Commun. Math. Phys. 168, 479 – 491 (1995)

Communications in Mathematical Physics

Statistical Mechanics of Nonlinear Wave Equations (4): Cubic Schrödinger

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Received: 23 June 1993/in revised form: 24 May 1994

Abstract: The cubic Schrödinger equation is considered on the circle, both in the de-focussing and the focussing case. The existence of the flow is proved together with the invariance of the appropriate Gibbsian measure, namely the petit canonical measure in the defocussing case and the micro-canonical measure in the focussing case.

1. Introduction

McKean–Vaninsky [1994(1)] discussed the petit canonical resemble for wave equations $\Box Q = \partial^2 Q/\partial t^2 - \partial^2 Q/\partial x^2 = -f(Q)$ of classical type, both on the circle $0 \leqq x < L$ and also for $L \uparrow \infty$. The force $f(Q)$ is odd and of the same signature as Q, *i.e.*, it is a restoring force; also, it is so large that $\int_0^\infty e^{-LF(h)}dh < \infty$ for $F(Q) = \int_0^Q f$. Let[1] $Q^\bullet = P$ and $H = (1/2)\int_0^L [P^2 + (Q')^2] + \int_0^L F(Q)$. Then, with a suitable interpretation of this object, the Gibbsian petit canonical measure $e^{-H}d^\infty P d^\infty Q$ is of total mass $Z < \infty$ and is invariant under the flow $Q^\bullet = P = \partial H/\partial P, P^\bullet = Q'' - f(Q) = -\partial H/\partial Q$ of $\Box Q = -f(Q)$; in particular, the flow exists for almost every choice of data from the petit ensemble.[2]

The present paper deals with nonclassical (dispersive) waves in the special case of the cubic Schrödinger equation:

$$Q^\bullet = -P'' \pm (P^2 + Q^2)P = \partial H/\partial P, \quad P'' = +Q'' \mp (P^2 + Q^2)Q = -\partial H/\partial Q$$

with

$$H = \frac{1}{2}\int_0^L [(P')^2 + (Q')^2]dx \pm \frac{1}{4}\int_0^L (P^2 + Q^2)^2 dx\,.$$

The work reported here was performed at the Courant Institute of Mathematical Sciences, with the partial support of the National Science Foundation under NSF Grant No. DMS-9112654, which is gratefully acknowledged.

[1] $\bullet$ signifies $\partial/\partial t$.

[2] Compare Friedlander [1985].

It owes its inception to Lebowitz–Rose–Speer [1989] who introduced the petit canonical ensemble:

$$e^{-H}d^{\infty}Pd^{\infty}Q = \frac{e^{-(1/2)\int(P')^2}}{(2\pi 0+)^{\infty/2}}d^{\infty}P \times \frac{e^{-(1/2)\int(Q')^2}}{(2\pi 0+)^{\infty/2}}d^{\infty}Q \times e^{\mp(1/4)\int(P^2+Q^2)^2}$$

and studied the possibility of phase changes in the temperature-dependent variant with $e^{-H/T}$ in place of e^{-H}. The meaning of this formal object is easy to explain: for example, the second factor signifies that Q is "circular Brownian motion," *i.e.*, it is standard Brownian motion starting at $Q(0) = m$, conditioned so as to come back to m at $x = L$, this common value being distributed over the line according to the law $(2\pi L)^{-1/2}dm$ of total mass $+\infty$: in symbols,

$$\int I(Q)\frac{e^{-(1/2)\int(Q')^2}}{(2\pi 0+)^{\infty/2}}d^{\infty}Q = \int_{-\infty}^{\infty} E_{00}[I(Q+m)]\frac{dm}{\sqrt{2\pi L}}$$

for nice functions $I(Q)$ of $Q(x); 0 \leqq x < L, E_{00}$ being the mean for the tied Brownian motion with $Q(0) = 0 = Q(L)$. The first factor has the same interpretation. The third is just a density, having a proper sense because the Brownian path is continuous. The ensemble is of total mass $Z < \infty$ if the upper (defocussing) sign is chosen. Contrariwise, the lower (focussing) sign produces infinite total mass, prompting Lebowitz–Rose–Speer [1989] to introduce a micro-canonical ensemble whereby the measure is restricted by fixing the value of the constant of motion $\int_0^L (P^2 + Q^2) = N$.[3] This ensemble, too, has total mass $Z < \infty$ even if $(P^2 + Q^2)^2$ is replaced by $(P^2 + Q^2)^{3-}$, as they found; see 4, below.

Thus far, the petit and micro-canonical ensembles. The rest of the paper is devoted to the proof that, for $L < \infty$, the flow makes sense in these ensembles and leaves them invariant. The method is novel: it relies upon a criterion for compactness of measures in path-space due to Kolmogorov–Čentsov, *viz.*, if for some fixed $\alpha > 0$ and $\beta > 1$, $E|Q(x+h) - Q(x)|^{\alpha} \leqq h^{\beta}$ for $0 \leqq x \leqq 1$ and $0 \leqq h < 1$, then $P[|Q(\bullet + h) - Q(\bullet)|_{\infty} \leqq ch^{\gamma}, 0 \leqq h < 1] > 1 - \varepsilon$ for fixed $\gamma < (\beta - 1)/\alpha$ and some universal constant c depending on ε alone; see, for example, McKean [1969:16].

