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Spectral Sequences and Adiabatic Limits

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Abstract: A Taylor series analysis of the Laplacian as the underlying manifold is deformed leads to a Hodge theoretic derivation of the Leray spectral sequence.

0. Introduction

Suppose (M, g) is a compact Riemannian manifold with a smooth distribution of n-planes A. Let B be the orthogonal distribution to A. Writing

$$g = g_A \oplus g_B ,$$

we define a 1-parameter family of metrics on M by setting, for $0 < \delta \leqq 1$,

$$g_\delta = g_A \oplus \delta^{-2} g_B .$$

In addition, let

$$V \to M$$

be a flat bundle,

In this paper we investigate a spectral sequence associated with A and B for the cohomology of M with values in V. We show in Sects. 2 and 3 how the spectral sequence arises naturally from a Taylor series analysis of the eigenvalues of $\square_\delta^p$ near $\delta = 0$ (where $\square_\delta^p$ denotes the Laplacian induced by the metric g_δ acting on p-forms of M with values in V). We demonstrate how the algebraic properties of the spectral sequence can be proved using standard Hodge theory. In Sect. 4 we show that our spectral sequence is intimately related to the Leray spectral sequence associated to a filtered differential complex. In addition, of A is integrable, the spectral sequence is isomorphic to the standard Leray spectral sequence associated to the foliation A. If A is integrable, and in addition satisfies certain geometric restrictions (see hypotheses (H1) and (H2)), we show in Sect. 5 that the leading order asymptotics of the small eigenvalues of $\square_\delta^p$, and the corresponding eigenspaces, are determined by

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information contained in the spectral sequence. In the special case that the splitting of TM arises from a fibration, our results overlap with those of [Ma-Me] and [Dai].

The limit of (M, g_δ) as $\delta \to 0$ is known as the "adiabatic limit." The adiabatic limit was introduced in this form by Witten in [Wi]. He considered the foliation F consisting of the fibers of a fibration

$$\mathscr{F} \hookrightarrow M \to N\,, \tag{0.1}$$

where $\mathscr{F}$ is compact, $N = S^1$, and the metric g makes (0.1) a Riemannian submersion. Witten investigated the limit of the eta-invariant of M and δ approached 0. This question was also considered in [Bi-Fr] and [Ch]. In [Bi-Ch] and [Dai], this investigation was extended to general base spaces N.

Our topic begins with (and this paper owes much to the ideas in) [Ma-Me] in which the authors, starting with a fibration as in (0.1), analyse the behavior of the space of harmonic forms on M as $\delta \to 0$. They show that, modulo a change of coordinates, the space of harmonic p-forms approaches a finite dimensional space $\overline{E}^p_\infty$. This space $\overline{E}^p_\infty$ can be identified from a Taylor series analysis as follows: Define a nested family of spaces

$$\overline{E}^p_0 \supseteq \overline{E}^p_1 \supseteq \overline{E}^p_2 \supseteq \dots$$

by

$$\begin{aligned}\overline{E}^p_k =&\{p\text{-forms } \omega | \exists p\text{-forms } \omega_1, \dots, \omega_j \text{ with} \\ &\Box^p_\delta(\omega + \delta\omega_1 + \dots + \delta^j\omega^j) \in 0(\delta^k)\}\,.\end{aligned}$$

They proved that there is an N such that

$$\overline{E}^p_N = \overline{E}^p_{N+1} = \dots = \overline{E}^p_\infty\,,$$

and this is the space referred to above. These results follow from their construction, using Melrose's calculus of pseudodifferential operators on manifolds with corners, of a parametrix for $\Box^p_\delta$ which has a uniform extension to the closed interval [0,1]. This implies that in the case of a fibration, the eigenvalues and eigenvectors have well-defined asymptotics as $\delta \to 0$.

In this paper, we take a simpler, and more general approach. We begin with any distribution. $A \subset TM$. In particular, we do not require that A arise from a fibration, or even that A be integrable. We show that a Taylor series analysis, motivated by the adiabatic limit, leads directly to a Hodge theoretic spectral sequence which converges to the cohomology of M. We observe that the spectral sequence structure, i.e. the associated differentials and bigradings (which do not appear in [Ma-Me]), arise naturally in our context.

We start with a rescaling map ρ_δ (see Lemma 1.2), which also appears implicitly in [Ma-Me], which is an isometry

$$\rho_\delta : (\Omega^p(V), g_\delta) \to (\Omega^p(V), g)$$

(where $\Omega^p(V)$ denotes p-forms on M with values in V). Let

$$d_\delta = \rho_\delta d \rho_\delta^{-1}, \quad d^*_\delta = \rho_\delta d^*_{(g_\delta)} \rho_\delta^{-1}\,,$$

and define a nested sequences of spaces

$$E_{-1} \supseteq E^p_0 \supseteq E^p_1 \supseteq E^p_2 \supseteq \dots \tag{0.2}$$

by

$$E_k^p = \{\omega \in \Omega^p(M,V) | \exists \omega_1,\dots,\omega_j \text{ with}$$
$$d_\delta(\omega + \delta\omega_1 + \cdots + \delta^j\omega^j) \in 0(\delta^k)$$
$$d_\delta^*(\omega + \delta\omega_1 + \cdots + \delta^j\omega_j) \in 0(\delta^k)\} \,. \tag{0.3}$$

Our first result (Theorem 1.3) is that the sequence (0.2) converges. That is, there is an N such that

$$E_N^p = E_{N+1}^p = \cdots = E_\infty^p$$

and, in fact, $\dim E_N^p < \infty$. More precisely, we show (Theorem 1.3) that $\dim E_{N'}^p < \infty$, where

$$N' = \min\{\dim A + 3, \dim B + 3\} \,.$$

We define a differential d_k on E_k^p by setting, for $\omega \in E_k^p$,

$$d_k\omega = \lim_{\delta \to 0} \delta^{-k} d_\delta(w + \delta w_1 + \cdots + \delta^j\omega_j) \,,$$

where the ω_i's are as in (0.3). Unfortunately, the map d_k depends on the ω_i's, but the map $\pi_k d_k$ does not, where π_k denotes the orthogonal projection onto E_k^p. In fact we show (Theorem 2.3)

(i) $(\pi_k d_k \pi_k)^2 = 0$.
(ii) The kernel of

$$\Delta_k = (\pi_k d_k \pi_k)(\pi_k d_k^* \pi_k) + (\pi_k d_k^* \pi_k)(\pi_k d_k \pi_k) : E_k^p \to E_k^p$$

is precisely E_{k+1}^p.

Statement (i) says that for each k

$$\pi_k d_k \pi_k : E_k^0 \to E_k^1 \to E_k^2 \to \cdots$$

forms a differential complex, and (ii) implies that the cohomology of the complex at the p^{th} stage is isomorphic to E_{k+1}^p. (There are some analytical subtleties here, but this is certainly true if $\dim E_k^p < \infty$, for example if $k \geqq N'$.) Thus the complexes $\{E_k^p, \pi_k d_k \pi_k\}$ form a spectral sequence.

Before leaving Sect. 2, we prove that the E_k^p spaces inherit a natural bigrading from the decomposition $TM = A + B$ which is compatible with the differential $\pi_k d_k \pi_k$. That is,

$$E_k^p = \bigoplus_{a+b=p} E_k^{a,b}, \text{ where } E_k^{a,b} = E_k^p \cap \Omega^{a,b} \,,$$

and

$$\pi_k d_k \pi_k (E_k^{a,b}) \subseteq E_k^{a-k+1,b+k} \,.$$

In Sect. 3 we make precise the sense in which (0.2) converges to $H^p(M,V)$, the cohomology of M with values in V. The main idea is to introduce the cohomology of the space of formal Laurent series of forms on M. This section essentially follows the ideas of Sects. 2, 5 and 6 of [Ma-Me], albeit in a more general setting. The conclusions follow from these observations:

(i) For every $\omega \in E_\infty^p$ there is a formal power series

$$\omega_\delta = \omega + \delta\omega_1 + \delta^2\omega_2 + \cdots$$

such that, formally,

$$d_\delta\omega_\delta = d_\delta^*\omega_\delta = 0 .$$

(ii) The ω_δ's arising in (i) form a basis, modulo the action of $\mathscr{L}$ (the ring of formal real Laurent series), for the cohomology of the complex $(\mathscr{L}[\Omega^p], d_\delta)$. Here, $\mathscr{L}[\Omega^p]$ denotes the space of formal Laurent series with coefficients in $\Omega^p(M, V)$.

(iii) The operator ρ_δ provides an isomorphism between $(\mathscr{L}[\Omega^p], d_\delta)$ and $(\mathscr{L}[\Omega^p], d)$.

(iv) The cohomology of $(\mathscr{L}[\Omega^p], d)$ is canonically isomorphic to $\mathscr{L}[H^p(M, V)]$ and hence, modulo $\mathscr{L}$, $H^p(M, V)$ provides a basis

Observations (i)–(iv) allow us to conclude, in particular, that for all p,

$$\dim E_\infty^p = \dim H^p(M, V) .$$

The topological nature of the spaces E_k^p is clarified in Sect. 4. We demonstrate that the spectral sequence $\{\mathscr{L}[E_k^p], \pi_k d_k \pi_k\}$ is isomorphic to the Leray spectral sequence arising from the natural infinite filtration of $\mathscr{L}[\Omega^p]$:

$$\mathscr{L}[\Omega^p] = \cdots \supseteq \delta^{-1}\Omega^p[[\delta]] \supseteq \Omega^p[[\delta]] \supseteq \delta\Omega^p[[\delta]] \supseteq \cdots$$

and the differential $\partial = \delta d_\delta$. As a corollary, we learn that the dimensions of the E_k^p are independent of the metrics g_A and g_B (since the metrics are not used in the construction of the Leray spectral sequence). In addition, if A is integrable we show that the spectral sequence $\{E_k^p, \pi_k d_k \pi_k\}$ is isomorphic to the standard Leray spectral sequence associated to the foliation A. In this case we learn that $\dim E_k^p$ depends only on A, that is, it is independent of the chosen complement B as well as the metrics g_A and g_B. In the special case of a fibration, this was proved in [Dai].

We make the relationship between E_∞^p and $H^p(M, V)$ more direct in Sect. 5. In this section we require, essentially, that A be a Riemannian foliation with compact leaves, and that g be a bundle-like metric. In particular, we require that (M, A, g) be given locally, by a fibration of the type (0.1). However, there are many interesting examples of foliations satisfying our hypotheses which are not globally of the form (0.1). We note that these restrictions are required only for the analysis of this section.

We study the behavior of the small eigenvalues of $\square_\delta^p$ and the corresponding eigenspaces. Let

$$\lambda_1^p(\delta) \leqq \lambda_2^p(\delta) \leqq \ldots$$

denote the eigenvalues of $\square_\delta^p$. Let

$$\mathrm{eig}_k^p = \mathrm{span}\{\omega \in \Omega^p | \rho_\delta \square_\delta^p \rho_\delta^{-1}\omega = \lambda_i^p(\delta)\omega \text{ and } \lambda_i^p(\delta) \in 0(\delta^{2k})\} .$$

Then we show (Theorem 5.17) that as $\delta \to 0$,

$$\mathrm{eig}_k^p = E_k^p + 0(\delta) .$$

As a corollary, if $\mathscr{H}_\delta^p(M, V)$ denotes the kernel of $\square_\delta^p$, then as $\delta \to 0$,

$$\rho_\delta \mathscr{H}^p_\delta(M, V) = E^p_\infty + 0(\delta)\,. \tag{0.4}$$

Now write $\lambda^p_i(\delta) \sim \delta^k$ if there is a c such that for all $\delta \in (0,1)$,

$$c\delta^k < \lambda^p_i(\delta) < \frac{1}{c}\delta^k\,.$$

Then every $\lambda^p_i(\delta)$ is $\sim \delta^{2k}$ for some k. The leading order term of such λ^p_i is determined by the Taylor series data. Namely, let $\widetilde{E}^p_k$ denote the orthogonal complement of E^p_{k+1} in E^p_k. We have seen that the kernel of

$$\Delta^p_k : E^p_k \to E^p_k$$

is E^p_{k+1}. Since Δ^p_k is self-adjoint, Δ^p_k must map $\widetilde{E}^p_k$ to itself. We have for $k \geqq 1$ (Theorem 5.20)

$$\{\lambda^p_i(\delta) | \lambda^p_i(\delta) \sim \delta^{2k}\} = \delta^{2k}\{\text{eigenvalues of } \Delta^p_k : \widetilde{E}^p_k \to \widetilde{E}^p_k\} + 0(\delta^{2k+1})\,.$$

The statement (0.4) implies, in particular, that the one-parameter family of spaces $\rho_\delta \mathscr{H}^p_\delta, \delta \in (0,1]$ has a continuous extension to the closed interval $\delta \in [0,1]$. We complete Sect. 5 by sharpening this statement. Namely, we show that a slight modification of our proof of (0.4) yields (Theorem 5.21) that this extension is, in fact, C^∞ on [0,1]. Equivalently, if

$$\omega_\delta = \omega + \delta\omega_1 + \delta^2\omega_2 + \cdots$$

is any formal power series in δ with values in $\Omega^p(M, V)$, which formally satisfies

$$L^p_\delta \omega_\delta = 0\,,$$

then ω_δ is the Taylor series at $\delta = 0$ of a C^∞ family of forms $\tilde{\omega}_\delta, \delta \in [0,1]$, which satisfy, for each $\delta \in (0,1]$,

$$L^p_\delta \omega_\delta = 0\,.$$

This extends Theorem 17 of [Ma-Me] to our setting.

In the special case of a fibration, the results in Sect. 5 overlap with those of [Dai] and [Ma-Me]. In [Dai], the results of [Ma-Me] were used to study the adiabatic limit of the eta-invariant of the fiber bundle M. In an analogous fashion, in another paper we will use these results to derive a formula for the analytic torsion of M.

In Sects. 1–4, we show that the adiabatic limit leads to a natural spectral sequence for any splitting of the tangent space $TM = A + B$. It remains an intriguing problem to find a more general context for the analysis of Sect. 5. That is, less restrictive geometric assumptions which still imply that the spectral data determines the precise asymptotics of the eigenvalues and eigenspaces.

1. Preliminaries

Let (M^m, g) be a compact Riemannian manifold, and $V \to M$ a flat vector bundle. That is, V comes equipped with a Euclidean inner product and a compatible flat connection. By $\Omega^p(V)$ we denote the space of p-forms on M with values in V. There is a natural extension of the usual deRham differential to a differential

$$d : \Omega^p(V) \to \Omega^{p+1}(V) .$$

Suppose $A \subset TM$ is a smooth distribution of n-planes, and $B = A^\perp$ is the orthogonal distribution. Then we have a decomposition

$$TM = A \oplus B \tag{1.1}$$

and a corresponding decomposition

$$T^*M = A^* \oplus B^* .$$

This decomposition induces a bigrading on $\Omega^p(V)$ by

$$\Omega^p(V) = \bigoplus_{i=0}^{p} \Omega^{i,p-i}(V) ,$$

where

$$\Omega^{i,p-i}(V) = \Gamma(\Lambda^i A^* \oplus \Lambda^j B^* \oplus V) .$$

Similarly, all operators on forms inherit a corresponding decomposition. In particular, the d operator inherits a bigrading

$$d = \sum_i d^{i,l-i} ,$$

where

$$d^{a,b} : \Omega^{i,j}(V) \to \Omega^{i+a,j+b}(V) .$$

For any such decomposition of TM we have

$$d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2} .$$

Note that $d^{1,0}$ and $d^{0,1}$ are first order differential operators, while $d^{2,-1}$ and $d^{-1,2}$ are zeroth order. Geometric properties of the distributions A and B are reflected in the analytic properties of these operators. For example

Lemma 1.1. *The operator $d^{2,-1} = 0$ if and only if A is integrable*

For a proof, see [Mo], page 58.
The identity $d^2 = 0$ yields the identities

$$\begin{aligned}
0 &= (d^2)^{4,-2} = (d^{2,-1})^2 \\
0 &= (d^2)^{3,-1} = d^{2,-1}d^{1,0} + d^{1,0}d^{2,-1} \\
0 &= (d^2)^{2,0} = d^{2,-1}d^{0,1} + (d^{1,0})^2 + d^{0,1}d^{2,-1} \\
0 &= (d^2)^{1,1} = d^{2,-1}d^{-1,2} + d^{1,0}d^{0,1} + d^{0,1}d^{1,0} + d^{-1,2}d^{2,-1} \\
0 &= (d^2)^{0,2} = d^{1,0}d^{-1,2} + (d^{0,1})^2 + d^{-1,2}d^{1,0} \\
0 &= (d^2)^{-1,3} = d^{0,1}d^{-1,2} + d^{-1,2})d^{0,1} \\
0 &= (d^2)^{-2,4} = (d^{-1,2})^2 .
\end{aligned}$$

The decomposition of the tangent space (1.1) induces a corresponding decomposition of the metric

$$g = g_A \oplus g_B .$$

We define a 1-parameter family of metrics $g_\delta, 0 < \delta \leqq 1$, by

$$g_\delta = g_A \oplus \delta^{-2} g_B \, .$$

For each p and δ we have an induced Lapalcian

$$\Box_\delta^p = d_{(g_\delta)}^* d + d d_{(g_\delta)}^* : \Omega^p(V) \to \Omega^p(V) \, ,$$

where $d_{(g_\delta)}^*$ is the adjoint of d with respect to the metric on $\Lambda^* T^* M$ induced by the metric g_δ.

Our goal is to study the behavior of the eigenvalues of $\Box_\delta^p$ as $\delta \to 0$. In our investigation, we will make use of the classical variational approach (see [Du-Sc], p. 908)

$$\lambda_i^p(\delta) = \sup_{v_1,\dots,v_{i-1} \in H_1^p} \quad \inf_{\substack{v_1 \in H_1^p \\ v_i \perp \{v_1,\dots,v_{i-1}\}}} \frac{\langle dv_i, dv_i \rangle_\delta + \langle d_{(g_\delta)}^* v_1, d_{(g_\delta)}^* v_i \rangle_\delta}{\langle v_i, v_i \rangle_\delta} \, . \tag{1.2}$$

Here we have numbered the eigenvalues of $\Box_\delta^p$ in increasing order, with each eigenvalue listed according to its multiplicity:

$$\lambda_1^p(\delta) \leqq \lambda_2^p(\delta) \leqq \dots \, .$$

Moreover, H_1^p denotes the completion, in the space of L^2 p-forms, of the C^∞ p-forms with respect to the norm

$$\|\omega\|_{H_1} = \|\nabla \omega\|_{L^2}^2 + \|\omega\|_{L^2}^2 \, .$$

Note that the space H_1^p is independent of the metric used to define the norms, and thus is independent of δ.

One difficulty in applying (1.2) is that both the operator and the inner product vary with δ. To simplify, we introduce the *isometry*

$$\rho_\delta : (\Omega^p(V), g_\delta) \to (\Omega^p(V), g) \, ,$$

where, for $\omega \in \Omega^{i,j}$,

$$\rho_\delta \omega = \delta^j \omega \, .$$

Then for all $\omega \in \Omega^p$,

$$\frac{\langle \Box_\delta^p \omega, \omega \rangle_\delta}{\langle \omega, \omega \rangle_\delta} = \frac{\langle (\rho_\delta \Box_\delta^p \rho_\delta^{-1})(\rho\delta\omega), (\rho\delta\omega) \rangle}{\langle \rho\delta\omega, \rho\delta\omega \rangle} \, ,$$

where we use $\langle \, , \, \rangle$ to denote the inner product induced by the original metric g.

Lemma 1.2. *Let* $L_\delta^p = \rho_\delta \Box_\delta^p \rho_\delta^{-1}$. *Then*

$$L_\delta^p = d_\delta d_\delta^* + d_\delta^* d_\delta \, ,$$

where

$$d_\delta = \delta^{-1} d^{2,-1} + d^{1,0} + \delta d^{0,1} + \delta^2 d^{-1,2} \tag{1.3}$$

and

$$d_\delta^* = \delta^{-1} (d^{2,-1})^* + (d^{1,0})^* + \delta (d^{0,1})^* + \delta^2 (d^{-1,2})^* \tag{1.4}$$

is the adjoint. Note that all adjoints are taken with respect to the metric g.

Proof. First note that

$$L_\delta^p = (\rho_\delta d\rho_\delta^{-1})(\rho_\delta d^*_{(g_\delta)}\rho_\delta^{-1}) + (\rho_\delta d^*_{(g_\delta)}\rho_\delta^{-1})(\rho_\delta d\rho_\delta^{-1}) .$$

The operator $\rho_\delta d^*_{(g_\delta)}\rho_\delta^{-1}$ is the adjoint of $\rho_\delta d\rho_\delta^{-1}$ with respect to the metric g, so (1.4) follows from (1.3). To prove (1.3) note that for all $\omega \in \Omega^{i,j}$,

$$\rho_\delta d^{a,b}\rho_\delta^{-1}\omega = \rho_\delta d^{a,b}\delta^{-j}\omega = \delta^{-j}\rho_\delta(d\omega)^{i+a,j+b} = \delta^{-j}\delta^{j+b}(d\omega)^{i+a,j+b} = \delta^b d^{a,b}\omega .$$

Thus, if we write

$$d = \sum d^{a,b} ,$$

then

$$\rho_\delta d\rho_\delta^{-1} = \sum \delta^b d^{a,b} . \qquad \square$$

Using Lemma 1.2 we can reformulate (1.2) as

$$\lambda_i^p(\delta) = \sup_{v_1,\dots,v_{i-1}\in H_1^p} \inf_{\substack{v_i\in H_1 \\ v_i\perp\{v_1,\dots,v_{i-1}\}}} \frac{|d_\delta v_i|^2 + |d_\delta^* v_i|^2}{|v_i|^2} . \tag{1.5}$$

This quotient motivates the following definition. Define the nested sequence of spaces

$$E_{-1}^p \supseteq E_0^p \supseteq E_1^p \supseteq E_2^p \supseteq \cdots . \tag{1.6}$$

where

$$\begin{aligned} E_k^p = \{\omega \in H_1^p | \exists j \text{ and } \omega_1,\dots,\omega_j \in H_1^p \text{ with } \\ d_\delta(\omega + \delta\omega_1 + \cdots + \delta^j\omega_j) \in \delta^k\Omega^{p+1}[\delta] \\ d_\delta^*(\omega + \delta\omega_1 + \cdots + \delta^j\omega_j) \in \delta^k\Omega^{p-1}[\delta]\} , \end{aligned}$$

where $\Omega^i[\delta]$ denotes the space of polynomials with δ with coefficients in $\Omega^i(E)$.

