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On the Relationship Between Monstrous Moonshine and the Uniqueness of the Moonshine Module

Michael P. Tuite

Department of Mathematical Physics, University College, Galway, Ireland and
Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland.
E-mail: mphtuite@bodkin.ucg.ie

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Abstract. We consider the relationship between the conjectured uniqueness of the Moonshine Module, $\mathscr{V}^\natural$, and Monstrous Moonshine, the genus zero property of the modular invariance group for each Monster group Thompson series. We first discuss a family of possible Z_n meromorphic orbifold constructions of $\mathscr{V}^\natural$ based on automorphisms of the Leech lattice compactified bosonic string. We reproduce the Thompson series for all 51 non-Fricke classes of the Monster group M together with a new relationship between the centralisers of these classes and 51 corresponding Conway group centralisers (generalising a well-known relationship for 5 such classes). Assuming that $\mathscr{V}^\natural$ is unique, we consider meromorphic orbifoldings of $\mathscr{V}^\natural$ and show that Monstrous Moonshine holds if and only Z_r if the only meromorphic orbifoldings of $\mathscr{V}^\natural$ are $\mathscr{V}^\natural$ itself or the Leech theory. This constraint on the meromorphic orbifoldings of $\mathscr{V}^\natural$ therefore relates Monstrous Moonshine to the uniqueness of $\mathscr{V}^\natural$ in a new way.

1. Introduction

The Moonshine Module, $\mathscr{V}^\natural$, of Frenkel, Lepowsky and Meurman (FLM) [1, 2, 3] is historically the first example of a Z_2 orbifold model [4] in Conformal Field Theory (CFT) [5, 6]. The orbifold construction is based on a reflection automorhism of the central charge 24 bosonic string which has been compactified [7] via the Leech lattice cf. [8]. The vertex operators (primary conformal fields) of $\mathscr{V}^\natural$ form a closed meromorphic Operator Product Algebra (OPA) [3, 9, 10] which is preserved by the Fischer-Griess Monster group, M [11]. By construction, $\mathscr{V}^\natural$ has no massless (conformal dimension 1) operators and has modular invariant partition function $J(\tau)$, the unique modular invariant meromorphic function with a simple pole at $\tau = \infty$ and no constant term. $J(\tau)$ is unique because the fundamental region for the full modular group is of genus zero cf. [12]. Conway and Norton [13] conjectured that this genus zero property extends to other modular functions called the Thompson series $T_g(\tau)$ for each conjugacy class of $g \in M$ [14]. Such a genus zero modular function is called a hauptmodul and this conjecture that each $T_g(\tau)$ is a hauptmodul is referred to as Monstrous Moonshine. Borcherds [15] has now proved the Moonshine conjectures but the origin of the genus zero property is still unclear. One of the main purposes of this paper is to provide a derivation of Monstrous Moonshine from a new principle related to the FLM uniqueness conjecture for $\mathscr{V}^\natural$ which states that $\mathscr{V}^\natural$ is the unique central charge 24 meromorphic CFT (up to isomorphisms) with partition function $J(\tau)$ [3]. Recently, Dong and Mason [16] have provided rigorous Z_p meromorphic orbifold constructions based on prime order p automorphisms of the Leech theory for $p = 3, 5, 7, 13$ each with partition function $J(\tau)$. The $p = 3$ case has also been considered by Montague [17]. The resulting CFTs have been proved to be isomorphic to $\mathscr{V}^\natural$ for $p = 3$ and almost certainly so for $p = 5, 7, 13$, lending weight to the FLM uniqueness conjecture.

This work is broadly divided into two parts. In the first part (Sects. 2 and 3) we discuss further evidence for the uniqueness of $\mathscr{V}^\natural$ where a family of Z_n meromorphic orbifoldings of the Leech theory (including the 5 prime ordered ones) which possibly reproduce $\mathscr{V}^\natural$ are described [18]. We also argue that each such candidate construction of $\mathscr{V}^\natural$ can be reorbifolded to reproduce the Leech theory again. In the second part of the paper, in Sect. 4, we discuss other meromorphic orbifoldings of $\mathscr{V}^\natural$ with respect to $g \in M$ [19]. We show that given the uniqueness of $\mathscr{V}^\natural$, then this orbifolding of $\mathscr{V}^\natural$ can give only $\mathscr{V}^\natural$ or the Leech theory if and only if the corresponding Thompson series is of genus zero. Thus, assuming the uniqueness of $\mathscr{V}^\natural$, Monstrous Moonshine can be derived from the constraints on the possible meromorphic orbifoldings of $\mathscr{V}^\natural$. The advantage of our approach is that a natural interpretation for a Thompson series is given and the origin of the modular invariance group for each series is clearly understood. Furthermore, when we show that Monstrous Moonshine is equivalent to the above constraints on the meromorphic orbifoldings of $\mathscr{V}^\natural$ (given the uniqueness of $\mathscr{V}^\natural$), a case by case study of the classes of M is not required.

We begin in Sect. 2 with a review of both the FLM construction of $\mathscr{V}^\natural$ [1, 2, 3] from the point of view of CFT [5, 6, 20, 21, 22] and Monstrous Moonshine [13]. In Sect. 3 a family of Z_n meromorphic orbifoldings of the Leech theory (including the 5 prime ordered ones) based on 38 automorphisms of the Leech lattice are described each with partition function $J(\tau)$ so that each orbifolding is a candidate construction of $\mathscr{V}^\natural$ [18]. Extensive use of non-meromorphic OPAs for various twisted operator sectors is made in both Sects. 2 and 3 since such algebras provide the most natural setting for describing orbifold constructions [20, 21]. However, it must be stated that a fully rigorous description of non-meromorphic OPAs has yet to be provided. We show that for each Z_n meromorphic orbifolding of the Leech theory there is a corresponding reorbifolding with respect to a "dual automorphism" which reproduces the Leech theory again so that the Leech theory is an orbifold partner to each such construction. Within these constructions, we naturally reproduce $T_g(\tau)$ of genus zero for all of the 51 non-Fricke elements of M, i.e. $T_g(\tau)$ is not invariant under the Fricke involution $\tau \to -1/nh\tau$, h an integer. We also find a generalisation of an observation of Conway and Norton [13] (for prime order $p = 2, 3, 5, 7, 13$) relating the centralisers of the non-Fricke elements in M to corresponding centralisers in the Conway group, the automorphism group of the Leech lattice. Finally, we explicitly find for 11 of the 38 orbifold constructions, a Z_2 reorbifolding which reproduces the Leech theory again and hence, as recently argued by Montague [23], these constructions must be equivalent to $\mathscr{V}^\natural$. All of these results strongly indicate that each Z_n construction reproduces $\mathscr{V}^\natural$ and that $\mathscr{V}^\natural$ is indeed unique and that the Leech theory is the orbifold partner to $\mathscr{V}^\natural$ for the meromorphic orbifoldings of $\mathscr{V}^\natural$ with respect to non-Fricke elements in M. In Sect. 4 we consider meromorphic orbifoldings of $\mathscr{V}^\natural$ with respect to the remaining Fricke elements of M. We show that assuming the FLM uniqueness conjecture for $\mathscr{V}^\natural$, then a meromorphic orbifolding of $\mathscr{V}^\natural$ with respect to an element $g \in M$ reproduces $\mathscr{V}^\natural$ (i.e. $\mathscr{V}^\natural$ is an orbifold partner to itself) if and only if $T_g(\tau)$ is of genus zero and is Fricke invariant. This result relies on the analysis of [19] where we related Monstrous Moonshine to the vacuum properties of $g \in M$ twisted operators. A standard construction of these twisted sectors is explicitly described for elements related to Leech lattice automorphisms but otherwise, we assume such twisted operator sectors exist. We also assume in all cases that these operators satisfy a closed non-meromorphic OPA. Together with the results of Sect. 3, we therefore find that, assuming $\mathscr{V}^\natural$ is unique, then $\mathscr{V}^\natural$ has either only itself or the Leech theory as a meromorphic orbifold partner if and only if Monstrous Moonshine holds for Thompson series. In Appendix A we review the modular groups required to described Monstrous Moonshine. In Appendix B we discuss a subgroup of the automorphism group for the OPA of a Z_n orbifolding of the Leech theory. This group is required to express the centraliser relationship between M and the Conway group described in Sect. 3.

2. The Moonshine Module and Monstrous Moonshine

2.1. Introduction. In this section we review the construction of the Moonshine Module, denoted by $\mathscr{V}^\natural$, of Frenkel, Lepowsky and Meurman (FLM) [1, 2, 3] in the language of conformal field theory (CFT) [5, 6, 22]. We emphasize certain aspects of this construction which we will later refer to both in considering possible alternative constructions of $\mathscr{V}^\natural$ in Sect. 3 and "reorbifoldings" of $\mathscr{V}^\natural$ of Sect. 4. We also review the main feature of this theory which is that the automorphism group of $\mathscr{V}^\natural$ is the Monster group M, the largest sporadic finite simple group. Finally, we introduce the Thompson series [14] for $g \in M$ which is the object of interest in the work of Conway and Norton known as "Monstrous Moonshine" [13].

The Moonshine Module is a Z_2 orbifold CFT [4] and is based on a Euclidean closed bosonic string compactified to a 24 dimensional torus T^{24} [7]. The torus T^{24} chosen is that defined by quotieting R^{24} with the Leech lattice which we denote throughout by Λ. Λ is the unique 24 dimensional even self-dual Euclidean lattice without roots, i.e. $\langle \alpha, \alpha \rangle \neq 2$ cf. [8, 24]. The Z_2 orbifolding construction is then based on the reflection automorphism of Λ.

2.2. The Leech Lattice String Construction. We begin with the usual left-moving closed bosonic string variables $X^i(z)$, where $z = \exp(2\pi(\sigma_0 + i\sigma_1))$ parametrises the string world sheet with "space" coordinate $0 \leq \sigma_1 \leq 1$ and "time" coordinate σ_0 [25]. On the torus T^{24} the closed string boundary condition is $X^i(e^{2\pi i} z) = X^i(z) + 2\pi\beta^i$ for $\beta \in \Lambda$. The standard mode expansion for $X^i(z)$ is

$$X^i(z) = q^i - ip^i \ln z + i \sum_{m \neq 0} \frac{\alpha^i_m}{m} z^{-m} \tag{2.1}$$

with commutation relations

$$\begin{aligned} [q^i, p^j] &= i\delta^{ij} , \\ [\alpha^i_m, \alpha^j_n] &= m\delta^{ij} \delta_{m+n,0} . \end{aligned} \tag{2.2}$$

A similar expansion holds for the right-moving part of the string $X^i(\bar{z})$. The 1-loop partition function corresponding to a world sheet torus $z \sim e^{2\pi i}z \sim e^{2\pi i\tau}z$ is parameterised by the modular parameter τ with $\mathrm{Im}\,\tau > 0$. Since Λ is even self-dual, the partition function factorizes into $Z(\tau)Z(\bar{\tau})$, where $Z(\tau)$ is a modular invariant function

$$Z(\tau) = \mathrm{Tr}(q^{L_0}) = \frac{\Theta_\Lambda(\tau)}{\eta^{24}(\tau)} \tag{2.3}$$

with $q = e^{2\pi i\tau}$ and where $\Theta_\Lambda(\tau) = \sum_{\beta\in\Lambda} q^{\beta^2/2}$ is the theta function associated with the Leech lattice Λ and is a modular form of weight 12 [12]. $L_0 = \frac{1}{2}p^2 + \sum_{m=1}^{\infty} \alpha^i_{-m}\alpha^i_m - 1$ is the normal ordered Virasoro Hamiltonian operator and $\eta(\tau) = q^{1/24}\prod_n(1-q^n)$ is the Dedekind eta function arising from the oscillator modes. The normal ordering constant gives the usual bosonic tachyonic vacuum energy -1 for central charge 24.

The set of primary conformal fields or vertex operators for this theory also factorizes into meromorphic in z (anti-meromorphic in $\bar{z}$) pieces which form a local meromorphic (anti-meromorphic) operator product algebra (OPA). We will consider the left-moving string which forms a meromorphic CFT [9]. The associated set of primary conformal fields, denoted by $\mathscr{V}^\Lambda$, consists of normal ordered vertex operators $\{\phi(z)\}$ of the form

$$\phi^{i_1\ldots i_r}_{n_1\ldots n_r}(\beta,z) =: \partial^{n_1}_y X^{i_1}(z)\ldots\partial^{n_r}_z X^{i_r}(z)e^{i\langle\beta,X(z)\rangle}: c(\beta) \tag{2.4}$$

with integer conformal dimension $h_\phi = n_1 + \ldots + n_r + \beta^2/2$, where $c(\beta)$ is the standard "cocycle factor" necessary for a local meromorphic OPA [3, 24, 10]

$$\phi_i(z)\phi_j(w) = \phi_j(w)\phi_i(z) \sim \sum_k C^{\phi\phi}_{ijk}(z-w)^{h_k-h_i-h_j}\phi_k(w) + \ldots . \tag{2.5}$$

The first equality in (2.5), which is the locality condition, relies on a suitable analytic continuation from $|z| > |w|$ to $|z| < |w|$. The cocycle factors in (2.4) are elements of a section of a central extension $\hat{\Lambda}$ of Λ by ± 1 and obey

$$c(\alpha)c(\beta)c^{-1}(\alpha)c^{-1}(\beta) = (-1)^{\langle\alpha,\beta\rangle}, \tag{2.6a}$$

$$c(\alpha)c(\beta) = \varepsilon(\alpha,\beta)c(\alpha+\beta), \tag{2.6b}$$

$$\varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma) = \varepsilon(\alpha,\beta+\gamma)\varepsilon(\beta,\gamma). \tag{2.6c}$$

The commutator (2.6a) defines the central extension whereas $\varepsilon(a,\beta) \in \{\pm 1\}$ of (2.6b) is a two-cocycle which depends on the section of $\hat{\Lambda}$ chosen and must obey the cocycle condition (2.6c). Let us denote the Hilbert space of states associated with $\mathscr{V}^\Lambda$, $\left\{|\phi\rangle = \lim_{z\to 0}\phi(z)\,|0\rangle\right\}$, by $\mathscr{H}_\Lambda$. These states can equivalently be constructed as a Fock space by the action of creation operators $\{\alpha^i_{-n}\}$, $n > 0$, on the highest weight states given by $\{|\beta\rangle\}$, where $p^i|\beta\rangle = \beta^i|\beta\rangle$. The trace in (2.3) is then performed over $\mathscr{H}_\Lambda$. $Z(\tau)$ is a meromorphic and modular invariant function of τ with a unique simple pole at $q = 0$ due to the tachyonic vacuum energy. $Z(\tau)$ is therefore given by the unique (up to an additive constant) modular invariant function $J(\tau)$ as follows:

$$Z(\tau) = J(\tau) + 24, \tag{2.7a}$$

$$J(\tau) = \frac{E_2^3(\tau)}{\eta^{24}(\tau)} - 744 = \frac{1}{q} + 0 + 196884q + \ldots , \tag{2.7b}$$

where $E_2(\tau)$ is the Eisenstein modular form of weight 4 [12]. Since Λ contains no roots, there are only 24 massless (conformal dimension 1) operators $\partial_z X^i(z)$.

The FLM Moonshine Module [1, 2, 3] is an orbifold CFT [4, 20] based on the Z_2 lattice reflection automorphism $\bar{r}:\beta \to -\beta$ for $\beta \in \Lambda$. The elements of $\mathscr{V}^\Lambda$ form a projective representation of the automorphism group of Λ, the Conway group Co_0, due to the cocycle factors of (2.6) [1, 3]. Thus the automorphism group of $\mathscr{V}^\Lambda$ which preserves the OPA (2.5) is a central extension $2^{24}.Co_0$ of Co_0 by Z_2^{24} (where 2^{24} denotes Z_2^{24} and where $A.B$ denotes a group with normal subgroup A and quotient group $B = A.B/B$). In particular, the lattice automorphism $\bar{r}$ lifts to a set of 2^{24} automorphisms of $\mathscr{V}^\Lambda$. With the cocycle factors chosen so that $\varepsilon(\alpha,\beta) = \varepsilon(-\alpha,-\beta)$ we can define a distinguished lifting of $\bar{r}$ to r by

$$rc(\beta)r^{-1} = c(-\beta)\,, \tag{2.8a}$$

$$r\partial_z X^i(z)r^{-1} = -\partial_z X^i(z)\,, \tag{2.8b}$$

which respects (2.5) and (2.6). Defining the projection operator $\mathscr{P}_r = (1+r)/2$, we let $\phi^{(+)}(z) = \mathscr{P}_r\phi(z)$ and $\phi^{(-)} = (1-\mathscr{P}_r)\phi(z)$ be ± 1 eigenvectors of r. The set of operators $\{\phi^{(+)}\} = \mathscr{P}_r\mathscr{V}^\Lambda$ then also form a meromorphic OPA. However, the corresponding partition function $\mathrm{Tr}_{\mathscr{P}_r\mathscr{H}_\Lambda}(q^{L_0}) = \frac{1}{2}\left(1\underset{1}{\square} + r\underset{1}{\square}\right)$ is not modular invariant, employing the standard notation for the world-sheet torus boundary conditions e.g. [6, 21]. Thus, under a modular transformation $S:\tau \to -1/\tau$,

$$r\underset{1}{\square} = \frac{1}{\eta_{\bar{r}}(\tau)} \to 1\underset{r}{\square} = 2^{12}\left[\frac{\eta(\tau)}{\eta(\tau/2)}\right]^{24}, \tag{2.9}$$

where $\eta_{\bar{r}}(\tau) = [\eta(2\tau)/\eta(\tau)]^{24}$. Therefore a "twisted" sector $\mathscr{H}_r$ is introduced to form a modular invariant theory [1, 4, 20].

2.3. The Twisted String Construction. Consider a closed string field $\tilde{X}^i(z)$ obeying the $\bar{r}$ twisted boundary condition (monodromy condition) $\tilde{X}(e^{2\pi i}z) = -\tilde{X}(z) + 2\pi\beta$, $\beta \in \Lambda$ with mode expansion

$$\tilde{X}^i(z) = \tilde{q}^i + i\sum_{m\in Z+\frac{1}{2}} \frac{\tilde{\alpha}^i_m}{m} z^{-m}\,, \tag{2.10}$$

where the oscillator modes obey the same commutator relations as given in (2.1) and $\tilde{q}^i \in L_{\bar{r}} = \Lambda/2\Lambda$, the $\bar{r}$ fixed point space of the torus. Then $L_{\bar{r}} = Z_2^{24}$ which we denote by 2^{24}. The states $\{|\psi\rangle\}$ of the twisted sector $\mathscr{H}_r$ can again be constructed from a set of vertex operators $\tilde{\mathscr{V}}^\Lambda$ acting, in this case, on a degenerate twisted vacuum. $\mathscr{H}_r$ can be also constructed as a Fock space from the action of creation operators $\{\tilde{\alpha}^i_{-m}\}$, $m > 0$, on this degenerate vacuum. These states are graded by the twisted Virasoro Hamiltonian $L_0 = \sum_{m\in Z+1/2}^{\infty} \tilde{\alpha}^i_{-m}\tilde{\alpha}^i_m + \frac{1}{2}$ with half integer energies where the normal ordering constant is now $\frac{1}{2}$. The resulting partition function is then $\mathrm{Tr}_{\mathscr{H}_r}(q^{L_0}) = 1\underset{r}{\square}$ of (2.9).

For each $\phi(z) \in \mathscr{V}^\Lambda$ there is a corresponding operator $\tilde{\phi}(z) \in \tilde{\mathscr{V}}^\Lambda$, with the same conformal dimension, which is physically interpreted as the emission of an untwisted state from the twisted vacuum. $\tilde{\mathscr{V}}^\Lambda$ then provides a representation of the OPA for $\mathscr{V}^\Lambda$ which is non-meromorphic because of half integer grading [3, 20, 10].

The construction of $\tilde{\phi}(z)$ is similar to (2.4), where the cocycle factors are replaced by a finite set of matrices, $\{c_T(\beta)\}$, acting on the degenerate twisted vacuum with β a representative element of $L_{\bar{r}}$, where $\beta \sim \alpha \Leftrightarrow \beta - \alpha \in 2\Lambda$. These cocycle matrices are defined as follows. The commutator map (2.6a) also defines a central extension $\hat{L}_{\bar{r}}$ of $L_{\bar{r}}$ by ± 1. Then $\hat{L}_{\bar{r}} = 2_+^{1+24}$, which denotes an extra-special group of the given order (with the defining property that the centre $\{\pm 1\}$ and commutator subgroup coincide). There exists a unique faithful irreducible 2^{12} dimensional representation π of $\hat{L}_{\bar{r}}$ in which the centre of $\hat{L}_{\bar{r}}$ is represented by ± 1 [3, 26]. The elements of $\pi(\hat{L}_{\bar{r}})$ are the twisted cocycle matrices $\{c_T(\beta)\}$ and the vacuum states, $\{|\sigma_r^l\rangle\}$, $l = 1, \ldots, 2^{12}$, form a basis for the vector space on which $\pi(\hat{L}_{\bar{r}})$ acts. These cocycle factors are again necessary for the twisted vertex operator modes to possess well-defined commutation relations [3, 27, 28, 10]. $\pi(\hat{L}_{\bar{r}})$ can be constructed from appropriate Dirac matrices since the elements of $\hat{L}_{\bar{r}}$ form a Clifford algebra [24, 10].

The defining characteristic of the operators $\{\tilde{\phi}(z)\}$ which act on the degenerate vacuum states $\{|\sigma_r^l\rangle\}$ is the monodromy condition associated with r

$$\tilde{\phi}^{(\pm)}(e^{2\pi i}z) = r^{-1}\tilde{\phi}^{(\pm)}(z)r\,, \tag{2.11}$$

where $r\tilde{\phi}^{(\pm)}(z)r^{-1} = \pm\tilde{\phi}^{(\pm)}(z)$ as defined above on the corresponding untwisted operator $\phi^{(\pm)}(z)$, e.g. $\partial_z \tilde{X}^i(e^{2\pi i}z) = -\partial_z \tilde{X}^i(z)$. Using the principles of CFT [5], each vacuum state $|\sigma_r^l\rangle$ is created from the untwisted vacuum by a primary "twist" conformal field (intertwining operator) $\sigma_r^l(z)$ with conformal dimension $\frac{3}{2}$, where $|\sigma_r^l\rangle = \lim_{z \to 0} \sigma_r^l(z)\,|0\rangle$ [3, 28, 10]. From (2.11), these operators form a non-meromorphic OPA with the vertex operators of $\mathscr{V}^\Lambda$ and $\tilde{\mathscr{V}}^\Lambda$ [20, 28, 10] as follows:

$$\tilde{\phi}^{(\pm)}(z)\sigma_r^l(w) = \sigma_r^l(w)\tilde{\phi}^{(\pm)}(z) \sim (z-w)^{h_\psi - h_\phi - 3/2}\psi^{(\mp)}(w) + \ldots \tag{2.12}$$

with a suitable analytic continuation assumed in the first equality. $\psi^{(\mp)}$ is a primary conformal operator which creates a higher conformal dimension h_ψ twisted state from the untwisted vacuum, where (2.11) implies that $h_{\psi^{(+)}} \in Z$, $h_{\psi^{(-)}} \in Z + 1/2$, e.g. the first excited twisted states $|\psi_{il}^{(+)}\rangle = \tilde{\alpha}_{-1/2}^i|\sigma_r^l\rangle$ with $h_\psi = 2$ are given by $\lim_{z \to 0} z^{1/2}\partial_y \tilde{X}^i(z)\,|\sigma_r^l\rangle$, the action of the first excited operators of $\mathscr{V}^\Lambda$. We denote the set of operators $\{\psi(z)\}$, which includes $\{\sigma_r^l(z)\}$, by $\mathscr{V}_r$.

