}$$

We could also assume that

$$C_1 = \left(-\frac{ik}{4\pi^2}\right)^{(\dim H^1)/2} . \tag{2.10}$$

However the 1-form zero modes of L_- that can not be extended to finite deformations of the flat connection, should not be simply dropped from Eq. (2.5). A non-zero mode of L_- contributes to Eq. (2.5) through a gaussian integral

$$\int_{-\infty}^{+\infty} \exp\left(\frac{ik}{4\pi}\lambda x^2\right) dx \sim \left(-\frac{ik}{4\pi^2}\right)^{-1/2} . \tag{2.11}$$

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[2] The factor $1/\mathrm{Vol}(H_i)$ appeared in slightly different circumstances in [11]. It also appeared in [12] and [13] where the Alexander polynomial was produced from a Chern–Simons theory based on a supergroup $U(1|1)$. It was shown there that $\mathrm{Vol}(U(1|1)) = 0$, so that the flat connections for which $H_i = U(1|1)$, gave infinite contributions to the partition function. These infinities helped to explain the nonmultiplicativity of the Alexander polynomial which distinguished it from the family of the $SU(N)$ Jones polynomials.

[3] Actually a partition function (1.2) has also a factor $(\frac{k}{4\pi^2})^{\#\text{ of all modes of }\Delta - \frac{1}{2}\#\text{ of all modes of }L_-}$ hidden in the integration measure $\mathscr{D}A_\mu$.

A zero mode 1-form that hits obstruction contributes through the integral

$$\int_{-\infty}^{+\infty} \exp\left(\frac{ik}{4\pi}\lambda x^4\right) dx \sim \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left(-\frac{ik}{4\pi}\right)^{-1/4} \tag{2.12}$$

Therefore a corrected version of the formula for C_1 is

$$C_1 = \left(-\frac{ik}{4\pi^2}\right)^{\frac{\dim H^1}{2}} \left(-\frac{ik}{4\pi^2}\right)^{-\frac{\dim H^1 - \dim X_i}{4}} = \left(-\frac{ik}{4\pi^2}\right)^{\frac{\dim H^1 + \dim X_i}{4}} \tag{2.13}$$

and the 1-loop formula (2.8) takes the form

$$\begin{aligned} Z^{(i)}(\mathcal{M}, k) &= \exp\left(2\pi i K S_{CS}^{(i)}\right) \frac{1}{\mathrm{Vol}(H_i)} \tau_R^{\frac{1}{2}} \left(\frac{k}{4\pi^2}\right)^{-\frac{\dim H^0}{2} + \frac{\dim H^1 + \dim X_i}{4}} \\ &\times \exp -i\frac{\pi}{4}\left[(1 + b^1)\dim G + 2I_i + \dim H^0 + \frac{\dim H^1 + \dim X_i}{2}\right]. \end{aligned} \tag{2.14}$$

Note that there are additional numerical factors coming from Eq. (2.12) if $\dim X_i \neq \dim H^1$, however the power of k in the preexponential factor of Eq. (2.14) is correct.

2.2. Surgery Calculus. Here we briefly present Witten's recipe of an exact calculation of the partition function (1.2). Witten used the fact that the Hilbert space of the Chern–Simons quantum field theory is isomorphic to the space of conformal blocks of the level k 2-dimensional WZW model based on the same group G. More specifically, a Chern–Simons Hilbert space corresponding to a 2-dimensional torus is equivalent to the space of affine characters of G (see, e.g. [14]).

Consider a path integral (1.2) calculated over a solid torus with a Wilson line carrying representation V_Λ of G going inside it. Λ denotes the shifted highest weight, i.e. the highest weight of V_Λ is $\Lambda - \rho, \rho$ being half the sum of positive roots of G: $\rho = \frac{1}{2}\sum_{\lambda_i \in \Delta_+} \lambda_i$. An inclusion of the Wilson line means that the integrand of Eq. (1.2) is multiplied by a trace of a holonomy $\mathrm{tr}_{V_\Lambda} \mathrm{P}\exp(\oint A_\mu dx^\mu)$. Such an integral is a function of the boundary conditions imposed on A_μ on the boundary of the solid torus. Therefore it is an element $|\Lambda\rangle$ of the Hilbert space of T^2. Witten claimed that this element corresponds to the affine character of level k built upon V_Λ and that all such elements corresponding to the integrable affine representations form an orthonormal basis in that Hilbert space.

The group $SL(2, \mathbf{Z})$ acting as modular transformations on T^2, generates canonical transformations in the phase space of the classical Chern–Simons theory. Therefore $SL(2, \mathbf{Z})$ can be unitarily represented in the Hilbert space. This representation is determined by the action of the matrices S and T

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.15}$$

on the affine characters. An action of a general unimodular matrix

$$M^{(p,q)} = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2, \mathbf{Z}) \tag{2.16}$$

is determined by its presentation as a product

$$M^{(p,q)} = T^{a_t} S \cdots T^{a_1} S \,. \tag{2.17}$$

The integer numbers a_t form a continued fraction expansion of p/q:

$$\frac{p}{q} = a_t - \cfrac{1}{a_{t-1} - \frac{1}{\cdots - \frac{1}{a_1}}} \,. \tag{2.18}$$

For more details on this construction, see e.g. [8] and [9].

A lens space $L(p,q)$ can be constructed by gluing the boundaries of two solid tori. The boundaries are identified trivially after a matrix $SM^{(p,-q)}$ acts on one of them, i.e. that matrix determines how the boundaries are glued together. Before the gluing, each solid torus produces an affine character V_ρ growing out of the trivial representation, as a state on its boundary. Hence according to the postulates of a quantum field theory, a partition function (1.2) is equal to a matrix element

$$Z(L(p,q),k) = (\tilde{S}\tilde{M}^{(p,-q)})_{\rho\rho} \,, \tag{2.19}$$

here the tilde denotes a representation of the $SL(2,\mathbf{Z})$ matrices in the space of affine characters.

Note that the lens space depends only on the numbers p and q. Different choices of the entries r and s of the matrix (2.16) correspond to different framings of the same lens space. A framing is a choice of three vector fields which form a basis in the tangent space at each point of the manifold. A phase of a partition function Z depends on a choice of framing. Formula (2.14) gives a 1-loop approximation to $Z(\mathcal{M},k)$ in the standard framing. The surgical formulas should also be reduced to the standard framing in order to yield a true invariant of the manifold. We will discuss this reduction in the end of this subsection and in Subsect. 3.3.

Consider now a manifold $S^2 \times S^1$. A Seifert manifold $X(\frac{p_1}{q_1},\ldots,\frac{p_n}{q_n})$ is constructed by cutting out the tubular neighborhoods of n strands going parallel to S^1 and then gluing them back after performing the $M^{(p_i,q_i)}$ transformations on their boundaries[4]. These transformations change the states on the surfaces of the solid tori from $|\rho\rangle$ into

$$|\Lambda_i'\rangle = \sum_{\Lambda} |\Lambda\rangle M^{(p_i,q_i)}_{\Lambda\rho} \,. \tag{2.20}$$

Therefore an invariant of a Seifert manifold is given by a multiple sum

$$Z(X,k) = e^{i\phi_{\mathrm{fr}}} \sum_{\Lambda_1,\ldots,\Lambda_n} M^{(p_1,q_1)}_{\Lambda_1\rho} \cdots M^{(p_n,q_n)}_{\Lambda_n\rho} N_{\Lambda_1\ldots\Lambda_n} \,. \tag{2.21}$$

Here the factor $e^{i\phi_{\mathrm{fr}}}$ reflects the dependence of Witten's invariant on the choice of framing of X, $\phi_{\mathrm{fr}} = 0$ for the framing coming directly from the surgeries $M^{(p_i,q_i)}$. $N_{\Lambda_1\ldots\Lambda_n}$ is a Verlinde number, which is equal to the invariant of the manifold $S^2 \times S^1$ containing n Wilson lines carrying representations V_{Λ_i} along S^1. This number is also "almost" equal to the number of times that a trivial representation appears in a decomposition of a tensor product $\bigotimes_{i=1}^n V_{\Lambda_i}$. $N_{\Lambda_1\ldots\Lambda_n}$ will be equal to that number if V_{Λ_i} are representations of the quantum algebra G_q. An expression for $N_{\Lambda_1\ldots\Lambda_n}$ in

[4] It is not hard to see that $X(\frac{p}{q})$ and $X(\frac{p_1}{q_1},\frac{p_2}{q_2})$ are the lens spaces.

the case of $n = 3$ and $G = SU(2)$ is presented in Subsect. 3.1, a general case will be considered in Subsect. 5.1.

As in the case of a lens space, the phase of a partition function (2.21) also depends on the choice of numbers r_i, s_i. Witten found in [1] that a change of framing by one unit is accompanied by a change in that phase by $\frac{\pi c}{12}$, c being a central charge of the level k WZW model.

To get a partition function $Z(X,k)$ in the standard framing, we should choose, according to [8],

$$\phi_{\mathrm{fr}} = \frac{\pi c}{12}\left[-3\sigma + \sum_{i=1}^{n}\sum_{j=1}^{t_i} a_j^{(i)}\right] \tag{2.22}$$

in Eq. (2.21). Here $a_j^{(i)}$ form a continued fraction expansion of p_i/q_i, while

$$\sigma = -\mathrm{sign}\left(\sum_{i=1}^{n}\frac{q_i}{p_i}\right) + \sum_{i=1}^{n}\mathrm{sign}\left(\frac{p_i}{q_i}\right) + \sum_{i=1}^{n}\sum_{j=1}^{t_i-1}\mathrm{sign}(a_j^{(i)}) . \tag{2.23}$$

2.3. $SU(2)$ Formulas and Poisson Resummation. Explicit formulas for the $SL(2,\mathbf{Z})$ representation in the space of affine $SU(2)$ characters of level k were derived in [9]. There are $k+1$ integrable $\widetilde{SU}(2)$ representations with spins $0 \leqq j \leqq \frac{k}{2}$. We use the shifted highest weight $\alpha = 2j+1$ instead of j and $K = k+2$ instead of K, so that $0 < \alpha < K$. The weight ρ is equal to 1 for $SU(2)$.

The formulas for $\tilde{S}_{\alpha\beta}$ and $\tilde{T}_{\alpha\beta}$ are well known:

$$\tilde{S}_{\alpha\beta} = \sqrt{\frac{2}{K}}\sin\frac{\pi\alpha\beta}{K}, \qquad \tilde{T}_{\alpha\beta} = e^{-\frac{i\pi}{4}} e^{\frac{i\pi}{2K}\alpha^2}\delta_{\alpha\beta} . \tag{2.24}$$

A substitution of these expressions in the r.h.s. of Eq. (2.17) turns it into a multiple finite gaussian sum. A summation over the intermediate indices goes from 1 to $K-1$. An application of the Poisson resummation formula in [9] converted that sum into another gaussian sum with a summation interval independent of K:

$$\tilde{M}_{\alpha\beta}^{(p,q)} = i\frac{\mathrm{sign}(q)}{\sqrt{2K|q|}} e^{-\frac{i\pi}{4}\Phi(M^{(p,q)})}$$
$$\times \sum_{\mu=\pm 1}\sum_{n=0}^{q-1}\mu \exp\frac{i\pi}{2Kq}\left[p\alpha^2 - 2\mu\alpha(\beta+2Kn) + s(\beta+2Kn)^2\right] . \tag{2.25}$$

Here $\Phi(M)$ is a Rademacher phi function defined as follows

$$\Phi\begin{bmatrix} p & r \\ q & s \end{bmatrix} = \begin{cases} \frac{p+s}{q} - 12s(s,q) & \text{if } q \neq 0 \\ \frac{r}{s} & \text{if } q = 0 \end{cases} , \tag{2.26}$$

a function $s(s,q)$ being a Dedekind sum:

$$s(m,n) = \frac{1}{4n}\sum_{j=1}^{n-1}\cot\frac{\pi j}{n}\cot\frac{\pi m j}{n} . \tag{2.27}$$

We illustrate the use of the Poisson formula

$$\sum_{n\in\mathbf{Z}} f(n) = \sum_{m\in\mathbf{Z}} \int_{-\infty}^{+\infty} e^{2\pi imz} f(x)dx \tag{2.28}$$

by explicitly deriving an expression for the matrix element

$$(\tilde{S}\tilde{T}^p\tilde{S})_{\alpha\beta} = -\frac{e^{-\frac{i\pi}{4}p}}{2K} \sum_{\gamma=1}^{K-1} \sum_{\mu_1,\mu_2=\pm1} \mu_1\mu_2 \exp\frac{i\pi}{2K}\left[p\gamma^2 + 2\gamma(\mu_1\alpha+\mu_2\beta)\right] . \tag{2.29}$$

This exercise will prepare us for the calculations that we will perform in Sect. 3.

We have to extend the summation range of γ to $\mathbf{Z}$ in order to be able to use Eq. (2.28). The summand in Eq. (2.29) is even and periodic with a period of $2K$. We first double the range of summation: $\sum_{\gamma=1}^{K-1} \to \frac{1}{2}\sum_{\gamma=-K}^{K-1}$. Then a formula

$$\sum_{n=0}^{N-1} f(n) = N \lim_{\varepsilon\to 0} \varepsilon^{1/2} \sum_{n\in\mathbf{Z}} e^{-\pi\varepsilon n^2} f(n), \quad \text{if } f(n) = f(n+N)\,, \tag{2.30}$$

and a Poisson resummation allow us to transform Eq. (2.29) into

$$\begin{aligned}(\tilde{S}\tilde{T}^p\tilde{S})_{\alpha\beta} &= -\frac{e^{-\frac{i\pi}{4}p}}{2K} \lim_{\varepsilon\to 0}(K\varepsilon^{1/2}) \sum_{\gamma\in\mathbf{Z}} \sum_{\mu_{1,2}=\pm1} \mu_1\mu_2 e^{-\pi\varepsilon\gamma^2} \exp\frac{i\pi}{2K}\left[p\gamma^2 + 2\gamma(\mu_1\alpha+\mu_2\beta)\right] \\ &= -\frac{e^{-\frac{i\pi}{4}p}}{2K} \lim_{\varepsilon\to 0}(K\varepsilon^{1/2}) \sum_{n\in\mathbf{Z}} \int_{-\infty}^{+\infty} d\gamma \sum_{\mu_{1,2}=\pm1} \mu_1\mu_2 e^{-\pi\varepsilon\gamma^2} \\ &\quad \times \exp\frac{i\pi}{2K}\left[p\gamma^2 + 2\gamma(\mu_1\alpha+\mu_2\beta+2Kn)\right] .\end{aligned} \tag{2.31}$$

Note that a change from a sum to an integral over γ has been essentially accomplished through a substitution

$$\beta \to \beta + 2Kn\,, \tag{2.32}$$

and a subsequent summation over n (actually we used $\mu_2 n$ rather than n, this makes no difference since we take a sum over $n \in \mathbf{Z}$). This is a trick that works for a general expression (2.17). Just one substitution like (2.32) for an initial or final index converts all the intermediate sums in Eq. (2.17) into gaussian integrals. This is, in fact, the origin of the expression $(\beta + 2Kn)$ in Eq. (2.25).

From this point we can proceed in two ways. A straightforward way is to integrate the r.h.s. of Eq. (2.31) over γ. Then, after neglecting some irrelevant terms we get a formula

$$\begin{aligned}(\tilde{S}\tilde{T}^p\tilde{S})_{\alpha\beta} &= -e^{-\frac{i\pi}{4}p}\sqrt{\frac{iK}{2p}} \lim_{\varepsilon\to 0} \sum_{n\in\mathbf{Z}} \sum_{\mu_{1,2}=\pm1} \mu_1\mu_2 \varepsilon^{1/2} e^{-4\pi\varepsilon\frac{K^2}{p^2}n^2} \\ &\quad \times \exp\left[-\frac{i\pi}{2Kp}(\mu_1\alpha+\mu_2\beta+2Kn)^2\right] .\end{aligned} \tag{2.33}$$

The second exponential here is periodic in n with a period p. Therefore a second application of Eq. (2.30), this time backwards, leads to the final expression

$$(\tilde{S}\tilde{T}^p\tilde{S})_{\alpha\beta} = -e^{-\frac{i\pi}{4}p}\sqrt{\frac{i}{8Kp}}\sum_{n=0}^{p-1}\sum_{\mu_{1,2}=\pm 1}\mu_1\mu_2 \exp\left[-\frac{i\pi}{2Kp}(\mu_1\alpha+\mu_2\beta+2Kn)^2\right]. \tag{2.34}$$

An equivalent way to treat the r.h.s. of Eq. (2.31) is to notice that since an integral over γ is gaussian, a stationary phase approximation is exact. An array of stationary phase points $\gamma_{\mathrm{st}}(n) = -2Kn/p$ and their contributions exhibit the same symmetry properties as a summand of Eq. (2.29). Therefore an inverse use of Eq. (2.30) shows that we can drop a factor $(K\varepsilon^{1/2})$ from the r.h.s. of Eq. (2.31) and restrict the sum there to those values of n for which

$$0 \leqq \gamma_{\mathrm{st}}(n) \leqq K. \tag{2.35}$$

The formula that we get this way is slightly different in its form from Eq. (2.34). For p-odd we get

$$\begin{aligned}(\tilde{S}\tilde{T}^p\tilde{S})_{\alpha\beta} = &-e^{-\frac{i\pi}{4}p}\sqrt{\frac{i}{2Kp}}\sum_{\mu_{1,2}=\pm 1}\mu_1\mu_2\left[\frac{1}{2}\exp\left(-\frac{i\pi}{2Kp}(\mu_1\alpha+\mu_2\beta)^2\right)\right. \\ &\left.+\sum_{n=1}^{\frac{p-1}{2}}\exp\left(-\frac{i\pi}{2Kp}(\mu_1\alpha+\mu_2\beta+2Kn)^2\right)\right]. \end{aligned}\tag{2.36}$$

The $1/2$ factor in front of the first exponential here is due to the fact that the stationary point $\gamma_{\mathrm{st}}(0) = 0$ is on the boundary of the interval (2.35). There is another such stationary point $\gamma_{\mathrm{st}}(p/2)$ for p-even.