Example. For $\omega_\delta = \omega + \delta\omega_1 + \cdots + \delta^j\omega_j$,

$$\begin{aligned} d_\delta\omega_\delta &= \delta^{-1}d^{2,-1}\omega + (d^{1,0}\omega + d^{2,-1}\omega_1) + \cdots , \\ d_\delta^*\omega_\delta &= \delta^{-1}(d^{2,-1})^*\omega + ((d^{1,0})^*\omega + (d^{2,-1})^*\omega_1) + \cdots . \end{aligned}$$

Therefore

$$\begin{aligned} E_0^p &= \{\omega \in H_1^p | d^{2,-1}\omega = (d^{2,-1})^*\omega = 0\} , \\ E_1^p &= \{\omega \in E_0^p | d^{1,0}\omega \in \mathrm{Image}(d^{2,-1}), (d^{1,0})^*\omega \in \mathrm{Image}(d^{2,-1})^*\} . \end{aligned} \tag{1.7}$$

This characterization of E_1^p follows from the orthogonal decomposition

$$\Omega^p = (\mathrm{Image}\, d^{2,-1}) \oplus (\mathrm{Kernel}\, d^{2,-1} \cap \mathrm{Kernel}(d^{2,-1})^*) \oplus (\mathrm{Image}(d^{2,-1})^*)$$

so that if

$$d^{1,0}\omega \in \mathrm{Image}\, d^{2,-1}, \quad (d^{1,0})^*\omega \in \mathrm{Image}(d^{2,-1})^*$$

there is a form ω_1^p with

$$d^{1,0}\omega + d^{2,-1}\omega_1 = (d^{1,0})^*\omega + (d^{2,-1})^*\omega_1 = 0 .$$

Thus, if we set $\omega_\delta = \omega + \delta\omega_1$ we have

$$d_\delta \omega_\delta, d_\delta^* \omega_\delta \in \delta\Omega[\delta] \Rightarrow \omega \in E_1^p .$$

In fact, from (1.7) we see that for $\omega \in E_0^p \subset \mathrm{Kernel}\, d^{2,-1}$,

$$\begin{aligned} d^{1,0}\omega \in \mathrm{Kernel}\, d^{2,-1} &= (\mathrm{Kernel}\, d^{2,-1} \cap \mathrm{Kernel}\,(d^{2,-1})^*) \oplus (\mathrm{Image}\, d^{2,-1}) \\ &= E_0^p \oplus (\mathrm{Image}\, d^{2,-1}) . \end{aligned}$$

Therefore

$$d^{1,0}\omega \in \mathrm{Image}\, d^{2,-1} \longleftrightarrow d^{1,0}\omega \in (E_0^{p+1})^\perp .$$

Similarly

$$(d^{1,0})^*\omega \in \mathrm{Image}\,(d^{2,-1})^* \longleftrightarrow (d^{1,0})^*\omega \in (E_0)^\perp .$$

Thus we can rewrite (1.7) as

$$E_1^p = \{\omega \in E_0^p | d^{1,0}\omega, (d^{1,0})^*\omega \in (E_0)^\perp\} .$$

We note here that (1.5) combined with the definition of the E_k^p yields

$$\#\{\lambda_i^p(\delta) \in \text{spectrum } (L_\delta^p) | \lambda_i^p(\delta) \in 0(\delta^{2k})\} \geqq \dim E_k^p . \tag{1.8}$$

Remark. In [Ma-Me], Mazzeo and Melrose define spaces

$$\bar{E}_k^p = \{\omega | \exists \omega_1, \dots, \omega_j \text{ with } L_\delta^p(\omega + \delta\omega_1 + \cdots + \delta^j \omega_j) \in \delta^k \Omega^p[\delta]\} .$$

It is clear that

$$E_k^p \subseteq \bar{E}_{k-1}^p \subseteq E_{\left[\frac{k-1}{2}\right]}^p .$$

So, in particular

$$E_\infty^p = \bigcap_{i=0}^\infty E_i^p = \bigcap_{i=0}^\infty \bar{E}_i^p = \bar{E}_\infty^p .$$

We conclude this section with a proof that the sequence (1.6) stabilizes. That is, there is an N such that

$$E_N^p = E_{N+1}^p = \cdots = E_\infty^p .$$

It is sufficient to prove that there is an N' with

$$\dim E_{N'}^p < \infty .$$

In fact, one can take

$$N' = \mathrm{Min}\{\dim A + 3,\ \dim B + 3\} . \tag{1.9}$$

Theorem 1.3. *With N' as in* (1.9),

$$\dim E_{N'}^p < \infty$$

for every p.

Proof. Let

$$\pi^{i,j} : \Omega^* \to \Omega^{i,j}$$

denote the canonical projection. We will show that for every i,

$$\dim \pi^{i,p-i} E_{N'}^p < \infty .$$

Suppose $\omega \in E^p_{N'}$. Then there are $\omega_1, \dots, \omega_j$ such that

$$d_\delta(\omega + \delta\omega_1 + \cdots + \delta^j\omega_j) \in \delta^{N'}\Omega[\delta]\,. \tag{1.10}$$

$$d^*_\delta(\omega + \delta\omega_1 + \cdots + \delta^j\omega_j) \in \delta^{N'}\Omega[\delta]\,. \tag{1.11}$$

Write $\eta^{i,j}$ for $\pi^{i,j}\eta$. From (1.9) we see that for every $0 \leqq i \leqq p$ and $k \geqq N' - 2$,

$$\omega^{i-k,p-i+k} = 0\,,$$

Thus from (1.10)

$$\begin{aligned} &d^{2,-1}\omega^{i,p-i} = 0 \\ &d^{1,0}\omega^{i,p-i} + d^{2,-1}\omega_1^{i-1,p-i+1} = 0 \\ &\vdots \\ &d^{-1,2}\omega_{N'-3}^{i-(N'-3),p-i+(N'-3)} = 0\,. \end{aligned} \tag{1.12}$$

Let

$$\bar{\omega} = \sum_{k<i} \omega_{i-k}^{k,p-k}\,.$$

Then (1.12) implies

$$d(\omega^{i,p-i} + \bar{\omega}) = 0$$

(where d is the usual d operator), so that

$$\omega^{i,p-i} + \bar{\omega} = du_1 + h_1$$

for some $u_1 \in \Omega^{p-1}$ and $h_1 \in \mathscr{H}^p(M, V)$, the (finite-dimensional) space of g harmonic p-forms.

Similarly, let

$$\bar{\bar{\omega}} = \sum_{k>i} \omega_{k-i}^{k,p-k}\,.$$

Then (1.11) implies

$$d^*(\omega^{i,j} + \bar{\bar{\omega}}) = 0\,,$$

so that

$$\omega^{i,p-i} + \bar{\bar{\omega}} = d^*u_2 + h_2$$

for some $u_2 \in \Omega^{p+1}, h_2 \in \mathscr{H}^p(M, E)$. Thus

$$\pi^{i,p-i}E^p_{N'} \subset \left[\left(\bigoplus_{k<i}\Omega^{k,p-k} \oplus \Omega^{p-1} \oplus \mathscr{H}^p\right) \cap \left(\bigoplus_{k>i}\Omega^{k,p-k} \oplus d^*\Omega^{p+1} \oplus \mathscr{H}^p\right)\right]. \tag{1.13}$$

Let $R^{i,p-i}$ denote the intersection on the right-hand side of (1.13). Define a linear map

$$\tau : R^{i,p-i} \to \mathscr{H}^p$$

as follows. For $\omega \in R^{i,p-i}$ we have

$$\omega = \tilde{\omega} + dv + h \tag{1.14}$$

for some $\tilde{\omega} \in \bigoplus_{k<i}\Omega^{k,p-k}, v \in \Omega^{p-1}, h \in \mathscr{H}^p$.

Set

$$\tau(\omega) = h\,.$$

The representation (1.14) may not be unique, in which case any such representation can be chosen on a basis and τ can be extended linearly to all of $R^{i,p-i}$.

The map τ is injective. To see this, suppose

$$\tau(\omega) = 0$$

so that $\omega \in \Omega^{i,p-i}$ satisfies

$$\omega = \tilde{\omega} + dv_1\,,$$
$$\omega = \tilde{\tilde{\omega}} + d^* v_2 + h$$

for $\tilde{\omega} \in \bigoplus_{k<i} \Omega^{k,p-k}, \tilde{\tilde{\omega}} \in \bigoplus_{k>i} \Omega^{k,p-k}$ and $h \in \mathscr{H}^p$. Then

$$|\omega|^2 = \langle \omega - \tilde{\omega}, \omega - \tilde{\tilde{\omega}} \rangle = \langle dv_1, d^* v_2 + h_2 \rangle = 0$$

so that

$$\omega = 0\,.$$

Thus

$$\dim \pi^{i,p-i} E^p_{N'} \leqq \dim R^{i,p-i} \leqq \dim \mathscr{H}^p < \infty\,.$$

as desired. □

2. The Sequence $\{E^p_k\}$ as a Spectral Sequence

The goal of this section is to show that the sequence (1.6) comes equipped with the algebraic structure of a spectral sequence, and that this structure arises naturally from the Taylor series analysis of Sect. 1. Moreover, we show that the E^p_k inherit a bigrading that is compatible with the differential.

We begin with some notation. In Theorem 1.3 we proved that the sequence (1.6) stabilizes. Denote by $N(p)$ the integer with the property that

$$E^p_{N(p)-1} \neq E^p_{N(p)} = E^p_\infty\,.$$

Let π_k denote the orthogonal projection onto E_k, with $\pi_k^\perp = 1 - \pi_k$. Let $\widetilde{E}^p_k \subset E^p_k$ denote the orthogonal complement in E^p_k of E^p_{k+1}, and $\tilde{\pi}_k$ the orthogonal projection onto $\widetilde{E}^p_k$ (so that $1 = \sum_{i=-1}^{N-1} \tilde{\pi}_i + \pi_N$).

Lastly, we define $\mathscr{E}^p_k \subset \Omega[\delta]$ by

$$\mathscr{E}^p_k = \{v(\delta) \in \Omega^p[\delta] | d_\delta v(\delta), d^*_\delta v(\delta) \in \delta^k \Omega[\delta]\}$$

so that $v \in E^p_k$ if and only if v has an extension $v(\delta) \in \mathscr{E}^p_k$, where we say $v(\delta)$ is an extension of v if $v(0) = v$.

Every $v \in E_k$ has an extension $v_\delta \in \mathscr{E}_k$. The set of such extensions is an affine space. It will be convenient, in what follows, to have a fixed origin for this space.

For $k = 0,\ 1, 2, \ldots, N-1$, let Φ be any linear extension map

$$\Phi : \widetilde{E}_k \to \mathscr{E}_k\,,$$

i.e. Φ is an assignment

$$v \in \widetilde{E}_k \to \Phi(v) = v + \delta v_1 + \delta^2 v_2 + \cdots + \delta^j v_j$$

such that $\Phi(v) \in \mathscr{E}_k$ and the map $v \to v_i$ is linear for every i.

For $k = -1$ and 0 we have

$$\widetilde{E}_k \subset \mathscr{E}_k .$$

So we can let Φ be identity map on $\widetilde{E}_{-1}$ and $\widetilde{E}_0$.

Now extend Φ linearly to all of $E_N^{\perp}$. That is, for all $v \in E_N^{\perp}$,

$$\Phi(v) = \sum_{i=0}^{N-1} \Phi(\tilde{\pi}_i v) .$$

We can express any extension $v_\delta \in \mathscr{E}_k$ of $v \in E_k$ in terms of our fixed extension Φ as follows

Lemma 2.1. *For $k \leqq N$, if $v \in \widetilde{E}_k$ and $v_\delta \in \mathscr{E}_k$ is any extension, then v_δ can be expressed uniquely as*

$$v_\delta = \Phi(v) + \delta r_\delta \tag{2.1}$$

for some $r_\delta \in \mathscr{E}_{k-1}$.

Conversely, if v_δ is defined by (2.1) for some $r_\delta \in \mathscr{E}_{k-1}$, then $v_\delta \in \mathscr{E}_k$ is an extension of v.

Proof. If $v_\delta \in \mathscr{E}_k$ is an extension of v then

$$v_\delta - \Phi(v) = \delta r_\delta$$

for some $r_\delta \in \Omega[\delta]$. Moreover, since v_δ and $\Phi(v) \in \mathscr{E}_k$ we must have

$$\delta r_\delta \in \mathscr{E}_k$$

which implies

$$r_\delta \in \mathscr{E}_{k-1}$$

as desired

The converse is clear. □

We now define the operators that will form the basis of this section. If

$$v_\delta = v_0 + \delta v_1 + \delta^2 v_2 + \cdots + \delta^j v_j$$

is a form-valued polynomial, define the operators $\tilde{d}_{-1}, \tilde{d}_0, \tilde{d}_1, \ldots$ by

$$d_\delta v_\delta = \delta^{-1}\tilde{d}_{-1} v_\delta + \tilde{d}_0 v_\delta + \delta \tilde{d}_1 v_\delta + \delta^2 \tilde{d}_2 v_\delta + \cdots .$$

Similarly, define $\tilde{d}^*_{-1}, \tilde{d}^*_0, \tilde{d}^*_1, \ldots$

Note that $v_\delta \in \mathscr{E}_k$ if and only if

$$\tilde{d}_i v_\delta = \tilde{d}^*_i v_\delta = 0$$

for all $i < k$.

Using the fixed extension map Φ we define linear maps

$$d_k : (E_N^p)^{\perp} \to \Omega^{p+1}$$

by

$$d_k v = \tilde{d}_k(\Phi(v))$$

and we similarly define their adjoints d_k^*. We extend these operators to all of Ω^p by setting for $v \in E_N^p$
$$d_k v + d_k^* v + 0 \quad \text{for all } k \, .$$
It is important to note that the maps d_k and d_k^* depend on our choice of the extension map Φ.

The basic properties of these operators are contained in the following lemma

Lemma 2.2.
(i) *For every $i > j$,*
$$\pi_i d_j = \pi_i d_j^* = d_j \pi_i = d_j^* \pi_i = 0 \, .$$
(ii) *For every i and j,*
$$\pi_i d_i d_j \pi_j = \pi_i d_i^* d_j^* \pi_j = 0 \, .$$

Proof. (i) For every v,
$$\pi_i v = \pi_N v + \sum_{k=i}^{N-1} \tilde{\pi}_k v \, .$$
By definition
$$d_j \pi_N v = d_j^* \pi_N v = 0 \tag{2.2}$$
for all j. For $i \leqq k \leqq N-1$, $\tilde{\pi}_k v \in E_k$ so that $\Phi(\tilde{\pi}_k v) \in \mathscr{E}_k$, which implies that (since $k \geqq i > j$)
$$\begin{aligned} d_j \tilde{\pi}_k v &= \tilde{d}_j \Phi(\tilde{\pi}_k v) = 0 \, , \\ d_j^* \tilde{\pi}_k v &= \tilde{d}_j \Phi(\tilde{\pi}_k v) = 0 \, . \end{aligned} \tag{2.3}$$
Adding (2.2) and (2.3) yields
$$d_j \pi_i = d_j^* \pi_i = 0 \quad \text{for } i > j \, .$$
Taking adjoints yields
$$\pi_i d_j = \pi_i d_j^* = 0 \quad \text{for } i > j$$
which proves (i).

(ii) From (i) we have
$$\pi_i d_i d_j \pi_j = \tilde{\pi}_i d_i d_j \tilde{\pi}_j \, .$$
For any $v \in \Omega^p, \omega \in \Omega^{p+2}$ we have, since $d_\delta^2 = 0$
$$0 = \langle d_\delta \Phi(\tilde{\pi}_i v), d_\delta^* \Phi(\tilde{\pi}_j w) \rangle = \delta^{i+j} \langle \tilde{d}_i \Phi(\tilde{\pi}_i v), \tilde{d}_j^* \Phi(\tilde{\pi}_j w) \rangle + 0(\delta^{i+j+1}) \, .$$
Thus
$$0 = \langle \tilde{d}_i \Phi(\tilde{\pi}_i v), \tilde{d}_j^* \Phi(\tilde{\pi}_j w) \rangle = \langle d_i \tilde{\pi}_i v, d_j^* \pi_j w \rangle = \langle \tilde{\pi}_j d_j d_i \tilde{\pi}_i v, w \rangle \, ,$$
which implies
$$\pi_j d_j d_i \pi_i = \tilde{\pi}_j d_j d_i \tilde{\pi}_i = 0 \, .$$
Taking adjoints yields
$$\pi_j d_j^* d_i^* \pi_i = \tilde{\pi}_j d_j^* d_i^* \tilde{\pi}_i = 0 \, . \quad \square$$

Lemma 2.2, part (i) has the following important corollary:

Lemma 2.3. *The maps $\pi_k d_k \pi_k$ and $\pi_k d_k^* \pi_k$ are independent of our choice of the extension map Φ.*

Proof. From Lemma 2.2 (i), $\pi_k d_k \pi_k = \tilde{\pi} d_k \tilde{\pi}_k$. For $v \in \Omega^p, \tilde{\pi}_k d_k \tilde{\pi}_k v = \tilde{\pi}_k \tilde{d}_k \Phi(\tilde{\pi}_k v)$. Suppose Φ' is any other extension map, then

$$\Phi(\tilde{\pi}_k v) - \Phi'(\tilde{\pi}_k v) = \delta\omega)\delta$$

for some $\omega_\delta \in \mathscr{E}_{k-1}$ (Lemma 2.1). Thus

$$\tilde{\pi}_k \tilde{d}_k(\Phi(\tilde{\pi}_k v) - \Phi'(\tilde{\pi}_k v)) = \tilde{\pi}_k d_{k-1}\omega_\delta = 0$$

by Lemma 2.2 (i). Therefore

$$\pi_k \tilde{d}_k \Phi(\tilde{\pi}_k v) = \pi_k \tilde{d}_k \Phi'(\tilde{\pi}_k v) ,$$

and this implies $\pi_k d_k \pi_k$ is independent of Φ. The same proof holds for $\pi_k d_k^* \pi_k$. $\square$

Our next goal is to show that for every $v \in E_k$ there is an extension v_δ^* such that

$$\tilde{d}_k v_\delta^* \in \tilde{E}_k, \quad \tilde{d}_k^* v_\delta^* \in \tilde{E}_k ,$$

so that

$$\tilde{d}_k v_\delta^* = (\pi_k d_k \pi_k)v ,$$
$$\tilde{d}_k^* v_\delta^* = (\pi_k d_k^* \pi_k)v .$$

It is from this fact that we derive all of our desired results.

Our proof of this fact begins with the definition of two more operators:

$$D_k : E_{k+1}^\perp \to E_{k+1}^\perp ,$$
$$\overline{D}_k : E_{k+1}^\perp \to E_{k+1}^\perp ,$$

where we set

$$D_k = \sum_{i=-1}^{k} \tilde{\pi}_i d_i ,$$

$$\overline{D}_k = \sum_{i=-1}^{k} d_i \tilde{\pi}_i .$$

These operators depend on the extension Φ. Note that Lemma 2.2 (ii) implies

$$D_k \overline{D}_k = \overline{D}_k^* D_k^* = 0$$

for all k. Now we define the "Laplacian" $\square_k$ by

$$\square_k = D_k D_k^* + \overline{D}_k^* \overline{D}_k = (D_k + \overline{D}_k^*)(D_k^* + \overline{D}_k) : E_{k+1}^\perp \to E_{k+1}^\perp .$$

Theorem 2.4.

(a) *Fix $v \in E_k^p$, then*

(i) *There is an extension $v_\delta^* \in \mathscr{E}$ of v with*

$$\tilde{d}_k v_\delta^* \in E_k^{p+1}, \quad \tilde{d}_k^* v_\delta^* \in E_k^{p-1} .$$

(ii) *The polynomial* v_δ^* *from part* (i) *is unique modulo* $\delta\mathscr{E}_k^p$.

(iii) *In terms of the extension* $\Phi, v_\delta^* = \Phi(v) + \sum_{i=-1}^{k-1} \delta^{k-i}\Phi(v_i)$, *where*

$$v_i = \tilde{\pi}_i \square_{k-1}^{-1}(\overline{D}_{k-1}^* d_k + D_{k-1} d_k^*) v \,.$$

(b) *For every* p *and* k, *the operator*

$$\square_k^p : (E_{k+1}^p)^\perp \to (E_{k+1}^p)^\perp$$

is invertible.

Proof. We will prove (a) and (b) simultaneously inductively on k.

$k = -1$: Part (a) is trivial since for every $v \in E_{-1}^p = \Omega^p$ and every extension v_δ of v we have

$$\tilde{d}_{-1} v_\delta = d^{2,-1} v \in \tilde{E}_{-1} \,,$$
$$\tilde{d}_{-1}^* v_\delta = (d^{2,-1})^* v \in \tilde{E}_{-1} \,.$$

Thus any extension will satisfy part (i). Parts (ii) and (iii) follow trivially since every two extensions of v differ by an element of $\delta\mathscr{E}_0 = \delta\Omega[\delta]$.

Part (b) follows from the observation

$$\begin{aligned} \mathrm{Kernel}\,\square_{-1} &= \mathrm{Kernel}((d^{2,-1})^* d^{2,-1} + d^{2,-1}(d^{2-1})^*) \\ &= \mathrm{Kernel}(d^{2,-1})^* \cap \mathrm{Kernel}(d^{2,-1}) = E_0 \,. \end{aligned}$$

Now assume (a) and (b) are true for $k-1$.

Proof of (*a*). Given $v \in E_k^p$ define an extension v_δ^* by the formula in part (iii), i.e.

$$v_\delta^* = \Phi(v) + \sum_{i=-1}^{k-1} \delta^{k-i}\Phi(v_i) \,,$$

where

$$v_i = \tilde{\pi}_i \square_{k-1}^{-1}(\overline{D}_{k-1}^* d_k + D_{k-1} d_k^*) v \,. \tag{2.4}$$

Note that for $v \in E_k$,

$$d_k v = d_k \tilde{\pi}_k v, \quad d_k^* v = d_k^* \tilde{\pi}_k v$$

(we have used Lemma 2.2 (i)). It follows from Lemma 2.2 (ii) that

$$\overline{D}_{k-1}^* d_k^* v = \overline{D}_{k-1}^* d_k^* \tilde{\pi}_k v = 0 \,,$$
$$D_{k-1} d_k v = D_{k-1} d_k \tilde{\pi}_k v = 0 \,.$$

Thus (2.4) can be written

$$v_\iota = \tilde{\pi}_i \square_{k-1}^{-1}(\overline{D}_{k-1}^* + D_{k-1})(d_k + d_k^*) v \,.$$

It is easy to check that ω_δ is an extension of v. Moreover

$$\begin{aligned} \tilde{d}_k v_\delta^* &= \tilde{d}_k \Phi(v) + \sum_{i=-1}^{k-1} \tilde{d}_i \Phi(v_i) = d_k v + \sum_{i=-1}^{k-1} d_i v_\iota \\ &= d_k v + \left(\sum_{i=-1}^{k-1} d_i \tilde{\pi}_i\right) \square_{k-1}^{-1}(\overline{D}_{k-1}^* + D_{k-1})(d_k + d_k^*) v \\ &= d_k v + \overline{D_{k-1}}\, \square_{k-1}^{-1}(\overline{D}_{k-1}^* + D_{k-1})(d_k + d_k)^* v \,. \end{aligned} \tag{2.5}$$

Similarly

$$\tilde{d}_k^* v_\delta^* = d_k^* v + D_{k-1}^* \Box_{k-1}^{-1}(\overline{D}_{k-1}^* + D_{k-1})(d_k + d_k^*)v \,. \tag{2.6}$$

Adding (2.5) and *2.6) yields

$$(\tilde{d}_k + \tilde{d}_k^*)v_\delta^* = (d_k + d_k^*)v + (\overline{D}_{k-1} + D_{k-1}^*)\Box_{k-1}^{-1}(\overline{D}_{k-1}^* + D_{k-1})(d_k + d_k^*)v \,. \tag{2.7}$$

Writing $\Box_{k-1} = (\overline{D}_{k-1}^* + D_{k-1})(\overline{D}_{k-1} + D_{k-1}^*)$ we see that

$$\begin{aligned}&(\overline{D}_{k-1} + D_{k-1}^*)(\Box_{k-1})^{-1}(\overline{D}_{k-1}^* + D_{k-1})\\ &\qquad = \text{orthogonal projection into } (\text{Kernel}\,\Box_{k-1})^\perp\\ &\qquad = \text{orthogonal projection onto } E_k^\perp\end{aligned}$$

(the inductive hypotheses implies $\text{Kernel}\,\Box_{k-1} = E_k$). It follows from Lemma 2.2 (i) that

$$(d_k + d_k^*)v \in E_{k+1}^\perp \,.$$

Thus (2.7) says

$$(\tilde{d}_k + \tilde{d}_k^*)v_\delta^* = \tilde{\pi}_k(d_k + d_k^*)v = \tilde{\pi}_k d_k v + \tilde{\pi}_k d_k^* v \,. \tag{2.8}$$

Now

$$\begin{aligned}&\tilde{d}_k v_\delta^*,\ \tilde{\pi}_k d_k v \in \Omega^{p+1} \,,\\ &\tilde{d}_k^* v_\delta^*,\ \tilde{\pi}_k d_k^* v \in \Omega^{p-1} \,.\end{aligned}$$

Therefore, (2.8) implies

$$\begin{aligned}\tilde{d}_k v_\delta^* &= \tilde{\pi}_k d_k v \in \tilde{E}_k \subset E_k \,,\\ \tilde{d}_k^* v_\delta^* &= \tilde{\pi}_k d_k^* v \in \tilde{E}_k \subset E_k \,.\end{aligned}$$

Thus v_δ^* as defined in part (iii) satisfies the conclusion of part (i).

