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Quantization and Representation Theory of Finite W Algebras

Jan de Boer [1] *, **Tjark Tjin** [2] **

[1] Institute for Theoretical Physics, Princetonplein 5, P.O. Box 80.006, 3508 TA Utrecht, The Netherlands

[2] Institute for Theoretical Physics, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

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Abstract: In this paper we study the finitely generated algebras underlying W algebras. These so called "finite W algebras" are constructed as Poisson reductions of Kirillov Poisson structures on simple Lie algebras. The inequivalent reductions are labeled by the inequivalent embeddings of sl_2 into the simple Lie algebra in question. For arbitrary embeddings a coordinate free formula for the reduced Poisson structure is derived. We also prove that any finite W algebra can be embedded into the Kirillov Poisson algebra of a (semi)simple Lie algebra (generalized Miura map). Furthermore it is shown that generalized finite Toda systems are reductions of a system describing a free particle moving on a group manifold and that they have finite W symmetry. In the second part we BRST quantize the finite W algebras. The BRST cohomology is calculated using a spectral sequence (which is different from the one used by Feigin and Frenkel). This allows us to quantize all finite W algebras in one stroke. Examples are given. In the last part of the paper we study the representation theory of finite W algebras. It is shown, using a quantum version of the generalized Miura transformation, that the representations of finite W algebras can be constructed from the representations of a certain Lie subalgebra of the original simple Lie algebra. As a byproduct of this we are able to construct the Fock realizations of arbitrary finite W algebras.

1. Introduction

It is only relatively recent that it was realized that nonlinear symmetry algebras play an important role in physics. The discovery of W algebras in Conformal Field theory [1] (see [2] for a recent review) made it clear that they would play an important role in string theory, field theory, integrable systems and the theory of 2D critical phenomena. One reason for their late discovery is that up to now they

* e-mail: deboer@ruunts.fys.ruu.nl

** e-mail: tjin@phys.uva.nl

are only known as infinitesimal symmetries. The global invariances of a system with a nonlinear symmetry are not known. This is of course related to the fact that nonlinear algebras don't exponentiate to groups like Lie algebras.

A lot of work has been done on trying to understand what the meaning of W algebras is. It turns out that many W algebras found in CFT are not completely unrelated to the linear theory of Lie algebras after all. This was first realized when it was shown in [3] that certain Poisson algebras occurring in the theory of integrable hierarchies of evolution equations were nothing but classical versions of the nonlinear algebras found in CFT. In particular the well known W_n algebras are related in this way to the second Hamiltonian structures of KdV like hierarchies. These hierarchies, and their Hamiltonian structures were however already shown to be reductions of a different class of integrable hierarchies which have a second Hamiltonian structure that is equal to the Kirillov Poisson structure on the dual of an affine Lie algebra [4]. This means on an algebraic level that classical W_n algebras can be obtained from affine Lie algebras by Poisson reduction. This picture was worked out in [5] where it was shown that a classical W_n algebra is nothing but the Dirac bracket algebra on a submanifold of the affine Lie algebra.

In the meantime many new W algebras were constructed by what can be called the "direct method," i.e. by imposing Jacobi identities on general nonlinear extensions of the Virasoro algebra. Since the Jacobi identities are themselves nonlinear algebraic equations the construction of W algebras in this manner is rather cumbersome. It was therefore a natural question to ask (also from the point of view of Poisson reduction of Poisson manifolds) whether more of these algebras can be obtained via Poisson reduction from affine Lie algebras. That this is the case was shown in [6] where the construction of [4, 5] was generalized to include many more W algebras besides W_n. Motivated by a similar situation encountered in the theory of dimensional reductions of selfdual Yang–Mills equations it was shown that to every embedding of sl_2 into the simple Lie algebra underlying the affine algebra there is associated a Poisson reduction leading to a W algebra. sl_2 embeddings that are related to one another by inner automorphisms lead to the same reductions, so in order to find out how many inequivalent reductions there are one needs to count the number of equivalence classes of sl_2 embeddings. For sl_n this number is $P(n)$, the number of partitions of n. The standard reductions leading W_n algebras turned out to be associated to the so-called "principal embeddings."

The fact that one knows that these W algebras have a linear origin helps a lot when one tries to analyse them. For example the construction of invariant chiral actions is facilitated as was shown in [7]. Also the construction of the classical covariant W gravities and their moduli spaces have been made possible by this [8, 9].

The procedure of Poisson reduction is of course purely classical. In order to really make contact with CFT one would like to quantize the W algebras obtained by Poisson reduction. The W_n algebras were quantized in [10] by (naively) quantizing the well known Miura transformation. In essence what one does is classically express the W_n generators in terms of classical harmonic oscillators via the Miura transform. One then quantizes the W_n algebra by quantizing the harmonic oscillators and normal ordering. This gives a quantum algebra that closes for A_n. It does not work for all algebras however [2], as was to be expected since this is in general not a valid quantization procedure. The quantization of the Poisson reduction leading to W_n algebras was made more precise in [11] where the BRST formalism was used to tackle this problem.

It was also attempted to construct the representation theory of W algebras from the representation theory of affine Lie algebras [13, 12, 11]. In principle the BRST procedure provides a functor from the representation theory of affine Lie algebras to the representation theory of W algebras. It turned out to be possible to obtain W_n characters from Kac–Moody characters. The general theory of W_n representations is however far from complete. Furthermore for the other reductions the quantizations (let alone the representation theories) are not known.

Up to now the study of W algebras has concentrated on the infinite dimensional case. This situation is comparable to trying to develop the theory of Lie algebras by starting with the infinite dimensional case. As the structure and representation theory of affine Lie algebras is largely determined by that of the finite dimensional simple Lie algebras that underly them it is our opinion that it might be helpful to look for and study the finitely generated structures underlying W algebras. This program was initiated in [14] and will be carried out in the present paper. It will turn out that the theory of "finite W algebras" is remarkably rich and contains already many of the features encountered in ordinary W algebras. It is therefore our expectation that much of what we will say in this paper will transfer without much alteration to the infinite dimensional case.

The paper is roughly split up into three parts. The first part deals with the classical theory. Classical finite W algebras are constructed as Poisson reductions of Kirillov Poisson structures on simple Lie algebras (in complete analogy with ordinary W algebras which are constructed as reductions of affine Lie algebras). The Poisson algebras thus obtained are nonlinear and finitely generated. We discuss their structure and show that in general they do have linear Poisson subalgebras that are isomorphic to Kirillov Poisson algebras. We also derive a coordinate free expression for the reduced Poisson structure of an arbitrary reduction. The Miura transformation turns out to have a finite dimensional analogue which can in fact be extended to arbitrary reductions. From this it follows that any finite W algebra can be embedded into the Kirillov Poisson algebra of a certain subalgebra of the simple Lie algebra with which we started. At the end of the first part of the paper we investigate which theories have finite W symmetry. It turns out that (as could have been expected) these are generalized finite Toda systems. In deriving this however we show that finite Toda systems are reductions of a system describing a free particle moving on a group manifold. This allows us to give general formulas for the solution space of such systems.