Bourgain [1992] proves the existence of the (individual) flow in H^0 by a clever mixture of trigonometrical series and number theory. Strichartz [1977] had proved

$$\int_0^1 dt \int_{-\infty}^{\infty} dx |e^{\sqrt{-1}tD^2} f(x)|^6 \leqq \text{ constant } \times \left[\int_{-\infty}^{\infty} |f(x)|^2 dx\right]^3,$$

and used it to verify the existence of the flow on the whole line. The estimate owes its validity to dispersion, and that does not operate on the circle. Bourgain [1992] replaced it by

$$\int_0^1 dt \int_0^L dx |e^{\sqrt{-1}tD^2} \mathrm{p} f(x)|^3 \leqq \exp[c \lg(\lg d) \times \left[\int_0^L |f(x)|^2 dx\right]^3$$

[3] LRS [1989] actually took $I = \int(P^2 + Q^2) \leqq N$ instead of $I = N$; in fact the latter, which is preferred here, is no more delicate than the former.

in which $\mathfrak{p}$ projects onto the span of $e^{2\pi\sqrt{-1}nx/L} : |n| < d$. In the present paper, an averaged version of Strichartz's estimate makes its appearance: if $I(PQ)$ is any polynomial of P and Q, of fixed degree $d \geqq 1$, then

$$M_0\left|\begin{matrix}\cos\\ \sin\end{matrix}(tD^2)I(PQ)\right|^6 \leqq c(d)[M_0|I(PQ)|^2]^3,$$

in which M_0 is the mean based upon the "free" ensemble $e^{-H_0}d^\infty P d^\infty Q$ with $H_0 = (1/2)\int[(P')^2 + (Q')^2 + P^2 + Q^2]$, and the constant depends on the degree only, growing like a factorial for $d \uparrow \infty$. The reader may deduce this estimate from the sample in Step 4.3 below. Unlike Bourgain's estimate, it is not destabilized by sharper trigonometrical approximation, and, as the basic non-linearity is of fixed degree (3), so it has an advantage. It is the key to the existence of the flow, with probability 1, in the petit ensemble. Naturally, Bourgain's individual flow is better, but the present method has much to recommend it: besides being easier, it is insensitive to the degree of the non-linearity, unlike the individual flow which must respect the "critical exponent" ($n = 2$) in $(P^2 + Q^2)^n(P - Q)$. Bourgain [1994] himself has considered the invariance of the petit ensemble and has made remarkable progress in the case of 2 spatial dimensions, as well.

The thermodynamic limit $L \uparrow \infty$ presents no difficoutly for the upper (defocussing) signature: the petit ensemble, normalized by Z^{-1}, tends to the law of the 2-dimensional stationary diffusion with infinitesimal operator $\mathfrak{G} = (1/2)(\partial^2/\partial P^2 + \partial^2/\partial Q^2) + m \cdot grad$, in which m is the logarithmic gradient of the ground state of $\mathfrak{G}_0 = -(1/2)(\partial^2/\partial P^2 + \partial^2/\partial Q^2) + (1/4)(P^2 + Q^2)^2$; compare McKean–Vaninsky [1994(1)]. The corresponding microcanonical ensemble with $\int(P^2 + Q^2) = N$, fixed $D = N/L$, and $L \uparrow \infty$ can also be described: the only change is that a constant multiple $P^2 + Q^2$ must be added to $\mathfrak{G}_0$ and the constant adjusted, in conformity with Gibbs' postulate, so that the stationary mean of $P^2 + Q^2$ has the prescribed value D; compare McKean–Vaninsky [1994(2)]. Contrariwise, the lower (focussing) signature seems really hard. Now $(1/4)\int(P^2 + Q^2)$ with its bad sign acts as a repulsive force, counteracted by the microcanonical fiat $\int(P^2 + Q^2) = N = DL$, and the outcome of this competition is not at all clear. Lebowitz–Rose–Speer [1989] present numerical evidence that the thermodynamic limit exists and suggest the fascinating possibility that the temperature-dependent microcanonical ensemble $e^{-H/T}d^\infty P d^\infty Q$ favors radiation/solitons at high/low temperature, i.e., that there is a phase change or "softening" here. No proof is known. Here, it is noted only that the ensemble can be reduced to a simpler one for $L \uparrow \infty$. The microcanonical law is based upon a 2-dimensional circular Brownian motion. This may be expressed in polar coordinates as a skew product $[R, \theta(T)]$ in which $R = \sqrt{P^2 + Q^2}$ is the (circular) 2-dimensional Bessel process BES(2) and the angle $\theta(T)$ is an independent 1-dimensional Brownian motion $BM(1)$, reduced modulo 2π and run with the clock $T = \int_0^x R^{-2}$. Now, conditional on the radial path, the return of the angle, at $x = L$, to its value at $x = 0$ costs[4]

$$\sum_{-\infty}^{\infty} \frac{e^{-(2\pi n)^2/2T(L)}}{\sqrt{2\pi T(L)}} = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{-n^2T(L)/2} = 1 + o(1)$$

in view of $\left[T(L) = \int_0^L R^{-2}\right] \times \left[\int_0^L R^2 = LD\right] \geqq L^2$, the moral being that the microcanonical ensemble behaves like the skew product of 1) BES(2) with weight

[4] Poisson summation is used.

$\exp\left[(1/4)\int R^4\right]$, microcanonical restriction $\int R^2 = N = DL$, and $L \uparrow \infty$, and 2) BM(1) reduced modulo 2π with clock $T = \int_0^x R^{-2}$. This reduced problem awaits investigation. It must also be confessed that the existence of the flow for $L = \infty$ is not known in any of these ensembles: for wave equations of classical type, propagation speed 1 does the trick, but now that is lost.