If v_δ and v_δ' are any two extensions of v in $\mathscr{E}_k$, then

$$v_\delta - v_\delta' = \delta r_\delta$$

for some $r_\delta \in \mathscr{E}_{k-1}$.

If

$$\tilde{d}_k v_\delta = \tilde{d}_k v_\delta', \quad \tilde{d}_k^* v_\delta = \tilde{d}_k^* v_\delta' \,,$$

then

$$0 = \tilde{d}_k(\delta r_\delta) = \tilde{d}_{k-1} r_\delta \,.$$

Similarly,

$$0 = \tilde{d}_{k-1}^* r_\delta \,.$$

Thus $r_\delta \in \mathscr{E}_k$. This shows that v_δ^* of part (i) is unique modulo $\delta\mathscr{E}_k$, which is part (ii).

Proof of (*b*). Note that

$$\begin{aligned}D_k &= D_{k-1} + \tilde{\pi}_k \tilde{d}_k \,,\\ \overline{D}_k &= \overline{D_{k-1}} + \tilde{d}_k \tilde{\pi}_k \,.\end{aligned}$$

Using the decomposition

$$E_k^\perp = E_{k-1}^\perp \oplus \tilde{E}_k ,$$

we express the operator $\Box_k$ as a 2×2 matrix of operators

$$\begin{pmatrix} \Box_{k-1} & \begin{matrix} D_{k-1}^* d_k + \\ \overline{D}_{k-1}^* d_k^* \end{matrix} \\ \begin{matrix} d_k D_{k-1} + \\ d_k^* \overline{D}_{k-1} \end{matrix} & \begin{matrix} d_k d_k^* + \\ d_k^* d_k \end{matrix} \end{pmatrix} .$$

For simplicity, relabel this matrix

$$\begin{pmatrix} A_k & B_k \\ B_k^* & C_k \end{pmatrix} .$$

By induction, A_k is invertible.

C_k is also invertible, since if $v \in E_k$ and $C_k v = 0$ then

$$\langle C_k v, v \rangle = 0$$

from which it follows that

$$d_k v = \tilde{d}_k \Phi(v) = 0, \quad d_k^* v = \tilde{d}_k^* \Phi(u) = 0 .$$

Thus $\Phi(v) \in \mathscr{E}_{k+1}$ which implies

$$v \in E_{k+1} \subset E_k^\perp$$

so $v = 0$.

Formally, the inverse of $\Box_k$ is given by the 2×2 matirx of operators

$$\begin{pmatrix} (A_k - B_k C_k^{-1} B_k^*)^{-1} & -A_k^{-1} B_k (C_k - B_k^* A_k^{-1} B_k)^{-1} \\ -C_k^{-1} B_k^* (A_k - B_k C_{-1} + k B_k^*)^{-1} & (C_k - B_k^* A_k^{-1} B_k)^{-1} \end{pmatrix} .$$

The invertibility of $\Box_k$ follows once we know

$$A_k - B_k C_k^{-1} B_k^*$$

and

$$C_k - B_k^* A_k^{-1} B_k$$

are invertible.

For $v \in E_k$

$$(C_k - B_k^* A_k^{-1} B_k) v$$
$$= (\tilde{\pi}_k d_k + \tilde{\pi}_k d_k^*)[1 - (\overline{D}_{k-1} + D_{k-1}^*) \Box_{k-1}^{-1} (\overline{D}_{k-1}^* + D_{k-1})](d_k \tilde{\pi}_k + d_k^* \tilde{\pi}_k) v . \tag{2.9}$$

But

$$1 - (\overline{D}_{k-1} + D_{k-1}^*) \Box_{k-1}^{-1} + D_{k-1}$$
$$= \text{orthogonal projection onto } (\mathrm{Kernel}\, \Box_{k-1})^\perp = E_k)$$

(by induction). Since

$$\mathrm{Image}\, d_k,\ \mathrm{Image}\, d_k^* \subset E_{k+1}^\perp$$

(from Lemma 2.2 (i)) (2.9) equals

$$(\tilde{\pi}_k d_k + \tilde{\pi}_k d_k^*)\tilde{\pi}_k(d_k\tilde{\pi}_k + d_k^*\tilde{\pi}_k)v\ .$$

If $(C_k - B_k^* A_{k-1}^{-1} B_k)v = 0$ then

$$0 = \langle (C_k - B_k^* A_{k-1}^{-1} B_k)v, v\rangle = |\tilde{\pi}_k(d_k + d_k^*)\tilde{\pi}_k v|^2 \Rightarrow (\tilde{\pi}_k d_k \tilde{\pi}_k + \tilde{\pi}_k d_k^* \tilde{\pi}_k)v = 0\ . \tag{2.10}$$

Since $\tilde{\pi}_k d_k \tilde{\pi}_k v \in \Omega^{p+1}, \tilde{\pi}_k d_k^* \tilde{\pi}_k v \in \Omega^{p-1}$ (2.10) implies

$$0 = \tilde{\pi}_k d_k \tilde{\pi}_k v = \tilde{d}_k v_\delta^*\ ,$$
$$0 = \tilde{\pi}_k d_k^* \tilde{\pi}_k v = \tilde{d}_k^* v_\delta^*\ .$$

Thus $v_\delta^* \in \mathscr{E}_{k+1} \Rightarrow v \in E_{k+1} \subset E_k^\perp$ so $v = 0$. Therefore $C_k - B_k^* A_k B_k$ in invertible.

Now we will see that $A_k - B_k C_k^{-1} B_k^*$ is invertible. Suppose $v \in E_k^\perp$ and let $z = -C_k^{-1} B_k^* v \in \tilde{E}_k$. Define an extension z_δ of z by

$$z_\delta = \Phi(z) + \sum_{i=-1}^{k-1} \delta^{k-1}\Phi(\tilde{\pi}_i v)\ .$$

Then

$$\tilde{d}_k z_\delta = \tilde{d}_k \Phi(z) + \sum_{i=-1}^{k-1} \tilde{d}_i \Phi(\tilde{\pi}_i v) = d_k z + \sum_{i=-1}^{k-1} d_i \tilde{\pi}_i v = d_k z + \overline{D}_{k-1} v\ .$$

Similarly

$$\tilde{d}_k^* z_\delta = d_k^* z + d_{k-1}^* v\ .$$

So we have

$$\begin{aligned}
&|\tilde{d}_k z_\delta|^2 + |\tilde{d}_k^* z_\delta|^2 \\
&\quad = |d_k z + \overline{D}_{k-1} v|^2 + |d_k^* z + d_{k-1}^* v|^2 \\
&\quad = \langle \overline{D}_{k-1}^* d_k z + \overline{D}_{k-1}^* \overline{D}_{k-1} v + D_{k-1} d_k^* z + D_{k-1} D_{k-1}^* v, v\rangle \\
&\qquad + \langle d_k^* d_k z + d_k^* \overline{D}_{k-1} v + d_k d_k^* z + d_k D_{k-1}^* v, z\rangle \\
&\quad = \langle B_k z + A_k v, v\rangle + \langle C_k z + B_k^* v, z\rangle\ .
\end{aligned} \tag{2.11}$$

Now use $z = -C_k^{-1} B_k^* v$ to find

$$\begin{aligned}
(2.11) &= \langle (A_k - B_k C_k^{-1} B_k^*)v, v\rangle + \langle (-C_k(C_k^{-1} B_k^*) + B_k^*)v, z\rangle \\
&= \langle (A_k - B_k C_k^{-1} B_k^*)v, v\rangle\ .
\end{aligned}$$

Therefore, if

$$(A_k - B_k C_k^{-1} b_k^*)v = 0\ ,$$

then

$$|\tilde{d}_k z_\delta|^2 + |\tilde{d}_k^* z_\delta|^2 = 0 \Rightarrow \tilde{d}_k z_\delta = \tilde{d}_k^* z_\delta = 0\ .$$

But this implies

$$z_\delta \in \mathscr{E}_{k+1} \Rightarrow z \in E_{k+1} \subset E_k^\perp\ .$$

Thus, we must have

$$0 = z = -C_k^{-1} B_k^* v\ .$$

But this implies

$$(A_k - B_k C_k^{-1} B_k^*) v = A_k v = 0\ .$$

This contradicts the invertibility of A_k. Therefore $A_k - B_k C_k^{-1} B_k^*$ is invertible. □

In what follows, a fundamental role will be played by the operator

$$\pi_k d_k \pi_k : E_k^p \to E_k^{p+1}$$

(which, by Lemma 2.3, is well-defined independent of our choice of the extension Φ) and the associated Laplacian

$$\Delta_k = \pi_k d_k \pi_k d_k^* \pi_k + \pi_k d_k^* \pi_k d_k^* \pi_k : E_k^p \to E_k^p\ .$$

The significance of these operators follows from:

Theorem 2.5.
a) $(\pi_k d_k\ \pi_k)^2 = (\pi_k d_k^* \pi_k)^2 = 0$.
b) *The kernel of the operator*

$$\Delta_k : E_k^p \to E_k^p$$

is precisely E_{k+1}^p.

Before we prove this theorem, we introduce a very useful definition. Up to this point, we have defined all operators with respect to an arbitrary extension map Φ. Theorem 2.4 provides us with a canonical choice for Φ.

Definition. *Define, for* $v \in \tilde{E}_k$. *The extension map* Φ' *by*

$$\Phi'(v) = v_\delta^*\ ,$$

where v_δ^* *is defined by the formula given in Theorem* 2.4 (a) (iii). *We let* d_k' *denote the operator* d_k *associated to the extension* Φ'. *That is, for* $v \in \tilde{E}_k$,

$$d_k' v = \tilde{d}_k v_\delta^*\ .$$

From Theorem 2.4 (a) (i) we know that for $v \in E_k$,

$$d_k' v,\ (d_k')^* v \in E_k\ .$$

This choice of extension greatly simplifies the proofs in this and the next section.

Proof of Theorem 2.5. a) Since, by Lemma 2.3, $\pi_k d_k \pi_k$ is independent of our extension so we can choose the extension Φ' defined above. This gives

$$\pi_k d_k \pi_k d_k \pi_k v = \pi_k d_k' \pi_k d_k' \pi_k v\ .$$

But the image of $d_k' \pi_k$ is contained in E_k so that

$$\pi_k d_k' \pi_k = d_k' \pi_k\ .$$

Thus

$$\pi_k d_k \pi d_k \pi_k = \pi_k d_k' d_k' \pi_k v = 0\ .$$

The last equality follows from Lemma 2.2 (ii) which states that $\pi_k d_k d_k \pi_k = 0$, where d_k is defined relative to any extension Φ. Applying this lemma to the extension Φ' yields the desired equality.

b) Clearly $E_{k+1} \subset \text{Kernel}\, \Delta_k$.

To prove the converse note that for $v \in E_k, v \in \text{Kernel}\, \Delta_k$ implies

$$\langle \Delta_k v, v \rangle = 0 ,$$

from which it follows that

$$\pi_k d_k \pi_k v = \pi_k d_k v = 0 ,$$
$$\pi_k d_k^* \pi_k v = \pi_k d_k^* v = 0 .$$

Thus

$$\tilde{d}_k \Phi'(v) = \pi_k d_k v = 0 ,$$
$$\tilde{d}_k^* \Phi'(v) = \pi_k d_k^* v = 0 .$$

Therefore $\Phi'(v) \in \mathscr{E}_{k+1}$ which implies $v \in E_{k+1}$ as desired. □

In addition to providing the fundamental properties of the operators $\tilde{\pi}_k d_k' \tilde{\pi}_k$ and Δ_k, Theorem 2.5 implies the following important observation concerning the higher order terms

$$d_{k+j}' v \equiv \tilde{d}_{k+j} \Phi'(v)$$

for $v \in E_k$ and $j > 0$.

Recall that Φ' was defined so that

$$d_k' v \equiv \tilde{d}_k \Phi'(v) \in E_k ,$$
$$(d_k')^* v \equiv \tilde{d}_k^* \Phi'(v) \in E_k .$$

In fact we have:

Corollary 2.6. *If $v \in E_k^p$ then there is a polynomial*

$$v_\delta = v + \delta v_1 + \ldots + \delta^i v_i$$

such that

$$\tilde{d}_j v_\delta = \tilde{d}_j^* v_\delta = 0 \quad \textit{for } j < k , \tag{2.12}$$

$$\tilde{d}_k v_\delta,\ \tilde{d}_k^* v_\delta \in E_k , \tag{2.13}$$

$$\tilde{d}_{k+j} v_\delta,\ \tilde{d}_{k+j}^* v_\delta \in E_{k+j-1} \quad \textit{for } j > 0 . \tag{2.14}$$

Proof. The extension Φ' defined earlier satisfies (2.12) and (2.13). We will modify Φ' to satisfy (2.14).

For any l and any $w \in \widetilde{E}_l$,

$$\begin{aligned} 0 &= \langle d_\delta \Phi'(v),\ d_\delta^* \Phi'(w) \rangle \\ &= \delta^{k+l} \langle d_k' v,\ (d_l')^* w \rangle \\ &\quad + \delta^{k+l+1} \langle d_k' v,\ (d_{l+1}')^* w \rangle + \langle d_{k+l}' v,\ (d_l')^* w \rangle \\ &\quad + 0(\delta^{k+l+2}) . \end{aligned} \tag{2.15}$$

In particular

$$\begin{aligned} 0 &= \langle d'_k v, (d'_{l+1})^* w\rangle + \langle d'_{k+l} v, (d'_l)^* w\rangle \\ &= \langle (\tilde{\pi}_l d'_{l+1} d'_k + \tilde{\pi}_l d'_l \tilde{\pi}_l d'_{k+l}) v, w\rangle . \end{aligned} \tag{2.16}$$

Suppose $l + 1 \leqq k$. Then $\tilde{\pi}_l d'_{l+l} d'_k = 0$. This can be seen as follows: If $l + 1 < k$, then since $d'_k v \in E_k$ we have $d'_{l+l} d'_k v = 0$ (Lemma 2.2 (i)).

If $l + 1 = k$, then

$$\tilde{\pi}_l d'_{l+l} d'_k v = \tilde{\pi}_{k-1} d'_k d'_k v .$$

We have

$$v \in E_k \Rightarrow d'_k v \in E_k \Rightarrow d'_k d'_k v \in E_k \Rightarrow \tilde{\pi}_{k-1} d'_k d'_k v = 0 .$$

Thus, from (2.16), for all $l \leqq k - 1$ and $w \in \widetilde{E}_l$,

$$\langle \tilde{\pi}_l d'_l \tilde{\pi}_l d'_{k+1} v, w\rangle = 0 ,$$

which implies for all $l \leqq k - 1$,

$$d'_{k+1} v \in \operatorname{Kernel} \tilde{\pi}_l d'_l \tilde{\pi}_l . \tag{2.17}$$

From Theorem 2.5(b), the sequence

$$\tilde{\pi}_l (d'_l)^* \tilde{\pi}_l : \widetilde{E}^n_l \to \widetilde{E}^{n-1}_l \to \widetilde{E}^{n-2}_l \to \cdots$$

is exact. That is,

$$\operatorname{Kernel} \tilde{\pi}_l d'_l \tilde{\pi}_l |_{\widetilde{E}_l} = \operatorname{Image} \tilde{\pi}_l d'_l \tilde{\pi}_l .$$

So that, for $l \leqq k - 1$,

$$\tilde{\pi}_l d'_{k+1} v \in \operatorname{Image} \tilde{\pi}_l d'_l \tilde{\pi}_l .$$

Similarly, for $l \leqq k - 1$,

$$\tilde{\pi}_l (d'_{k+1})^* v \in \operatorname{Image} \tilde{\pi}_l (d'_l)^* \tilde{\pi}_l .$$

Thus, for every $l \leqq k - 1$ we can find a $u_l \in E_l$ such that

$$\begin{aligned} d'_l u_l &= \tilde{\pi}_l d_{k+1} v , \\ (d'_l)^* u_l &= \tilde{\pi}_l d^*_{k+1} v . \end{aligned}$$

Let

$$\Phi'_1(v) = \Phi'(v) - \sum_{i=-1}^{k-1} \delta^{k-\imath+1} \Phi(u_l) .$$

Then Φ'_1 satisfies (2.12), (2.13) and (2.14) for $j = 1$. Continuing inductively, setting the coefficient of δ^{k+1+j} in (2.15) equal to 0 yields the desired extension. $\square$

Remark. Theorem 2.5(a) implies that for each $k \geqq -1$ we have a complex

$$0 \longrightarrow E^0_k \xrightarrow{\pi_k d_k \pi_k} E^1_k \xrightarrow{\pi_k d_k \pi_k} E^2_k \longrightarrow \cdots .$$

Theorem 2.5(b) implies that the cohomology of this sequence at the i^{th} step is isomorphic to E^i_{k+1}.

Thus the sequence of complexes has the structure of a spectral sequence. In Sect. 3 of this paper we prove that this spectral sequence converges to the cohomology of M. In the case that M is a fiber bundle, and the vector bundle is trivial, this is Theorem 7 of [Ma–Me]. Dai has observed ([Dai]) that in this case the above analytic sequence is isomorphic to the Leray spectral sequence. We also prove this in Sect. 4 as a special case of a more general theorem.

Before leaving this section, we prove that the spectral sequence comes equipped with more structure, namely a bigrading inherited from the bigrading on $\Omega^*(M,V)$. Let

$$E_k^{a,b} = \pi^{a,b} E_k \ ,$$

where

$$\pi^{a,b} : \Omega^*(M,V) \to \Omega^{a,b}(M,V)$$

denotes the canonical projection. We will prove that

$$E_k^p = \bigoplus_{a+b=p} E_k^{a,b} \tag{2.18}$$

and that

$$\pi_k d_k \pi_k \left(E_k^{a,b}\right) \subseteq E_k^{a-k+1,b+k} \ ,$$
$$\pi_k d_k^* \pi_k \left(E_k^{a,b}\right) \subseteq E_k^{a+k-1,b-k} \ . \tag{2.19}$$

We begin with a lemma which will also play a crucial role in Sect. 4 when we prove the equivalence between our spectral sequence and the Leray spectral sequence.

Let

$$\mathscr{D}_k = \{\omega_\delta \in \Omega[\delta] | d_\delta \omega_\delta \in \delta^k \Omega[\delta]\} \ . \tag{2.20}$$

We define a map

$$\tilde{d}_k : \mathscr{D}_k \to \Omega$$

by

$$\tilde{d}_k \omega_\delta = \lim_{\delta \to 0} \delta^{-k} d_\delta \omega_\delta \ . \tag{2.21}$$

Furthermore, let

$$D_k = \{\omega \in \Omega | \exists \omega_\delta \in \mathscr{D}_k \text{ with } \omega_\delta(0) = \omega\} \ . \tag{2.22}$$

Then we have

Lemma 2.7. *For all k*

(i) $$D_k = \bigoplus_{i=-1}^{k-1} \operatorname{Image} \pi_i d_i \pi_i + E_k \ ,$$

(ii) $$\tilde{d}_k \mathscr{D}_k = \bigoplus_{i=-1}^{k} \operatorname{Image} \pi_i d_i \pi_i \ .$$

Proof. Proof of (i): We have

$$E_{-1} = \Omega^* = \operatorname{Image} \pi_{-1} d_{-1} \pi_{-1} + \operatorname{Image} \pi_{-1} d_{-1}^* \pi_{-1} + E_0 \ .$$

Similarly,

$$E_0 = \mathrm{Image}\,\pi_0 d_0 \pi_0 + \mathrm{Image}\,\pi_0 d_0^* \pi_0 + E_1 \,.$$

Continuing in this fashion, for every k

$$\Omega^* = \bigoplus_{i=-1}^{k-1} \mathrm{Image}\,\pi_i d_i \pi_i + \bigoplus_{i=-1}^{k-1} \mathrm{Image}\,\pi_i d_i^* \pi_i + E_k \,.$$

To prove (i) we first show

$$\bigoplus_{i=-1}^{k-1} \mathrm{Image}\,\pi_i d_i \pi_i + E_k \subseteq D_k \,. \tag{2.23}$$

Clearly $E_k \subseteq D_k$ since every $w \in E_k$ has an extension ω_δ with

$$d_\delta \omega_\delta \in \delta^k \Omega[\delta] \,.$$

Moreover, if $\omega \in \mathrm{Image}\,\pi_i d_i \pi_i$, then

$$\omega = \lim_{\delta \to 0} \delta^{-i} d_\delta \Phi'(v)$$

for some $v \in \widetilde{E}_i$. Thus

$$\omega_\delta = \delta^{-i} d_\delta \Phi'(v)$$

is an extension of ω and $d_\delta \omega_\delta = 0$ so that for all i,

$$\mathrm{Image}\,\pi_i d_i \pi_i \subseteq D_\infty \subseteq D_k \,.$$

To prove the converse of (2.23) we will show

$$\left[\bigoplus_{i=-1}^{k-1} \mathrm{Image}\,\pi_i d_i^* \pi_i\right] \cap D_k = 0 \,.$$

Suppose, for simplicity

$$\omega \in \mathrm{Image}\,\pi_i d_i^* \pi_i \cap D_k$$

for some $i \leqq k-1$. Since

$$\omega \in \mathrm{Image}\,\pi_i d_i^* \pi_i \,,$$

we have

$$\omega = \lim_{\delta \to 0} \delta^{-i} d_\delta^* \Phi'(v)$$

for some $v \in \widetilde{E}_i$, so that

$$\omega_\delta = \delta^{-i} d_\delta^* \Phi'(v)$$

is an extension of ω. Since $\omega \in D_k$, ω has another extension $\tilde{\omega}_\delta \in \mathscr{D}_k$, so that

$$\omega_\delta - \tilde{\omega}_\delta = \delta z_\delta$$

for some $z_\delta \in \Omega[\delta]$. Thus

$$\begin{aligned} 0(\delta^k) \ni \langle d_\delta \tilde{\omega}_\delta, \Phi'(v)\rangle &= \delta^{-i}\langle d_\delta d_\delta^* \Phi'(v), \Phi'(v)\rangle - \delta\langle d_\delta z_\delta, \Phi'(v)\rangle \\ &= \delta^{-i}|d_\delta^* \Phi'(v)|^2 - \delta\langle z_\delta, d_\delta^* \Phi'(v)\rangle = \delta^i|\omega|^2 + 0(\delta^{i+1}) \end{aligned}$$

which implies, since $i \leqq k-1$, $\omega = 0$. The proof for general

$$\omega \in \left[\bigoplus_i \text{Image}\, \pi_i d_i^* \pi_i \right] \cap D_k$$

is the same.