The lattice automorphism $\bar{r}$ also lifts to a set of automorphisms of $\mathscr{V}_r$. Since $L_{\bar{r}}$ is invariant under $\bar{r}, \bar{r}$ is lifted to ± 1 in its action on the degenerate vacuum. We choose the lifting, which we also denote by r, to be

$$r\sigma_r^l(z)r^{-1} = -\sigma_r^l(z)\,, \tag{2.13a}$$

$$r\psi^{(\pm)}(z)r^{-1} = \pm\psi^{(\pm)}(z) = e^{-2\pi i h_\psi}\psi^{(\pm)}(z)\,, \tag{2.13b}$$

which preserves the OPA (2.12) so that the operators with integer valued conformal weights are invariant under r. Then (2.11) and (2.12) imply that the twisted operators $\mathscr{V}_r$ when acting on the vacuum $|0\rangle$ obey the monodromy condition

$$\psi(e^{2\pi i}z) = e^{-2\pi i h_\psi}\psi(z) = r\psi(z)r^{-1}\,. \tag{2.14}$$

Equation (2.14) implies that under the modular transformation $T\colon \tau \longrightarrow \tau + 1$, $1\underset{r}{\square} \rightarrow r^{-1}\underset{r}{\square}$. Thus the lifting of $\bar{r}$ chosen in (2.13) is compatible with the twisted vacuum energy of 1/2 and ensures that no extra phase occurs in this transformation.

The OPA (2.12) can be generalised by replacing $\sigma^l_r(w)$ by any twisted state $\psi(w) \in \mathscr{V}_r$. Likewise, we may define for each $\psi(z) \in \mathscr{V}_r$ a vertex operator $\tilde{\psi}(z) \in \tilde{\mathscr{V}}_r$ which acts on the twisted vacuum to give an untwisted state. The set of such operators forms a closed non-meromorphic OPA [3, 20, 28, 10]

$$\tilde{\phi}_i(z)\psi_j(w) \sim \sum_k C^{\phi\psi}_{ijk}(z-w)^{h_k-h_i-h_j}\psi_k(w) + \dots , \tag{2.15a}$$

$$\tilde{\psi}_i(z)\psi_j(w) \sim \sum_k C^{\psi\psi}_{ijk}(z-w)^{h_k-h_i-h_j}\phi_k(w) + \dots . \tag{2.15b}$$

$\mathscr{V}^\Lambda$ is thus enlarged by the inclusion of the twist fields $\{\sigma^l_r(z)\}$ to $\mathscr{V}' = \mathscr{V}^\Lambda \oplus \mathscr{V}_r$ which forms a closed non-meromorphic OPA. Furthermore, the r invariant set $\mathscr{P}_r\mathscr{V}'$ forms a closed meromorphic OPA and defines a modular invariant meromorphic CFT. This is the FLM Moonshine Module $\mathscr{V}^\natural$ [1, 2, 3]. As far as we are aware, a completely rigorous construction of (2.15) does not yet exist except for this $\mathscr{P}_r$ projection which forms a meromorphic OPA. This projection ensures the absence of the 24 massless (conformal dimension 1) operators $\partial_y X^i(z)$ whereas the twisted sector operators are all massive since the twisted vacuum energy is $1/2$. Therefore, the modular invariant partition function for the associated Hilbert space of states $\mathscr{H}^\natural$ is

$$\mathrm{Tr}_{\mathscr{H}^\natural}(q^{L_0}) = \mathscr{P}_r \underset{1}{\square}_r + \mathscr{P}_r \underset{r}{\square} = J(\tau) , \tag{2.16}$$

where $J(\tau)$ is the unique modular invariant of (2.7b) without a constant term.

The absence of any massless operators in $\mathscr{V}^\natural$ is the crucial feature that sets the Moonshine Module apart from any other string theory. Normally such operators are present and form a Kac-Moody algebra. However, in the present case, the 196884 conformal dimension 2 operators, including the energy-momentum tensor $T(z) = -\frac{1}{2}{:}\partial_z X^i(z)\partial_z X^i(z){:}$, can be used to define a closed non-associatve commutative algebra. FLM [1, 2, 3] showed that this algebra is an affine version of the 196883 dimensional Griess algebra [11] together with the energy-momentum tensor. The automorphism group of the Griess algebra is the Monster finite simple group M of order $2^{46}.3^{20}.5^9.7^6.11^2.13^3.17.19.23.29.31.41.47.59.71 \sim 8\times 10^{53}$. FLM further showed that M is the automorphism group for the full OPA of $\mathscr{V}^\natural$, where $T(z)$ is a singlet. Thus the operators of $\mathscr{V}^\natural$ of a given conformal weight form (reducible) representations of M. This explains an earlier observation of McKay and Thompson [14] that the coefficients of the modular function $J(\tau)$ are positive sums of dimensions of the irreducible representations of M, e.g. the coefficient of q is $196884 = 1 + 196883$, the sum of the trivial and adjoint representation formed by the Griess algebra.

2.4. A Monster Group Centraliser and Z_2 Reorbifolding $\mathscr{V}^\natural$. We may identify an involution (order two) automorphism $i \in M$, defined like a "fermion number," under which all untwisted (twisted) operators have eigenvalue $+1(-1)$. i clearly also respects the larger non-local OPA of (2.5) and (2.15). The centraliser $C(i \mid M) = \{g \in M \mid ig = gi\}$ may also be determined since this is given by all OPA automorphisms which map $\mathscr{P}_r\mathscr{V}^\Lambda$ and $\mathscr{P}_r\mathscr{V}_r$ into themselves. As stated earlier, the automorphism group of $\mathscr{V}^\Lambda$ consists of all liftings of the Conway group Co_0 to automorphisms of the OPA (2.5) and contains $L_{\bar r}Co_0$, where $L_{\bar r} = 2^{24}$. The fixed point space $L_{\bar r}$ is invariant under $\bar r$ and the automorphism group of the twisted sector $\mathscr{V}_r$ is $\hat{L}_{\bar r}.Co_1$, an extension of the Conway simple group $Co_1 = Co_0/\{1,\bar r\}$ by $\hat{L}_{\bar r} = 2^{1+24}_+$.

The extension is determined by the automorphism group of the twisted cocycle matrices $c_T(\alpha) \in \pi(\hat{L}_{\bar{r}})$. In particular, the inner automorphisms of $\pi(\hat{L}_{\bar{r}})$ defined by $c_T(\alpha): c_T(\beta) \to c_T(\alpha)c_T(\beta)c_T(\alpha)^{-1} = (-1)^{\alpha,\beta}c_T(\beta)$ describe the liftings of the identity element of Co_1 and the given extension. The automorphism group for $\mathscr{V}_r$ then follows from (2.12). Putting these automorphism groups together, one can show that the corresponding automorhism group for the projected set of operators $\mathscr{P}_r\mathscr{V}'$ is $C(i\,|\,M) = 2_+^{1+24}.Co_1$ (see Appendix B). This result is an essential part of the FLM construction since Griess showed that M is generated by $2_+^{1+24}.Co_1$ and a second involution σ. FLM constructed σ, which mixes the untwisted and twisted sectors, from a hidden triality OPA symmetry in the theory [1, 2, 3, 29] and so demonstrated that the automorphism group of $\mathscr{V}^\natural$ is M.

The automorphisms i and r can be said to be "dual" to each other in the sense that both are automorphisms of the non-meromorphic OPA for $\mathscr{V}' = \mathscr{V}^\Lambda \oplus \mathscr{V}_r$ and that the subsets invariant under i and r, $\mathscr{V}^\Lambda$ and $\mathscr{V}^\natural$ respectively, form meromorphic OPAs. Then we may "reorbifold" $\mathscr{V}^\natural$ with respect to i by employing the 24 massless operators $\{\partial_z X^i(z)\}$ to re-introduce the $r=-1$ eigenvalue operators $\{\phi^{(-)}\}\oplus\{\psi^{(-)}\}$, where (schematically) $\phi^{(+)}\partial_z X \sim \phi^{(-)}$, $\psi^{(+)}\partial_z X \sim \psi^{(-)}$ from (2.5) and (2.15), i.e. the operators $\{\partial_z X^i(z)\}$ create the states of the i twisted vacuum. Similarly, monodromy conditions analogous to (2.11) and (2.14) also hold with r replaced with i, $\mathscr{V}^\Lambda$ replaced by $\mathscr{V}^\natural$ in (2.11) and $\mathscr{V}_r$ replaced by $\{\phi^{(-)}\}\oplus\{\psi^{(-)}\}$ in (2.12). From this point of view the two meromorphic constructions $\mathscr{V}^\Lambda$ and $\mathscr{V}^\natural$ are placed on an equal footing with each contained in the enlarged set $\mathscr{V}'$ and each related to the other by an appropriate Z_2 orbifolding procedure. Equivalently, we can define $\mathscr{V}'$ to be the set of all operators which form a meromorphic OPA with $\mathscr{P}_r\mathscr{V}^\Lambda = \mathscr{P}_i\mathscr{V}^\natural$, i.e. $\mathscr{V}'$ is "dual" to $\mathscr{P}_r\mathscr{V}^\Lambda = \mathscr{P}_i\mathscr{V}^\natural$ in the sense suggested by Goddard [9]. The orbifolding of $\mathscr{V}^\Lambda$ with respect to r is then $\mathscr{V}^\natural = \mathscr{P}_r\mathscr{V}'$ and the orbifolding of $\mathscr{V}^\natural$ with respect to i is $\mathscr{V}^\Lambda = \mathscr{P}_i\mathscr{V}'$, where $\mathscr{V}^\Lambda$ and $\mathscr{V}^\natural$ are self-dual, i.e.

$$
\begin{tikzcd}
 & \mathscr{V}' \arrow[dl, "\mathscr{P}_i"'] \arrow[dr, "\mathscr{P}_r"] & \\
\mathscr{V}^\Lambda \arrow[rr, shift left, "r"] \arrow[dr, "\mathscr{P}_r"'] & & \mathscr{V}^\natural \arrow[ll, shift left, "i"] \arrow[dl, "\mathscr{P}_i"] \\
 & \mathscr{P}_r\mathscr{V}^\Lambda = \mathscr{P}_i\mathscr{V}^\natural &
\end{tikzcd}
\tag{2.17}
$$

where the horizontal (diagonal) arrows denote an orbifolding (projection).

2.5. Thompson Series, Hauptmoduls and Monstrous Moonshine. The states of $\mathscr{H}^\natural$ of a given conformal weight form reducible representations of the Monster group M. The Thompson series $T_g(\tau)$ for $g \in M$ is then defined by

$$
\begin{aligned}
T_g(\tau) &= \mathrm{Tr}_{\mathscr{H}^\natural}(g q^{L_0}) \\
&= \frac{1}{q} + 0 + (1 + \chi_A(g))q + \ldots
\end{aligned}
\tag{2.18}
$$

which depends only on the conjugacy class of g, where χ_A is the character of the 196883 adjoint representation and where the other coefficients are similarly positive sums of irreducible characters e.g. for the involution i, $T_i(\tau) = 1/\eta_{\bar{r}}(\tau)+24$. Likewise, an explicit formula may be found for $g \in C(i\,|\,M) = 2_+^{1+24}.Co_1$ [2, 3] (see Sect. 4.4).

The Thompson series for the identity element is the partition function $J(\tau)$. The compactification $\bar{\mathscr{F}}$ of the fundamental region $\mathscr{F} = H/\Gamma$ (where Γ is the full modular group and H is the upper half plane) is isomorphic to the Riemann sphere of genus zero. The function $J(\tau)$ explicitly realises this isomorphism by providing a one to one map between $\bar{\mathscr{F}}$ and the Riemann sphere. Such a function is called a hauptmodul for the genus zero modular group Γ. A modular invariant meromorphic function is a hauptmodul if and only if it possesses a unique simple pole on $\bar{\mathscr{F}}$. Once the location of this pole is specified, this function is itself unique up to constant. Thus $J(\tau)$ is the unique (up to a constant) modular invariant meromorphic function with a simple pole at $q = 0$ e.g. [12, 19].

Based on "experimental" evidence, Conway and Norton suggested in their famous paper "Monstrous Moonshine," that the Thompson series for each $g \in M$ is a hauptmodul (with a simple pole at $q = 0$) for some genus zero modular group Γ_g under which $T_g(\tau)$ is invariant. Γ_g was explicitly found by Conway and Norton as follows [13].

Monstrous Moonshine. Let $g \in M$, g of order n.
(a) *The Thompson series $T_g(\tau)$ is invariant up to h roots of unity under a subgroup of $\mathscr{N}(\Gamma_0(N))$ of the form $\Gamma_0(n \mid h) + e_1, e_2, \ldots$, where $h \mid 24$, $h \mid n$ and $N = nh$.*
(b) *The subgroup Γ_g of these transformations which fixes $T_g(\tau)$ (and contains $\Gamma_0(N)$) is of genus zero where $T_g(\tau)$ is the corresponding hauptmodul.*

The modular groups $\Gamma_0(n \mid h) + e_1, e_2, \ldots$ and $\mathscr{N}(\Gamma_0(N))$, the normalizer of $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \middle| \det = 1 \right\}$ in $SL(2, R)$ are described in Appendix A. This result has been rigorously demonstrated by Borcherds [15] by identifying each Thompson series with a Weyl-Kac determinant for an associated generalised Kac-Moody algebra. The proof of Monstrous Moonshine then ultimately relies on a case by case study of these formulae so that the origin of the genus zero property remains obscure. Apart from two classes of order 27, the Thompson series and corresponding genus zero modular group is unique to each class of M. Following Conway and Norton, we will abbreviate the notation denoting the modular groups above and the corresponding Monster group class in the following way: $\Gamma_0(n \mid h) + e_1, e_2, \ldots$ is abbreviated to $n \mid h + e_1, e_2, \ldots$ and to $n + e_1, e_2, \ldots$ when $h = 1$. If all AL possible involutions are adjointed, these groups are denoted by $n \mid h+$ and $n+$, respectively, whereas if no AL involutions are adjoined, then they are denoted by $n \mid h-$ and $n-$, respectively. Thus each class of M will be denoted by $g = n \mid h + e_1, e_2, \ldots$ corresponding to the modular group for $T_g(\tau)$ in this notation. As an example, for the involution i, $T_i(\tau)$ is a hauptmodul for the genus zero modular group $\Gamma_0(2)$ and i is a member of the class $2-$.

3. Other Constructions of the Moonshine Module

3.1 The FLM *Uniqueness Conjecture.* In the last section we reviewed the FLM construction of the Moonshine Module $\mathscr{V}^\natural$. There we saw that $\mathscr{V}^\natural$ is a modular invariant meromorphic CFT without any massless states with partition function $J(\tau)$. FLM have conjectured that $\mathscr{V}^\natural$ is characterised (up to isomorphism) as follows [3]:

FLM *Uniqueness Conjecture.* $\mathscr{V}^\natural$ *is the unique meromorphic conformal field thory with modular invariant partition function* $J(\tau)$ *and central charge* 24.

This uniqueness conjecture is analogous to the uniqueness property of the Leech lattice as being the only even self-dual lattice in 24 dimensions without roots. In this section we will discuss some evidence to support this conjecture by considering alternative orbifold constructions which are modular invariant meromorphic CFTs without massless operators and with partition function $J(\tau)$. Within these constructions, we will recognise known properties of the Monster group and will also find a new relationship between 51 centralisers of the Conway and Monster groups generalising an observation made by Conway and Norton [13]. In the next section we will also link this uniqueness conjecture to the Monstrous Moonshine properties of Conway and Norton [13].

3.2. Z_n *Orbifoldings of* $\mathscr{V}^\natural$ *with Partition Function* $J(\tau)$. Let us now consider orbifold models based on other order n automorphisms $\{a\}$ of the untwisted Leech lattice theory $\mathscr{V}^\Lambda$ [3, 18, 16]. a will be chosen so that each model contains no massless operators, has a meromorphic OPA and is modular invariant with partition function $J(\tau)$ as in (2.16) and hence, according to the uniqueness conjecture, reproduces $\mathscr{V}^\natural$. In each construction, we will also be able to identify an automorphism g_n, where g_n (or a power of g_n) is "dual" to a. We will find a total of 51 such automorphisms which we will argue are representatives of the complete list of 51 Monster group classes $n\,|\,h + e_1, e_2, \ldots$ with $e_i \neq n/h$, i.e. elements whose Thompson series are not invariant under the Fricke involution $w_n : \tau \to -1/nh\tau$. Such elements of M are called non-Fricke. Each stage of the original construction reviewed in Sect. 2 will be appropriately generalised but a rigorous treatment along the lines of FLM is not yet available in general with the exception of the prime ordered cases recently described by Dong and Mason [16].

Let us consider an OPA automorphism a of $\mathscr{V}^\Lambda$ lifted from an automorphism $\bar{a} \in Co_0$ of Λ given by

$$ac(\beta)a^{-1} = e^{2\pi i f_a(\beta)} c(\bar{a}\beta)\,, \tag{3.1a}$$

$$a\partial_z X^i(z) a^{-1} = \omega^{s_i} \partial_z X^i(z)\,, \tag{3.1b}$$

where we choose a diagonal basis for $\bar{a} = \mathrm{diag}(\omega^{s_1}, \ldots, \omega^{s_{24}})$ with $\omega = e^{2\pi i/n}$. $f_a(\beta) \in Z/2$ describes the lifting of $\bar{a}$ to an automorphism a which preserves (2.6). We only consider lattice automorhisms $\bar{a}$ without fixed points in order to ensure that no untwisted massless states $\partial_z X^i(z)$ survive projection under $\mathscr{P}_a = (1 + a + \ldots + a^{n-1})/n$. This condition also guarantees that a and $\bar{a}$ are of the same order n throughout [30]. Each conjugacy class of Co_0 is parameterised by the characteristic equation for a representative element $\bar{a}$ as follows:

$$\det(x - \bar{a}) = \prod_{k\,|\,n} (x^k - 1)^{a_k}\,. \tag{3.2}$$

$k\,|\,n$ denotes that k divides n, the order of $\bar{a}$ and each a_k is a not necessarily positive integer where

$$\sum_{k\,|\,n} k a_k = 24\,, \qquad \sum_{k\,|\,n} a_k = 0\,. \tag{3.3}$$

The absence of fixed points for $\bar{a}$ implies the second condition and also that $a_1 \leq 0$, e.g. $\bar{r}$ is parameterised by $r_2 = -r_1 = 24$ with $\det(x - \bar{r}) = (x+1)^{24}$. For $n = p$ prime, the parameters are given by $a_p = -a_1 = 2d$, where $(p-1)2d = 24$ with $d = 12, 6, 3, 2, 1$ for $p = 2, 3, 5, 7, 13$.

Since a is an OPA automorphism for $\mathscr{V}^\Lambda, \mathscr{P}_a \mathscr{V}^\Lambda$ also forms a meromorphic OPA which closes. The associated partition function $\mathrm{Tr}_{\mathscr{P}_a \mathscr{H}_\Lambda}(q^{L_0})$ is not modular invariant, as before, necessitating the introduction of $b = a^r$ twisted sectors where b is lifted from $\bar{b} = \bar{a}^r$ of order $m = n/(n,r)$ with characteristic equation parameters $\{b_k\}$. Thus we find that under the modular transformation $S: \tau \to -1/\tau$ [7, 19]

$$b\underset{1}{\square} = \frac{\Theta_{\Lambda_{\bar{b}}}(\tau)}{\eta_{\bar{b}}(\tau)} \to 1\underset{b}{\square} = \frac{D_{\bar{b}}^{1/2}}{V_{\bar{b}}} \frac{\Theta_{\Lambda_{\bar{b}}^*}(\tau)}{\eta_{\bar{b}}^*(\tau)} = \frac{D_{\bar{b}}^{1/2}}{V_{\bar{b}}} q^{E_0^b}(1 + O(q^{1/m})), \tag{3.4}$$

where

$$\Theta_{\Lambda_{\bar{b}}}(\tau) = \sum_{\beta \in \Lambda_{\bar{b}}} q^{\beta^2/2}, \quad \eta_{\bar{b}}(\tau) = \prod_{k \mid m} \eta(k\tau)^{b_k}, \quad \eta_{\bar{b}}^*(\tau) = \prod_{k \mid m} \eta(\tau/k)^{b_k}, \tag{3.5a}$$

$$D_{\bar{b}} = \prod_{k \mid m} k^{b_k} = \det{}_T(1 - \bar{b}), \quad E_0^b = -\frac{1}{24} \sum_{k \mid m} \frac{b_k}{k}. \tag{3.5b}$$

Here we have chosen the lifting b of $\bar{b}$ to an automorphism of $\hat{\Lambda}$, where $bc(\beta)b^{-1} = c(\beta)$ for $\beta \in \Lambda_{\bar{b}}$, the sublattice of Λ fixed by $\bar{b}$. $\Lambda_{\bar{b}}$ has dual lattice $\Lambda_{\bar{b}}^* = \Lambda_{\|} \equiv \mathscr{P}_{\bar{b}}\Lambda_{\bar{b}}$ and is of volume $V_{\bar{b}} = |\Lambda_{\|}/\Lambda_{\bar{b}}|^{1/2}$. The determinant of (3.5b) denotes the exclusion of all unit eigenvalues of $\bar{b}$. These expressions simplify for $b = a$ lifted from $\bar{a}$ in which case $\Theta_{\Lambda_{\bar{a}}}(\tau) = 1$ and $V_{\bar{a}} = 1$.