In the next section we will apply Eqs. (2.28) and (2.30) to formula (2.21). Meanwhile we use Eq. (2.34) together with Eq. (2.19) in order to get an expression for $Z(L(p,-1),k)$ and check a factor of $1/\mathrm{Vol}(H)$ in the 1-loop approximation formula (2.14). In the large k limit for odd p

$$\begin{aligned} Z(L(p,-1),k) &= (\tilde{S}\tilde{T}^p\tilde{S})_{11} \\ &\approx e^{-\frac{i\pi}{4}p}\left[\sqrt{2}\pi\left(\frac{i}{Kp}\right)^{3/2} + 4\sqrt{\frac{i}{2Kp}}\sum_{n=1}^{\frac{p-1}{2}} e^{-2\pi i Kn^2/p}\sin^2\frac{2\pi n}{p}\right]. \end{aligned}\tag{2.37}$$

The first term in the square brackets is a contribution of the trivial connection, while the remaining sum goes over nontrivial maps $\pi_1(L(p,-1)) = Z_p \to SU(2)$. The $SU(2)$ subgroup H commuting with the image of π_1 is $SU(2)$ and $U(1)$ respectively.

According to [8], a square root of the Reidemeister torsion of the trivial connection is $p^{-3/2}$ and that of a nontrivial one is $4p^{-1}\sin^2(2\pi n/p)$. Therefore if Eq. (2.14) is correct, then

$$\mathrm{Vol}\,(SU(2)) = 4\sqrt{2}\pi^2, \quad \mathrm{Vol}\,(U(1)) = 2\sqrt{2}\pi. \tag{2.38}$$

The same value of $\mathrm{Vol}(SU(2))$ is predicted by a large k limit of $Z(S^3,k)$, which is equal to $\sqrt{2}\pi k^{-3/2}$. These values are what we expect since the group $SU(2)$ is a 3-dimensional sphere and $U(1)$ is its big circle. The radius is equal to $\sqrt{2}$.

3. A Large k Limit of the Invariants of 3-Fibered Seifert Manifolds

3.1. Stationary Phase Points. Let us try to apply the Poisson formula (2.28) to the partition function (2.21) for the case of $n = 3$ in order to put it in the form (2.3) or (2.4).

We start by giving an explicit expression for $N_{\alpha_1,\alpha_2,\alpha_3}$ inside the fundamental cube

$$0 < \alpha_1, \alpha_2, \alpha_3 < K\,. \tag{3.1}$$

According to its definition, $N_{\alpha_1,\alpha_2,\alpha_3} = 1$ iff $\alpha_1 + \alpha_2 + \alpha_3$ is odd and the following 4 inequalities are satisfied:

$$\begin{aligned} \alpha_1 + \alpha_2 - \alpha_3 &> 0\,, \\ \alpha_1 - \alpha_2 + \alpha_3 &> 0\,, \\ -\alpha_1 + \alpha_2 + \alpha_3 &> 0\,, \\ \alpha_1 + \alpha_2 + \alpha_3 &< 2K\,. \end{aligned} \tag{3.2}$$

Otherwise, $N_{\alpha_1,\alpha_2,\alpha_3} = 0$. We can drop a restriction on the parity of $\alpha_1 + \alpha_2 + \alpha_3$ if we change the formula (2.21) into

$$Z\left(X\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right), k\right) = e^{i\phi_{\mathrm{fr}}} \sum_{0<\alpha_1,\alpha_2,\alpha_3<K} \sum_{\lambda=0,\frac{1}{2}} \frac{1}{2} e^{2\pi i\lambda(1-\alpha_1-\alpha_2-\alpha_3)}$$
$$\times \tilde{M}^{(p_1,q_1)}_{\alpha_1 1} \tilde{M}^{(p_2,q_2)}_{\alpha_2 1} \tilde{M}^{(p_3,q_3)}_{\alpha_3 1} \tilde{N}_{\alpha_1,\alpha_2,\alpha_3}\,, \tag{3.3}$$

here $\tilde{N}_{\alpha_1,\alpha_2,\alpha_3}$ is a modified Verlinde number, $\tilde{N}_{\alpha_1,\alpha_2,\alpha_3} = 1$ in the whole region defined by the inequalities (3.2). This region is a tetrahedron within the fundamental cube (3.1) (see Fig. 1). To simplify this picture we draw its section by a plane $\alpha_3 = \text{const}$ in Fig. 2 (region 1_+).

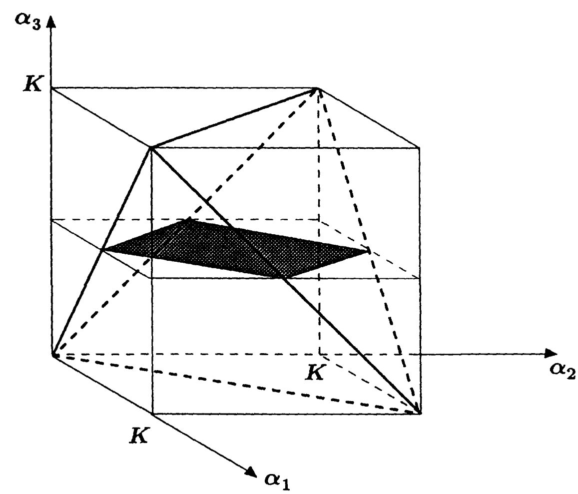

Fig. 1. A fundamental tetrahedron.

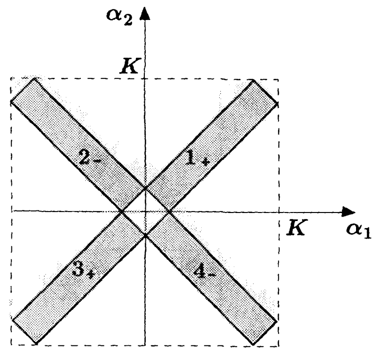

Fig. 2. A section of the fundamental tetrahedron and its Weyl reflections by a plane $\alpha_3 =$const.

At this stage we could use a discrete stationary phase approximation method described in Appendix A of [8], in order to get the 1-loop approximation of Z. The stationary phase points inside the tetrahedron would correspond to the irreducible flat connections. The conditional stationary phase points on the faces of the tetrahedron would correspond to the reducible flat connections.

We use a different approach close to the one in Subsect. 2.3 in order to get the full $1/k$ expansion of $Z(X,k)$. We want to extend the sum in Eq. (3.3) from the fundamental cube to the whole 3-dimensional space. We do this in two steps by using the symmetries of the matrices $\tilde{M}^{(p,q)}_{\alpha,\beta}$ under the affine Weyl transformations. These transformations include the ordinary Weyl reflections as well as the shifts by the root lattice multiplied by K:

$$\tilde{M}^{(p,q)}_{-\alpha,\beta} = -\tilde{M}^{(p,q)}_{\alpha,\beta}, \qquad \tilde{M}^{(p,q)}_{\alpha+2K,\beta} = \tilde{M}^{(p,q)}_{\alpha,\beta}. \tag{3.4}$$

The easiest way to see these symmetries is to use Eq. (2.17) and expressions (2.24) for $\tilde{S}_{\alpha\beta}$ and $\tilde{T}_{\alpha\beta}$.

The first of Eqs. (3.4) enables us to extend the sum over α_i in Eq. (3.3) to a bigger cube: $\sum_{0<\alpha_1,\alpha_2,\alpha_3<K} \to \frac{1}{8}\sum_{-K<\alpha_1,\alpha_2,\alpha_3<K}$ if we extend $\tilde{N}_{\alpha_1,\alpha_2,\alpha_3}$ as an antisymmetric function inside that cube (see the regions 2_-, 3_+ and 4_- in Fig. 2). The translational invariance of $\tilde{M}_{\alpha\beta}$ together with Eq. (2.30) brings us to another formula for Z:

$$Z = \lim_{\varepsilon\to 0} \frac{(2K\varepsilon^{1/2})^3}{8} e^{i\phi_{\mathrm{fr}}} \sum_{\alpha_1,\alpha_2,\alpha_3\in\mathbf{Z}} \sum_{\lambda=0,\frac{1}{2}} \frac{1}{2} e^{2\pi i\lambda(1-\alpha_1-\alpha_2-\alpha_3)-\pi\varepsilon(\alpha_1^2+\alpha_2^2+\alpha_3^2)}$$
$$\times \tilde{M}^{(p_1,q_1)}_{\alpha_1 1}\tilde{M}^{(p_2,q_2)}_{\alpha_2 1}\tilde{M}^{(p_3,q_3)}_{\alpha_3 1}\tilde{N}_{\alpha_1,\alpha_2,\alpha_3} \tag{3.5}$$

if we require $\tilde{N}_{\alpha_1,\alpha_2,\alpha_3}$ to be a periodic function of its indices with the period of $2K$ (see Fig. 3).

A Poisson formula (2.28) transforms the sum over α_i into an integral:

$$\sum_{\alpha_1,\alpha_2,\alpha_3\in\mathbf{Z}} = \int d\alpha_1 d\alpha_2 d\alpha_3 \sum_{n_1,n_2,n_3\in\mathbf{Z}} \prod_{i=1}^{3}\delta(\alpha_i - n_i)$$
$$= \int d\alpha_1 d\alpha_2 d\alpha_3 \sum_{m_1,m_2,m_3\in\mathbf{Z}} \exp\left(2\pi i\sum_{i=1}^{3}\alpha_i m_i\right). \tag{3.6}$$

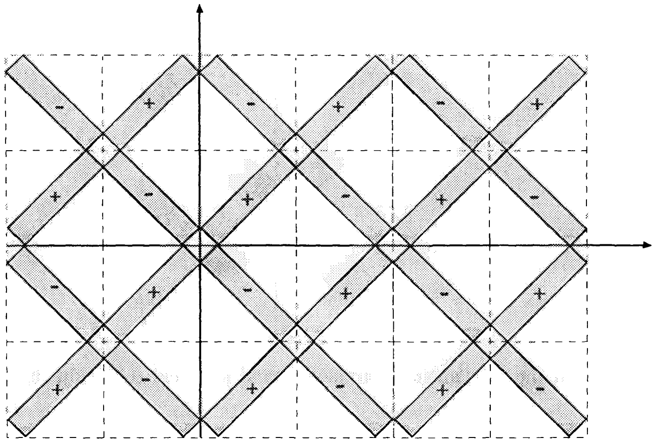

Fig. 3. A section of the full array of tetrahedra by a plane α_3 =const.

On the other hand,

$$e^{2\pi i\alpha m}\tilde{M}_{\alpha\beta} = \tilde{M}_{\alpha,\beta+2Kqm}\,, \tag{3.7}$$

so the sum over m_i and the exponential in the r.h.s. of Eq. (3.6) can be absorbed by extending the sum over n in Eq. (2.25) to all integer numbers. Finally

$$Z(X,k) = e^{i\phi_{\mathrm{fr}}} Z_1 Z_2\,, \tag{3.8}$$

$$Z_1 = i^3 \operatorname{sign}(q_1q_2q_3)\exp\left[-\frac{i\pi}{4}\sum_{i=1}^{3}\Phi(M^{(p_i,q_i)})\right], \tag{3.9}$$

$$Z_2 = \lim_{\varepsilon\to 0}\frac{(2K\varepsilon^{1/2})^3}{8}\frac{1}{2}\sum_{\lambda=0,\frac{1}{2}} e^{2\pi i\lambda}\int\tilde{N}_{\alpha_1\alpha_2\alpha_3}\prod_{i=1}^{3}\frac{d\alpha_i}{\sqrt{2K|q_i|}}\sum_{\mu_i=\pm1}\sum_{n_i\in\mathbf{Z}}\mu_i e^{-\pi\varepsilon\alpha_i^2}$$
$$\times\exp\frac{i\pi}{2Kq_i}\left[p_i\alpha_i^2 - 2\alpha_i(2K(n_i+q_i\lambda)+\mu_i) + s_i(2Kn_i+\mu_i)^2\right]. \tag{3.10}$$

We separated explicitly the phase factor Z_1 in order to simplify our formulas.

The integral in Eq. (3.10) is gaussian, but the function $\tilde{N}_{\alpha_1\alpha_2\alpha_3}$ carves a rather complicated region out of the 3-dimensional α-space. A slice of that region for $\alpha_3 = \text{const}$ is depicted in Fig. 3. Fortunately, this region can be represented as a linear combination of positive strips (double wedges in 3-dimensional space)

$$\alpha_1 - \alpha_3 + 2Kl < \alpha_2 < \alpha_1 + \alpha_3 + 2Kl, \quad l\in\mathbf{Z} \tag{3.11}$$

and negative strips

$$\alpha_3 - \alpha_1 + 2Kl < \alpha_2 < -\alpha_3 - \alpha_1 + 2Kl, \quad l\in\mathbf{Z}\,. \tag{3.12}$$

Each strip (double wedge) is in turn a difference between two half-planes (half-spaces). Overall we have a superposition of half-spaces

$$\sum_{i=1}^{3}\nu_i\alpha_i + 2Kl > 0, \quad \nu_1 = -1, \quad \nu_2,\nu_3 = \pm 1\,. \tag{3.13}$$

These half-spaces are related to the Bernstein–Gelfand–Gelfand resolution of affine modules. We will use this relation in Subsect. 5.1.

A sign of the contribution coming from a half-space (3.13) is determined by the product $v_2 v_3$. Therefore we can change $\int \tilde{N}_{\alpha_1\alpha_2\alpha_3}$ in Eq. (3.10) for

$$\sum_{l\in\mathbf{Z}} \sum_{v_{2,3}=\pm 1} v_1 v_2 v_3 \int_{\sum_{i=1}^3 v_i\alpha_i+2Kl>0} . \tag{3.14}$$

The stationary points of the phase in Eq. (3.10) are

$$\alpha_i^{(\mathrm{st})} = 2K\frac{\tilde{n}_i}{p_i}, \quad \text{here } \tilde{n}_i = n_i + q_i\lambda . \tag{3.15}$$

There are also conditional stationary points on the boundary planes

$$\sum_{i=1}^3 v_i\alpha_i + 2Kl = 0 . \tag{3.16}$$

They are

$$\alpha_i^{(\mathrm{cst})} = \frac{2K}{p_i} v_i(n_i - q_i c_0) , \tag{3.17}$$

here

$$c_0 = \frac{H}{P}\sum_{j=1}^3 \left(\frac{n_j}{p_j} + l\right), \quad P = p_1 p_2 p_3,$$

$$H = P\sum_{j=1}^3 \frac{q_i}{p_i} = p_1 p_2 q_3 + p_1 q_2 p_3 + q_1 p_2 p_3 . \tag{3.18}$$

$|H|$ is the order of homology group of the Seifert manifold $X(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$. The points (3.17) form a 2-dimensional lattice on the plane (3.16). Note that $\alpha_i^{(\mathrm{cst})}$ are not changed under a simultaneous shift

$$n_i \to n_i + q_i m, \quad m \in \mathbf{Z} . \tag{3.19}$$

Consider now an integral from Eq. (3.10) with a substitution (3.14):

$$\int_{\sum_{i=1}^3 v_i\alpha_i+2Kl>0} \prod_{i=1}^3 \frac{d\alpha_i}{\sqrt{2K|q_i|}} \exp\frac{i\pi}{2Kq_i}\left[p_i\alpha_i^2 - 2\alpha_i(2K(n_i + q_i\lambda) + \mu_i)\right.$$
$$\left. + s_i(2Kn_i + \mu_i)^2\right] . \tag{3.20}$$

We dropped a regularization factor $\exp(-\pi\varepsilon\sum_{i=1}^3 \alpha_i^2)$, while keeping in mind that it will suppress a contribution of the stationary points (3.15) and (3.17) by its value at those points. If a point (3.15) does not belong to the half-space of (3.20), then the integral is equal to a contribution of the conditional point (3.17). If, however, a point (3.15) is within the half-space, then we use an obvious relation

$$\int_{\sum_{i=1}^3 v_i\alpha_i+2Kl>0} = \int_{\mathbf{R}^3} - \int_{\sum_{i=1}^3 v_i\alpha_i+2Kl<0} \tag{3.21}$$

The first integral in the r.h.s. of this equation is purely gaussian, it is determined by the point (3.15). The second integral is again determined by a conditional point

(3.17). We will calculate both integrals in the next subsection. Here we just note that as it follows from Eq. (3.21), a contribution of a point (3.15) to the integral (3.20) is either zero or a quantity which does not depend on the half-space to which it belongs. Therefore if a point (3.15) belongs to an array of tetrahedra whose slice is depicted in Fig. 3, then its contribution to the whole expression (3.10) is equal to the first integral in the r.h.s. of Eq. (3.21). If the point does not belong to the array, then its contribution is zero.

