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Combinatorial Expression for Universal Vassiliev Link Invariant

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Abstract: A general model similar to R-matrix-type models for link invariants is constructed. It contains all R-matrix invariants and is a generating function for "universal" Vassiliev link invariants. This expression is simpler than Kontsevich's expression for the same quantity, because it is defined combinatorially and does not contain any integrals, except for an expression for "the universal Drinfeld's associator."

1. Introduction

Vassiliev knot invariants were invented in attempts to construct some natural basis for the space of all knot invariants (this space can be described as the cohomology space H^0 (Embeddings: $S^1 \to R^3$)). For this purpose Vassiliev used certain stratification of the discriminant set of nonembeddings: $S^1 \to R^3$ and some finite-dimensional approximations of the space of all knots. (We recommend the reader [Va1, Va2] and especially [BN1] for a very detailed introduction to the theory of Vassiliev invariants).

Although the question whether Vassiliev knot invariants can distinguish any two knots is still open, this language seems to be the most appropriate in studying classical knot and link invariants.

All known classical knot and link invariants: Alexander polynomial, Jones polynomial, Kauffman polynomial, HOMFLY polynomial and all their generalizations, as well as the Milnor μ-invariants (see [Ro, Co, Jo1, Ka1, Ka2, HOMFLY, Tu1, Tu2, Re1, RT, Mi1, Mi2] for a precise definitions), can be incorporated into this scheme (see [BL, Li1, Li2, BN5]).

The space of Vassiliev knot invariants of fixed degree n (divided by the space of invariants of degree $n-1$) has a purely combinatorial description. It is isomorphic to a certain linear subspace in the space of functions on the set of "Vassiliev $[n]$-diagrams" (or combinatorial types of n pairs of points of S^1). The linear relations, defining this subspace in the space of all functions on the set of "Vassiliev $[n]$-diagrams" were first written explicitly by Birman and Lin [BL]. The fact, that the set of relations written in [BL] is complete and there are no extra relations, was proved by Kontsevich [Ko1].

To prove the isomorphism between the space of Vassiliev knot invariants of degree n (divided by the space of invariants of degree $n-1$) and the linear space F_n^*, defined purely combinatorially, Kontsevich used an explicit integral presentation of "the universal Vassiliev invariant of degree n." This "universal invariant" I_n takes values in the linear space F_n, dual to F_n^*.

The space F_n has another very nice description in terms of Feynman diagrams of perturbative Chern–Simons theory [BN2]. The graded linear space $F = \oplus_n F_n$ admits a Hopf algebra structure [Ko1, BN3] (Kontsevich Hopf algebra). The space of primitive elements in this Hopf algebra is generated by connected Feynman diagrams [Pi2].

The generating function $I = \sum_{n=0}^{\infty} h^n I_n$ of "the universal Vassiliev invariants of degree n" gives us "the universal Vassiliev invariant" I taking its values in Kontsevich Hopf algebra F. Here h is a formal parameter, $I_n(K)$ is a certain n-fold integral over the knot K (Kontsevich integral [Ko1, Ar2]). At the moment nobody is able to calculate explicitly $I(K)$ for any non-trivial knot K.

The aim of the present paper is to give a simpler expression for this quantity, which can be calculated explicitly to all orders in h if one can calculate "the universal Drinfeld associator" [Dr1]. This expression models the state sum expression for knot polynomials $P_{g,V}(q^{\pm 1})$ (here $q = e^h$) constructed from a simple Lie algebra g and its irreducible representation V (see [Re1, Tu1, Jo2] for an explicit form of this state sum expression).

The connection between $P_{g,V}(q^{\pm 1})$ and Vassiliev knot invariants was found in the most general form by Lin [Li1]: If $P_{g,V}(h) = \sum_{n=0}^{\infty} P_{g,V,n} h^n$ then $P_{g,V,n}$ is Vassiliev invariant of degree n. The explicit state sum expression for $P_{g,V,n} \in F_n$ was deduced in [Pi1].

The question is, whether it is possible to forget about the Lie algebra g and the representation V and to write the "universal" state sum expression $P = \sum_{n=0}^{\infty} h^n P_n$ with values in Kontsevich Hopf algebra F.

There are three ways to do this. The first way is to use perturbative Chern–Somons theory as in [Ko1, BN3] (see also [AS] and [GMM]). The second way (Kontsevich integrals [Ko1]) is to use the perturbative expansion of the monodromy of the KZ-equation found by Kontsevich. The third way (combinatorial) is presented here. Our approach, unlike the first two, does not use complicated integrals.

The paper is organized as follows:

In Sect. 2 the basic facts about Vassiliev link invariants are presented.

In Sect. 3 Drinfeld's construction of "the universal prounipotent" braid group representation is presented.

In Sect. 4 the F-valued "Markov trace" in this representation is constructed and the fact that it is a generating function for "universal Vassiliev link invariants" is proved. A multiplicative property of this "universal invariant" with respect to connected sums is proved. A generalization for string link invariants is also given.

In Sect. 5 some examples of calculations with out formula are given.

In Sect. 6 some open problems are discussed.

2. Preliminaries

Definition. *We shall call a trivalent graph consisting of several oriented circles (called Wilson lines) and several dashed lines (called propagators) a* **CS-diagram**.

The propagators and Wilson loops are allowed to meet in two types of vertices: one type (called R^2g-vertices) in which a propagator ends on one of the Wilson lines; and another type (called g^3-vertices) connecting three propagators.

We assume, that our graph has no connected components which contain only dashed lines.

We also assume, that one of two possible cyclic orders of propagators meeting in any vertex is specified.

Each CS-diagram can be presented by its plane projection (see Fig. 1 as an example).

We assume, that the counterclockwise cyclic degree in each vertex is fixed. For instance, the cyclic orders of propagators on graphs in Fig. 2a and Fig. 2b are different.

To avoid confusion with the cyclic order in R^2g-vertices, let us make the following convention: We assume that besides the cyclic order in any R^2g-vertex **the actual order** of the three lines incident to this vertex is fixed. The first line in this order is the dashed line, the second line is "the ingoing" solid line and the third one is "the outgoing" solid line (see Fig. 2c as an example).

So, we have two different cyclic orders in any R^2g-vertex one which came from the plane projection of the diagram and another which came from the actual order of the lines described above.

If these two cyclic orders agree then we will say that the given R^2g-vertex has a **positive orientation**. If they do not agree then we will say that the given R^2g-vertex has a **negative orientation**.

Note that the positive-oriented R^2g-vertex becomes negative-oriented if we will change orientation of the Wilson line on which this R^2g-vertex lies.

Let K be some ring $Z \subset K \subset C$.

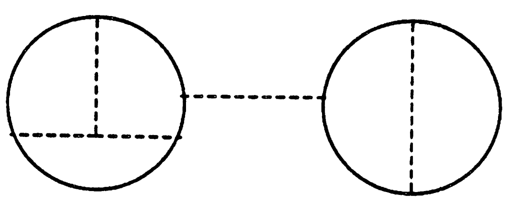


Fig. 1.

Fig. 2a–2b.

Fig. 2c–2d.

