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Universal Estimate of the Gap for the Kac Operator in the Convex Case

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Abstract: The aim of this paper is to prove that if V is a strictly convex potential with quadratic behavior at ∞, then the quotient μ_2/μ_1 between the largest eigenvalue and the second eigenvalue of the Kac operator defined on $L^2(\mathbb{R}^m)$ by $\exp -V(x)/2 \cdot \exp \Delta_x \cdot \exp -V(x)/2$, where Δ_x is the Laplacian on $\mathbb{R}^m$ satisfies the condition:

$$\mu_2/\mu_1 \leq \exp - \cosh^{-1}(\sigma + 1)/2 ,$$

where σ is such that $\operatorname{Hess} V(x) \geq \sigma > 0$.

1. Introduction

In some problems in statistical mechanics on a lattice $\mathbb{Z}^2$, a mechanism of reduction to a one dimensional lattice permits to reduce the general questions about correlations or thermodynamic limit to corresponding spectral properties for a compact operator K_V associated to a C^∞ potential V by the formula:

$$K_V = \exp -V/2 \cdot \exp \Delta \cdot \exp -V/2 ,$$

where Δ is the usual Laplacian on $\mathbb{R}^m$. It was proved in [22], that in the case of the Schrödinger operator, the assumption that V is strictly convex uniformly in $\mathbb{R}^m$, that is satisfying for some $\sigma > 0$,

$$\inf_x (\operatorname{Hess} V)(x) = \sigma > 0 , \tag{1.1}$$

permits to get a minoration of the splitting between the second eigenvalue λ_2 and the first eigenvalue λ_1:

$$\lambda_2 - \lambda_1 \geq \sqrt{2\sigma} . \tag{1.2}$$

This condition appears to be optimal in the case of the harmonic oscillator in the sense that we get equality. So it is natural to ask for the same type question in the case of the Kac operator. Under condition (1.1) (and some conditions on the derivatives), the operator K_V is compact and its spectrum is given by a sequence of eigenvalues tending to $0\,\mu_j$ that we order by the conditon: $\mu_j \geq \mu_{j+1}$.

Moreover μ_1 is simple and so it is again interesting to study the quotient: μ_2/μ_1 and to have a universal estimate of $1 - \mu_2/\mu_1$ in the case when V is strictly convex.

This question was to our knowledge open (at least in this general framework) but there are results of this type in quantum field theory for particular cases (see the book of Glimm and Jaffe [3]), as was mentioned to us by T. Spencer in 1991. The main goal of this article is the proof of the following:

Theorem 1.1. *Let V be a C^∞ potential satisfying* (1.1) *and the condition*

$$|(D_x^\alpha V)(x)| \leq C_\alpha \quad \textit{for} \quad |\alpha| \geq 2\,,$$

then the quotient (μ_2/μ_1) *satisfies:*

$$\mu_2/\mu_1 \leq \exp -(\cosh^{-1}(1+\sigma)/2)\,. \tag{1.3}$$

Remark 1.2. This result is not optimal and we think it is possible to improve it in

$$\mu_2/\mu_1 \leq \exp -(\cosh^{-1}(1+\sigma))\,, \tag{1.4}$$

which is what we hope from the study of the "harmonic" Kac operator (see Appendix A).

Remark 1.3. In the case of the specific problem posed by M. Kac in [13], the potential was:

$$(1/4)\sum_{k=1}^{m}(x_k)^2 - \sum_{k=1}^{m}\ln\cosh(\sqrt{\nu/2}(x_k + x_{k+1}))\,, \tag{1.5}$$

and the assumptions of the theorem are satisfied if

$$\sigma = (1-4\nu)/2 > 0\,. \tag{1.6}$$

In particular, the majoration given by Theorem 1.1 is independent of the dimension m.

We recall that in this Kac model the splitting appears in the computation of the correlation between two lines which is given in this Kac model (see [13]) by:

$$\varrho(r) = \lim_{m\to\infty}\sum_{j=2}^{\infty}(\mu_j/\mu_1)^r\left(\int u_1^m(x)\cdot\tanh[\sqrt{\nu/2}(x_1+x_2)]\,u_j^m(x)\,d^m x\right)^2\,, \tag{1.7}$$

where $u_j^m(x)$ is the eigenvector corresponding to the eigenvalue μ_j.

As r tends to ∞, the behavior depends heavily on (μ_2/μ_1). In the convex case we find, using the inequality

$$\|u_1^m(x)\cdot\tanh[\sqrt{\nu}(x_1+x_2)]\|_{L^2} \leq 1\,,$$

the majoration:

$$(\mu_2/\mu_1)^r\left(\int u_1^m(x)\cdot\tanh[\sqrt{\nu}(x_1+x_2)]\,u_2^m(x)\,d^m x\right)^2 \leq \varrho(r) \leq (\mu_2/\mu_1)^r\,. \tag{1.8}$$

So we get in the convex case

$$\lim_{r\to\infty}\varrho(r) = 0\,,$$

uniformly with respect to m, and more precisely the exponential decay is controlled in function of μ_2/μ_1. We do not know if the quantity

$$\int u_1^m(x) \cdot \tanh[\sqrt{\nu}(x_1 + x_2)]\, u_2^m(x)\, d^m x$$

vanishes or not but this is irrelevant for the majoration.