The overall picture is this: we have two lattices (3.15) and (3.17). Z_2 is equal to the sum of the contributions of the points of these lattices. The lattices and the contributions of their points exhibit the same symmetry under the affine Weyl group transformations, as the summand of Eq. (3.5). Therefore by using the inverse Eq. (2.30) in exactly the same way as we did in deriving Eq. (2.36), we drop the factor $(2K\varepsilon^{1/2})^3/8$ from the integral in Eq. (3.10). At the same time we restrict the sum over n_i to those stationary points (3.15) which belong to the fundamental tetrahedron (3.2) and to those conditional stationary points (3.17) which lie on its faces.

In other words,

$$Z(X,k) = \sum_{\lambda=0,\frac{1}{2}} \sum_{(n_1,n_2,n_3)\in\mathrm{St}} Z_{\mathrm{st}}^{(n_1,n_2,n_3;\lambda)} + \sum_{(n_1,n_2,n_3)\in\mathrm{Cst}} Z_{\mathrm{cst}}^{(n_1,n_2,n_3)},$$

$$Z_{\mathrm{st}}^{(n_1,n_2,n_3;\lambda)} = e^{i\phi_{\mathrm{fr}}} Z_1 Z_{2,\mathrm{st}}^{(n_1,n_2,n_3;\lambda)}, \qquad Z_{\mathrm{cst}}^{(n_1,n_2,n_3)} = e^{i\phi_{\mathrm{fr}}} Z_1 Z_{2,\mathrm{cst}}^{(n_1,n_2,n_3)}. \tag{3.22}$$

Here St is the set of all triplets of integer numbers (n_1, n_2, n_3) such that the points $\alpha_i^{(\mathrm{st})}$ of Eq. (3.15) belong to the fundamental tetrahedron. $Z_{2,\mathrm{st}}^{(n_1,n_2,n_3;\lambda)}$ is equal to the gaussian integral

$$Z_{2,\mathrm{st}}^{(n_1,n_2,n_3;\lambda)} = \frac{1}{2} e^{2\pi i\lambda} \int \prod_{i=1}^{3} \left(\frac{d\alpha_i}{\sqrt{2K|q_i|}} \sum_{\mu_i=\pm 1} \mu_i \exp \frac{i\pi}{2Kq_i} \right.$$
$$\left. \times \left[p_i\alpha_i^2 - 2\alpha_i(2K(n_i + q_i\lambda) + \mu_i) + s_i(2Kn_i + \mu_i)^2 \right] \right). \tag{3.23}$$

The set Cst consists of all the triplets of integer numbers (n_1, n_2, n_3) such that the points $\alpha_i^{(\mathrm{cst})}$ of Eq. (3.17) belong to the faces of the fundamental tetrahedron. $Z_{2,\mathrm{cst}}^{(n_1,n_2,n_3)}$ is the contribution of the stationary point $\alpha_i^{(\mathrm{cst})}$ to the sum of integrals

$$Z_{2,\mathrm{cst}}^{(n_1,n_2,n_3)} = -v_1 v_2 v_3 \frac{1}{2} \sum_{\lambda=0,\frac{1}{2}} e^{2\pi i\lambda} \sum_{m\in\mathbf{Z}} \int_{\substack{[\alpha_i^{(\mathrm{cst})}] \\ \sum_{i=1}^{3} v_i\alpha_i + 2Kl > 0}}$$
$$\times \prod_{i=1}^{3} \left(\frac{d\alpha_i}{\sqrt{2K|q_i|}} \sum_{\mu_i=\pm 1} \mu_i \exp \frac{i\pi}{2Kq_i} \right.$$
$$\left. \times \left[p_i\alpha_i^2 - 2\alpha_i(2K(n_i + q_i\lambda + q_i m) + \mu_i) + s_i(2Kn_i + \mu_i)^2 \right] \right), \tag{3.24}$$

here $\int_{[\alpha_i^{(\mathrm{cst})}]}$ denotes a contribution of the conditional stationary phase point $\alpha_i = \alpha_i^{(\mathrm{cst})}$ to the integral. We take the sum over m because of the invariance of $\alpha_i^{(\mathrm{cst})}$ under the transformation (3.19).

The stationary points that belong to the intersection of the planes (3.16) require special care. Their contribution to the integral over the region carved by $\tilde{N}_{\alpha_1\alpha_2\alpha_3}$ is proportional to the number of planes to which they belong. However the reduction to the fundamental tetrahedron should also account for the fact that these points are invariant under the action of a subgroup W' of the affine Weyl group. Therefore the total contribution of such points is equal to their contribution to the integral (3.10) times a factor

$$\frac{\#\text{ of planes}}{\#\text{ of elements in } W'} \,. \tag{3.25}$$

This factor is similar to the factor 1/2 in Eq. (2.36). It is equal to $\frac{2}{2} = 1$ for the points on the edges of the tetrahedron and to $\frac{4}{8} = \frac{1}{2}$ for the points on the vertices of the tetrahedron.

We can "unfold" the surface of the tetrahedron and require the points (3.17) to belong to the intersection of the plane $\alpha_1 + \alpha_2 + \alpha_3 = 0$ with the cube $-2K < \alpha_1 < 0$, $-2K < \alpha_2 < 0$, $0 < \alpha_3 < 2K$. This intersection is an equilateral triangle (see Fig. 4) consisting of 4 smaller triangles that can be mapped by Weyl reflections onto the faces of the fundamental tetrahedron.

There is yet another way to view the fundamental set of conditional stationary phase points. As we have noted, the triplets n_i related by transformation (3.19) define the same point through Eqs. (3.17). A transformation

$$n_i \to n_i + m p_i, \qquad n_j \to n_j - m p_j, \quad i \neq j \tag{3.26}$$

does not change c_0 and shifts $\alpha_i^{(\mathrm{cst})}$ by $2Km$ and $\alpha_j^{(\mathrm{cst})}$ by $-2Km$, thus leaving them within the same equivalence class of affine Weyl transformations. We can describe a fundamental region of the conditional stationary phase points as a factor of a lattice of all integer triplets n_i over a lattice generated by three vectors

$$\vec{v}_1 = (q_1, q_2, q_3), \qquad \vec{v}_2 = (p_1, -p_2, 0), \qquad \vec{v}_3 = (0, p_2, -p_3)\,. \tag{3.27}$$

The number of triplets n_i inside that factor is equal to the volume of a parallelopiped formed by the vectors $\vec{v}_i$

$$\#\text{ of conditional points} = \begin{vmatrix} q_1 & q_2 & q_3 \\ p_1 & -p_2 & 0 \\ 0 & p_2 & -p_3 \end{vmatrix} = |H|\,. \tag{3.28}$$

We should be interested only in approximately half of the triplets, because the volume of the prism built upon a triangle of Fig. 4 is twice as small as that of the parallelopiped which is built upon the whole parallelogram. Thus the number of the conditional stationary phase points within the fundamental domain is approximately equal to half the rank of the homology group. This result is not surprising since we intend to identify the conditional stationary phase points with reducible $SU(2)$ flat connections. The number of these connections is also approximately equal to $|H|/2$.

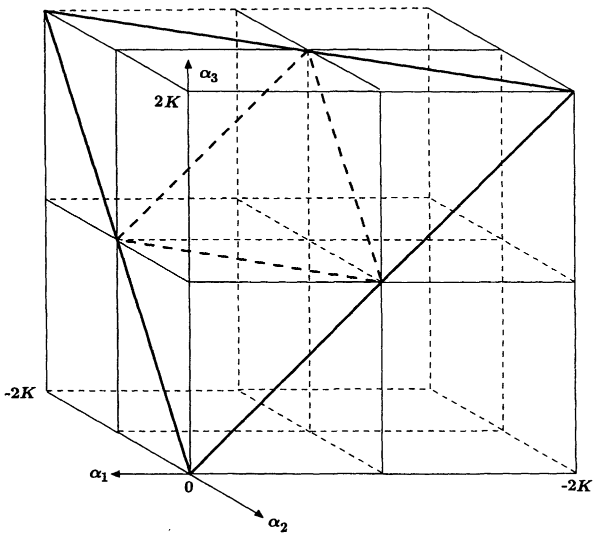

Fig. 4. A fundamental triangle for reducible connections.

3.2. The Integrals

Stationary Phase Points. We start with the simplest case of a contribution of the point (3.15) which is inside the fundamental tetrahedron (3.2) to the integral (3.23). As we saw in the previous subsection, it is equal to the gaussian integral taken over the whole α-space:

$$\begin{aligned} Z_{2,\mathrm{st}}^{(n_1,n_2,n_3;\lambda)} &= \frac{1}{2} e^{2\pi i\lambda} \prod_{i=1}^{3} \sum_{\mu_i=\pm 1} \mu_i \int_{-\infty}^{+\infty} \frac{d\alpha_i}{\sqrt{2K|q_i|}} \\ &\times \exp \frac{i\pi}{2Kq_i} \left[p_i\alpha_i^2 - 2\alpha_i(2K\tilde{n}_i + \mu_i) + s_i(2Kn_i + \mu_i)^2 \right] \\ &= Z_3 \frac{1}{2} e^{2\pi i\lambda} \prod_{i=1}^{3} \frac{1}{\sqrt{|p_i|}} 2i \sin 2\pi \left(\frac{r_i}{p_i}\tilde{n}_i + s_i\lambda \right) \exp 2\pi i K \left[\frac{r_i}{p_i}\tilde{n}_i^2 - q_i s_i \lambda^2 \right] . \end{aligned} \tag{3.29}$$

Here Z_3 is a factor that will be present in all the subsequent expressions for the contributions to Z_2:

$$Z_3 = \prod_{i=1}^{3} e^{\frac{i\pi}{4}\operatorname{sign}(p_i q_i)} \exp \left(\frac{i\pi}{2K} \frac{r_i}{p_i} \right) . \tag{3.30}$$

Conditional Stationary Phase Points. Consider now a contribution of the points

$$\alpha_i^{(\mathrm{cst})} = \frac{2K}{p_i}(n_i - q_i c_0), \quad c_0 = \frac{P}{H} \sum_{i=1}^{3} \frac{n_i}{p_i} \tag{3.31}$$

which belong to the plane

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 . \tag{3.32}$$

The integral (3.24) can be extended to the whole α-space if we add an extra factor

$$\theta(-\alpha_1-\alpha_2-\alpha_3)=\int_0^{+\infty} dx \int_{-\infty}^{+\infty} dc \exp\,[2\pi i c(\alpha_1+\alpha_2+\alpha_3+x)] \tag{3.33}$$

to its integrand. As a result, the full contribution of the point $\alpha_i^{(\mathrm{cst})}$ to $Z_{2,\mathrm{cst}}^{(n_1,n_2,n_3)}$ is

$$\begin{aligned}
Z_{2,\mathrm{cst}}^{(n_1,n_2,n_3)} &= \frac{1}{2}\sum_{\lambda=0,\frac{1}{2}} e^{2\pi i\lambda}\sum_{m\in\mathbf{Z}}\int_{\substack{0\\ [0]}}^{+\infty} dx \int_{-\infty}^{+\infty} dc\; e^{2\pi i c x}\prod_{i=1}^{3}\sum_{\mu_i=\pm1}\mu_i\int_{-\infty}^{+\infty}\frac{d\alpha_i}{\sqrt{2K|q|}}\\
&\quad\times \exp\frac{i\pi}{2Kq_i}\,\big[p_i\alpha_i^2-2\alpha_i(2K(n_i+q_i(\lambda+m-c))+\mu_i)\\
&\qquad\qquad + s_i(2K(n_i+q_im)+\mu_i)^2\big]\\
&= Z_3\frac{e^{-\frac{i\pi}{4}\operatorname{sign}(H/P)}}{\sqrt{2K|H|}}\;\exp\,2\pi i K\left[\sum_{i=1}^{3}\frac{r_i}{p_i}n_i^2+\frac{H}{P}c_0^2\right]\\
&\quad\times\sum_{\mu_{1,2,3}=\pm1}\left\{\prod_{i=1}^{3}\mu_i\exp\left[2\pi i\mu_i\frac{r_in_i+c_0}{p_i}\right]\right\}\frac{1}{2}\sum_{\lambda=0,\frac{1}{2}}e^{2\pi i\lambda}\sum_{m\in\mathbf{Z}}I(m)\,,
\end{aligned} \tag{3.34}$$

here

$$I(m)=\int_{\substack{0\\ [0]}}^{+\infty} dx\,\exp\left[2\pi i x(c_0+m+\lambda)+\frac{i\pi}{2K}\frac{P}{H}\left(x+\sum_{i=1}^{3}\frac{\mu_i}{p_i}\right)^2\right]. \tag{3.35}$$

We calculate the integral (3.35) in the spirit of remarks preceding Eq. (3.21). If $\frac{P}{H}(c_0+m+\lambda)<0$, then the stationary point of the phase in Eq. (3.35) lies outside the integration region, and the dominant contribution comes from the boundary point $x=0$:

$$\begin{aligned}
I(m)&=\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{i\pi}{2K}\frac{P}{H}\right)^j\int_0^{\infty}dx\left(x+\sum_{i=1}^{3}\frac{\mu_i}{p_i}\right)^{2j}\exp\,[2\pi i x(c_0+m+\lambda)]\\
&=\sum_{j=0}^{\infty}\frac{1}{j!}(8\pi i K)^{-j}\left(\frac{P}{H}\right)^j\partial_\varepsilon^{(2j)}\\
&\quad\times\left[\int_0^{\infty}dx\,\exp\left(2\pi i x(c_0+m+\lambda+\varepsilon)\;+2\pi i\varepsilon\sum_{i=1}^{3}\frac{\mu_i}{p_i}\right)\right]\Bigg|_{\varepsilon=0}.
\end{aligned} \tag{3.36}$$

If however $\frac{P}{H}(c_0+m+\lambda)>0$, then we use a relation similar to Eq. (3.21):

$$\int_0^{\infty}dx=\int_{-\infty}^{+\infty}dx-\int_{-\infty}^{0}dx\,. \tag{3.37}$$

The integral $\int_{-\infty}^{+\infty}dx$ is fully determined by the stationary phase point and therefore has been accounted for in Eq. (3.29). The integral $-\int_{-\infty}^{0}dx$ is dominated by the boundary point $x=0$ and leads to the same expression (3.36). Thus Eq. (3.36) is valid if $c_0+m+\lambda\neq0$.

A Poisson formula (2.28) allows us to convert a sum over m back into a "discretization" of the integral over x:

$$
\begin{aligned}
&\frac{1}{2}\sum_{\lambda=0,\frac{1}{2}} e^{2\pi i\lambda}\sum_{m\in\mathbf{Z}} I(m)\\
&= \sum_{j=0}^{\infty}\frac{1}{j!}(8\pi iK)^{-j}\left(\frac{P}{H}\right)^{j}\partial_{\varepsilon}^{(2j)}\left[\frac{1}{2}\sum_{\lambda=0,\frac{1}{2}} e^{2\pi i(\lambda+\varepsilon\sum_{i=1}^{3}\frac{\mu_i}{p_i})}\sum_{x=0}^{\infty} e^{2\pi ix(c_0+\lambda+\varepsilon)}\right]\Bigg|_{\varepsilon=0}\\
&= -\sum_{j=0}^{\infty}\frac{1}{j!}(8\pi iK)^{-j}\left(\frac{P}{H}\right)^{j}\partial_{\varepsilon}^{(2j)}\left[\frac{e^{2\pi i\varepsilon\sum_{i=1}^{3}\frac{\mu_i}{p_i}}}{2i\sin 2\pi(c_0+\varepsilon)}\right]\Bigg|_{\varepsilon=0}.
\end{aligned}
\tag{3.38}
$$

By substituting this expression into Eq. (3.34) and taking a sum over μ_i we get the final expression for the contribution of the stationary point (3.31) to Z_2:

$$
\begin{aligned}
Z_{2,\mathrm{cst}}^{(n_1,n_2,n_3)} &= -Z_3\frac{e^{-i\frac{\pi}{4}\operatorname{sign}(\frac{H}{P})}}{\sqrt{2K|H|}}\exp 2\pi iK\left[\sum_{i=1}^{3}\frac{r_i}{p_i}n_i^2+\frac{H}{P}c_0^2\right]\\
&\times\sum_{j=0}^{\infty}\frac{1}{j!}(8\pi iK)^{-j}\left(\frac{P}{H}\right)^{j}\left[\partial_c^{(2j)}\frac{\prod_{i=1}^{3}2i\sin\left(2\pi\frac{r_in_i+c}{p_i}\right)}{2i\sin 2\pi c}\right]\Bigg|_{c=c_0}.
\end{aligned}
\tag{3.39}
$$

A Stationary Phase Point on the Boundary. A stationary phase point (3.15) presents a special case when it belongs to the boundary of one of the planes (3.13). Suppose, that $\alpha_i^{(\mathrm{cst})}$ satisfy conditions (3.32). Comparing Eqs. (3.15) and (3.31), we see that this may happen if

$$
c_0+\lambda=0\,. \tag{3.40}
$$

or, in other words, $c_0=0,-\frac{1}{2}$.