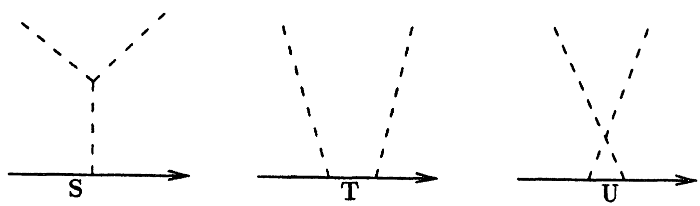


Fig. 3.

Definition. *A function* C : (*CS-diagrams*) $\to K$ *is called a* **weight system** *if*

$$C(S) = C(T) - C(U)\,, \tag{2.1}$$

where S, T, U *are CS-diagrams, identical everywhere except in some small ball, where they look as in Fig.* 3.

Definition. *Following Vassiliev* [*Va*1, *Va*2] *and Birman-Lin* [BL] *we shall call a CS-diagram with* $2n$ R^2g*-vertices and without* g^3*-vertices a* **Vassiliev [*n*]-diagram,** *and a* CS*-diagram with* $2n2R^2g$*-vertices and with one* g^3*-vertex a* **Vassiliev**$\langle n\rangle$**-diagram–**$\langle n\rangle$**-solid**.

Let D be Vassiliev $\langle n\rangle$-diagram. Let z_1, z_2 and z_3 be three R^2g-vertices connected by propagators to (the unique) g^3-vertex in D. Let us define Vassiliev $[n]$-diagrams D_{1+}, D_{1-} as Vassiliev $[n]$-diagrams, obtained from D by the local procedure shown in Fig. 4.

(The Vassiliev $[n]$-diagrams $D_{2+}, D_{2-}, D_{3+}, D_{3-}$ can be defined in the same way by changing z_1 to z_2 and to z_3 respectively.)

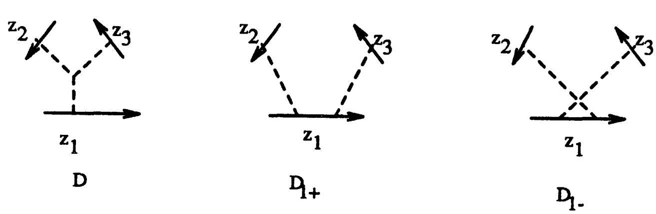


Fig. 4.

Definition. $[Bl, Va1, Va2, Ko1]$.

Let $W_n^s (s \in N)$ be a free K-module, generated by the set of s-Wilson-loop Vassiliev $[n]$-diagrams. (We assume that if two Vassiliev diagrams differ from each other by the cyclic order in one of their vertices, then the corresponding elements of the module W_n^s will differ by a minus sign. The same convention will be for more general CS-diagrams.) Let F_n^s be the quotient of W_n^s by the ideal, generated by the relations

$$D_{1+} - D_1 = D_{2+} - D_{2-} \tag{2.2}$$

(D runs over Vassiliev $\langle n \rangle$-diagrams).

Let us denote $F_0^s = K; F^s = \bigoplus_n F_n^s$, and let us identify $1 \in K = F_0^s$ with (the unique) s-Wilson-loop Vassiliev $[0]$-diagram.

Theorem 2.1. $[Ar1, BN1, Ko2]$.

K-module F_n^s is isomorphic to the quotient of the free module D_n^s, generated by **s**-Wilsons-loop CS-diagrams with $2n$ vertices by the ideal, generated by relations (2.3)–(2.5):

$$S = T - U\,, \tag{2.3}$$

where S, T and U are CS-diagrams, identical everywhere except inside some small ball, where they look as in Fig. 3.

$$I = H - X\,, \tag{2.4}$$

where I, H and X are CS-diagrams, identical everywhere except inside some small ball, where they look as in Fig. 5

$$Y + Z = 0\,, \tag{2.5A}$$

where Y and Z are CS-diagrams, identical everywhere except some small ball, where they look as in Figs. 2a and 2b respectively.

$$Y + Z = 0\,, \tag{2.5B}$$

where Y and Z are CS-diagrams, identical everywhere except some small ball, where they look as in Figs. 2c and 2d respectively.

In fact F^1 can be equipped with a structure of a graded Hopf algebra [Ko1] and we shall call it the Kontsevich Hopf algebra. When it will not lead to confusion, we will omit the superscript 1 and write $F = F^1$.

The Kontsevich Hopf algebra F acts on F^s (taking the connected sum along the Wilson loop) in s different mutually commuting ways [BN5], thus we have a graded action of $F^{\otimes s}$ on F^s.

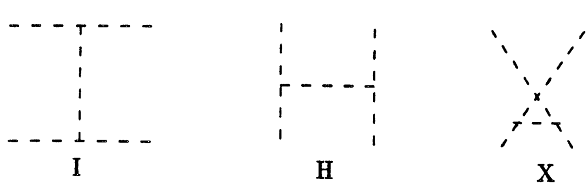


Fig. 5.

Let $A = \bigoplus_n A_n$ be the quotient of the Kontsevich algebra $F = \bigoplus_n F_n^1$ by the ideal generated by F_1^1. (The K-module F_1 has rank one and is generated by a single Vassiliev [1]-diagram. Let us denote this diagram by $t \in F_1$.) Since the element t is primitive, A is also a Hopf algebra.

It is well-known [Ko1] that the space A_n^* dual to A_n is canonically isomorphic to the space V_n of Vassiliev knot invariants of degree n factored by the space V_{n-1}. The map $V_n/V_{n-1} \to A_n^*$ is the evaluation of a knot invariant on singular embeddings with n double points [Va1, Va2, BL] which gives a linear function $V_n/V_{n-1} \otimes A_n \to C$.

The inverse map $I_n : A_n^* \to V_n \to V_n/V_{n-1}$ was first constructed in [Ko1] and is called "Kontsevich integral." The aim of this paper is to construct (formally another) inverse map $P_n : A_n^* \to V_n$ which has a simple combinatorial description.

Definition. *Let $X^m (m \in N)$ be the graded completion of Lie algebra $\bigoplus_n X_n^m$ (here the subscript n stands for the grading), wiith homogenous generators t^{ij} $(1 \leqq i < j \leqq m)$ of degree 1 and with relations*

$$\left[t^{ij}; t^{kl}\right] = 0 \quad (i \neq j \neq k \neq l)\,, \tag{2.6A}$$

$$\left[t^{ij}; t^{ik} + t^{jk}\right] = 0\,. \tag{2.6B}$$

The universal enveloping algebra UX^m of this Lie algebra is the prounipotent completion of the group algebra of the pure braid group (see [K2] and references therein). Kohno [K1] used this algebra in order to write the most general form of the Knizhnik–Zamolodchikov equation [KZ],

$$\frac{d\Psi}{dz_i} = \hbar \sum_{j \neq i} \frac{t^{ij}}{z_i - z_j} \Psi\,, \tag{2.7}$$

where ψ is a UX^m-valued meromorphic function on (C^m *diagonals*), $\hbar = \frac{h}{2\pi i}$. Relations (2.6) are imposed in degree to preserve the zero-curvature condition

$$\left[\frac{d}{dz_i} - \hbar \sum_{j \neq i} \frac{t^{ij}}{z_i - z_j}; \frac{d}{dz_k} - \hbar \sum_{l \neq k} \frac{t^{kl}}{z_k - z_l}\right] = 0\,, \tag{2.8}$$

which allows us to construct monodromy representation of pure braid group in the group $exp(X^m) \subset UX^m$. This representation is nonlocal and its matrix elements are certain hypergeometric-type integrals (see [Ao] and [K2] for more detailed exposition). In our approach we don't use this complicated technique.