In the second part of the paper we BRST quantize the finite W algebras. The nontrivial part of this is of course calculating the BRST cohomology and its algebraic structure. Since the BRST differential is a sum of two other differentials one can associate a double complex to the BRST complex. In order to calculate the BRST cohomology one can then use the theory of spectral sequences. There is a choice to be made between one out of two spectral sequences that one can associate to a double complex. These spectral sequences must give the same final answer for the BRST cohomology, as is well known from the theory of spectral sequences, but for the calculation it is crucial which one one takes. The choice we make is different from the one made by Feigin and Frenkel and allows us to quantize any finite W algebra and reconstruct its algebraic structure.

In the third and last part of the paper we discuss the representation theory of finite W algebras. Crucial for this is a quantum version of the generalized Miura transformation which embeds any finite W algebra into the universal enveloping algebra of some (semi)simple Lie algebra. An arbitrary representation of this Lie algebra therefore immediately yields a representation of the finite W algebra. This also allows us to derive Fock realizations for arbitrary finite W algebras since Fock realizations for simple Lie algebras are well known. This replaces the cumbersome construction of W algebras as commutants of screening operators. As an illustrative example we realize the finite dimensional representations of the finite W algebra $\bar{W}_3^{(2)}$ as a subrepresentation of certain Fock realizations. In principle this provides the first term of a Fock space resolution of these representations [15].

2. Classical Finite W Algebras

In this section we develop the theory of classical finite W algebras. As mentioned before these will be certain reductions of Kirillov Poisson structures on simple Lie algebras. This Poisson structure is well known to be given by

$$\{F, G\}(\alpha) = (\alpha, [\mathrm{grad}_\alpha F, \mathrm{grad}_\alpha G]) , \tag{2.1}$$

where $F, G \in C^\infty(g)$, $\alpha \in g$, $(\cdot,\cdot)$ is the Cartan–Killing form on g and $\mathrm{grad}_\alpha F$ is defined by

$$\frac{d}{d\varepsilon} F(\alpha + \varepsilon\beta)|_{\varepsilon=0} = (\beta, \mathrm{grad}_\alpha F) . \tag{2.2}$$

More explicitly if $\{t_a\}$ is a basis of g, J^a is the dual basis ($J^a(t_b) = \delta_b^a$) and $[t_a, t_b] = f_{ab}^c t_c$, then $\{J^a, J^b\} = f_c^{ab} J^c$. Here we used the Cartan–Killing metric $g_{ab} = (t_a, t_b)$ to raise and lower indices.

Let us now briefly discuss what we mean by reduction. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and $\{\phi_i\}_{i=1}^n$ a set of second class constraints, then on the zero set $\bar{M}$ of the constraints $\{\cdot,\cdot\}$ induces a Poisson structure known as the Dirac bracket [18]

$$\{\bar{f}, \bar{g}\}^* = \overline{\{f, g\} - \{f, \phi_i\}\Delta^{ij}\{\phi_j, g\}} , \tag{2.3}$$

where $f, g \in C^\infty(M)$, the bar denotes restriction to $\bar{M}$ and Δ^{ij} is the inverse of $\Delta_{ij} = \{\phi_i, \phi_j\}$. If some constraints are first class then the Dirac bracket is not defined as one can easily see from (2.3). This is caused by the fact that first class constraints generate gauge invariances [18]. These can be fixed by adding gauge fixing constraints in such a way that the total set of constraints is completely second class.

The constraints we impose will be determined by a certain sl_2 embedding [19]. The motivation for this was given in [6] in which it was shown that to every sl_2 embedding into the simple Lie algebra underlying a certain Kac–Moody algebra there is associated a reduction leading to a W algebra. Since we are interested in constructing the finite W algebras associated to those W algebras we will mimic the constraints used in [6]. Before this let us introduce some notations and prove some lemmas that we shall need later on.

Let $i: sl_2 \hookrightarrow g$ be an embedding of sl_2 into a simple Lie algebra g and let $\{t_0, t_+, t_-\}$ be the standard generators of $i(sl_2)$. The Cartan element t_0, called the defining vector of the embedding, can always be chosen to be an element of the Cartan subalgebra of g [19]. Therefore it defines a $\frac{1}{2}\mathbf{Z}$ gradation of g given by

$$g = \bigoplus_{m \in \frac{1}{2}\mathbf{Z}} g^{(m)} ; \tag{2.4}$$

where $g^{(m)} = \{x \in g \,|\, [t_0, x] = mx\}$. We can choose a basis

$$\{t_{j,m}^{(\mu)}\}_{j \in \frac{1}{2}\mathbf{N};\, -j \leqq m \leqq j;\, 1 \leqq \mu \leqq n_j^{\mathrm{ad}}} \tag{2.5}$$

of $g = sl_n$ (where n_j^{ad} is the multiplicity of the spin j representation of sl_2) such that

$$\begin{aligned} [t_3, t_{j,m}^{(\mu)}] &= m t_{j,m}^{(\mu)} , \\ [t_\pm, t_{j,m}^{(\mu)}] &= c(j,m) t_{j,m\pm 1}^{(\mu)} , \end{aligned} \tag{2.6}$$

where $c(j, m)$ are standard normalization factors. We will always take the labeling to be such that $t_{1,\pm 1}^{(1)} \equiv t_\pm$ and $t_{1,0}^{(1)} \equiv t_0$. We have the following

Lemma 1. *The spaces $g^{(m)}$ and $g^{(n)}$ are orthogonal* w.r.t. *the Cartan–Killing form on g,* i.e.

$$(g^{(m)}, g^{(n)}) = 0 \tag{2.7}$$

iff $m \neq -n$.

Proof. Let $x \in g^{(m)}$ and $y \in g^{(n)}$, then obviously $([t_0, x], y) = m(x, y)$ but also $([t_0, x], y) = -(x, [t_0, y]) = -n(x, y)$, where we used the invariance property of the Cartan–Killing form. It follows immediately that $(n + m)(x, y) = 0$. Therefore if $n \neq -m$ we must have $(x, y) = 0$. This proves the lemma.

For notational convenience we shall sometimes denote the basis elements $t_{j,m}^{(\mu)}$ simply by t_a, where a is now the multi-index $a \equiv (j, m; \mu)$. Let K_{ab} denote the matrix components of the Cartan–Killing form in this basis, i.e. $K_{ab} = (t_a, t_b)$ and let K^{ab} denote its matrix inverse.

From the above lemma and the fact that the Cartan–Killing form is non-degenerate on g follows immediately that $g^{(k)}$ and $g^{(-k)}$ are non-degenerately paired. This implies that if $t_a \in g^{(k)}$ then $K^{ab} t_b \in g^{(-k)}$ (where we used summation convention).

We then have the following lemma which we shall need later.