This last question will be more important in the non-convex case where it is conjectured by [14] that:

$$\lim_{m \to \infty} \mu_2(m)/\mu_1(m) = 1\,. \tag{1.9}$$

Finally, let us mention that different questions on the splitting are solved, in general in the semiclassical context, in the case of the Schrödinger equation ([15, 8, 19, 16, 20, 21, 10]) and in the case of the Kac operator ([13, 2 and 4]).

2. Splitting and Thermodynamic Limit

Let us start from the well known property

$$\int (\mu_1)^{-p} (K_V^{(p)})\,(x, x)\, dx \to \int u_1(x)^2\, dx = 1 \tag{2.1}$$

as p tend to ∞, where $(K_V^{(p)})\,(x, y)$ denotes the distribution kernel of $(K_V)^p$. Let us analyze in more detail the convergence;

$$\int (\mu_1)^{-p} (K_V^{(p)})\,(x, x)\, dx - 1 = \sum_{j \geq 2} (\mu_j/\mu_1)^p\,. \tag{2.2}$$

Let us suppose that μ_2 has multiplicity k. We then get:

$$\int (\mu_1)^{-p} (K_V^{(p)})\,(x, x)\, dx - 1 = k(\mu_2/\mu_1)^p + (\mu_{k+2}/\mu_1)^p \left(\sum_{j \geq k+2} (\mu_j/\mu_{k+2})^p \right). \tag{2.3}$$

We observe now that $\left(\sum_{j \geq k+2} (\mu_j/\mu_{k+2})^p \right)$ can be interpreted as $\operatorname{tr} \tilde{K}^p$ where $\tilde{K}$ is an operator of Trace class, and of norm 1 in $\mathscr{L}(L^2)$. We then write that:

$$\operatorname{tr} \tilde{K}^p \leq \|\tilde{K}^p\|_{\mathrm{Tr}} \leq \|\tilde{K}\|_{\mathrm{Tr}}\,,$$

so we get:

$$\int (\mu_1)^{-p} (K_V^{(p)})\,(x, x)\, dx - 1 - k(\mu_2/\mu_1)^p \leq C(\mu_{k+2}/\mu_1)^p\,, \tag{2.4}$$

where C is independent of p.

Let us now take the logarithm and divide by p; we get

$$-\ln \mu_1 + \left(\frac{1}{p} \ln \int (K_V^{(p)})\,(x, x)\, dx \right) = (k/p)\,(\mu_2/\mu_1)^p\,(1 + \mathscr{O}(\exp -\delta p)) \tag{2.5}$$

with $\delta > 0$.

Consequently a control **by other means**, of the following type:

$$\left| -\ln \mu_1 + \left(\frac{1}{p} \ln \int (K_V^{(p)})\,(x, x)\, dx \right) \right| \leq C \exp -\beta p$$

for some C and some $\beta > 0$, will give the following majoration:

$$\mu_2/\mu_1 \leq \exp -\beta\,. \tag{2.6}$$

In the spirit of what has been done in [22, 9 and 11], it is then natural to try to analyze if we can obtain a universal estimate of some (possibly) optimal β, and as a consequence we shall get a universal estimate for the splitting. Let us recall that, in the case of the Schrödinger operator, a universal estimate was obtained for the splitting between the second eigenvalue λ_2 and the first eigenvalue λ_1 in [21] (by the maximum principle and a direct computation) and later in [9] as a consequence of the Brascamp-Lieb inequalities [1] (see formula (1.2)). Here we have to think of the correspondence $\lambda_j = -\ln \mu_j$ (which is asymptotically correct in the semiclassical limit) between the eigenvalues μ_j of the Kac operator and the eigenvalues λ_j of the Schrödinger operator. In order to prove the analogue of (1.2) for the Kac operator, that is formula (1.3), we shall now observe that $\int (K_V^{(p)})\,(x,x)\,dx$ can be written as:

$$\int (K_V^{(p)})\,(x,x)\,dx = \int \exp -\Phi(x_1,\,\ldots,\,x_p)\,dx_1,\,\ldots,\,dx_p \tag{2.7}$$

with:

$$\Phi(x_1,\,\ldots,\,x_p) = \sum_{j=1}^{p} \left(V(x_j) + \tfrac{1}{4}\,|x_{j+1} - x_j|^2\right), \tag{2.8}$$

where $x_j \in \mathbb{R}^m$ and where we take the convention that $x_{p+1} = x_1$.