3.3. 51 Automorphisms of the Leech Lattice. We may anticipate some features of a b twisted sector $\mathscr{H}_b$ with the partition function $1\underset{b}{\square}$. We expect $\mathscr{H}_b$ to have vacuum degeneracy $D_{\bar{b}}^{1/2}/V_{\bar{b}}$ and vacuum energy E_0^b. From (3.4), $1\underset{b}{\square}$ is invariant up to a phase $\exp(2\pi i m E_0^b)$ under $T^m: \tau \to \tau + m$, i.e. the action of b on the twisted sector is of order m up to this phase. However, to construct a meromorphic orbifold CFT with a modular invariant partition function we must have $mE_0^b = 0 \bmod 1$, i.e. there is no global phase anomaly [31, 32]. Equivalently, there is no such anomaly provided $b\underset{1}{\square}$ is invariant under the modular group $\Gamma_0(m)$ [31, 19]. Lastly, if $E_0^b \leq 0$ then the b twisted sector may reintroduce massless states. Let us initially consider the a twisted sector here and study those automorphisms with $nE_0^a = 0 \bmod 1$ and $E_0^a > 0$ [18]. As we will see below, these conditions are sufficient to ensure the absence of a global phase anomaly and massless states in any of the $b = a^r$ twisted sectors in the full orbifold construction. We therefore restrict ourselves to the study of automorphisms $\bar{a}$ obeying [18]

$$\sum_{k \mid n} a_k = 0, \qquad E_0^a > 0, \tag{3.6a}$$

$$nE_0^a = 0 \bmod 1 \tag{3.6b}$$

Table 1. The 38 conjugacy classes of Co_0 obeying (3.6). The first column gives $\bar{g}$ in Frame shape notation. The corresponding modular group Γ_a appears in column 2. The group appearing in columns 3, 4 and 5 are expressed in terms of standard Atlas groups [42], where n^k denotes the direct product of k cyclic groups of order n and $[p_1^a.p_2^b\ldots]$ denotes an unknown group of the given order. $A\times B$ denotes a direct product group and $A.B$ denotes a group with normal subgroup A, where $B=A.B/A$

$\bar{a}\in Co_0$	Γ_a	$\hat{L}_{\bar{a}}$	G_n	$C(g_n\mid M)$	
$2^{24}/1^{24}$	$2-$	2^{1+24}	Co_1	$2^{1+24}.Co_1$	†
$3^{12}/1^{12}$	$3-$	3^{1+12}	$2.Sz$	$3^{1+12}.2.Sz$	*
$4^8/1^8$	$4-$	4.4^8	$2.2^6.S_6(2)$	$4.2^{15}.2^8.S_6(2)$	‡
$5^6/1^6$	$5-$	5^{1+6}	$2.HJ$	$5^{1+6}.2.HJ$	*
$2^6 6^6/1^6 3^6$	$6+3$	$2^{1+12}\times 3$	$3.U_4(3).2$	$2^{1+12}.3^2.U_4(3).2$	†
$3^4 6^4/1^4 2^4$	$6+2$	$2\times 3^{1+8}$	$2^{1+6}.U_4(2)$	$2.3^{1+8}.2^{1+6}.U_4(2)$	*
$2.6^5/1^5 3$	$6-$	$2^{1+6}\times 3^{1+4}$	$2.U_4(2)$	$2.3^{1+4}.2^{1+6}.U_4(2)$	‡
$7^4/1^4$	$7-$	7^{1+4}	$2.A_7$	$7^{1+4}.2.A_7$	*
$2^2 8^4/1^4 4^2$	$8-$	$8.(8^2\times 4^2)$	$[2^9.3]$	$[2^{22}.3]$	‡
$9^3/1^3$	$9-$	$9.(9^2\times 3^2)$	$[2^4.3^3]$	$[2^4.3^{11}]$	*
$2^4 10^4/1^4 5^4$	$10+5$	$5\times 2^{1+8}$	$(A_5\times A_5).2$	$5\times 2^{1+8}.(A_5\times A_5).2$	†
$5^2 10^2/1^2 2^2$	$10+2$	$2\times 5^{1+4}$	$2^{1+4}.A_5$	$2.5^{1+4}.2^{1+4}.A_5$	*
$2.10^3/1^3 5$	$10-$	$2^{1+4}\times 5^{1+2}$	$2.A_5$	$2.5^{1+2}.2^{1+4}.A_5$	‡
$2^4 3^4 12^4/1^4 4^4 6^4$	$12+4$	$4\times 3^{1+4}$	$2.2^4.S_6$	$[2^{11}.3^7.5]$	‡
$4^2 12^2/1^2 3^2$	$12+3$	$4.4^4\times 3$	$[2^5.3^2]$	$[2^{15}.3^3]$	‡
$2^2 3.12^3/1^3 4.6^2$	$12-$	$4.4^2\times 3^{1+2}$	$[2^4.3]$	$[2^{10}.3^4]$	‡
$13^2/1^2$	$13-$	13^{1+2}	$2.A_4$	$13^{1+2}.2.A_4$	*
$2^3 14^3/1^3 7^3$	$14+7$	$7\times 2^{1+6}$	$L_2(7)$	$[2^{10}.3.7^2]$	†
$3^2 15^2/1^2 5^2$	$15+5$	$5\times 3^{1+4}$	$2.A_5$	$[2^3.3^6.5^2]$	*
$2.16^2/1^2 8$	$16-$	16.8^2	$[2^3]$	$[2^{13}]$	‡
$9.18/1.2$	$18+2$	2×9.9^2	$[2^3.3]$	$[2^4.3^7]$	*
$2^3 3^2 18^3/1^3 6^2 9^3$	$18+9$	$2^{1+4}\times 9$	$[2.3^3]$	$[2^6.3^5]$	†
$2.3.18^2/1^2 6.9$	$18-$	$2^{1+2}\times 9.3^2$	2.3	$[2^4.3^5]$	‡
$2^2 5^2 20^2/1^2 4^2 10^2$	$20+4$	$4\times 5^{1+2}$	$2.S_4$	$[2^6.3.5^3]$	‡
$7.21/1.3$	$21+3$	$3\times 7^{1+2}$	2.3	$[2.3^2.7^3]$	*
$2^2 22^2/1^2 11^2$	$22+11$	$2^{1+4}\times 11$	S_3	$[2^6.3.11]$	†
$2.3^2 4.24^2/1^2 6.8^2 12$	$24+8$	$8\times 3^{1+2}$	$[2^4]$	$[2^7.3^3]$	‡
$4.28/1.7$	$28+7$	$4.4^2\times 7$	2	$[2^7.7]$	‡
$2^3 3^3 5^3 30^3/1^3 6^3 10^3 15^3$	$30+6,10,15$	$2\times 3\times 5$	A_6	$[2^4.3^3.5^2]$	†
$2.6.10.30/1.3.5.15$	$30+3,5,15$	$2^{1+4}\times 3\times 5$	S_3	$[2^6.3^2.5]$	†
$2^2 3.5.30^2/1^2 6.10.15^2$	$30+15$	$2^{1+2}\times 3\times 5$	2	$[2^4.3.5]$	†
$3.33/1.11$	$33+11$	$3^{1+2}\times 11$	2	$[2.3^3.11]$	*
$2.9.36/1.4.18$	$36+4$	4×9.3^2	2	$[2^3.3^4]$	‡
$2^2 3^2 7^2 42^2/1^2 6^2 14^2 21^2$	$42+6,14,21$	$2\times 3\times 7$	A_4	$[2^3.3^2 7]$	†
$2.46/1.23$	$46+23$	$2^{1+2}\times 23$	1	$[2^3.23]$	†
$3.4.5.60/1.12.15.20$	$60+12,15,20$	$4\times 3\times 5$	2	$[2^3.3.5]$	‡
$2.5.7.70/1.10.14.35$	$70+10,14,35$	$2\times 5\times 7$	1	$[2.5.7]$	†
$2.3.13.78/1.6.26.39$	$78+6,26,39$	$2\times 3\times 13$	1	$[2.3.13]$	†

In column 1 of Table 1 we give a complete list of the 38 classes of Co_0 [33] that obey the constraints (3.6). $\bar{a}$ with parameters $a_k, \dots, a_l, -a_m, \dots, -a_n > 0$ is denoted by $k^{a_k} \dots l^{a_l}/m^{a_m} \dots n^{a_n}$, called the Frame shape notation. In each case a_k obeys the symmetry relation $a_k = -a_{n/k}$ and therefore, from (3.3) and (3.5b), $E_0^a = 1/n$. One may also check that $b = a^r$ of order m obeys $mE_0^b = 0 \bmod 1$ and hence no global phase anomaly occurs in the b twisted sector. Under a general modular transformation $\tau \to (a\tau + b)/(c\tau + d)$ we also find that $a\underset{1}{\square} \to a^d \underset{a^{-c}}{\square}$ in the usual way. Therefore for $\gamma \in \Gamma_0(n)$, where $\gamma: \tau \to (a\tau + b)/(cn\tau + d)$, $a\underset{1}{\square} \to a^d\underset{1}{\square} = a\underset{1}{\square}$ since $(d, n) = 1$, i.e. n and d are relatively prime and hence $\eta_{\bar{a}^d} = \eta_{\bar{a}}$. In column 2 we give the full modular invariance group Γ_a of $a\underset{1}{\square}$ in the notation described in Sect. 2.4. In general, Γ_a does not uniquely specify a class of Co_0 but does do so for classes obeying (3.6a). In Table 2 we give a complete list of the remaining 13 classes of Co_0 that obey the constraints (3.6a) only. Each of these classes is characterised by the existence of an integer $h \neq 1$ with $h \mid k$ for all $a_k \neq 0$ where, from (3.3), $h \mid 24$. In each case the parameters $\{a_k\}$ obey the symmetry relation $a_k = -a_{nh/k}$ and therefore $E_0^a = 1/nh$ violating (3.6b) for $h \neq 1$. Column 2 shows the modular group Γ_a under which $a\underset{1}{\square}$ is invariant up to phases of order h (and hence forms a projective representation of Γ_a). This set of classes cannot be employed to construct a meromorphic orbifold CFT but is of interest since for each $\bar{a}$ in Table 2, $\bar{a}^h$ appears in Table 1. In general, Table 2 contains all the remaining classes of Co_0 with some power in Table 1.

Table 2. The 13 conjugacy classes of Co_0 obeying (3.6a) only. For such each $\bar{a}$ there is an integer $h \mid n$, $h \mid 24$, where $\bar{a}^h$ appears in Table 1

$\bar{a} \in Co_0$	Γ_a	$\hat{L}_{\bar{a}}$	G_n	$C(g_n \mid M)$
$4^{12}/2^{12}$	$4 \mid 2-$	4.2^{12}	$G_2(4).2$	$4.2^{12}.G_2(4).2$
$6^8/3^8$	$6 \mid 3-$	$3 \times 2^{1+8}$	A_9	$3 \times 2^{1+8}.A_9$
$8^4/2^4$	$8 \mid 2-$	8.4^4	$2.2^4.A_6$	$8.2^9.2^4.A_6$
$8^6/4^6$	$8 \mid 4-$	8.2^6	$U_3(3)$	$8.2^6.U_3(3)$
$6^2 12^2/2^2 4^2$	$12 \mid 2+2$	$4 \times 3^{1+4}$	$[2^7.3]$	$[2^9.3^6]$
$12^4/6^4$	$12 \mid 6-$	3×4.2^4	$A_5 \times 2$	$[2^9.3^2 5]$
$15^2/3^2$	$15 \mid 3-$	$3 \times 5^{1+2}$	$2.A_4$	$[2^3.3^2.5^3]$
$4^2 20^2/2^2 10^2$	$20 \mid 2+5$	$4.2^4 \times 5$	A_5	$[2^8.3.5^2]$
$8.24/2.6$	$24 \mid 2+3$	3×8.4^2	$[2.3]$	$[2^8.3^2]$
$12.24/4.8$	$24 \mid 4+2$	$8 \times 3^{1+2}$	$[2^2]$	$[2^5.3^3]$
$24^2/12^2$	$24 \mid 12-$	3×8.2^2	3	$[2^5.3^2]$
$6.42/3.21$	$42 \mid 3+7$	$2^{1+2} \times 3 \times 7$	1	$[2^3.3.7]$
$4.6.14.84/2.12.28.42$	$84 \mid 2+6, 14, 21$	$4 \times 3 \times 7$	1	$[2^2.3.7]$

The modular groups Γ_a appearing in Tables 1 and 2 are amongst the list of genus zero groups considered by Conway and Norton [13], i.e. for each Γ_a there is a corresponding $g_n \in M$ with a Thompson series $T_{g_n}(\tau)$ of (2.18) invariant under Γ_a (up to phases of order h). Furthermore, $a\underset{1}{\square} = 1/\eta_{\bar{a}}(\tau)$ is the hauptmodul for Γ_a $\Big($or, for $h \neq 1$, the subgroup of Γ_a that leaves $a\underset{1}{\square}$ invariant$\Big)$ and hence $T_{g_n}(\tau) = 1/\eta_{\bar{a}}(\tau) - a_1$, where the constant is fixed by the absence of massless states in $\mathscr{V}^\natural$. We will identify such an element g_n explicitly below. Also note that none of these modular groups includes the Fricke involution $w_n : \tau \to -1/nh\tau$ since $\eta_{\bar{a}}(\tau)$ is inverted under w_n with $\eta_{\bar{a}}(\tau) \to D_a^{-1/2}/\eta_{\bar{a}}(\tau)$ and hence $1\underset{a}{\square} = D_{\bar{a}}^{1/2}\eta_{\bar{a}}(\tau/nh)$. In fact, column 2 of Tables 1 and 2 gives an exhaustive list of all the modular groups for Thompson series which are not invariant under the Fricke involution, i.e. the corresponding elements $g_n \in M$ are the non-Fricke elements.

3.4. The $\bar{a}$ Twisted String Construction. Let us now consider the construction of the a twisted sector, which is similar to that of Sect. 2, for the automorphisms of both Table 1 and 2. We will briefly discuss the construction of the general $b = a^r$ twisted sector later on and in Appendix B. We introduce $\tilde{X}^i(z)$ obeying the twisted monodromy condition $\tilde{X}^i(e^{2\pi i}z) = \omega^{-s_i}\tilde{X}^i(z) + 2\pi\beta^i$ (with $\bar{a}$ in the diagonal basis) with mode expansion [30, 26, 4, 34]

$$\tilde{X}(z) = \tilde{q}^i + i \sum_{m \in Z + s_i/n} \frac{\tilde{\alpha}^i_m}{m} z^{-m}, \tag{3.7}$$

where $\tilde{\alpha}^i_m$ obey the commutation relations (2.2). $\tilde{q}^i \in L_{\bar{a}} = \Lambda/(1-\bar{a})\Lambda$ is the $\bar{a}$ fixed point space of the torus and is a finite abelian group of order $D_{\bar{a}} = \det(1-\bar{a})$.

The twisted states $\mathscr{H}_a$ with Virasoro Hamiltonian $L_0 = \sum_m \tilde{\alpha}^i_m\tilde{\alpha}^i_{-m} + E^a_0$ and partition function $1\underset{a}{\square}$ of (3.4) can be again constructed from a set of vertex operators $\tilde{\mathscr{V}}_a^{\Lambda}$ which form a representation of the untwisted set $\mathscr{V}^{\Lambda}$. These act on a degenerate vacuum of dimension $D_{\bar{a}}^{1/2}$ and their OPA forms a representation of the OPA (2.5) which is a non-meromorphic OPA due to Z/n grading. The construction of $\tilde{\phi}(z) \in \tilde{\mathscr{V}}_a^{\Lambda}$ is similar to (2.4) where now the cocycle factors are replaced by $\{c_T(\alpha)\}$ defined as follows [30, 26]. Consider a central extension $\hat{L}_{\bar{a}}$ of $L_{\bar{a}}$ by $\langle\omega\rangle$, the cyclic group generated by $\omega = e^{2\pi i/n}$, given by

$$c(\alpha)c(\beta)c(\alpha)^{-1}c(\beta)^{-1} = \exp(2\pi i S_a(\alpha,\beta)), \tag{3.8a}$$

$$S_a(\alpha,\beta) = -S_a(\beta,\alpha) = \langle\alpha,(1-\bar{a})^{-1}\beta\rangle \bmod 1, \tag{3.8b}$$

where α, β are representative elements of $L_{\bar{a}}$ and $S_a(\alpha,\beta) \in Z/n$. Associated with each section $\{c(\alpha)\}$ is a 2-cocycle $\varepsilon(\alpha,\beta) \in \langle\omega\rangle$ as in (2.6b) obeying the cocycle condition (2.6c). In general, the commutator subgroup of $\hat{L}_{\bar{a}}$ is a subgroup of the center $\langle\omega\rangle$ and for $n = p$, prime, is equivalent to $\langle\omega\rangle$ in which case $\hat{L}_{\bar{a}} = p_+^{1+2d}$, an extra-special p group cf. [3]. For the full set of automorphisms obeying (3.6a), $\hat{L}_{\bar{a}}$ is given in column 3 of Tables 1 and 2. The group $\hat{L}_{\bar{a}}$ has a unique irreducible faithful representation π of dimension $D_{\bar{a}}^{1/2}$ in which the center is represented by the roots of unity $\langle\omega\rangle$ [30, 3, 26]. The elements of $\pi(\hat{L}_{\bar{a}})$ are then the cocycle matrices $\{c_T(\alpha)\}$ which act on a vector space with basis formed by the a twisted vacuum states $\{|\sigma_a^l\rangle\}$, $l = 1, \ldots, D_{\bar{a}}^{1/2}$.

For each operator $\phi(z) \in \mathscr{V}^{\Lambda}$ there is a corresponding operator $\tilde{\phi}(z) \in \tilde{\mathscr{V}}_a^{\Lambda}$ which acts on the a twisted vacuum states $\{|\sigma_a^l\rangle\}$ and obeys the monodromy condition associated with the automorphism a as follows.

$$\tilde{\phi}^{(k)}(e^{2\pi i}z) = a^{-1}\tilde{\phi}^{(k)}(z)a = \omega^{-k}\tilde{\phi}^{(k)}(z)\,, \tag{3.9}$$

where $\phi^{(k)}(z) \in \mathscr{V}^{\Lambda}$ is an ω^k eigenstate of a. The twisted vacuum states are in turn generated by twist operators $\{\sigma_a^l(z)\}$ which act on the untwisted vacuum. For the automorhisms of Table 1 which lead to a modular consistent theory, these twist operators are of conformal dimension $h_\sigma = 1 + E_0^a = 1 + 1/n$. The remaining constructions based on the automorphisms of Table 2 are discussed below. The construction of $\{\sigma_a^l(z)\}$ can be explicitly performed [34] where these operators form a non-meromorphic OPA with the vertex operators of $\mathscr{V}^{\Lambda}$ and $\tilde{\mathscr{V}}_a^{\Lambda}$,

$$\tilde{\phi}^{(k)}(z)\sigma_a^l(w) = \sigma_a^l(w)\phi^{(k)}(z) \sim (z-w)^{h_\psi - h_\phi - h_\sigma}\psi_a^{(k-1)}(w) + \ldots\,, \tag{3.10}$$

with a suitable analytic continuation assumed in the first equality [34]. $\psi_a^{(k)}(z)$ denotes a conformal field that creates a twisted state from the untwisted vacuum where (3.9) implies that the conformal dimension $h_\psi \in Z - k/n$. Thus the first excited twisted states $|\psi^{il}\rangle = \tilde{a}^i_{-1/n}|\sigma_a^l\rangle$ with energy $2/n$ are given by $\lim\limits_{z\to 0} z^{(n-1)/n}\partial_z \tilde{X}^i(z)\,|\sigma_a^l\rangle$ for $i = 1, \ldots, a_1$, i.e. they are created by the lowest conformal dimension operators $\partial_z \tilde{X}^i(z)$ of $\tilde{\mathscr{V}}^{\Lambda}$ which are ω^{n-1} eigenvectors under $\bar{a}$. We denote the set of operators $\{\psi_a^{(k)}(z)\}$, including $\{\sigma_a^l(z)\}$, by $\mathscr{V}_a$.

The lattice automorphism $\bar{a}$ acts as the identity on the fixed point space $L_{\bar{a}}$. This allows us to choose a lifting of $\bar{a}$ as an automorphisms of $\pi(\hat{L}_{\bar{a}})$, which we also denote by a, given by $ac_T(\alpha)a^{-1} = \omega^{-1}c_T(a)$ which is the appropriate choice for $E_0^a = 1/n$. We may then define the following automorphism of the OPA (3.10)

$$a\sigma_a^l(z)a^{-1} = \omega^{-1}\sigma_a^l(z)\,, \tag{3.11a}$$

$$a\psi_a^{(k)}(z)a^{-1} = \omega^k\psi_a^{(k)}(z) = e^{-2\pi i h_\psi}\psi_a^{(k)}(z)\,. \tag{3.11b}$$

From (3.10), the twisted operators of $\mathscr{V}_a$ therefore obey the twisted monodromy condition when acting on the vacuum $|0\rangle$

$$\psi_a(e^{2\pi i z}z) = e^{-2\pi i h_\psi}\xi_a(z) = a\psi_a(z)a^{-1}\,. \tag{3.12}$$

Thus $e^{2\pi i L_0}|\psi_a\rangle = a^{-1}|\psi_a\rangle$ which implies that under $T{:}\,\tau \to \tau + 1$, $1\underset{a}{\Box} \to a^{-1}\underset{a}{\Box}$ in the expected way e.g. [6]. The lifting of $\bar{a}$ chosen therefore ensures that no extra phase occurs in this transformation and that there is no global phase anomaly [31, 32].

For the automorphisms of Table 2, the twist operators have conformal dimension $h_\sigma = 1 + 1/nh$ and $\psi_a^{(k)}(z)$ has conformal dimension $h_{\psi^{(k)}} \in Z - (k+1)/n + 1/nh$. Equation (3.11) must therefore be modified where now $\sigma_a^l(z)$ and $\psi_a^{(k)}(z)$, are, respectively, unit and $\omega^{k+1} = \omega^{1/h}e^{-2\pi i h_\psi}$ eigenstates under a. Likewise, an extra phase of $\omega^{-1/h}$ appears on the RHS of (3.12). Hence $1\underset{a}{\Box}$ is invariant under T^n only up to an overall global phase of $e^{2\pi i/h}$ giving the global phase anomaly anticipated earlier.