Lemma 2. *If t_a is a highest weight vector* (i.e. $a = (j, j; \mu)$ *for some j and μ*) *then $K^{ab} t_b$ is a lowest weight vector (and vice versa). In particular if $t_a \in C(i)$ then $K^{ab} t_b \in C(i)$.*

Proof. Since t_a is a highest weight vector we have $0 = ([t_a, t_+], x) = (t_a, [t_+, x])$ for all $x \in g$ which means that t_a is orthogonal to $\mathrm{Im}(\mathrm{ad}_{t_+})$ or put differently $\mathrm{Ker}(\mathrm{ad}_{t_+}) \perp \mathrm{Im}(\mathrm{ad}_{t_+})$. It is easy to do the same thing for t_-. One therefore has the following decomposition of g into mutually orthogonal spaces:

$$g = (\mathrm{Ker}(\mathrm{ad}_{t_+}) + \mathrm{Ker}(\mathrm{ad}_{t_-})) \oplus (\mathrm{Im}(\mathrm{ad}_{t_+}) \cap \mathrm{Im}(\mathrm{ad}_{t_-})) \tag{2.8}$$

where

$$(\mathrm{Ker}(\mathrm{ad}_{t_+}) + \mathrm{Ker}(\mathrm{ad}_{t_-})) \perp (\mathrm{Im}(\mathrm{ad}_{t_+}) \cap \mathrm{Im}(\mathrm{ad}_{t_-})) . \tag{2.9}$$

Also one has the following decomposition:

$$\mathrm{Ker}(\mathrm{ad}_{t_+}) + \mathrm{Ker}(\mathrm{ad}_{t_-}) = C(i) \bigoplus_{k>0} \mathrm{Ker}(\mathrm{ad}_{t_+})^{(k)} \bigoplus_{k>0} \mathrm{Ker}(\mathrm{ad}_{t_-})^{(-k)} . \tag{2.10}$$

As we saw above $K^{ab}t_b \in g^{(-k)}$ iff $t_a \in g^{(k)}$. Therefore

$$\begin{aligned} \mathrm{Ker}(\mathrm{ad}_{t_\pm})^{(\pm k)} &\perp \mathrm{Ker}(\mathrm{ad}_{t_\pm})^{(\pm l)} \quad \text{for all } k, l > 0 \\ \mathrm{Ker}(\mathrm{ad}_{t_+})^{(k)} &\perp \mathrm{Ker}(\mathrm{ad}_{t_-})^{(-l)} \quad \text{for } k \neq l \\ \mathrm{Ker}(\mathrm{ad}_{t_+})^{(k)} &\perp C(i) \quad \text{for all } k > 0 \\ \mathrm{Ker}(\mathrm{ad}_{t_-})^{(-k)} &\perp C(i) \quad \text{for all } k > 0 \,, \end{aligned}$$

from which the lemma follows.

Note that from the proof of this lemma follows that the spaces $\mathrm{Ker}(\mathrm{ad}_{t_+})^{(k)}$ and $\mathrm{Ker}(\mathrm{ad}_{t_-})^{(-k)}$ are nondegenerately paired. The same is true for the centralizer $C(i)$ with itself.

The procedure of Poisson reduction can be applied to Kirillov Poisson structures to give new and in general nonlinear Poisson algebras. If the Lie algebra is a KM algebra then, as was shown in [6] many W-algebras appearing in Conformal Field theory can be constructed in this way. These Poisson reductions are associated to inequivalent sl_2 embeddings into the finite underlying algebra of the KM algebra. In [14] it was announced that the same procedure can be applied to finite dimensional Lie algebras and an interesting special example was considered in detail. In this section the general theory of these reductions will be developed.

Let there be given a certain sl_2 subalgebra $\{t_0, t_+, t_-\}$ of g and let $\{t_{j,m}^{(\mu)}\}$ be the basis of g introduced in the previous section. Associated with this basis is a set of C^∞ functions $\{J_{(\mu)}^{j,m}\}$ on g with the property $J_{(\mu)}^{j,m}(t_{j',m'}^{(\mu')}) = \delta_\mu^{\mu'}\delta_{j'}^{j}\delta_{m'}^{m}$. These can be called the (global) coordinate functions on g in the basis $\{t_{j,m}^{(\mu)}\}$.

Let's now (motivated by [6]) impose the following set of constraints

$$\{\phi_{(\mu)}^{j,m} \equiv J_{(\mu)}^{j,m} - \delta_1^j\delta_1^m\delta_\mu^1\}_{j\in\frac{1}{2}\mathbb{N};\, m>0} \tag{2.11}$$

(remember that $t_{1,\pm 1}^{(1)} \equiv t_\pm$; $t_{1,0}^{(1)} \equiv t_0$). Denote the "zero set" in g of these constraints by g_c. Its elements have the form

$$\alpha = t_+ + \sum_j \sum_{m\leqq 0} \sum_\mu \alpha_{(\mu)}^{j,m} t_{j,m}^{(\mu)} \,, \tag{2.12}$$

where $\alpha_{(\mu)}^{j,m}$ are real or complex numbers (depending on which case we want to consider).

The constraints postulated above are motivated in the infinite dimensional case by the requirement that the Poisson algebra which we obtain after reduction must be a W algebra [5, 6]. In principle, from a mathematical point of view, one could consider more general sets of constraints, however since we are primarily interested in applications of our theory in conformal field theory we shall restrict ourselves to the constraints (2.11).

As discussed earlier we need to find out which of the constraints (2.11) are first class for they will generate gauge invariances on g_c. This is the subject of the next lemma.

Lemma 3. *The constraints* $\{\phi_{(\mu)}^{j,m}\}_{m\geqq 1}$ *are first class.*

Proof. First we show that

$$\{J_{(\mu)}^{j,m}, J_{(\mu')}^{j',m'}\} = \sum_{j'',\mu''} \alpha_{j''}^{\mu''} J_{(\mu'')}^{j'',m+m'} \tag{2.13}$$

for some coefficients $\alpha_j^{(\mu)}$. Let $x \in g^{(m'')}$, then

$$\{J_{(\mu)}^{j,m}, J_{(\mu')}^{j',m'}\}(x) = (x, [\mathrm{grad}_x J_{(\mu)}^{j,m}, \mathrm{grad}_x J_{(\mu')}^{j',m'}]) . \tag{2.14}$$

Now let y be an element of $g^{(k)}$, then

$$(y, \mathrm{grad}_x J_{(\mu)}^{j,m}) = \frac{d}{d\varepsilon} J_{(\mu)}^{j,m}(x + \varepsilon y)|_{\varepsilon = 0} , \tag{2.15}$$

which is zero except when $k = m$ (see definition of $J_{(\mu)}^{j,m}$). From this and the lemmas of the previous section one concludes that $\mathrm{grad}_x J_{(\mu)}^{j,m} \in g^{(-m)}$ and that the Poisson bracket (2.14) is nonzero if and only if $m'' = m + m'$. From this Eq. (2.13) follows.

Consider now the Poisson bracket between two constraints in the set $\{\phi_{(\mu)}^{j,m}\}_{m \geqq 1}$,

$$\begin{aligned}\{\phi_{(\mu)}^{j,m}, \phi_{(\mu')}^{j',m'}\} &= \{J_{(\mu)}^{j,m}, J_{(\mu')}^{j',m'}\} \\ &= \sum_{j'',\mu''} \alpha_{j'',\mu''} J_{(\mu'')}^{j'',m+m'} \\ &= \sum_{j'',\mu''} \alpha_{j'',\mu''} \phi_{(\mu'')}^{j'',m+m'} ,\end{aligned}$$

which is obviously equal to zero on g_c. Note that the fact $m, m' \geqq 1$ was used in the last equality sign. This proves the lemma.

Note that in general the set $\{\phi_{(\mu)}^{j,m}\}_{m \geqq 1}$ is not equal to the total set of constraints since the constraints with $m = \frac{1}{2}$ are not included. These constraints will turn out to be second class.