Proof of (ii). The proof is by induction. $k = -1 : \mathscr{D}_{-1} = \Omega[\delta]$ and $\tilde{d}_{-1}(\omega + \delta\omega_1 + \cdots) = d^{2,-1}\omega = \pi_{-1} d_{-1} \pi_{-1}\omega$. Thus

$$\tilde{d}_{-1}\mathscr{D}_{-1} = \text{Image}\, \pi_{-1} d_{-1} \pi_{-1} \ .$$

General k: For any $\omega_\delta \in \mathscr{D}_k, \omega_\delta(0) \in D_k$, so from part (i)

$$\omega_\delta(0) \in \bigoplus_{i=-1}^{k-1} \text{Image}\, \pi_i d_i \pi_i + E_k \ .$$

As seen in the proof of part (i), every $v \in \text{Image}\, \pi_i d_i \pi_i$ has an extension v_δ with

$$d_\delta v_\delta = 0$$

so, in particular

$$\tilde{d}_k v_\delta = 0 \ .$$

Every $v \in E_k$ has an extension $v_\delta = \Phi'(v) \in \mathscr{E}_k \subseteq \mathscr{D}_k$ such that

$$\tilde{d}_k v_\delta = \pi_k d_k \pi_k v \ .$$

Thus

$$\tilde{d}_k \mathscr{D}_k \supseteq \text{Image}\, \pi_k d_k \pi_k \ .$$

On the other hand, for any $v \in D_k$ the set of all extensions is precisely

$$v_\delta + \delta\mathscr{D}_{k-1} \ ,$$

where v_δ is the extension given above, and

$$\tilde{d}_k(v_\delta + \delta\mathscr{D}_{k-1}) = \tilde{d}_k v_\delta + \tilde{d}_{k-1}\mathscr{D}_{k-1} \ .$$

Thus

$$\tilde{d}_k \mathscr{D}_k = \text{Image}\, \pi_k d_k \pi_k + \tilde{d}_{k-1}\mathscr{D}_{k-1} \ .$$

By induction, this is equal to

$$\bigoplus_{i=-1}^{k} \text{Image}\, \pi_i d_i \pi_i \ ,$$

as desired □

We are now prepared to prove (2.18) and (2.19)

Theorem 2.8. *For every p and k*

(i) $$E_k^p = \bigoplus_{a+b=p} E_k^{a,b} \ ,$$

(ii)
$$\pi_k d_k \pi_k(E_k^{a,b}) \subseteq E_k^{a-k+1,b+k} ,$$
$$\pi_k d_k^* \pi_k(E_k^{a,b}) \subseteq E_k^{a+k-1,b-k} ,$$

Proof. We first note that
$$E_k^p \subseteq \bigoplus_{a+b=p} E_k^{a,b} ,$$
so to prove (i) it is sufficient to show that if $a+b=p$,
$$E_k^{a,b} \subseteq E_k^p .$$

We will prove (i) and (ii) simultaneously inductively in k

$k=-1$: Part (i). Suppose
$$E_{-1}^p \ni \omega = \sum_{a+b=p} \omega^{a,b} ,$$
where
$$\omega^{a,b} \in \Omega^{a,b} .$$
We need to see that for all a and b,
$$\omega^{a,b} \in E_{-1}^p .$$
Since $\omega \in E_{-1}^p$ we have
$$d^{2,-1}\omega = (d^{2,-1})^*\omega = 0 ,$$
but
$$0 = d^{2,-1}\omega = \sum_{a,b} d^{2,-1}\omega^{a,b} . \tag{2.20}$$
Since
$$d^{2,-1}\omega^{a,b} \in \Omega^{a+2,b-1} ,$$
(2.20) implies that for all a and b
$$d^{2,-1}\omega^{a,b} = 0 .$$
Similarly,
$$(d^{2,-1})^*\omega = 0$$
implies that for all a and b
$$(d^{2,-1})^*\omega^{a,b} = 0$$
and thus
$$\omega^{a,b} \in E_{-1}^P .$$

Part (ii). We note simply that for
$$\omega^{a,b} \in E_{-1}^{a,b} ,$$
We have
$$\pi_{-1} d_{-1} \pi_{-1} \omega^{a,b} = d^{2,-1}\omega^{a,b} \in E_{-1}^{a+2,b-1} .$$
Similarly
$$\pi_{-1} d_{-1}^* \pi_{-1} \omega^{a,b} = (d^{2,-1})^*\omega^{a,b} \in E_{-1}^{a-2,b-+1}$$

as desired

General k: Part (i). Suppose

$$\omega^p = \sum_{a+b=p} \omega^{a,b} \in E_k^p .$$

Then certainly $\omega \in E_{k-1}^p$, so by induction on part (i)

$$\omega^{a,b} \in E_{k-1}^p .$$

For any $v \in E_{k-1}^p$,

$$v \in E_k^p \longleftrightarrow \pi_{k-1} d_{k-1} \pi_{k-1} v = \pi_{k-1} d_{k-1}^* \pi_{k-1} v = 0 .$$

But $\omega \in E_k^p$ implies

$$0 = \pi_{k-1} d_{k-1} \pi_{k-1} w = \sum_{a,b} \pi_{k-1} d_{k-1} \pi_{k-1} \omega^{a,b} . \tag{2.21}$$

By induction on part (ii),

$$\pi_{k-1} d_{k-1} \pi_{k-1} \omega^{a,b} \in \Omega^{a-k+2,b+k-2} .$$

So that (2.21) implies

$$\pi_{k-1} d_{k-1} \pi_{k-1} \omega^{a,b} = 0 . \tag{2.22}$$

Similarly

$$\pi_{k-1} d_{k-1}^* \pi_{k-1} \omega = 0$$

implies that

$$\pi_{k-1} d_{k-1}^* \pi_{k-1} \omega^{a,b} = 0 . \tag{2.23}$$

Together, (2.22) and (2.23) imply that for all a and b,

$$\omega^{a,b} \in E_k^p$$

as desired.

Part (ii). Suppose $\omega^{a,b} \in E_k^{a,b}$ so that there exists a polynomial

$$\omega_\delta = \omega^{a,b} + \delta\omega_1 + \delta^2\omega_2 + \cdots$$

with

$$d_\delta \omega_\delta, d_\delta^* \omega_\delta \in \delta^k \Omega[\delta] .$$

Then

$$d_\delta \omega_\delta = d_\delta \left(\omega^{a,b} + \sum_{i>0} \delta^i \omega_i \right)$$

$$d_\delta \left(\omega^{a,b} + \sum_{i>0} \delta^i \omega_i^{a-i,b+1} \right) + \sum_{c \neq 0} d_\delta \left(\sum_{i>0} \delta^i \omega_i^{a+c-i,b-c+i} \right) . \tag{2.24}$$

Let

$$\tilde{\omega}_\delta = \omega^{a,b} + \sum_{i>0} \delta^i \omega_i^{a-i,b+i} .$$

The first term on the right-hand side of (2.24) is equal to

$$d_\delta \tilde{\omega}_\delta = \delta^{-1}(d^{2,-1}\omega^{a,b}) + (d^{1,0}\omega^{a,b} + d^{2,-1}\omega_1^{a-1,b+1}) + \cdots .$$

Writing, for any $v_\delta \in \Omega[\delta]$,

$$d_\delta v_\delta = \delta^{-1}\tilde{d}_{-1}v_\delta + \tilde{d}_0 v_\delta + \delta \tilde{d}_1 v_\delta + \cdots .$$

we see that

$$\tilde{d}_j \tilde{\omega}_\delta \in \Omega^{a-j+1,b+j} . \tag{2.25}$$

Similarly, for all $c \neq 0$,

$$\tilde{d}_j \left(\sum_{i>0} \delta^i \omega_i^{a+c-i,b-c+i} \right) \in \Omega^{a+c-j+1,b-c+j-1} . \tag{2.26}$$

Since $d_\delta \omega_\delta \in 0(\delta^k)$ we have that for $j < k$,

$$0 = \tilde{d}_j \omega_\delta = \tilde{d}_j \tilde{\omega}_\delta + \sum_c \tilde{d}_j \left(\sum_{i>0} \delta^i \omega_i^{a+c-i,b-c,+i} \right) .$$

From (2.25) and (2.26) we learn that for $j < k$ and every $c \neq 0$,

$$0 = \tilde{d}_j \tilde{\omega}_\delta = \tilde{d}_j \left(\sum_{i>0} \delta^i \omega_i^{a+c-i,b-c+i} \right) .$$

In particular, for $c \neq 0$,

$$\sum_{i>0} \delta^i \omega^{a+c-i,b-c+i} \in \mathscr{D}_k$$

which implies

$$\sum_{i>0} \delta^{i-1} \omega_i^{a+c-i,b-c+i} \in \mathscr{D}_{k-1} .$$

Therefore, by Lemma 2.7, for $c \neq 0$,

$$\tilde{d}_k \left(\sum_{i>0} \delta^i \omega_i^{a+c-i,b-c+i} \right) = \tilde{d}_{k-1} \left(\sum_{i>0} \delta^{i-1} \omega_i^{a+c-i,b-c+i} \right)$$
$$\in \bigoplus_{i=-i}^{k-1} \operatorname{Image} \pi_i d_i \pi_i \subseteq E_k^\perp . \tag{2.27}$$

By induction on part (i), for $i < k$,

$$E_i^p = \bigoplus_{a+b=p} E_i^{a,b} .$$

So that for $i < k$, and every a and b,

$$v \in \Omega^{a,b} \Longrightarrow \tilde{\pi}_i v \in \Omega^{a,b} . \tag{2.28}$$

Together, (2.25), (2.27) and (2.28) imply

$$\pi_k d_k \pi_k \omega^{a,b} = \pi_k \tilde{d}_k \omega_\delta = \pi_k \tilde{d}_k \tilde{\omega}_\delta = \tilde{d}_k \tilde{\omega}_\delta - \sum_{i<k} \tilde{\pi}_i \tilde{d}_k \tilde{\omega}_\delta \in \Omega^{a-k+1,b+k} .$$

The same argument shows

$$\pi_k d_k^* \pi_k \omega^{a,b} \in \Omega^{a+k-1,b-k} ,$$

as desired. □

3. The Convergence of the Sequence $\{E_k^p\}$

We now describe the sense in which this spectral sequence converges to the cohomology of M with values in V. (This section essentially follows the ideas of Sects. 2, 5 and 6 of [Ma-Me]).

In Theorem 3.3 we prove that the space of formal Laurent series in δ with values in $\Omega^p(V)$ has a Hodge decomposition

$$(d_\delta \text{ exact}) \oplus (d_\delta \text{ coexact}) \oplus (d_\delta \text{ harmonic}) .$$

This implies that the space of harmonic Laurent series is isomorphic to the Laurent cohomology

$$(d_\delta \text{ closed Laurent series})/(d_\delta \text{ exact Laurent series}) .$$

This isomorphism is as modules over $\mathscr{L}$, the space of real formal Laurent series.

On the one hand, we observe (3.4) that the Laurent cohomology is isomorphic to the space of Laurent series with values in the usual deRham cohomology $H^p(M, V)$.

On the other hand we show (Theorem 3.1) that every $v \in E_\infty^p$ has an extension to a formal power series which is formally harmonic, and that these form a basis, modulo $\mathscr{L}$, of the harmonic Laurent series.

Thus, combining the above observations, we have, in particular, that

$$\dim E_\infty^p = \dim H^p(M, V) .$$

In Sect. 5 of this paper we show, with some additional geometric hypotheses, that as $\delta \to 0$ the space of g_δ harmonic forms approaches the space E_∞^p.

We begin by proving that every $v \in E_\infty^p$ is the value at $\delta = 0$ of a formally harmonic power series.

Theorem 3.1. *Suppose $v \in E_\infty^p$. Then there is a unique formal power series*

$$v_\delta = v + \delta v_1 + \delta^2 v_2 + \delta^3 v_3 + \dots \tag{3.1}$$

such that

$$v_i \perp E_\infty^p \quad \textit{for all } i \geqq 1 \tag{3.2}$$

and, formally,

$$d_\delta v_\delta = d_\delta^* v_\delta = 0 . \tag{3.3}$$

Proof. Uniqueness: If v_δ and v'_δ are two power series of the form (3.1), then $v_\delta - v'_\delta$ is a formal power series satisfying (3.3). Thus, the first non-zero coefficient must be in E_∞^p. On the other hand all coefficients of $v_\delta - v'_\delta$ are in $(E_\infty^p)^\perp$. Thus $v_\delta - v'_\delta$ must be 0.

Existence: For every $v \in E_\infty^p$ and every $K > N$, there is a polynomial

$$v_{\delta,K} = v + \delta v_1 + \dots + \delta^j v_j$$

such that

$$d_\delta v_\delta,\ d_\delta^* v_\delta \in 0(\delta^K)\,.$$

The form v_1 is not uniquely determined since we can add any form-valued polynomial $\delta\omega_\delta$ with $\omega_\delta \in \mathscr{E}_{K-1}$. Any such polynomial must satisfy $\omega_0 \in E^p_{K-1} = E^p_N = E^p_\infty$. Thus, the non-uniqueness of v_1 is given precisely by E^p_N. Hence we can choose a $v_{\delta,K}$ with $v_1 \in E^\perp_\infty$. This value of v_1 is unique and independent of K. If $K > N+1$, then our choice of v_2 is again determined up to E_∞, so we can choose $v_{\delta,K}$ with $v_1, v_2 \in E^\perp_\infty$. Continuing in this fashion, we can find a $v_{\delta,K}$ with $v_1, \ldots, v_{K-N} \in E^\perp_\infty$.

These forms are uniquely determined. For fixed j, the values of v_i, $1 \leqq i \leqq j$ are fixed for $K \geqq N+j$. Hence, as $K \to \infty$, the $v_{\delta,K}$ chosen in this fashion converge to a power series satisfying (3.2) and (3.3). □

For simplicity, it is useful to introduce the notion of a Laurent series of forms. Define $\mathscr{L}[\Omega^p]$ to be the space of Laurent series with coefficients in Ω^p, i.e. and element of $\mathscr{L}[\Omega^p]$ is of the form

$$v_\delta = \sum_{j \geqq a} \delta^j v_j$$

with $a \in \mathbf{Z}$ and $v_j \in \Omega^p$ for all j. The operator d_δ maps $\mathscr{L}[\Omega^p]$ to $\mathscr{L}[\Omega^{p+1}]$ and satisfies $d_\delta^2 = 0$. Thus we can define, for $0 \leqq p \leqq \dim M$,

$$H^p_{\mathscr{L}} = Z^p_{\mathscr{L}} / B^p_{\mathscr{L}}\,,$$

where $Z^p_{\mathscr{L}}$ is the kernel of the map

$$d^p_\delta : \mathscr{L}[\Omega^p] \to \mathscr{L}[\Omega^{p+1}]$$

and $B^p_{\mathscr{L}}$ is the image of d^{p-1}_δ. At first it is simpler to consider instead the space

$$\overline{H}^p_{\mathscr{L}} = \overline{Z}^p_\delta / \overline{B}^p_\delta\,,$$

where

$$\overline{Z}^p_{\mathscr{L}} = \text{kernel of } d \subset \mathscr{L}[\Omega^p]\,,$$

where d is the usual d operator acting term by term on Laurent series, and

$$\overline{B}^p_{\mathscr{L}} = \text{image of } d \subset \mathscr{L}[\Omega^p]\,.$$

Clearly

$$\overline{Z}^p_{\mathscr{L}} = \mathscr{L}[Z^p]\,,$$

where $Z^p \subset \Omega^p$ is the kernel of d, i.e. the usual space of closed forms, and $\mathscr{L}[Z^p]$ is the space of Laurent series with coefficients in Z.

Similarly,

$$\overline{B}^p_{\mathscr{L}} = \mathscr{L}[B^p]\,,$$

where $B^p \subset \Omega^p$ is the image of d, i.e. the usual space of exact forms. Thus

$$\overline{H}^p_{\mathscr{L}} = \overline{Z}^p_\delta / \overline{B}^p_\delta \cong \mathscr{L}[H^p(M, V)]\,.$$

Note that $\overline{H}^p_{\mathscr{L}}$ is an $\mathscr{L}$ denotes the ring of formal Laurent series with real coefficients, isomorphic (as $\mathscr{L}$-modules) to

$$H^p(M, V) \otimes_{\mathbf{R}} \mathscr{L}$$

So that, in particular,

$$\dim_{\mathscr{L}} \overline{H}^p_{\mathscr{L}} = \dim_{\mathbf{R}} H^p(M, V) .$$

Now observe that the map ρ_δ defined in Sect. 1 induces an isomorphism

$$\rho_\delta : \mathscr{L}[\Omega^p] \to \mathscr{L}[\Omega^p] .$$

Moreover, ρ_δ maps $\overline{Z}^p_\delta$ to $Z^p_\delta, \overline{B}^p_\delta$ to B^p_δ, and commutes with the action of $\mathscr{L}$. Thus ρ_δ induces an isomorphism as $\mathscr{L}$ modules

$$\rho_\delta : \overline{H}^p_{\mathscr{L}} \to H^p_{\mathscr{L}} . \tag{3.4}$$

Our goal now is to relate the space E^p_∞ to $H^p_{\mathscr{L}}$. From Lemma 3.1 there is a map

$$v \in E^p_\infty \mapsto v_\delta = v + \delta v_1 + \delta^2 v_2 + \ldots$$

with v_δ satisfying (3.2) and (3.3). The linearity of the map $v \mapsto v_\delta$ follows from the uniqueness of v_δ. That is, if $v_1 + v_2 = v$, then $(v_1)_\delta + (v_2)_\delta$ satisfies (3.2) and (3.3) so we must have

$$(v_1)_\delta + (v_2)_\delta = v_\delta .$$

For $v \in E^p_\infty$, we have $d_\delta v_\delta = 0$, so $[v_\delta]$ represents a class in $H^p_{\mathscr{L}}$. This map extends to a map

$$E^p_\infty \otimes \mathscr{L} \to H^p_{\mathscr{L}} , \tag{3.5}$$

where the element

$$\sum_{j \geqq a} \delta^j v_j \in E^p_\infty \otimes \mathscr{L}$$

(so that $v_j \in E^p_\infty$ for all j), is mapped to the element

$$\sum_{j \geqq a} \delta^j [(v_j)_\delta] \in H^p_{\mathscr{L}}$$

In fact, the map (3.5) is an isomorphism. This will be proved in 2 steps

(i) The map (3.5) is an injection:
This follows from the following lemma

Lemma 3.2. *If* $w \in \mathscr{L}[\Omega]$ *satisfies*

$$d^*_\delta d_\delta w = 0 ,$$

then

$$d_\delta w = 0 .$$

Proof. Suppose $w = \sum_{j \geqq a} \delta^j w_j$, and let

$$w^k = \sum_{a \leqq j \leqq k} \delta^j w_j .$$

Then

$$d^*_\delta d_\delta w^k = d^*_\delta d_\delta (w^k - w) \in \delta^{k-1} \Omega^p[[\delta]] ,$$

so that

$$\langle d_\delta w^k, d_\delta w^k \rangle = \langle d^*_\delta d_\delta w^k, w^k \rangle \in \delta^{k+a-1} \Omega^p[[\delta]] .$$

This implies

$$d_\delta w^k \in \delta^{\left[\frac{k+a-1}{2}\right]}\Omega^p[[\delta]]\,. \tag{3.6}$$

We also know

$$d_\delta w - d_\delta w^k \in \delta^k \Omega^p[[\delta]]\,. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$d_\delta w \in \delta^{l(k)}\Omega^p[[\delta]]\,,$$

where

$$l(k) = \min\left\{\left[\frac{k+a-1}{2}\right], k\right\} = \left[\frac{k+a-1}{2}\right] \quad \text{for } k \text{ large enough}\,.$$

Letting $k \longrightarrow \infty$ proves the lemma. □

If v_δ is harmonic and $v_\delta = d_\delta w_\delta$, then

$$0 = d_\delta^* v_\delta = d_\delta^* d_\delta w_\delta$$

which implies, by Lemma 3.2, that

$$v_\delta = d_\delta w_\delta = 0\,.$$

Thus, for $0 \neq v \in E_\infty^p$, $[v_\delta] \in H_{\mathscr{L}}^p$ is non-zero, so the map (3.5) is an injection.

(ii) The map (3.5) is a surjection:

This follows from the existence of a Hodge decomposition for Laurent series of forms:

Theorem 3.3. *If $v \in \mathscr{L}[\Omega^p]$ then there exist $v_1, v_2, v_3 \in \mathscr{L}[\Omega^p]$ such that*

a) *v_1 is harmonic,*

b) $v_2 \in d_\delta \mathscr{L}[\Omega^{p-1}]$,

c) $v_3 \in d_\delta^* \mathscr{L}[\Omega^{p+1}]$,

d) $v = v_1 + v_2 + v_3$.

Moreover, v_1, v_2 and v_3 are uniquely determined by v. (Note that if $d_\delta v = 0$ then we have $v_3 = 0$ by Lemma 3.2, and this proves (3.5) is a surjection).

Proof. The uniqueness of the v_i is clear. To prove existence we work modulo harmonic Laurent series.

Write

$$v = \sum_{j \geqq a} \delta^j v_j\,.$$

By adding the harmonic series $-\delta^a(\pi_\infty v_a)_\delta$ to v we can assume $v_a \in E_\infty^\perp$. Then by adding $-\delta^{a+1}(\pi_\infty v_{a+1})_\delta$ to v we can assume $c_{a+1} \in E_\infty^\perp$. Continuing in this fashion, modulo harmonic series we can write

$$v = \sum_{j \geqq a} \delta^j \sum_{k=-1}^{N-1} v_{j,k}$$

with $v_{j,k} \in \tilde{E}_k^p$ for all j, k.