2. De-focussing Case

A cut-off tames the nonlinearity by suppressing high wave-numbers; later, it will be removed with the help of Kolmogorov–Čentsov. Let the perimeter of the circle be fixed at $L = 1$ (it plays no role), and (by abuse of notation) let $\mathfrak{p}$ be the Féjer operator:[5]

$$\mathfrak{p}: Q \to \frac{1}{d}\sum_{m=0}^{d-1} \sum_{|n|\leqq m} \widehat{Q}(n) e^{2\pi\sqrt{-1}nx} = \int_{-1/2}^{1/2} Q(x+x')\frac{\sin^2(\pi d x')}{d \times \sin^2(\pi x')} dx',$$

noting $\mathfrak{p}^\ddagger = \mathfrak{p}$ and also $\mathfrak{p}1 = 1$. Introduce, too, the cutoff Hamiltonian

$$H_d =^6 \frac{1}{2}\int_0^1 [(P')^2 + (Q')^2 + P^2 + Q^2]\, dx + \frac{1}{4}\int_0^1 \mathfrak{p}^\downarrow (P^2 + Q^2 - 1)^2 dx,$$

with its associated vector field

$$X_d =^7 [1 - D^2 + \mathfrak{p}^\uparrow(P^2 + Q^2 - 1)]J: \begin{array}{l} Q \to -P'' + P + \mathfrak{p}^\uparrow(P^2 + Q^2 - 1)P \\ P \to Q'' - Q - \mathfrak{p}^\uparrow(P^2 + Q^2 - 1)Q \end{array}$$

and petit canonical measure

$$M_d = (2\pi 0+)^{-\infty} e^{-(1/2)\int [(P')^2 + (Q')^2 + P^2 + Q^2]} d^\infty P d^\infty Q \times e^{-(1/4)\int \mathfrak{p}^\downarrow (P^2+Q^2-1)^2},$$

and observe that everything splits according to the splitting of P and Q, by projection onto low harmonics $e^{2\pi\sqrt{-1}nx} : |n| < d$, and by co-projection onto high harmonics $e^{2\pi\sqrt{-1}nx} : |n| \geqq d$. The low harmonics form a classical $(2d-1)$-dimensional Hamiltonian system with its private Hamiltonian

$$H_- = \frac{1}{2}\sum_{|n|<d} (1 + 4\pi^2 n^2)\left[|\widehat{P}(n)|^2 + |\widehat{Q}(n)|^2\right] + \frac{1}{4}\int \mathfrak{p}^\downarrow (p^2 + Q^2 - 1)^2 dx :$$

it preserves the associated classical petit ensemble and also its contribution to the constant of motion $N = \int (P^2 + Q^2)$. The high harmonics have the simpler Hamiltonian

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[5] $\widehat{Q}(n) = \int_0^1 Q(x) e^{-2\pi\sqrt{-1}nx} dx$ *etc.* The present $\mathfrak{p}$ is preferred to the projection because it reduces $|Q|_\infty$.

[6] $\mathfrak{p}^\downarrow I(PQ)$ means $I(\mathfrak{p}P, \mathfrak{p}Q)$.

[7] $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathfrak{p}^\uparrow$ has an extra $\mathfrak{p}$ in front, *i.e.*, $\mathfrak{p}^\uparrow I = \mathfrak{p}(\mathfrak{p}^\downarrow I)$.

$$H_+ = \frac{1}{2} \sum_{|n| \geqq d} (1 + 4\pi^2 n^2) \left[|\widehat{P}(n)|^2 + |\widehat{Q}(n)|^2 \right] ;$$

here also the associated petit ensemble is preserved since $\widehat{P}(n)$ and $\widehat{Q}(n)$: $n \geqq d$ are independent isotropic Gaussian variables with (absolute) mean squares $(1 + 4\pi^2 n^2)^{-1}$ and these statistics are undisturbed by the individual flows induced by $\exp t(1 - D^2)J$:

$$\begin{matrix} \widehat{Q}(n) \\ \widehat{P}(n) \end{matrix} \rightarrow \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} [(1 + 4\pi^2 n^2)t] \begin{matrix} \widehat{Q}(n) \\ \widehat{P}(n) \end{matrix} .$$

The preservation of the full ensemble M_d under the flow of X_d will be plain from these remarks. It remains only to remove the cut-off with the help of Kolmogorov–Čentsov. The free (Gaussian) measure

$$M_0 = (2\pi 0+)^{-\infty} e^{-(1/2)\int [(P')^2 + (Q')^2 + P^2 + Q^2]} d^\infty P d^\infty Q$$

is used for reference; it exceeds M_d for every $d \geqq 1, d = \infty$ included. The constants c_1, c_2, *etc.* appearing below are independent of d, t, x – whatever.

Step 1.

$$M_d \left| e^{tX_d} \frac{Q}{P}(x+h) - e^{tX_d} \frac{Q}{P}(x) \right|^4 \leqq c_1 h^2 .$$

Proof. M_d is preserved by the flow and also by rotations of the circle, so you may reduce to $t = x = 0$; also, $M_d \leqq M_0$, and as the latter is Gaussian and symmetric in Q and P, so the mean in question is overestimated by a universal multiple of $M_0|Q(h) - Q(0)|^2$ *squared.*[8] Now compute

$$\begin{aligned} & M_0|Q(h) - Q(0)|^2 \\ & = \int_{-\infty}^{\infty} \frac{dm}{\sqrt{2\pi}} E_{00}[e^{-(1/2)\int (Q+m)^2} Q^2(h)] \\ & = E_{00}[e^{-(1/2)\int (Q-\overline{Q})^2} Q^2(h)] \\ & \leqq E_{00}[Q^2(h)] = h(1-h) , \end{aligned}$$

the substitution $m \rightarrow m - (\overline{Q} = \int_0^1 Q)$ being used *inside* the expectation in line 2.

Step 2.

$$M_d \left| e^{(t+h)X_d} \frac{Q}{P}(x) - e^{tX_d} \frac{Q}{P}(x) \right|^6 \leqq c_2 h^{3/2} .$$

Proof. Reduce to $t = x = 0$ as before and note that

$$(e^{hX_d} - 1)\frac{Q}{P} = (e^{hX_0} - 1)\frac{Q}{P} +^9 \int_0^h e^{(h-t)X_0} (e^{tX_d})^{\downarrow} \mathfrak{p}^{\uparrow} (P^2 + Q^2 - 1)] J \frac{Q}{P} dt$$

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[8] M_0 is Gaussian and $\int_{-\infty}^{\infty} x^4 e^{-x^2/2\sigma^2} (2\pi\sigma^2)^{-1/2} = 3\sigma^4$.