We can proceed with the same analysis as for an ordinary conditional stationary phase point up to Eq. (3.36). The integral $I(0)$ requires a separate calculation. According to Eq. (3.35),

$$
\begin{aligned}
I(0) &= \int_0^{\infty} dx\,\exp\left[\frac{i\pi}{2K}\frac{P}{H}\left(x+\sum_{i=1}^{3}\frac{\mu_i}{p_i}\right)^2\right]\\
&= \sqrt{\frac{K}{2}\left|\frac{H}{P}\right|}\,e^{i\frac{\pi}{4}\operatorname{sign}(H/P)} - \int_0^{\sum_{i=1}^{3}\frac{\mu_i}{p_i}} dx\,\exp\left[\frac{i\pi}{2K}\frac{P}{H}x^2\right]\\
&= \sqrt{\frac{K}{2}\left|\frac{H}{P}\right|}\,e^{i\frac{\pi}{4}\operatorname{sign}(H/P)} - \sum_{j=0}^{\infty}\frac{1}{j!}(8\pi iK)^{-j}\left(\frac{P}{H}\right)^{j}\left[\partial_{\varepsilon}^{(2j)}\frac{e^{2\pi i\sum_{i=1}^{3}\frac{\mu_i}{p_i}}-1}{2\pi i\varepsilon}\right]\Bigg|_{\varepsilon=0}.
\end{aligned}
\tag{3.41}
$$

The remaining part of Eq. (3.38) is equal to

$$
-\sum_{j=0}^{\infty}\frac{1}{j!}(8\pi iK)^{-j}\left(\frac{P}{H}\right)^{j}e^{2\pi ic_0}\partial_{\varepsilon}^{(2j)}\left[e^{2\pi i\varepsilon\sum_{i=1}^{3}\frac{\mu_i}{p_i}}\left(\frac{1}{2i\sin 2\pi\varepsilon}-\frac{1}{4\pi i\varepsilon}\right)\right]\Bigg|_{\varepsilon=0}. \tag{3.42}
$$

Adding $I(0)$ with an extra factor $\frac{1}{2}e^{2\pi i c_0}$ to this expression and substituting it into Eq. (3.34) brings us to the formula

$$Z_{2,\mathrm{cst}}^{(n_1,n_2,n_3)} = Z_3 \frac{e^{-i\frac{\pi}{4}\mathrm{sign}(\frac{H}{P})}}{\sqrt{2K|H|}} \exp 2\pi i K \left[\sum_{i=1}^{3} \frac{r_i}{p_i} n_i^2 + \frac{H}{P} c_0^2\right] e^{2\pi i c_0}$$

$$\times \left\{ e^{i\frac{\pi}{4}\mathrm{sign}(\frac{H}{P})} \sqrt{\frac{K}{8}\left|\frac{H}{P}\right|} \prod_{i=1}^{3} 2i \sin\left(2\pi \frac{r_i n_i + c_0}{p_i}\right) - \sum_{j=0}^{\infty} \frac{1}{j!} (8\pi i K)^{-j} \left(\frac{P}{H}\right)^j \partial_\varepsilon^{(2j)} \right.$$

$$\left. \times \left[\frac{\prod_{i=1}^{3} 2i \sin\left(2\pi \frac{r_i n_i + c_0 + \varepsilon}{p_i}\right)}{2i \sin(2\pi\varepsilon)} - \frac{\prod_{i=1}^{3} 2i \sin\left(2\pi \frac{r_i n_i + c_0}{p_i}\right)}{4\pi i \varepsilon} \right] \Bigg|_{\varepsilon=0} \right\}. \tag{3.43}$$

Trivial Connection. In the case of $n_1 = n_2 = n_3 = c_0 = 0$ the formula (3.43) requires an extra 1/2 factor coming from the ratio (3.25). After some simplifications it becomes

$$Z_{2,\mathrm{cst}}^{(\mathrm{triv})} = -Z_3 \frac{e^{-i\frac{\pi}{4}\mathrm{sign}(\frac{H}{P})}}{\sqrt{8K|H|}} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\pi}{2iK} \frac{P}{H}\right)^j \left[\partial_\varepsilon^{(2j)} \frac{\prod_{i=1}^{3} 2i \sin\left(\frac{\varepsilon}{p_i}\right)}{2i \sin \varepsilon} \right] \Bigg|_{\varepsilon=0}. \tag{3.44}$$

3.3. Framing Corrections. Finally we have to simplify the product of the phase factors $e^{i\phi_{\mathrm{fr}}} Z_1 Z_3$ which appeared at different stages of our calculations in the contributions of all stationary points. We start with Eq. (2.22). According to [9],

$$\sum_{j=1}^{t_i} a_j^{(i)} - 3 \sum_{j=1}^{t_i - 1} \mathrm{sign}\left(a_j^{(i)}\right) = \Phi(M^{(p_i,q_i)}). \tag{3.45}$$

Also note that

$$\Phi(M^{(p_i,q_i)}) - 3\mathrm{sign}(p_i q_i) = \Phi(SM^{(p_i,q_i)}). \tag{3.46}$$

The central charge of the $SU(2)$ WZW model is

$$c = \frac{3(K-2)}{K}. \tag{3.47}$$

Therefore the full framing correction is

$$e^{i\phi_{\mathrm{fr}}} = \exp \frac{i\pi}{4} \left[\sum_{i=1}^{3} (\Phi(M^{(p_i,q_i)}) - 3\mathrm{sign}(p_i q_i)) + 3\mathrm{sign}\left(\frac{H}{P}\right) \right]$$

$$\times \exp - \frac{i\pi}{2K} \left[\sum_{i=1}^{3} \Phi(SM^{(p_i,q_i)}) + 3\mathrm{sign}\left(\frac{H}{P}\right) \right], \tag{3.48}$$

and the product of all three phase factors is

$$e^{i\phi_{\mathrm{fr}}} Z_1 Z_3 = e^{i\frac{3\pi}{4}\mathrm{sign}\left(\frac{H}{P}\right)} \prod_{i=1}^{3} \mathrm{sign}\,(p_i) \exp$$

$$- \frac{i\pi}{2K}\left[3\mathrm{sign}\left(\frac{H}{P}\right) + \sum_{i=1}^{3}\left(12s(q_i, p_i) - \frac{q_i}{p_i}\right)\right]. \qquad (3.49)$$

We used the following property of the Dedekind sum in order to derive this formula:

$$s(m^*, n) = s(m, n), \quad \text{if } mm^* = 1(\mathrm{mod}\ n). \qquad (3.50)$$

Equation (3.49) together with Eqs. (3.30), (3.39), (3.43) and (3.44) leads to the final formulas (1.3)–(1.8).

4. One-Loop Approximation Formulas

4.1. Irreducible Flat Connections. Irreducible flat connections on a Seifert manifold $X(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n})$ are the ones for which the subgroup H commuting with the image of the homomorphism (2.2) does not have continuous parameters. In the case of $G = SU(2)$ this simply means that the image of (2.2) is noncommutative.

The fundamental group π_1 of the Seifert manifold is known to be generated by the elements $x_1, \ldots, x_n, h$ satisfying relations

$$x_i^{p_i} h^{q_i} = 1, \qquad hx_i = x_i h, \qquad \prod_{i=1}^{n} x_i = 1. \qquad (4.1)$$

The elements x_i go around the solid tori that make up the manifold, while h goes along the S^1 cycle of the "mother-manifold" $S^1 \times S^2$ (see [15 and 8] for details).

Suppose that the image of h does not belong to the center of $SU(2)$. Since h commutes with all the elements of π_1, then the whole image of π_1 belongs to $U(1) \subset SU(2)$. Therefore $H \supset U(1)$, so this is a reducible case. An irreducible connection is produced only if the image of h belongs to the center of $SU(2)$:

$$h \overset{A}{\mapsto} \begin{pmatrix} e^{2\pi i\lambda} & 0 \\ 0 & e^{-2\pi i\lambda} \end{pmatrix}, \qquad \lambda = 0, \frac{1}{2}. \qquad (4.2)$$

The images of the elements x_i belong to the conjugation classes of diagonal matrices whose phases we denote as u_i:

$$x_i \overset{A}{\mapsto} g_i^{-1} \begin{pmatrix} e^{2\pi i u_i} & 0 \\ 0 & e^{-2\pi i u_i} \end{pmatrix} g_i. \qquad (4.3)$$

The first of relations (4.1) determines the possible values of these phases:

$$u_i = \frac{\tilde{n}_i}{p_i}, \qquad (4.4)$$

here the numbers $\tilde{n}_i$ are defined in Eq. (3.15). The phases u_i determine the map (2.2) uniquely up to an overall conjugation if the number of the solid tori is $n = 3$. If $n > 3$, then each particular choice of the phases u_i corresponds to a connected component of the $2(n-3)$ dimensional moduli space of these maps, which is a moduli space of flat connections on a Seifert manifold. Such a connected component is isomorphic to the space of flat connections on an n-holed sphere if the holonomies around the holes are fixed by Eq. (4.3). We will discuss this subject further in Subsect. 5.2. Here we specialize to the case $n = 3$ and present the expressions for the manifold invariants entering Eq. (2.14). A Chern–Simons invariant of an irreducible flat connection on a Seifert manifold was computed in [17]:

$$S_{CS} = \sum_{i=1}^{3} \frac{1}{p_i}(r_i n_i^2 - 2\lambda n_i - q_i \lambda^2) = \sum_{i=1}^{3} \left(\frac{r_i}{p_i} \tilde{n}_i^2 - q_i s_i \lambda^2 \right) \pmod 1 . \tag{4.5}$$

According to [16], a square root of the corresponding Reidemeister torsion is

$$\tau_X^{1/2} = \prod_{i=1}^{3} \frac{2}{\sqrt{p_i}} |\sin(2\pi\phi_i)| , \tag{4.6}$$

here

$$\phi_i = \frac{r_i n_i - \lambda}{p_i} = \frac{r_i}{p_i}\tilde{n}_i + s_i \lambda \pmod 1 \tag{4.7}$$

are the phases of the conjugation classes of the holonomies along the central fibers of the solid tori that make up the Seifert manifold.

The spectral flow was calculated in [15]:

$$I_A = -3 + 8S_{CS} + \sum_{i=1}^{3} \frac{2}{p_i} \sum_{l=1}^{p_i - 1} \cot\left(\frac{\pi r_i l}{p_i}\right) \cot\left(\frac{\pi l}{p_i}\right) \sin^2\left[\frac{2\pi l}{p_i}(r_i n_i - \lambda)\right] . \tag{4.8}$$

L. Jeffrey presented in her paper [9] a proof by D. Zagier of an equation

$$(-i)\mathrm{sign}\left(\sin\frac{2\pi r n}{p}\sin\frac{2\pi n}{p}\right) = \exp\frac{i\pi}{2}\left[\frac{8rn^2}{p} - \frac{2}{p}\sum_{l=1}^{p-1}\cot\frac{\pi l}{p}\cot\frac{\pi q l}{p}\sin^2\frac{2\pi n l}{p}\right] , \tag{4.9}$$

here $n, p, q, r \in \mathbf{Z}$, $qr = -1 \pmod p$. A slight modification of that proof shows that Eq. (4.9) works also for a half-integer n if we multiply its l.h.s. by an extra factor of $e^{2\pi i n}$. Then an application of Eq. (4.9) with a substitution

$$q = r_i, \qquad r = q_i, \qquad n = r_i n_i - \lambda \tag{4.10}$$

to the r.h.s. of Eq. (4.8) leads to a formula for the exponential of the spectral flow:

$$\begin{aligned} \exp\left(-\frac{i\pi}{2} I_A\right) &= -\prod_{i=1}^{3} e^{2\pi i \lambda} \mathrm{sign}\left(\sin\frac{2\pi\tilde{n}_i}{p_i}\sin 2\pi\phi_i\right) \\ &= -e^{2\pi i\lambda}\prod_{i=1}^{3}\mathrm{sign}\,(\sin\, 2\pi\phi_i); \end{aligned} \tag{4.11}$$

here we used the fact that $0 < \frac{\tilde{n}_i}{p_i} < \frac{1}{2}$ because $0 < \alpha_i^{(\mathrm{st})} < K$. Apparently the role of the factor $(-i)^{I_A}$ is to remove the absolute value from the sines in the square root of the Reidemeister torsion (4.6). A similar effect was observed for lens spaces in [9].

For an irreducible connection on a 3-fibered Seifert manifold $X(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$, $\dim H^0 = \dim H^1 = b^1 = 0$ and $\dim SU(2) = 3$ while H is a center of $SU(2)$

which consists of 2 elements, $\mathrm{Vol}(H) = 2$. Therefore according to Eq. (2.14), the 1-loop contribution of an irreducible flat connection should be equal to

$$-\frac{1}{2}e^{-i\frac{3\pi}{4}}e^{2\pi i\lambda}\prod_{i=1}^{3}\frac{2}{\sqrt{|p_i|}}\sin 2\pi\left(\frac{r_i}{p_i}\tilde{n}_i + s_i\lambda\right)\exp 2\pi iK\left(\frac{r_i}{p_i}\tilde{n}_i^2 - q_i s_i\lambda^2\right). \quad (4.12)$$

If we compare this expression to Eq. (1.3), then we see that[5] the exact contribution differs from Eq. (4.12) only by a phase factor

$$\exp -\frac{i\pi}{2K}\left[3\mathrm{sign}\left(\frac{H}{P}\right) + \sum_{i=1}^{3}\left(12s(q_i, p_i) - \frac{q_i}{p_i}\right)\right]. \quad (4.13)$$

It comes from the overall phase factor $Z_1Z_3Z_f$ and can be interpreted as a 2-loop correction according to Eq. (2.4). Note that this factor is the same for all the stationary phase contributions. It does not "feel" the background gauge field and seems to be of "gravitational" origin. A similar 2-loop phase factor has been found in [10 and 9] for the lens spaces $L(p,q)$ to be equal to exp $[\frac{i\pi}{2K}12s(q,p)]$. S. Garoufalidis noted in [10], that this phase is proportional to Casson's invariant extended by K. Walker to rational homology spheres. According to [18], this invariant is equal to $s(q,p)$ for the lens space $L(p,q)$. C. Lescop computed the Casson–Walker invariant for n-fibered Seifert manifolds in [19]:

$$\lambda_{CW}\left(X\left(\frac{p_1}{q_1},\ldots,\frac{p_n}{q_n}\right)\right) = \frac{1}{12}\frac{P}{H}\left(2 - n + \sum_{i=1}^{n}p_i^{-2}\right)$$
$$-\frac{1}{12}\left[3\mathrm{sign}\left(\frac{H}{P}\right) + \sum_{i=1}^{n}\left(12s(q_i, p_i) - \frac{q_i}{p_i}\right)\right]. \quad (4.14)$$

We see that the phase of (4.13) is indeed proportional to the second term in the r.h.s. of Eq. (4.14), however the first term (which is dominating in the limit of large p_i) is missing. We will see that the missing part appears in the total 2-loop correction to the trivial connection, which includes some terms of the asymptotic series together with the phase (4.13).

4.2. General Reducible Flat Connections. As we noted in the previous subsection, the image of π_1 under the homomorphism (2.2) belongs to the $U(1)$ subgroup of $SU(2)$ for the reducible flat connections. This generally happens when the image of h does not belong to the center of $SU(2)$:

$$h \overset{A}{\mapsto} \begin{pmatrix} e^{2\pi i c_0} & 0 \\ 0 & e^{-2\pi i c_0} \end{pmatrix}, \qquad c_0 \neq 0, \frac{1}{2}. \quad (4.15)$$

Since h commutes with all x_i, their images should also be diagonal. The first of Eqs. (4.1) again determines the phases:

$$x_i \overset{A}{\mapsto} \begin{pmatrix} e^{2\pi i u_i} & 0 \\ 0 & e^{-2\pi i u_i} \end{pmatrix}, \qquad u_i = \frac{n_i - q_i c_0}{p_i}, \quad (4.16)$$

[5] In assumption of $p_1, p_2, p_3, H > 0$.

while c_0 is determined by the second equation in (3.31). The phases of the holonomies going along the fibers of the solid tori are

$$\phi_i = \frac{r_i n_i + c_0}{p_i}\,. \tag{4.17}$$

A Chern–Simons action and a square root of the Reidemeister torsion are known to be[6]

$$S_{CS} = \sum_{i=1}^{3} \frac{r_i}{p_i} n_i^2 + \frac{H}{P} c_0^2, \qquad \tau_X^{1/2} = |H|^{-1/2} \left| \frac{\prod_{i=1}^{3} 2\sin(2\pi\phi_i)}{2\sin(2\pi c_0)} \right| . \tag{4.18}$$

Reducibility of connection means that this time $\dim H^0 = 1$, $H = U(1)$, $\mathrm{Vol}(H) = 1/\sqrt{2}$.