This integral looks like the integral we have considered in [11] but the assumptions given in this paper are not satisfied; we shall however try to follow the strategy given in this paper (and earlier in [22]). The theory developed more recently in [25] can be applied modulo small modifications. Actually the theory of [11] works directly in the case $m = 1$, so we have in some sense to find a version ot these theorems where $\mathbb{R}$ is replaced by $\mathbb{R}^m$ with m fixed. We emphasize that it is p which will tend to ∞ and we do not worry about convergence which was already proved.

3. Link with the Computation of a Correlation

So we begin to study the potential introduced in (2.8). For each coordinate of $x_i \in \mathbb{R}^m$, we shall use the notation $x_i = (x_{ij}; j = 1,\,\ldots,\,m)$. Let us recall the general strategy chosen in [11] in order to analyze the quantity:

$$\delta^{(p)} = \int \exp -\Phi^{(p)}(x_1,\,\ldots,\,x_p)\,dx^{(p)}\,. \tag{3.1}$$

Because we can consider a subsequence for a particular problem we shall take $p = 2i$ which simplifies some notations.

We just introduce a t-dependent family:

$$\Phi(x_1,\,\ldots,\,x_p;t) = \sum_{j=1}^{p} \left(V(x_j) + \frac{t}{4}\,|x_{j+1} - x_j|^2\right), \tag{3.2}$$

where $x_j \in \mathbb{R}^m$ and where we take the convention that $x_{p+1} = x_1$.

We are interested more generally with the rapid convergence of $\ln \delta^{(p)}(t)/p$ with:

$$\delta^{(p)}(t) = \int \exp -\Phi^{(p)}(x_1,\,\ldots,\,x_p;t)\,dx^{(p)}\,. \tag{3.3}$$

Here we observe that for $t = 0$,

$$\delta^{(p)}(0) = \left(\int \exp -V(x_1)\, dx_1 \right)^p .$$

The sequence $\ln \delta^{(p)}(0)/p$ is consequently stationary and we conclude that it is natural to look for the logarithmic derivative with respect to t and to study the convergence of:

$$\begin{aligned} \varrho(p,t) &= \langle |x_1 - x_2|^2 \rangle_\Phi \\ &= \int |x_1 - x_2|^2 \exp -\Phi^{(p)}(x_1, \ldots, x_p; t)\, dx^{(p)} \Big/ \\ &\qquad \int \exp -\Phi^{(p)}(x_1, \ldots, x_p; t)\, dx^{(p)} , \end{aligned} \tag{3.4}$$

as p tends to ∞.

Here we observe that by the structure of the phase $\Phi^{(p)}$ we have:

$$\begin{aligned} \varrho(p,t) = \int |x_i - x_{i+1}|^2 \exp -\Phi^{(p)}(x_1, \ldots, x_p; t)\, dx^{(p)} \Big/ \\ \int \exp -\Phi^{(p)}(x_1, \ldots, x_p; t)\, dx^{(p)} . \end{aligned} \tag{3.5}$$

We shall use this property and choose $i = p/2$. In order to study the convergence with respect to p, we introduce as in [22] or later in [11] a new family depending on p, interpolating for example between what we want for p and what we want for $2p$. Let us take

$$\Phi^{(p,p)}(x', x''; t, s) = s\Phi^{(2p)}(x', x''; t) + (1-s)\,(\Phi^{(p)}(x'; t) + \Phi^{(p)}(x''; t)) . \tag{3.6}$$

We are interested in a control of the convergence of $\varrho(2p,t) - \varrho(p,t)$, with $\varrho(p,t)$ defined ion (3.5). Similarly we introduce now:

$$\begin{aligned} \varrho(p,p,t,s) = \int |x_i - x_{i+1}|^2 \exp -\Phi^{(p,p)}(x', x''; t, s)\, dx^{(2p)} \Big/ \\ \int \exp -\Phi^{(p,p)}(x', x''; t, s)\, dx^{(2p)} . \end{aligned} \tag{3.7}$$

We observe that:

$$\varrho(2p,t) = \varrho(p,p,t,1)$$

and

$$\varrho(p,t) = \varrho(p,p,t,0) .$$

We have consequently to analzye the derivative with respect to s, of the expression above. We observe indeed that if we get a uniform control with respect to (t,s) of $(\partial_s \varrho)(p,p,t,s)$ of the type:

$$|(\partial_s \varrho)(p,p,t,s)| \leq C \exp -\beta p , \tag{3.8}$$

we get also:

$$|\varrho(2p,t) - \varrho(p,t)| \leq C \exp -\beta p , \tag{3.9}$$

and consequently:

$$|\varrho(\infty,t)-\varrho(p,t)|\leq \tilde{C}\exp-\beta p\,, \tag{3.10}$$

where $\varrho(\infty,t)=\lim_{p\to\infty}\varrho(p,t)$ and $\tilde{C}=C\Big(\sum_{k=0}^{\infty}\exp-\beta 2^k\Big)$.