Algebra UX^m can be embedded in the algebra A_{kz}^m of Feynman diagrams (see [BN1, BN5]) of the form depicted in Fig. 6.

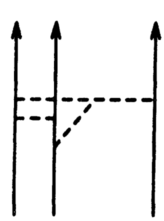


Fig. 6.

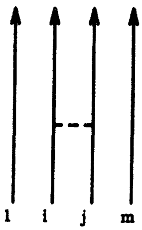


Fig. 7.

These diagrams are defined in the same way as usual for CS-diagrams, but they have m upward pointed Wilson lines instead of several Wilson loops. Here the generator t^{ij} is presented by the diagram on Fig. 7.

A diagram with $2n$ vertices is siad to be of degree n. The multiplication of two diagrams in the algebra A^m_{kz} is just putting the second diagram over the first one. It is easy to see that the grading and multiplication in A^m_{kz} defined above are compatible with those in UX^m.

3. Explanations of Drinfeld's Construction

Let K be come filed. Let $\phi(A,B)$ be some formal power series in two non-commuting variables A and B with coefficients in K and let

$$\Phi = \phi\left(\hbar t^{12}, \hbar t^{23}\right) \in UX^3 . \tag{3.1}$$

Definition. *A formal noncommutative power series $\phi(A,B)$ will be called an* **associator** *if* $\log(\phi(A,B))$ *belongs to the graded completion of the free Lie algebra with two generators A and B and if Eqs.* (3.2)–(3.5) *hold:*

$$\begin{aligned}&\phi\left(\hbar t^{12}, \hbar\left(t^{23}+t^{24}\right)\right)\phi\left(\hbar\left(t^{12}+t^{13}\right), \hbar t^{34}\right)\\ &\quad = \phi\left(\hbar t^{23}, \hbar t^{34}\right)\phi\left(\hbar\left(t^{12}+t^{13}\right), \hbar\left(t^{24}+t^{34}\right)\right)\phi\left(\hbar t^{12}, \hbar t^{23}\right) \in \exp\left(X^4\right),\end{aligned} \tag{3.2}$$

$$e^{\frac{\hbar t^{13}+\hbar t^{23}}{2}} = \Phi^{312} e^{\frac{\hbar t^{13}}{2}} (\Phi^{132})^{-1} e^{\frac{\hbar t^{23}}{2}} \Phi \in \exp\left(X^3\right), \tag{3.3}$$

$$e^{\frac{\hbar t^{13}+\hbar t^{12}}{2}} = (\Phi^{231})^{-1} e^{\frac{\hbar t^{13}}{2}} \Phi^{213} e^{\frac{\hbar t^{12}}{2}} (\Phi)^{-1} \in \exp\left(X^3\right), \tag{3.4}$$

$$\Phi^{321} = \Phi^{-1} \in \exp\left(X^3\right). \tag{3.5}$$

Here Φ^{ijk} (ijk is a permutation of 123) is the image of $\Phi \in UX^3$ under automorphism

$$s_{ijk} : UX^3 \to UX^3 ,$$

which maps t^{12} to t^{ij}; t^{13} to t^{ij} and t^{23} to t^{jk}.

Theorem 3.1 (Drinfeld). *Such an "associator" exists for any field K such that $Q \subset K \subset C$.*

We will give here an explicit construction of associator for $K = C$ due to Drinfeld. This construction will not be used later. We'll need for our purposes only formal properties (3.2)–(3.5) of "associator" $\phi(A,B)$ but not an explicit form of this "associator."

Following Drinfeld [D1], let us write a differential equation

$$\frac{dG(x)}{dx} = \hbar\left(\frac{A}{x} + \frac{B}{x-1}\right)G(x)\,. \tag{3.6}$$

Let G_1 and G_2 be solutions of (3.6) defined when $0 < x < 1$ with the asymptotic behavior

$$G_1(x) \approx x^{\hbar A}(x \to 0)$$

and

$$G_2(x) \approx (1-x)^{\hbar B}(x \to 1)\,.$$

Then

$$G_1 = G_2\phi_{kz} \tag{3.7}$$

for some formal noncommutative power series ϕ_{kx}.

Theorem 3.2 (Drinfeld). *ϕ_{kz} is an "associator."*

Everywhere below we will fix some choice of "associator" ϕ once and for all (for instance, let us put $\phi = \phi_{kz}$). All our constructions will work for any choice of ϕ.

We will need for our purposes to define a semi-direct product Y^m of the group algebra KS_m of the symmetric group S_m, and A^m_{kz} as follows: Y^m is generated as a linear space by pairs (x, s), where x is a diagram from A^m_{kz}; $s \in S_m$. Multiplication on Y^m is defined as follows:

$$(x_1, s_1)(x_2, s_2) = (s_2(x_1)x_2, s_1 s_2)\,.$$

Here we suppose that the symmetric group acts on A^m_{kz} by permutations of strings. The algebra Y^m has an important subgroup

$$G^m = S_m * \exp(X^m) \subset Y^m\,.$$

Let $\sigma_i (1 \leqq i \leqq m-1)$ be the standard generators of the braid group B_m satisfying relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad if(i-j) > 1\,, \tag{3.8}$$

and

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \tag{3.9}$$

(if it will not lead to confusion, we'll denote the elementary transpositions $s_i \in S_m$ by the same symbols as the braid group generators).

Let us define a map $\rho : B_m \to G^m \subset Y^m$ (which we will later prove to be a group homomorphism) as follows:

$$\rho(\sigma_1) = \left(e^{\frac{ht^{12}}{2}}; s_1\right)\,, \tag{3.10}$$

$$\rho(\sigma_i) = \phi^{-1}\left(h\sum_{s=1}^{i-1} t^{s,i}; ht^{i,i+1}\right)\left(e^{\frac{ht^{i,i+1}}{2}}; s_i\right)\phi\left(h\sum_{s=1}^{i-1} t^{s,i}; ht^{i,i+1}\right) \tag{3.11}$$

$$\text{if } 1 < i.$$

This construction of the map ρ (which is actually a representation of the braid group B_m into the algebra Y^m) is due to Drinfeld (the second formula in the proof of Proposition 5.1. of [Dr2]). It may be called "the universal prounipotent" braid group representation since the group G^m can be interpreted as a prounipotent completion of B_m.

We will prove in this section why ρ really gives us a braid group representation since it was not explained in [Dr2] or anywhere else. Representation ρ constructed above is a generalization of the braid group action on quasitensor category [Re1].

For the convenience of the reader let me give some conceptual explanation of the origin of the Drinfeld's formulas (3.10) and (3.11). I will explain how one can think about these formulas in such a way that it should be clear that they really give us some braid group representation.