We define a linear ordering on the pairs $(j, k), j \in \mathbf{Z}, -1 \leqq k \leqq N-1$, as follows:

Say $(j, k) < (j', k')$ if and only if

$$j + k < j' + k' \text{ or } j + k = j' + k' \text{ and } j < j' .$$

Let $(j_0, k_0) = \min\{(j,k) | v_{j,k} \neq 0\}$. From Theorem 2.5(b), we know that

$$\Delta_{k_0} : \tilde{E}^p_{k_0} \to \tilde{E}^p_{k_0}$$

is an isomorphism. Therefore there exist $w_1 \in \tilde{E}^{p-1}_{k_0}, w_2 \in \tilde{E}^{p+1}_{k_0}$ such that

$$v_{j_0,k_0} = d_{k_0} w_1 + d^*_{k_0} w_2 . \tag{3.8}$$

Let

$$v' = v - d_\delta(\delta^{j_0}\Phi'(w_1)) - d^*_\delta(\delta^{j_0}\Phi'(w_2)) .$$

As before, modulo harmonic series we can write

$$v' = \sum_{j \geqq a} \delta^j \sum_{k=-1}^{N-1} v'_{j,k}$$

with $v'_{j,k} \in \tilde{E}_k$ for all j, k. Formula (3.8) implies $v'_{j_0,k_0} = 0$. Corollary 2.6 implies that if

$$(j_1, k_1) = \min\{(j,k) | v'_{j,k} \neq 0\} ,$$

then $(j_1, k_1) > (j_0, k_0)$. Thus we can repeat the operation and in this fashion construct the desired v_2 and v_3 term by term. $\square$

4. The Leray Spectral Sequence

In this section we show that the spectral sequence $\{E^p_k, \pi_k d_k \pi_k\}$, or more precisely $\{\mathscr{L}(E^p_k), \pi_k d_k \pi_k\}$, is isomorphic to the Leray spectral sequence associated to a filtered differential complex constructed from M and the splitting $TM = A + B$. As a corollary, we learn that for all p and k the dimension of E^p_k is independent of the metric $g = g_A + g_B$.

In addition, in the case that A is integreable, we show that $\{E^p_k, \pi_k d_k \pi_k\}$ is isomorphic to the standard Leray spectral sequence associated to the corresponding foliation.

Let us briefly recall Leray's construction of a spectral sequence from a filtered differential complex. (See [McC] Sect. 2.2 for general discussion.)

Let $\{K, \partial\}$ be a differential complex. That is,

$$K = \bigoplus_i K^i, \partial(K^i) \subseteq K^{i+1} \quad \text{and} \quad \partial^2 = 0 .$$

Suppose further that

$$K = \cdots \supseteq K_{-1} \supseteq K_0 \supseteq K_1 \supseteq \ldots$$

is a filteration by subcomplexes, i.e.

$$K = \bigcup_i K_i \quad \text{and} \quad \partial(K_i) \subseteq K_i \, .$$

Then we get a spectral sequence as follows: Define

$$Z_r^{p,q} = K_p^{p+q} \cap \partial^{-1}(K_{p+r}^{p+q+1}) \, , \tag{4.1}$$

$$B_r^{p,q} = K_p^{p+q} \cup \partial(K_{p-r}^{p+q+1}) \, , \tag{4.2}$$

and

$$e_r^{p,q} = Z_r^{p,q} / Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q} \, . \tag{4.3}$$

Then ([McC] Theorem 2.1) the operator ∂ induces a differential ∂_r on the spaces $e_r^{p,q}$ such that

$$e_{r+1} = H^*(e_r, \partial_r) \, ,$$

and thus the complexes $\{e_r, \partial_r\}$ form a spectral sequence. Lastly, we define

$$e_r^l = \bigcup_\alpha \bigoplus_{p \geqq \alpha} e_r^{p,l-p} \, ,$$

so that

$$\partial_r : e_r^l \to e^{l+1} \, .$$

To place the spectral sequence $\{E_k^p, \pi_k d_k \pi_k\}$ in this general framework, we borrow an idea from Sect. 3 and let

$$K^p = \mathscr{L}[\Omega^p]$$

where $\mathscr{L}[\Omega^p]$ denotes the space of Laurent series in δ with coefficients in Ω^p. Let

$$K_i^p = \delta^i \Omega^p[[\delta]]$$

so that

$$K^p = \cdots \supseteq K_{-1}^p \supseteq K_0^p \supseteq K_1^p \supseteq \dots \, .$$

For the differential, we take

$$\partial = \delta d_\delta = d^{2,-1} + \delta d^{1,0} + \delta^2 d^{0,1} + \delta^3 d^{-1,2} \, ,$$

so that

$$\partial(K_i) \subseteq K_i \, .$$

Note also that

$$H^p(K, \partial) = H_p(K, d_\delta) \cong \mathscr{L}[E_\infty^p] \, .$$

In fact, we have

Theorem 4.1. *With* K_i *and* ∂ *defined as in* (4.1), (4.2) *and* (4.3), *the induced spectral sequence satisfies*:

For every r

$$\{e_r, \partial_r\} \cong \{\mathscr{L}[E_{r-1}], \pi_{r-1} d_{r-1} \pi_{r-1}\} \, ,$$

where $\pi_{r-1} d_{r-1} \pi_{r-1}$ *acts term by term on* $\mathscr{L}[E_{r-1}]$.

Proof. We begin by defining the space

$$\overline{\mathscr{D}}_k^p = \{\omega_\delta \in \Omega^p[[\delta]] | d_\delta \omega_\delta \in \delta^k \Omega^{p+1}[[\delta]]\} \, .$$

Note that $\overline{\mathscr{D}}_k^p$ is just a completion of the space $\mathscr{D}_k^p$ defined in (2.20).

Then

$$Z_r^{p,q} = \delta^p \Omega^{p+q}[[\delta]] \cap \partial^{-1}(\delta^{p+r}\Omega^{p+q+1}[[\delta]]) = \delta^p \overline{\mathscr{D}}_{r-1}^{p+q},$$

$$B_r^{p,q} = \delta^p \Omega^{p+q}[[\delta]] \cap \partial(\delta^{p-r}\Omega^{p+q-1}[[\delta]]) = \delta^{p-r}\partial\overline{\mathscr{D}}_{r-1}^{p+q-1},$$

and

$$e_r^{p,q} = \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q+1} + B_{r-1}^{p,q}} = \frac{\delta^p \overline{\mathscr{D}}_{r-1}^{p+q}}{\delta^{p+1}\overline{\mathscr{D}}_{r-2}^{p+q} + \delta^{p-r+1}\partial\overline{\mathscr{D}}_{r-2}^{p+q-1}}, \tag{4.4}$$

$$\cong \delta^p \left[\frac{D_{r-1}^{p+q}}{\tilde{d}_{r-1}\overline{\mathscr{D}}_{r-2}^{p+q-1}}\right] \tag{4.5}$$

(where D_{r-1}^{p+q} and $\tilde{d}_{r-2}$ are defined in (2.21) so that

$$e_r^l = \bigcup_\alpha \bigoplus_{p \geqq \alpha} e_r^{p,l-p} \cong \mathscr{L}\left[\frac{D_r^l}{\tilde{d}_{r-2}\overline{\mathscr{D}}_{r-2}^{l-1}}\right].$$

The theorem follows from the following two facts:

(i) For all l and r

$$D_r^l = \bigoplus_{i=-1}^{r-1} \operatorname{Image}(\pi_i d_i \pi_i|_{E_i^{l-1}}) + E_r^l.$$

(ii) For all l and r

$$\tilde{d}_r(\overline{\mathscr{D}}_r^{l-1}) = \bigoplus_{i=-1}^{r} \operatorname{Image}(\pi_i d_i \pi_i|_{E_i^{l-1}})$$

which is precisely the content of Lemma 2.7. (The slight difference between $\mathscr{D}_r^k$ and $\overline{\mathscr{D}}_r^k$ does not affect the lemma or the proof).

Now, we note that (i) and (ii), combined with (4.5) clearly yield

$$e_r^l \cong \mathscr{L}[E_{r-1}^l].$$

Moreover, the correspondence between ∂_r and $\pi_{r-1}d_{r-1}\pi_{r-1}$ follows as well:

For a class $\alpha \in e_r^{p,q}$, let $\delta^p\overline{\alpha} \in \delta^p\mathscr{D}_{r-1}^{p=q}$ be any representative of α (using the formulation (4.5)), and $\overline{\alpha}_\delta \in \overline{\mathscr{D}}_{r-1}^{p+q}$ any extension of $\overline{\alpha}$.

Then $\delta^p\overline{\alpha}_\delta$ represents α in (4.4), and by definition

$$\partial_r\alpha = [\partial\delta^p\overline{\alpha}_\delta],$$

where $[\partial\delta^p\overline{\alpha}_\delta]$ denotes the class in $e_r^{p+1,q-1}$ represented by $\partial\delta^p\overline{\alpha}_\delta$. From (i) and (ii) there is a unique $\overline{\alpha} \in \delta^p E_{r-1}^{p+q}$ which represents α. We can then choose $\overline{\alpha}_\delta = \Phi'(\alpha)$ (where Φ' is the extension defined in Theorem 2.4), so that

$$\partial_r\alpha = [\delta^p\partial\overline{\alpha}_\delta] = [\delta^{p+1}d_\delta\Phi'(\overline{\alpha})] = [\delta^{p+r}\pi_{r-1}d_{r-1}\pi_{r-1}\overline{\alpha}] \tag{4.6}$$

as claimed. □

The spectral sequence $\{e_k^p, \partial_k\}$ is defined independently of the metric $g = g_A + g_B$. Moreover, it is clear from the proof of Theorem 4.1 that the isomorphism

$$(e_k^p, \partial_k) \cong (\mathscr{L}[E_{k-1}^p], \pi_{k-1} d_{k-1} \pi_{k-1})$$

is as $\mathscr{L}$-modules. In particular, we learn

Corollary 4.2. *For all p and k, $\dim E_k^p$ is independent of the metrics $g_A$andg_B in the decomposition*

$$g = g_A + g_B .$$

For the remainder of this section we assume that A is integrable. In this case we know that

$$d^{2,-1} = 0$$

so that, in particular

$$d = d^{1,0} + d^{0,1} + d^{-1,2} . \tag{4.7}$$

This implies that our spectral sequence satisfies

$$\Omega^* = E_{-1} = E_0 .$$

There is a canonical filtration of the deRham complex associated to an integrable distribution. Namely, let

$$K = \Omega^*(V)$$

and

$$K_i = \bigoplus_{\substack{a \in z \\ b \geqq i}} \Omega^{a,b}(V) .$$

Then

$$K = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_{\dim B} \supseteq K_{\dim B+1} = 0$$

and (4.7) shows

$$d(K_i) \subseteq K_i .$$

This data gives rise to a Leray spectral sequence defined by (4.1), (4.2) and (4.3).

In analogy with our earlier analysis, to help identify the spaces $e_r^{p,q}$ we define

$$\mathscr{D}_r^{p,q} = \{\omega \in \Omega^{p+q} \cap K_q | d\omega \in \Omega^{p+q+1} \cap K_{q+r}\}$$

and

$$\mathscr{D}_r^{p,q} = \{\omega \in \Omega^{p,q} | \tilde{\omega} \in \mathscr{D}_r^{p,q} \text{ with } \pi^{p,q}\tilde{\omega} = \omega\} .$$

We also define the operator

$$\tilde{d}_r : \mathscr{D}_r^{p,q} \to D_r^{p-r+1,q+r}$$

by

$$\tilde{d}_r\omega : \pi^{p-r+1,q+r} d\omega .$$

(Note that

$$d(d\omega) = 0 \quad \text{and} \quad (d\omega) \in K_{q+r} \cap \Omega^{p+q+1}$$

imply

$$\pi^{p-r+1,q+r}(d\omega) \in D^{p-r+1,q+r} \subseteq D_r^{p-r+1,q+r}) .$$

Then

$$e_r^{p,q} \cong D_r^{p,q}/\tilde{d}_{r-1}\mathscr{D}_{r-1}^{p+r-2,q-r+1} . \tag{4.8}$$

The expression on the right-hand side of (4.8) can be simplified with the following lemma

Lemma 4.3. *For all* p, q *and* r

(i)
$$D_r^{p,q} = \left[\bigoplus_{i=0}^{r-1}(\mathrm{Image}\,\pi_i d_i \pi_i \cap \Omega^{p,q})\right] \oplus E_r^{p,q} ,$$

(ii)
$$\tilde{d}_r\mathscr{D}_r^{p,q} = \left[\bigoplus_{i=0}^{r}(\mathrm{Image}\,\pi_i d_i \pi_i \cap \Omega^{p-r+1,q+r})\right] .$$

Before proving this lemma we state two corollaries. Lemma 4.3 and (4.8) imply

$$e_r^{p,q} \cong E_r^{p,q} .$$

Moreover, the proof of (4.6) can easily be adapted to our context to prove that the differentials coincide. Thus we have

Corollary 4.4. *If* A *is integrable*

$$\{e_r^{p,q}, \partial_r\} \cong \{E_r^{p,q}, \pi_r d_r \pi_r\} ,$$

where $\{e_r^{p,q}, \partial_r\}$ *is the Leray spectral sequence associated with* A

We note that the Leray spectral sequence $\{e_r^{p,q}, \partial_r\}$ is defined without reference to the metric $g = g_A + g_B$. Moreover, it can be defined without reference to the distribution B. This can be done by observing

$$K_i = \bigoplus_p \{\omega \in \Omega^p | i_{a_1} i_{a_2} \dots i_{a_{p-i+1}}\omega = 0 \text{ (where } i = \text{interior product}$$
$$\text{for all } a_1, \dots, a_{p-i+1} \in A\} .$$

This implies

Corollary 4.5. *If* A *is integrable then for all* p, q *and* r

$$\dim E_r^{p,q}$$

depends only on the distribution A. *In particular*, $\dim E_r^{p,q}$ *is independent of the complementary distribution* B *as well as the metrices* g_A *and* g_B.

Thus it only remains to prove Lemma 4.3

Proof of Lemma 4.3:

(i) We first show

$$\left[\bigoplus_{i=0}^{r-1}(\mathrm{Image}\,\pi_i d_i \pi_i \cap \Omega^{p,q})\right] \oplus E_{\subseteq}^{p,q} D_r^{p,q} . \tag{4.9}$$

To prove $E_r^{p,q} \subseteq D_r^{p,q}$ we let

$$\omega^{p,q} \in E_r^{p,q} .$$

Then there exist $\omega_1, \omega_2, \dots$ with

$$d_\delta(\omega^{p,q} + \delta\omega_1 + \delta^2\omega_2 + \dots) \in \delta^r \Omega[\delta] ,$$

so that

$$\begin{aligned} &d^{1,0}\omega^{p,q} = 0 , \\ &d^{0,1}\omega^{p,q} + d^{1,0}\omega_1^{p-1,q+1} = 0 , \\ &d^{-1,2}\omega_{r-3}^{p-r+3,q-r-3} + d^{0,1}\omega_{r-2}^{p-r+2,q+r-2} + d^{1,0}\omega_{r-1}^{p-r+1,q+r-1} = 0 , \\ &\vdots \end{aligned}$$

From this it follows that

$$d\left(\omega^{p,q} + \sum_{i>0} \omega_i^{p-i,q+i}\right) \in K_{q+r}$$

which implies

$$\omega^{p,q} \in D_r^{p,q} .$$

From Theorem 2.8 we know

$$\mathrm{Image}\, \pi_i d_i \pi_i = \bigoplus_{p,q} \mathrm{Image}\, \pi_i d_i \pi_i |_{E_i^{p,q}}$$

and

$$\pi_i d_i \pi_i |_{E_i^{p,q}} \subset E_i^{p-i+1,q-i} .$$

For $\omega^{p,q} \in \mathrm{Image}\, \pi_i d_i \pi_i \cap \Omega^{p,q}$ there is a

$$v \in E_i^{p+i-1,q-i} .$$

with

$$\omega^{p,q} = \lim_{\delta \to 0} \delta^{-i} d_\delta \Phi'(v) ,$$

where

$$\Phi'(v) = v + \delta v_1 + \delta^2 v_2 + \dots$$

for some $v_1, v_2, \dots$.

Then

$$\begin{aligned} &d_\delta[v^{p+i-1,q-i} + \delta v_1^{p+i-2,q-i+1} + \delta^2 v_2 v^{p+i-3,q-i+2} + \dots] \\ &= \delta^i \omega^{p,q} + \delta^{i+1}\omega_1^{p-1mq+1} + \delta^{i+2}\omega_2^{p-2,q+2} \end{aligned}$$

for some

$$\omega_i^{p-i,q+i} \in \Omega^{p-i,q+i} .$$

Thus

$$d_\delta \left[\omega^{p,q} + \sum_{i>0} \delta^i \omega_i^{p-i,q+i}\right] = 0$$

or, setting $\delta = 1$

$$d\left[\omega^{p,q} + \sum_{i>0} \omega_i^{p-i,q+i}\right] = 0 . \tag{4.10}$$

From (4.10) it follows that

$$\omega^{p,q} \in D_{\infty}^{p,q} \subseteq D_{r}^{p,q} ,$$

which implies

$$\operatorname{Image} \pi_i d_i \pi_i \cap \Omega^{p,q} \subseteq D_r^{p,q} .$$

The converse of (4.9), as well as part (ii) of this lemma are proved similarly. That is, one simply follows the proof of Lemma 2.7 using the minor modifications demonstrated in our proof of (4.9). □

5. Small Eigenvalues and Eigenspaces

In this section we assume that A is integrable, and hence defines a foliation of M (so that $d^{2,-1} = d^{2,-1*} = 0$ and $E_{-1} = E_0$). In addition we make assumptions restricting the local geometry (see hypothesis (H1) and the global geometry (see hypothesis (H2)) of the foliation. We essentially assume that A is a Reimannian foliation with compact leaves, and g is a bundle-like metric (these terms will be defined shortly). There are many examples of such foliations. For example, if

$$F \hookrightarrow M \to N \tag{5.1}$$

is any fibration of M with compact fiber F then M can be given a metric g such that (5.1) is a Reimannian submersion. Then the foliation of M given by the fibers of (5.1) satisfies the hypotheses. Moreover, if G is a compact Lie group acting on M such that all orbits have the same dimension, and if the metric G is invariant under the action of G (such metrices always exist), then the foliation of M by the orbits of the action satisfy the hypotheses. See [Re] and [Mo] for other examples.

Under these restrictions we prove that for all p,

$$\dim E_2^p < \infty .$$

This was proven under more general hypotheses in [Sa].

We can then make precise statements about the behavior of the small eigenvalues of L_δ, as well as the corresponding eigenspaces, as $\delta \to 0$. A key role is played by the spaces $\tilde{E}_k^p$ and the operators δ_k^p defined in Sect. 2.

In particular, we show that the number of eigenvalues which are $\sim \delta^{2k}$ as $\delta \to 0$ is precisely $\dim \tilde{E}_k^p$ (Theorem 5.15), that the corresponding eigenspaces converge to $\tilde{E}_k^p$ (Theorem 5.17), and that the eigenvalues are asymptotic to the eigenvalues of

$$\delta^{2k} \Delta_k^p : \tilde{E}_k^p \to \tilde{E}_k^p \tag{5.2}$$

(Theorem 5.20). Recall that the operator Δ_k^p and the spaces $\tilde{E}_k^p$ in (5.2) are independent of δ, and defined purely in terms of the Taylor series analysis of Sect. 2. That is, loosely speaking, we demonstrate that under the hypotheses (H1) and (H2), the formal Taylor series analysis of Sect. 2 enables one to conclude precise quantitative statements concerning the asymptotics of the eigenvalues and eigenspaces of L_δ^p as $\delta \to 0$.

We begin this section with a statement of our hypotheses. We first assume

(H1) (M, A, g) is a Riemannian foliation with a bundle-like metric.

That is, we assume (M, A, g) satisfies one of the following equivalent conditions (see [Re] Proposition 4.2):

1) (M, A, g) locally has the structure of a Riemannian submersion.
2) For every vector field X tangent to $A, \mathscr{L}_X(g_B) = 0$ (where we have written $TM = A + B$, and $g = g_A + g_B$, and $\mathscr{L}_X$ denotes the Lie derivative in the direction X).
3) The distribution B is totally geodesic, i.e. for every $p \in M$, $b \in B_p$ and every extension of b to a neighborhood of p

$$(\nabla_b b)^A(p) = 0 ,$$

(Note that $(\nabla_b b)^A(p)$ is independent of the extension.) Let ∇_A denote the map

$$\nabla_A : \Omega^{i,j} \to T^*A \otimes \Omega^{i,j}$$

given by

$$\nabla_A \omega = \sum_{i=1}^{n} a^i \otimes \nabla^{0,0}_{a_i} \omega ,$$

where $\{a_i\}_{i=1,\dots,n}$ denotes a basis of A and $\{a^i\}$ is the dual basis of T^*A. The hypothesis on the local structure of A implies the following:

Lemma 5.1. *Assuming the hypothesis* (H1),

a) *The operator* $\square^{1,0} = d^{1,0*}(d^{1,0}) + (d^{1,0})^* d^{1,0}$ *has the form*

$$\square^{1,0} = (\nabla_A)^* \nabla_A + C_1$$

where C_1 *is a zero*$^{\text{th}}$ *order operator.*

b) *The operator* $d^{1,0}(d^{0,1})^* + (d^{0,1})^* d^{1,0}$ *has the form*

$$d^{1,0}(d^{0,1})^* + (d^{0,1})^* d^{1,0} = C_2 \circ \nabla_A + C_3$$

for zero$^{\text{th}}$ *order operators* C_2 *and* C_3. *That is, for a basis* $\{a_i\}$ *of* A *there are zero*$^{\text{th}}$ *order operators* $\{C_{2,i}\}$ *and* C_3 *such that*

$$d^{1,0}(d^{0,1})^* + (d^{1,0})^* d^{1,0} = \sum_i C_{2,i} \nabla^{0,0}_{a_i} + C_3 .$$

The same is true for the operator $(d^{1,0})^* d^{0,1} + d^{0,1}(d^{1,0})^*$.

c) *The operators* $d^{1,0}(d^{-1,2})^* + (d^{-1,2})^* d^{1,0}$ *and* $(d^{1,0})^* d^{-1,2} + d^{-1,2}(d^{1,0})^*$ *are zero*$^{\text{th}}$ *order. (This is true independent of* (H1).)