But, for any function f on $\mathbb{R}^{2p}\times\mathbb{R}^2_{(t,s)}$, we have:

$$\partial_s(\langle f(\,\cdot\,;s,t)\rangle_{s,t})=\langle\partial_s f(\,\cdot\,;s,t)\rangle_{s,t}-\mathrm{Cor}(f,\partial_s\Phi^{(p,p)})\,,$$

where $\langle\,\cdot\,\rangle_{s,t}$ is the mean value with respect to the measure

$$\exp-\Phi^{(p,p)}dx^{(2p)}\Big/\int\exp-\Phi^{(p,p)}dx^{(2p)}\,,$$

and where for two functions f and g, $\mathrm{Cor}(f,g)$ is by definition given by:

$$\mathrm{Cor}(f,g)=\langle(f-\langle f\rangle)(g-\langle g\rangle)\rangle\,.$$

Here our specific f is independent of (s,t), and let us recall that:

$$\partial_s\Phi=2x_px_1+2x_{2p}x_{p+1}-2x_px_{p+1}-2x_{2p}x_1\,.$$

We have consequently to control the correlation of f and g where $f=|x_i-x_{i+1}|^2$ depends only of the variables x_i,x_{i+1} and $g=\partial_s\Phi$ depends only of the variables x_p,x_1,x_{2p},x_{p+1} and we recall that $p=2i$. So we have to analyze:

$$(\partial_s\varrho)(t,s)=\mathrm{Cor}(|x_i-x_{i+1}|^2,\partial_s\Phi)\,. \tag{3.11}$$

In the spirit of [22] or [11], we are waiting consequently for some exponential decay. In order to simplify the notations we observe that the quantity we have to estimate is a finite (independent of p) sum of correlations of the type $\mathrm{Cor}(f,g)$ with:

$$f=x_{ij}\cdot x_{kj}\,,\qquad g=x_{lv}\cdot x_{nv}\,,$$

where $|i-k|\leq 1$, $i=p/2$ and l,n are near 1 or p in $\mathbb{Z}/(2p\mathbb{Z})=\mathbb{Z}^{(2p)}$. If f and g were with bounded gradient, we should only have to follow the strategy of [11] and analyze the mean value $\langle v\cdot\nabla g\rangle$, where $v=\nabla u$ was the solution of the so-called basic equation:

$$(*)\nabla f=(-\Delta+\nabla\Phi\cdot\nabla)v+(\mathrm{Hess}\,\Phi)v\,.$$

Let us remark that the same type of computations was performed for the study of higher correlations in [11]. As in [11] we are now looking for a vector field $v^{(i,j;p)}$ defined on $E^{2p}\times[0,1]^2:(x,t,s)\to v^{(i,j;p)}(x,t,s)\in E^{2p}$ (with $E=\mathbb{R}^m$) solution of:

$$x_{ij}=\langle x_{ij}\rangle_{(t,s)}+v\cdot\nabla_x\Phi-\mathrm{div}_x\,v\,. \tag{3.12}$$

Note that $(*)$ is simply obtained by derivation of (3.12). We omit sometimes the reference to p,t,s in what follows in order to simplify the notations. Let us compute $\mathrm{Cor}(f^{ikj},g^{lnr})$ with: $f^{ikj}=x_{ij}\cdot x_{kj}$; $g^{lnr}=x_{lr}\cdot x_{nr}$.

We then write:

$$\begin{aligned}\mathrm{Cor}(f^{ikj},g^{lnr})&=\langle v^{ij}_{kj}(x)\cdot x_{lr}\cdot x_{nr}\rangle-\langle v^{ij}_{kj}(x)\rangle\cdot\langle x_{lr}\cdot x_{nr}\rangle\\&\quad+\langle v^{ij}_{lr}(x)\cdot x_{kj}\cdot x_{nr}\rangle+\langle v^{ij}_{nr}(x)\cdot x_{kj}\cdot x_{lr}\rangle\,.\end{aligned} \tag{3.13}$$

We shall prove in Sect. 4 that there exists β and C s.t.

$$\sup_x |v_{lr}^{ij}(x)| \leq C \exp -\beta p \tag{3.14}$$

$\forall i, j, l, r$ such that $i = p/2$, $|i-j| \leq 1$, $\inf(|l-p|, |l-1|, |l-2p|) \leq 1$.

We then conclude that $\langle v_{lr}^{ij}(x) \cdot x_{kj} \cdot x_{nr} \rangle$ and $\langle v_{nr}^{ij}(x) \cdot x_{kj} \cdot x_{lr} \rangle$ have the same majoration using the property that:

$$|\langle v_{lr}^{ij}(x) \cdot x_{kj} \cdot x_{nr} \rangle| \leq \sup_x |v_{lr}^{ij}(x)| \left(\langle (x_{kj})^2 \rangle + \langle (x_{nr})^2 \rangle \right)$$

and we have:

$$\sup_x |v_{lr}^{ij}(x)| \left(\langle (x_{kj})^2 \rangle + \langle (x_{nr})^2 \rangle \right) \leq \text{const},$$

using either the Brascamp-Lieb inequality [1] or again the trick that:

$$\langle (x_{kj})^2 \rangle = \langle v_{kj}^{kj} \rangle .$$

This permits to deduce the control of the two last terms from the control of $\sup_x |v_{kj}^{kj}(x)|$ which will be also obtained in Sect. 4.