[9] $e^{(h-t)X_0}$ acts *outside* in the conventional way. $(e^{tX_d})^{\downarrow}$ acts *inside, i.e.,* it replaces P and Q by $e^{tX_d}P$ and $e^{tX_d}Q$ throughout.

with $X_0 =$ the free field $(1-D^2)J$, and $M_d \leqq M_0$. The first piece of the right-hand sum is overestimated by a multiple of $M_0|(e^{hX_0}-1)Q|^2$ *cubed.*[10] Now compute

$$\frac{1}{2}M_0|(e^{hX_0}-1)Q(0)|^2 = \sum_{Z}\frac{1-\cos(1+4\pi^2n^2)h}{1+4\pi^2n^2} \leqq c_3 h^{1/2}\,.$$

It remains to estimate

$$M_d\left|\int_0^h e^{(h-t)X_0}(e^{tX_d})^{\downarrow}\mathfrak{p}^{\uparrow}(P^2+Q^2-1)J\frac{Q}{P}dt\right|^6$$
$$\leqq h^5\int_0^h M_d\left|e^{(h-t)X_0}(e^{tX_d})^{\downarrow}\mathfrak{p}^{\uparrow}(P^2+Q^2-1)J\frac{Q}{P}\right|^6 dt$$
$$\leqq^{11}\ h^5\int_0^h M_0\left|e^{tX_0}\mathfrak{p}^{\uparrow}(P^2+Q^2-1)J\frac{Q}{P}\right|^6 dt\,.$$

The discussion is broken off at this point to prepare the averaged version of Strichartz's inequality cited above.

Step 3. The Gaussian character of M_0 permits the universal estimate

$$M_0\left|e^{tX_0}\frac{Q}{P}\right|^6 \leqq c_4\left[M_0\left|e^{tX_0}\frac{Q}{P}\right|^2\right]^3 = c_4\left[M_0\left|\frac{Q}{P}\right|^2\right]^3,$$

the final equality being the result of the rotation invariance of M_0 and the fact that e^{tX_0} is an orthogonal transformation:

$$M_0\left|e^{tX_0}\frac{Q}{P}(0)\right|^2 = M_0\int_0^1\left|e^{tX_0}\frac{Q}{P}(x)\right|^2dx = M_0\int_0^1\left|\frac{Q}{P}(x)\right|^2dx = M_0\left|\frac{Q}{P}(0)\right|^2\,.$$

It is desired to extend this type of thing to cubics such as $\mathfrak{p}^{\uparrow}(P^2+Q^2)Q$ so as to confirm that $M_0|e^{tX_0}\mathfrak{p}^{\uparrow}(P^2+Q^2)-1)J\frac{Q}{P}|^6$ lies under a fixed bound, independently of $t \geqq 0$ and $d \geqq 1$. This will complete step 2.

Sample Proof. $\cos[t(1-D^2)]\,\mathfrak{p}^{\uparrow}Q^3$ is treated: *in extenso,*

$$\cos[t(1-D^2)]\,\mathfrak{p}^{\uparrow}Q^3(0) = \sum\frac{d-|n|}{d}\cos\,t(1+4\pi^2n^2)\sum_{\substack{p_1+p_2+p_3\\=n}}\prod_{i=1}^{3}\frac{d-|p_i|}{d}\widehat{Q}(p_i)\,,$$

in which n and the p's run from $-d+1$ to $d-1$. The (absolute) sixth power of this (real) quantity takes the form of a sum

$$\sum\prod_{i=1}\frac{d-|n_i|}{d}\cos t(1+4\pi^2n_i^2)\prod_{j=1}^{18}\frac{d-|p_j|}{d}\widehat{Q}(P_j)$$

over certain restricted values of $(p_1,\dots,p_{18})\in Z^{18}$ and now, if the mean M_0 be taken, the only summands that survive have the numbers $p_1,\dots,p_{18}$ paired,

[10] $\int_{-\infty}^{\infty}x^6e^{-x^2/2\sigma^2}dx(2\pi\sigma^2)^{-1/2} = 15\sigma^6$.

[11] M_d is preserved by the flow of X_d and $M_0 \geqq M_d$.

p' to $-p''$,[12] the reason being that under the free measure, the several coefficients $\widehat{Q}(p)$ are independent isotropic Gaussian variables of the form $(1+4\pi^2p^2)^{-1/2}\int_0^1 e^{-2\pi\sqrt{-1}px}$ (white noise) dx. The mean is now expressed as a sum of products

$$\prod_{i=1}^{6}\frac{d-|n_i|}{d}\cos t(1+4\pi^2n_i^2)\prod_{j=1}^{9}\left(\frac{d-|p_j|}{d}\right)^2 M_0|\widehat{Q}(p_j)|^2$$

over certain restricted values of $(p_1,\ldots,p_9)\in Z^9$, and striking out the cosines produces the averaged variant of Strichartz's inequality:

$$M_0|\cos[t(1-D^2)]\mathfrak{p}^{\uparrow}Q^3(0)|^6 \leqq M_0|\mathfrak{p}^{\uparrow}Q^3(0)|^6 \leqq M_0Q^{18}(0)\,.$$

The details are not important now: you need only carry away the fact that the bound is independent of both $t \geqq 0$ and $d \geqq 1$.