All these formulas are compatible with the leading term in the $1/k$ expansion of the conditional stationary phase contribution (1.6) at least up to a phase factor. Indeed, we see that the 1-loop part of Eq. (1.6) (assuming that $p_1, p_2, p_3, H > 0$) is equal to

$$\frac{i}{\sqrt{2KH}} \cdot \frac{\prod_{i=1}^{3} 2\sin(2\pi\phi_i)}{2\sin(2\pi c_0)} \exp 2\pi i K \left[\sum_{i=1}^{3} \frac{r_i}{p_i} n_i^2 + \frac{H}{P} c_0 \right] . \tag{4.19}$$

4.3. Special Reducible Connections. The special reducible flat connections are those for which one or more sines in Eq. (4.18) are equal to zero. This amounts to a condition that a stationary phase point $\alpha^{(\mathrm{st})}$ defined by Eq. (3.15) belongs to a face, an edge or a vertex of the fundamental tetrahedron (3.2).

Point on a Face. Suppose that a condition (3.40) is satisfied for some value of λ. This means that the element $h \in \pi_1$ is mapped, according to Eq. (4.15), to $e^{2\pi i\lambda} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that the conditional stationary phase point on a face of the tetrahedron is, in fact, unconditional. The approximation (4.19) breaks down, the reason being that Eq. (1.7) should be used instead of Eq. (1.6). Then the leading contribution to a partition function is equal to one-half of Eq. (4.12):

$$\frac{1}{4} e^{-i\frac{3\pi}{4}} e^{2\pi i\lambda} \prod_{i=1}^{3} \frac{2}{\sqrt{|p_i|}} \sin 2\pi \left(\frac{r_i}{p_i} \tilde{n}_i + s_i \lambda \right) \exp 2\pi i K \left(\frac{r_i}{p_i} \tilde{n}_i^2 - q_i s_i \lambda^2 \right) . \tag{4.20}$$

Let us reconcile this expression with Eq. (2.14). The fact that a denominator of Eq. (4.18) is zero for $c_0 = 0, \frac{1}{2}$ indicates a presence of 1-form zero modes in the operator L_-. Indeed, the first two conditions (4.1) fix the images of x_i in $SU(2)$ only up to arbitrary conjugations because the image of h again belongs to the center of $SU(2)$. The last condition (4.1) says that the points x_1, x_2x_1 and $x_3x_2x_1 = 1$ form a "curved" triangle inside $SU(2)$. The size of the sides of this triangle is fixed by the first condition (4.1), but their orientation is constrained only by a condition that the triangle is closed. Such a triangle is a rigid object and can be rotated into

[6] See e.g. [13], where these quantities were calculated by using a $U(1|1)$ Chern–Simons–Witten theory.

a predetermined position by an overall conjugation. This is why the irreducible connections on a 3-fibered Seifert manifold have no moduli.

The rigidity of the triangle is considerably decreased if all three of its vertices belong to the same big circle (see Fig. 4), i.e. if the images of all x_i belong to the same $U(1)$ subgroup of $SU(2)$, as it happens for a stationary phase point on a face. In this case, say, a middle vertex can be infinitesimally shifted in the plane perpendicular to the line of the triangle, with the sizes of the sides of the triangle changing only to the second order in the shift. Thus the operator L_- has two zero modes, $\dim H^1 = 2$, however there are still no moduli, because an obstruction prevents an extension of those modes to a 1-parameter family of flat connections.

The two zero modes form a 2-dimensional representation of $U(1)$ which is a symmetry group of the reducible flat connections. This means that the modes are gauge equivalent. However a procedure of dropping the zero modes of Δ and L_- from the determinants of Eq. (2.5) amounts to neglecting the global $U(1)$ gauge transformations (see the Appendix). The integration in path integral (1.2) includes the directions along both zero modes. The exponent corresponding to these directions has no quadratic terms, the cubic terms are prohibited by the $U(1)$ symmetry. Therefore the dominating term in the exponent is generally of the fourth order in coordinates along the zero modes. Each direction contributes an integral (2.12) which results in the formula (2.13) for the factor C_1. Since in our case $\dim H^0 = 1$, $\dim H^1 = 2$, $\dim X = 0$ we see that Eq. (2.14) predicts an overall power of K to be equal to zero in agreement with the surgery asymptotics (4.20).

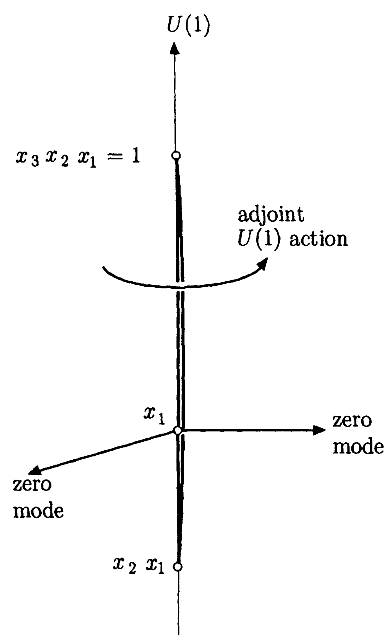

Fig. 5. The zero modes of deformations of a degenerate triangle.

Point on an Edge. Suppose that in addition to the previous conditions, one of the phases ϕ_i is equal to zero or $\frac{1}{2}$. This means that $\alpha_i^{(\mathrm{st})}$ equals either 0 to K, so that a stationary phase point belongs to an edge of the tetrahedron (3.2).

Let, for example, $\phi_1 = \alpha_1^{(\mathrm{st})} = u_1 = 0$. Then the image of x_1 in $SU(2)$ is the identity matrix. The triangle $x_1, x_2x_1, x_3x_2x_1 = 1$ is even more degenerate, because its first side has shrunk. The rigidity of the construction is restored, the zero modes of L_- disappeared, $\dim H^1 = 0$. Since $\dim H^0 = 1$, then according to Eq. (2.14) we expect a contribution to be proportional to $k^{-1/2}$. Indeed, the approximation (4.20) breaks down and the first subleading term in Eq. (1.7) contributes

$$i\frac{e^{2\pi i\lambda}}{\sqrt{2KH}}\frac{1}{p_1}\left(\prod_{i=2}^{3} 2\sin(2\pi\phi_i)\right)\exp 2\pi iK\left(\sum_{i=1}^{3}\frac{r_i}{p_i}n_i^2 + \frac{H}{P}\lambda^2\right). \tag{4.21}$$

This expression is very similar to Eq. (4.19). We easily recognize the same construction blocks assuming that now

$$\tau_R^{1/2} = \frac{1}{\sqrt{H}}\frac{1}{p_1}\left|\prod_{i=2}^{3} 2\sin(2\pi\phi_i)\right|. \tag{4.22}$$

Point on a Vertex. This is the case when the image of π_1 is a subgroup of the center of $SU(2)$, that is, $c_0, \phi_1, \phi_2, \phi_3 = 0, \frac{1}{2}$. Let us take a particular case of a trivial connection, for which $c_0 = \phi_1 = \phi_2 = \phi_3 = 0$. The Chern–Simons invariant is zero, the square root of the Reidemeister torsion is known to be (see, e.g. [8])

$$\tau_X^{1/2} = H^{-\frac{\dim G}{2}} = H^{-3/2}. \tag{4.23}$$

The group H is the whole $SU(2)$, its volume in the proper normalization is $1/(\sqrt{2}\pi)$ (see Eq. (2.38)). We also know that $\dim H^0 = 3$, $\dim H^1 = 0$. As for the phase factor in Eq. (2.14), it is shown in [8] that for the trivial connection

$$\exp -\frac{i\pi}{4}[2I_\alpha + 3(1 + b^1) + \dim H^0 + \dim H^1] = 1. \tag{4.24}$$

Therefore, according to Eq. (2.14), the 1-loop contribution of the trivial connection should be

$$Z = \sqrt{2}\pi(KH)^{-3/2}. \tag{4.25}$$

We get the same expression from the surgery calculus if we take the term with $j = 1$ in Eq. (1.8).

2-Loop Correction. Let us use Eq. (1.8) to calculate the next subleading correction to the formula (4.25)[7]. In other words, we are looking for a 2-loop term S_2 as defined by Eq. (2.4) (note, however, that we are using now $K = k + 2$ instead of k as an expansion parameter). One obvious source of S_2 is the 2-loop phase $-i\frac{\pi}{2}\phi$ (see Eqs. (1.5) or (4.13)). The other source is the $j = 2$ term in

[7] I am indebted to D. Freed for turning my attention to the 2-loop correction. I learned later that a similar calculation was performed by J.E. Anderson.

the asymptotic series of (1.8). Actually, we have to take a logarithm of that series to bring it to the form (2.4). At the 2-loop level of approximation this amounts to dividing the $j = 2$ term by the leading $j = 1$ term. Since

$$\left[\partial_\varepsilon^{(4)} \frac{\prod_{i=1}^{3} 2i\sin\left(\frac{\varepsilon}{p_i}\right)}{2i\sin\varepsilon}\right]\Bigg|_{\varepsilon=0} = \frac{16}{P}\left[\sum_{i=1}^{3} p_i^{-2} - 1\right], \tag{4.26}$$

then the whole 2-loop correction S_2 is

$$\begin{aligned} S_2 &= \frac{\pi}{2}\left[\frac{P}{H}\left(\sum_{i=1}^{3} p_i^{-2} - 1\right) - 3\,\mathrm{sign}\left(\frac{H}{P}\right) - \sum_{i=1}^{3}\left(12s(q_i, p_i) - \frac{q_i}{p_i}\right)\right] \\ &= 6\pi\lambda_{CW}\,, \end{aligned} \tag{4.27}$$

here λ_{CW} is a Casson–Walker invariant (see Eq. (4.14)).

4.4. Identification of Flat Connections. Throughout this section we identified the flat connections of the Seifert manifold with the stationary points $\alpha_i^{(\mathrm{st})}$ and $\alpha_i^{(\mathrm{cst})}$ of the surgery formula (2.21) by comparing the already known Chern–Simons invariants (as well as the Reidemeister torsion and spectral flow) with the leading (in K^{-1} expansion) part of the stationary phase contributions with the help of Eq. (2.14). However there is a more direct method of identifying a stationary point of the surgery formula with a flat connection of a manifold. Consider the Witten's invariant of the 3-fibered Seifert manifold equipped with the Wilson line going around the i^{th} fiber (i.e., equivalent to the element $x_i \in \pi_1(X)$) and carrying the γ-dimensional representation of $SU(2)$. According to [1], the surgery formula for this invariant is

$$\begin{aligned} Z\left(X\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right), k\right) &= e^{i\phi_{\mathrm{fr}}} \sum_{0<\alpha_1,\alpha_2,\alpha_3<K} \sum_{\lambda=0,\frac{1}{2}} \frac{1}{2} e^{2\pi i\lambda(1-\alpha_1-\alpha_2-\alpha_3)} \\ &\times \frac{\sin\left(\frac{\pi\alpha_i\gamma}{K}\right)}{\sin\left(\frac{\pi\alpha_i}{K}\right)} \tilde{M}_{\alpha_1 1}^{(p_1,q_1)} \tilde{M}_{\alpha_2 1}^{(p_2,q_2)} \tilde{M}_{\alpha_3 1}^{(p_3,q_3)} \tilde{N}_{\alpha_1,\alpha_2,\alpha_3}\,. \end{aligned} \tag{4.28}$$

As a result, a contribution of a particular special point α_i^* (i.e., either a stationary point $\alpha_i^{(\mathrm{st})}$ or a conditional stationary point $\alpha_i^{(\mathrm{cst})}$) will acquire an extra factor

$$\frac{\sin\left(\frac{\pi\alpha_i^*\gamma}{K}\right)}{\sin\left(\frac{\pi\alpha_i^*}{K}\right)} \tag{4.29}$$

up to the corrections of the higher order in K^{-1}. On the other hand, according to the quantum field theory loop expansion, the 1-loop contribution (2.14) of a particular flat connection A_μ^* acquires an extra factor

$$\mathrm{Tr}_\gamma \mathrm{Pexp}\left(i\int_{x_i} A_\mu^* dx^\mu\right) = \frac{\sin(2\pi\gamma u_i^*)}{\sin(2\pi u_i^*)}\,, \tag{4.30}$$

here we used Eq. (4.3) for the holonomy along x_i as well as the Weyl character formula to calculate the trace. Comparing Eqs. (4.29) and (4.30) we conclude that for the flat connection corresponding to the special point α_i^*,

$$u_i^* = \pm \frac{\alpha_i^*}{2K} . \tag{4.31}$$

This relation is compatible with Eqs. (3.15), (4.4) and (3.17), (4.16).

From the physical point of view, we "measure" the observable $\mathrm{Tr}_\gamma \mathrm{Pexp}(i \int_{x_i} A_\mu dx^\mu)$ for the contour x_i and compare it to its classical value. By performing this procedure for various contours we may reconstruct the holonomies of the flat connection corresponding to a particular special point of the surgery formula.

5. A Large k Limit of the Invariants of General Seifert Manifolds

5.1. A Bernstein–Gelfand–Gelfand Resolution and Verlinde Numbers. A calculation of Witten's invariant for a general Seifert manifold can proceed along the same lines as that for the 3-fibered one, described in Sect. 3. We will try again to convert the sums in Eq. (2.21) into the integrals over the n-dimensional half-spaces. We need a representation for Verlinde numbers $N_{\alpha_1,\ldots,\alpha_n}$ similar to that of Subsect. 3.1. We will do this with the help of the Bernstein–Gelfand–Gelfand resolution, which presents a representation space of a Lie group G as a cohomology over a complex of certain vector spaces (see e.g. a review [20] and references therein).

We introduce the following notation. Δ with various subscripts will denote the sets of weights of G coming with multiplicities. In other words, the elements of Δ are pairs (v, m), where v is a weight and $m \in \mathbf{Z}$ is its multiplicity. The weights form an abelian group. Consider its group algebra $\mathscr{A}$ with the coefficients in $\mathbf{Z}$. There is a one-to-one correspondence between the sets Δ and the elements of $\mathscr{A}$:

$$\Delta \longleftrightarrow \sum_{(v,m)\in\Delta} mv . \tag{5.1}$$

We define the sums and products of the sets Δ which parallel the operations in $\mathscr{A}$. The sum $\Delta_1 + \Delta_2$ consists of weights belonging to either of the sets Δ_1, Δ_2 and coming with the multiplicities which are sums of their multiplicities in Δ_1 and Δ_2. To build a product $\Delta_1 \circ \Delta_2$ we take all the pairs of weights $v_1 \in \Delta_1$, $v_2 \in \Delta_2$. Their sums $v_1 + v_2$ appear in the product $\Delta_1 \circ \Delta_2$ with multiplicities $m_1 m_2$. If the same weight v appears more than once as a sum $v_1 + v_2$, then we add all its multiplicities in order to account for the similar terms. A sum $\sum_{v\in\Delta} F(v)$ is a shorthand for $\sum_{(v,m)\in\Delta} mF(v)$, in the same way a product $\prod_{v\in\Delta} F(v)$ is equivalent to $\prod_{(v,m)\in\Delta} F^m(v)$.

Consider a representation space V_Λ of a Lie group G with the shifted highest weight Λ (we recall that a highest weight of V_Λ is $\Lambda - \rho$, $\rho = \frac{1}{2}\sum_{\lambda_i\in\Delta_+} \lambda_i$, Δ_+ is the set of positive roots of G). A Weyl formula for the character of V_Λ as a function of the element x of a Cartan subalgebra, is a ratio

$$\chi_\Lambda(x) \equiv \sum_{v\in\Delta_\Lambda} e^{iv\cdot x} = \frac{\sum_{w\in W}(-1)^{|w|}e^{i[w(\Lambda)-\rho]\cdot x}}{\prod_{\lambda_i\in\Delta_+}(1-e^{-i\lambda_i\cdot x})}. \tag{5.2}$$

Here Δ_Λ is the set of weights of V_Λ, W is a Weyl group and $|w|$ is a number of elementary Weyl reflections (mod 2) whose product is equal to w.

An individual term in the formula (5.2) can be presented as a sum

$$(-1)^{|w|}\frac{e^{i[w(\Lambda)-\rho]\cdot x}}{\prod_{\lambda_i\in\Delta_+}(1-e^{-i\lambda_i\cdot x})} = (-1)^{|w|}\sum_{v\in\Delta^{(1)}_{w(\Lambda)}} e^{iv\cdot x}, \tag{5.3}$$

here $\Delta^{(1)}_\Lambda$ is the set of all the weights of the form

$$v = \Lambda - \rho - \sum_i n_i\lambda_i, \quad n_i \geqq 0 \tag{5.4}$$

with multiplicity 1 (in fact, many weights may ultimately have a bigger multiplicity, because generally the positive roots λ_i are not linearly independent). It follows from Eqs. (5.2) and (5.3) that

$$\Delta_\Lambda = \sum_{w\in W}(-1)^{|w|}\Delta^{(1)}_{w(\Lambda)}. \tag{5.5}$$

The infinite dimensional modules of the Bernstein–Gelfand–Gelfand resolution consist of vectors with the weights and multiplicities of $\Delta^{(1)}_{w(\Lambda)}$.