To construct representation of the braid group B_m using the machinery of the quasitensor categories one has to choose some configuration of parentheses in the (nonassociative) product of m symbols $x_1,\dots,x_m$. Each transition from one configuration of parentheses to another configuration of parentheses can be decomposed (in a non-unique way) in the product of "the elementary transitions" of the form (3.12) where only one pair of parentheses changes:

$$\begin{aligned}&\dots((x_\imath\dots x_{\jmath-1}((x_j\dots x_{k-1}(x_k\dots x_{l-1})))\dots\\&\quad\to\dots(((x_i\dots x_{j-1})(x_j\dots x_{k-1}))(x_k\dots x_{l-1}))\dots\,.\end{aligned}\tag{3.12}$$

Let us associate to "the elementary transition" (3.12) "the elementary transition operator" Φ_{ijkl}

$$\Phi_{ijkl}=\phi\left(h\sum_{s=\imath}^{j-1}\sum_{p=j}^{k-1}t^{s,p};h\sum_{p=\jmath}^{k-1}\sum_{r=k}^{l-1}t^{p,r}\right)\,.\tag{3.13}$$

Then, to any transition from one configuration of parentheses to another configuration of parentheses we can associate "transition operator" Φ_{trans} by functoriality. The "pentagon identity" (3.2) insures that Φ_{trans} is independent of the choice of decomposition in the product of the elementary transitions.

Then, in degree to define the action of the braid group generator $s_\imath$, we should:

a) change the configuration of parentheses in degree to have $\dots(x_ix_{i+1})\dots$ inside one pair of parentheses (this gives us some "transition operator" Φ_{trans}),

b) apply the Drinfeld's R-matrix $\left(e^{\frac{ht^{i,i+1}}{2}};s_\imath\right)$, and

c) return back to our initial configuration of parentheses (this gives us an inverse operator to the operator Φ_{trans}).

Formulas (3.10) and (3.11) correspond to one particular choice of configuration of parentheses, namely, $(\dots((x_1x_2)\dots)x_{m-1})x_m)$ but any other choice is possible as well and gives us an equivalent representation with the transition operator between these two configurations of parentheses as an intertwiner. (If it will not lead to confusion, we'll denote all "transition operators" corresponding to transitions between different configurations of parentheses, by the same gymbol Φ_{trans}.)

Lemma 3.3. *If* $(i-j)>1$ *then Eqs.* (3.14)–(3.17) *hold:*

$$\left[t^{i,i+1};t^{j,j+1}\right]=0\,,\tag{3.14}$$

$$\left[t^{\imath,\imath+1};\phi\left(h\sum_{p=1}^{j-1}t^{p,j};ht^{j,j+1}\right)\right]=0\,,\tag{3.15}$$

$$\left[t^{j,j+1};\phi\left(h\sum_{s=1}^{i-1}t^{s,\imath};ht^{i,i+1}\right)\right]=0\,,\tag{3.16}$$

$$\left[\phi\left(h\sum_{s=1}^{i-1}t^{s,i};ht^{i,i+1}\right);\phi\left(h\sum_{p=1}^{j-1}t^{p,j};ht^{j,j+1}\right)\right]=0\,. \tag{3.17}$$

Proof. Relations (3.14) and (3.15) follow directly from (2.6).

Relation (3.16) follows from the fact that $\left[t^{j,j+1};h\sum_{s=1}^{i-1}t^{s,i}\right]=0$, from (3.14) and from the Leibnitz rule.

Relation (3.17) follows from (3.14), (3.15), (3.16), from the fact that

$$\left[\sum_{s=1}^{i-1}t^{s,i};\sum_{p=1}^{j-1}t^{p,j}\right]=0 \tag{3.18}$$

and from the Leibnitz rule.

To prove (3.18) it is sufficient to notice that $\left[\sum_{s=1}^{i-1}t^{s,i};t^{p,j}\right]=0$ for any p and then take the sum over the index p. The lemma is proved.

Lemma 3.4. *If* $(i-j)>1$ *then:*

$$\rho(\sigma_i)\rho(\sigma_j)=\rho(\sigma_j)\rho(\sigma_i)\,. \tag{3.19}$$

Proof. It follows immediately from the definition of ρ given by (3.10) and (3.11), and from Lemma 3.3.

Lemma 3.5.

$$\rho(\sigma_1)\rho(\sigma_2)\rho(\sigma_1)=\rho(\sigma_2)\rho(\sigma_1)\rho(\sigma_2)\,. \tag{3.20}$$

Proof. If we use the definition of ρ given by (3.10) and (3.11), then (3.20) can be rewritten in the following form:

$$e^{\frac{ht^{12}}{2}}(\Phi^{213})^{-1}e^{\frac{ht^{13}}{2}}\Phi^{231}e^{\frac{ht^{23}}{2}}=(\Phi^{123})^{-1}e^{\frac{ht^{23}}{2}}\Phi^{123}e^{\frac{ht^{13}}{2}}(\Phi^{312})^{-1}e^{\frac{ht^{12}}{2}}\Phi^{321}\,. \tag{3.20A}$$

Using (3.5) several times and multiplying both the l.h.s and the r.h.s of (3.20A) by Φ on the right, we obtain another equivalent form of (3.20):

$$e^{\frac{ht^{12}}{2}}\Phi^{312}e^{\frac{ht^{13}}{2}}(\Phi^{132})^{-1}e^{\frac{ht^{23}}{2}}\Phi=\Phi^{321}e^{\frac{ht^{23}}{2}}(\Phi^{132})^{-1}e^{\frac{ht^{13}}{2}}\Phi^{213}e^{\frac{ht^{12}}{2}}\,. \tag{3.20B}$$

Using (3.3) we see that the l.h.s of (3.20B) is equal to $\left(e^{\frac{ht^{12}}{2}};s_1\right)\left(e^{\frac{ht^{13}+ht^{23}}{2}};1\right)$ and the r.h.s of (3.20B) is equal to $\left(e^{\frac{ht^{13}+ht^{23}}{2}};1\right)\left(e^{\frac{ht^{12}}{2}};s_1\right)$.

The equality of these two expressions follows from (2.6B) which proves the lemma.

Let $\Phi_i=\phi\left(h\sum_{p=1}^{i-1}t^{p,i}+t^{p,i+1};ht^{i,i+2}+ht^{i+1,i+2}\right)\phi\left(h\sum_{s=1}^{i-1}t^{s,i};ht^{i,i+1}\right)$.