Step 4. The variable t is now limited (temporarily) to the interval $[0, 1]$, say, and Steps 1 and 2 are used to guarantee that, if $0\leqq c<\infty$ is large enough, then, *independently of* $d \geqq 1, M_d$ is nearly concentrated on the *compact* class of paths $Z\colon [0,1]^2\to R^2$ with

$$|Z(t_2,x_2)-Z(t_1,x_1)| \leqq c[|t_2-t_1|^{1/4-}+|x_2-x_1|^{1/12-}]\,,$$

exemplified by $Z(t,x)=e^{tX_d}\,{}^{Q}_{P}(x)$.[13] Now change the point of view, regarding the paths as secondary and the measure M_d^* induced on them as primary, *i.e.*, fix the paths and encode the flow into the measure. Pick $d=d_1<d_2<etc.\uparrow\infty$ so that M_d^* tends weakly to some measure M_∞^* and let $I(QP)$ be a nice function of $(QP)=Z(0,x): 0\leqq x\leqq 1$. Then

$$\begin{aligned} M_\infty^* I[Z(0,\bullet)] &= \lim_{d\uparrow\infty} M_d^* I[Z(0,\bullet)] = \lim_{d\uparrow\infty} M_d I(PQ) \\ &= \lim_{d\uparrow\infty} M_d (e^{tX_d})^{\downarrow} I(PQ) = \lim_{d\uparrow\infty} M_d^* I[Z(t,\bullet)] = M_\infty^* I[Z(t,\bullet)]\,, \end{aligned}$$

i.e., M_∞^* is invariant under the shift $Z(0,\bullet)\to Z(t,\bullet)$. But what is that?

Step 5 provides the answer: it is nothing but the flow $Z^\bullet = X_\infty Z = (-D^2+Z^2)JZ$ in a weak form. Fix $d\leqq\infty$. Then the weak flow $Z^\bullet = X_dZ = [1-D^2+\mathfrak{p}^{\uparrow}(Z^2-1)]JZ$ is expressed by the vanishing of

$$\int_{-\infty}^{\infty} dt\int_0^1 dx\left[\frac{\partial\phi}{\partial t}+\left(1-\frac{\partial^2}{\partial x^2}+\mathfrak{p}^{\uparrow}(Z^2-1)\right)\phi J\right]Z$$

for any (sure) compact test function $\phi(t,x)$ of class $C^\infty\ R\times(0,1)\to R^2$, and now the statement follows from

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[12] The recipe will be found in Wiener [1958:11].

[13] This is Kolmogorov–Čentsov. $1/4-$ stands for any fixed number $<1/4=(2-1)/4$; similarly, $1/12-$ is any fixed number $<1/12=(3/2-1)/6$.

$$
\begin{aligned}
M_\infty^* &\left| \int_{-\infty}^{\infty} dt \int_0^1 dx \left[\frac{\partial\phi}{\partial t} Z + \left(-\frac{\partial^2}{\partial x^2} + Z^2 \right) \phi JZ \right] \right|^2 \\
&= \lim_{d\uparrow\infty} M_d^\infty |ditto|^2 \\
&= \lim_{d\uparrow\infty} M_d \left| \int_{-\infty}^{\infty} dt \int_0^1 dx (e^{tX_d})^\downarrow \left[\frac{\partial\phi}{\partial t} Z + \left(-\frac{\partial^2}{\partial x^2} + Z^2 \right) \phi JZ \right] \right|^2 \\
&= \lim_{d\uparrow\infty} M_d \left| \int_{-\infty}^{\infty} dt \int_0^1 dx (e^{tX_d})^\downarrow (1 - \mathfrak{p}^\uparrow)(Z^2) JZ \right|^2 \\
&\leqq c_5 \lim_{d\uparrow\infty} M_0 |(1 - \mathfrak{p}^\uparrow)(Z^2 - 1) JZ|^2 \\
&= 0 \,,
\end{aligned}
$$

for which the elementary estimate $|\mathfrak{p}|_\infty \leqq 1$ provides a domination.

Step 6 improves upon Step 5. M_∞^* lives on paths $Z \in C[0,\infty] \times [0, 1] \to R^2$, so

$$
\frac{\partial}{\partial t} \int_0^1 \phi Z \, dx = \int_0^1 \left[-\frac{\partial^2 \phi}{\partial x^2} + \phi Z^2 \right] JZ \, dx
$$

for any (sure) test function $\phi(x)$ of class $C^\infty(0, 1) \to R^2$. Now put

$$
Z^*(t, \bullet) = e^{-tD^2 J} Z(0, \bullet) + \int_0^t e^{-(t-t')D^2 J} Z^2 J \, Z(t', \bullet) \, dt' \,,
$$

interpreting everything in $L^2[0, 1) \times L^2[0, 1)$, as you may in view of $\int_0^1 Z^6 < \infty$. Then

$$
\int_0^1 \phi Z^* = \int_0^1 e^{tD^2 J} \phi \left[Z(0, \bullet) + \int_0^t e^{t' D^2 J} Z^2 JZ(t', \bullet) \, dt' \right] \,,
$$

so that

$$
\frac{\partial}{\partial t} \int_0^1 \phi Z^* = -\int_0^1 \frac{\partial^2 \phi}{\partial x^2} JZ^* + \int_0^1 \phi Z^2 JZ \,;
$$

in particular,[14] $c_n(t) = (Z - Z^*)^\wedge(n)$ satisfies $c_n^\bullet = 4\pi^2 n^2 c_n$, and as $c_n(0) = 0$, so $c_n(t)$ vanishes for $t \geqq 0$, which is to say that the shift $Z(0, \bullet) \to Z(t, \bullet)$ satisfies the identity

$$
Z(t, \bullet) = e^{-tD^2 J} Z(0, \bullet) + \int_0^t e^{-(t-t')D^2 J} Z^2 J \, Z(t', \bullet) \, dt'
$$

in $L^2[0, 1) \times L^2[0, 1)$. The fact that $Z(t, x)$ is of class $C[0,\infty) \times [0, 1) \to R^2$ is now re-invoked to justify the replacement of $Z^2 JZ$ by $K^\downarrow Z^2 JZ$ with a cut-off such as $KZ = [k(Q), k(P)]$ in which $k(Q) = -N$, or Q, or $+N$ according as $Q \leqq -N$, or $N \leqq Q \leqq +N$, or $Q > +N$. This has no effect on the identity for the shift up to time $t = 1$, say, if $N > max|Z(t, x)| : 0 \leqq t, x \leqq 1$, but now you may solve the cutoff identity for $Z(t, \bullet)$ in terms of $Z(0, \bullet)$ as the fixed point of a contraction in

[14] $\phi(x)$ is taken as $exp(-2\pi\sqrt{-1} nx)$.