Let us now determine the group of gauge transformations on g_c generated by the first class constraints. Again we use the multi-index notation where now Roman letters $a, b, \ldots$ run over all j, m and μ, and Greek letters $\alpha, \beta \ldots$ over all j, μ but only $m > 0$ (those are the indices associated with the constraints). Let ϕ^α be one of the first class constraints (i.e. $\alpha \equiv (j, m; \mu)$ with $m \geqq 1$), then the gauge transformations associated to it are generated by its Hamiltonian vectorfields $X_\alpha \equiv \{\phi^\alpha, \cdot\}$. Let $x = x^a t_a \in g$, then the change of x under a gauge transformation generated by ϕ^α is given by

$$\begin{aligned}\delta_\alpha x &= \varepsilon\{\phi^\alpha, J^a\}(x) t_a = \varepsilon\{J^\alpha, J^a\}(x) t_a \\ &= \varepsilon g^{\alpha a} f_{ab}^c x^b t_c = [\varepsilon g^{\alpha a} t_a, x] .\end{aligned} \tag{2.16}$$

Since $g^{\alpha a} t_a \in g^{(-k)}$ iff $t_\alpha \in g^{(k)}$ (see lemmas in the previous section) we find that the Lie algebra of gauge transformations is given by

$$h = \bigoplus_{k \geqq 1} g^{(-k)} . \tag{2.17}$$

This is obviously a nilpotent Lie subalgebra of g and can be exponentiated to a group H. This is the gauge group generated by the first class constraints. Note that from Eq. (2.16) it follows that H acts on g_c in the adjoint representation, i.e. the gauge orbit of a point $x \in g_c$ is given by $\mathcal{O} = \{gxg^{-1} \mid g \in H\}$.

Now that we have identified the gauge group we can come to the matter of constructing the space g_c/H, or equivalently, gauge fixing. Of course this can be done in many ways, however in [5, 6] it was argued that there are certain gauges which are the most convenient from the conformal field theoretic point of view.

These are the so-called "lowest weight gauges." Define the subset g_{fix} of g_c as follows:

$$g_{\text{fix}} = \left\{ t_+ + \sum_{j,\mu} x^j_{(\mu)} t^{(\mu)}_{j,-j} \,|\, x^j_{(\mu)} \in \mathbf{C} \right\} . \tag{2.18}$$

We then have the following

Theorem 1.

$$H \times g_{\text{fix}} \simeq g_c . \tag{2.19}$$

Proof. Note first the obvious fact that $g^{(-l)} = g^{(-l)}_{lw} \oplus g^{(-l)}_0$, where $g_0^{(-l+1)} = [t_+, g^{(-l)}]$ and $[t_-, g^{(-l)}_{lw}] = 0$. Let now $x \in g_c$ which means that $x = t_+ + x^{(0)} + \cdots + x^{(-p)}$, where $x^{(-k)} \in g^{(-k)}$ and $p \in \frac{1}{2}\mathbf{N}$ is the largest j value in the decomposition of the adjoint representation w.r.t. the embedding i. Of course each $x^{(-k)}$ can be written as a sum $x_0^{(-k)} + x_{lw}^{(-k)}$, where $x^{(-k)} \in g_0^{(-k)}$ and $x^{(-k)} \in g_{lw}^{(-k)}$. Let also $\alpha^{(-k)}$ be an element of grade $-k$ in h, i.e. $\alpha^{(-k)} \in g^{(-k)} \subset h$. Then

$$(e^{\alpha^{(-k)}} x e^{-\alpha^{(-k)}})^{(-k+1)} = (x_0^{(-k+1)} + x_{lw}^{(-k+1)}) - \mathrm{ad}_{t_+}(\alpha^{(-k)}) ,$$

$$(e^{\alpha^{(-k)}} x e^{-\alpha^{(-k)}})^{(-k+i)} = x^{(-k+i)} \quad \text{for } 1 < i \leqq k ,$$

i.e. only the elements $x^{(-k+1)}, \ldots, x^{(-p)}$ are changed by this gauge transformation. Now, since $\mathrm{ad}_{t_+}: g^{(-k)} \to g_0^{(-k+1)}$ is bijective by definition, there exists a unique $\alpha^{(-k)} \in g^{(-k)}$ such that $\mathrm{ad}_{t_+}(\alpha^{(-k)}) = x_0^{(-k+1)}$, i.e. the element $x_0^{(-k+1)}$ can be gauged away by choosing $\alpha^{(-k)}$ appropriately. From this follows immediately that there exist unique elements $\alpha^{(-l)} \in g^{(-l)}$ $(l = 1, \ldots, p)$ such that

$$e^{\alpha^{(-p)}} \ldots e^{\alpha^{(-1)}} x e^{-\alpha^{(-1)}} \ldots e^{-\alpha^{(-p)}} = y \in g_{\text{fix}} . \tag{2.20}$$

This provides, as one can now easily see, a bijective map between $H \times g_{\text{fix}}$ and g_c given by

$$(e^{-\alpha^{(-1)}} \ldots e^{-\alpha^{(-p)}}, y) \to x . \tag{2.21}$$

This proves the theorem.

So starting from g, after imposing constraints and fixing gauge invariances we have arrived at a submanifold g_{fix} of g. We now want to determine the Poisson algebra structure of $C^\infty(g_{\text{fix}})$. For this we need to calculate the Dirac brackets

$$\{ J^{j,-j}_{(\mu)}, J^{j',-j'}_{(\mu')} \}^* \tag{2.22}$$

between the generators $\{J^{j,-j}_{(\mu)}\}$ of $C^\infty(g_{\text{fix}})$. We shall first address a slightly more general problem and then specialize to reductions associated to sl_2 embeddings.

Let $\{t_i\}_{i=1}^{\dim(g)}$ be a basis of the Lie algebra g, let k be a positive integer smaller or equal to $\dim(g)$ and denote $t_{k+1} \equiv \Lambda$. Consider then the following subset of g

$$g_f = \left\{ \Lambda + \sum_{i=1}^{k} \alpha^i t_i \,\middle|\, \alpha^i \in \mathbf{C} \right\} , \tag{2.23}$$

which can be seen as the zero set of the constraints $\phi^1 = J^{k+1} - 1$ and $\phi^i = J^{k+i}$ for $1 < i \leq \dim(g) - k$. Also suppose that the Kirillov bracket on g induces a Dirac bracket $\{.,.\}^*$ on g_f (i.e. all constraints are second class).

Denote by $\mathcal{R}$ the set of smooth functions R: $\mathbf{C}^k \times g_f \to g$ of the form

$$R(\vec{z}; y) = \sum_{i=1}^{k} z_i R^i(y) , \tag{2.24}$$

where $\vec{z} \equiv \{z_i\}_{i=1}^k \in \mathbf{C}^k$, $y \in g_f$ and $R^i(y) \in sl_n$. To any element $R \in \mathcal{R}$ one can associate a map Q_R: $\mathbf{C}^k \times g_f \to C^\infty(g)$ defined by

$$Q_R(\vec{z}; y) = \sum_{i=1}^{\dim(g)} (R(\vec{z}; y), t_i) J^i \tag{2.25}$$

(where as before $J^i \in C^\infty(g)$ is defined by $J^i(t_j) = \delta^i_j$). We are going to use certain elements of the set $\mathcal{R}$ in order to explicitly calculate the Dirac brackets on g_f. We have the following theorem.