Lemma 3.6. *Equations* (3.21) *and* (3.22) *hold:*

$$\Phi_i\rho(\sigma_i)\Phi_i^{-1}=\left(e^{\frac{ht^{i,i+1}}{2}};s_i\right)\,, \tag{3.21}$$

$$\Phi_i\rho(\sigma_{i+1})\Phi_i^{-1}=\phi^{-1}\left(ht^{i,i+1};ht^{i+1,i+2}\right)\left(e^{\frac{ht^{i+1,i+2}}{2}};s_{i+1}\right)\phi\left(ht^{i,i+1};ht^{i+1,i+2}\right)\,. \tag{3.22}$$

Proof. Since

$$\left[\left(e^{\frac{ht^{i,i+1}}{2}};s_i\right);\phi\left(h\sum_{p=1}^{i-1}t^{p,i}+t^{p,i+1};ht^{i,i+2}+ht^{i+1,i+2}\right)\right]=0\,,$$

then $\Phi_i^{-1}\left(e^{\frac{ht^{i,i+1}}{2}};s_i\right)\Phi_i$ is equal to the r.h.s of (3.11). Thus,

$$\rho(\sigma_i)=\Phi_i^{-1}\left(e^{\frac{ht^{i,i+1}}{2}};s_i\right)\Phi_i\,,$$

which is equivalent to (3.21)

To prove (3.22) let us use "the pentagon identity" (3.2) in the form

$$\begin{aligned}&\phi\left(ht^{i,i+1};ht^{i+1,i+2}\right)\phi\left(h\sum_{p=1}^{i-1}t^{p,i}+t^{p,i+1};ht^{i,i+2}+ht^{i+1,i+2}\right)\phi\left(h\sum_{s=1}^{i-1}t^{s,i};ht^{i,i+1}\right)\\&\quad=\phi\left(h\sum_{r=1}^{i}t^{r,i+1}+t^{r,i+2};ht^{i,i+1}+ht^{i,i+2}\right)\phi\left(h\sum_{s=1}^{i-1}t^{s,i+1};ht^{i+1,i+2}\right)\,,\end{aligned}\tag{3.23}$$

or, equivalently

$$\begin{aligned}&\phi\left(ht^{i,i+1};ht^{i+1,i+2}\right)\Phi_i\\&\quad=\phi\left(h\sum_{r=1}^{i}t^{r,i+1}+t^{r,i+2};ht^{i,i+1}+ht^{i,i+2}\right)\phi\left(h\sum_{s=1}^{i-1}t^{s,i+1};ht^{i+1,i+2}\right)\,,\end{aligned}\tag{3.23A}$$

Equation (3.23A) implies that

$$\begin{aligned}&\Phi_i^{-1}\phi^{-1}\left(ht^{i,i+1};ht^{i+1,i+2}\right)\left(e^{\frac{ht^{i+1,i+2}}{2}};s_{i+1}\right)\phi\left(ht^{i,i+1};ht^{i+1,i+2}\right)\Phi_i\\&\quad=\phi^{-1}\left(h\sum_{s=1}^{i-1}ht^{s,i+1}+t^{i+1,i+2}\right)\phi^{-1}\left(h\sum_{r=1}^{i}t^{r,i+1}+t^{r,i+2};ht^{i,i+1}+ht^{i,i+2}\right)\\&\quad\times\left(e^{\frac{ht^{i+1,i+2}}{2}};s_{i+1}\right)\phi\left(h\sum_{r=1}^{i}t^{r,i+1}+t^{r,i+2};ht^{i,i+1}+ht^{i,i+2}\right)\\&\quad\times\phi\left(h\sum_{s=1}^{i-1}t^{s,i+1};ht^{i+1,i+2}\right)\,.\end{aligned}\tag{3.24}$$

Since

$$\left[\left(e^{\frac{ht^{i+1,i+2}}{2}};s_{i+1}\right);\phi\left(h\sum_{r=1}^{i}t^{p,i+1}+t^{p,i+2};ht^{i,i+1}+ht^{i,i+2}\right)\right]=0\,,$$

then the r.h.s of (3.24) can be rewritten in the form

$$\phi^{-1}\left(h\sum_{s=1}^{i-1}t^{s,i+1};ht^{i+1,i+2}\right)\left(e^{\frac{ht^{i+1,i+2}}{2}};s_{i+1}\right)\phi\left(h\sum_{s=1}^{i-1}t^{s,i+1};ht^{i+1,i+2}\right)\,.\tag{3.24A}$$

But the expression (3.24A) is equal to $\rho(\sigma_{i+1})$ which implies

$$\Phi_i^{-1}\phi^{-1}\left(ht^{i,i+1};ht^{i+1,i+2}\right)\left(e^{\frac{ht^{i+1,i+2}}{2}};s_{i+1}\right)\phi\left(ht^{i,i+1};ht^{i+1,i+2}\right)\Phi_i=\rho(\sigma_{i+1})\,.\tag{3.25}$$

But (3.25) is equivalent to (3.22). The lemma is proved.

Lemma 3.7.

$$\rho(\sigma_i)\rho(\sigma_{i+1})\rho(\sigma_i) = \rho(\sigma_{i+1})\rho(\sigma_i)\rho(\sigma_{i+1}) . \tag{3.26}$$

Proof. Equations (3.21) and (3.22) reduce the statement of the lemma to the case $i = 1$. But this case was already proved in Lemma 3.5. The lemma is proved.

4. Taking the Trace

It is well-known (see, for instance, [Bi]) that any oriented s-component link L can be presented as a closed braid. Two braids $b_1 \in B_{m1}$ and $b_2 \in B_{m2}$ give under closure the same link iff they can be obtained from each other by a finite sequence of Markov moves of two types:

$$b_1 b_2 \approx b_2 b_1 \in B_m \tag{4.1}$$

and

$$b \in B_m \approx b\sigma_m^{\pm 1} \in B_{m+1} . \tag{4.2}$$

Thus, any function $f : \cup_m B_m \to \cup_s F^s$ gives rise to some link invariant iff f takes equal values on braids equivalent with respect to (4.1) and (4.2).

Any framed link also can be presented as a closed braid. The analogues of Markov moves for braids which give under closure the same framed link (with blackboard framing [Tu, Pi3, Pi4]) can also be described explicitly (see [Re2]). Here we give sufficient conditions (4.1A) and (4.2A) for a function $f : \cup_m B_m \to \cup_s F^s$ to descend to some framed link invariant:

$$f(b_1 b_2) = f(b_2 b_1) \quad (b_1; b_2 \in B_m) , \tag{4.1A}$$

and

$$f\left(b s_m^{\pm 1}\right) = q^{\pm 1} * f(b) \quad b \in B_m . \tag{4.2A}$$

Here $*$ is the action of F on F^s (on the s^{th} component), $q = e^{\frac{ht}{2}} \in F$, t is the standard generator in F_1 (see Fig. 8).

Let us fix a configuration of parentheses in the (nonassociative) product of $2m$ symbols $x_1, \ldots, x_m, y_m, \ldots, y_1$ as follows:

$$((x_1((x_2(\ldots((x_{m-1}(x_m y_m))y_{m-1})\ldots))y_2))y_1) . \tag{4.3}$$

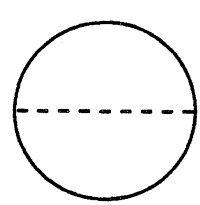


Fig. 8.

Let us define, using this configuration of parentheses, formulas (3.10)–(3.13) and remarks following them, a representation $\hat{\rho} : b_m \to G^{2m}$ as the restriction to $B_m \subset B_{2m}$ of the representation

$$\Phi_{\mathrm{trans}}^{-1} \rho \Phi_{\mathrm{trans}} : B_{2m} \to G^{2m} \subset Y^{2m} ,$$

where Φ_{trans} is the transition operator between "the standard" configuration of parentheses on the set of $2n$ elements and the configuration (4.3).