Examining the twisted partition function for these cases, we also notice that it is related to that for a^h with $1\underset{a^h}{\Box}(\tau) = \left[1\underset{a}{\Box}(h\tau)\right]^h$, where $D_{\bar{a}^h} = D_{\bar{a}}^h$ and $\eta_{\bar{a}^h}(\tau) = [\eta_{\bar{a}}(\tau/h)]^h$ in (3.4). This observation leads us to an isomorphism between the corresponding twisted Hilbert spaces with

$$\mathscr{H}_{a^h} \cong \mathscr{H}_a \otimes \ldots \otimes \mathscr{H}_a\,, \tag{3.13}$$

where the RHS denotes a tensor product over h copies of $\mathscr{H}_a$. The explicit form of this isomorphism is found by first noting that $L_{\bar{a}^h} \cong L_{\bar{a}} \times \ldots \times L_{\bar{a}}$ for each automorphism $\bar{a}$ of Table 2. Since $\bar{a}^h$ has no fixed points we have $(1-\bar{a})^{-1} = (1-\bar{a}^h)^{-1}(1+\bar{a}+\ldots+\bar{a}^{h-1})$ so that the commutator subgroup of $\hat{L}_{\bar{a}}$ obeys $[\hat{L}_{\bar{a}}, \hat{L}_{\bar{a}}] \subseteq \langle \omega^h \rangle$ from (3.8b). The representation $\pi(\hat{L}_{\bar{a}})$ acts on a vector space $T^{\bar{a}}$ of dimension $D_a^{1/2}$, where the centre is represented by the cyclic group of phases $\langle \omega \rangle$. Thus $T^{\bar{a}}$ defines the vector space for a projective representation for $L_{\bar{a}}$ with phases in $\langle \omega^h \rangle$. Taking the tensor product of h copies of $T^{\bar{a}}$ we obtain the vector space $T^{\bar{a}} \otimes \ldots \otimes T^{\bar{a}}$ for the representation $\pi(\hat{L}_{\bar{a}^h})$ of dimension $D_{a^h}^{1/2} = D_{\bar{a}}^{h/2}$ which forms a projective representation for $L_{\bar{a}^h} \cong L_{\bar{a}} \times \ldots \times L_{\bar{a}}$ with phases in $\langle \omega^h \rangle$. Thus the vacuum states of the twisted Hilbert spaces of (3.13) are isomorphic. Now define $\tilde{\Phi}_{i_1 \ldots i_h}(z) = \tilde{\phi}_{i_1}(z^h) \otimes \ldots \otimes \tilde{\phi}_{i_h}(z^h)$ which acts on these twisted vacuum states. Then $\tilde{\Phi}_{i_1 \ldots i_h}(z)$ obeys the monodromy condition (3.9) for a^h. The operators $\{\Phi(z)\}$ obey a non-meromorphic OPA due to hZ/n grading and create Virasoro eigenstates in $\mathscr{H}_{a^h}$ (but are not primary conformal fields in $\mathscr{V}_{a^h}$). The vacuum states of $\mathscr{H}_{a^h}$, which are created by the twist operators $\Sigma_{a^h}^{l_1 \ldots l_h}(z) = \sigma_a^{l_1}(z^h) \otimes \ldots \otimes \sigma_a^{l_h}(z^h)$ have energy $h_\Sigma = h/n$ and hence the global phase anomaly disappears by taking this tensor product. Thus the isomorphism between Hilbert spaces in (3.13) follows.

We may repeat the $\mathscr{V}_a$ construction above for the remaining sectors $\mathscr{V}_b$ with $X^i(z)$ twisted by $\bar{b} = \bar{a}^r$ in (3.7b). This is briefly reviewed in Appendix B. For r relatively prime to n, $\bar{b}$ is of order n also and $\mathscr{V}_b$ is isomorphic to $\mathscr{V}_a$. Otherwise, $\bar{b}$ may have unit eigenvalues and (3.7b) must be modified to include a momentum component belonging to $\Lambda_{\|}$ and where now $\tilde{q}^i$ lies in the $\bar{b}$ fixed point space of the torus $L_{\bar{b}} = \Lambda_{\bar{b}}^T/(1-\bar{b})\Lambda_T$ with $\Lambda_{\bar{b}}^T = \{\beta \in \Lambda \,|\, \mathscr{P}_b\beta = 0\}$, $\Lambda_T = (1-\mathscr{P}_b)\Lambda$. $L_{\bar{b}}$ is a finite abelian group of order $D_{\bar{b}}/V_{\bar{b}}^2$. The construction of the $D_{\bar{b}}^{1/2}/V_{\bar{b}}$ twisted vacuum states $\{|\sigma_b^l\rangle\}$ can be similarly defined [30, 34] together with vertex operators $\tilde{\mathscr{V}}_b^{\Lambda}$ which create $\mathscr{H}_b$ with partition function $1\underset{b}{\square}$ of (3.4). Likewise, the non-meromorphic OPA of (3.10) and monodromy conditions of (3.9) and (3.12) are generalised with a replaced by b throughout and $\mathscr{V}_a$ replaced by $\mathscr{V}_b$. These other twisted sectors are required for modular invariance and for the expected closure of the corresponding meromorphic OPA. In particular, we expect the original a twisted operators $\{\sigma_a^l(z)\}$ to form an intertwining non-local OPA with the operators of each sector where

$$\tilde{\psi}_b^{(k)}(z)\sigma_a(w) \sim (z-w)^{h_\chi - h_\psi - h_\sigma}\chi_{ab}^{(k-1)}(w) + \ldots, \tag{3.14}$$

where $\psi_b^{(k)} \in \mathscr{V}_b$, $\chi_{ab}^{(k)} \in \mathscr{V}_{ab}$ are ω^k eigenstates of a and where for each $\psi_b(z) \in \mathscr{V}_b$, there is an operator $\tilde{\psi}(z)$ which acts on the a twisted vacuum creating a state in the ab twisted sector. The $b = a^r$ monodromy condition (generalised from (3.12)) implies that $\psi_b^{(k)}$ has conformal dimension $h_\psi \in Z - kr/n$.

We therefore enlarge the meromorphic set of operators $\mathscr{V}^\Lambda$ by the introduction of the twisted operators $\{\sigma_a^l\}$ to the set of operators $\mathscr{V}' = \mathscr{V}^\Lambda \oplus \mathscr{V}_a \oplus \ldots \oplus \mathscr{V}_{a^{n-1}}$ which forms a closed but non-meromorphic OPA. $\mathscr{V}'$ consists of all operators which form a meromorphic OPA with $\mathscr{P}_a\mathscr{V}^\Lambda$, i.e. $\mathscr{V}'$ and $\mathscr{P}_a\mathscr{V}^\Lambda$ are dual [9]. Then $\mathscr{V}_{\text{orb}}^a = \mathscr{P}_a\mathscr{V}'$ forms a closed meromorphic OPA which is self-dual. Note that only this meromorphic $\mathscr{P}_a$ projection of the intertwining OPA (3.14) has been rigorously constructed and then only in the prime ordered cases $p = 2$ in [1, 28, 10] and for $p = 3, 5, 7, 13$ in [16]. We will assume that (3.14) is true in general. The partition function for the corresponding space of states $\mathscr{H}^a_{\rm orb}$ is modular invariant with a unique simple pole at $q = 0$ as before and is therefore given by $Z_{\rm orb}(\tau) = J(\tau) + N_0$, where N_0 is the number of massless operators. The condition $E^a_0 > 0$ ensures that no massless operators occur in the a twisted sector, i.e. there is no a invariant operator $\psi^{(0)}(z)$ with $h_\psi = 1$ which satisfies a meromorphic monodromy condition $\psi^{(0)}(e^{2\pi i}z) = \psi^{(0)}(z)$ from (3.12). Nevertheless, there may be a massless operator $\psi^{(0)}(z)$ present in one of the other $b = a^r$ twisted sectors, where $\psi^{(0)}(z)$ is b invariant from the b monodromy condition (e.g. for $\bar{a} = 4^8/1^8$, the twisted sector corresponding to $\bar{b} = \bar{a}^2 = 2^{16}/1^8$ has a massless vacuum from (3.5b)). Taking the a invariant projection we find $\mathscr{P}_a\psi^{(0)} = 0$ unless $\psi^{(0)}(z)$ is also a invariant and therefore contradicts our assumption. Thus no massless operators that may occur in the other twisted sectors can survive the $\mathscr{P}_a$ projection and hence the condition $E^a_0 > 0$ is sufficient to ensure the absence of massless operators in $\mathscr{V}^a_{\rm orb}$ and the partition function is $Z_{\rm orb}(\tau) = J(\tau)$ once again. Therefore, according to the FLM uniqueness conjecture, we expect $\mathscr{V}^a_{\rm orb} \equiv \mathscr{V}^\natural$ for each of the 38 automorphisms of Table 1. Let us now consider some evidence to support this.

3.5. Centralisers, Thompson Series and $\mathbf{Z}_n$ *Reorbifolding* $\mathscr{V}^a_{\rm orb}$. Let $M^a_{\rm orb}$ be the automorphism group of the OPA for $\mathscr{V}^a_{\rm orb}$ which, from the FLM uniqueness conjecture, we expect to be M, the Monster group. For the prime ordered cases $p = 3, 5, 7$ and 13, Dong and Mason have recently demonstrated that $M^a_{\rm orb} \equiv M$ for $p = 3$ and very nearly so for $p = 5, 7, 13$ [16]. We may identify an automorphism $a^* \in M^a_{\rm orb}$ of order n (which generalises the fermion number involution i in the original FLM construction) under which the operators of $\mathscr{V}_{a^k}$ are eigenvectors with eigenvalue ω^k. From (3.14), a^* is also an automorphism of the non-meromorphic OPA for the enlarged set of operators $\mathscr{V}' = \mathscr{V}^\Lambda \oplus \mathscr{V}_a \oplus \ldots \oplus \mathscr{V}_{a^{n-1}}$ and a^* is "dual" to the automorphism a, i.e. the a invariant subset of $\mathscr{V}'$ is $\mathscr{V}^a_{\rm orb}$ whereas the a^* invariant subset is $\mathscr{V}^\Lambda$. Furthermore, $\mathscr{V}'$ is the set of all operators which form a meromorphic OPA with $\mathscr{P}_a\mathscr{V}^\Lambda = \mathscr{P}_{a^*}\mathscr{V}^a_{\rm orb}$ ($\mathscr{V}'$ is dual to $\mathscr{P}_a\mathscr{V}^\Lambda$) and hence we may reorbifold $\mathscr{V}^a_{\rm orb}$ with respect to a^* to reproduce $\mathscr{V}^\Lambda$. We can see this explicitly as follows. Consider the massless states $\{\alpha^i_{-1}|0\rangle\}$, $i = 1, \ldots -a_1$, which are ω eigenstates of a. The operators of $\mathscr{V}^a_{\rm orb}$ obey the a^* twisted monodromy condition when acting on these states:

$$\psi^{(0)}_b(e^{2\pi i}z) = \omega^{-r}\psi^{(0)}_b(z) = a^{*-1}\psi^{(0)}_b(z)a^* \tag{3.15}$$

which is analogous to (3.9), i.e. the $-a_1$ massless operators $\{\partial_z X^i(z)\}$, $i = 1, \ldots -a_1$ implement the a^* monodromy condition for $\mathscr{V}^a_{\rm orb}$ and create the a^* twisted vacuum states. The resulting non-meromorphic OPA closes once again in the enlarged set $\mathscr{V}'$ of which the a^* invariant subset is $\mathscr{V}^\Lambda$. Thus

$$\begin{tikzcd}
 & \mathscr{V}' \arrow[dl, "\mathscr{P}_{a^*}"'] \arrow[dr, "\mathscr{P}_a"] & \\
\mathscr{V}^\Lambda \arrow[rr, shift left, "a"] \arrow[dr, "\mathscr{P}_a"'] & & \mathscr{V}^a_{\rm orb} \arrow[ll, shift left, "a^*"] \arrow[dl, "\mathscr{P}_{a^*}"] \\
 & \mathscr{P}_a\mathscr{V}^\Lambda &
\end{tikzcd} \tag{3.16}$$

where the horizontal (diagonal) arrows represent orbifolding (projecting) with respect to the denoted automorphism.

We may also compute the Thompson series $T^{\rm orb}_{a^*}(\tau)$ for $a^* \in M^a_{\rm orb}$ by taking the trace over $\mathscr{H}^a_{\rm orb}$, the Hilbert space of states created by $\mathscr{V}^a_{\rm orb}$, as follows:

$$T^{\rm orb}_{a^*}(\tau) = \mathrm{Tr}_{\mathscr{H}^a_{\rm orb}}(a^* q^{L_0}) = \mathscr{P}_a \underset{1}{\square} + \omega \mathscr{P}_a \underset{a}{\square} + \ldots + \omega^{n-1} \mathscr{P}_a \underset{a^{n-1}}{\square} \ . \tag{3.17}$$

For $n = p$, prime, a^* is of prime order and hence $\sum_{k=1}^{p} T^{\rm orb}_{a^{*k}}(\tau) = J(\tau) + (p-1)T^{\rm orb}_{a^*}(\tau)$. This is also equal to $1\underset{1}{\square} + (p-1)a\underset{1}{\square}$ from (3.17), where $\Sigma_k a^{*k}$ vanishes on each twisted sector. Therefore we find that $T^{\rm orb}_{a^*}(\tau) = a\underset{1}{\square} + 24/(p-1) = 1/\eta_{\bar{a}}(\tau) + 2d$. Thus $a^* \in M^a_{\rm orb}$ has the same Thompson series as $p- \in M$ with genus zero modular group $\Gamma_0(p)$. We can show that this generalizes to all orbifoldings generated by the elements of Table 1, where

$$T^{\rm orb}_{a^*}(\tau) = \frac{1}{\eta_{\bar{a}}(\tau)} - a_1 \tag{3.18}$$

which is the hauptmodul for the genus zero modular group $n + e_1, e_2, \ldots$. This result follows from a consideration of the singularities of $T^{\rm orb}_{a^*}(\tau)$ and showing that they agree with those of $1/\eta_{\bar{a}}(\tau)$ [18]. Thus each $a^* \in M^a_{\rm orb}$, the automorphism of $\mathscr{V}^a_{\rm orb}$ dual to a, has the same Thompson series as the non-Fricke elements $n + e_1, e_2, \ldots \in M$, $e_i \neq n$.

Equation (3.18) may be generalized to include the other automorphisms $\{\bar{a}\}$ of Table 2. As already described, such automorphisms cannot be used to construct a meromorphic orbifold CFT. However, $\bar{a}' = \bar{a}^h$, of order $m = n/h$, can be employed to construct an orbifold with partition function $J(\tau)$. Let g_n denote the lifting of $\bar{a}$, where $g_n^h = a'^*$ is dual to $a' = a^h$, a lifting of $\bar{a}'$. g_n then acts on each twisted space and is in the centraliser of a'^* in $M^{a^h}_{\rm orb}$ (see below). We may compute the Thompson series for g_n as a trace over $\mathscr{H}^{a^h}_{\rm orb}$ by a similar trick to the prime ordered cases above. $g_n^{1+hk} = g_n a'^{*k}$ is of order n for each $k = 1, 2, \ldots m$ and has the same Thompson series as g_n. Likewise, for each k, $g_n \underset{1}{\square} = g_n^{1+hk} \underset{1}{\square}$ and therefore $T_{g_n}(\tau) = \dfrac{1}{m} \sum_k T_{g_n^{1+hk}}(\tau) = g_n \underset{1}{\square} = 1/\eta_{\bar{a}}(\tau)$, where $\sum_k a'^{*k}$ vanishes on each twisted sector. Thus (3.18) also holds for the automorphism g_n (since $a_1 = 0$ for $h \neq 1$) and g_n has the same Thompson series as $n \,|\, h + e_1, e_2, \ldots$, with $e_i \neq n/h$ and $h \neq 1$, a non-Fricke element.

We may next compute the centraliser $C(g_n \,|\, M^{a^h}_{\rm orb}) = \{g \in M^{a^h}_{\rm orb} \,|\, g_n^{-1} g g_n = g\}$. For the 38 automorphisms with $h = 1$ this consists of all OPA automorphisms that do not mix the various projected sectors $\mathscr{P}_a \mathscr{V}_{a^k}$ of $\mathscr{V}^a_{\rm orb}$. For the remaining 13 automorphisms g_n with $h \neq 1$, $C(g_n \,|\, M^{a^h}_{\rm orb}) \subset C(a'^* \,|\, M^{a^h}_{\rm orb})$. Every element $g \in C(a^* \,|\, M^a_{\rm orb})$ must commute with a in order to preserve the $\mathscr{P}_a$ projection. Thus $C(a^* \,|\, M^a_{\rm orb})$ is some extension of $G_n = C(\bar{a}|\, Co_0)/\langle \bar{a} \rangle$, the non-trivial part of the Conway group centraliser, which is reproduced from [35] in column 4 of Tables 1 and 2. The nature of this extension can be seen by considering the automorphism group preserving the twisted sector $\mathscr{P}_a \mathscr{V}_a$ [18]. Let g and g' be two inequivalent liftings of $\bar{g}$ to automorphisms of $\pi(\hat{L}_{\bar{a}})$, the faithful representation of $\hat{L}_{\bar{a}}$ whose elements are the a twisted cocycle matrices $\{c_T(\alpha)\}$. Thus $g' g^{-1}$ is a lifting of the identity lattice automorphism. However, the inner automorphisms of $\pi(\hat{L}_{\bar{a}})$ given by $c_T(\alpha){:}c_T(\beta) \to c_T(\alpha)c_T(\beta)c_T(\alpha)^{-1} = \exp(2\pi i S_{\bar{a}}(\alpha,\beta))c_T(\beta)$ describe the inequivalent liftings of the identity and hence the inequivalent liftings g of $\bar{g}$. As discussed above in (3.11), the lifting a of $\bar{a}$ to an automorphism of $\pi(\hat{L}_{\bar{a}})$ is $ac_T(\alpha)a^{-1} = \omega^{-1}c_T(\alpha)$. Hence g commutes with a and in turn, defines an automorphism for $\mathscr{P}_a\mathscr{V}_a$ through (3.10). Thus we find that the group of inequivalent OPA automorphisms preserving $\mathscr{P}_a\mathscr{V}_a$ is $\hat{L}_{\bar{a}}.G_n$, an extension of G_n. The same result also holds for the isomorphic twisted sectors $\mathscr{P}_a\mathscr{V}_{a^k}$, where a^k is of order n, i.e. k is relatively prime to n. In Appendix B we discuss the contribution of the remaining sectors to $C(a^* \mid M^a_{\text{orb}})$. There we also consider the other 13 automorphisms with $h \neq 1$ and demonstrate that for all 51 atuomorphisms g_n,

$$C(g_n \mid M^{a^h}_{\text{orb}}) = \hat{L}_{\bar{a}}.G_n\,. \tag{3.19}$$

In column 5 of Tables 1 and 2 we have reproduced $C(g_n \mid M)$ from [13] which may be compared with $\hat{L}_{\bar{a}}$ and G_n in columns 3 and 4 to verify (3.19) assuming that $M^{a^h}_{\text{orb}} \equiv M$ and $g_n \equiv n \mid h + e_1, e_2, \ldots$, a non-Fricke element of M. Equation (3.19) is a new generalisation of the original observation of Conway and Norton concerning the five $n = p$, prime, cases where $C(p- \mid M) = p^{1+2d}_+.G_p$ with $a^* = p-$ [13]. For the other 46 automorphisms of Tables 1 and 2, there are only 11 cases for which (3.19) can be explicitly checked using the available information about these centralisers in [13, 35]. However, the order of these groups agrees with (3.19) in each case supporting the very likely validity of the result in general.

From (3.10) we may observe that $\hat{L}_{\bar{a}}.G_n$ must be an extension of $\hat{G}_n = C(a \mid 2^{24}.Co_0)/\langle a\rangle$, the subgroup of automorphisms of $\mathscr{V}^\Lambda$ which preserve $\mathscr{P}_a\mathscr{V}^\Lambda$, where the extension contains the central cyclic group generated by g_n. This extension is due to the presence of the $D^{1/2}_{\bar{a}}$ twist operators $\{\sigma^l_a\}$ which form a representation of $\hat{G}_n$. Thus for the prime ordered cases $\hat{G}_2 = 2^{24}.Co_1$ and $\hat{G}_p = G_p$ for $p = 3, 5, 7, 13$. In particular, we also note that if the a twisted vacuum is unique, then $\hat{L}_{\bar{a}}.G_n$ is isomorphic to $n.\hat{G}_n$. A similar observation will be useful in Sect. 4 when we consider other possible orbifoldings of $\mathscr{V}^\natural$.

3.6. A $\mathbf{Z}_2$ Reorbifolding of $\mathscr{V}^a_{\text{orb}}$. Recently, Montague made the interesting suggestion [23] that a CFT, such as $\mathscr{V}^a_{\text{orb}}$, with partition function $J(\tau)$ can be shown to be isomorphic to $\mathscr{V}^\natural$ by the existence of an involution i of $\mathscr{V}^a_{\text{orb}}$ and a set of twisted operators $\mathscr{V}_i$ with non-negative vacuum energy (see Sect. 4). Then the Thompson series $T^{\text{orb}}_i(\tau)$ is $\Gamma_0(2)$ invariant with a unique simple pole at $q = 0$ and must be hauptmodul $1/\eta_{\bar{r}}(\tau) + 24$. Therefore, assuming that we can reorbifold $\mathscr{V}^a_{\text{orb}}$ with respect to i, we obtain a CFT with partition function $J(\tau) + 24$. But $\mathscr{V}^\Lambda$ is now known to be the unique CFT with this partition function [23] and hence this reorbifolding reproduces $\mathscr{V}^\Lambda$. If we consider the involution i^* dual to i which acts on $\mathscr{V}^\Lambda$, then the 24 massless operators of $\mathscr{V}^\Lambda$ are -1 eigenvectors under i^* and hence i^* can be identified with the involution r introduced in the original FLM construction. Thus $\mathscr{V}^a_{\text{orb}}$ can be obtained from $\mathscr{V}^\Lambda$ by orbifolding with respect to $i^* \equiv r$ and must be isomorphic to $\mathscr{V}^\natural$.

We will now consider the constructions of $\mathscr{V}^a_{\text{orb}}$ given above and find an involution i with the correct Thompson series in 11 cases in addition to the standard FLM construction. We will only consider here an involution in the centraliser $C(a^* \mid M^a_{\text{orb}}) = \hat{L}_{\bar{a}}.G_n$ which is lifted from the reflection atuomorphism $\bar{r}$ of Λ. This restriction excludes 13 automorphisms of even order n (including the original automorphism $\bar{r}$

adopted by FLM!) denoted by † in the last column of Table 1 for which $\bar{a}^{n/2} = \bar{r} = -1$ so that $\bar{r} \notin G_n$. For the remaining automorphisms we can compute $T_i^{\mathrm{orb}}(\tau)$ similarly to (3.17). Under $S: \tau \to -1/\tau$ we obtain given the usual modular transformation properties

$$T_i^{\mathrm{orb}}\left(-\frac{1}{\tau}\right) = \mathscr{P}_a \underset{i}{\square} + \mathscr{P}_a \underset{ia}{\square} + \ldots + \mathscr{P}_a \underset{ia^{n-1}}{\square}\,, \tag{3.20}$$

which is T^2 invariant and hence $T_i^{\mathrm{orb}}(\tau)$ is $\Gamma_0(2)$ invariant. We can determine whether $T_i^{\mathrm{orb}}(\tau)$ is a hauptmodul for $\Gamma_0(2)$ by considering the behaviour at $\tau = 0$ via (3.20). The sector twisted by i has vacuum energy $+1/2$ because i is lifted from $\bar{r}$ and therefore contributes no singularity. Each sector twisted by ia^k, of order m, has vacuum energy which always obeys $E_0 \geq -1/m$ (see Sect. 4.4) and therefore contributes no singularity unless $m = 2$ with $E_0 = -1/2$ since (3.20) is T^2 invariant. This occurs when ia^k is lifted from $-\bar{a}^k$ with Frame shape $1^8.2^8$, i.e. $\bar{a}$ is of even order $n = 2k$ and $\bar{a}^k$ has Frame shape $2^{16}/1^8$ which is the case for the 14 automorphisms denoted by ‡ in Table 1. Otherwise, for the 11 remaining automorphisms, denoted by $*$ in Table 1, $T_i^{\mathrm{orb}}(\tau)$ has a unique simple pole at $q = 0$ and is therefore a hauptmodul for $\Gamma_0(2)$. These consist of 3 even ordered automorphisms and all the odd ordered automorphisms including the odd prime ones considered by Dong and Mason [16]. Thus, in these 11 cases, one can construct the required involution. In the remaining cases, a more technical construction is required and is currently under investigation.