$L^2[0,1) \times L^2[0,1)$, and that tells you two things: 1) the shift is nothing but the flow e^{tX} of $Z^\bullet = (-D^2 + Z^2)JZ$, not only in the weak version of Step 5, but in the present more constructive version; 2) $Z(t,\bullet)$ is measurable over the field of $Z(0,\bullet)$. The latter fact was the aim of Step 6.

Step 7 is the punch-line: $\lim_{d\uparrow\infty} M_d = M_\infty$ in variation since $\exp -\frac{1}{4}\int_0^1 \mathfrak{p}^\downarrow (P^2 + Q^2 - 1)^2$ tends to $\exp -\frac{1}{4}\int_0^1 (P^2 + Q^2 - 1)^2$ under the bound 1 where the free measure lives; in particular, $M_\infty^* = M_\infty$ on functions of $Z(0,\bullet) = QP$. Now let I be any nice function of QP, note that $(e^{tX})^\downarrow I(QP)$ is a measurable function I' of QP by Step 6, and compute as follows:

$$
\begin{aligned}
M_\infty I(QP) &= M_\infty^* I(QP) && \text{since } M_\infty = M_\infty^* \text{ on the fixed of } QP\\
&= M_\infty^* (e^{tX})^\downarrow I(QP) && \text{by shift invariance of } M_\infty^*\\
&= M_\infty^* I'(QP) && \text{by definition of } I'(QP)\\
&= M_\infty I'(QP) && \text{by the first reason}\\
&= M_\infty (e^{tX})^\downarrow I(QP) && \text{by the third reason}\,.
\end{aligned}
$$

The discussion is finished: the flow of $Z^\bullet = (-D^2 + Z^2)JZ$ exists (with probability 1) in the petit ensemble and preserves the canonical mean-value $M = M_\infty$. Now on to the focussing case. The difference is surprisingly little.

3. Focussing Case: Preparations

The present section prepares some tricks for the evaluation of micro-canonical averages. Fix $1 \leqq d \leqq \infty$. The micro-canonical measure is

$$M_d = (2\pi 0+)^{-\infty} e^{-(1/2)\int [(P')^2 + (Q')^2]} d^\infty P d^\infty Q \times e^{(1/4)\int \mathfrak{p}^\downarrow (P^2+Q^2)^2}$$

conditioned on the value of the constant of motion $\int (P^2 + Q^2) = N$, i.e.,

$$M_d K(QP) = \frac{M_0[K(QP) e^{(1/4)\int \mathfrak{p}^\downarrow (P^2+Q^2)^2}, \int (P^2+Q^2) = N]}{Z_d = M_0[e^{(1/4)[\mathfrak{p}^\downarrow (P^2+Q^2)^2}, \int (P^2+Q^2) = N]}$$

with the understanding that $\mathfrak{p} = 1$ if $d = \infty$. The reference measure M_0 is now the joint law of 2 independent (circular) Brownian motions, and the peculiar but helpful notation used top and bottom indicates a density, as in $M_0[K(QP), \int (P^2 + Q^2) = N] = (\partial/\partial N) M_0[K(QP), \int (P^2 + Q^2) \leqq N]$.

Step 1. Z_d is bounded from 0 and ∞, independently of $1 \leqq d \leqq \infty$, for any fixed N. The fact is due to Lebowitz–Rose–Speer [1989] for $d = \infty$; they even show that $Z < \infty$ with a small constant $\times (P^2 + Q^2)^3$ in place of $(P^2 + Q^2)^2$, as will appear below, and that these limitations (on constant and power) are sharp.

Item 1. Let E be the double tied Brownian mean $E_{00} \times E_{00}$. Z_d is the ratio of the infinitesimal dN to

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} db\, E\left[e^{(1/4)\int \mathfrak{p}^\downarrow [(P+a)^2 + (Q+b)^2]^2}, \int (P+a)^2 + (Q+b)^2\right] \in dN]\\
&= \int_0^\infty r\,dr \frac{1}{2\pi}\int_0^{2\pi} d\theta\, E[e^{(1/4)\int \mathfrak{p}^\downarrow [(P-\overline{P}+r\cos\theta)^2 + (Q-\overline{Q}+r\sin\theta)^2]}, I + r^2 \in dN]
\end{aligned}
$$

with $\overline{P} = \int_0^1 P$, $\overline{Q} = \int_0^1 Q$, and $I = \int_0^1[(P-\overline{P})^2 + (Q-\overline{Q})^2]$, as you will check by putting $\int\int da\,db$ inside E and changing variables $ab \to \overline{P}\,\overline{Q} + (r,\theta)$, and this expression, in turn, is equal to

$$\frac{dN}{2} \times \frac{1}{2\pi}\int_0^{2\pi} d\theta\, E[e^{(1/4)\int \mathfrak{p}^{\downarrow}[(P-\overline{P}+\sqrt{N-I}\cos\theta)^2+(Q-\overline{Q}+\sqrt{N-I}\sin\theta)^2]^2}, I \leqq N] .$$

$Z_d \geqq \frac{1}{2}P(I \leqq N)$ is immediate; it is the desired lower bound.