Now let us suppose that the first m Wilson lines in any "diagram" $(x,s) \in Y^{2m}$ are oriented "up" and the second m Wilson lines are oriented "down." Then for any $m \in N$ let us consider a map $\tau : Y^{2m} \to \cup_{s=1}^{m} F^s$ of graded linear spaces, defined as follows:

For any diagram $(x,s) \in Y^{2m}$ we have $4m$ free ends on it. Let us mark each of these free ends with a natural number from 1 to m as it is shown on Fig. 9a.

Then let us connect by (directed) line each pair of free ends on the top of the diagram with the same markings, and let us do the same on the bottom of the diagram (see Fig. 9b as an example. In this example $m = 3$)

The result of this procedure will be, by definition, $\tau(x,s)$.

If it will not lead to confusion, we will not distinguish braids in B_m and their images in Y^{2m}.

Let $b_1 \in B_{m_1}; b_2 \in B_{m_2}$ be two braids, let b_2 give a knot under closure, and let $(b_1 * b_2) \in B_{m_1+m_2-1}$ be the braid, obtained from b_1 and b_2 by the procedure shown on Fig. 10.

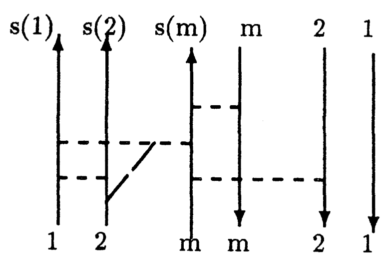


Fig. 9a.

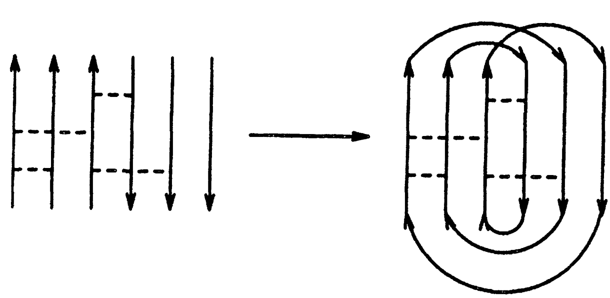


Fig. 9b.

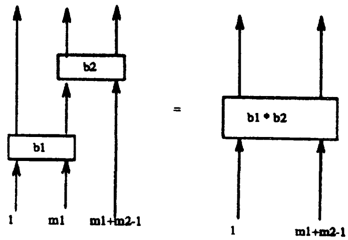


Fig. 10.

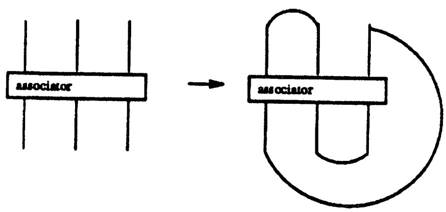


Fig. 11.

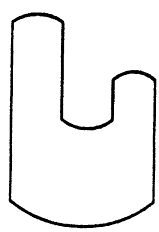


Fig. 12.

Theorem 4.1. *The following identitiy holds:*

$$\tau(b_1 * b_2) = \tau(b_2) * \tau(b_1) \tag{4.4}$$

where $*$ is the action of F on F^s (on the s^{th} component).

Proof. Geometrically obvious from (4.3), Fig. 9b and Fig. 10.

Let $q = e^{\frac{ht}{2}}$ and let $\mu \in F$ be the image of associator $\Phi \in$ exp $(X^3) \subset UX^3$ under "the closure map" shown on Fig. 11.

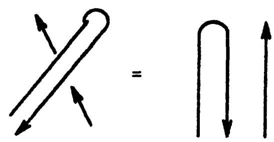
Remark. If $\Phi = \Phi_{kz}$, the μ is equal to the value of the generating function of Kontsevich integrals on the Morse knot shown on Fig. 12.

Lemma 4.2. *The identity* (4.5) *shown on Fig.* 13 *holds:*

Proof. It follows from (3.3) that the l.h.s of (4.5) is equal to the expression shown on Fig. 14

But due to our sign convention (change of the cyclic order in any vertex implies the change of sign) the following formula (4.6) holds

The formula (4.6) implies that the expression on Fig. 14 is equal to identity



(4.5)

Fig. 13.

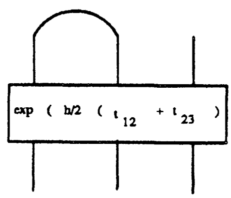
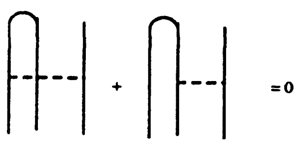


Fig. 14.



(4.6)

Fig. 15.

Lemma 4.3. *Let s_1 be the standard generator of B_2. Then $\tau(s_1^{\pm 1}) = q^{\pm 1}\mu$,*

We give here a pictorial proof (see Fig. 16):
The first identity in Fig. 16 follows from Lemma 4.2.
Let $P : B_m \longrightarrow \cup_s F^s$ be equal to $(\mu)^{1-m}\tau : B_m \longrightarrow \cup_s F^s$.

Lemma 4.4. *The map P is a "Markov trace", i.e., it satisfies* (4.1A) *and* (4.2A).

Property (4.1A) is geometrically obvious. Property (4.2A) follows from Theorem 4.1 and from Lemma 4.3.

Let P above be defined framed link invariant. Let us consider its perturbative expansion : $P = \sum_{n=0}^{\infty} h^n P_n$.

Lemma 4.5. *P_n is F_n^s-valued Vassiliev framed link invariant of degree n.*

Proof. Let $b \in B_m$ be a braid and let $\tilde{\rho}(b) = \sum_{n=0}^{\infty} x_n(b)h^n \subset Y^{2m}$.

Then $x_n(b) \subset Y^{2m}$ has degree n in Y^{2m} (since this fact is true for the generators $\sigma_\iota \in B_m$ we can deduce it, for any $b \in B_m$). Thus, for any framed oriented link $LP_n(L)$ also has degree n, which implies that $P_n(L) \in F_n^s \subset F^s$.

Let L be a singular embedding of $(S^1)^s$ into R^3 with $(n+1)$ double crossing points. Then L can be presented as a closure of a "generalized braid" [Pi1, Ba] (braid where in some places the generators σ_i are changed to the generators a_ι with double crossing on i place. The generators a_i are depicted on Fig. 17).

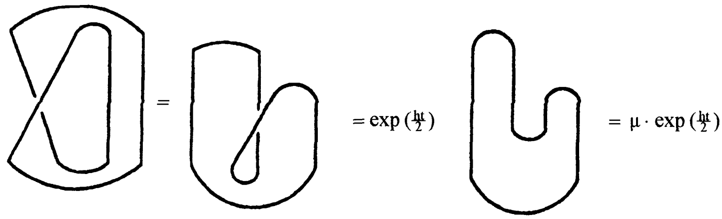


Fig. 16.

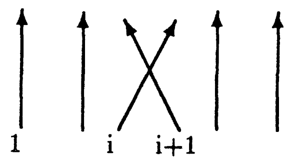


Fig. 17.