To summarise this section, we have described 38 meromorphic orbifold constructions $\mathscr{V}_{\mathrm{orb}}^{a}$ (including the original one of FLM and the prime ordered constructions of Dong and Mason) with partition function $J(\tau)$. Amongst these constructions, we have found 51 automorphisms $\{g_n\}$ that can be identified with the 51 non-Fricke Monster group classes, where g_n satisfies $g_n^h = a'^*$, the automorphism dual to $a' = a^h$. For each g_n, the Thompson series agrees with the corresponding Monster group Thompson series and the centraliser in (3.19) also agrees explicitly in many cases (and very probably in all cases). For 11 of these new constructions, an involution can also be found which is dual to the involution r of $\mathscr{V}^{\Lambda}$ used in the FLM construction of $\mathscr{V}^{\natural}$ and so $\mathscr{V}_{\mathrm{orb}}^{a} \equiv \mathscr{V}^{\natural}$ for these cases (assuming that the various twisted sectors obey the OPAs (3.10) and (3.14)). We also note that we may in general compute the Thompson series within $\mathscr{V}_{\mathrm{orb}}^{a}$ for each element of $C(g_n \mid M_{\mathrm{orb}}^a)$ as a sum of traces over each sector $\mathscr{P}_a \mathscr{H}_b$ (in [18] we give an explicit formula for the prime ordered constructions). In particular, it is straightforward to show that $T_{g_n^k(\tau)}$ agrees with the expected result in each case. All of these results support the conjecture that $\mathscr{V}_{\mathrm{orb}}^{a} \equiv \mathscr{V}^{\natural}$ as expected from the FLM uniqueness conjecture. Finally, we expect a generalised version of the hidden triality symmetry in the FLM construction which mixes the untwisted and twisted sectors to exist [1, 3, 29]. Thus there should exist some symmetry group Σ_n which mixes the various sectors of $\mathscr{V}_{\mathrm{orb}}^{a}$, where $C(g_n \mid M)$ and Σ_n generate M. In the prime cases $p = 3, 5, 7, 13$, Σ_p has been constructed by Dong and Mason [16].

4. Orbifolding the Moonshine Module and Monstrous Moonshine

4.1. Monstrous Moonshine and Orbifolding $\mathscr{V}^{\natural}$. Let us now consider one of the main objectives of this paper which is to discuss the relationship of the FLM uniqueness conjecture to Monstrous Moonshine, the genus zero property for Thompson series [13]. Our main result is as follows: Assuming the FLM uniqueness conjecture holds, then the Thompson series for $g \in M$ is a hauptmodul if and only if the only meromorphic orbifoldings of $\mathscr{V}^\natural$ with respect to g are $\mathscr{V}^\Lambda$ or $\mathscr{V}^\natural$.

We will assume throughout this section that the FLM uniqueness conjecture is correct. Therefore $\mathscr{V}^a_{\text{orb}} \cong \mathscr{V}^\natural$ for each of the orbifoldings described in Sect. 3 and $\mathscr{V}^\Lambda$ can be reconstructed by reorbifolding $\mathscr{V}^\natural$ with respect to the non-Fricke dual automorphisms $a^* = n{+}e_1, e_2, \ldots$ with $e_i \neq n$. The Thompson series for a^* of (3.15) is then recognised as a contribution to the partition function for this reorbifolding. It is natural to interpret all the Thompson series $T_g(\tau)$ in this way and to construct an orbifolding of $\mathscr{V}^\natural$ with respect to each $g \in M$ [19]. In particular, we expect that under $S: \tau \to -1/\tau$, $T_g(\tau) = \mathrm{Tr}_{\mathscr{H}^\natural}(g q^{L_0})$ transforms to the partition function for a g twisted sector as follows:

$$T_g(\tau) = g\underset{1}{\square}^{\natural} \to 1\underset{g}{\square}^{\natural} + \ldots, \tag{4.1}$$

where the superscript $\natural$ denotes a trace contribution to the orbifolding of $\mathscr{V}^\natural$ (in distinction to orbifoldings of $\mathscr{V}^\Lambda$) and where the g twisted sector $\mathscr{V}^\natural_g$ has vacuum energy E_0^g and degeneracy N_g. For the 38 automorphisms a^* dual to a we find from (3.16) that $1\underset{a^*}{\square}^{\natural} = -a_1 + D_{\bar{a}}^{1/2}\eta_{\bar{a}}(\tau/n)$ with vacuum energy $E_0^{a^*} = 0$ and degeneracy $N_{a^*} = -a_1$. In these cases, $\mathscr{V}^\natural_{a^*} = \{\phi^{(1)}\} \oplus \{\psi^{(1)}_a\} \oplus \ldots \oplus \{\psi^{(1)}_{a^{n-1}}\}$, the subspace of $\mathscr{V}^\Lambda \oplus \mathscr{V}_a \oplus \ldots \oplus \mathscr{V}_{a^{n-1}}$ with eigenvalue ω under a where, as noted in Sect. 3, the a^* twisted vacuum is created by the massless operators $\partial_z X^i(z)$, $i = 1, \ldots -a_1$. Likewise, the other 13 non-Fricke automorphisms g_n with $g_n^h = b^*$ (where b^* is dual to a^h and $h \neq 1$) have vacuum energy $E_0^{g_n} = 1/nh$ and degeneracy $N_{g_n} = D_{\bar{a}}^{1/2}$ and therefore possesses a global phase anomaly leading to an orbifold construction which is not meromorphic and not consistent with modular symmetry [31, 32]. The twisted space of operators $\mathscr{V}^\natural_{g_n}$ will be discussed in Sects. 4.3 and 4.4 below. For the remaining Fricke classes of M, $g = n \,|\, n{+}\dfrac{n}{h}, e_2, \ldots$ $\left(\text{i.e. } e_1 = \dfrac{n}{h}\right)$, we will assume that the twisted operator sector $\mathscr{V}^\natural_g$, with a corresponding Hilbert space of states $\mathscr{H}^\natural_g$, can always be constructed. There are a total of 120 of these classes (including two classes 27A, 27B which have the same Thompson series) of which 82 classes have $h = 1$ [13]. For many of these classes, the method of construction of these sectors is not known since the origin of the automorphism is not geometrical as was the case for the lattice automorphisms of Sect. 3. However, for automorphisms in the centraliser $C(i \,|\, M) = 2^{1+24}.Co_1$ which are associated with Leech lattice automorphisms, a method of construction is given later on Sect. 4.4.

The q^n coefficients of the trace on the RHS of (4.1) must all be non-negative since this is the partition function $\mathrm{Tr}_{\mathscr{H}^\natural_g}(q^{L_0})$ for the Hilbert space $\mathscr{H}^\natural_g$ associated with $\mathscr{V}^\natural_g$. (In fact, from the point of view of the representation theory of Virasoro algebras, $\mathrm{Tr}_{\mathscr{H}^\natural_g}(q^{L_0})$ is the characteristic function and is arguably a more natural object to study than the original Thompson series). For the Fricke classes $T_g(\tau) = 1\underset{g}{\square}^{\natural}(nh\tau)$ whereas for the non-Fricke classes $T_{g_n}(\tau) = 1/\eta_{\bar{a}}(\tau) - a_1 = -a_1 + D_{\bar{a}}^{1/2}\Big/\Big(a_1 + 1\underset{g_n}{\square}^{\natural}(nh\tau)\Big)$. Therefore the q^n coefficients of $T_g(\tau)$ must be non-negative for the

Fricke classes and of mixed sign for the non-Fricke classes. These properties are indeed observed for all Thompson series.

For orbifold constructions leading to a theory with modular consistency, the vacuum energy E_0^g must also satisfy $nE_0^g = 0 \bmod 1$ and $1\underset{g}{\square}^{\natural}$ is T^n invariant. Assuming the usual orbifold trace modular transformation properties, for all $\gamma \in \Gamma_0(n)$, where $\gamma : \tau \to (a\tau + b)/(cn\tau + d)$ we find $T_g(\tau) \to g^d \underset{1}{\square}^{\natural} = T_g(\tau)$ since $(d, n) = 1$, i.e. n and d are relatively prime so that g and g^d are in the same conjugacy class and hence have the same Thompson series. Thus, in the absence of a global phase anomaly, $T_g(\tau)$ is $\Gamma_0(n)$ invariant and hence $h = 1$. Let us consider, for the present, only Thompson series with this property.

In general, we assume that there exists a set of operators $\{\sigma_g^l(z)\}$, $l = 1, \ldots, N_g$ of conformal dimension $h_\sigma = 1 + j/n$ which create the vacuum operators of $\mathcal{V}_g^\natural$. We also assume that for each operator $\psi(z) \in \mathcal{V}^\natural$, there is an operator $\tilde{\psi}(z)$, which acts on this twisted vacuum and creates a state in $\mathcal{H}_g^\natural$. If $\psi^{(k)}(z) \in \mathcal{V}^\natural$ is an ω^k eigenstate of g, then we assume that when acting on the vacuum states $\{|\sigma_g^l\rangle\}$, $\tilde{\psi}^{(k)}(z)$ satisfies the following monodromy condition:

$$\tilde{\psi}^{(k)}(e^{2\pi i} z) = \omega^{-k}\tilde{\psi}^{(k)}(z) = g^{-1}\tilde{\psi}^{(k)}(z) g\,. \tag{4.2}$$

Similarly to (3.9) and (3.10), (4.2) follows from a non-meromorphic OPA which the twisted operators $\{\sigma_g^l(z)\}$ satisfy with $\mathcal{V}^\natural$, where

$$\tilde{\psi}^{(k)}(z)\sigma_g^l(w) = \sigma_g^l(w)\psi^{(k)}(z) \sim (z-w)^{h_\chi - h_\psi - h_\sigma}\chi_g^{(k-j)}(w)\ldots\,, \tag{4.3}$$

where the operators $\{\sigma_g^l(z)\}$ are ω^{-j} eigenvectors of g and $\chi_g^{(k)}(z) \in \mathcal{V}_g^\natural$ has conformal dimension $h_\chi \in Z - k/n$ and is an ω^k eigenvector of g. Then each $\chi_g(z) \in \mathcal{V}_g^\natural$ obeys the usual monodromy condition

$$\chi_g(e^{2\pi i} z) = g\chi_g(z) g^{-1} = e^{-2\pi i h_\chi}\chi_g(z) \tag{4.4}$$

when acting on the untwisted vacuum $|0\rangle$ so that $T : 1\underset{g}{\square}^{\natural} \to g^{-1}\underset{g}{\square}^{\natural}$ as expected, without any global phase anomaly. Likewise, the twisted sectors $\{\mathcal{V}_{g^k}^\natural\}$ are assumed to exist with vacuum energy $E_0^{g^k}$ and degeneracy N_{g^k}, where together $\mathcal{V}' = \mathcal{V}^\natural \oplus \mathcal{V}_g^\natural \oplus \ldots \oplus \mathcal{V}_{g^{n-1}}^\natural$ forms a closed non-meromorphic OPA. Taking the projection we define $\mathcal{V}_{\mathrm{orb}}^g = \mathcal{P}_g \mathcal{V}'$, the CFT constructed from $\mathcal{V}^\natural$ by orbifolding with respect to g. The operators of $\mathcal{V}_{\mathrm{orb}}^g$ form a meromorphic OPA and the partition function is again $Z_{\mathrm{orb}}(\tau) = J(\tau) + N_0$, where N_0 is the number of massless operators. For each of the 38 non-Fricke automorphisms a^* dual to a, this construction gives us $\mathcal{V}^\Lambda$ with $N_0 = 24$. Assuming that Thompson series are hauptmoduls, we will show below that $N_0 = 0$ for the remaining 82 global phase anomaly free Fricke classes (which we denote by $f = n + n, e_2, \ldots$) so that $\mathcal{V}_{\mathrm{orb}}^f \cong \mathcal{V}^\natural$ again i.e. every meromorphic orbifolding of $\mathcal{V}^\natural$ with respect to $g \in M$ either produces $\mathcal{V}^\Lambda$ or reproduces $\mathcal{V}^\natural$ again. Conversely, we will show in Sect. 4.2 that given this result then $T_g(\tau)$ must be a hauptmodul for some genus zero modular group.

We begin by describing how $T_g(\tau)$ can be a hauptmodul in terms of the vacuum properties of $\mathcal{V}_{g^k}^\natural$ for a meromorphic orbifolding of $\mathcal{V}^\natural$ with respect to g. We assume that under a general modular transformation $\gamma(\tau) = (a\tau + b)/(c\tau + d)$, $T_g(\gamma(\tau)) = g^{-d}\underset{g^c}{\Box}^\natural$. Thus any possible singular behaviour of $T_g(\tau)$ at a cusp point $a/c = \lim_{\tau\to\infty} \gamma(\tau)$ is governed by the vacuum energy and degeneracy of the g^c twisted sector. In [19] we showed that for $g = n + e_1, e_2, \ldots \in M$, $T_g(\tau) = g\underset{1}{\Box}^\natural$ is a hauptmodul for the modular group $\Gamma_g = \Gamma_0(n) + e_1, e_2, \ldots$ if and only if the vacuum energies and degeneracies of the twisted sectors $\mathscr{V}^\natural_{g^k}$ obey the following properties.

Vacuum Properties

(I) *The vacuum energy* $E_0^{g^k}$ *for* $\mathscr{V}^\natural_{g^k}$ *is non-negative unless* g^k *is of order* $e \in \{e_1, e_2, \ldots\}$ *in which case* $E_0^{g^k} = -1/e$ *(*$\mathscr{V}^\natural_{g^k}$ *is tachyonic) and the vacuum degeneracy* $N_{g^k} = 1$.

(II) *(Atkin-Lehner Closure) If both sectors* $\mathscr{V}^\natural_{g^{k_1}}$ *and* $\mathscr{V}_{g^{k_2}}$ *are tachyonic (with vacuum energies* $-1/e_1, -1/e_2$*) then the sector* $\mathscr{V}^\natural_{g^{k_3}}$ *is also tachyonic (with vacuum energy* $-1/e_3$*) where* g^{k_3} *is of order* $e_3 = e_1e_2/(e_1, e_2)^2$.

Condition (I) is required to ensure that $T_g(\tau)$ has the correct residue and pole strength at any singular cusps whereas condition (II) ensures that the composition of two Atkin-Lehner involution invariances of $T_g(\tau)$ is another Atkin-Lehner invariance as in (A.4).

The Vacuum Properties are easily understood for g of prime order p as follows. As described above, $T_g(\tau)$ is always $\Gamma_0(p)$ invariant. The fundamental region for this group, $\mathscr{F}_p = H/\Gamma_0(p)$, has two cusp points at $\tau = \infty$ ($q = 0$), where $T_g(\tau)$ has a simple pole and $\tau = 0$, at which $T_g(\tau)$ may have a second pole determined by the sign of the vacuum energy E_0^g and residue given by N_g from (4.1). Thus E_0^g is non-negative if and only if $T_g(\tau)$ has a unique simple pole at $q = 0$, i.e. $T_g(\tau)$ is a hauptmodul for $\Gamma_0(p)$ and $g = p-$. For $g = p+$ where $T_g(\tau)$ is invariant under the Fricke involution $W_p\colon \tau \to -1/p\tau$, then $g\underset{1}{\Box}^\natural(\tau) = 1\underset{g}{\Box}^\natural(p\tau)$ and we have $N_g = 1$ and $E_0^g = -1/p$, as given in the Vacuum Properties. Conversely, if $N_g = 1$, $E_0^g = -1/p$ then $f(\tau) = T_g(\tau) - T_g(W_p(\tau))$ is $\Gamma_0(p)$ invariant without any poles. Therefore $f(\tau)$ is holomorphic on the compactification of $\mathscr{F}_p$ (a compact Riemann surface) which is impossible unless f is constant. But $f(W_p(\tau)) = -f(\tau)$ implies $f = 0$. Therefore, $T_g(\tau)$ is $\Gamma_0(p)+$ invariant and has a unique simple pole at $q = 0$ on $H/\Gamma_0(p)+$ and thus $\Gamma_0(p)+$ is a genus zero modular group with hauptmodul $T_g(\tau)$. A similar argument to this applies in the more general situation where g is not of prime order and $T_g(\tau)$ can be invariant under other Atkin-Lehner involutions [19]. In addition, the Vacuum Properties imply that Thompson series obey the power-map formula which relates Γ_g to Γ_{g^k}. This is an empirical observation in [13] not derivable from the genus zero property [19].

For the 82 Fricke classes $f = n + n,\ e_2, \ldots, 1\underset{f}{\Box}^\natural(\tau) = f\underset{1}{\Box}^\natural(\tau/n) = q^{-1/n} + 0 + O(q^{1/n})$. Thus, despite the fact that $E_0^f = -1/n$, $\mathscr{V}_f^\natural$ contains no massless operators because the first excited states of $\mathscr{H}_f$ with energy $1/n$ are created by the action of conformal weight 2 operators of $\mathscr{V}^\natural$ on the f twisted vacuum. We may then repeat the argument of Sect. 3 to conclude that no massless operator $\psi^{(0)}(z)$ present in any other twisted sector $\mathscr{V}^\natural_{f^k}$ can be invariant under the $\mathscr{P}_f$ projection (otherwise $\psi^{(0)}(e^{2\pi i}z) = f\psi^{(0)}(z)f^{-1} = \psi^{(0)}(z)$ obeys the defining monodromy condition for a massless operator twisted by f which is impossible). Hence, for these Fricke classes, $\mathscr{V}^f_{\text{orb}}$ contains no massless operators so that $Z_{\text{orb}}(\tau) = J(\tau)$ again. Therefore, given the uniqueness of $\mathscr{V}^\natural$, we find that $\mathscr{V}^f_{\text{orb}} \cong \mathscr{V}^\natural$. We have therefore shown that orbifolding $\mathscr{V}^\natural$ with respect to the 38 non-Fricke classes $\{a^*\}$ gives $\mathscr{V}^\Lambda$ whereas orbifolding $\mathscr{V}^\natural$ with respect to the 82 Fricke classes $\{f\}$ reproduces $\mathscr{V}^\natural$, assuming that $\mathscr{V}^\natural$ is unique and the Vacuum Properties hold (i.e. the Thompson series are hautpmoduls). Thus we have

$$\mathscr{V}^\Lambda \underset{\xleftarrow{a^*}}{\xrightarrow{a}} \mathscr{V}^\natural \xleftrightarrow{f} \mathscr{V}^\natural , \tag{4.5}$$

where each arrow represents an orbifolding with respect to the denoted automorphism. We will refer to (4.5) as the Unique Orbifold Partner Property for $\mathscr{V}^\natural$.

4.2 Monstrous Moonshine from the Unique Orbifold Partner Property. We will now argue that the converse to the statement above is also true, i.e. assuming that $\mathscr{V}^\natural$ is unique and (4.5) holds for all meromorphic orbifoldings of $\mathscr{V}^\natural$ with respect to $g \in M$, then the Vacuum Properties hold and hence each Thompson series $T_g(\tau)$ is a hauptmodul for a genus zero modular group.

We begin with an orbifolding of $\mathscr{V}^\natural$ with respect to an automorphism, which we denote by a^*, which produces the Leech theory $\mathscr{V}^\Lambda$. a^* is dual to an automorphism a of $\mathscr{V}^\Lambda$ which must belong to one of the 38 classes described in Sect. 3. However, assuming the uniqueness of $\mathscr{V}^\natural$, then there must be exactly 38 different corresponding classes of automorphisms $\{a^*\}$ of $\mathscr{V}^\natural$ with Thompson series $T_{a^*}(\tau) = 1/\eta_{\bar{a}}(\tau) - a_1$. The associated twisted sector $\mathscr{V}^\natural_{a^*}$ therefore has vacuum energy $E_0^{a^*} = 0$ (and degeneracy $N_{a^*} = -a_1$) in agreement with the Vacuum Properties concerning $\mathscr{V}^\natural_{a^*}$. Furthermore, $T_{a^*}(\tau)$ is known to be a hauptmodul for the genus zero modular group $\Gamma_0(n)+e_1, e_2, \ldots, e_i \neq n$ and hence a^* is a non-Fricke element of type $n+e_1, e_2, \ldots$. Thus the remaining Vacuum Properties concerning $\mathscr{V}^\natural_{a^{*k}}$ must also hold for these elements. We will briefly consider further reasons for this result later on in the light of our discussion of the Fricke elements.

Let us now consider the remaining allowed orbifoldings of $\mathscr{V}^\natural$ with respect to automorphisms, which we denote by $\{f\}$, which are assumed to reproduce $\mathscr{V}^\natural$. Each orbifolding is necessarily free of global phase anomalies and hence, as described above, $T_f(\tau)$ is $\Gamma_0(n)$ invariant where f is of order n. We will show that the Vacuum Properties hold for these automorphisms and that $T_f(\tau)$ is a hauptmodul which is Fricke invariant.