Theorem 2. *If there exists an $R \in \mathcal{R}$ such that for all $\vec{z} \in \mathbf{C}^k$ and $y \in g_f$ we have*

$$\Lambda + [R(\vec{z}; y), y] \in g_f \tag{2.26}$$

and

$$Q_R(\vec{z}; y)|_{g_f} = \sum_{i=1}^{k} z_i J^i + (R(\vec{z}; y), \Lambda) \tag{2.27}$$

(i.e. *the restriction of $Q_R(\vec{z}; y)$ to g_f is equal to the right-hand side of* (2.27)), *then*

$$\sum_{i,j=1}^{k} z_i \{J^i, J^j\}^*(y) t_j = [R(\vec{z}; y), y] \tag{2.28}$$

for all $\vec{z} \in \mathbf{C}^k$ and $y \in g_f$.

The proof of this theorem has been given for the affine case and for the principal embedding in [5]. It is however straightforward to prove the more general statement given above using the same arguments.

Note that from Eq. (2.28) one can read off all the Dirac brackets between the generators $\{J^i\}_{i=1}^k$ of $C^\infty(g_f)$ since the formula holds for all $\vec{z} \in \mathbf{C}^k$ and the elements t_j are all independent. The only thing that one therefore needs to determine is the map R. Also note that from Eq. (2.27) it follows that within the Dirac bracket $\{\cdot, \cdot\}^*$ the function $Q_R(\vec{z}; y)$ is equal to $\sum_{i=1}^k z_i J^i$, i.e.

$$\{Q_R(\vec{z}; y), \cdot\}^* = \sum_{i=1}^{k} z_i \{J^i, \cdot\}^* , \tag{2.29}$$

since constants commute with everything and restriction to g_f is always implied within the Dirac bracket.

Note that conversely in order to show the existence of the Dirac bracket on g_f it is sufficient to prove that Eqs. (2.26) and (2.27) of the above theorem are solvable within $\mathcal{R}$. We will now show that this is the case when $g_f \equiv g_{\text{fix}}$ associated to an arbitrary sl_2 embedding.

Theorem 3. *Let $|i|$ be the total number of sl_2 multiplets in the branching of the adjoint representation of g. There is a unique $R \in \mathcal{R}$ such that*

$$t_+ + [R(\vec{z}; y), y] \in g_{\text{fix}} ,$$
$$Q_R(\vec{z}; y)|_{g_{\text{fix}}} = \sum_{j;\mu} z_j^{(\mu)} J^{j,-j}_{(\mu)} + (R(\vec{z}; y), t_+)$$

for all $\vec{z} \in \mathbf{C}^{|i|}$ and $y \in g_{\text{fix}}$.

The proof of this theorem for the principal embedding (and in the affine case) has been given in [5]. It is straightforward however to generalize the proof to arbitrary embeddings.

It is possible to derive a general formulae for $R(\vec{z}; y)$ using arguments similar to those used in [8]. Let again

$$g_{lw} = \bigoplus_{j \in \frac{1}{2}\mathbb{N}} g_{lw}^{(-j)} = \mathrm{span}(\{t_{j,-j}^{(\mu)}\}_{j;\mu} , \tag{2.30}$$

and let Π be the orthogonal (w.r.t. the Cartan–Killing form) projection onto $\mathrm{Im}(\mathrm{ad}_{t_+})$. Obviously the map ad_{t_+}: $\mathrm{Im}(\mathrm{ad}_{t_-}) \to \mathrm{Im}(\mathrm{ad}_{t_+})$ is invertible. Denote the inverse of this map, extended by 0 to g by L. As before what we want to do is solve the equation $[R, y] = x \in g_{lw}$ for $y \in g_{\mathrm{fix}}$. Noting that $y = t_+ + w$, where $w \in g_{lw}$ and applying Π this equation reduces to

$$\Pi \circ \mathrm{ad}_{t_+} R(\varepsilon) = \varepsilon \Pi([R(\varepsilon), w]) , \tag{2.31}$$

where we introduced a parameter ε in the right-hand side which we want to put to 1 later. Note that the left-hand side is equal to $\mathrm{ad}_{t_+} R(\varepsilon)$ since this is already an element of $\mathrm{Im}(\mathrm{ad}_{t_+})$. Assume now that $R(\varepsilon)$ can be (perturbatively) written as

$$R(\varepsilon) = \sum_{k=0}^{\infty} R_k \varepsilon^k \tag{2.32}$$

(we shall have to justify this later). The zeroth order part of Eq. (2.31) reads

$$\mathrm{ad}_{t_+} R_0 = 0 . \tag{2.33}$$

This means that R_0 is of the form $\sum_{j,\mu} z_j^{(\mu)} t_{(\mu)}^{j,j} \equiv F(\vec{z})$. The first order equation is equal to $\mathrm{ad}_{t_+} R_1 = \Pi([F, w])$. Obviously this equation is solved by $R_1 = -L \circ \mathrm{ad}_w F$. Proceeding with higher orders we find

$$R_{k+1}(\vec{z}; y) = -L \circ \mathrm{ad}_w (R_k(\vec{z}; y)) , \tag{2.34}$$

which means that

$$R(\vec{z}; y; \varepsilon) = \frac{1}{1 + \varepsilon L \circ \mathrm{ad}_w} F(\vec{z}) . \tag{2.35}$$

There are no convergence problems with this series since the operator L lowers the degree by one which means that after $2p$ steps it must cancel. Taking $\varepsilon = 1$ we find

$$\sum_{j,j';\mu\mu'} z_j^{(\mu)} \{ J_{(\mu)}^{j,-j}, J_{(\mu')}^{j',-j'} \}^*(y) t_{j',-j'}^{(\mu')} = -\mathrm{ad}_y \left(\frac{1}{1 + L \mathrm{ad}_w} F(\vec{z}) \right) . \tag{2.36}$$

Now let $w \in g_{lw}$ and $Q \in C^\infty(g_{lw})$. We then define $\mathrm{grad}_w Q \in \mathrm{Ker}(\mathrm{ad}_{t_+}) \equiv g_{hw}$ by

$$(x, \mathrm{grad}_w Q) \equiv \frac{d}{d\varepsilon} Q(w + \varepsilon x)|_{\varepsilon=0} \quad \text{for all } x \in g_{lw} . \tag{2.37}$$

Note that this uniquely defines $\mathrm{grad}_w Q$ because, as we saw before, g_{lw} and g_{hw} are nondegenerately paired. Using Eq. (2.36) we can now give the following general description of the classical finite W algebra: it is nothing but the Poisson algebra $(C^\infty(g_{lw}), \{\cdot,\cdot\}^*)$, where

$$\{Q_1, Q_2\}^*(w) = \left(w, \left[\mathrm{grad}_w Q_1, \frac{1}{1 + L\circ \mathrm{ad}_w}\mathrm{grad}_w Q_2\right]\right) \tag{2.38}$$

for all $Q_1, Q_2 \in C^\infty(g_{lw})$ and $w \in g_{lw}$. For the so-called trivial embedding $(t_0 = t_\pm = 0)$ the map L is equal to the zero map and the above formula reduces to the ordinary Kirillov bracket as it should (because in that case $g_{\mathrm{fix}} \equiv g$). For nontrivial sl_2 embeddings however (2.38) is a new and highly nontrivial Poisson structure.

From now on denote the Kirillov Poisson algebra associated to a Lie algebra g by $K(g)$ and the Poisson algebra $(C^\infty(g_{lw}), \{\cdot,\cdot\}^*)$ by $\mathscr{W}(i)$, where i is again the sl_2 embedding in question.