The "classical" Verlinde numbers appear in the decomposition of the tensor product

$$V_{\Lambda_1}\otimes V_{\Lambda_2} = \sum_{\Lambda_3} N^{\Lambda_3}_{\Lambda_1\Lambda_2} V_{\Lambda_2}. \tag{5.6}$$

A product of characters decomposes as

$$\chi_{\Lambda_1}(x)\chi_{\Lambda_2}(x) = \sum_{\Lambda_3\in\Delta_{\Lambda_1,\Lambda_2}} \chi_{\Lambda_3}(x), \tag{5.7}$$

here $\Delta_{\Lambda_1,\Lambda_2}$ is the set of shifted highest weights of all representations appearing in the decomposition (5.6) and coming with the multiplicities $N^{\Lambda_3}_{\Lambda_1\Lambda_2}$. Note that we raised a third index of Verlinde numbers. The indices are raised and lowered by the metric $N_{\Lambda_1\Lambda_2}$ which is equal to 1 if V_{Λ_1} and V_{Λ_2} are conjugate representations, and is zero otherwise. For $G = SU(2)$ there is no distinction between the upper and lower indices since $N_{\alpha_1\alpha_2} = \delta_{\alpha_1\alpha_2}$.

If we use the r.h.s. of Eq. (5.2) for χ_{Λ_1} and χ_{Λ_3}, and the middle expression of Eq. (5.2) for χ_{Λ_2}, then we see that

$$\sum_{w\in W}\sum_{v\in\Delta_{\Lambda_2}}(-1)^{|w|}e^{i[w(\Lambda_1)+v]\cdot x} = \sum_{w\in W}\sum_{\Lambda_3\in\Delta_{\Lambda_1,\Lambda_2}}(-1)^{|w|}e^{iw(\Lambda_3)\cdot x}. \tag{5.8}$$

Let us denote by $\Delta_{W(\Lambda)}$ the set containing all the weights $w(\Lambda)$, $w\in W$ with multiplicities $(-1)^{|w|}$. Then it is easy to translate Eq. (5.8) into a statement about sets:

$$\Delta_{W(\Lambda_1)}\circ\Delta_{\Lambda_2} = \sum_{\Lambda_3\in\Delta_{\Lambda_1,\Lambda_2}}\Delta_{W(\Lambda_3)}. \tag{5.9}$$

Finally applying Eq. (5.5) to Δ_{Λ_2} we get

$$\sum_{w\in W}(-1)^{|w|}\Delta_{W(\Lambda_1)}\circ\Delta^{(1)}_{w(\Lambda_2)}=\sum_{\Lambda_3\in\Delta_{\Lambda_1,\Lambda_2}}\Delta_{W(\Lambda_3)}\,, \tag{5.10}$$

or equivalently,

$$\sum_{w_{1,2}\in W}(-1)^{|w_1|+|w_2|}\Delta^{(1)}_{w_1(\Lambda_1)+w_2(\Lambda_2)}=\sum_{\Lambda_3\in\Delta_{\Lambda_1,\Lambda_2}}\Delta_{W(\Lambda_3)}\,. \tag{5.11}$$

It is easy to generalize this relation to a tensor product of $n-1$ vector spaces:

$$\sum_{w_1,\ldots,w_{n-1}\in W}(-1)^{|w_1|+\cdots+|w_{n-1}|}\Delta^{(n-2)}_{w_1(\Lambda_1)+\cdots+w_{n-1}(\Lambda_{n-1})}=\sum_{\Lambda_n\in\Delta_{\Lambda_1,\ldots,\Lambda_{n-1}}}\Delta_{W(\Lambda_n)}\,, \tag{5.12}$$

here $\Delta_{\Lambda_1,\ldots,\Lambda_{n-1}}$ is the set of weights Λ_n taken with multiplicities $N^{\Lambda_n}_{\Lambda_1\ldots\Lambda_{n-1}}$, while $\Delta^{(n)}_{\Lambda}$ contains all the weights

$$v=\Lambda-n\rho-\sum_i\sum_{j=1}^{n}n_{i,j}\lambda_i,\quad n_{i,j}\geqq 0 \tag{5.13}$$

coming with multiplicities 1 (before the counting of similar terms). There are $\begin{pmatrix} n-1 \\ m+n-1\end{pmatrix}$ ways in which a number m can be represented as a sum of n non-negative numbers. Therefore we can say that $\Delta^{(n)}_{\Lambda}$ consists of the weights

$$v=\Lambda-n\rho-\sum_i n_i\lambda_i\,, \tag{5.14}$$

coming with multiplicities $\prod_i\begin{pmatrix} n-1 \\ n_i+n-1\end{pmatrix}$.

We need an analog of Eq. (5.12) for the case of affine Lie algebra (or a quantum group G_q). The affine Weyl group $\tilde{W}$ is a semidirect product of W and an abelian group T of translations by the elements of the root lattice multiplied by K. We can not simply substitute $\tilde{W}$ for W in Eq. (5.12), because the previous reasoning does not quite apply to the case of affine algebras (e.g. Eq. (5.7) is no longer valid). Still it turns out that a simple modification of Eq. (5.12) makes it work for affine algebras or quantum groups:

$$\begin{aligned}\sum_{t\in T}\sum_{w_1,\ldots,w_{n-1}\in W}&(-1)^{|w_1|+\cdots+|w_{n-1}|}\Delta^{(n-2)}_{t(w_1(\Lambda_1)+\cdots+w_{n-1}(\Lambda_{n-1}))}\\ &=\sum_{\Lambda_n\in\Delta_{\Lambda_1,\ldots,\Lambda_{n-1}}}\Delta_{\tilde{W}(\Lambda_n)}\,,\end{aligned} \tag{5.15}$$

here $\Delta_{\tilde{W}(\Lambda)}$ is the set of weights $w(\Lambda)$, $w \in \tilde{W}$ coming with multiplicities $(-1)^{|w|}$, $|w|$ counts only the number of reflections[8].

The l.h.s. of Eq. (5.15) consists of all the weights

$$v = \sum_{i=1}^{n-1} w_i(\Lambda_i) - (n-2)\rho - \sum_i n_i\lambda_i + K\sum_i m_i\lambda_i, \quad n_i, m_i \in \mathbf{Z}, \quad n_i \geqq 0 \tag{5.17}$$

coming with multiplicities

$$(-1)^{\sum_{i=1}^{n-1}|w_i|}\prod_i \binom{n-3}{n_i+n-3}. \tag{5.18}$$

For a given set of numbers m_i and Weyl reflections w_i, these vectors form a half-space similar to that of Eq. (3.13). The r.h.s. of Eq. (5.15) consists of the highest weights Λ_n of integrable representations of an affine Lie algebra coming with multiplicities $N_{\Lambda_1\ldots\Lambda_{n-1}}^{\Lambda_n}$ together with all their images $w(\Lambda_n)$, $w \in \tilde{W}$, whose multiplicities have an extra factor $(-1)^{|w|}$. In other words, the r.h.s. of Eq. (5.15) consists of all the weights Λ_n coming with the multiplicities $\tilde{N}_{\Lambda_1\ldots\Lambda_{n-1}}^{\Lambda_n}$ which are Verlinde numbers $N_{\Lambda_1\ldots\Lambda_{n-1}}^{\Lambda_n}$ extended to all the weights of G by the affine Weyl group: the extended numbers $\tilde{N}_{\Lambda_1\ldots\Lambda_{n-1}}^{\Lambda_n}$ are invariant under the shifts of T and they are antisymmetric under the Weyl reflections. Since the matrices $\tilde{M}$ of Eq. (2.21) exhibit the same properties under the action of $\tilde{W}$, we can use the extended Verlinde numbers as defined by Eqs. (5.15)–(5.18) in order to extend the sums in Eq. (2.21) from the integrable highest weights to the whole weight lattice of G and to transform it into a sum over the "half-spaces" (5.17), (5.18).

The Case of $SU(2)$. Let us study specifically the case of $G = SU(2)$. We recall that the variables α play the role of shifted highest weights, $\rho = 1$ and the only positive root is equal to 2. Thus according to Eq. (5.15), we can drop Verlinde

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[8] The same equation can be derived directly from Verlinde's formula

$$N_{\Lambda_1,\ldots,\Lambda_n} = \sum_{\Lambda\in\Delta}\left(\prod_{i=1}^{n} S_{\Lambda\Lambda_i}\right)\Big/ S_{\rho\Lambda}^{n-2}$$

(Δ being the set of integrable highest weights), by expanding its denominator in a geometric series similar to that of Eq. (5.3) and performing a Poisson resummation on Λ. In fact, Λ plays a role very similar to c in Eq. (5.27).

A generalised Verlinde's formula

$$N_{\Lambda_1,\ldots,\Lambda_n}^{(g)} = \sum_{\Lambda\in\Delta}\left(\prod_{i=1}^{n} S_{\Lambda\Lambda_i}\right)\Big/ S_{\rho\Lambda}^{n-2+2g} \tag{5.16}$$

for the number of conformal blocks on a g-handled Riemann surface Σ_g with n primary fields $\mathscr{V}_{\Lambda_i}$, allows us to generalize the results of our calculations to the case of a Seifert manifold constructed by a surgery on circles in $S^1 \times \Sigma_g$. Equation (5.16) suggests that the presence of handles can be accounted for by a simple substitution $n \to n + 2g$ in Eqs. (5.17)–(5.20) and (5.24). Thus only a multiplicity factor $N_n(x)$ is affected (it is substituted by N_{n+2g}), while other quantities, such as Chern–Simons action of flat connections, remain unchanged, so that Eq. (4.5) is still valid in agreement with [17]. We hope to discuss this subject further in a forthcoming paper.

numbers $N_{\alpha_1..\alpha_n}$ from Eq. (2.21) if we take the sum there over all the n-dimensional vectors $(\alpha_1,\ldots,\alpha_n)$ satisfying an equation

$$\sum_{i=1}^{n} v_i\alpha_i = (n-2) + 2m + 2Kl \tag{5.19}$$

and coming with multiplicities

$$\binom{n-3}{m+n-3}\prod_{i=1}^{n} v_i\,. \tag{5.20}$$

Here

$$v_1,\ldots,v_{n-1} = \pm 1,\ v_n = -1,\ m,l\in\mathbf{Z},\quad m \geqq 0\,. \tag{5.21}$$

The multiplicity appears as a new factor in Eq. (2.21) taking the place of Verlinde numbers.

We can make a substitution

$$m = \frac{1}{2}(x-n+2)\,, \tag{5.22}$$

so that Eq. (5.19) transforms into

$$\sum_{i=1}^{n} v_i\alpha_i = x + 2Kl \tag{5.23}$$

and the multiplicity factor is

$$N_n(x) = -\frac{\prod_{i=1}^{n} v_i}{2^{n-3}(n-3)!}\prod_{\substack{-n+4\leqq i\leqq n-4\\ i+n \text{ even}}}(x-i)\,. \tag{5.24}$$

The substitution (5.22) requires that $x-n$ is even and $x \geqq n-2$. In fact we may demand only that $x \geqq 0$, because the fixed parity together with the last factor of Eq. (5.24) eliminates all possible extra values of x.

5.2. A Contribution of Irreducible Flat Connections. The Witten's invariant of an n-fibered Seifert manifold is equal to a sum

$$Z\left(X\left(\frac{p_1}{q_1},\ldots,\frac{p_n}{q_n}\right)\right) = e^{i\phi_{\mathrm{fr}}}\sum_{l\in\mathbf{Z}}\sum_{\substack{v_i=\pm 1\\ 1\leqq i\leqq n-1}} Z_{v_1,\ldots,v_{n-1},-1;l}\,, \tag{5.25}$$

$$Z_{v_1,\ldots,v_n;l} = \sum_{\substack{x>0\\ x-n \text{ even}}}\ \sum_{\alpha_i:\ \sum_{i=1}^{n} v_i\alpha_i = x+2Kl} N_n(x)\prod_{i=1}^{n}\tilde{M}^{(p_i,q_i)}_{\alpha_i 1}\,, \tag{5.26}$$

here $e^{i\phi_{\mathrm{fr}}}$ is a framing correction given by Eq. (3.48) with a substitution $\sum_{i=1}^{3}\longrightarrow\sum_{i=1}^{n}$.

Similarly to Subsect. 3.1 we turn a sum over α_i into an integral by extending the sum over n in Eq. (2.25) to all integer numbers. We take care of a condition (5.23) by adding a factor

$$\int_{-\infty}^{+\infty} dc\exp 2\pi i c\left(x+2Kl-\sum_{i=1}^{3} v_i\alpha_i\right) \tag{5.27}$$

to Eq. (5.26). Another familiar factor

$$\frac{1}{2}\sum_{\lambda=0,\frac{1}{2}} \exp 2\pi i\lambda\left(\sum_{i=1}^{n}\alpha_i + n\right) \tag{5.28}$$

guarantees together with the factor (5.27) that $x - n$ is even. Finally since the factor (5.27) makes x an integer, we can take an integral over x rather than a sum:

$$\begin{aligned} Z_{\nu_1,\dots,\nu_n;l} &= \frac{1}{2}\sum_{\lambda=0,\frac{1}{2}} e^{2\pi i\lambda n}\int_0^{\infty} dx\, N_n(x)\int_{-\infty}^{+\infty} dc\, \exp 2\pi ic(x+2Kl) \\ &\times \prod_{i=1}^{n}\sum_{n_i\in\mathbf{Z}}\sum_{\mu_i=\pm 1}\mu_i\int_{-\infty}^{+\infty} d\alpha_i i\frac{\operatorname{sign}(q_i)}{\sqrt{2K|q_i|}}e^{-i\frac{\pi}{4}\Phi(M^{(p_i,q_i)})} \\ &\times \exp\frac{i\pi}{2Kq_i}[p_i\alpha_i^2 + 2\alpha_i(2K(\tilde{n}_i+\nu_iq_ic)+\mu_i) + s_i(2Kn_i+\mu_i)^2]. \end{aligned} \tag{5.29}$$

We fix the numbers n_i in order to study a contribution of a particular point (3.15). After integrating over α_i and c, a partition function becomes a product of two factors Z_1 and Z_2:

$$Z^{(n_1,\dots,n_n;\lambda)}_{\mathrm{st}|\nu_1,\dots,\nu_n;l} = Z_1 Z^{(n_1,\dots,n_n;\lambda)}_{2,\mathrm{st}}, \tag{5.30}$$

$$\begin{aligned} Z_1 &= e^{i\frac{3\pi}{4}\operatorname{sign}(\frac{H}{P})}\left(\prod_{i=1}^{n}\operatorname{sign} p_i\right) \\ &\times \exp -\frac{i\pi}{2K}\left[3\operatorname{sign}\left(\frac{H}{P}\right)+\sum_{i=1}^{n}\left(12s(q_i,p_i)-\frac{q_i}{p_i}\right)\right], \end{aligned} \tag{5.31}$$

$$\begin{aligned} Z^{(n_1,\dots,n_n;\lambda)}_{2,\mathrm{st}} &= \frac{e^{-i\frac{\pi}{4}\operatorname{sign}(\frac{H}{P})}}{\sqrt{2K|H|}}\frac{1}{2}\sum_{\lambda=0,\frac{1}{2}} e^{2\pi i\lambda n} \\ &\times \sum_{\mu_1,\dots,\mu_n=\pm 1}\left[\prod_{i=1}^{n}\mu_i \exp 2\pi iK\left(\frac{r_i}{p_i}\tilde{n}_i^2 - s_iq_i\lambda^2\right)\exp 2\pi i\mu_i\left(\frac{r_i}{p_i}\tilde{n}_i + s_i\lambda\right)\right] \\ &\times I(\nu_1,\dots,\nu_n;\mu_1,\dots,\mu_n;l), \end{aligned} \tag{5.32}$$

here

$$I(\nu_1,\dots,\nu_n;\mu_1,\dots,\mu_n;l) = \int_{\substack{x_0\\ [0]}}^{\infty} dx\, N_n(x-x_0)\exp\left[\frac{i\pi}{2K}\frac{P}{H}\left(x-\sum_{i=1}^{n}\nu_i\frac{\mu_i}{p_i}\right)^2\right], \tag{5.33}$$

$$x_0 = 2Kl - \sum_{i=1}^{n}\nu_i\alpha_i^{(\mathrm{st})}, \tag{5.34}$$

and we remind that $\alpha_i^{(\mathrm{st})}$ are defined by Eq. (3.15). We made a change of variables $x \longrightarrow x - x_0$ in deriving Eq. (5.33).

The integral (5.33) is gaussian apart from a polynomial factor $N_n(x-x_0)$. This integral is similar to the integral (3.35) and should be treated in a similar way.

Here we are concerned with a contribution of a stationary phase point $x = 0$, which contributes to the integral (5.33) if $x_0 < 0$.