$\mathscr{V}^f_{\text{orb}} \cong \mathscr{V}^\natural$ implies the absence of massless operators in $\mathscr{V}^\natural_f$. Therefore the twisted vacuum energy obeys either $E_0^f > 0$ or $E_0^f = -1/n$ (so that $\mathscr{V}^\natural_f$ is tachyonic). The first case is the only possibility in a regular lattice orbifolding as in Sect. 3. $E_0^f = -1/n$ is also possible for an orbifolding of $\mathscr{V}^\natural$ because the lowest excited energy operators $\{\psi_2(z)\}$ of $\mathscr{V}^\natural$ are of conformal dimension 2. Based on our experience with lattice orbifoldings, we expect the first excited states of $\mathscr{H}_f$ to be created by the action of some of these operators on the twisted vacuum as in (4.3). These excited states can then have minimum energy $1/n$ so that the absence of any massless operators in $\mathscr{V}_f^{\natural}$ is directly due to a similar absence in $\mathscr{V}^{\natural}$. On the other hand, any other negative value of E_0^f would result in massless operators in $\mathscr{V}_f^{\natural}$. We will directly observe this situation below in Sect. 4.4 when we consider automorphisms based on lattice automorphisms for which $\mathscr{V}_f^{\natural}$ can be explicitly constructed. Thus, we have determined that for any $f \in M$ either $E_0^f > 0$ or $E_0^f = -1/n$ ($\mathscr{V}_f^{\natural}$ is tachyonic) whereas for $a^* \in M$, $E_0^{a^*} = 0$. (Later on we will eliminate the possibility of $E_0^f > 0$ by studying the singularities and modular properties of $T_f(\tau)$.) As described before, the behaviour of $T_f(\tau)$ at a cusp point a/c is determined by $f^{-d}\underset{f^c}{\square}^{\natural}$ (where $ad - bc = 1$) with singular behaviour when $E_0^{f^c} < 0$, where f^c is of order n'. Therefore $\mathscr{V}_{\text{orb}}^{f^c} \cong \mathscr{V}^{\natural}$ with $E_0^{f^c} = -1/n'$ and the residue of this pole is N_{f^c}, the vacuum degeneracy of the twisted sector $\mathscr{V}_{f^c}$. We will next show that $N_{f^c} = 1$.

As was the case for the lattice orbifold constructions of Sects. 2 and 3, we may identify an automorphism f^*, which is dual to the automorphism f, where the operators of $\mathscr{V}_{f^k}^{\natural}$ are eigenvectors with eigenvalue ω^k for $\omega = e^{2\pi i/n}$. f^* is then an automorphism of the OPA for $\mathscr{V}_{\text{orb}}^f$, where $\mathscr{V}_{\text{orb}}^f \cong \mathscr{V}^{\natural}$ by assumption, i.e. $f^* \in M$ and $\mathscr{V}_{\text{orb}}^{f^*} \cong \mathscr{V}^{\natural}$. We can then calculate the Thompson series $T_{f^*}(\tau) = \mathrm{Tr}_{\mathscr{H}_{\text{orb}}^f}(f^* q^{L_0}) = \sum\limits_{k=1}^{n} \omega^k \mathscr{P}_f \underset{f^k}{\square}^{\natural}$, which is $\Gamma_0(n)$ invariant using the usual modular transformation properties of these traces. Furthermore, we can show that $T_{f^*}(\tau) = T_f(\tau)$ by considering the sum of Thompson series $\sum\limits_{k=1}^{n} T_{f^{*k}}(\tau) = \sum\limits_{k=1}^{n} \mathrm{Tr}_{\mathscr{H}_{\text{orb}}^f}(f^{*k} q^{L_0})$. Since only the untwisted sectors contribute we find

$$\sum_{r \mid n} d_r T_{f^{*r}}(\tau) = \sum_{r \mid n} d_r T_{f^r}(\tau)\,, \tag{4.6}$$

where d_r is the number of integers $k \in \{1, \ldots, n\}$ with $(k,n) = r$ so that $T_{f^r}(\tau) = T_{f^k}(\tau)$ and likewise for f^*. For $n = p$, prime, we have $d_1 = p - 1$, $d_p = 1$ and (4.6) implies that $T_{f^*}(\tau) = T_f(\tau)$. For n not prime we may identify the singularities of $T_f(\tau)$ and $T_{f^*}(\tau)$ as follows. Consider the modular function $\phi(\tau) = d_1(T_{f^*} - T_f)$. As described above, the behaviour of $\phi(\tau)$ at $\tau = 0$ can only be singular if either $E_0^{f^*} = -1/n$ or $E_0^f = -1/n$ or both where $\phi(-1/\tau) = Aq^{-1/n} + 0 + \ldots$ for $A = N_{f^*}$ or N_f or $N_{f^*} - N_f$ respectively. But from (4.6), $\phi(\tau) = \sum\limits_{r>1} d_r(T_{f^r} - T_{f^{*r}})$ has singular behaviour at $\tau = 0$ determined by $\phi(-1/\tau) = Bq^{-r/n} + \ldots$ which is inconsistent unless $E_0^{f^*} = E_0^f = -1/n$ and $N_{f^*} = N_f$ for all tachyonic sectors. Therefore $\phi(\tau)$ is $\Gamma_0(n)$ invariant without singularities and defines a holomorphic function on the compactification of $H/\Gamma_0(n)$ (a compact Riemann surface). This is impossible unless $\phi(\tau)$ is a constant which must be zero since Thompson series contain no constant term. Therefore $T_{f^*}(\tau) = T_f(\tau)$ and so f and f^* can be identified as members of the same conjugacy class of M (apart from the classes $27A$, $27B$ where possibly f and f^* are in different classes).

We next examine the centraliser $C(f^* \mid M)$ by a similar analysis to that of Sect. 3 and Appendix B. Define Aut($\mathscr{P}_f \mathscr{V}_f^{\natural}$) to be the automorphism group of the OPA for $\mathscr{V}_{\text{orb}}^f$

which maps $\mathscr{P}_f\mathscr{V}_f^{\natural}$ into itself. Then $n.\,\mathrm{Aut}(\mathscr{P}_f\mathscr{V}_f^{\natural}) \subseteq C(f^* \mid M)$, where the extension is the central cyclic group generated by f^*. From (4.3), the vacuum operators $\{\sigma_f^r(z)\}$ of $\mathscr{V}_f^{\natural}$ must form a N_f dimensional representation for $C(f \mid M)/\langle f\rangle$ which defines some extension L_σ so that $\mathrm{Aut}(\mathscr{P}_f\mathscr{V}_f^{\natural}) = L_\sigma(C(f \mid M)/\langle f\rangle)$. Therefore we find that $n.L_\sigma.(C(f|M)/\langle f\rangle) \subseteq C(f^*| M)$. However, this is impossible since f^* and f are in the same conjugacy class of M unless $L_\sigma = 1$ so that the twisted vacuum of $\mathscr{V}_f^{\natural}$ is unique where $N_f = 1$. (For the two classes $27A$, $27B$, the centralisers are of the same order so that again $L_\sigma = 1$).

We have shown that for any $f \in M$, where $\mathscr{V}_{\mathrm{orb}}^{f} \cong \mathscr{V}^{\natural}$, $\mathscr{V}_f^{\natural}$ has vacuum energy $E_0^f > 0$ or $E_0^f = -1/n$ with degeneracy $N_f = 1$. We will now eliminate the possibility of $E_0^f > 0$. If $E_0^{f^k} \geq 0$ for all $k \neq n$, then $T_f(\tau)$ has a unique simple pole at $q = 0$ and is therefore a hauptmodul for $\Gamma_0(n)$. This is only possible for $2 \leq n \leq 10$, $n = 12, 13, 16, 18$ with hauptmodul $T_f(\tau) = 1/\eta_{\bar{a}}(\tau) - a_1$ for the corresponding automorphism $\bar{a} \in Co_0$ in Table 1 with modular group $\Gamma_0(n) = n-$. Then under $S{:}\,\tau \to -1/\tau$ we get $E_0^f = 0$ with $N_f = -a_1 \neq 0$ in contradiction so that $E_0^f = -1/n$ in these cases which includes all the prime ones. For the remaining non-prime cases with some $E_0^{f^k} < 0$, we consider the composition of two orbifoldings of $\mathscr{V}^{\natural}$ which will allow us to determine the location and strength of any singularities of $T_f(\tau)$.

Choose $f \in M$ of non-prime order n (where either $E_0^f > 0$ or $E_0^f = -1/n$) such that for any $f_1 \in M$ of order $n_1 < n$, where $\mathscr{V}_{\mathrm{orb}}^{f_1} \cong \mathscr{V}^{\natural}$ then $E_0^{f_1} = -1/n_1$. This choice includes the automorphism f of least order with $E_0^f > 0$ which we will show cannot exist. With this choice of f, if $\mathscr{V}_{\mathrm{orb}}^{f^r} \cong \mathscr{V}^{\natural}$ for f^r of order $e = n/r$, $\mathscr{V}_{f^r}^{\natural}$ must be tachyonic with $E_0^{f^r} = -1/e$. (We may assume that $r \mid n$ since $\mathscr{V}_{f^r}^{\natural}$ and $\mathscr{V}_{r^{r'}}^{\natural}$ are isomorphic for $(n,r) = (n,r')$ in general). We will show that $\mathscr{V}_{f^e}^{\natural}$ must also be tachyonic with $E_0^{f^e} = -1/r$, where $(e,r) = 1$, i.e. $e \,\|\, n$. This corresponds to the singularities given in (I) of the Vacuum Properties and will also lead to the closure property in (II) once we have shown that $E_0^f = -1/n$. In constructing $\mathscr{V}_{\mathrm{orb}}^{f} \cong \mathscr{V}^{\natural}$ we employ twisted operators which are also involved in constructing $\mathscr{V}_{\mathrm{orb}}^{f^r} \cong \mathscr{V}^{\natural}$. The contribution to $\mathscr{V}_{\mathrm{orb}}^{f}$ from these operators is

$$\mathscr{P}_f(\mathscr{V}^{\natural} \oplus \mathscr{V}_{f^r}^{\natural} \oplus \ldots \oplus \mathscr{V}_{f^{r(e-1)}}^{\natural}) = \frac{1}{r}(1 + f + \ldots + f^{r-1})\mathscr{V}_{\mathrm{orb}}^{f^r} \cong \mathscr{P}_{f'}\mathscr{V}^{\natural}, \tag{4.7}$$

where f' is an automorphism of $\mathscr{V}^{\natural}$ of order r defined by the automorphism f acting on $\mathscr{V}_{\mathrm{orb}}^{f^r}$ (since f^r acts as unity on $\mathscr{V}_{\mathrm{orb}}^{f^r}$). But $\mathscr{P}_{f'}\mathscr{V}^{\natural}$ is the untwisted contribution to the orbifolding of $\mathscr{V}_{\mathrm{orb}}^{f^r}$ with respect to f'. Furthermore, the orbifolding of $\mathscr{V}^{\natural}$ with respect to f is a composition of the orbifolding of $\mathscr{V}^{\natural}$ with respect to f^r and the orbifolding of $\mathscr{V}_{\mathrm{orb}}^{f^r} \cong \mathscr{V}^{\natural}$ with respect to f' as follows

$$\begin{tikzcd} & \mathscr{V}^{\natural} \arrow[dr, "f'"] & \\ \mathscr{V}^{\natural} \arrow[ur, "f^r"] \arrow[rr, "f"] & & \mathscr{V}^{\natural} \end{tikzcd}, \tag{4.8}$$

where the arrows represent an orbifolding with respect to the denoted automorphism. Thus $\mathscr{V}^{f'}_{\rm orb} \cong \mathscr{V}^\natural$ and therefore $\mathscr{V}^\natural_{f'}$ is also tachyonic with $E_0^{f'} = -1/r$ by our choice of f since f' is of order $r < n$. We can check for the consistency of this composition of orbifoldings by considering the Thompson series $T_{f'}(\tau)$ for f' as a trace over $\mathscr{V}^{f^r}_{\rm orb}$. Under $S:\tau \to -1/\tau$ this becomes

$$T_{f'}\left(-\frac{1}{\tau}\right) = \sum_{k=1}^{e} \mathscr{P}_{f^r} \underset{f^{1+rk}}{\square}^{\natural} \tag{4.9}$$

which must have leading behaviour $q^{-1/r} + \ldots$ from (4.1). Therefore, at least one of the twisted sectors contributing to the RHS of (4.9) must be tachyonic with vacuum energy $-1/r$ and f^{1+rk} of order $r < n$. Thus $r(1+rk) = nl$ for some l so that $el - rk = 1$ which implies that $(e, r) = 1$. Therefore, $e \,\|\, n$ (and $r \,\|\, n$) and $\mathscr{V}^\natural_{f^e}$ is tachyonic with vacuum energy $-1/r$ (as is the isomorphic twisted sector $\mathscr{V}^\natural_{f^{el}}$ since $(l, r) = 1$). Thus orbifolding $\mathscr{V}^\natural$ with respect to f^e also reproduces $\mathscr{V}^\natural$. To summarise, for f of order n as chosen, if $\mathscr{V}^{f^r}_{\rm orb} \cong \mathscr{V}^\natural$ (so that $\mathscr{V}^\natural_{f^r}$ is tachyonic), where f^r is of order $e = n/r$ then $e \,\|\, n$ and $\mathscr{V}^\natural_{f^e}$ must also be tachyonic with $\mathscr{V}^{f^e}_{\rm orb} \cong \mathscr{V}^\natural$.

This translates into information about the singularity structure of $T_f(\tau)$ [19]. If we choose the representative form for the Atkin-Lehner (AL) involution $W_e = \begin{pmatrix} e & b \\ n & de \end{pmatrix}$, for $e \neq n$, as in Appendix A. Then $T_f(W_e(\tau)) = f^{-ed} \underset{f^r}{\square}^{\natural}(e\tau) = q^{-1} + 0 + O(q)$ when $\mathscr{V}^{f^r}_{\rm orb} \cong \mathscr{V}^\natural$. Note that the constant term is zero since $\mathscr{V}^\natural_{f^r}$ contains no massless operators. We define $\tau_e = W_e(\infty) = 1/r$ which we call an AL cusp. On the fundamental region $\mathscr{F}_n = H/\Gamma_0(n)$, the singularity at τ_e is then a simple pole since W_e is an automorphism of $\mathscr{F}_n$. In addition, $T_f(\tau)$ also has a simple pole at the AL cusp $\tau_r = 1/e = W_r(\infty)$ since $\mathscr{V}^{f^e}_{\rm orb} \cong \mathscr{V}^\natural$. Thus $T_f(\tau)$ has simple poles with residue 1 at $\tau = \infty$ $(q = 0)$ and possibly at $\tau = 0$ (if $E_0^f = -1/n$) and at the AL cusps τ_e and τ_r.

We next show that $T_f(\tau)$ must always be singular at $\tau = 0$ with $E_0^f = -1/n$. Suppose that $E_0^f > 0$, then under the Fricke involution $W_n : \tau \to -1/n\tau$, τ_e and τ_r are interchanged. Then $\phi(\tau) = T_f(\tau) - T_f(W_n(\tau))$ is a $\Gamma_0(n)$ invariant meromorphic function on $\mathscr{F}_n$ with two simple poles at $\tau = \infty$ $(q = 0)$ and $\tau = 0$. $\phi(\tau)$ also has zeros at τ_e and τ_r since $\phi(W_e(\tau)) = q^{-1} - q^{-1} + 0 + O(q)$, where it is essential that the AL poles have the same strength and residue and that $\mathscr{V}^\natural_{f^r}$ and $\mathscr{V}^\natural_{f^e}$ contain no massless operators. Likewise, $\phi(\tau)$ has zeros at any other such pairs of singular AL cusps. But $\phi(\tau)$ is odd under W_n and therefore also has a zero at the W_n fixed point $i/\sqrt{n}$. Thus ϕ has two simple poles and at least three zeros on the compactification of $\mathscr{F}_n$ which is a compact Riemann surface. But every meromorphic function on a compact Riemann surface has an equal number of zeros as poles. Therefore, there is a contradiction and hence $E_0^f = -1/n$.

We have now derived condition (I) of the Vacuum Properties for f. In addition, a restricted version of the AL closure condition (II) has also been demonstrated. Namely, if $\mathscr{V}^\natural_f$ and $\mathscr{V}^\natural_{f^r}$ are tachyonic (where f^r is of order $e \,\|\, n$), then so is $\mathscr{V}^\natural_{f^e}$ where f^e is of order $r = ne/e^2$. We can use this to generate the general AL closure property as follows. Suppose that $\mathscr{V}^\natural_{f_1}$ and $\mathscr{V}^\natural_{f_2}$ are both tachyonic with $f_1 = f^{r_1}$ and $f_2 = f^{r_2}$ with $r_1 \neq n/r_2$, where f_i is of order e_i where $e_i \| n$. (We can take $r_i \,|\, n$, as before, since $\mathscr{V}^\natural_{f^r}$ and $\mathscr{V}^\natural_{f^{r'}}$ are isomorphic for $(n,r) = (n,r')$). Then the sectors twisted by f^{e_i} of order r_i are also tachyonic. By interchanging e_i with r_i if necessary we can assume that $(e_1, e_2) = 1$. This is easily shown by observing that the order n of every element of M has at most 3 distinct prime divisors ($n < 2.3.5.7 = 210$). Then $e_3 \,\|\, n$ for $e_3 = e_1 e_2$ and $r_3 = r_1 r_2/(r_1, r_2)^2 = n/e_3$ with $r_1 = r_3 e_2$ and $r_2 = r_3 e_1$. Consider $g = f^{r_3}$ of order e_3. Then $g^{e_1} = f^{r_2}$ and $g^{e_2} = f^{r_1}$ are of order e_2 and e_1, respectively, so that the corresponding twisted sectors are tachyonic. Therefore, by taking the composition of orbifoldings with respect to g^{e_i}, as in (4.8), we find that $\mathscr{V}^\natural_g$ is also tachyonic with $g = f^{r_3}$ of order $e_3 = e_1 e_2$. As before, the sector twisted by f^{e_3} of order $r_3 = r_1 r_2/(r_1, r_2)^2$ must also then be tachyonic. Thus the general AL closure condition (II) is derived.

We have now demonstrated that the genus zero property for Thompson series can be derived from (4.5) assuming that $\mathscr{V}^\natural$ is unique and so we have:

Monstrous Moonshine is Equivalent to the Unique Orbifold Partner Property. Assume that the FLM *uniqueness conjecture holds. Then $T_g(\tau)$ for $g \in M$ is a hauptmodul for a genus zero modular group $\Gamma_0(n) + e_1, e_2, \ldots$ if and only if the only meromorphic orbifoldings of $\mathscr{V}^\natural$ with respect to g are $\mathscr{V}^\Lambda$ and $\mathscr{V}^\natural$.*

We note that we may also understand the Vacuum Properties already found for the non-Fricke elements a^* dual to a in a similar fashion to this derivation for the Fricke elements. Suppose that $f = a^{*r}$ of order $e = n/r$ is Fricke so that $\mathscr{V}^f_{\text{orb}} \cong \mathscr{V}^\natural$. We can then deduce that $e \,\|\, n$ and that a^{*e} is non-Fricke as follows. The orbifolding of $\mathscr{V}^\natural$ with respect to a^* (which gives $\mathscr{V}^\Lambda$) is the composition of the orbifolding of $\mathscr{V}^\natural$ with respect to f and the orbifolding of $\mathscr{V}^f_{\text{orb}} \cong \mathscr{V}^\natural$ with respect to b^* of order r, where b^* is the action of a^* on $\mathscr{V}^{f^r}_{\text{orb}}$. Thus b^* is dual to b, one of the 38 automorphisms of $\mathscr{V}^\Lambda$ discussed in Sect. 3. It is straightforward to then see that $b = a^e$ (lifted from $\bar{a}^e$) has the correct action on $\mathscr{V}^{f^r}_{\text{orb}}$ to be dual to b^*. If we examine the 38 automorphisms listed in Table 1, we find that $\bar{a}^e$ is contained in Table 1 if and only if $e \,\|\, n$ and $\eta_{\bar{a}}(\tau)$ is invariant under the AL involution W_e (but is inverted by W_r). In fact, in each such case this follows from the symmetry properties of the characteristic equation parameters, where $a_k = -a_{n/k} = a_{ek_r/k_e} = -a_{rk_e/k_r}$ (with $k_e = (k,e)$ and $k_r = (k,r)$) so that $\bar{b} = \bar{a}^e$ has parameters $b_k = -b_{r/k}$. Similarly, the closure condition (II) follows directly from these parameter relationships.

4.3 Moonshine for $n\,|\,h + e_1, e_2, \ldots, h \neq 1$. Let us now consider the Thompson series for the classes of M which cannot be employed to construct a meromorphic modular invariant orbifold due to a global phase anomaly. These classes consist of the 13 non-Fricke classes of Sect. 3 and 38 Fricke classes. The twisted sector $\mathscr{V}^\natural_g$ for the non-Fricke classes and some of the Fricke classes can be constructed since they belong to the centraliser $C(i\,|\,M) = 2^{1+24}.Co_1$ as described below in Sect. 4.4. We find that $E_0^g = 1/nh$ for the non-Fricke classes and $E_0^g = -1/nh$ for the Fricke classes where $h\,|\,n$. We will assume that this latter property is also correct for the remaining Fricke classes. The integer $h \neq 1$ parameterises the global phase anomaly present in these cases where $T^n : 1\underset{g}{\Box}^{\natural} \to e^{\pm 2\pi i/h} 1\underset{g}{\Box}^{\natural}$. In Sect. 3 we considered the 13 Leech lattice automorphisms with a global phase anomaly, where we found an isomorphism between $\mathscr{H}_a \otimes \ldots \otimes \mathscr{H}_a$ and $\mathscr{H}_{a^h}$ in (3.13). A similar isomorphism is also expected here between the twisted Hilbert spaces $\mathscr{H}_g^\natural$ and $\mathscr{H}_{g^h}^\natural$ as follows [19]. Let $\tilde{\psi}_i(z)$ create a twisted state in $\mathscr{H}_g^\natural$ by acting on the twisted vacuum states $\{|\sigma_g^l\rangle\}$. Then $\tilde{\psi}_{i_1\ldots i_h}(z) = \tilde{\psi}_{i_1}(z^h)\otimes\ldots\otimes\tilde{\psi}_{i_h}(z^h)$ which acts on $|\Sigma_{g^h}^{l_1\ldots l_h}\rangle = |\sigma_g^{l_1}\rangle\otimes\ldots\otimes|\sigma_g^{l_h}\rangle$ obeys the monodromy condition (4.2) for g^h of order n/h. $\tilde{\Psi}(z)$ creates a state in $\mathscr{H}_{g^h}^\natural$ but is not a primary conformal field. For the non-Fricke classes the states $|\Sigma_{g^h}\rangle$ are of energy h/n whereas for the Fricke classes, $|\Sigma_{g^h}\rangle$ is unique and is of energy $-h/n$ and reproduces the vacuum of $\mathscr{H}_{g^h}^\natural$. Thus as before, the global phase anomaly disappears by taking such a tensor product. Thus an identification can be made between the non-massless states of $\mathscr{H}_{g^h}^\natural$ and $\mathscr{H}_g^\natural\otimes\ldots\otimes\mathscr{H}_g^\natural$. For the non-Fricke classes, $\mathscr{H}_{g^h}^\natural$ always contains $N_{g^h} > 0$ massless states whereas the energies of all the states of $\mathscr{H}_g^\natural\otimes\ldots\otimes\mathscr{H}_g^\natural$ are positive. On the other hand, for the Fricke classes $\mathscr{H}_{g^h}^\natural$ contains no massless states but $\mathscr{H}_g^\natural\otimes\ldots\otimes\mathscr{H}_g^\natural$ contains hN_1 massless states where N_1 is the number of operators of $\mathscr{H}_g^\natural$ with first excited energy level $-1/nh + 1/n$. Therefore the partition functions are expected to be related as follows:

$$\left[1\underset{g}{\square}^\natural(h\tau)\right]^h = 1\underset{g^h}{\square}^\natural(\tau) + C\,, \tag{4.10}$$

where $C = -N_{g^h}$ for the non-Fricke classes and $C = hN_1$ for the Fricke classes. In terms of the Thompson series this is the harmonic formula of Conway and Norton [13]

$$[T_g(\tau/h)]^h = T_{g^h}(\tau) + C\,. \tag{4.11}$$

This relationship implies that $T_g(\tau)$ is $\Gamma_0(n\,|\,h) + e_1, e_2, \ldots$ invariant up to h roots of unity. We also know that $1\underset{g}{\square}^\natural(\tau)$ is T^{nh} invariant from which we may show that $T_g(\tau)$ is $\Gamma_0(N)$ invariant with $N = nh$. Thus $\Gamma_0(n\,|\,h)$ must be in the normaliser of $\Gamma_0(N)$ and hence $h\,|\,24$ from Appendix A. The invariance group Γ_g for $T_g(\tau)$ of index h in $\Gamma_0(n\,|\,h) + e_1, e_2, \ldots$ can then be shown to be of genus zero with hauptmodul $T_g(\tau)$ because the invariance group $\Gamma_0\left(\frac{n}{h}\right) + e_1, e_2, \ldots$ of $T_{g^h}(\tau)$ is of genus zero [19].