Proof. At $p \in M$, let $\{a_i\}$ be an orthonormal basis for A, $\{b_j\}$ an orthonormal basis for B, and $\{a^j\}$ and $\{b^j\}$ the dual bases. We now extend these bases to orthonormal bases in a neighborhood of p. Since B is totally geodesic, we can choose these extensions to be parallel in the B direction so that at p,

$$\nabla_{b_j} a_i = \nabla_{b_j} b_k = 0 \quad \text{for all } i, j, k .$$

We can choose the extension so that, in addition, at p

$$\nabla^{0,0}_{a_i} a_k = \nabla^{0,0}_{a_i} b_j = 0 \quad \text{for all } i, j, k .$$

Then
$$d^{1,0} = \sum_i a^i \nabla^{0,0}_{a_i} + \sum_j b^j \nabla^{1,-1}_{b_j} .$$
We observe that at p $\nabla^{1,-1}_{b_j} = 0$. Since
$$(d^{1,0})^* = -\sum \nabla^{0,0}_{a_k} i_{a_k} + C_1 ,$$
where C_1 is a zero$^{\text{th}}$ order operator which vanishes at p, we have at p,
$$\begin{aligned} d^{1,0}(d^{1,0})^* + (d^{1,0})^* d^{1,0} &= -\sum_{i,k} a^i \nabla^{0,0}_{a_i} \nabla^{0,0}_{a_k} i_{a_k} + \nabla^{0,0}_{a_k} i_{a_k} a^i \nabla^{0,0}_{a_i} \\ &\quad + (\text{zero}^{\text{th}} \text{ order term}) \\ &= (\text{using } \nabla^{0,0}_{a_i} a_j = 0 \text{ for all } i,j) \\ &\quad - \sum_{i,k} a^i a_k \nabla^{0,0}_{a_i} \nabla^{0,0}_{a_k} + i_{a_k} a^i \nabla^{0,0}_{a_k} \nabla^{0,0}_{a_i} + (\text{z.o.t.}) \\ &= -\sum_{i,k} (a^i i_{a_k} + i_{a_k} a^i) \nabla^{0,0}_{a_i} \nabla^{0,0}_{a_k} \\ &\quad - \sum_{i,k} i_{a_k} a^i (\nabla^{0,0}_{a_k} \nabla^{0,0}_{a_i} - \nabla^{0,0}_{a_i} \nabla^{0,0}_{a_k}) + \text{z.o.t.}). \end{aligned} \tag{5.3}$$
Since $a^i i_{a_k} + i_{a_k} a^i = \delta_{ik}$, the first term in (5.3) is, at p
$$-\sum_i \nabla^{0,0}_{a_i} \nabla^{0,0}_{a_i} = \nabla^*_A \nabla_A .$$
Since
$$\nabla_{a_k} \nabla_{a_i} - \nabla_{a_i} \nabla_{a_k} = \nabla_{[a_k, a_i]} + R(a_k, a_i) ,$$
where $R(a_k, a_i)$ is a zero$^{\text{th}}$ order curvature operator, we have
$$\begin{aligned} \nabla^{0,0}_{a_k} \nabla^{0,0}_{a_i} - \nabla^{0,0}_{a_i} \nabla^{0,0}_{a_k} &= R(a_i, a_k)^{0,0} + \nabla^{0,0}_{[a_k, a_i]} \\ &\quad - (\nabla^{1,-1}_{a_k} \nabla^{-1,1}_{a_i} + \nabla^{-1,1}_{a_k} \nabla^{1,-1}_{a_i} - \nabla^{1,-1}_{a_i} \nabla^{-1,1}_{a_k} - \nabla^{-1,1}_{a_i} \nabla^{-1,-1}_{a_k}) . \end{aligned}$$
For every $v, \nabla^{a,b}_v$ is 0^{th} order if $(a,b) \neq (0,0)$. Furthermore, $[a_k, a_i] \subset A$ so
$$[a_k, a_i] = \nabla^{0,0}_{a_k} a_i - \nabla^{0,0}_{a_i} = 0 \quad \text{at } p .$$
Thus, at p
$$\square^{1,0} = \nabla^*_A \nabla_A + \text{ zero}^{\text{th}} \text{ order terms.}$$
Since p is arbitrary, this proves (a).

Parts (b) and (c) are proved similarly. $\square$

Corollary 5.2. *There are constants c_1 and c_2 so that for all $\omega \in \Omega^*$,*
$$\begin{aligned} |(d^{1,0}(d^{0,1})^* + (d^{0,1})^* d^{1,0})\omega| &\leqq c_1(|\nabla_A \omega| + |\omega|) \\ &\leqq c_2(\langle \square^{1,0}\omega, \omega\rangle^{\frac{1}{2}} + |\omega|) . \end{aligned}$$

The assumption (H1) is a restriction on the local geometry of (M, A, g). Our second, and last, assumption restricts the global geometry.

(H2) We assume that the positive spectrum of the operator $\Box^{1,0} = d^{1,0}(d^{1,0})^* + (d^{1,0})^* d^{1,0}$ is bounded away from 0 by a positive constant.

That is, we assume there is a $c > 0$ such that $\omega \in (\ker \Box^{1,0})^{\perp}$ implies

$$\langle \Box^{1,0}\omega, \omega \rangle \geqq c|\omega|^2 .$$

Lemma 5.3. *If* (M, A, g) *satisfies* (H1) *and, in addition, all leaves of* A *are compact, then* (M, A, g) *satisfies* (H2).

Proof. First suppose A consists of the fibers of a Riemannian submersion $\pi : M \to N$ with compact fibers. Then $\omega \in \Omega^{i,j}(M, V)$ can be considered as a j-form on N with values in the infinite-dimensional bundle

$$\begin{array}{c} \Omega^i(F, V) \\ \downarrow \\ N \end{array}$$

(where $\Omega^i(A, V)$ denotes the bundle whose fiber at x is $\Omega^i(\pi^{-1}(x), V_{\pi^{-1}(x)})$.) For $\omega = \alpha \otimes \beta, \alpha \in \Omega^i(A, V)$ and $\beta \in \Omega^j(N)$, (or, more precisely, $\omega = \alpha \wedge \pi^* \beta$) we have

$$\begin{aligned} d^{1,0}\omega &= (d_A \alpha) \otimes \beta , \\ (d^{1,0})^* \omega &= (d_A^* \alpha) \otimes \beta , \\ \Box^{1,0}\omega &= (\Box_A \alpha) \otimes \beta , \end{aligned}$$

where d_A, d_A^* and $\Box_A$ are the differential, codifferential and Laplacian, resp., as an element of $\Omega^i(\pi^{-1}(x), V_{\pi^{-1}(x)})$. Thus

$$\ker \Box^{1,0} = \Gamma(N, \Lambda^* T^* N \otimes \mathscr{H}^*(A, V)) ,$$

where $\mathscr{H}^*(A, V)$ denotes the vector bundle over N whose fiber at x are the harmonic forms on $\pi^{-1}(x)$ with values in V. Moreover

$$\begin{aligned} &\inf\{\lambda \in \operatorname{spec} \Box^{1,0} \mid \lambda > 0\} \\ &\quad = \inf_{x \in N} [\inf\{\lambda \in \operatorname{spec} \Box_N : \Omega^*(\pi^{-1}(x), V_{\pi^{-1}(x)}) \supset| \lambda > 0\}] . \end{aligned} \tag{5.4}$$

The spectrum of $\Box_A$ varies continuously over N, and the multiplicity of 0 is constant. Thus the smallest positive eigenvalue is a continuous function on N and therefore achieves a positive minimum. This implies the infimum in (5.4) is positive, which is precisely (H2).

More generally, suppose A is any foliation with compact leaves satisfying (H1). Then, by Proposition 3.7 of [Mo], the leaf space $N = M/A$ has the structure of a compact Satake V-manifold (see [Mo] for a precise definition). We will show that the infimum in (5.4) is positive by showing that the function

$$\lambda(x) = \inf\{\lambda \in \operatorname{spec} \Box_A : \Omega^*(\pi^{-1}(x), V) \supset| \lambda > 0\} \tag{5.5}$$

is locally bounded away from 0 (from which (H2) follows by the compactness of N). M is a locally trivial fibration over the dense set U of non-singular points in N, and at any $x \in U$ the local boundedness of $\lambda(x)$ follows as before.

If x is a singular point of N, then all nearby leaves cover $\pi^{-1}(x)$, with the degree of the covers bounded above by some k. Suppose $\{\mathscr{C}_1,\ldots,\mathscr{C}_k\}$ are all the Riemannian covers of $\pi^{-1}(x)$ with degree $\leqq k$, and

$$\lambda(\mathscr{C}_i) = \inf\{\lambda \in \operatorname{spec}\square : \Omega^*(\mathscr{C}, V) \supset | \, \lambda > 0\}\,, \tag{5.6}$$

where V in (5.6) denotes the pull-back to $\mathscr{C}_i$ of the bundle V over $\pi^{-1}(x)$.

There is a neighborhood of x which can be stratified by sets $\sum_i$ such that $\{\pi^{-1}(x') \mid x' \in \sum_i\}$ are all diffeomorphic and as $x' \in \sum_i$ approaches $x, \pi^{-1})(x')$ approaches some $\mathscr{C}_j$. Thus, as $x' \in \sum_i$ approaches $x, \lambda(x')$ approaches $\lambda(\mathscr{C}_j) > 0$, which proves λ is locally bounded away from 0. $\square$

From now on, we assume hypotheses (H1) and (H2). The significance of these assumptions is hinted at in the following lemma.

Lemma 5.4. *There is a* $c > 0$ *such that for all* δ *small enough,*

$$L_\delta \geqq \frac{1}{2}\square^{1,0} + \delta^2(\square^{0,1} - C) \geqq \delta^2(\square^{1,0} + \square^{0,1} - C)\,,$$

where $\square^{a,b} = d^{a,b}(d^{a,b})^* + (d^{a,b})^* d^{a,b}$.

Proof.

$$\begin{aligned} L_\delta = \square^{1,0} &+ \delta^2\square^{0,1} + \delta^4\square^{-1,2} + \delta(d^{1,0}(d^{0,1})^* \\ &+ (d^{0,1})^* d^{1,0} + (d^{1,0})^* d^{0,1} + d^{0,1}(d^{1,0})^*) \\ &+ \delta^2(d^{1,0}(d^{-1,2})^* + (d^{-1,2})^* d^{1,0} + (d^{1,0})^* d^{-1,2} + d^{-1,2}(d^{1,0})^*) \\ &+ \delta^3(d^{0,1}(d^{-1,2})^* + (d^{-1,2})^* d^{0,1} + (d^{0,1})^* d^{-1,2} + d^{1,2}(d^{0,1})^*)\,. \end{aligned}$$

To simplify notation we will write

$$L_\delta = \square^{1,0} + \delta^2\square^{0,1} + \delta^4\square^{-1,2} + \delta K_1 + \delta^2 K_2 + \delta^3 K_3\,.$$

We observe that $\square^{-1,2}$ is positive, and from Lemma 5.1 (c), K_2 is a zero$^{\text{th}}$ order operator, so for some c_1,

$$L_\delta \geqq \square^{1,0} + \delta^2\square^{0,1} + \delta K_1 + \delta^3 K_3 - c_1\delta^2\,.$$

We now observe that for any ω

$$\langle \delta^3 K_3\omega, \omega\rangle = 2\langle \delta^{3/2} d^{0,1}\omega, \delta^{3/2} d^{-1,2}\omega\rangle + 2\langle \delta^{3/2}(d^{0,1})^*\omega, \delta^{3/2}(d^{-1,2})^*\omega\rangle\,,$$

so that

$$|\langle \delta^3 K_3\omega, \omega\rangle| \leqq \delta^3\langle \square^{0,1}\omega, \omega\rangle + \delta^3\langle \square^{-1,2}\omega, \omega\rangle\,.$$

Thus, for $\delta \leqq \frac{1}{2}$ there is a c_2 so that

$$L_\delta \geqq \square^{1,0} + \frac{1}{2}\delta^2\square^{0,1} + \delta K_1 - c_2\delta^2\,. \tag{5.7}$$

Lastly, we write

$$\delta\langle K_1\omega, \omega\rangle = \delta(\langle K_1\tilde{\pi}_0\omega, \tilde{\pi}_0\omega\rangle + 2\langle K_1\pi_1\omega, \tilde{\pi}_0\omega\rangle + \langle K_1\pi_1\omega, \pi_1\omega\rangle)\,. \tag{5.8}$$

We note that

$$\langle K_1 \pi_1 \omega, \pi_1 \omega \rangle = 0 \tag{5.9}$$

and, from Corollary 5.2, there is a c_3 such that

$$|K_1 \pi_1 \omega| \leqq c_3 |\pi_1 \omega| \,.$$

Thus, for any $\varepsilon > 0$,

$$\begin{aligned} |\delta \langle K_1 \pi_1 \omega, \tilde{\pi}_0 \omega \rangle| &\leqq 2\delta c_3 |\pi_1 \omega| |\tilde{\pi}_0 \omega| = 2 |\varepsilon \tilde{\pi}_0 \omega| \left| \frac{\delta c_3}{\varepsilon} \pi_1 \omega \right| \\ &\leqq \varepsilon^2 |\tilde{\pi}_0 \omega|^2 + \delta^2 \left(\frac{c_3}{\varepsilon^2} |\pi_1 \omega|^2 \right) \,. \end{aligned}$$

From (H2) we can choose ε small enough so that for all ω,

$$\frac{1}{4} \langle \square^{1,0} \omega, \omega \rangle \geqq \varepsilon^2 |\pi_0 \omega|^2 \,.$$

For such ε

$$2 \langle \delta K_1 \pi_1 \omega, \tilde{\pi}_0 \omega \rangle \geqq - \left(\frac{1}{4} \langle \square^{1,0} \omega, \omega \rangle + \delta^2 \left(\frac{c_3}{\varepsilon^2} \right) |\omega|^2 \right) \,. \tag{5.10}$$

From Corollary 5.2 there is a c_4 such that

$$|\langle \delta K_1 \tilde{\pi}_0 \omega, \tilde{\pi}_0 \omega \rangle| \leqq \delta c_4 \left(\langle \square^{1,0} \tilde{\pi}_0 \omega, \tilde{\pi}_0 \omega \rangle^{1/2} |\tilde{\pi}_0 \omega| + |\tilde{\pi}_0 \omega|^2 \right) \,. \tag{5.11}$$

We have from (H2),

$$\delta c_4 |\pi_0 \omega|^2 \leqq \delta c_5 \langle \square^{1,0} \omega, \omega \rangle \,,$$

so for δ small enough

$$\delta c_4 |\pi_0 \omega|^2 \leqq \frac{1}{8} \langle \square^{1,0} \omega, \omega \rangle \,. \tag{5.12}$$

Moreover

$$\delta c_4 \langle \square^{1,0} \tilde{\pi}_0 \omega, \tilde{\pi}_0 \omega \rangle^{1/2} |\tilde{\pi} \omega| = 2 \left(\frac{1}{\sqrt{8}} \langle \square^{1,0} \tilde{\pi}_0 \omega, \tilde{\pi}_0 \omega \rangle^{1/2} \right) \left(\frac{\delta c_4 \sqrt{8}}{2} |\pi_0 \omega|^2 \right)$$

$$\leqq \frac{1}{8} \langle \square^{1,0} \tilde{\pi}_0 \omega, \tilde{\pi} \omega \rangle + \delta^2) 2 c_4^2) |\pi_0 \omega|^2 \leqq \frac{1}{8} \langle \square^{1,0} \omega, \omega \rangle + \delta^2 (2 c_4^2) |\omega|^2 \,. \tag{5.13}$$

Combining (5.7)–(5.13) yields the theorem. □

We observe that $\square^{1,0} + \square^{0,1}$ is an elliptic 2nd order operator with positive symbol. Thus, for any $k \in \mathbf{R}$, only finitely many eigenvalues of $\square^{1,0} + \square^{0,1} - C$ are less than k.

Combining this observation with Lemma 5.4 we see that at most finitely many eigenvalues of L^p_δ can be in $0(\delta^4)$. Therefore, from (1.8) we learn

Corollary 5.5. *For all p*

$$\dim E_2^p < \infty \,.$$

Before leaving this topic, we present an implication of (H1) and (H2) which will play an important role in the analysis in this section.

Lemma 5.6. *There is a* $c > 0$ *such that*

$$|\tilde{\pi}_0 d^{0,1}\pi_1|, |\tilde{\pi}_0 (d^{0,1})^* \pi_1|, |\pi_1 d^{0,1}\tilde{\pi}_0|, |\pi_1 (d^{0,1})^* \tilde{\pi}_0|$$

are all bounded above by c.

Proof. We prove the lemma for $\tilde{\pi}_0 d^{0,1}\pi_1$. The bound for the other operators follows similarly. Suppose $\omega \in \Omega^p$ satisfies $\omega \in E_1 = \mathrm{Kernel}\,\square^{1,0} = \mathrm{Kernel}\, d^{1,0} \cap \mathrm{Kernel}(d^{1,0})^*$. Write $d^{0,1}\omega = \alpha + \beta$ with

$$\alpha \in E_1,$$
$$\beta \in \tilde{E}_0 = E_1^{\perp} = (\mathrm{Kernel}\,\square^{1,0})^{\perp} = \mathrm{Image}\, d^{1,0} \oplus \mathrm{Image}(d^{1,0})^* .$$

Since

$$d^{1,0}(d^{0,1}\omega) = -d^{0,1}(d^{1,0}\omega) = 0$$

and

$$d^{1,0}\alpha = 0$$

we must have

$$\beta \in \mathrm{Kernel}\, d^{1,0} .$$

Then, from Corollary 5.2,

$$\begin{aligned} c_1|\omega| &\geqq |(d^{0,1}(d^{1,0})^* + (d^{1,0})^* d^{0,1})\omega| \\ &= |(d^{1,0})^* d^{0,1}\omega| = |(d^{1,0})^*\beta| \\ &= |(d^{1,0} + (d^{1,0})^*)\beta| \geqq c_2|\beta| \end{aligned}$$

(from (H2))

$$= c_2|\tilde{\pi}_0 d^{0,1}\omega| .$$

Thus, we have proven Lemma 4.6 with $c = \frac{c_1}{c_2}$. $\square$

We now commence our study of the small eigenvalues of L_δ. We begin by showing that the number of eigenvalues which are in $o(\delta^2)$ as $\delta \to 0$ is precisely $\dim E_2^p$.

The first step is to show that the corresponding eigenspaces converge to E_2^p and $\delta \to 0$.

Lemma 5.7. *Given any* $c_1 > 0$ *there is a* $c_2 > 0$ *such that*:

If $\delta_i, i = 1, 2, 3, \ldots,$ *is any sequence with* $\delta_i \to 0$, *and* ω_i *any sequence of p-forms with* $|\omega_i| = 1$ *and*

$$\langle L_{\delta_i}\omega_i, \omega_i\rangle \leqq c_1\delta^2 ,$$

then

$$|\tilde{\pi}_0\omega_\imath| \leqq c_2\delta_\imath .$$

Proof. This follows directly from Lemma 5.4 and (H2) which imply

$$c_1\delta^2 \geqq \langle L_{\delta_i}\omega_i, \omega_i\rangle \geqq \langle \square^{1,0}\omega, \omega\rangle - c_3\delta^2 \geqq c_4|\tilde{\pi}_0\omega|^2 - c_3\delta^2 . \quad \square$$

Theorem 5.8. *If* $\delta_i, i = 1, 2, 3, \ldots,$ *is any sequence with* $\delta_i \to 0$, *and* ω_i *any sequence of p-forms satisfying* $|\omega_i| = 1$ *and*

$$\langle L_{\delta_i}\omega_i, \omega_i\rangle \in o(\delta^2)\,,$$

then a subsequence of the ω_i's converges (strongly in L^2) to an element $\omega \in E_2^p$.

Proof. Write $\omega_i = \alpha_i + \beta_i$ with

$$\alpha_i \in \tilde{E}_0^p = (\ker \square^{1,0})^\perp$$

and

$$\beta_i \in E_1^p = \ker \square^{1,0}\,.$$

From Lemma 5.7 it follows that there is a $c > 0$ with

$$|\alpha_i| < c\delta_i\,,$$

so that

$$|\beta_i| = 1 + O(\delta_i)\,.$$

The lemma follows once we show that a subsequence of the β_i's converges to an element in E_2^p.

(i) First we show that the β_i's are bounded in H_1. By definition we have

$$|d^{1,0}\beta_i| = |(d^{1,0})^*\beta| = 0\,,$$

so it is enough to show that for some $k > 0$,

$$|d^{0,1}\beta_i| < k,\ \ |(d^{0,1})^*\beta_i| < k\,.$$

Now $|d_\delta\omega|^2 \in o(\delta^2)$ implies

$$|d^{1,0}\omega + \delta d^{0,1}\omega + \delta^2 d^{-1,2}\omega| \in o(\delta^2) \Rightarrow \pi_1(d^{0,1}\omega + \delta d^{-1,2}\omega) \to 0 \Rightarrow \pi_1 d^{0,1}\omega \to 0\,.$$

But

$$\pi_1 d^{0,1}\omega = \pi_1 d^{0,1}(\alpha + \beta) = (\pi_1 d^{0,1}\tilde{\pi}_0)\alpha + \pi_1 d^{0,1}\beta\,.$$

From Lemma 5.6

$$|\pi_1 d^{0,1}\tilde{\pi}_0\alpha| \leqq c|\alpha| \in O(\delta)\,.$$

Thus

$$|\pi_1 d^{0,1}\beta| \to 0\,. \tag{5.14}$$

On the other hand

$$|\tilde{\pi}_0 d^{0,1}\beta| = |\tilde{\pi}_0 d^{0,1}\pi_1\beta| \leqq c|\beta| \leqq c|\omega| = c\,. \tag{5.15}$$

Together, (5.14) and (5.15) imply

$$|d^{0,1}\beta| < k\,.$$

Similarly, one can prove

$$|(d^{0,1})^*\beta| < k\,.$$

Thus the β_i's are bounded in H_1 and hence a subsequence of the β_i's converges weakly in H_1 ($\Rightarrow$ strongly in L^2) to some $\beta \in H_1$.

(ii) $\beta \in E_1^p$: Since $\beta_i \to \beta$ strongly in L^2, (2) implies $|\beta| = 1$. Now

$$0 = \tilde{\pi}_0\beta_i \to \tilde{\pi}_0\beta$$

implies $\tilde{\pi}_0\beta = 0$ so $\beta \in E_1^p$.

(iii) $\beta \in E_2^p$: Since $\beta_i \to \beta$ weakly in H_1 we have $d^{0,1}\beta_i \to d^{0,1}\beta$ weakly in L^2. Thus

$$\pi_1 d^{0,1}\beta_i \to \pi_1 d^{0,1}\beta$$

weakly in L^2. By (5.14) $|\pi_1 d^{0,1}\beta_i| \to 0$. This implies $\pi_1 d^{0,1}\beta_i \to 0$ strongly in L^2. Therefore (by the uniqueness of weak limits)

$$\pi_1 d^{0,1}\beta = 0 . \tag{5.16}$$

Similarly

$$\pi_1 (d^{0,1})^*\beta = 0 . \tag{5.17}$$

Together, (5.16) and (5.17) imply $\beta \in E_2^p$. □

Corollary 5.9.

$$\#\{\lambda_i^p \in \operatorname{spec}(L_\delta^p) \mid \liminf_{\delta\to 0} \delta^{-2}\lambda_i^p = 0\} = E_2^p .$$

Proof. The inequality $\geqq$ follows from (1.8). If there were a strict inequality, then we could find $\delta_i \to 0$ and $\omega_i \in \Omega^p$ with

$$|\omega_i| = 1,\ \langle L_{\delta_i}\omega_i, \omega_i\rangle \in o(\delta^2)$$

and $\langle \omega_i, \omega\rangle = 0$ for every $\omega \in E_2^p$, but this contradicts Theorem 5.8. □

We saw in (1.8) that

$$\#\{\lambda_i^p \in O(\delta^4)\} \geqq \dim E_2^p .$$

Combining this with Corollary 5.9 we have

Corollary 5.10. *There are constants $c_1, c_2 > 0$ such that for every i either*

$$\lambda_i^p \geqq c_1\delta^2 \quad \textit{for all } \delta$$

or

$$\lambda_i^p \leqq c_2\delta^4 \quad \textit{for all } \delta .$$

Our next goal is to prove analogous statements about eigenvalues which are $O(\delta^{2k})$.