In general $\mathscr{W}(i)$ is a non-linear Poisson algebra as we shall see when we consider examples. However it can happen that $\mathscr{W}(i)$ contains a subalgebra that is isomorphic to a Kirillov algebra. The next theorem deals with this.

Theorem 4. *The finite W algebra $\mathscr{W}(i)$ has a Poisson subalgebra which is isomorphic to $K(C)$, where C is the centralizer of the sl_2 subalgebra in g.*

Proof. Since all elements of the centralizer are lowest (and highest) weight vectors w.r.t. $i(sl_2)$ the function J^a is not constrained if $t_a \in C(i)$, i.e. all the elements J^a associated to the centralizer survive the reduction. It is not obvious however that they still form the same algebra w.r.t. the Poisson bracket (refPA). This is what we have to show.

The part of the equation

$$\sum_{j,j';\mu,\mu'} z_j^{(\mu)}\{J_{(\mu)}^{j,-j}, J_{(\mu')}^{j',-j'}\}^*(y)t_{j',-j'}^{(\mu')} = [R(\vec{z}; y), y] \tag{2.39}$$

that determines the Poisson relations between the currents associated to the centralizer is

$$\sum_{\mu,\mu'} z_0^{(\mu)}\{J_{(\mu)}^{0,0}, J_{(\mu')}^{0,0}\}^*(y)t_{0,0}^{(\mu')} = [R(\vec{z}; y), y]^{(0)}\,. \tag{2.40}$$

The right-hand side of this equation reads in more detail

$$\begin{aligned}[R, y]^{(0)} = [R_{lw}^{(0)}, y^{(0)}] + [R_0^{(0)}, y^{(0)}] + \cdots + [R^{(p)}, y^{(-p)}]\\ + [R^{(-1)}, t_+]\,.\end{aligned} \tag{2.41}$$

Note that $R_{lw}^{(0)}$ and $y^{(0)}$ are both elements of $C(i)$ which means that the first term in the right-hand side is also an element of $C(i)$. We will now show that $R_0^{(0)}, R^{(1)}, \ldots, R^{(p)}$ and $R^{(-1)}$ do not depend on $\{z_0^{(\mu)}\}_\mu$ which means that all but the first term in the right-hand side of Eq. (2.41) are irrelevant for the Poisson brackets $\{J_{(\mu)}^{0,0}, J_{(\mu')}^{0,0}\}^*$.

One can easily see that $R_0^{(k)}$ is only a function of $z_p^{(\mu)}, \ldots, z_{k+1}^{(\mu)}$ for $k \geqq 0$ and therefore $R^{(k)} = R^{(k)}(z_p^{(\mu)}, \ldots, z_k^{(\mu)})$. Now, $R^{(-1)}$ is determined by the equation

$$[t_+, R^{(-1)}] = [R^{(0)}, y^{(0)}]_0 + \cdots + [R^{(p)}, y^{(-p)}]_0\,. \tag{2.42}$$

Note that the terms $[R^{(l)}, y^{(-l)}]$ for $l > 0$ certainly do not contain $z_0^{(\mu)}$ as we just saw. Also note that

$$[R^{(0)}, y^{(0)}]_0 = [R_0^{(0)} + R_{lw}^{(0)}, y^{(0)}]_0 = [R_0^{(0)}, y^{(0)}] \tag{2.43}$$

because of the reason we mentioned earlier that $R_{hw}^{(0)}$ and $y^{(0)}$ are both in $C(i)$. However, as we have seen above $R_0^{(0)}$ does not contain $z_0^{(\mu)}$. From this we conclude that the only term in $[R, y]^{(0)}$ that contains $z_0^{(\mu)}$ is $[R_{lw}^{(0)}, y^{(0)}]$, i.e.

$$\begin{aligned}\sum_{\mu\mu'} z_0^{(\mu)} \{J_{(\mu)}^{0,0}, J_{(\mu')}^{0,0}\}^*(y) t_{0,0}^{(\mu)} &= [R_{lw}^{(0)}, y^{(0)}] \\ &= \left[R_{lw}^{(0)}, \sum_\mu J_{(\mu)}^{0,0}(y) t_{0,0}^{(\mu)}\right].\end{aligned} \tag{2.44}$$

The generators $\{t_{0,0}^{(\mu)}\}$ are a basis of $C(i)$. From this, Eq. (2.44) and Theorem 2 follows immediately that the Poisson algebra generated by $\{J_{(\mu)}^{0,0}\}$ w.r.t. the Dirac bracket is isomorphic to Kirillov algebra of $C(i)$. This proves the theorem.

2.1. Generalized Finite Miura Transformations. In this section we will present a generalized version of the Miura transformation. Roughly this is a Poisson homomorphism of the finite W algebra $\mathscr{W}(i)$ in question to a certain Kirillov algebra. In order to be able to describe this map for arbitrary embeddings however we first have to concern ourselves with the cases when in the decomposition of g into sl_2 multiplets there appear half integer grades. As we have seen, in those cases the constraints $\phi_{(\mu)}^{j,\frac{1}{2}}$ are second class. In what follows it will be necessary to be able to replace the usual set of constraints by an alternative set which contains only first class constraints but which gives rise to the same reduction [20]. Roughly what one does is impose only half of the constraints that turned out to be second class in such a way that they become first class. The other constraints that were second class can then be obtained as gauge fixing conditions. In this way g_{fix} stays the same but g_c is different. Since the resulting Poisson algebra only depends on g_{fix} it is clear that we obtain the same algebra $\mathscr{W}(i)$.

Let's now make all of this more precise for $g = sl_n$. We describe the sl_n algebra in the standard way by traceless $n \times n$ matrices; E_{ij} denotes the matrix with a one in its (i, j) entry and zeroes everywhere else. As we said earlier embeddings of sl_2 into sl_n are in one-to-one correspondence with partitions of n. Let $(n_1, n_2, \ldots)$ be a partition of n, with $n_1 \geqq n_2 \geqq \ldots$. Define a different partition $(m_1, m_2, \ldots)$ of n, with m_k equal to the number of i for which $n_i \geqq k$. Furthermore let $s_t = \sum_{i=1}^{t} m_i$. Then we have the following

Lemma 1. *An embedding of sl_2 in sl_n under which the fundamental representation branches according to $n \to \bigoplus n_i$ is given by*

$$\begin{aligned} t_+ &= \sum_{l \geqq 1} \sum_{k=1}^{n_l - 1} E_{l+s_{k-1}, l+s_k}, \\ t_0 &= \sum_{l \geqq 1} \sum_{k=1}^{n_l} \left(\frac{n_l + 1}{2} - k\right) E_{l+s_{k-1}, l+s_{k-1}}, \\ t_- &= \sum_{l \geqq 1} \sum_{k=1}^{n_l - 1} k(n_l - k) E_{l+s_k, l+s_{k-1}}. \end{aligned} \tag{2.45}$$

The proof is by direct computation[1]. If the fundamental representation of sl_n is spanned by vectors $v_1, \ldots, v_n$, on which sl_n acts via $E_{ij}(v_k) = \delta_{jk} v_i$, then the irreducible representations n_l of sl_2 to which it reduces are spanned by $\{v_{l+s_{k-1}}\}_{1 \leqq k \leqq n_l}$. Previously we decomposed the Lie algebra g into eigenspaces of ad_{t_0}. However, for our present purposes it is convenient to introduce a different grading of the Lie algebra g, that we need to describe the generalized Miura map and the BRST quantization. The grading is defined by the following element of the Cartan subalgebra of sl_n:

$$\delta = \sum_{k \geqq 1} \sum_{j=1}^{m_k} \left(\frac{\sum_l l m_l}{\sum_l m_l} - k \right) E_{s_{k-1}+j, s_{k-1}+j} \,. \tag{2.46}$$