4.4. Twisted Operators for $c \in C(i\,|\,M)$. We will now discuss the construction of the twisted sector $\mathscr{V}_c^\natural$ for $c \in C(i\,|\,M)$, where c is lifted from a Leech lattice automorphism $\bar{c} \in Co_0$ and is therefore geometrical in origin. Because c does not interchange the sectors $\mathscr{P}_r\mathscr{V}^\Lambda$ and $\mathscr{P}_r\mathscr{V}_r$ in the original FLM construction, the Thompson series for c can be explicitly computed [1, 2, 3] to be

$$\begin{aligned} T_c(\tau) &= c\mathscr{P}_r\underset{1}{\square} + c\mathscr{P}_r\underset{r}{\square} \\ &= \frac{1}{2}\left\{\frac{\Theta_{\Lambda_{\bar{c}}}(\tau)}{\eta_{\bar{c}}(\tau)} + \frac{\Theta_{\Lambda_{-\bar{c}}}(\tau)}{\eta_{-\bar{c}}(\tau)} + \mathrm{Tr}(c_T)\,\frac{\eta_{\bar{c}}(\tau)}{\eta_{\bar{c}}(\tau/2)} - \mathrm{Tr}(c_T)\,\frac{\eta_{-\bar{c}}(\tau)}{\eta_{-\bar{c}}(\tau/2)}\right\}, \end{aligned} \tag{4.12}$$

where $\Theta_{\Lambda_{\pm\bar{c}}}$ is the theta function for the sublattice $\Lambda_{\pm\bar{c}}$ of Λ invariant under $\pm\bar{c}$ and $\eta_{\pm\bar{c}}$ is the eta function as in (3.5a). The lifting of $\bar{c}$ to an automorphism c of $\mathscr{V}^\Lambda$ is chosen so that $cc(\beta)c^{-1} = c(\beta)$ for all $\beta \in \Lambda_{\bar{c}}$ (see (3.1a)) and similarly for rc lifted from $-\bar{c}$ (where r and c commute). c_T is the action of the lifting of $\bar{c}$ on the vacuum of $\mathscr{V}_r$. Given the usual modular transformation properties for the traces of (4.12), $T_c(\tau)$ is automatically $\Gamma_0(m)$ invariant (up to possible phases) where m is the order of $\pm\bar{c}$ in Co_1. We also find that under $S:\tau \to -1/\tau$,

$$\begin{aligned} T_c(-1/\tau) &= \frac{1}{2}\left\{1\underset{c}{\square} + r\underset{c}{\square} + 1\underset{rc}{\square} + r\underset{rc}{\square}\right\} \\ &= \frac{1}{2}\left\{\frac{D^{1/2}}{V_{\bar{c}}}\,\frac{\Theta_{\Lambda^*_{\bar{c}}}(\tau)}{\eta^*_{\bar{c}}(\tau)} + \frac{\mathrm{Tr}(c_T)}{2^{d_+/2}}\,\frac{\eta^*_{\bar{c}}(\tau)}{\eta^*_{\bar{c}}(2\tau)}\right. \\ &\quad \left. + \frac{D^{1/2}_{-\bar{c}}}{V_{-c}}\,\frac{\Theta_{\Lambda^*_{-\bar{c}}}(\tau)}{\eta^*_{-\bar{c}}(\tau)} - \frac{\mathrm{Tr}(-c_T)}{2^{d_-/2}}\,\frac{\eta^*_{-\bar{c}}(\tau)}{\eta^*_{-\bar{c}}(2\tau)}\right\}, \end{aligned} \tag{4.13}$$

where $\Theta_{\Lambda^*_{\pm\bar{c}}}(\tau)$, $\eta^*_{\pm\bar{c}}(\tau)$ and $D_{\pm\bar{c}}$ are defined as (3.5) and $V_{\pm\bar{c}}$ is the volume of $\Lambda_{\pm\bar{c}}$. $d_\pm = \sum_k c_k^\pm$ determines the number of unit eigenvalues of $\pm\bar{c}$ with characteristic equation parameters $\{c_k^\pm\}$ as in (3.2). From (4.1), we may therefore define the twisted sector for each such $c \in C(i \mid M)$ to be

$$\mathscr{V}_c^\natural = \mathscr{P}_r\mathscr{V}_c \oplus \mathscr{P}_r\mathscr{V}_{rc}, \tag{4.14}$$

where $\mathscr{V}_c$ and $\mathscr{V}_{rc}$ are the twisted sectors constructed in the standard way from the Λ compactified string as described in Sect. 3.4 and Appendix B [30, 26, 4, 34], where $\tilde{X}(e^{2\pi i}z) = \pm(\bar{c})^{-1}\tilde{X}(e^{2\pi i}z) + 2\pi\beta$. Then $\chi_c \in \mathscr{V}_c$ and $\chi_{rc} \in \mathscr{V}_{rc}$ obey the monodromy conditions $\chi_c(e^{2\pi i}z) = c\chi_c(z)c^{-1}$ and $\chi_{rc}(e^{2\pi i}z) = rc\chi_{rc}(z)\,(rc)^{-1}$ as in (3.12). For $\psi_r \in \mathscr{V}_r$ we expect the (schematic) OPAs $\psi_r\chi_c \sim \chi_{rc}$ and $\psi_r\chi_{rc} \sim \chi_c$ to hold together with the usual OPAs of (2.5), (2.15) and (3.14). Since r and c commute, r preserves these OPAs and hence the projection with respect to $\mathscr{P}_r$ can be taken. Then for $\psi \in \mathscr{V}^\natural = \mathscr{P}_r(\mathscr{V}^\Lambda \oplus \mathscr{V}_r)$, the monodromy conditions and OPA (4.2) and (4.3) follow where $\{\sigma_c\}$ denotes the vacuum operators for $\mathscr{P}_r(\mathscr{V}_c \oplus \mathscr{V}_{rc})$. Thus $\mathscr{V}_c^\natural$ given in (4.14) satisfies the defining relations for the c twisted sector.

We may check for the other properties satisfied by $\mathscr{V}_c^\natural$ (particularly when c is a Fricke element of M) which lead to Thompson series which are hauptmoduls as described in Sects. 4.2 and 4.3. In [36] a survey is presented of the modular functions $c\underset{1}{\square} = \Theta_{\Lambda_{\bar{c}}}/\eta_{\bar{c}} = q^{-1} + c_1 + \dots$ for all $\bar{c} \in Co_0$. It is shown that $\Theta_{\Lambda_{\bar{c}}}/\eta_{\bar{c}}$ is a hauptmodul for a genus zero fixing group $n|\,h + e_1, e_2 \dots$ for all but 15 classes of Co_0 (thereby falsefying a conjecture of Conway and Norton [13]). We will return to these anomalous classes below. For the remaining classes, we may describe some general properties of the vacuum of $\mathscr{V}_c$, similar to the vacuum properties of $\mathscr{V}_g^\natural$ above. Thus $\Theta_{\Lambda_{\bar{c}}}/\eta_{\bar{c}}$ is Fricke invariant under $\tau \to -1/nh\tau$ if and only if the vacuum energy of $\mathscr{V}_c$ obeys $E_0^c = -1/nh$ and the vacuum degeneracy $N_c = D_{\bar{c}}^{1/2}/V_{\bar{c}} = 1$. (We will call the corresponding class of Co_0 a Fricke class). Otherwise, $E_0^c \geq 0$ and the vacuum may be degenerate. Likewise, the other vacuum properties of Sect. 4.1 must hold.

For all the Fricke classes, the characteristic equation parameters c_k are observed to obey the symmetry condition $c_k = c_{nh/k}$, where $h \mid k$ for all $c_k \neq 0$ [36]. Therefore $D_{\bar{c}} = (nh)^d$, $\eta^*_{\bar{c}}(\tau) = \eta_{\bar{c}}(\tau/nh)$ and hence $E_0^c = -1/nh$. Similarly, from (4.13) we find that since $N_c = 1$, $V_{\bar{c}} = (nh)^{d/2}$ and $\Theta_{\Lambda^*_{\bar{c}}}(\tau) = \Theta_{\Lambda_{\bar{c}}}(\tau/nh)$, where $\beta^2 \in 2hZ$, $\beta^2 \geq 4$ for $\beta \in \Lambda_{\bar{c}} \subset \Lambda$. Thus for $h = 1$, $\beta^{*2} \geq 4/n$ whereas for $h \neq 1$, $\beta^{*2} \geq 2/n$ for all $\beta^* \in \Lambda^*_{\bar{c}}$. Furthermore we can observe from [37] that $\Lambda_{\bar{c}} \equiv \sqrt{nh}\Lambda_{\bar{c}^*}$ in many such cases (e.g. for $\bar{c} = 1^4 8^4/2^2 4^2$ of order $n = 8$, $\Lambda_{\bar{c}} = \sqrt{2}D_4$ and

$\Lambda_{\bar{c}}^* = D_4^*/\sqrt{2} \equiv D_4/2$, after a $\pi/4$ rotation). This non-trivial property for $\Lambda_{\bar{c}}$ is very likely to be true for all such Fricke automorphisms of the Conway group.

From (4.13), the uniqueness of the c twisted vacuum $|\sigma_c\rangle$ for the Fricke classes implies that $\mathrm{Tr}(c_T) = \varepsilon_r 2^{d/2}$, where $r|\sigma_c\rangle = \varepsilon_r|\sigma_c\rangle$ with $\varepsilon_r = \pm 1$. For $h = 1$, when $E_0^c = -1/n$, the first excited (massless) states of this sector are given by $|\psi_c^i\rangle = \tilde{\alpha}_{-1/n}^i|\sigma_c\rangle = \lim_{z\to 0} z^{1-1/n}\partial_z \tilde{X}^i(z)\,|\sigma_c\rangle$ for $i = 1,\ldots,c_n$, where $\partial_z \tilde{X}^i(z)$ is an ω^{-1} eigenvector of $\bar{c}$ which implies $r|\psi_c^i\rangle = -\varepsilon_r|\psi_c^i\rangle$. Since $\beta^{*2} \geq 4/n$, no massless states are associated with the dual lattice $\Lambda_{\bar{c}}^*$. Hence, for any Fricke class $\bar{c} \in Co_0$ with $h = 1$, we have either $\mathscr{P}_r \underset{c}{\square} = q^{-1/n} + 0 + O(q^{1/n})$ for $\varepsilon_r = 1$ or $\mathscr{P}_r \underset{c}{\square} = c_1 + O(q^{1/n})$ for $\varepsilon_r = -1$. For the Fricke classes with $h \neq 1$, the first excited states of $\mathscr{V}_c^{\cdot}$ with energy $-1/nh + 1/n$ are given by $|\psi_c^i\rangle$ above together with states $|\beta^*\rangle$ created by $e^{i\langle\beta,\tilde{X}(0)\rangle}$ for $\beta^{*2} = 2/n$. Thus for $h = 1$, $\mathscr{P}_r\mathscr{V}_c^{\cdot}$ contains either a unique vacuum with energy $E_0^c = -1/n$ but no massless operators ($\varepsilon_r = 1$) or else has a massless vacuum ($\varepsilon_r = -1$). Similarly, for $h \neq 1$, $\mathscr{P}_r\mathscr{V}_c^{\cdot}$ contains either a unique vacuum with $E_0^c = -1/nh$ with first excited operators of energy $-1/nh + 1/n$ or else has a vacuum of energy $-1/nh + 1/n$.

We may use these observations to describe the corresponding properties of $\mathscr{V}_c^{\natural}$ defined in (4.14). Consider $\bar{c}$ any Fricke element of Co_0 of order n with $h = 1$. If n is odd then $-\bar{c}$ is of order $2n$ and $\mathscr{V}_{rc}^{\cdot}$ has vacuum energy $E_0^{rc} = 1/2n > 0$. If n is even then $-\bar{c}$ is of order n or $n/2$ and we can observe from [36] that $E_0^{rc} \geq 0$ in all cases. For $-\bar{c}$ of order n with $E_0^{rc} = 0$, one can check from (4.13) and [36] that $r\,|\,\sigma_{rc}\rangle = -\varepsilon_r\,|\,\sigma_{rc}\rangle$, with ε_r as above, so that $\mathscr{P}_r\mathscr{V}_{rc}^{\cdot}$ contains no massless operators for $\varepsilon_r = 1$. If $-\bar{c}$ is of order $n/2$ then $\bar{c}^{n/2} = \bar{r}$ and $r = c^{n/2}$ so that $\varepsilon_r = -1$ from (4.13) (by considering invariance under $\tau \to \tau + n/2$). Thus, for any Fricke element $\bar{c} \in Co_0$ with $h = 1$, $\mathscr{V}_c^{\natural}$ contains either a unique vacuum of energy $-1/n$ and no massless operators so that $c \in M$ is Fricke or $\mathscr{V}_c^{\natural}$ contains a massless vacuum and $c \in M$ is non-Fricke. One can similarly show for a Fricke class $\bar{c} \in Co_0$ with $h \neq 1$ that $\mathscr{V}_c^{\natural}$ either contains a unique vacuum with energy $-1/nh$ and first excited operators with energy $-1/nh + 1/n$ (c is Fricke in M) or else has a vacuum of energy $-1/nh + 1/n$ (c is non-Fricke in M). Likewise, if $\bar{c}$ and $-\bar{c}$ are both non-Fricke then c is non-Fricke in M and $\mathscr{V}_c^{\natural}$ has the required properties. Thus $\mathscr{V}_c^{\natural}$ defined in (4.14) possesses all the properties for a Monster group twisted sector as described in Sects. 4.2 and 4.3.

Let us now discuss the 15 anomalous automorphisms $\{\bar{c}\}$ mentioned earlier for which $c\,\underset{1}{\square} = \Theta_{\Lambda_{\bar{c}}}/\eta_{\bar{c}}$ is not a hauptmodul but is fixed by a genus zero modular group [36]. These classes fall into 5 families of the form $\{\bar{c}_1, \bar{c}_2, \bar{c}_3\}$ with each $\bar{c}_i$ of the same order $n = 6, 10, 12, 18$ or 30. For each such $\bar{c}$, part (I) of the vacuum properties Sect. 4.1 is satisfied but the Atkin-Lehner closure condition (II) fails and so $\Theta_{\bar{c}}/\eta_{\bar{c}}$ is not a hauptmodul. For example, for $n = 6$, $\{\bar{c}_1, \bar{c}_2, \bar{c}_3\}$ have Frame shapes $\{1^4 2.6^5/3^4,\ 2^5 3^4 6/1^4,\ 1^5 3.6^4/2^4\}$ (where $\bar{c}_1 = -\bar{c}_2$). Then $\Theta_{\bar{c}_1}/\eta_{\bar{c}_1}$ has simple poles with residue 1 at the cusps $\tau = \infty$, 0 and the AL cusp $\tau_2 = 1/3$ but not at the AL cusp $\tau_3 = 1/2$. Likewise, for $\bar{c}_2$ and $\bar{c}_3$, the poles occur at $\{\infty, 1/2, 1/3\}$ and $\{\infty, 0, 1/2\}$. The other anomalous families have very similar properties [36]. Despite this behaviour, one can repeat the analysis above to show that $\mathscr{V}_c^{\natural}$ of (4.14) possesses all the required properties given in Sect. 4.2.

We will end this section with some remarks concerning the reorbifolding of $\mathscr{V}^\natural$ with respect to Fricke elements of M. For a Fricke element $c \in C(i\,|\,M)$ of order m with $\mathscr{V}_c^\natural$ as in (4.14), then given (4.5), we find $\mathscr{V}^\natural = \mathscr{V}_{\mathrm{orb}}^c = \mathscr{P}_c(\mathscr{V}^\natural \oplus \mathscr{V}_c^\natural \oplus \ldots \oplus \mathscr{V}_{c^{m-1}}^\natural)$ is just a $Z_2 \times Z_m$ orbifolding of $\mathscr{V}^\Lambda$ with respect to the abelian group generated by r and c. We can similarly expect that the observations of this subsection can be generalised to the other assumed constructions of $\mathscr{V}^\natural$ given in Sect. 3 based on the 38 automorphisms $\bar{a}$ of Table 1. Thus for $c \in C(a^*\,|\,M)$, we can define $\mathscr{V}_c^\natural = \mathscr{P}_a\mathscr{V}_c \oplus \ldots \oplus \mathscr{P}_a\mathscr{V}_{ca^{n-1}}$, where a and c commute, which satisfies the monodromy conditions and OPA of (4.2) and (4.3). Then reorbifolding $\mathscr{V}_{\mathrm{orb}}^c$ with respect to an element of $C(a^*\,|\,M)$ is equivalent to a $Z_n \times Z_m$ orbifolding of $\mathscr{V}^\Lambda$ with respect to the abelian group generated by a and c. Thus, assuming (4.5) so that $\mathscr{V}_{\mathrm{orb}}^c = \mathscr{V}^\natural$ for a Fricke element $c \in C(a^*\,|\,M)$, we can, in principle, provide a large family of $Z_n \times Z_m$ orbifold constructions of $\mathscr{V}^\natural$ from $\mathscr{V}^\Lambda$.

5. Concluding Remarks

We conclude with a number of observations concerning various open questions and some generalisations of the constructions considered above. We begin with a few remarks about Norton's Generalised Moonshine [38] which concerns Moonshine for modular functions associated with centraliser groups of elements in the Monster. In [22] it was suggested that these correspond to orbifold traces of the form $g_1\underset{g_2}{\square}^\natural$ for $g_1 \in C(g_2\,|\,M)$. Given the usual modular transformation properties for such traces, then the structure of the vacuum of the Monster twisted sectors $\mathscr{V}_{g_2}^\natural$ described here should be sufficient to show that each such trace is a hauptmodul. A general discussion of this will appear elsewhere [39] but we make three brief observations here. Firstly, for g_2 a non-Fricke element, the vacuum of $\mathscr{V}_{g_2}^\natural$ is degenerate in most cases so that each g_1 is actually an element of an extension of $C(g_2\,|\,M)$ in these cases, as observed by Norton [40]. For the remaining non-Fricke and all the Fricke classes, no such extension of the centraliser is required. Secondly, $g_1\underset{g_2}{\square}^\natural$ can be easily shown to be a hauptmodul for g_1 and g_2 of relatively prime order n_1 and n_2 with associated modular groups $n_1 + e_1, e_2, \ldots$ and $n_2 + e_1', e_2', \ldots$, i.e. $h_1 = h_2 = 1$, where the corresponding twisted sectors are global phase anomaly free. Since $(n_1, n_2) = 1$ we have $n_1 b + n_2 a = 1$ for some a, b. Define $g = g_1^a g_2^b$ of order $n = n_1 n_2$ so that $g_1 = g^{n_2}$ and $g_2 = g^{n_1}$. Then under a modular transformation with respect to $\gamma = \begin{pmatrix} a & b \\ -n_1 & n_2 \end{pmatrix}$ we find

$$T_g(\tau) = g\underset{1}{\square}^\natural \to g^{n_2}\underset{g^{n_1}}{\square}^\natural = g_1\underset{g_2}{\square}^\natural . \tag{5.1}$$

Therefore $g_1\underset{g_2}{\square}^\natural$ is a hauptmodul for $\Gamma_g = n + \hat{e}_1, \hat{e}_2, \ldots$, where $e_1, e_1', e_2, e_2' \in \{\hat{e}_1, \hat{e}_2, \ldots\}$ and if $e_i = n_1$ and $e_j' = n_2$ for some i, j then $n \in \{\hat{e}_1, \hat{e}_2, \ldots\}$ also, i.e. g is Fricke if both g_1 and g_2 are Fricke. This property is observed for all the appropriate modular functions associated with the centralisers of the limited number of elements of M discussed in [41]. Our last observation concerns Moonshine for $C(g_2\,|\,M)$, where $T_{g_2}(\tau)$ has modular invariance group $n_2\,|\,h + e_1, e_2, \ldots$ with $h \neq 1$. From Sect. 4.3 we expect that the following harmonic formula should hold for each g_1:

$$\Big[g_1 \underset{g_2}{\Box}^{\natural}(h\tau)\Big]^h = g_1 \underset{g_2^h}{\Box}^{\natural}(\tau) + C\,, \tag{5.2}$$

where C is a constant. For the case $g_2 = 3|\,3$, this formula can be verified [41].

The use of non-meromorphic OPAs has been central in our discussion. Such algebras were employed both in defining the properties of twisted operators and in considering reorbifoldings. From this point of view, the two meromorphic CFTs which are orbifold partners are embedded in a larger set of operators $\mathscr{V}'$ obeying a non-meromorphic OPA. However, a rigorous construction of such a non-meromorphic OPA has yet to be given even in the simplest Z_2 case. Another interesting question is to ask what form does the automorphism group for $\mathscr{V}'$ take? This has not even been determined in the original FLM Z_2 construction with OPAs (2.5) and (2.15). We know that in this case this group contains the original reflection involution r together with the dual involution i and other extensions of elements of the Conway group Co_0. Furthermore, the triality symmetry [3, 29] interchanging the untwisted and twisted sectors may also still hold. Given this, then we can speculate that the automorphism group for $\mathscr{V}'$ may be the "Bimonster" or wreath square of the Monster [42]. Similarly, for the other orbifold constructions, the automorphism group for $\mathscr{V}'$ may provide other enlargements of the Monster which would be of obvious interest. Finally, apart from these more general considerations, the Monster Fricke element twisted sectors not related to Leech lattice automorphisms have also yet to be constructed explicitly.