To analyze the small eigenvalues, it is convenient to modify the $\tilde{E}_k$ spaces by using the extension map Φ' defined in Sect. 2.

Recall that for each k, Φ' is a linear map

$$\Phi' : \tilde{E}_k^p \to \Omega^p[\delta] ,$$

such that for $v \in \tilde{E}_k^p$

$$d_\delta \Phi'(v) \in \delta^k \tilde{E}_k^{p+1} + \delta^{k+1}\Omega^{p+1}[\delta] ,$$

$$d_\delta^* \Phi'(v) \in \delta^k \tilde{E}_k^{p-1} + \delta^{k+1}\Omega^{p-1}[\delta] .$$

For each $v \in E_N^p$, there is a formal power series

$$v_\delta = v + \delta v_1 + \delta^2 v_2 + \dots . \tag{5.18}$$

such that the map $v \mapsto v_\delta$ is linear, and, formally,

$$d_\delta v_\delta = d_\delta^* v_\delta = 0\,.$$

Define, for $v \in E_N^p$

$$\Phi'(v) = v + \delta v_1 + \delta^2 v_2 + \cdots + \delta^{N+2} v_{N+2}$$

(where the v_i's are as in (5.18)), so that

$$d_\delta \Phi'(v) \in \delta^{N+3}\Omega^{p+1}[\delta]\,,$$
$$d_\delta^* \Phi'(v) \in \delta^{N+3}\Omega^{p-1}[\delta]\,.$$

Now extend Φ' linearly to a map

$$\Phi' : \Omega^p \longrightarrow \Omega^p[\delta]\,.$$

For every $\delta \in [0,1]$ we can evaluate $\Phi'(v)$ at δ to get a map

$$\Phi' \mid_\delta : \Omega^p \longrightarrow \Omega^p\,.$$

Let $E_{k,\delta}^p$ be the image of E_k^p under this map, and let $\pi_{k,\delta}^\perp$ denote the orthogonal projection onto the complement of $E_{k,\delta}^p$. Define $\tilde{E}_{k,\delta}^p$ to be the orthogonal complement of $E_{k+1,\delta}^p$ in $E_{k,\delta}^p$, i.e.

$$\tilde{E}_{k,\delta}^p = \pi_{k+1,\delta}^\perp E_{k,\delta}^p$$

and $\tilde{\pi}_{k,\delta}$ the orthogonal projection onto $\widetilde{E}_{k,\delta}^p$.

Suppose $v_1, \ldots, v_r$ is the basis of E_{k+1}^p, so that $\Phi'(v_1)|_\delta, \ldots, \Phi'(v_r)|_\delta$ span $E_{k+1,\delta}^p$. If $v \in \tilde{E}_k^p$, the

$$\tilde{\pi}_{k,\delta}\Phi'(v) = \Phi'(v) - \sum_{j=1}^r \langle \Phi'(v), \Phi'(v_j)\rangle \Phi'(v_j)\,.$$

Each $\langle \Phi'(v), \Phi'(v_j)\rangle$ is in $O(\delta)$, since $\Phi'(v) = v + O(\delta), \Phi'(v_j) = v_j + O(\delta)$ and $\langle v, v_j\rangle = 0$. Therefore,

$$\tilde{\pi}_{k,\delta}\Phi'(v) = v + O(\delta)\,,$$
$$d_\delta \tilde{\pi}_{k,\delta}\Phi'(v) = d^k d_k' v + O(d^{k+1})$$

(since $d_\delta \Phi'(v_j) \in O(\delta^{k+1})$ for all j), and $d_\delta^* \tilde{\pi}_{k,\delta}\Phi'(v) = \delta^k (d_k')^* v + O(\delta^{k+1}$. Furthermore, we have that for every $\delta \in [0,1]$,

$$\tilde{E}_{k,\delta}^p = \{\tilde{\pi}_{k,\delta}\Phi'(v) \mid v \in \tilde{E}_p^k\}\,.$$

Suppose $k > 0$, and $v_\delta \in \tilde{E}_{k,\delta}^p$ for $\delta \in [0,1]$. Thus

$$v_\delta = \tilde{\pi}_{k,\delta}\Phi'(\tilde{v}_\delta)$$

for some 1-parameter family of elements

$$\tilde{v}_\delta \in \tilde{E}_k^p\,.$$

Then

$$\langle L_\delta v_\delta, v_\delta\rangle = \delta^{2k}\langle \Delta_k \tilde{v}_\delta, \tilde{v}_\delta\rangle + O(\delta^{2k+1}) \tag{5.19}$$

(where $\Delta_k = d_k'(d_k')^* + (d_k')^* d_k'$).

From Theorem 2.5 (b)

$$\Delta_k : \tilde{E}_k \to \tilde{E}_k$$

in invertible. Since $\tilde{E}_k$ is finite dimensional for $k > 1$, there are positive constants c_1 and c_2 such that for all $v \in \tilde{E}_k$,

$$c_1\delta^{2k}|v|^2 < \langle \Delta_k v, v\rangle < c_2\delta^{2k}|v|^2 . \tag{5.20}$$

Combining (5.19) and (5.20) we find

Lemma 5.11. *For every $k > 1$ there are constants $c_1, c_2 > 0$ such that for all $v_\delta \in \tilde{E}_{k,\delta}$,*

$$c_1\delta^{2k}|v_\delta|^2 < \langle L_\delta v_\delta, v_\delta\rangle < c_2\delta^{2k}|v_\delta|^2 .$$

In addition, we know from the definition of Φ' that if $v \in \tilde{E}_k$, then

$$d_k' v, (d_k')^* v \in \tilde{E}_k .$$

Thus, if follows for $v_1 \in \tilde{E}_k, v_2 \in \tilde{E}_l, k \neq l$, we have that

$$\langle d_\delta \tilde{\pi}_{k,\delta}\Phi'(v_1), d_\delta \tilde{\pi}_{l,\delta}\Phi'(v_2)\rangle \in O(\delta^{k+l+1}) ,$$

$$\langle d_\delta^* \tilde{\pi}_{k,\delta}\Phi'(v_1), d_\delta^* \tilde{\pi}_{l,\delta}\Phi'(v_2)\rangle \in O(\delta^{k+l+1}) .$$

This implies that for any v and w,

$$\langle \tilde{\pi}_{l,\delta} L_\delta \tilde{\pi}_{k,\delta} v, w\rangle \in O(\delta^{k+l+1}) . \tag{5.21}$$

If $k > 1$ or $l > 1$, then $\tilde{\pi}_{l,\delta} L_\delta \tilde{\pi}_{k,\delta}$ has finite rank, so the bound in (5.21) is uniform in v and w. That is

Lemma 5.12. *There is a $c > 0$ such that for all k and l with*

$$\max\{k, l\} \geqq 2 .$$

we have

$$|\tilde{\pi}_{l,\delta} L_\delta \tilde{\pi}_{k,\delta}| \leqq \begin{cases} c\delta^{k+l} & \text{if } k = 1 \\ c\delta^{k+l+1} & \text{if } k \neq l . \end{cases}$$

This brings us to the theorem from which our main results will be derived.

Theorem 5.13.

a) *There is a $c > 0$ such that for $k \geqq 1$ restricted to $E_{k,\delta}^\perp$,*

$$\pi_{k,\delta}^\perp L_\delta \pi_{k,\delta}^\perp > c\delta^{2(k-1)} .$$

b) *For every $c_2 > 0$ there is a $c_2 > 0$ such that for all $k \geqq 1, \delta \in (0, 1]$ and $v_\delta \in E_{k,\delta}^\perp$ if $|v_\delta| = 1$ and $\langle L_\delta v_\delta, v_\delta\rangle \leqq c_1\delta^{2(k-1)}$ then for all $0 \leqq i \leqq k - 1$,*

$$|\tilde{\pi}_{i,\delta} v_\delta| \leqq c_2\delta^{k-i-1} .$$

c) *There is a* $c > 0$ *such that for every* $k \geqq 1$, *and every* i, j *with* $0 \leqq i, j \leqq k-1$,

$$|\tilde{\pi}_{i,\delta}(\pi_{k,\delta}^{\perp} L_\delta \pi_{k,\delta}^{\perp})^{-1} \tilde{\pi}_{j,\delta}| \leqq c\delta^{-i-j} . \tag{5.22}$$

Proof. Parts (a), (b) and (c) are intimately related, and we prove them simultaneously, inductively in k

$k = 1$: Part (a) follows from Lemma 5.4 and Hypothesis (H2)
Part (b) is vacuous
Part (c) follows directly from (a)

$k = 2$: Part (a) is Theorem 5.8)
Part (b) is Lemma 5.7.

We now proceed inductively.

Proof of (*c*). Assume (a) (b) and (c) have been proved for $k-1$, and (a) and (b) have been proved for $k \geqq 2$. The proof of part (c) will be by downward induction on max $\{i, j\}$.

max{i, j} = k − 1. Assume $j = k-1$. We will show that for all $i \leqq k-1$ (leaving off the δ subscripts)

$$|\tilde{\pi}_i(\pi_k^{\perp} L \pi_k^{\perp})^{-1} \tilde{\pi}_{k-1}| \leqq c\delta^{-(k-1)-i} .$$

Since

$$\left(\tilde{\pi}_i(\pi_k^{\perp} L \pi_k^{\perp})^{-1} \tilde{\pi}_j\right)^* = \tilde{\pi}_j(\pi_k^{\perp} L \pi_k^{\perp})^{-1} \tilde{\pi}_i$$

and taking adjoints preserves norms, this implies the same inequality for $j \leqq i = k-1$.

Suppose $v \in \tilde{E}_{k-1}$ and $|v| = 1$, and let

$$(\pi_k^{\perp} L \pi_k^{\perp})^{-1} v = \omega .$$

Then, by part (a), $|\omega| \leqq c\delta^{-2(k-1)}$. Let

$$\tilde{\omega} = c^{-1}\delta^{2(k-1)}\omega$$

so that $|\tilde{\omega}| \leqq 1$. Then $\tilde{\omega} \in E_{k,\delta}^{\perp}$ and

$$\langle L\tilde{\omega}, \tilde{\omega}\rangle = \langle(\pi_k^{\perp} L \pi_k^{\perp})\tilde{\omega}, \tilde{\omega}\rangle = c^{-1}\delta^{2(k-1)}\langle v, \tilde{\omega}\rangle \leqq c^{-1}\delta^{2(k-1)} .$$

Therefore, by part (b) of this theorem, there is a $\bar{c} > 0$ with

$$|\tilde{\pi}_i\tilde{\omega}| \leqq \bar{c}\delta^{k-1-i}$$

which implies

$$|\tilde{\pi}_i(\pi_k^{\perp} L \pi_k^{\perp})\tilde{\pi}_{k-1} v| = |\tilde{\pi}_\imath \omega| \leqq (c\bar{c})\delta^{-(k-1)-i} ,$$

as desired.

1 ≦ max {i, j} = k−r, r > 1. We assume (5.16) has been proven for $\max\{i, j\} > k-r$. As above, it is sufficient to assume $i \leqq j = k-r$. Suppose $v \in \widetilde{E}_{j,\delta}, |v| = 1$ and write

$$\left(\pi_k^\perp L \pi_k^\perp\right)^{-1} v = \omega\,,$$

so that

$$\pi_{k-r+1}^\perp L \pi_k^\perp \omega = v\,.$$

Since $v \in \widetilde{E}_{k-r} \subset E_{k-r+1}^\perp$ we have

$$\pi_{k-r+1}^\perp L \pi_k^\perp \omega = v\,.$$

Therefore

$$\pi_{k-r+1}^\perp L \pi_{k-r+1}^\perp \omega = v - \sum_{l=k-r+1}^{k-1} \pi_{k-r+1}^\perp L \tilde{\pi}_l \omega\,,$$

so for $i \leqq k-r$

$$\tilde{\pi}_i \omega = \sum_{m=0}^{k-r} \tilde{\pi}_i (\pi_{k-r+1}^\perp L \pi_{k-r+1}^\perp)^{-1} \tilde{\pi}_m \left[v - \sum_{l=k-r+1}^{k-1} \pi_{k-r+1}^\perp L \tilde{\pi}_l \omega \right]$$

$$= \tilde{\pi}_i (\pi_{k-r+1}^\perp L \pi_{k-r+1}^\perp)^{-1} \tilde{\pi}_{k-r} v - \sum_{m=0}^{k-r} \sum_{l=k-r+1}^{k-1} \tilde{\pi}_i (\pi_{k-r+1}^\perp L \pi_{k-r+1}^\perp)^{-1} \tilde{\pi}_m L \tilde{\pi}_m L \tilde{\pi}_l \omega\,. \tag{5.23}$$

By induction on part (c),

$$\left|\tilde{\pi}_i (\pi_{k-r+1}^\perp L \pi_{k-r+1}^\perp) \tilde{\pi}_j\right| \leqq c\delta^{-i-j}\,.$$

Thus

$$|\tilde{\pi}_i (\pi_{k-r+1}^\perp L \pi_{k-r+1}^\perp)^{-1} \tilde{\pi}_{k-r} v| \leqq c\delta^{-(k-r)-i} \tag{5.24}$$

and

$$|\sum_m \sum_l \tilde{\pi}_i (\pi_{k-r+1}^\perp l \pi_{k-r+1}^\perp)^{-1} \tilde{\pi}_m L \tilde{\pi}_l \omega| \leqq \sum_m \sum_l c\delta^{-i-m} |\tilde{\pi}_m L \tilde{\pi}_l|\; |\tilde{\pi}_l \omega|\,. \tag{5.25}$$

Since $l > k-r$, it follows by the induction on $\max\{i,j\}$ that

$$|\tilde{\pi}_l \omega| = |\tilde{\pi}_l (\pi_k^\perp L \pi_k^\perp)^{-1} \tilde{\pi}_{k-r} v| \leqq c\delta^{-l-(k-r)}\,. \tag{5.26}$$

From Lemma 5.12, since $l \geqq k-r+1 \geqq 2$ and $l \geqq k-r+1 > m$,

$$|\tilde{\pi}_m L \tilde{\pi}_l| \in O(\delta^{m+l+1})\,. \tag{5.27}$$

Substituting (5.26) and (5.27) into (5.25) yields

$$|\sum_m \sum_l \tilde{\pi}_i (\pi_{k-r+1}^\perp L \pi_{k-r+1}^\perp)^{-1} \tilde{\pi}_l \omega| \leqq c\delta^{-(k-r)-i+1}\,. \tag{5.28}$$

Combining (5.23), (5.24) and (5.28) yields the desired inequality.

i =j =0: The only case not previously covered is $i = j = 0$. Suppose $v \in \tilde{E}_0, |v| = 1$ and

$$(\pi_k^\perp L \pi_k^\perp)^{-1} v = w\,,$$

so that $w \in E_k^\perp$ and

$$\pi_k^\perp L \pi_k^\perp w = v\,.$$

Then
$$\langle Lw, w\rangle = \langle{}_k^{\perp}L\pi_k^{\perp}w, w\rangle \leqq |\tilde{\pi}_0 w| \, . \tag{5.29}$$
On the other hand, we have
$$\langle Lw, w\rangle = \langle L\pi_2^{\perp}w, \pi_2^{\perp}w\rangle + \sum_{\substack{i,j \\ \max\{i,j\}\geqq 2}} \langle L\tilde{\pi}_i w, \tilde{\pi}_j w\rangle \, . \tag{5.30}$$

If $\max\{i,j\} \geqq 2$
$$|\langle L\tilde{\pi}_i w, \tilde{\pi}_j w\rangle| = |\langle(\tilde{\pi}_j L, \tilde{\pi}_i)\tilde{\pi}_i w, \tilde{\pi}_j w\rangle| \leqq |\tilde{\pi}_j L\tilde{\pi}_i| \; |\tilde{\pi}_j w|| \, .$$
From Lemma 5.12,
$$|\tilde{\pi}_j L\tilde{\pi}_i| \leqq c\delta^{i+j} \, .$$
By the downward induction on $\max\{i,j\}$
$$|\tilde{\pi}_i w| = (\tilde{\pi}_i(\pi_i(\pi_{k-1}L\pi_{k-1})^{-1}\pi_0)v| \leqq c\delta^{-i}$$
Similarly
$$|\tilde{\pi}_j w| \leqq c\delta^{-j} \, .$$
Thus, there is a c_1 such that if $\max\{i,j\} \geqq 2$,
$$|\langle L\tilde{\pi}_i w, \tilde{\pi}_j w\rangle| \leqq c_1 \, . \tag{5.31}$$
From Lemma 5.4 there is a c such that
$$\langle \pi_2^{\perp}w, \pi_2^{\perp}w\rangle \geqq \frac{1}{2}\langle \Box^{1,0}\pi_2^{\perp}w, \pi_2^{\perp}w\rangle - c\delta^2|\pi_2^{\perp}w|^2 \, .$$
From (H2)
$$\langle \Box^{1,0}\pi_2^{\perp}w, \pi_2^{\perp}w\rangle c|\tilde{\pi}_0 w|^2 \, .$$
Moreover
$$c\delta^2\pi_2^{\perp}w|^2 = c\delta^2|\tilde{\pi}_0 w|^2 + c\delta^2|\tilde{\pi}_1 w|^2 \, .$$
From the downward induction on $\max\{i,j\}$ there is a c such that
$$|\tilde{\pi}_1 w| = \tilde{\pi}_1(\pi_{k-1}L\pi_{k-1})^{-1}\pi_0 v| \leqq c\delta^{-1} \, .$$
Thus for δ small enough
$$\langle L\pi_2^{\perp}w, \pi_2^{\perp}w\rangle \geqq c_2|\tilde{\pi}_0 w|^2 - c_3 \, . \tag{5.32}$$
From (5.29), (5.30), (5.31) and (5.32) there is a constant c_4 such that
$$c_2|\tilde{\pi}_0 w|^2 - c_4 \leqq |\tilde{\pi}_0 w| \, .$$
which implies there is a c_5 with
$$|\tilde{\pi}_0 w| = |\tilde{\pi}_0(\pi_{k-1})\tilde{\pi}_0 v| \leqq c_5$$
as desired.

Proof of Part (b). Assume that (a) (b) and (c) have been proved for $k-1$ (where $k \geqq 3$). Suppose $v \in E_{k,\delta}^{\perp}$, $|v| = 1$ and

$$\langle Lv, v\rangle \leqq c_1\delta^{2(k-1)} .$$

Write $v = \alpha + \beta$ with $\alpha \in E^{\perp}_{k-1,\delta}, \beta \in \widetilde{E}_{k-1,\delta}$. Then

$$\begin{aligned} c_1\delta^{2(k-1)} &\geqq \langle Lv, v\rangle = \langle L\alpha, \beta\rangle + \langle \beta, \beta\rangle \\ &\geqq \langle L\alpha, \alpha\rangle + 2\langle L\alpha, \beta\rangle \\ &\geqq |L^{\frac{1}{2}}\alpha|^2 - 2|L^{\frac{1}{2}}\alpha|\ |L^{\frac{1}{2}}\beta| . \end{aligned}$$

By the definition of $\widetilde{E}_{k-1,\delta}$,

$$|L^{\frac{1}{2}}\beta| = (|d_\delta\beta|^2 + d^*_\delta\beta|^2)^{\frac{1}{2}} \in O(\delta^{k-1}) .$$

Since $\widetilde{E}_{k-1,\beta}$ is dinite dimensional for $k \geqq 3$ there is a $c_2 > 0$ such that

$$|L^{\frac{1}{2}}\beta| \leqq c_2\delta^{k-1} .$$

Therefore

$$c_1\delta^{2(k-1)} \geqq |L^{\frac{1}{2}}\alpha|^2 - 2c_2\delta^{k-1}|L^{\frac{1}{2}}\alpha|^2 ,$$

which implies there is a c_3 such that

$$|L^{\frac{1}{2}}\alpha| \leqq c_3\delta^{k-1} .$$

That is

$$\langle L\alpha, \alpha\rangle \leqq c_3^2\delta^{2(k-1)} . \tag{5.33}$$

By induction on part (a)

$$\langle L\alpha, \alpha\rangle \leqq c_4\delta^{2(k-2)}|\alpha|^2 . \tag{5.34}$$

Together, (5.33) and (5.34) imply

$$|\alpha|^2 \leqq \frac{c_3^2}{c_4}\delta^2 .$$

Therefore, $\frac{\alpha}{\delta} \in E^{\perp}_{k-1,\delta}$ satisfies

$$\left|\frac{\alpha}{\delta}\right| \leqq \frac{c_3^2}{c_4}, \left\langle L\left(\frac{\alpha}{\delta}\right), L\left(\frac{\alpha}{\delta}\right)\right\rangle \leqq c_3^2\delta^{2(k-2)} .$$

By induction on part (b), for $0 \leqq j \leqq k-2$ there is a $c > 0$ s.t.

$$\left|\tilde{\pi}_i\left(\frac{\alpha}{\delta}\right)\right| \leqq c\delta^{(k-1)-i-1} .$$

Therefore

$$|\tilde{\pi}_{i,\delta}v| = |\tilde{\pi}_{i,\delta}\alpha| \leqq c\delta^{k-i-1}$$

for $0 \leqq i \leqq k-2$. For $i = k-1$, the estimate follows from $|\tilde{\pi}_{k-1}v| \leqq |v| = 1$. This proves part (b).