This leads to the alternative decomposition $g = g_- \oplus g_0 \oplus g_+$ of g into spaces with negative, zero and positive eigenvalues under the adjoint action of δ respectively (note that in case the grading of ad_{t_0} is an integral grading then we have $t_0 = \delta$ so in those cases nothing happens. In general however $t_0 \neq \delta$ and also $g^{(m)} \neq g_m$). The subalgebra g_0 consists of matrices whose nonzero entries are in square blocks of dimensions $m_1 \times m_1$, $m_2 \times m_2$, etc. along the diagonal of the matrix and is therefore a direct sum of sl_{m_k} subalgebras (modulo $u(1)$ terms). The nilpotent subalgebra g_+ is spanned by $\{E_{l+s_{k-1}, r+s_k}\}_{l \geqq 1; 1 \leqq k \leqq n_l - 1; r > 0}$, and the nilpotent subalgebra g_- by the transpose of these. Let $\pi_\pm$ denote the projections onto $g_\pm$. Then the following theorem describes the replacement of the mixed system of first and second class constraints by a system of first class constraints only.

Theorem 1. *The constraints $\{J^{l+s_{k-1}, r+s_k} - \delta^{r,l}\}_{l \geqq 1; 1 \leqq k \leqq n_l - 1; r > 0}$ are first class. The gauge group they generate is $\hat{H} = \exp(g_-)$ acting via the adjoint representation on g. The resulting finite W algebra is the same as the one obtained by imposing the constraints* (2.11).

Proof. Decompose g in eigenvalues of ad_δ, $g = \bigoplus_k g_k$. Note that ad_δ has only integral eigenvalues. Using the explicit form of t_+ in (2.45), one easily verifies that $[\delta, t_+] = t_+$. Thus, $t_+ \in g_1$. Again since $[g_+, g_+] = [g_{\geqq 1}, g_{\geqq 1}] \subset g_{\geqq 2}$ it follows (exactly like in Lemma 3) that the constraints $\{J^{l+s_{k-1}, r+s_k} - \delta^{r,l}\}_{l \geqq 1; 1 \leqq k \leqq n_l - 1; r > 0}$ are first class. The gauge group can be determined similarly as in Sect. 3.2, and the analogue of Theorem 1 of Sect. 3.2 can be proven in the same way with the same choice of g_{fix}, if one uses the decomposition of g in eigenspaces of ad_δ rather than ad_{t_0}. Therefore the resulting finite W algebra is the same, because Theorem 2 and 3 of Sect. 3.2 show that it only depends on the form of g_{fix}.

Let us explain this theorem in words. The number of second class constraints in any system is necessarily even. If one switches to the ad_δ grading the set of second class constraints is split into half: one half gets grade 1 w.r.t. ad_δ while the other half gets grade 0. Now what one does is impose only that half that has obtained grade 1 w.r.t. the ad_δ grading. These constraints are then first class. The gauge transformations they generate can be completely fixed by imposing the constraints that were in the other half. Having done that we are back in exactly the same situation as before. The only difference is that we now know of a system of first class constraints that in the end leads to the same reduction.

[1] The commutation relations are $[t_0, t_\pm] = \pm t_\pm$, and $[t_+, t_-] = 2t_0$

Note that the number of generators of the finite W algebra is equal to $\dim(g_0) = (\sum_i m_i^2) - 1$. This is indeed the same as the number of irreducible representations of sl_2, minus one, one obtains from $(\bigoplus_i n_i) \otimes (\bigoplus_i n_i)$, as the latter number equals $(\sum_j (2j-1)n_j) - 1$, and one easily checks that $\sum_i m_i^2 = \sum_j (2j-1)n_j$.

The generalized Miura transformation can now be formulated as

Theorem 5. *There exists an injective Poisson homomorphism from $\mathscr{W}(i)$ to $K(g_0)$.*

Proof. First we show that for every element $x \in t_+ + g_0$ there exist a unique element h in the gauge group $\hat{H}$ such that $h \cdot x \cdot h^{-1} \in g_{\mathrm{fix}}$. For this note that there exists a unique element $h' \in \hat{H}$ such that $h' \cdot x \cdot h'^{-1} \in g_c$. This follows from the previous theorem and the remarks made after it. In fact it follows from those remarks that h' is an element of that subgroup of $\hat{H}$ that is generated by the "ex" second class constraints (the ones that were made into first class constraints by not imposing the other half). It was shown earlier however that $g_{\mathrm{fix}} \times H \simeq g_c$ which means that there exists a unique element $h'' \in H$ such that $h''h' \cdot x \cdot (h''h')^{-1} \in g_{\mathrm{fix}}$. We conclude from this that there is a surjective map from $x \in t_+ + g_0$ to g_{fix}. The pull back of this map $C^\infty(g_{\mathrm{fix}}) \equiv \mathscr{W}(i) \to C^\infty(t_+ + g_0)$ (which will therefore be injective) is then the Miura map. Of course we still have to check whether this map is a Poisson homomorphism. This we address next.

What is the Poisson structure on $C^\infty(t_+ + g_0)$? Since the constraints of which the space $t_+ + g_0$ is the zero set are obviously second class the Kirillov bracket on g induces a Dirac bracket on it. It is not difficult to see that the Dirac term in the Dirac bracket cancels in this case which means that the Poisson algebra $C^\infty(t_+ + g_0)$ with the induced Poisson structure is isomorphic to the Poisson algebra $K(g_0)$. Since the transformation from $\mathscr{W}(i)$ to $K(g_0)$ corresponds to a gauge transformation the map is necessarily a homomorphism.

2.2. Examples. The simplest examples of finite W algebras are those associated to the so-called "principal sl_2 embeddings." These embeddings are associated to the trivial partition of the number n: $n = n$. The fundamental representation of $g = sl_n$ therefore becomes an irreducible representation of the sl_2 subalgebra, i.e. $\underline{n}_n \simeq \underline{n}_2$. The branching rule for the adjoint representation of g therefore reads

$$\underline{\mathrm{ad}}_n \simeq \underline{3}_2 \oplus \underline{5}_2 \oplus \cdots \oplus \underline{2n-1}_2 \,. \tag{2.47}$$

From this follows immediately that the finite W algebra will have $n-1$ generators (since there are $n-1$ sl_2 multiplets). Without going into details we can immediately predict the Poisson relations between these generators from the generalized Miura transformation for in this case the subalgebra $g^{(0)} = g_0$ coincides with the Cartan subalgebra of sl_n. Since the Cartan subalgebra is an abelian algebra, and since the Kirillov algebra of an abelian Lie algebra is a Poisson commutative algebra we find that a finite W algebra associated to a principal embedding must also be Poisson commutative since it is isomorphic to a Poisson subalgebra of $K(g_0)$. We conclude therefore that the principal sl_2 embedding into sl_n leads to the abelian Poisson algebras with $(n-1)$ generators.