Remark 3.1. Let us remark also that if the so-called Lebowitz inequality (see for example [18] or [3]) were satisfied, then we would have the inequalities:

$$0 \leq \mathrm{Cor}(f^{ikj} g^{lnr}) \leq \left(\langle v_{lr}^{ij} \rangle \langle v_{nr}^{kj} \rangle + \langle v_{nr}^{ij} \rangle \langle v_{lr}^{kj} \rangle \right)$$

and we would obtain a control in $\exp -2\beta p$ using (3.13). But we do not know if these inequalities, which are related to ferromagnetic properties, are satisfied in our case.

Because i and k are near, we can not use (3.14) for the two first terms in the left-hand side of (3.13):

$$\langle v_{kj}^{ij}(x) \cdot x_{lr} \cdot x_{nr} \rangle - \langle v_{kj}^{ij}(x) \rangle \cdot \langle x_{lr} \cdot x_{nr} \rangle , \tag{3.15}$$

which is the correlation $\mathrm{Cor}(v_{kj}^{ij}, g^{lnr})$.

Let w^{ikj} be the solution of $(*)$ with $f = v_{kj}^{ij}$. We shall have to prove that w^{ikj} has essentially the same properties as ∇v_{kj}^{ij}. On the other hand we have:

$$\mathrm{Cor}(v_{kj}^{ij}, g^{lnr}) = \langle w_{lr}^{ikj} \cdot x_{nr} \rangle + \langle w_{nr}^{ikj} \cdot x_{lr} \rangle . \tag{3.16}$$

If we prove that:

$$\sup_x |w^{ikj}|_{lr} \leq C \exp -\beta p \tag{3.17}$$

under the same conditions as in (3.14), we shall have the following control for $\mathrm{Cor}(v_{kj}^{ij}, g^{lnr})$:

$$|\,\mathrm{Cor}(v_{kj}^{ij}, g^{lnr})| \leq C \exp -\beta p . \tag{3.18}$$

Finally we have proved, that if (3.14) and (3.17) are satisfied, then (3.7) is satisfied and Theorem 1.1 is proved for some β. The next section will be devoted to the proof of (3.14) and (3.17) with the computation of a very explicit β. Let us finally observe that in order to get (2.6) for some β_0, it is sufficient to have (3.18) for any $\beta < \beta_0$.

4. Maximum Principle

In the preceding sections we have seen that, as in [11], the control of the correlation will be a consequence of the fact that $v_{lr}^{ij}(x)$ and w_{nr}^{ikj} are small if l or n and i are far from each other. For the first term, it will be obtained by proving that the vector v^{ij} belongs to a spcae $l_p^{\infty}(E)$ associated to a weight ϱ defined on $\mathbb{Z}^{(2p)} = \mathbb{Z}/2p\mathbb{Z}$ such that

$$\varrho(i) = \exp -\kappa(p/2)\,, \qquad \varrho(p) = \varrho(p+1) = \varrho(1) = \varrho(2p) = 1$$

and whose logarithm is controlled. These weights were introduced and then used in different papers [22, 9, 11, 25]. The main difference is here that we need a kind of vector-valued version: $\mathbb{R}$ is replaced by $E = \mathbb{R}^m$. In our case, we can apply almost directly (with only small modifications) the results of [25] but we repeat the argument in order to be complete. The only new technical point is to define properly a suitable norm on these weighted spaces. In order to see how the maximum principle can be used, let us introduce the normed spaces:

$$l_{\varrho}^{\infty}(E) = \{x = (x_1, \ldots, x_{2p}) \in E^{2p};\, |||x|||_{\infty,\varrho,E} = \sup_j \varrho(j)\, ||x_j||_E\}$$

and

$$l_{\varrho}^{1}(E) = \{y = (y_1, \ldots, y_{2p}) \in E^{2p};\, |||y|||_{1,\varrho,E} = \sum_j \varrho(j)\, ||y_j||_E\}\,.$$

We remark here that there are different ways to express this norm. We can for example write that

$$|||x|||_{\infty,\varrho,E} = \sup_{u \in l^1_{1/\varrho}(\mathbb{Z}^{(2p)},\mathbb{R});\, |u|_{1,1/\varrho} \leq 1} \sum u_j ||x_j||_E\,,$$

or another way is to use the duality between $l_{\varrho}^{\infty}(E)$ and $l_{1/\varrho}^{1}(E)$ and to write:

$$|||x|||_{\infty,\varrho,E} = \sup_{u \in l^1_{1/\varrho}(\mathbb{Z}^{(2p)},E);\, |||u|||_{1,1/\varrho,E} \leq 1} \sum \langle u_j, x_j \rangle_E\,.$$