The representation $\rho : B_m \to Y^m$ can be extended to these "generalized braids" by the formula

$$\rho(a_i) = \rho(\sigma_i) - \rho(\sigma_i^{-1}) \tag{4.8}$$

(and the representation $\hat{\rho} : B_m \to Y^{2m}$ can also be extended to "the generalized braids" by the same formula).

The formulas (4.8), (3.10) and (3.11) imply that

$$\rho(a_1) = \left(2 \sinh \frac{ht^{12}}{2}; s_1\right), \tag{4.9}$$

and

$$\rho(\sigma_i) = \phi_{kz}^{-1}\left(h\sum_{s=1}^{i-1} t^{s,i}; ht^{i,i+1}\right)\left(2 \sinh \frac{ht^{i,i+1}}{2}; s_i\right)\phi_{kz}\left(h\sum_{s=1}^{i-1} t^{s,i}; ht^{i,i+1}\right) \tag{4.10}$$

if $1 < i$.

Thus we have shown that $\hat{\rho}(a_i)$ are divisible by h in $Y^{2m} \otimes C[h]$. This fact implies, that $\hat{\rho}(b)$ is divisible by h^{n+1} for any "generalized braid" $b \in B_m$ with $(n+1)$ double crossing points.

Thus $P(L)$ is divisible by h^{n+1}. This means that $P_n(L) = 0$ for any singular embedding L with $(n+1)$ double crossing points, or, equivalently, that P_n is a Vassiliev invariant of degree n. The result of the lemma follows.

Let V_n^s be the space of Vassiliev invariants of framed s-component links of order n. Then there is a natural map $f_n : V_n^s \to V_n^s/V_{n-1}^s \to (F_n^s)^*$, defined as follows: Let v be some Vassiliev invariant of degree n and let D be Vassiliev $[n]$-diagram. Then

$$(P_n(v); D) = (v; L(D)), \tag{4.11}$$

where $L(D)$ is some singular embedding $(S^1)^s \to R^3$ with n double crossing points for which the underlying configuration of n points on $(S^1)^s$ is given by the diagram D.

Theorem 4.6. *The map $\langle P_n; \ldots\rangle : (F_n^s)^* \to V_n^s$ is left inverse to f_n, and differs from its right inverse on some Vassiliev invariant of degree $n-1$.*

Proof. It is sufficient to prove that for any singular embedding L from $(S^1)^s$ to R^3 with precisely n double points the following equation (4.12) holds:

$$P_n(L) = D(L)\,, \tag{4.12}$$

Here $D(L)$ is a CS-diagram with n propagators which join those points on $(S^1)^s$, which are identified under L.

At this point it is time to mention that we need to specify cyclic orders in all $R^s g$-vertices of the diagram $D(L)$. Different cyclic orderings of vertices will lead to the (possible) change of the overall sign before the diagram $D(L)$.

The correct choice will be to take the s disjoint circles on the plane, to join n pairs of vertices according to combinatorics of the singular embedding L and then **to take the overall minus sign before the diagram.**

One can check that the above prescription can be deduced from the closed braid presentation of the link diagram.

Our sign conventions (corresponding to positive or negative cyclic order in $R^2 g$-vertices is different from the sign conventions used by Kontsevich [Ko1].

Kontsevich used embedding of the knot in 3-space, decomposition of the 3-space in the product of a complex plane C and a real line R and used sign convention in $R^2 g$-vertices according to the sign of the scalar product of the tangent vector to the knot and (the positive unit) tangent vector to the real line R.

By closer examination of both sign conventions, one can observe that they are in fact equivalent.

Let us present L as a closure of some "generalized braid" $b \in B_m$. Then $\hat{\rho}(b)$ is product of some terms of the form

$$\left(e^{\frac{ht^{i,i+1}}{2}}; s_i\right)\,, \tag{4.13}$$

$$\Phi_{\mathrm{trans}}^{\pm 1}\,, \tag{4.14}$$

and

$$\left(2sh\frac{ht^{i,i+1}}{2}; s_i\right)\,. \tag{4.15}$$

There are precisely n terms of forms of the form (4.15).

Let us observe that the following statements hold:

A) the terms (4.13) and (4.14) have the form

$$1 + hX \tag{4.16}$$

for some $X \in Y^{2m}$;

B) $\mu^{\pm 1}$ also has the form (4.16) for some $X \in F$; and

C) the terms (4.15) have the form

$$ht^{i,i+1} + h^2 X \tag{4.17}$$

for some $X \in Y^{2m}$.

From these facts we can deduce that the expression for the coefficient in h^n in perturbative expansion of $\hat{\rho}(b)$ consists of the single term. This term is the product of n terms of the form (4.15). This fact implies that $P_n(L) = D(L)$, as desired. The theorem is proved.

Kontsevich Hopf algebra F has a (graded) quotient $A = F/F_1F$. then A_n^* is canonically identified with the space of Vassiliev unframed knot invariants of degree n factored by the space of invariants of degree $n-1$. In the basis of Vassiliev

$[n]$-diagrams in F_n the projector $Pr : F_n \to A_n$ can be described explicitly [Pi2],

$$Pr(D) = \sum_{k=0}^{n}(-t)^n \sum_I D_I \,, \tag{4.18}$$

where t is the generator of F_1; the second sum in (4.18) is taken over all $[k]$-subdiagrams D of D. The quantity $Pr(P) = \sum_{n=0}^{\infty} h^n Pr(P_n)$ which is the map: *Knots* $\to F$ is the generating function for "universal" (order n)-Vassiliev knot invariants and has the same formal properties as the generating function $I = \sum_{n=0}^{\infty} h^n I_n$ of Kontsevich integrals [Ko1].

Theorem 4.7. *Let K_1 and K_2 be two oriented framed knots; $K_1 * K_2$ be their connected sum. Then $P(K_1 * K_2) = P(K_1)P(K_2)$.*

Proof. It follows immediately from Theorem 3.1 and the definition of P.

5. Examples and Applications

To illustrate how this algorithm works we will calculate the second Vassiliev invariant V_2 for torus knots of type $(2, 2n+1)$. We will do this in two different ways: in the classical way and in the way using the above-developed machinery.

Let us first recall that the space of Vassiliev knot invariants of order two which vanish on the trivial knot is one-dimensional. The generator of this vector space can be chosen as (the unique) Vassiliev knot invariant V_2 of order two which vanishes on the trivial knot and which takes value one on the trefoil knot [Va1]. This invariant takes value one on any singular knot with two double transversal crossings and with underlying Vassiliev [2]-diagram as shown on Fig. 18a. Let us denote this Vassiliev [2]-diagram D_2^n.

The other (trivial) Vassiliev [2]-diagram is shown on Fig. 18b. Let us denote this Vassiliev [2]-diagram D_2^t.

The knot invariant V_2 is equal to the classical Robertello knot invariant. Its value on the given knot K is equal to the value of the second coefficient of the Taylor expansion of the normalized Alexander polynomial of this knot [Ro].