Item 2 prepares the way for the upper bound. Z_d is controlled from above by[15]

$$E[e^{3\int \mathfrak{p}^{\downarrow}[(P-\overline{P})^4+(Q-\overline{Q})^4+(N-I)^2]}, I \leqq N]$$
$$\leqq e^{3N^2} \times \left| E\left[e^{3N\max_{0\leqq x\leqq 1}|Q(x)-\overline{Q}|^2}, \int_0^1 (Q-\overline{Q})^2 \leqq N\right]\right|^2$$

in view of the independence of P and Q and the simple estimate[16]

$$\int \mathfrak{p}^{\downarrow}(Q-\overline{Q})^4 \leqq \max \mathfrak{p}^{\downarrow}|Q-\overline{Q}|^2 \times \int |\mathfrak{p}(Q-\overline{Q})|^2 \leqq \max|Q-\overline{Q}|^2 \times N .$$

Step 3. [17] $\int (Q-\overline{Q})^2 \leqq N$ implies that meas(x: $|Q(x)-\overline{Q}| \geqq n/2$) cannot exceed $4N/n^2$, so $|Q-\overline{Q}|_\infty \geqq n$ only if[18] Q suffers a displacement $\geqq n/2$ in some small subinterval of $[0,1)$ of length $4N/n^2$. Let $n^2/4N = m = 1,2,3$, *etc.* for convenience. Then $|Q(x) - Q(j/m)]| \geqq n/4$ for some $0 \leqq j \leqq m$ and some x at distance $\leqq 1/m$ from j/m. Let n, and so also m, be large. Then both x and j/m are distant $\geqq 1/3$ either from $x = 0$ or from $x = 1$. But if B is an event from the field of $Q(x)$: $x \leqq 2/3$, say, then the tried Brownian probability $P_{00}(B) = \sqrt{3/2}\,E_0[e^{-3Q^2(2/3)/4}, B]$ is less than $\sqrt{3/2}P_0(B)$, in which E_0 and P_0 refer to the standard Brownian motion with $Q(1)$ free, and now the (tied) automorphism $Q(x) \to Q(1-x)$ permits you to control $P_{00}[|Q-\overline{Q}|_\infty \geqq n]$ by

$$mP_0\left[\max_{x\leqq 1/m}|Q(x)| \geqq \frac{n}{4}\right] = \sqrt{\frac{2}{\pi}}m^{3/2}\int_{n/4}^{\infty} e^{-mx^2/2}\,dx \leqq \frac{2}{N}e^{-n^4/128N} .$$

The rest will be plain, and you will understand why $Z < \infty$ not only for $(P^2+Q^2)^2$ but for a small constant $\times(P^2+Q^2)^3$, as well.

Step 2. The procedure of Step 1 supplies a nice upper bound for the general microcanonical mean $M_d(K)$: it is overestimated by a fixed multiple of

$$\int_0^{2\pi} d\theta\, E[K(P-\overline{P}+\sqrt{N-I}\cos\theta, Q-\overline{Q}+\sqrt{N-I}\sin\theta)$$
$$\times e^{3\int \mathfrak{p}^{\downarrow}[(P-\overline{P})^4+(Q-\overline{Q})^4]+3N^2}$$
$$\times \text{ the indicator of } I = \int[(P-\overline{P})^2+(Q-\overline{Q})^2] \leqq N]$$

[15] $\mathfrak{p}$ is an *average*; *also* $[(a+b)^2+(c+d)^2]^2 \leqq 12[a^4+c^4+(b^2+d^2)^2]$.
[16] $\mathfrak{p}$ is an *average*, *esp.* $\mathfrak{p}\overline{Q} = \overline{Q}$.
[17] Varadhan helped with this [private communication].
[18] $Q - \overline{Q} = 0$ some place.

$$\leqq \int_0^{2\pi} d\theta \sqrt{E}[K^2(P-\overline{P}+\sqrt{N-I}\cos\theta\, etc.), I \leqq N]$$
$$\times e^{3N^2}$$
$$\times E\left[e^{6[\mathfrak{p}^{\downarrow}(Q-\overline{Q})^4}, \int (Q-\overline{Q})^2 \leqq N\right],$$

in which the last piece may be overestimated, independently of d, in the manner of Step 1.

4. Focussing Case: Micro-Canonical Ensemble

The discussion is not much changed from Sect. 2, only the estimates are a little more elaborate. The existence of the cutoff flow $Z^{\bullet} = (-D^2 + \mathfrak{p}^{\uparrow} Z^2)JZ$ is proved as before: it splits into a free flow at high wave numbers and a $(2d-1)$-dimensional classical Hamiltonian flow at low wave numbers, each preserving its private Hamiltonian and also its contribution to the constant of motion $\int(P^2+Q^2)$. The preservation of the (cutoff) micro-canonical ensemble is obvious from that. Now come the estimates for the Kolmogorov–Čentsov trick, parallel to Steps 2.1, 2, and 3.

Step 1. [19]

$$M_0\left[e^{tX_d}\frac{Q}{P}(x+h) - e^{tX_d}\frac{Q}{P}(x)\right|^4 \leqq c_6 h^2 .$$

Proof. By item 3.2, you have only to control, e.g., $\sqrt{E_{00}}Q^8(h)$ which is proportional to $[E_{00}Q^2(h)]^2 = h^2(1-h)^2$.

Step 2.

$$M_d\left|e^{(t+h)X_d}\frac{Q}{P}(x) - e^{tX_d}\frac{Q}{P}(x)\right|^6 \leqq c_7 h^{3/2} .$$

Proof. Now $(e^{hX_d}-1)\frac{Q}{P} = (e^{hX_0}-1)\frac{Q}{P} + \int_0^h e^{(h-t)X_0}(e^{tX_d})^{\downarrow}\mathfrak{p}^{\uparrow}(P^2+Q^2)J\frac{Q}{P}$. The rest of the proof is divided into 3 small items.