Appendix A. Modular Groups in Monstrous Moonshine

In this appendix we describe the modular groups relevant to the Moonshine properties of Thompson series described by Conway and Norton [13].

$\Gamma_0(N)$: The group of matrices contained in the full modular group of the form

$$\begin{pmatrix} a & b \\ cN & d \end{pmatrix}, \qquad \det = 1\,, \tag{A.1}$$

where $a, b, c, d \in Z$.

The normaliser $\mathscr{N}(\Gamma_0(N)) = \{\varrho \in PSL(2,R) \,|\, \varrho\Gamma_0(N)\varrho^{-1} = \Gamma_0(N)\}$, is also required to describe Monstrous Moonshine. Let h be an integer where $h^2 \,|\, N$ (h^2 divides N) and let $N = nh$. Then we define the following sets of matrices.

$\Gamma_0(n\,|\,h)$: The group of matrices of the form

$$\begin{pmatrix} a & \frac{b}{h} \\ cn & d \end{pmatrix}, \qquad \det = 1\,, \tag{A.2}$$

where $a, b, c, d \in Z$. For h the largest divisor of 24 for which $h^2 \,|\, N$, $\Gamma_0(n\,|\,h)$ forms a subgroup of $\mathscr{N}(\Gamma_0(N))$. For $h = 1$, $\Gamma_0(n\,|\,h) = \Gamma_0(n)$.

W_e: The set of matrices for a given integer e

$$\begin{pmatrix} ae & b \\ cN & de \end{pmatrix}, \qquad \det = e, \qquad e \,\|\, N\,, \tag{A.3}$$

where $a, b, c, d \in Z$. $e \| N$ denotes the property that $e \mid N$ and the greatest common divisor $(e, N/e) = 1$. The set W_e forms a single coset of $\Gamma_0(N)$ in $\mathcal{N}(\Gamma_0(N))$ with $W_1 = \Gamma_0(N)$. It is straightforward to show that (up to scale factors)

$$\begin{aligned} W_e^2 &= 1 \bmod(\Gamma_0(N)), \\ W_{e_1} W_{e_2} &= W_{e_2} W_{e_1} = W_{e_3} \bmod(\Gamma_0(N)), \qquad e_3 = \frac{e_1 e_2}{(e_1, e_2)}. \end{aligned} \tag{A.4}$$

The coset W_e is referred to as an Atkin-Lehner (AL) involution for $\Gamma_0(N)$. The simplest example is the Fricke involution W_N with coset representative $\begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}$ which generates $\tau \to -1/N\tau$ and interchanges the cusp points at $\tau = \infty$ and $\tau = 0$. For $e \neq n$ we can choose the coset representative $\begin{pmatrix} e & b \\ N & de \end{pmatrix}$, where $ed - bN/e = 1$ which interchanges the cusp points at $\tau = \infty$ and $\tau = e/N$.

w_e: The set of matrices for a given integer e of the form

$$\begin{pmatrix} ae & \dfrac{b}{h} \\ cn & de \end{pmatrix}, \qquad \det = e, \qquad e \left\| \frac{n}{h} \right. \tag{A.5}$$

where $a, b, c, d \in Z$. The set w_e is called an Atkin-Lehner (AL) involution for $\Gamma_0(n \mid h)$. The properties (A.4) are similarly obeyed by w_e with $\Gamma_0(N)$ replaced by $\Gamma_0(n \mid h)$.

$\mathcal{N}(\Gamma_0(N))$: The Normalizer of $\Gamma_0(N)$ in $PSL(2, R)$ is constructed by adjoining to $\Gamma_0(n \mid h)$ all its AL involutions $w_{e_1}, w_{e_2}, \ldots$, where h is the largest divisor of 24 with $h^2 \mid N$ and $N = nh$.

$\Gamma_0(n \mid h) + e_1, e_2, \ldots$: This denotes the group obtained by adjoining to $\Gamma_0(n \mid h)$ a particular subset of AL involutions $w_{e_1}, w_{e_2}, \ldots$ and forms a subgroup of $\mathcal{N}(\Gamma_0(N))$.

Appendix B. Automorphism Groups for Twisted Sectors

In this appendix we will derive the centraliser formula (3.19) by describing the automorphism group which preserves the OPA of $\mathscr{V}_{\text{orb}}^{a}$ where no mixing between the various sectors $\mathscr{P}_a \mathscr{V}_b$ is considered where $b = a^r$ is lifted from $\bar{b} = \bar{a}^r$ of order $m = n/r'$ with $r' = (n, r)$. In general, $\bar{b}$ may have unit eigenvalues (for $r' \neq 1$) so that Λ contains a $\bar{b}$ invariant sublattice $\Lambda_{\bar{b}}$ which has dual lattice $\Lambda_{\bar{b}}^* = \Lambda_{\|} \equiv \mathscr{P}_{\bar{b}} \Lambda$. Likewise, we define $\Lambda_{\bar{b}}^T$ to be the sublattice of Λ orthogonal to $\Lambda_{\bar{b}}$, where the dual lattice is $\Lambda_{\bar{b}}^{T*} = \Lambda_T \equiv (1 - \mathscr{P}_{\bar{b}})\Lambda$. It is then easy to show that $\Lambda_{\|}/\Lambda_{\bar{b}} \cong \Lambda_T/\Lambda_{\bar{b}}^T$ so that the volume of $\Lambda_{\bar{b}}$ is given by $V_{\bar{b}} = |\Lambda_{\|}/\Lambda_{\bar{b}}|^{1/2} = |\Lambda_T/\Lambda_{\bar{b}}^T|^{1/2}$.

The b twisted states are constructed from a set of vertex operators $\tilde{\mathscr{V}}_\Lambda$ which form a representation of the original untwisted OPA (2.5) with a non-meromorphic OPA. These operators act on a b twisted vacuum which from (3.4) we expect to have degeneracy $D_{\bar{b}}^{1/2}/V_{\bar{b}}$. The construction of $\tilde{\mathscr{V}}^{\Lambda}$ follows from considering a string with twisted boundary condition $\tilde{X}(e^{2\pi i} z) = \bar{b}^{-1}\tilde{X}(z) + 2\pi\beta$, where $\beta \in \Lambda$ [30, 26, 34] with a mode expansion similar to (3.7). The corresponding states are graded by $L_0 = \sum_m \tilde{\alpha}_m^i \tilde{\alpha}_{-m}^i + \frac{1}{2} p_{\|}^2 + E_0^b$, where $p_{\|}$ has eigenvalues in $\Lambda_{\|}$ and E_0^b is the vacuum energy given in (3.5e) which obeys $mE_0^b = 0 \bmod 1$. As before, cocycle factors $\{c_T(\alpha)\}$ are required for a local OPA. These are defined as follows. Consider the central extension $\hat{\Lambda}$ of Λ by $\langle(-1)^m\varrho\rangle$ (where $\varrho = \omega^r$, $\omega = e^{2\pi i/n}$) given by the following commutator [30]:

$$c(\alpha)c(\beta)c(\alpha)^{-1}c(\beta)^{-1} = \exp(2\pi i S_{\bar{b}}(\alpha,\beta))\,, \tag{B.1a}$$

$$S_{\bar{b}}(\alpha,\beta) = -S_{\bar{b}}(\beta,\alpha) = \left[\tfrac{1}{2}\langle\alpha_{\|},\beta_{\|}\rangle + \langle\alpha_T,(1-\bar{b})^{-1}\beta_T\rangle\right] \bmod 1\,, \tag{B.1b}$$

where $\{c(\alpha)\}$ is a section of $\tilde{\Lambda}$ and where $\alpha_{\|} = \mathscr{P}_{\bar{b}}\alpha \in \Lambda_{\|}$, $\alpha_T = (1-\mathscr{P}_{\bar{b}})\alpha \in \Lambda_T$. Equation (B.1b) reduces to (2.6a) when $\bar{b} = 1$ and to (3.8b) when $\bar{b}$ is without unit eigenvalues. Equation (B.1) also defines a central extension $\hat{\Lambda}_{\bar{b}}^T$ of the sublattice $\Lambda_{\bar{b}}^T$ by $\langle\varrho\rangle$ with centre determined by the lifting of $(1-\bar{b})\Lambda \subset \Lambda_{\bar{b}}^T$. Taking the quotient of these two groups we obtain a central extension $\hat{L}_{\bar{b}}$ of $L_{\bar{b}} = \Lambda_{\bar{b}}^T/(1-\bar{b})\Lambda$ by $\langle\varrho\rangle$ with centre $\langle\varrho\rangle$. $L_{\bar{b}}$ is a finite group of order $|\Lambda_{\bar{b}}^T/(1-\bar{b})\Lambda| = |\Lambda_{\bar{b}}^T/\Lambda_T|\,|\Lambda_T/(1-\bar{b})\Lambda| = D_{\bar{b}}/V_{\bar{b}}^2$. In addition, $\hat{L}_{\bar{b}}$ has a unique irreducible faithful representation $\pi(\hat{L}_{\bar{b}})$ of dimension $D_{\bar{b}}^{1/2}/V_{\bar{b}}$ in which the centre is represented by phases $\langle\varrho\rangle$ [30, 3]. Let $T^{\bar{b}}$ denote the vector space on which $\pi(\hat{L}_{\bar{b}})$ acts. Then the states $\{|\sigma_b\rangle\}$ of the degenerate b twisted vacuum form a basis for $T^{\bar{b}}$ and the cocycle factors $\{c_T(\alpha)\}$ are $\alpha_{\|}$ valued matrices acting on $T^{\bar{b}}$ which obey (B.1).

Let us now describe the group of inequivalent automorphisms $\mathrm{Aut}(\tilde{\mathscr{V}}^{\Lambda})$ of the OPA of $\tilde{\mathscr{V}}^{\Lambda}$ which act on the vector space $T^{\bar{b}}$. This group is an extension of the centraliser $C(\bar{b}|\,Co_0)$, where each lattice automorphism $\bar{g} \in C(\bar{b}|\,Co_0)$ acts on $\tilde{X}(z)$ in the usual way but is lifted to a set of automorphisms $\{g\}$ of $\tilde{\Lambda}$ where

$$gc(\alpha)g^{-1} = e^{2\pi i f_g(\alpha)}c(\bar{g}\alpha)\,, \tag{B.2}$$

where $f_g(\alpha)$ parametrises the liftings of $\bar{g}$. Let g and g' be two inequivalent liftings of $\bar{g}$. Then $e = g'g^{-1}$ is a lifting of the identity lattice automorphism. The group of liftings of the identity automorphism form a normal subgroup of $\mathrm{Aut}(\tilde{\mathscr{V}}^{\Lambda})$ and is parameterised by $f_e(\alpha)$ obeying $f_e(\alpha+\beta) = f_e(\alpha) + f_e(\beta)$ and $f_e(0) = 0$. Let $\lambda^{(i)}$ be a basis for Λ and $\lambda^*_{(j)}$ a dual basis, where $\langle\lambda^{(i)},\lambda^*_{(j)}\rangle = \delta^i_j$. Then define $\mu^i = f_e(\lambda^{(i)})$ so that $f_e(\alpha) = \mu^i\alpha_i = \langle\mu,\alpha\rangle$ with $\alpha = \alpha_i\lambda^{(i)}$ and $\mu = \mu^j\lambda^*_{(j)}$ i.e. each lifting is parameterised by μ. We may determine μ by considering the inner automorphisms of $\hat{\Lambda}$, where $c(\beta){:}\,c(\alpha) \to \exp[2\pi i S_{\bar{b}}(\beta,\alpha)]c(\alpha)$ from (B.1) and hence $\mu = -\beta_{\|}/2 - (1-\bar{b})^{-1}\beta_T$ for $\beta \in \Lambda$. As described above, the cocycle factors $\{c_T(\alpha)\}$ used in constructing the vertex operators $\tilde{\mathscr{V}}^{\Lambda}$ are defined to act on the twisted vacuum space $T^{\bar{b}}$. Hence only the inner automorphisms generated by $c_T(\Lambda_{\bar{b}}^T) = \{c_T(\beta_T)\,|\,\beta_T \in \Lambda_{\bar{b}}^T\} \equiv \pi(\hat{L}_{\bar{b}})$, with $\mu \in (1-\bar{b})^{-1}\Lambda_{\bar{b}}^T$, give the inequivalent liftings of the identity to automorphisms of $\tilde{\mathscr{V}}^{\Lambda}$ since $c_T(\Lambda_{\bar{b}}^T)$ maps $T^{\bar{b}}$ onto itself. Furthermore, from (B.2), the liftings of $\bar{b}$ itself are themselves equivalent to liftings of the identity lattice automorphism to automorphisms of $\{c_T(\alpha)\}$. (In particular, we may define one distinguished lifting in the centre of $\pi(\hat{L}_{\bar{b}})$, denoted by $b = \exp(-2\pi i E_0^b) \in \langle\varrho\rangle$. b then describes the twisting of the vacuum states with $\exp(2\pi i L_0)\,|\,\sigma_b\rangle = b^{-1}|\,\sigma_b\rangle$). Thus we find that the group of inequivalent automorphisms $\mathrm{Aut}(\tilde{\mathscr{V}}^{\Lambda})$ is given by $\hat{L}_{\bar{b}}.(C(\bar{b}|\,Co_0)/\langle\bar{b}\rangle)$.

We next describe $\mathrm{Aut}(\mathscr{P}_a\tilde{\mathscr{V}}^{\Lambda})$ where a is the lifting of $\bar{a} \in C(\bar{b}\,|\,Co_0)$ to an automorphism of the OPA of $\mathscr{P}_a\tilde{\mathscr{V}}^{\Lambda}$ with $a^r = b$. a acts as the identity on $\mathscr{P}_a\tilde{\mathscr{V}}^{\Lambda}$

and hence each $g \in \mathrm{Aut}(\mathscr{P}_a\tilde{\mathscr{V}}^{\Lambda})$ must commute with a. Therefore, g is lifted from $\bar{g} \in G_n = C(\bar{a}\,|\,Co_0)/\langle\bar{a}\rangle$. The inequivalent liftings of $\bar{g}$ are given by the inequivalent liftings, e, of the identity which commute with a. Using the parameterisation above, this implies that $\langle\mu,\alpha\rangle = \langle\mu,\bar{a}\alpha\rangle$ mod 1 for all $\alpha \in \Lambda$ and hence $\mu \in (1-\bar{a})\Lambda^{-1}$. From above we also know that $\mu \in (1-\bar{b})^{-1}\Lambda_{\bar{b}}^T$. Together, we find that $\mu \in (1-\bar{a})^{-1}\Lambda_{\bar{b}}^T$ so that the inequivalent liftings of the identity that commute with a are given by the inner automorphisms generated by $\hat{K} \equiv c_T\left(\left(\frac{1-\bar{b}}{1-\bar{a}}\right)\Lambda_{\bar{b}}^T\right) \subseteq \pi(\hat{L}_{\bar{b}})$. Two elements $c_T\left(\left(\frac{1-\bar{b}}{1-\bar{a}}\right)\alpha_T\right), c_T\left(\left(\frac{1-\bar{b}}{1-\bar{a}}\right)\beta_T\right)$ of $\hat{K}$ are equivalent $\Leftrightarrow \left(\frac{1-\bar{b}}{1-\bar{a}}\right)(\alpha_T-\beta_T) = (1-\bar{b})\lambda$ for $\lambda \in \Lambda \Leftrightarrow \alpha_T - \beta_T = (1-\bar{a})\lambda$ with $\lambda \in \Lambda_{\bar{b}}^T$. Thus $\hat{K} = m.K$, a central extension by $\langle\varrho\rangle$ of $K \equiv \Lambda_{\bar{b}}^T/(1-\bar{a})\Lambda_{\bar{b}}^T$. We therefore find that

$$\mathrm{Aut}(\mathscr{P}_a\tilde{\mathscr{V}}^{\Lambda}) = \hat{K}.G_n\,, \tag{B.3}$$

where $\hat{K}$ is the normal subgroup of automorphisms lifted from the lattice identity automorphism.

In the case where $r' = (r,n) = 1$ we have $\hat{K} = \hat{L}_{\bar{a}}$ and so $\mathrm{Aut}(\mathscr{P}_a\tilde{\mathscr{V}}^{\Lambda}) = \hat{L}_{\bar{a}}.G_n$. For all the other sectors, including the untwisted sector, the corresponding automorphism group can always be expressed as a quotient of $\hat{L}_{\bar{a}}.G_n$ by some normal subgroup. In the untwisted case when $r = 0$, the elements of $\mathrm{Aut}(\mathscr{P}_a\mathscr{V}^{\Lambda})$ must commute with a and are determined by $\mu \in (1-\bar{a})^{-1}\Lambda$ as above. Thus $\mathrm{Aut}(\mathscr{P}_a\mathscr{V}^{\Lambda}) = L_{\bar{a}}.G_n = (\hat{L}_{\bar{a}}.G_n)/\langle\omega\rangle$ i.e. $\mathrm{Aut}(\mathscr{P}_a\mathscr{V}^{\Lambda})$ is a quotient group of $\hat{L}_{\bar{a}}.G_n$. For $\bar{b} = \bar{a}^r$ and $r' \neq 0, 1$, $\hat{L}_{\bar{a}}.G_n$ contains a normal subgroup $\hat{J} = r'.J$ with $J \equiv \Lambda_{\bar{b}}/(1-\bar{a})\Lambda_{\bar{b}}$. $\hat{J}$ is the group of automorphisms of $\mathscr{P}_a\tilde{\mathscr{V}}^{\Lambda}$ lifted from the identity lattice automorphism and given by the inner automorphisms generated by $c_T(\Lambda_{\bar{b}})$. We therefore find that $\mathrm{Aut}(\mathscr{P}_a\tilde{\mathscr{V}}^{\Lambda}) = \hat{K}.G_n = (\hat{L}_{\bar{a}}.G_n)/\hat{J}$. Thus for all sectors $\mathscr{P}_a\tilde{\mathscr{V}}^{\Lambda}$, including the untwisted one, we may describe the OPA automorphism group $\mathrm{Aut}(\mathscr{P}_a\tilde{\mathscr{V}}^{\Lambda})$ by $\hat{L}_{\bar{a}}.G_n$, where some normal subgroup may act as the identity on $\mathscr{P}_a\tilde{\mathscr{V}}^{\Lambda}$, namely $\langle\omega\rangle$ for $r = 0$ and $\hat{J}$ for $r' \neq 0, 1$.

The operators of $\mathscr{P}_a\tilde{\mathscr{V}}^{\Lambda}$ create b twisted states from the twisted vacuum vector space $T^{\bar{b}}$. We may then define vertex operators $\{\psi_b\} = \mathscr{P}_a\mathscr{V}_b$ which create these states from the untwisted vacuum where (schematically) $\tilde{\phi}\sigma_b \sim \psi_b$ with $\tilde{\phi} \in \mathscr{P}_a\tilde{\mathscr{V}}^{\Lambda}$ and σ_b creates a twisted vacuum state. This OPA algebra is also invariant under $\hat{L}_{\bar{a}}.G_n$ with an appropriate identity action under a normal subgroup as described above. Likewise, the intertwining OPA between the various twisted sectors $\mathscr{P}_a\mathscr{V}_b$ as in (3.14), which is expected to exist, is invariant under $\hat{L}_{\bar{a}}.G_n$. Note that we are not considering here mixing (triality) automorphisms between the various sectors which are expected as in the usual Moonshine constructions [3, 29, 16]. We therefore find that the OPA of $\mathscr{V}_{\mathrm{orb}}^a = \mathscr{P}_a(\mathscr{V}^{\Lambda} \oplus \mathscr{V}_a \oplus \ldots \mathscr{V}_{a^{n-1}})$ is invariant under $\hat{L}_{\bar{a}}.G_n$, where no mixing between the various twisted sectors is considered. With a^* defined on $\mathscr{V}_{\mathrm{orb}}^a$ as in Sect. 3 (the operators of $\mathscr{V}_{a^k}$ are eigenvectors with eigenvalue ω^k) we have

$$C(a^*\,|\,M_{\mathrm{orb}}^a) = \hat{L}_{\bar{a}}.G_n\,, \tag{B.4}$$

where $M_{\mathrm{orb}}^a = \mathrm{Aut}(\mathscr{V}_{\mathrm{orb}}^a)$ is the complete automorphism group for $\mathscr{V}_{\mathrm{orb}}^a$. This is the result given in (3.19) for the 38 modular invariant orbifold constructions from the lattice automorphisms of Table 1.

We may also compute the centraliser $C(g_n \mid M_{\text{orb}}^{a^h})$, where g_n is lifted from one of the 13 lattice automorphisms $\bar{a}$ of Table 2. Orbifolding $\mathscr{V}^{\Lambda}$ with respect to $\bar{a}' = \bar{a}^h$ gives a modular consistent construction $\mathscr{V}_{\text{orb}}^{a'}$ and $g_n^h = a'^*$, where a'^* is dual to the lifting of $\bar{a}'$. From (B.4) we have $C(g_n | M_{\text{orb}}^{a^h}) \subset C(a'^* \mid M_{\text{orb}}^{a^h}) = \hat{L}_{a'}.G_{n'}$, where $G_{n'} = C(\bar{a}' \mid Co_0)/\langle \bar{a}' \rangle$. We may next repeat most of the argument given above to firstly find the automorphism group for the OPA of the vertex operators $\mathscr{P}_{a'}\mathscr{V}_{a'}$. Each automorphism $g \in C(g_n | \operatorname{Aut}(\mathscr{P}_{a'}\mathscr{V}_{a'}))$ is lifted from a lattice automorphism $\bar{g} \in C(\bar{a} \mid Co_0) = n.G_n$, where the inequivalent liftings are determined by the group of liftings of the identity lattice automorphism which commute with g_n. This forms a normal subgroup of $\operatorname{Aut}(\mathscr{P}_{a'}, \mathscr{V}_{a'})$, as before, generated by the inner automorphisms with respect to $c_T\left(\left(\dfrac{1-\bar{a}^h}{1-\bar{a}}\right)\Lambda\right) \subset \pi(\hat{L}_{a'})$. This group together with g_n itself generates $\hat{L}_{\bar{a}}$. Thus the group of automorphisms of $\mathscr{P}_{a'}\mathscr{V}_{a'}$ that commute with g_n is $\hat{L}_{\bar{a}}.G_n$. By following an argument similar to that above, we can also show that the automorphisms of $\mathscr{P}_{a'}\mathscr{V}_{b'}$ which commute with g_n are given by the quotient group of $\hat{L}_{\bar{a}}.G_n$ by a normal subgroup. Thus the centraliser is $C(g_n \mid M_{\text{orb}}^{a^h}) = \hat{L}_{\bar{a}}.G_n$ as in (3.19).

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