Proof of Part (*a*). We assume (a), (b), (c) have been proved $k-1$, and part (b) has been proved for $k(\geqq 3)$. We will show that, restricted to $E^{\perp}_{k,\delta}$,

$$|\left(\pi_k^\perp L\pi_k^\perp\right)^{-1}| \leqq c\delta^{-2(k-1)}\,.$$

Let λ_δ be the smallest eigenvalue of $\pi_k^\perp L\pi_k^\perp$ restricted to $E_{k,\delta}^\perp$. Since

$$\langle Lv, v\rangle \in O(\delta^{2(k-1)})$$

for $v \in \widetilde{E}_{k-1}, \lambda_\delta \in O(\delta^{2(k-1)})$. Thus if w is the corresponding eigenfunction, $|w| = 1$, then by part (b)

$$|(1-\tilde{\pi}_{k-1})w| \in O(\delta)\,.$$

Now we have

$$\begin{aligned}
\lambda_\delta^{-1} &= \left\langle(\pi_k^\perp L\pi_k^\perp)^{-1}w, w\right\rangle \\
&= \left\langle(\pi_k^\perp L\pi_k^\perp)^{-1}w, \tilde{\pi}_{k-1}w\right\rangle + O(\delta)\lambda_\delta^{-1} \\
&= \left\langle w, (\pi_k^\perp L\pi_k^\perp)^{-1}\tilde{\pi}_{k-1}w\right\rangle + O(\delta)\lambda_\delta^{-1} \\
&= \left\langle\tilde{\pi}_{k-1}w, (\pi_k^\perp L\pi_k^\perp)^{-1}\tilde{\pi}_{k-1}w\right\rangle + O(\delta)|(\pi_k^\perp L\pi_k^\perp)^{-1}\tilde{\pi}_{k-1}w| + O(\delta)\lambda_\delta^{-1}\,.
\end{aligned} \tag{5.35}$$

Note that

$$|(\pi_k^\perp L\pi_k^\perp)^{-1}\tilde{\pi}_{k-1}w| \leqq \lambda_\delta^{-1}|\tilde{\pi}_{k-1}w| + \lambda_\delta^{-1}(1+O(\delta))\,.$$

Therefore, (5.35) implies

$$\left\langle\tilde{\pi}_{k-1}w, (\pi_k^\perp L\pi_k^\perp)^{-1}\tilde{\pi}_{k-1}w\right\rangle = \lambda_\delta^{-1}(1+O(\delta))\,.$$

The desired estimate on λ_δ^{-1} follows once we see that

$$|\tilde{\pi}_{k-1}(\pi_k^\perp L\pi_k^\perp)^{-1}\tilde{\pi}_{k-1}| \leqq c\delta^{-2(k-1)}\,.$$

Writing $E_{k,\delta}^\perp = E_{k-1,\delta}^\perp \oplus \widetilde{E}_{k-1,\delta}$ we deompose $\pi_k^\perp L\pi_k^\perp$ into a 2×2 matrix of operators

$$\pi_k^\perp L\pi_k^\perp = \begin{pmatrix} \pi_{k-1}^\perp L\pi_{k-1}^\perp & \pi_{k-1}^\perp L\tilde{\pi}_{k-1}^\perp \\ \tilde{\pi}_{k-1}L\pi_{k-1}^\perp & \tilde{\pi}_{k-1}L\tilde{\pi}_{k-1} \end{pmatrix}.$$

Then we can write

$$\tilde{\pi}_{k-1}(\pi_k^\perp L\pi_k^\perp)^{-1}\tilde{\pi}_{k-1} = \left(\tilde{\pi}_{k-1}L\tilde{\pi}_{k-1} - \tilde{\pi}_{k-1}L\pi_k^\perp(\pi_{k-1}^\perp L\pi_{k-1}^\perp)^{-1}\pi_{k-1}^\perp L\tilde{\pi}_{k-1}\right)^{-1}. \tag{5.36}$$

From Lemma 5.11, there is a $c > 0$ with

$$\tilde{\pi}_{k-1}L\tilde{\pi}_{k-1} \geqq c\delta^{2(k-1)}\,. \tag{5.37}$$

On the other hand

$$\tilde{\pi}_{k-1}L\pi_{k-1}^\perp\left(\pi_{k-1}^\perp\right)^{-1}\pi_{k-1}^\perp L\tilde{\pi}_{k-1} = \sum_{i,j=-1}^{k-2}\tilde{\pi}_{k-1}L\tilde{\pi}_\imath\left(\pi_{k-1}^\perp L\pi_{k-1}^\perp\right)^{-1}\tilde{\pi}_j L\tilde{\pi}_{k-1}\,.$$

Thus, applying Lemma 5.12 and part (b) for $k-1$,

$$\begin{aligned}
&|\tilde{\pi}_{k-1}L\pi_{k-1}^{\perp}(\pi_{k-1}^{\perp}(\pi_{k-1}^{\perp}L\pi_{k-1}^{\perp}l\tilde{\pi}_{k-1}| \\
&\leqq \sum_{i,j=-1}^{k-2} |\tilde{\pi}_{k-1}L\tilde{\pi}_i\left(\pi_{k-1}^{\perp}L\pi_{k-1}^{\perp}\right)^{-1}\tilde{\pi}_jL\tilde{\pi}_{k-1}| \\
&\leqq \sum_{i,j} |\tilde{\pi}_{k-1}L\tilde{\pi}_i|\ |\tilde{\pi}_i\left(\pi_{k-1}^{\perp}L\pi_{k-1}^{\perp}\right)^{-1}\tilde{\pi}_j|\ |\tilde{\pi}_jL\tilde{\pi}_{k-1}| \\
&\leqq c_1\sum_{i,j}\delta^{i+k-1+1}\cdot\delta^{-i-j}\cdot\delta^{j+k-1+1} \\
&\leqq c_2\delta^{2(k-1)+2}\,.
\end{aligned} \tag{5.38}$$

Combining (5.36), (5.37) and (5.38) we see that for δ small enough,

$$\tilde{\pi}_{k-1}(\pi_k^{\perp}L\pi_k^{\perp})^{-1}\tilde{\pi}_{k-1} = (\tilde{\pi}_{k-1}L\tilde{\pi}_{k-1}(1+O(\delta^2)))^{-1}\,.$$

Now $\tilde{\pi}_{k-1}L\tilde{\pi}_{k-1} \geqq c\delta^{2(k-1)}$ implies

$$|\tilde{\pi}_{k-1}(\pi_k^{\perp}(\pi_k^{\perp})^{-1}\tilde{\pi}_{k-1}| \leqq c^{-1}\delta^{-2(k-1)}\,,$$

is desired. $\square$

This theorem provides enough information for us to deduce the desired results.

Corollary 5.14.

$$\begin{aligned}
&\#\{\lambda_i(\delta)\in \mathrm{spec}\square_\delta^p\,|\,\liminf_{\delta\to 0}\delta^{-2k}\lambda_i(\delta)=0\} \\
&\qquad = \#\{\lambda_i(\delta)\in\mathrm{spec}\square_\delta^p\,|\,\lambda_i(\delta)\in O(\delta^{2k+2})\} = \dim E_{k+1}^p\,.
\end{aligned}$$

Proof. The first quantity is clearly $\geqq$ the second, is $\geqq$ the third, by (1.8). If either of these inequalities were strict, we could find a sequence $\delta_i\to 0$ and a sequence $\omega_i\in\Omega^p$ such that $|\omega_i|=1, \omega\perp E_{k+1,\delta}^p$, and for every $c>0$,

$$\langle L_{\delta_i}\omega_i,\omega_i\rangle < c\delta^{2k}$$

for δ small enough. This contradicts Theorem 5.13 (a). $\square$

Now write $\lambda_i(\delta)\sim\delta^k$ if

$$\lambda_i(\delta)\in O(\delta^k)\quad\text{and}\quad(\lambda_i(\delta)))^{-1}\in O(\delta^{-k})\,.$$

Theorem 5.15.

$$\#\{\lambda_i(\delta)\in\mathrm{spec}\square_\delta^p\,|\,\lambda_i(\delta)\sim\delta^{2k}\} = \dim\widetilde{E}_k^p\,.$$

Proof. From Corollary 5.14

$$\#\{\lambda_i(\delta)\in O(\delta^{2k})\} = \dim E_k^p\,.$$

Moreover,

$$(\lambda_i(\delta))^{-1}\notin O(\delta^{-2k})\leftrightarrow\delta^{2k}(\lambda_i(\delta))^{-1}\notin O(1)\leftrightarrow\liminf_{\delta\to 0}\delta^{-2k}\lambda_i(\delta)=0\,.$$

From Corollary 5.14, all eigenvalues λ_i with

$$(\lambda_i(\delta))^{-1} \notin O(\delta^{2k})$$

are $O(\delta^{2k+2})$ and the number of such eigenvalues is $\dim E^p_{k+1}$.

Thus

$$\begin{aligned}\#\{\lambda_i(\delta) \sim O(\delta^{2k})\} &= \#\{\lambda_i(\delta) \in O(\delta^{2k})\} - \#\{\lambda_i(\delta) \in O(\delta^{2k+2})\} \\ &= \dim E^p_k - \dim E^p_{k+1} = \dim \widetilde{E}^p_k \ . \quad \square\end{aligned}$$

Corollary 5.16. *If $\lambda^p_i(\delta) \in O(\delta^k)$ for all k, then $\lambda^p_i(\delta) \equiv 0$ for all δ.*

Proof. Clearly $\{\lambda_i(\delta) \in O(\delta^k)$ for all $\delta\} \supseteq \{\lambda_i(\delta) \equiv 0$ for all $\delta\}$. From Corollary 5.14 $\#\{\lambda^p_i(\delta) \in O(\delta^k)$ for all $k\} = \dim E^p_N = \dim E^p_\infty$. In addition, by the results of Sect. 3,

$$\begin{aligned}\dim E^p_N &= \dim H^p(M,E) \\ &= \#\{\lambda^p_i(\delta) \equiv 0 \text{ for all } \delta\} \ .\end{aligned}$$

This proves the corollary. $\square$

We now investigate the behavior of the eigenspaces as $\delta \to 0$. Let

$$\mathrm{eig}^p_{k,\delta} = \mathrm{span}\{\omega_i(\delta) = \lambda_i(\delta)\omega_i(\delta) \text{ with } \lambda_i(\delta) \in (\delta^{2k})\} \ .$$

Theorem 5.17. *For $k > 1$,*

$$\mathrm{eig}^p_{k,\delta} = E^p_k + O(\delta) \ .$$

By which we mean, if $v_\delta \in \mathrm{eig}^p_{k,\delta}, |v_\delta| = 1$, then we can write

$$v_\delta = \alpha_\delta + \beta_\delta$$

with $\alpha_\delta \in E^p_k$, and $|\beta| \in O(\delta)$. Equivalently, if $\rho_{k,\delta}$ is the orthogonal projection onto $\mathrm{eig}^p_{k,\delta}$ then

$$|\rho_{k.\delta} - \pi_k| \in O(\delta) \ .$$

Proof. If $v_\delta \in \mathrm{eig}^p_{k,\delta}, |v - \delta| = 1$ then

$$\langle L_\delta v_\delta, v_\delta\rangle \leqq c_1\delta^{2k} \ .$$

Write $v_\delta = \alpha_\delta + \beta_\delta$ with

$$\alpha_\delta \in E^p_{k,\delta}, \quad \beta_\delta \in (E^p_{k,\delta})^\perp \ .$$

Then

$$\langle Lv, v\rangle = \langle L\alpha, \alpha\rangle + 2\langle L\alpha, \beta\rangle + \langle L\beta, \beta\rangle = |L^{\frac{1}{2}}\alpha|^2 + 2\left\langle L^{\frac{1}{2}}\alpha, L^{\frac{1}{2}}\beta\right\rangle + L^{\frac{1}{2}}\beta|^2 \ .$$

Since $k > 1$, there is a $c > 0$ with

$$|L^{\frac{1}{2}}\alpha| \leqq c|\delta^k \leqq c_2\delta^k \ ,$$

so that

$$|L^{\frac{1}{2}}\beta|^2 - 2c_2\delta^k|L^{\frac{1}{2}}\beta| \leqq (c_1 - c_2)\delta^{2k} \ ,$$

which implies

$$\langle L\beta, \beta\rangle = |L^{\frac{1}{2}}\beta|^2 \leqq c_3\delta^{2k} .$$

It follows from theorem 5.13 (a), that $|\beta| \in O(\delta)$. Thus $v_\delta = \alpha_\delta + O(\delta)$, with $\delta \in R^p_{k,\delta}$, but

$$E^p_{k,\delta} = E^p_k + O(\delta) ,$$

which proves the theorem. $\square$

The following two corollaries are immediate

Corollary 5.18. *Let $\mathscr{H}^p_\delta(M, V)$ denote the kernel of $\square^p_\delta$ (= the space of g_δ-harmonic p-forms), then*

$$\rho_\delta \mathscr{H}^p_\delta(M, V) = \ker L^p_\delta = E^p_\infty + O(\delta) .$$

Corollary 5.19. *Let*

$$\widetilde{\text{eig}}^p_{k,\delta} = \text{span}\{\omega_i(\delta) | \square^p_\delta \omega_i(\delta) = \lambda_i(\delta)\omega_i(\delta) \textit{ with } \lambda_i(\delta) \sim \delta^{2k}\} ,$$

then

$$\widetilde{\text{eig}}^p_{k,\delta} = \widetilde{E}^p_k + O(\delta) .$$

We are now ready to make precise statements about the asymptotics of the eigenvalues. We have already seen that

$$\{\lambda^p_i(\delta) | \dim E^p_{k+1} + 1 \leqq i \leqq \dim E^p_k\}$$

are the eigenvalues of L^p_δ which are $\sim \delta^{2k}$, and the corresponding eigenspaces converge to $\widetilde{E}^p_k$. We now prove

Theorem 5.20. *Fix p. Then for every i,*

$$\dim E^p_{k+1} + 1 \leqq i \leqq \dim E^p_k \quad \textit{if } k > 1 ,$$

$$\dim E^p_2 + 1 \leqq i < \infty \quad \textit{if } k = 1 ,$$

we have

$$\lambda_i(\delta) = \delta^{2k}\overline{\lambda}_i + O(\delta^{2k+1})$$

for some $\overline{\lambda}_i$. These $\dim \widetilde{E}^p_k$ values of $\overline{\lambda}_i$ are given by the eigenvalues of the operator

$$\Delta^p_k : \widetilde{E}^p_k \to \widetilde{E}^p_k .$$

[In particular, for $k > 1$,

$$\{\lambda_i(\delta) | \lambda_i(\delta) \sim \delta^{2k}\} = \delta^{2k}\{\text{eigenvalues of } \Delta^p_k : \widetilde{E}^p_k \to \widetilde{E}^p_k\} + O(\delta^{2k+1})] .$$

Proof. Assume $k > 1$. First we prove that the eigenvalues of L^p_δ which are $\sim \delta^{2k}$ are closely approximated by the eigenvalues of $\pi^\perp_{k+1,\delta} L_\delta \pi^\perp_{k+1,\delta}$. If $v \in \Omega^p, |v| = 1$ then write

$$v = \alpha + \beta$$

with

$$\alpha \in (E^p_{k+1,\delta})^\perp, \quad \beta \in E^p_{k+1,\delta} .$$

Then

$$\langle Lv, v\rangle = \langle L\alpha, \alpha\rangle + 2\left\langle L^{\frac{1}{2}}\alpha, L^{\frac{1}{2}}\beta\right\rangle + \langle L\beta, \beta\rangle .$$

Note that

$$\left|2\left\langle L^{\frac{1}{2}}\alpha, L^{\frac{1}{2}}\beta\right\rangle\right| \leqq \delta\langle L\alpha, \alpha\rangle + \delta^{-1}\langle L\beta, \beta\rangle ,$$

and there is a c such that for all δ and $\beta \in E^p_{k+1,\delta}$,

$$\langle L\beta, \beta\rangle \leqq c\delta^{2k+2}|\beta|^2 \leqq c\delta^{2k+2} .$$

Thus, since $\pi^\perp_{k+1,\delta}v = \alpha$, we have

$$\langle Lv, v\rangle = \left\langle \pi^\perp_{k+1}L\pi^\perp_{k+1}v, v\right\rangle + c_1\left\langle \pi^\perp_{k+1}L\pi^\perp_{k+1}v, v\right\rangle + c_2 , \tag{5.39}$$

where

$$|c_1| \leqq \delta, |c_2| \leqq c\delta^{2k+1} .$$

Let $\mu_1(\delta) \leqq \mu_2(\delta) \leqq \cdots$ be the eigenvalues of $\pi^\perp_{k+1}L^p\pi^\perp_{k+1}$, so that, in particular,

$$\mu_1(\delta) \equiv \mu_2(\delta) \equiv \cdots \equiv \mu_{\dim E^p_{k+1}}(\delta) \equiv 0 .$$

Then (5.39) implies that for all i

$$|\lambda_i - \mu_i| \leqq c(\delta\mu_i + \delta^{2k+1}) .$$

This shows that for

$$\dim E^p_{k+1} + 1 \leqq i \leqq \dim E^p_k$$

the eigenvalues $\mu_i(\delta)$ are $\sim \delta^{2k}$, and for such i

$$|\lambda_i, -\mu_i| \leqq c\delta^{2k+1} .$$

The theorem follows from proving that these μ_i's have the form

$$\mu_i = \delta^{2k}\overline{\mu}_i + O(d^{2k+1})$$

with the $\overline{\mu}_i$'s the eigenvalues of Δ^p_k. The inverses of the non-zero μ_i's are given by the eigenvalues of $(\pi^\perp_{k+1}L\pi^\perp_{k+1})^{-1}$ restricted to $(E^p_k)^\perp$. From Theorem 5.13 (c) it follows that, restricted to $(E^p_{k+1})^\perp$

$$|(\pi^\perp_{k+1}L\pi^\perp_{k+1})^{-1} - \tilde{\pi}_k(\pi^\perp_{k+1}L\pi^1_{k+1})^{-1}\tilde{\pi}_k| \leqq c_1\delta^{-2k+1} .$$

Thus, if $v_1(\delta), \ldots, v_{\dim \widetilde{E}^p_k}(\delta)$ are the inverses of the non-zero eigenvalues of $\tilde{\pi}_k(\pi^\perp_{k+1}L\pi^\perp_{k+1})^{-1}\ \tilde{\pi}_k$, we have, for $1 \leqq i \leqq \dim \widetilde{E}^p_k$,

$$|\mu_{\dim E_{k+1}+i}(\delta) - v_i(\delta)| \leqq c_2\delta^{2k+1} .$$

From (5.36) we see that

$$(\tilde{\pi}_k(\pi^\perp_{k+1}L\pi^\perp_{k+1})^{-1}\tilde{\pi}_k)^{-1} = \tilde{\pi}_kL\tilde{\pi}_k + \tilde{\pi}_kL\pi^\perp_{k+1}(\pi^\perp_{k+1}L\pi^\perp_{k+1})^{-1}\pi^\perp_{k+1}L\tilde{\pi}_k .$$

From (5.19) we have

$$\tilde{\pi}_kL\tilde{\pi}_k = \delta^{2k}\Delta_k + O(\delta^{2k+1}) ,$$

and from Lemma 5.12 combined with Theorem 5.13 (c) we find

$$|\tilde{\pi}_k L \pi_{k+1}^{\perp} (\pi_{k+1}^{\perp} L \pi_{k+1}^{\perp})^{-1} \pi_{k+1}^{\perp} L \tilde{\pi}_k| \leqq c_2 \delta^{2k+2} ,$$

which proves the theorem in the case $k > 1$.

If $k = 1$ then $\dim E_k^p = \infty$, and we do not have uniform bounds for the errors which appear. However, this is not a problem. Fix $c > 0$ and consider

$$\Gamma_c = \{\lambda_i(\delta) | \limsup_{\delta \to 0} \delta^{-2} \lambda_i(c) < c\} .$$

From Lemma 5.4 we know $L_\delta > \delta^2 k$ for a 2nd order operator k with positive symbol, so there is an upper bound N_c, independent of δ, for the number of eigenvalues of L_δ which are $< c\delta^2$. Then the same argument as for $k > 1$ shows that for $\dim E_2^p + 1 \leqq i \leqq \# \Gamma_c$,

$$\lambda_i(\delta) = \delta^2 \overline{\lambda}_i + O(\delta) ,$$

where the $\overline{\lambda}_i$ are the eigenvalues of

$$\Delta_0^p : \widetilde{E}_0^p \to \widetilde{E}_0^p$$

which are $< c$. Letting $c \to \infty$ completes the proof. □

So far, we have shown that the eigenspaces of L_δ^p approach the spaces E_k^p continuously. We complete this section by showing that our previous analysis actually implies the C^∞ convergence of the kernel of L_δ^p.

Theorem 5.21. *The spaces*

$$\mathrm{Ker} L_\delta^p = \rho_\delta \mathscr{H}_\delta^p(M, V)$$

form a C^∞ map from $[0,1]$ to the space of $(\dim H^p(M,V))$-dimensional subspaces of the L^2 p-forms on M.

(Note : This generalizes Theorem 17 of [Ma-Me].)

Proof. Fix $M > 0$. We will show that $\rho_\delta \mathscr{H}_\delta^p$ is C^M. We follow the proof of Theorem 5.17, with one modification. In defining $E_{N,\delta}^p = E_{\infty,\delta}^p$, we truncated the formal power series (5.18) at the δ^{N+2} term. To prove C^M convergence, we truncate the power series at δ^{N+M}. That is, let

$$\Phi'(v) = v + \delta v_1 + \cdots + \delta^{N+M-1} v_{N+M-1} ,$$

so that

$$d_\delta \Phi'(v) \in \delta^{N+M} \Omega^{p+1}[\delta] ,$$

$$d_\delta^* \Phi'(v) \in \delta^{N+M} \Omega^{p-1}[\delta] . \tag{5.40}$$

Let $E_{\infty,\delta}^p$ denote the image of Φ' applied to the space E_∞^p. Then $E_{\infty,\delta}^p$ is C^∞ on $[0, 1]$ (in fact is polynomial). Now we continue as before. Suppose

$$v_\delta \in \rho_\delta \mathscr{H}_\delta^p ,$$

so that

$$\langle L_\delta v_\delta, v_\delta\rangle = 0\ ,$$

and

$$|v_\delta| = 1\ .$$

Write

$$v_\delta = \alpha_\delta + \beta_\delta$$

with

$$\alpha_\delta \in E^p_{\infty,\delta}, \beta_\delta \in (E^p_{\infty,\delta})^\perp\ .$$

Then

$$0 = \langle Lv, v\rangle = |L^{\frac{1}{2}}\alpha|^2 + 2\left\langle L^{\frac{1}{2}}\alpha, L^{\frac{1}{2}}\beta\right\rangle + |L^{\frac{1}{2}}\beta|^2\ ,$$

so that

$$\begin{aligned} 0 &\geqq |L^{\frac{1}{2}}\beta|^2 + 2\left\langle L^{\frac{1}{2}}\alpha, L^{\frac{1}{2}}\beta\right\rangle \\ &\geqq |L^{\frac{1}{2}}\beta|^2 - 2|L^{\frac{1}{2}}\alpha||L^{\frac{1}{2}}\beta| \\ &= |L^{\frac{1}{2}}\beta|\left(|L^{\frac{1}{2}}\beta| - 2|L^{\frac{1}{2}}\alpha|\right)\ . \end{aligned}$$

Therefore

$$|L^{\frac{1}{2}}\beta| \leqq 2|L^{\frac{1}{2}}\alpha|\ .$$

Now, $\alpha \in E^p_{\infty,\delta}$ so from (5.40),

$$|L^{\frac{1}{2}}\alpha| \leqq c\delta^{N+M}|\alpha| \leqq c\delta^{N+M}\ .$$

Moreover, $\beta \in (E^p_{\infty,\delta})^\perp$ implies there is a $c > 0$ such that

$$|L^{\frac{1}{2}}\beta| \geqq c\delta^{N-1}|\beta|\ .$$

This yields

$$|\beta| \leqq c\delta^{m+1}\ .$$

Thus

$$\rho_\delta \mathscr{H}^p_\delta = E^p_{\infty,\delta} + O(\delta^{M+1})\ ,$$

which implies $\rho_\delta \mathscr{H}^p_\delta$ is C^M as desired. □

As noted in [Ma-Me], this implies that any formally harmonic power series (5.12) is, in fact, the Taylor series at $\delta = 0$ of a C^∞ family of forms ω_δ satisfying,

$$\text{for every } \delta \in [0,1]\ \omega_\delta \in \rho_\delta \mathscr{H}^p_\delta(M, V)\ .$$

Applying the map ρ_δ^{-1}, we learn

Corollary 5.22 (Corollary 18 of [Ma-Me]): *The space $H^p_\delta(M, V)$ defines a C^∞ map from* [0, 1] *to the space of* ($\dim H^p(M, V)$)*-dimensional subspaces of the L^2 p-forms on M.*

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