Item 1.

$$M_d\left|(e^{hX_0}-1)\frac{Q}{P}\right|^6$$
$$\leqq^{20} c_8 \int_0^{2\pi} d\theta \sqrt{E}\left|(e^{hX_0}-1)\frac{Q-\overline{Q}+\sqrt{N-I}\cos\theta}{P-\overline{P}+\sqrt{N-I}\sin\theta}\right|^{12}$$
$$\leqq^{21} c_9\sqrt{E}\left|(e^{hX_0}-1)\frac{Q}{P}\right|^{12}$$
$$\leqq c_{10}\left[E\left|(e^{hX_0}-1)\frac{Q}{P}\right|^2\right]^3,$$

by the usual Gaussian trick.

[19] X_d is now $-D^2J + \mathfrak{p}^{\uparrow}(P^2+Q^2)J$.
[20] see Item 3.2.
[21] $e^{hX_0}-1$ kills constants.

Item 2. $E|(e^{hX_0}-1)\frac{Q}{P}|^2 \leqq c_{11}h^{1/2}$, much as in Step 2.2. The point is that the *tied* Brownian motion Q is identical in law to $Q(x)-xQ(1)$ with a *free* Brownian motion Q starting at 0, so $\widehat{Q}_{tied}(n)=(2\pi\sqrt{-1}n)^{-1}\times\int_0^1 e^{-2\pi\sqrt{-1}nx}dQ_{free}$ for $n\neq 0$, and the only change from Step 2.2 is that the correlation $E_{00}[Q(0)e^{hX_0}Q(0)]$ is now

$$2\sum_{n\neq 0}\frac{1-\cos 4\pi^2n^2h}{4\pi^2n^2} \quad \text{instead of} \quad 2\sum_{-\infty}^{\infty}\frac{1-\cos(1+4\pi^2n^2)h}{1+4\pi^2n^2}\,.$$

Item 3.

$$\begin{aligned} &M_d\left|\int_0^h e^{(h-t)X_0}(e^{tX_d})^{\downarrow}\mathrm{p}^{\uparrow}(P^2+Q^2)J\frac{Q}{P}dt\right|^6 \\ &\leqq h^5\int_0^h dt\,M_d\left|e^{tX_0}\,p^{\downarrow}(P^2+Q^2)J\frac{Q}{P}\right|^6 \\ &\leqq^{22} h^5\int_0^h dt\,c_{12}\int_0^{2\pi}d\theta\sqrt{E}\int e^{tX_0}\mathrm{p}^{\uparrow}(P-\overline{P}+\sqrt{N-I}\cos\theta)^2 etc.|^{12}\,, \end{aligned}$$

and this is controlled by a sum of means typified by $E_{00}|\cos(tD^2)\mathrm{p}^{\uparrow}(Q-\overline{Q})^3|^{12}$. It remains to bound such things independently of $t\geqq 0$ and $d\geqq 1$.

Step 3 does that. $Q_{tied}=Q_{free}-xQ_{free}(1)$, as noted in Item 2, and $\hat{Q}_{tied}(n): n\neq 0$ are independent isotropic Gaussian variables. This is the basis of the computation:

$$\begin{aligned} &|\cos(tD^2)\mathrm{p}^{\uparrow}(Q-\overline{Q})|^{12} \\ &=^{23}\left|\sum\frac{d-|n|}{d}\cos(4\pi^2n^2t)\sum_{\substack{p_1+p_2+p_3\\=n}}\prod_{j=1}^{3}\frac{d-|p_j|}{d}(Q-\overline{Q})^{\wedge}(p_j)\right|^{12} \\ &=\sum\prod_{i=1}^{12}\frac{d-|n_i|}{d}\cos(4\pi^2n_i^2t)\sum\prod_{j=1}^{36}\frac{d-|p_j|}{d}(Q-\overline{Q})^{\wedge}(p_j) \end{aligned}$$

with $|n_i|<d$, $|p_j|<d$, and $p_1+p_2+p_3=n_1$, *etc.* in blocks of 3, and if now the expectation E_{00} be taken, the p's must be paired, p' to $-p''$, or else that summand is killed, with the result that the whole sum is overestimated by

$$\sum_{\substack{p\in Z^{18}\\ p_1\cdots p_{18}\neq 0}}\prod_1^{18}\frac{1}{4\pi^2p_j^2}=\left(\sum_{p\neq 0}\frac{1}{4\pi^2p^2}\right)^{18}.$$

The rest is plain: just put this back into Item 3, bounding the expression considered there by $c_{13}h^6$.

The remainder of Sect. 2 is now repeated *verbatim*. The proof of the existence of the flow in the micro-canonical ensemble and of the invariance of the latter under the former is finished.

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22 see Item 3.2.

23 n and the p's run from $-d+1$ to $d-1$.

Note Added in Proof. P.E. Zhidkov has kindly pointed out the following references to his work: *Dokl. Akad. Nauk CCCP* **317** (1991); *Sov. Math. Dokl.* **43**, 431–433 (1991); *Nonlinear Anal.* **22**, 319–325 (1994). The first establishes the invariant measure for the classical wave equation $\partial^2 Q/\partial t^2 - \partial^2 Q/\partial x^2 + f(Q) = 0$, as in McKean–Vaninsky [1994 (1)], but only for restoring force $f(Q)$ comparable to Q. The second announces the result of the present paper with a like restriction on the nonlinearity, *esp.*, the cubic is excluded.

Acknowledgements. It is a pleasure to thank S.R.S. Varadhan for encouragement on several occasions and also J. Shatah for his continued friendly interest in this work.

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Communicated by A. Jaffe