We shall use also the identification of E^{2p} with $\oplus E_j$ with E_j isometric to a finite dimensional real Hilbert space E (actually $E = \mathbb{R}^m$). We want now to work with the Maximum Principle applied to the basic equation:

$$w = (-\Delta + \nabla\Phi \cdot \nabla)v + (\mathrm{Hess}\,\Phi)v\,. \tag{4.1}$$

Let us consider a point x_0, where $|||v(x)|||_{\infty,\varrho,E}$ is maximal. As in [11] or [25] we can by using cutoff functions reduce to a situation where we suppose that w and v, ∇w and ∇v tend to 0 at ∞. Then there exists y in $l_{1/\varrho}^{1}(E)$ with $|||y|||_{1,1/\varrho,E} = 1$ such that:

$$\langle v(x_0), y\rangle = \sum_j \langle v_j(x_0), y_j\rangle_{E_j} = |||v(x_0)|||_{\infty,\varrho,E} = \sup_x |||v(x)|||_{\infty,\varrho,E}\,.$$

Let us use this property in (4.1); we get:

$$\langle w(x), y\rangle = (-\Delta + \nabla\Phi \cdot \nabla)\,\langle v(x), y\rangle + \langle(\mathrm{Hess}\,\Phi(x))\,v(x), y\rangle\,.$$

If we observe now that $\langle v(x), y\rangle$ takes it maximum at x_0, we get:

$$\langle(\mathrm{Hess}\,\Phi(x_0))\,v(x_0), y\rangle \leq \langle w(x_0), y\rangle\,. \tag{4.2}$$

We rewrite this equation using the natural decomposition of Hess Φ which will be considered as a $2p \times 2p$ block matrix where each coefficient $(\mathrm{Hess}\,\Phi)_{jk}$ is an element in $\mathscr{L}(E_k, E_j)$. We then use the decomposition in the sum of a diagonal part (elements in $\mathscr{L}(E_j)$) and a non-diagonal part, and we get:

$$\sum_j \langle M_{jj} v_j(x_0), y_j \rangle_{E_j} + \sum_{j \neq k} \langle M_{jk} v_k(x_0), y_j \rangle_{E_j} \leq |||w(x_0)||| \,. \tag{4.3}$$

Here we observe that y_j is necessary colinear to $v_j(x_0)$: $v_j = \beta_j y_j$ with $\beta_j > 0$, and using the assumption of strict positivity of M_{jj}:

$$M_{jj} \geq \sigma Id_E \quad \text{with} \quad \sigma > 0 \,, \tag{4.4}$$

we get:

$$\begin{aligned} \sigma \sum_j \langle v_j(x_0), y_j \rangle_{E_j} - \sum_{j \neq k} \| \varrho(j)\, \varrho(k)^{-1} M_{jk} \|_{\mathscr{L}(E_k, E_j)} \| \varrho(k) v_k \| & \\ \times \| \varrho(j)^{-1} y_j \| \leq |||w(x_0)||| \,, & \end{aligned}$$

and finally we have obtained:

$$(\sigma - \delta) \, |||v(x_0)||| \leq |||w(x_0)||| \,, \tag{4.5}$$

where δ is the norm in $\mathscr{L}(l^\infty)$ of the $2p \times 2p$ block matrix whose coefficients are 0 on the diagonal and equal to $\|\varrho(j)\,\varrho(k)^{-1} M_{jk}\|_{\mathscr{L}(E_k, E_j)}$ for $j \neq k$. So we get an estimation if: $(\sigma - \delta) > 0$. We now verify the assumption in our case. We have (where the indices are considered as elements of $\mathbb{Z}^{(2p)}$),

$$\begin{aligned} M_{jj} &= \mathrm{Hess}\, V + tI \,, \\ M_{j,j\pm 1} &= -\, tId/2 \,, \\ M_{jk} &= 0 \quad \text{if} \quad k \neq j, j \pm 1 \,. \end{aligned} \tag{4.6}$$

It is then clear that the assumptions are verified for $\sigma > 0$ and $\varrho = 1$. But we shall have to use weights ϱ such that

$$\exp -\kappa \leq \varrho(j+1)/\varrho(j) \leq \exp \kappa \,,$$

and we get the condition:

$$\sigma + t > (t/2) \sup_j (\varrho(j)/\varrho(j+1) + \varrho(j)/\varrho(j-1)) \,,$$

recalling the convention that ϱ is a weight satisfying

$$\varrho(p) = \varrho(p+1) = \varrho(1) = \varrho(2p) = 1 \,.$$

(If we want later to work with l^1 norms, we will need the stronger condition:

$$\begin{aligned} \max \Big(& \sup_j (\varrho(j)/\varrho(j+1) + \varrho(j)/\varrho(j-1)), \\ & \sup_j (\varrho(j-1)/\varrho(j) + \varrho(j+1)/\varrho(j)) \Big) < 2(\sigma + t)/t \,.) \end{aligned}$$