Let $T(p,q)$ be torus knot of type (p,q). The normalized Alexander polynomial of this knot is well-known and is equal to $\dfrac{\left(t^{\frac{pq}{2}} - t^{\frac{-pq}{2}}\right)\left(t^{\frac{1}{2}} - t^{\frac{-1}{2}}\right)}{\left(t^{\frac{p}{2}} - t^{\frac{-p}{2}}\right)\left(t^{\frac{q}{2}} - t^{\frac{-q}{2}}\right)}$.

For the case of knot $T(2, 2n+1)$ the normalized Alexander polynomial is equal to $\dfrac{\left(t^{\frac{2n+1}{2}} + t^{\frac{-2n-1}{2}}\right)}{\left(t^{\frac{1}{2}} + t^{\frac{-1}{2}}\right)}$. For the case of trefoil knot $T(2, \pm 3)$, this number is equal to one as expected.

The second derivative of this Laurent polynomial at $t = 1$ is equal to $n(n+1)$. Thus, the value of the Robertello invariant on the knot $T(2, 2n+1)$ is equal to $\frac{n(n+1)}{2}$.

Now we are going to recover this classical formula using the machinery developed in the main body of our paper.

To calculate explicitly for any given knot all its Vassiliev invariants up to order n orders using our method we should know an explicit expression for some "associator" $\phi(hA, hB)$ (a formal power series satisfying (3.2) - (3.5)) up to order n in h.

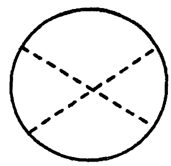


Fig. 18a.

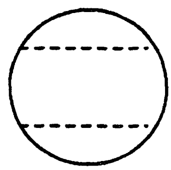


Fig. 18b.

Thus, to calculate the second order Vassiliev invariant we should know an explicit expression for "associator" $\phi(hA, hB)$ up to order two in h. This expression is known [Dr1, Dr2, BN6]. The formula for the associator is:

$$\phi(hA, hB) = 1 + h^2 \frac{(AB - BA)}{24} + O(h^4)\,, \tag{5.1A}$$

$$\phi^{-1}(hA, hB) = 1 - h^2 \frac{(AB - BA)}{24} + O(h^4)\,. \tag{5.1B}$$

Let us take our torus knot $T(2, 2n+1)$ as a closure of 2-briad b_{2n+1} with $2n+1$ over-crossings.

By our prescription from Sect. 4 we have

$$P(T(2, 2n+1)) = (\tau(b_{2n+1}))\mu^{-1}\,. \tag{5.2}$$

Then we should expand the r.h.s. of (5.1) in the formal power series in h and take the coefficient in h^2. This coefficient (let us call it $P_2(T(2, 2n+1)))$ will be an element of 2-dimensional vector space F_2^1 generated by Vassiliev [2]-diagrams D_2^n and D_2^t.

Since we are computing the Vassiliev invariant of **unframed knots**, the diagram D_2^t is equal to zero and what we need to compute is the coefficient in D_2^n in the expansion of $P_2(T(2, 2n+1))$ in the basis $\{D_2^n; D_2^t\}$ of the vector space F_2^1. We should prove that this coefficient is equal to $\frac{n(n+1)}{2}$.

To do this it is sufficient to calculate $\tau(b_{2n+1})$ and μ^{-1} up to order two in the h expansion.

Let us cut the plane projection of $T(2, 2n+1)$ (presented as a closed 2-braid) by four horizontal lines into five "elementary" pieces (see Fig. 19).

Only three of these pieces "in the middle" will give a non-trivial contribution to $\tau(b_{2n+1})$. The coefficient in h^2 in $\tau(b_{2n+1})$ is equal to the sum of three terms corresponding to these three pieces on Fig. 19.

1) Contribution from the associator at the bottom is equal to $-\frac{h^2}{24} D_2^n + \frac{h^2}{24} D_2^t$ (we derived this expression from (5.1A)).
2) Contribution from the exponent is equal to $\frac{h^2}{2}\left(\frac{2n+1}{2}\right)^2 D_2^n$.

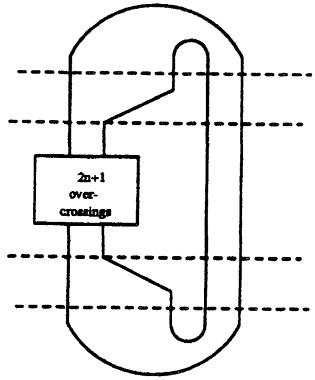


Fig. 19.

3) Contribution from the $(associator)^{-1}$ at the top is equal to $-\frac{h^2}{24}D_2^n + \frac{h^2}{24}D_2^t$ (we derived this expression from (5.1B)).

Thus, we see that

$$\tau(b_{2n+1}) = 1 + h^2\left[\frac{(2n+1)^2}{8} - \frac{1}{12}\right]D_2^n + \frac{h^2}{12}D_2^t + O(h^3)\,. \tag{5.3A}$$

But (5.1B) implies that

$$\tau(\mu)^{-1} = -\frac{h^2}{24}\,. \tag{5.3B}$$

Combining (5.2) and (5.3) together we see that $P_2(T(2,2n+1))$ is equal to $\frac{n(n+1)}{2}D_2^n$ (plus the term corresponding to D_n^t which should be dropped) which matches with the classical formula.

6. Discussions

At the moment, there are three different expressions for the universal Vassiliev knot invariant, (the quantity, which satisfies conditions of theorems 4.3 and 4.6). The first one is constructed from perturbative expansion of the monodromy of the KZ-equation (Kontsevich integrals [Ko1]), the second one is constructed from perturbative Chern–Simons theory [Ko1, BN3] (see also [AS] and [GMM]). The third construction is presented here (and also independently in [Car]).

The "universal Vassiliev invariant" in the form presented here can be evaluated purely combinatorially for any particular link L, if we know an explicit expression for the "Drinfeld's associator" ϕ_{kz} as a formal noncommutative power series in $\hbar t^{12}$ and $\hbar t^{23}$. An "iterated integral" expression for the "associator" was proposed in [BN6], which proves immediately the equivalence of our approach with Kontsevich's one (see also [LM] for some related results).

The analogous problem for "Kontsevich integrals" is much more complicated and involves calculations with hypergeometric type integrals [TK, Ao] and with multiple zeta-functions [Ko5]. In our approach only 2^n such integrals (for each $n \in N$) should be calculated. The calculations in perturbative Chern–Simons theory are even more complicated.

The above defined construction τ of the universal Vassiliev invariant of a link which can be presented as a closure of braid has a straightforward generalization to an arbitrary link diagram, and even to a string link diagram [BN5]. Roughly speaking, τ is a decomposition of the generating function of Kontsevich integrals (before inserting the correction factor μ^{1-m}) in the product of "the elementary" factors corresponding to the decomposition of the link diagram into "the elementary" pieces.

Drinfeld's construction of the representation $\rho : B_m \to G^m$ depends on the choice of "associator" ϕ. We can construct explicitly only one such "associator" (namely, ϕ_{kz}) but we would like construct explicitly "the universal Q-valued Vassiliev invariant."

There are two possibilities how one could do this. The first one is to try to calculate explictly Q-valued "associator" (which nobody knows how to do). The second possibility is to prove that all these formally different "universal Vassiliev invariants" (with different ϕ) are equal. We conjecture that it is so.

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