Consider a specific point $\alpha_i^{(\mathrm{st})}$ of Eq. (3.15). As we know from Subsect. 3.1, we should limit our attention to the points belonging to the fundamental cube

$$0 \leqq \alpha_i^{(\mathrm{st})} < K\,. \tag{5.35}$$

We have to determine a set $\mathscr{S}$ of arrays $(v_1, \ldots, v_{n-1}; l)$ for which a stationary phase point $\alpha_i^{(\mathrm{st})}$ contributes to the integral (5.33). Then we should substitute a sum

$$N_n^{(\mathrm{tot})}(x) = \sum_{\mathscr{S}} N_n(x + \sum_{i=1}^{n} v_i \alpha_i^{(\mathrm{st})} - 2Kl) \tag{5.36}$$

instead of $N_n(x - x_0)$ in Eq. (5.33) and extend the integral over x to $\int_{-\infty}^{+\infty}$ in order to get a full contribution of the point $\alpha_i^{(\mathrm{st})}$ to the partition function (5.30).

Let $\mathscr{S}$ be the set of all arrays $(v_1, \ldots, v_{n-1}; l)$ for which

$$2Kl - \sum_{i=1}^{n} v_i \alpha_i^{(\mathrm{st})} \leqq 0\,, \tag{5.37}$$

then the sum (5.36) will contain infinitely many terms. However most of these terms will cancel each other. Therefore we propose another procedure that will express $N_n^{(\mathrm{tot})}$ as a finite sum. Suppose for simplicity that $\alpha_n^{(\mathrm{st})} \neq 0$. Consider a line in the α-space

$$\alpha_i(t) = t\alpha_i^{(\mathrm{st})}, \quad i = 1, \ldots, n-1; \quad \alpha_n(t) = \alpha_n^{(\mathrm{st})}\,. \tag{5.38}$$

Obviously, $\alpha_i(1) = \alpha_i^{(\mathrm{st})}$. Remember now that $N_n^{(\mathrm{tot})}$ is a Verlinde number. Therefore $N_n^{(\mathrm{tot})} = 0$ for $\alpha_i(0)$. This means that all the terms in Eq. (5.36) cancel each other and we may drop them altogether. As t starts to grow, suppose that for some value t_*,

$$2Kl - \sum_{i=1}^{n} v_i \alpha_i(t_*) = 0\,. \tag{5.39}$$

If for $t > t_*$ the l.h.s. of Eq. (5.39) is negative, then by passing $t = t_*$ we gained a contribution of the array $(v_1, \ldots, v_{n-1}; l)$ to Eq. (5.36) and the corresponding term should be added there. If, however, for $t > t_*$ the l.h.s. of Eq. (5.39) is positive, then we lost the contribution of the array $(v_1, \ldots, v_{n-1}; l)$, and the corresponding term should be subtracted. As a result,

$$N_n^{(\mathrm{tot})}(x) = -\sum_{\mathscr{S}'} \mathrm{sign}\left(\sum_{i=1}^{n-1} v_i \alpha_i\right) N_n\left(x + \sum_{i=1}^{n} v_i \alpha_i^{(\mathrm{st})} - 2Kl\right), \tag{5.40}$$

here a set $\mathscr{S}'$ consists of all arrays $(v_1, \ldots, v_{n-1}; l)$ such that for some $t_* \in [0, 1]$ Eq. (5.39) is satisfied.

Since $\alpha_i^{(\mathrm{st})}$ are proportional to K, the function $N_n^{(\mathrm{tot})}(x)$ is a polynomial in K and x:

$$N_n^{(\mathrm{tot})}(x) = \sum_{j=0}^{n-3} C_j K^{n-3-j} x^j . \qquad (5.41)$$

A Verlinde number $N_{\alpha_1^{(\mathrm{st})}\ldots\alpha_n^{(\mathrm{st})}} = N_n^{(\mathrm{tot})}(0) = C_0$ is a number of the WZW conformal blocks of n primary fields $\mathcal{V}_{\alpha_i^{(\mathrm{st})}}$ on a sphere[9]. This number to the leading power in K is proportional to the volume of the moduli space of flat connections on a sphere with n punctures, the holonomies around which are fixed by Eq. (4.3). This moduli space coincides with a connected component of the moduli space of flat connections on the Seifert manifold, for which the map (2.2) is determined by Eq. (4.3). Therefore the coefficient C_0 is equal to the volume of that component of the moduli space (calculated with the proper measure).

Let us substitute Eq. (5.41) into the integral (5.33) modified in order to get the contribution of the stationary phase point $\alpha_i^{(\mathrm{st})}$:

$$I = \sum_{j=0}^{n-3} C_j K^{n-3-j} \int_{-\infty}^{+\infty} dx \left(x + \sum_{i=1}^{n} v_i \frac{\mu_i}{p_i} \right)^j \exp\left(\frac{i\pi}{2K} \frac{P}{H} x^2 \right) . \qquad (5.42)$$

The dominant contribution comes from the term with $j = 0$. It is proportional to K^{n-3}:

$$I \approx e^{i\frac{\pi}{4}\mathrm{sign}\left(\frac{H}{P}\right)} \sqrt{\frac{2K|H|}{|P|}}\, C_0 K^{n-3} , \qquad (5.43)$$

so that the whole 1-loop partition function contribution coming from the point (3.15) is (for $p_1, p_2, p_3, H > 0$)

$$\begin{aligned} Z_{\mathrm{st}}^{(n_1,\ldots,n_n;\lambda)} &\approx \frac{1}{2} e^{i\frac{3\pi}{4}} \exp - \frac{i\pi}{2K} \left[3 + \sum_{i=1}^{n} \left(12 s(q_i, p_i) - \frac{q_i}{p_i} \right) \right] K^{n-3} C_0 \\ &\times \sum_{\lambda=0,\frac{1}{2}} e^{2\pi i \lambda n} \prod_{i=1}^{n} \exp 2\pi i K \left(\frac{r_i}{p_i} \tilde{n}_i^2 - s_i q_i \lambda^2 \right) \frac{2i}{\sqrt{p_i}} \sin 2\pi \left(\frac{r_i}{p_i} \tilde{n}_i + s_i \lambda \right) . \end{aligned} \qquad (5.44)$$

Comparing this expression with Eq. (2.14) we note that $\dim H^1 = 2(n-3)$, hence the factor K^{n-3}. Also an integral over the moduli space (which is included in the sum over flat connections in Eq. (2.14)) produces its volume C_0.

In contrast to the results of Subsect. 4.1, the contribution of the irreducible flat connection on an n-fibered Seifert manifold ($n \geqq 4$) contains higher loop corrections coming from the sum in Eq. (5.42). However the number of these corrections is finite. The highest order correction is of the order of K^0, so the number of loop corrections is equal to half the dimension of the moduli space.

5.3. A Contribution of Reducible Flat Connections. A contribution of a conditional stationary phase point (3.17) is easier to calculate than that of an unconditional one $\alpha^{(\mathrm{st})}$, because the former involves an integral only over one half-space (5.23), to which boundary it belongs. We will use an expression for $Z_{2,\mathrm{cst}}^{(n_1,\ldots,n_n)}$ which is slightly

[9] The numbers $\alpha_i^{(\mathrm{st})}$ are not necessarily integer, so in fact, we should take the closest integer numbers. This does not change a conclusion that C_0 is the volume of the moduli space.

different from that of Eq. (5.32) and is a generalization of Eq. (3.34). We express the multiplicity factor $N_n(x)$ of Eq. (5.24) as a derivative:

$$N_n(x) = -\frac{\prod_{i=1}^n v_i}{(n-3)!}\, \partial_a^{(n-3)} a^{\frac{1}{2}(x+n-4)}\Big|_{a=1} . \tag{5.45}$$

We also shift the integration range of x from $x \geqq 0$ to $x \geqq 4-n$. The contribution of the extra values of x is killed by the zeros of $N_n(x)$ (recall that x is actually an even integer). After a shift in the integration variable $x \to x+n-4$ we get the following expression for Z_2 (the other factor Z_1 is defined by Eq. (5.31)):

$$\begin{aligned}
&Z_{2,\mathrm{cst}}^{(n_1,\dots,n_n)} \\
&= -\left[\prod_{i=1}^n e^{-i\frac{\pi}{4}\operatorname{sign}(p_i q_i)} \exp\left(-\frac{i\pi}{2K}\frac{r_i}{p_i}\right)\right] \sum_{\lambda=0,\frac{1}{2}} e^{2\pi i \lambda n} \\
&\quad \times \sum_{m\in\mathbf{Z}} \int_{\substack{0\\ [0]}}^{\infty} dx \left[\frac{1}{(n-3)!}\partial_a^{(n-3)} a^{\frac{x}{2}}\Big|_{a=1}\right] \int_{-\infty}^{+\infty} dc\, e^{-2\pi i c(x+4-n+2Kl)} \\
&\quad \times \prod_{i=1}^n \sum_{\mu_i=\pm 1} \mu_i \int_{-\infty}^{+\infty} \frac{d\alpha_i}{\sqrt{2K|q_i|}} \\
&\quad \times \exp \frac{i\pi}{2Kq_i}[p_i\alpha_i^2 - 2v_i\alpha_i(2Kn_i + 2Kq_i(\lambda+m-c)+\mu_i) \\
&\qquad\qquad + s_i(2K(n_i+q_i m)+\mu_i)^2] \\
&= -\frac{e^{-i\frac{\pi}{4}\operatorname{sign}(\frac{H}{P})}}{\sqrt{2K|H|}} \exp 2\pi i K\left[\sum_{i=1}^n \frac{r_i}{p_i}n_i^2 + \frac{H}{P}c_0^2\right] \\
&\quad \times \sum_{\mu_1,\dots,\mu_n=\pm 1}\left[\prod_{i=1}^n \mu_i \exp\left(2\pi i \mu_i \frac{r_i n_i + c_0}{p_i}\right)\right] \frac{1}{2}\sum_{\lambda=0,\frac{1}{2}}\sum_{m\in\mathbf{Z}} I(m,\lambda),
\end{aligned} \tag{5.46}$$

here c_0 is defined by Eq. (3.18), while

$$\begin{aligned}
I(m,\lambda) &= \frac{e^{2\pi i c_0(n-4)}}{(n-3)!} \int_{\substack{0\\ [0]}}^{\infty} dx \left[\partial_a^{(n-3)} a^{\frac{x}{2}}\Big|_{a=1}\right] \\
&\quad \times \exp\left[-2\pi i x(c_0+m+\lambda) + \frac{i\pi}{2K}\frac{P}{H}\left(x-n+4-\sum_{i=1}^n\frac{\mu_i}{p_i}\right)^2\right] \\
&= \frac{e^{2\pi i c_0(n-4)}}{(n-3)!}\sum_{j=0}^{\infty}\frac{1}{j!}(8\pi i K)^{-j}\left(\frac{P}{H}\right)^j \partial_\varepsilon^{(2j)}\partial_a^{(n-3)} \\
&\quad \times \left\{ e^{2\pi i \varepsilon\left(\sum_{i=1}^n \frac{\mu_i}{p_i}+n-4\right)} \int_0^{\infty} dx\, a^{\frac{x}{2}} \exp[-2\pi i x(c_0+m+\lambda+\varepsilon)]\right\}\Bigg|_{\substack{a=1\\ \varepsilon=0}} .
\end{aligned} \tag{5.47}$$

General Reducible Connection. A sum over m and λ converts an integral over x in Eq. (5.47) into a sum over even x:

$$\frac{1}{2}\sum_{\lambda=0,\frac{1}{2}}\sum_{m\in\mathbf{Z}} I(m) = \frac{e^{2\pi i c_0(n-4)}}{(n-3)!}\sum_{j=0}^{\infty}\frac{1}{j!}(8\pi i K)^{-j}\left(\frac{P}{H}\right)^j$$
$$\times\,\partial_\varepsilon^{(2j)}\partial_a^{(n-3)}\left[e^{2\pi i\varepsilon\left(\sum_{i=1}^n\frac{\mu_i}{p_i}+n-4\right)}\sum_{\substack{x\geq 0\\ x-\text{ even}}} a^{\frac{x}{2}}\,e^{-2\pi i x(c_0+\varepsilon)}\right]\Bigg|_{\substack{a=1\\ \varepsilon=0}}$$
$$=\sum_{j=0}^{\infty}\frac{1}{j!}(8\pi i K)^{-j}\left(\frac{P}{H}\right)^j\partial_\varepsilon^{(2j)}\left.\frac{e^{2\pi i\varepsilon\sum_{i=1}^n\frac{\mu_i}{p_i}}}{[2i\sin 2\pi(c_0+\varepsilon)]^{n-2}}\right|_{\varepsilon=0}. \tag{5.48}$$

Therefore the total contribution of a general reducible connection is

$$Z_{\text{cst}}^{(n_1,\ldots,n_n)} = -\frac{e^{i\frac{\pi}{2}\operatorname{sign}\left(\frac{H}{P}\right)}}{\sqrt{2K|H|}}\operatorname{sign}(P)\exp-\frac{i\pi}{2K}\left[3\operatorname{sign}\left(\frac{H}{P}\right)+\sum_{i=1}^n\left(12s(q_i,p_i)-\frac{q_i}{p_i}\right)\right]$$
$$\times\exp 2\pi i K\left(\sum_{i=1}^n\frac{r_i}{p_i}n_i^2+\frac{H}{P}c_0^2\right)\sum_{j=0}^{\infty}\frac{1}{j!}(8\pi i K)^{-j}\left(\frac{P}{H}\right)^j$$
$$\times\,\partial_c^{(2j)}\left[\frac{\prod_{i=1}^n 2i\sin\left(2\pi\frac{r_i n_i+c}{p_i}\right)}{[2i\sin(2\pi c)]^{n-2}}\right]\Bigg|_{c=c_0}. \tag{5.49}$$

Special Reducible Connection. If $c_0+m_0+\lambda_0=0$ for some values $\lambda_0=0,\frac{1}{2}$, $m_0\in\mathbf{Z}$, then a calculation of $I(m_0,\lambda_0)$ has to be performed separately:

$$I(m_0,\lambda_0)$$
$$=\frac{e^{2\pi i c_0 n}}{(n-3)!}\int_0^\infty dx\left[\partial_a^{(n-3)}a^{\frac{x}{2}}\Big|_{a=1}\right]\exp\left[\frac{i\pi}{2K}\frac{P}{H}\left(x-n+4-\sum_{i=1}^n\frac{\mu_i}{p_i}\right)^2\right]$$
$$=\frac{e^{2\pi i c_0 n}}{(n-3)!}\left(\int_0^\infty-\int_0^{n-4+\sum_{i=1}^n\frac{\mu_i}{p_i}}\right)dx\left[\partial_a^{(n-3)}a^{\frac{1}{2}\left(x+n-4+\sum_{i=1}^n\frac{\mu_i}{p_i}\right)}\Big|_{a=1}\right]\exp\left(\frac{i\pi}{2K}\frac{P}{H}x^2\right)$$
$$=\frac{e^{2\pi i c_0 n}}{(n-3)!}\left\{\int_0^\infty dx\left[\partial_a^{(n-3)}a^{\frac{1}{2}\left(x+n-4+\sum_{i=1}^n\frac{\mu_i}{p_i}\right)}\Big|_{a=1}\right]\exp\left(\frac{i\pi}{2K}\frac{P}{H}x^2\right)\right.$$
$$-\sum_{j=0}^{\infty}\frac{1}{j!}(8\pi i K)^{-j}\left(\frac{P}{H}\right)^j\partial_\varepsilon^{(2j)}\partial_a^{(n-3)}$$
$$\left.\times\left.\frac{e^{2\pi i\varepsilon\left(n-4+\sum_{i=1}^n\frac{\mu_i}{p_i}\right)}-a^{\frac{1}{2}\left(n-4+\sum_{i=1}^n\frac{\mu_i}{p_i}\right)}}{\frac{1}{2}\log a-2\pi i\varepsilon}\right|_{\substack{\varepsilon=0\\ a=1}}\right\}. \tag{5.50}$$