Here we follow the arguments in [22] (see also [11]). Let $\mu(j) = \varrho(j+1)/\varrho(j)$. If we assume that

$$\exp -\xi \leq \mu(j+1)/\mu(j) \leq \exp \xi$$

for a small $\xi > 0$, then we get the condition that:

$$t \cdot \exp \xi \sup_k (1/2)\,(\mu(k) + \mu(k)^{-1}) < \sigma + t$$

or equivalently:

$$|\ln(\varrho(k+1)/\varrho(k))| < \cosh^{-1}((\exp -\xi)\,(\sigma + t)/t)\,.$$

But we can take ξ as small as we want in the future. So we meet the condition:

$$\cosh \kappa < (1 + \sigma/t) \quad \text{for} \quad 0 \le t \le 1\,.$$

This is at $t = 1$ that the condition is the strongest, so we get:

$$\cosh \kappa < (1 + \sigma)\,. \tag{4.7}$$

This suggests the idea that the exponential convergence in our problem will be directly measured by σ which measures the strict convexity of V. Actually, we have verified all the properties for $s = 1$ but all the arguments go through for the phase $\Phi^{(p,p)}(x', x''; t, s)$ and uniformly with respect to the parameters s, t.

Remark 4.1. Sjöstrand's Maximum Principle. In order to verify that we are in the framework of the theory developed by [25], let us recall some definitions introduced by the author. Let us consider a finite dimensional real Banach space B and B^* is the dual space.

If $A : B \to B$ is a linear map and if $\varepsilon \ge 0$, we say that A satisfies $mp(\varepsilon)$ (with respect to the space B), if the following property is satisfied:

$$\begin{aligned} &\text{If} \quad v \in B,\ y \in B^* \quad \text{and} \quad \langle v, y\rangle = \|v\|_B\,\|y\|_{B^*}\,,\\ &\text{then} \quad \langle Av, y\rangle \ge \varepsilon \|v\|_B\,\|y\|_{B^*}\,. \end{aligned} \tag{4.8}$$

What we have used in our proof is exactly the property that $\operatorname{Hess}\Phi^{p,p}$ satisfies $(mp(\sigma - \delta))$ with $B = l^\infty_\varrho(E)$ and uniformly with respect to the different parameters p, t, s. In order to relate with the proof we give above, we have just to observe that if v and y satisfy $\langle v, y\rangle = \|v\|_B\,\|y\|_{B^*}$, then: $v_j = \beta_j y_j$ with $\beta_j \ge 0$ (see above).

Proof of 3.14. Let us now apply the result to the estimate of $v^{ij}_{lr}(x)$ with $i = p/2$ and $l = 1$, $p, p+1$ or $2p$. We observe that in the basic equation, the gradient of the function $x \to x_{ij}$ is bounded by $C \exp -\kappa p/2$ in $l^\infty_\varrho(E)$ for $\varrho(q) = \exp -\kappa d(q, \{p, p+1, \dots, 2p, 1\})$. Under the condition (4.7) on κ we get the same property for $\sup_x \varrho(q)\,\|v^{ij}_q(x)\|_E$. In particular we get for $q = l$ the following property that for any κ satisfying (4.7) there exists a constant C_κ such that

$$\sup_x |v^{ij}_{lr}(x)| \le C_\kappa \exp -\kappa p/2\,. \tag{4.9}$$

Proof of 3.17. In order to analyze w^{ikj}_{nr}, we observe that, according to the equation satisfied by w^{ikj}:

$$(*)\nabla(v^{ij}_{kj}) = (-\Delta + \nabla\Phi \cdot \nabla)\,w^{ikj} + (\operatorname{Hess}\Phi)\,w^{ikj}\,. \tag{4.10}$$

We have only to verify that $\nabla(v^{ij}_{kj})$ is bounded by $C \exp -\kappa p/2$ in $l^\infty_\varrho(E)$ with the same ϱ's.

But this leads to the control of the $2p \times 2p$ block matrix ∇v^{ij} which is again controlled by the maximum principle once one remarks that ∇v is a solution of the second basic equation:

$$(-\Delta + \nabla\Phi \cdot \nabla)\nabla v + (\mathrm{Hess}\,\Phi) \circ \nabla v + \nabla v \circ (\mathrm{Hess}\,\Phi) = \langle \Phi^{(3)} \mid v \rangle\,, \tag{4.11}$$

where $\langle \Phi^{(3)} \mid v \rangle$ is the contraction of $\Phi^{(3)}$ and v. Using the property $mp(\sigma - \delta)$ we obtain as in [25] (we omit the cutoff argument) that:

$$\sup_x \|\nabla v(x)\|_{\mathscr{L}(B)} \leq C_\kappa \sup_x \|\langle \Phi^{(3)}(x) \mid v(x) \rangle\|_{\mathscr{L}(B)}\,, \tag{4.12}$$

where $B = l^\infty_\varrho(E)$. Let us now compute:

$$\|\langle \Phi^{(3)} \mid v^{ij} \rangle\|_{\mathscr{L}(B)} = \sup_{(a,b)\in B\times B^*, \|a\|\leq 1, \|b\|\leq 1} |\langle \Phi^{(3)}(x), v^{ij}(x) \otimes a \otimes b \rangle|\,.$$