A remaining part of the sum (5.48) is

$$\frac{1}{2}\sum_{\substack{\lambda,m\\ \lambda+m+c_0\neq 0}} I(m,\lambda) = e^{2\pi i n c_0}\sum_{j=0}^{\infty}\frac{1}{j!}(8\pi iK)^{-j}\left(\frac{P}{H}\right)^{j}\partial_\varepsilon^{(2j)}$$
$$\times\left\{e^{2\pi i\varepsilon\sum_{i=1}^{n}\frac{\mu_i}{p_i}}\left[\frac{1}{(2i\sin 2\pi\varepsilon)^{n-2}}+\frac{e^{2\pi i\varepsilon(n-4)}}{(n-3)!}\partial_a^{(n-3)}\frac{1}{\frac{1}{2}\log a-2\pi i\varepsilon}\bigg|_{a=1}\right]\right\}\bigg|_{\varepsilon=0}, \quad (5.51)$$

so that the whole expression for the contribution of a special reducible connection is

$$\begin{aligned}
Z_{\mathrm{cst}}^{(n_1,\ldots,n_n)} &= -\frac{e^{i\frac{\pi}{2}\mathrm{sign}(\frac{H}{P})}}{\sqrt{2K|H|}}e^{2\pi i n c_0}\mathrm{sign}(P)\exp -\frac{i\pi}{2K}\left[3\mathrm{sign}\left(\frac{H}{P}\right)+\sum_{i=1}^{n}\left(12s(q_i,p_i)-\frac{q_i}{p_i}\right)\right]\\
&\times \exp 2\pi iK\left(\sum_{i=1}^{n}\frac{r_i}{p_i}n_i^2+\frac{H}{P}c_0^2\right)\\
&\times\left\{\frac{1}{2}\sum_{\mu_1,\ldots,\mu_n=\pm 1}\left[\prod_{i=1}^{n}\mu_i\exp\left(2\pi i\mu_i\frac{r_in_i+c_0}{p_i}\right)\right]\right.\\
&\times\frac{1}{(n-3)!}\int_0^{\infty}dx\left[\partial_a^{(n-3)}a^{\frac{1}{2}\left(x+n-4+\sum_{i=1}^{n}\frac{\mu_i}{p_i}\right)}\bigg|_{a=1}\right]\exp\left(\frac{i\pi}{2K}\frac{P}{H}x^2\right)\\
&+\sum_{j=0}^{\infty}(8\pi iK)^{-j}\left(\frac{P}{H}\right)^{j}\partial_\varepsilon^{(2j)}\left[\frac{\prod_{i=1}^{n}2i\sin\left(2\pi\frac{r_in_i+c_0+\varepsilon}{p_i}\right)}{[2i\sin(2\pi\varepsilon)]^{n-2}}\right.\\
&\left.\left.+\frac{1}{(n-3)!}\partial_a^{(n-3)}\frac{a^{\frac{n-4}{2}}\prod_{i=1}^{n}\sin\left(2\pi\frac{r_in_i+c_0+\frac{i}{2}\log a}{p_i}\right)}{\frac{1}{2}\log a-2\pi i\varepsilon}\bigg|_{a=1}\right]\bigg|_{\varepsilon=0}\right\}.
\end{aligned}\quad (5.52)$$

It is not hard to see that the term

$$\frac{1}{(n-3)!}\partial_a^{(n-3)}\frac{a^{\frac{n-4}{2}}\prod_{i=1}^{n}\sin\left(2\pi\frac{r_in_i+c_0+\frac{i}{2}\log a}{p_i}\right)}{\frac{1}{2}\log a-2\pi i\varepsilon}\bigg|_{a=1} \quad (5.53)$$

contains only the negative powers of ε in its Laurent series expansion. Therefore the only purpose of this term is to cancel the negative powers of ε in the expansion of the term

$$\frac{\prod_{i=1}^{n}2i\sin\left(2\pi\frac{r_in_i+c_0+\varepsilon}{p_i}\right)}{[2i\sin(2\pi\varepsilon)]^{n-2}}, \quad (5.54)$$

so that the whole expression has a smooth limit of $\varepsilon\to 0$.

Trivial Connection. When all $n_i=0$, Eq. (5.53) can be simplified. In particular, the integral over x and the term (5.53) are both equal to zero after taking a sum over

μ_i. After adding a factor (3.25), a total contribution of the trivial connection is

$$Z_{\rm cst}^{(\rm triv)} = -\frac{e^{i\frac{\pi}{2}\operatorname{sign}(\frac{H}{P})}}{2\sqrt{2K|H|}} e^{2\pi i n c_0} \operatorname{sign}(P) \exp -\frac{i\pi}{2K}\left[3\operatorname{sign}\left(\frac{H}{P}\right) + \sum_{i=1}^{n}\left(12 s(q_i, p_i) - \frac{q_i}{p_i}\right)\right]$$

$$\times \sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\pi}{2iK}\frac{P}{H}\right)^j \partial_\varepsilon^{(2j)} \frac{\prod_{i=1}^{n} 2i\sin\left(\frac{\varepsilon}{p_i}\right)}{(2i\sin\varepsilon)^{n-2}}\Bigg|_{\varepsilon=0} . \tag{5.55}$$

As we have noted in the end of Subsect. 4.3, a ratio of the $j=2$ and $j=1$ terms contributes together with the phase ϕ of Eq. (1.5) to the 2-loop correction S_2 as defined by Eq. (2.4). A simple calculation similar to that of Eq. (4.26) shows again that S_2 is proportional to Casson's invariant (4.14):

$$S_2 = 6\pi\lambda_{CW} . \tag{5.56}$$

6. Discussion

In this paper we calculated a full asymptotic large k expansion of the exact surgery formula for Witten's invariant of Seifert manifolds. We found a complete agreement between our results and the 1-loop quantum field theory predictions thus extending the results of the papers [8, 9] on this subject. To achieve this agreement we had to modify slightly the previous 1-loop formulas for the case of reducible flat connections and for the case of obstructions in extending the elements of H^1 to the moduli of flat connections.

It seems that the method of Poisson resummation used in our calculations can be applied to Witten's invariants of graph manifolds, i.e. manifolds constructed by "plumbing" the Seifert manifolds (the solid tori parallel to the fibers are cut out of Seifert manifolds and the corresponding 2-dimensional boundaries are glued together after the modular transformations are performed). This method can also be applied to the invariants built upon simple Lie groups other than $SU(2)$. We showed that the applicability of the Poisson resummation is based on Bernstein–Gelfand–Gelfand resolution.

A rather surprising result of our calculations is the finite loop exactness of the contributions of irreducible flat connections. This exactness in somewhat reminiscent of the formulas of paper [11] in which Witten applied a localization principle to the 2-dimensional gauge theory. The order of the highest loop corrections is equal to half the dimension of the moduli space of flat connections. This may suggest that these corrections are related to some intersection numbers in the moduli space.

The contributions of all flat connections have a specific 2-loop phase correction ϕ (see Eq. (1.5)). This phase is the same for all flat connections of a given manifold. In case of a 3-fibered Seifert manifold, ϕ is the only 2-loop correction for the contribution of irreducible flat connections. The phase ϕ looks similar to Casson's invariant, however certain terms are missing there. It seems, however, that ϕ is a manifold invariant in its own right. It would be interesting to understand its topological nature.

The "missing terms" appear when the full 2-loop correction to the contribution of the trivial connection is calculated. The trivial connection is reducible and its contribution contains an asymptotic series in K^{-1}. The whole 2-loop correction is a combination of ϕ and the second term in this series, and it turns to be proportional to Casson's invariant. We checked this observation for n-fibered Seifert manifold and found a full agreement with the formula (4.14).

It is worth noting that the change in Casson's invariant under a surgery on a knot depends on the second derivative of the Alexander polynomial of that knot (see e.g. [18] and references therein). This derivative is a second order Vassiliev invariant of the knot and comes as a 2-loop correction to any (i.e., say, either Alexander or Jones) knot polynomial if the latter is calculated through Feynman diagrams. Thus a change in the 2-loop correction under a surgery on a knot depends on a 2-loop invariant of that knot. We could go a step backwards and observe a similar relationship between the self-linking number of a framed knot and the order of homology group (or its logarithm) of the manifold[10] (see e.g. [21]). Both objects can be interpreted as 1-loop corrections. It would be interesting to derive a formula (if it exists) expressing the change in the n-loop correction to the contribution of the trivial connection under a surgery on a knot through Vassiliev invariants of this knot up to order n.

Let us try to conjecture the formula relating Vassiliev invariants of a knot to loop corrections of a trivial connection, basing on our formula (1.8). The higher loop corrections to the contributions of the reducible connections (see Eqs. (1.6) and (5.49)) are proportional to the "derivatives" of the $U(1)$ Reidemeister torsion. This looks rather strange in view of the fact that the flat $U(1)$ connections on Seifert manifolds do not have any moduli along which they could be changed. We therefore propose a different interpretation of these formulas.

Note that an equation

$$\sum_{j=0}^{\infty}\frac{1}{j!}(8\pi iK)^{-j}\left(\frac{P}{H}\right)^{j}\partial_c^{(2j)}f(c)\Big|_{c=c_0} \tag{6.1}$$

$$= e^{i\frac{\pi}{4}\operatorname{sign}\left(\frac{H}{P}\right)}\sqrt{2K\frac{H}{P}}\int_{-\infty}^{+\infty}dc\,f(c)\exp\left[-2\pi iK\frac{H}{P}(c-c_0)^2\right] \tag{6.2}$$

can transform the derivatives in Eqs. (1.6) and (5.47) into an integral over c. Equation (6.1) can be derived either by expanding $f(c)$ in Taylor series at $c=c_0$ and checking it for every term separately, or by noting that the l.h.s. of Eq. (6.1) is an exponential of the 1-dimensional Laplacian, which can be expressed through the heat kernel. In particular, a contribution of the trivial connection (1.8) can be cast in a form

$$-\frac{e^{i\frac{3}{4}\pi\operatorname{sign}\left(\frac{H}{P}\right)}}{2\sqrt{|P|}}\operatorname{sign}(P)e^{-\frac{i\pi}{2K}\phi}\int_{-\infty}^{+\infty}d\beta\, e^{-\frac{i\pi}{2K}\frac{H}{P}\beta^2}\frac{\prod_{l=1}^{3}2i\sin\frac{\pi\beta}{p_l}}{2i\sin\frac{\pi\beta}{K}}\,. \tag{6.3}$$

Here we made a substitution $\beta=2Kc/\pi$.

The formula (6.3) can be given a following interpretation. We can construct a Seifert manifold $X(\frac{p_1}{q_1},\frac{p_2}{q_2},\frac{p_3}{q_3})$ by a surgery on a link consisting of 3 "fiber" loops linked to a "base" loop (see, e.g. [8], [19]). A surgery of the fiber loops produces a connected sum of three lens spaces. We apply a Witten–Reshetikhin–Turaev formula related to the final surgery on the base loop in order to find Witten's invariant

[10] I am thankful to N. Reshetikhin for discussing the results of his research on this subject with me.

of $X(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$. The Jones polynomial of the base loop can be expressed as a sum over flat connections in the connected sum of lens spaces. A contribution of the trivial connection turns out to be proportional to the integrand of (6.3) up to a factor $\sin(\pi\beta/K)$. Therefore expression (6.3) is actually a Witten–Reshetikhin–Turaev formula in which only a trivial connection part of the Jones polynomial is taken and a sum over an integrable weight $\sum_{\beta=1}^{K-1}$ is substituted by an integral $\frac{1}{2}\int_{-\infty}^{+\infty} d\beta$. We conjecture that these two changes in the Witten–Reshetikhin–Turaev formula produce a trivial connection contribution to Witten's invariant of the manifold constructed by a simple T^pS surgery on a knot belonging to some other manifold (in case of a general surgery (2.16), only the $n = 0$ term should be retained in the sum of Eq. (2.25)).

Consider now a logarithm of the trivial connection part of the Jones polynomial of a knot. According to [6], the coefficients in its expansion in powers of $1/K$ are Vassiliev's invariants of the knot. A 1-loop piece in this expansion, which is proportional to $\pi\beta^2/K$, comes from the self-linking number of the knot. This number can be fractional if the original manifold is nontrivial (for example, it is equal to H/P in (6.3)). We conjecture that in the other terms appearing in the logarithm, the power of β is less or equal to the negative power of K. Therefore, if we split off the self-linking exponential factor and expand the remaining part of the Jones polynomial in $1/K$, then the gaussian integral in the modified Witten–Reshetikhin–Turaev formula will produce the $1/K$ expansion of the trivial connection contribution to Witten's invariant of the new manifold. Each term in this expansion will be expressed through a finite number of Vassiliev invariants of the knot. We hope to present this calculation in more details in a forthcoming paper. Here we just want to mention that its result seems to agree with Walker's formula [18] for the Casson invariant (if we assume that the Casson invariant is indeed proportional to a 2-loop correction). This is partly due to the fact that the second derivative of the Alexander polynomial is also a 2-loop correction to the Jones polynomial, as established by D. Bar-Natan in his paper [6].

Acknowledgements. I want to thank Profs. D. Auckly, R. Gompf, C. Gordon, N. Reshetikhin and H. Saleur for many valuable discussions. I am especially indebted to Profs. D. Freed and A. Vaintrob for their help, consultations and encouragement. I am thankful for the referee's comments which helped me to improve the text.

Appendix

We present a simple finite dimensional example which illustrates the appearance of the factor $1/\mathrm{Vol}(H)$ in the gauge invariant theory. Consider a 2-dimensional integral

$$I_{\text{gauge}} = \int_{-\infty}^{+\infty} dX_1 \int_{-\infty}^{+\infty} dX_2 \exp[2\pi i k\, f(\sqrt{X_1^2 + X_2^2})] \tag{A.1}$$

for some function $f(r)$. The integrand of this integral is obviously invariant under the $U(1)$ rotation around the origin. Let us treat it as a gauge symmetry. Then a "physical" quantity would be an integral I_{gauge} divided by the volume of the gauge group $\mathrm{Vol}(U(1)) = 2\pi$. A full machinery of Faddeev–Popov gauge fixing will lead to a well known expression for the "physical" integral:

$$I_{\text{phys}} = \frac{I_{\text{gauge}}}{\text{Vol}(U(1))} = \int_0^\infty dr\, r\, \exp(2\pi i k f(r))\,. \tag{A.2}$$

In order to parallel the discussion of Subsect. 2.1 we use a stationary phase approximation. Suppose that the function $f(r)$ has a critical point at $r = r_0$, i.e. $f'(r_0) = 0$. We want to take an integral (A.1) over the vicinity of that point. We take a representative point on the gauge orbit:

$$X_1 = r_0, \quad X_2 = 0\,. \tag{A.3}$$

This is our "background gauge field." A simplest choice for the gauge fixing condition to impose on the fluctuations (x_1, x_2) around the background (A.3), would be $x_2 = 0$. However we make a different choice:

$$kx_i v^i(X_0, X_1) = 0\,, \tag{A.4}$$

here v^i is a vector field representing the infinitesimal gauge transformation:

$$v^i(X_1, X_2) = \varepsilon^{ij} X_j\,. \tag{A.5}$$

The gauge fixing (A.4) closely resembles a background covariant gauge fixing $D_\mu a_\mu$ of the Chern–Simons theory used in [1]. Indeed, for an infinitesimal gauge transformation ϕ, the analog of v^i is $D_\mu \phi$ and Eq. (A.4) is similar to a condition

$$\int (D_\mu \phi) a_\mu\, d^3x = 0 \tag{A.6}$$

for any function ϕ, which is equivalent to $D_\mu a_\mu = 0$.

By substituting Eqs. (A.5) and (A.3) into Eq. (A.4) we get an explicit form of the covariant gauge fixing condition:

$$kx_2 r_0 = 0\,. \tag{A.7}$$

A Faddeev–Popov ghost determinant is a variation of the gauge fixing condition with respect to the gauge transformation:

$$\Delta_{\text{gh}} = kr_0^2\,. \tag{A.8}$$

We supplement a quadratic term $i\pi k f''(r_0) x_1^2$ in the exponent of Eq. (A.1) with a gauge fixing term $2\pi i k r_0 y x_2$. An integral over y produces a δ-function of the condition (A.7). Therefore an operator corresponding to a quadratic form in the exponent of Eq. (A.1) is

$$L_- = ik \begin{pmatrix} f''(r_0) & 0 & 0 \\ 0 & 0 & r_0 \\ 0 & r_0 & 0 \end{pmatrix}\,. \tag{A.9}$$

The 1-loop "field-theoretic" prediction of the physical integral I_{phys} is

$$I_{\text{phys}} = \frac{\det \Delta_{\text{gh}}}{\sqrt{\det(-L_-)}} e^{2\pi i k f(r_0)} = e^{i\frac{\pi}{4}} \frac{r_0}{\sqrt{k f''(r_0)}} e^{2\pi i k f(r_0)} \tag{A.10}$$

in full agreement with the stationary phase approximation of the integral in the r.h.s of Eq. (A.2).

Let us see now what happens in the special case when $r_0 = 0$. A background point (A.3) lies at the origin and is invariant under the action of $U(1)$. In other words, a "background field" has a $U(1)$ symmetry, so it is similar to a reducible gauge connection of the Chern–Simons theory. A ghost determinant (A.8) has a zero mode. The operator L_- is different from its ordinary form (A.9):

$$L_- = ik \begin{pmatrix} f''(0) & 0 & 0 \\ 0 & f''(0) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.11}$$

and it also has one zero mode. The prescription for the Reidemeister torsion (2.5) in a similar situation was to drop the zero modes. Since no modes are left for the ghosts, we have

$$\Delta_{\mathrm{gh}} = 1\,, \tag{A.12}$$

while

$$\det(-L_-) = -[kf''(0)]\,. \tag{A.13}$$

A non-degenerate part of the operator (A.11) is the same as if we were calculating the stationary phase approximation of the integral (A.1) at the origin without remembering the $U(1)$ symmetry. The same is suggested by the ghost determinant (A.12). In other words, we see that by dropping the zero modes of ghost determinant and covariant gauge fixing we "forgot" about the $U(1)$ symmetry. Therefore if we substitute expressions (A.11) and (A.12) in the middle part of Eq. (A.10), then we will get the whole integral I_{gauge} rather than its physical "gauge fixed" counterpart I_{phys}. So in order to get I_{phys} from the determinants (A.11) and (A.12) we have to add a factor $1/\mathrm{Vol}(U(1))$ "by hands" as we did in Eq. (2.9).

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Communicated by N.Yu. Reshetikhin