But we now observe that the components $\Phi^{(3)}_{p'q'r'}$ of $\Phi^{(3)}$ vanish unless $p' = q' = r'$ and therefore we get:

$$\|\langle \Phi^{(3)} \mid v^{ij} \rangle\|_{\mathscr{L}(B)} \leq C$$

using the property that $\|v^{ij}\|_{l^\infty_{\varrho_0}(E)}$ is bounded for the weight $\varrho_0 = 1$. So we get:

$$\sup_x \|\nabla v^{ij}(x)\|_{\mathscr{L}(B)} \leq C\,.$$

We then deduce immediately by chosing $a = a_k = (\delta_{jj'}) \in E_k$, which is bounded in $l^\infty_\varrho(E)$ by $C \exp -\kappa p/2$, that $(\nabla v^{ij})a = (\nabla v^{ij}_{kj})$ is also bounded by $C \exp -\kappa p/2$ in $l^\infty_\varrho(E)$, and using Eq. (4.10) the same property for w_{ikj}.

Remark 4.2. If we come back to our estimate of the convergence, we have obtained the estimate

$$(\mu_2/\mu_1) \leq \exp -(1-\varepsilon)(\cosh^{-1}(1+\sigma))/2\,, \qquad \forall\varepsilon\,,$$

with $\sigma = \inf_x \lambda_{\min}(\mathrm{Hess}\,V(x))$. So we get that:

$$(\mu_2/\mu_1) \leq \exp -(\cosh^{-1}(1+\sigma))/2\,.$$

If we compare with the result we can obtain in the case $m = 1$, $V(x) = \sigma x^2$ (see Appendix A), we have already mentioned that we have probably lost a factor 1/2. This loss can probably be eliminated by using the techniques of the second part of [22] associated with techniques of [25]. It will probably be useful to improve our estimates on the correlations introduced in Sect. 3. We shall probably need for that other "basic equations" deduced of the first one by differentiation. We hope to come back later to this point.

A. Harmonic Oscillator and Harmonic Kac Operator

We want to compare:

$$\exp(-x^2/2)\cdot\exp(t(d/dx)^2)\cdot\exp(-x^2/2)\,, \tag{A.1}$$

with

$$\exp-(-t(d/dx)^2+x^2) \quad \text{for} \quad t>0\,. \tag{A.2}$$

As is well known, everything can be computed explicitly (see for example [26]) and we reproduce some of the formulas for completeness. The distribution kernel of the operator defined in (A.1) is explicitly given by:

$$2^{-1}\pi^{-1/2}\cdot t^{-1/2}\exp(-(x^2+y^2)/2-(x-y)^2/4t)\,. \tag{A.3}$$

On the other hand, it is well known that:

$$\pi^{-1/2}\cdot K^{1/2}\exp(-K[z(x^2+y^2)-2xy]) \tag{A.4}$$

is the distributional kernel of:

$$\exp[-[\ln(z+\sqrt{z^2-1})/4K\sqrt{z^2-1}]\cdot[-d^2/d^2+4K^2(z^2-1)x^2]]\,. \tag{A.5}$$

We observe now that the two kernels coincide if $K=1/4t$ and $z=(2t+1)$. We have consequently:

$$\begin{aligned}\exp(-x^2/2)\cdot\exp(t(d/dx)^2)\cdot\exp(-x^2/2)\\ =\exp[-[\ln(z+\sqrt{z^2-1})/4K\sqrt{z^2-1}]\\ \times[-d^2/d^2+4K^2(z^2-1)x^2]]\end{aligned} \tag{A.6}$$

with: $K=1/4t$ and $z=(2t+1)$.

We then obtain the explicit computation of the eigenvalues of

$$\exp(-x^2/2)\cdot\exp(t(d/dx)^2)\cdot\exp(-x^2/2)$$

as: $\exp(-(n-1/2))\ln(z+\sqrt{z^2-1}))\,(n\geq 1, n\in\mathbb{N})$.

In order to come back to our notation above we compute the formula for the operator $\exp(-\sigma x^2/4)\cdot\exp(h^2(d/dx)^2)\cdot\exp(-\sigma x^2/4)$ and we get for the eigenvalues μ_n:

$$\mu_n=\exp(-((2n-1)/2)\cosh^{-1}(\sigma h^2+1))\,, \tag{A.7}$$

and the splitting is given by:

$$\mu_2/\mu_1=\exp-\cosh^{-1}(1+\sigma h^2)\,. \tag{A.8}$$

So in the case when $h=1$, we see that we have lost one factor 1/2 inside the exponential. In the semi-classical limit, we recover the result on the harmonic oscillator that:

$$\ln(\mu_2/\mu_1)(h)=h\sqrt{2\sigma}+\mathcal{O}(h^2)\,.$$

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