The simplest nontrivial case of a finite W algebra is associated to the (only) nonprincipal embedding of sl_2 into sl_3. This embedding is associated to the following partition of 3: $3 \to 2 + 1$. The branching rule of the fundamental representation of sl_3 is therefore $\underline{3}_3 \simeq \underline{2}_2 \oplus \underline{1}_2$. From this we find the following branching rule for the adjoint representation

$$\underline{\mathrm{ad}}_3 \simeq \underline{3}_2 \oplus 2.\underline{2}_2 \oplus \underline{1}_2 \tag{2.48}$$

from which follows immediately that the finite W algebra associated to this embedding will have 4 generators. We shall go through the construction of this finite W algebra in some detail in order to illustrate the theory discussed above.

The explicit form of the sl_2 embedding is $t_+ = E_{1,3}$; $t_0 = \mathrm{diag}(\frac{1}{2}, 0, -\frac{1}{2})$; $t_- = E_{3,1}$, where as before E_{ij} denotes the matrix with a one in its (i,j) entry and zeros everywhere else. The (sl_3 valued) function $J = t^{(\mu)}_{j,m} J^{j,m}_{(\mu)}$ (where we used summation convention) reads

$$J = \begin{pmatrix} \frac{1}{2} J^{1,0}_{(1)} + J^{0,0}_{(1)} & J^{\frac{1}{2},\frac{1}{2}}_{(1)} & J^{1,1}_{(1)} \\ J^{\frac{1}{2},-\frac{1}{2}}_{(2)} & -2J^{0,0}_{(1)} & J^{\frac{1}{2},\frac{1}{2}}_{(2)} \\ J^{1,-1}_{(1)} & J^{\frac{1}{2},-\frac{1}{2}}_{(1)} & J^{0,0}_{(1)} - \frac{1}{2} J^{1,0}_{(1)} \end{pmatrix} . \tag{2.49}$$

According to the general prescription the constraints are

$$J^{1,1}_{(1)} - 1 = J^{\frac{1}{2},\frac{1}{2}}_{(1)} = J^{\frac{1}{2},\frac{1}{2}}_{(2)} = 0 , \tag{2.50}$$

the first one being the only first class constraint. As was shown earlier the gauge invariance generated by this constraint can be completely fixed by adding the "gauge fixing condition" $J^{1,0}_{(1)} = 0$. The Dirac brackets between the generators $\{J^{0,0}_{(1)}, J^{\frac{1}{2},-\frac{1}{2}}_{(1)}, J^{\frac{1}{2},-\frac{1}{2}}_{(2)}\}$ and $J^{1,-1}_{(1)}$ can now easily be calculated. In order to describe the final answer in a nice form introduce

$$\begin{aligned} C &= -\frac{4}{3}(J^{1,-1}_{(1)} + 3(J^{0,0}_{(1)})^2) , \\ E &= J^{\frac{1}{2},-\frac{1}{2}}_{(1)} , \\ F &= \frac{4}{3} J^{\frac{1}{2},-\frac{1}{2}}_{(2)} , \\ H &= 4J^{0,0}_{(1)} \end{aligned} \tag{2.51}$$

(note that this is an invertible basis transformation). The Dirac bracket algebra between these generators reads [14]

$$\begin{aligned} \{H, E\}^* &= 2E , \\ \{H, F\}^* &= -2F , \\ \{E, F\}^* &= H^2 + C , \end{aligned} \tag{2.52}$$

and C Poisson commutes with everything. This algebra which is called $\bar{W}_3^{(2)}$ was first constructed in [21] as a nonlinear deformation of $su(2)$. In [14] it was shown to be a reduction of sl_3 and its representation theory was explicitly constructed.

Let's now consider the finite Miura transformation for this algebra. Since the grading of sl_3 by ad_{t_0} is half integer we have to switch to the grading by ad_δ. The explicit form of δ is $\delta = \frac{1}{3}\mathrm{diag}(1, 1, -2)$. It is easily checked that this defines an integer grading of sl_3. The crucial change is that the elements E_{23} and E_{12} have grade 1 and 0 w.r.t. the ad_δ grading while they have grade $\frac{1}{2}$ w.r.t. ad_{t_0}. According to the prescription given in the previous section the alternative set of constraints that one now imposes in order to reduce the mixed system of first and second class constraints to a system of first class constraints only is

$$J^{1,1}_{(1)} - 1 = J^{\frac{1}{2},\frac{1}{2}}_{(2)} = 0\,. \tag{2.53}$$

As we already mentioned in the previous section, what has happened here is that one has now imposed only half of the constraints that turned out to be second class. The result is that both constraints in (2.53) are first class. The point however is that the gauge symmetry induced by the second constraint in Eq. (2.53) can be completely fixed by adding the gauge fixing condition $J^{\frac{1}{2},\frac{1}{2}}_{(1)} = 0$ which then leaves us with exactly the same set of constraints and gauge invariances as before.

We can now describe the generalized Miura map for this case. An arbitrary element of $t_+ + g_0$ is given by

$$J_0 \equiv \begin{pmatrix} h+s & e & 1 \\ f & s-h & 0 \\ 0 & 0 & -2s \end{pmatrix}. \tag{2.54}$$

Note that $g_0 \simeq sl_2 \oplus u(1)$ which means that the Poisson relations in $K(g_0)$ between the generators, h, e, f and s (viewed as elements of $C^\infty(g_0)$) are given by

$$\begin{aligned} \{h, e\} &= e\,, \\ \{h, f\} &= -f\,, \\ \{e, f\} &= 2h\,, \end{aligned} \tag{2.55}$$

and s commutes with everything. As shown in the previous section the equation that we have to solve in order to get explicit formulas for the Miura map is the following equation for $h \in \hat{H}$ (where $\hat{H}$ is again the group of gauge transformations generated by the two first class constraints (2.53)),

$$hJ_0h^{-1} = \begin{pmatrix} J^{0,0}_{(1)} & 0 & 1 \\ J^{\frac{1}{2},-\frac{1}{2}}_{(2)} & -2J^{0,0}_{(1)} & 0 \\ J^{1,-1}_{(1)} & J^{\frac{1}{2},-\frac{1}{2}}_{(1)} & J^{0,0}_{(1)} \end{pmatrix}. \tag{2.56}$$

The unique solution of this equation is given by

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{3}{2}s + \frac{1}{2}h & e & 1 \end{pmatrix}. \tag{2.57}$$

Inserting this back into Eq. (2.56) one finds certain expressions for $J^{0,0}_{(1)}$, $J^{\frac{1}{2},-\frac{1}{2}}_{(1)}$, $J^{\frac{1}{2},-\frac{1}{2}}_{(2)}$ and $J^{1,-1}_{(1)}$ in terms of the functions $\{e, f, h, s\}$. In terms of the generators (2.51) these read

$$\begin{aligned} H &= 2h - 2s\,, \\ E &= (3s - h)e\,, \\ F &= \frac{4}{3}f\,, \\ C &= -\frac{4}{3}(h^2 + 3s^2 + fe)\,. \end{aligned} \tag{2.58}$$

It is easy to check, using the relations (2.55) that these satisfy the algebra (2.52). Therefore we find that indeed (2.58) provides an injective Poisson homomorphism from $W_3^{(2)}$ into the Kirillov Poisson algebra of the Lie algebra $g_0 = sl_2 \oplus u(1)$.

3. Finite W Symmetries in Generalized Toda Theories

It is well known [22] that infinite W algebras arise as algebras of conserved currents of Toda theories. The infinite W algebras related to arbitrary sl_2 embeddings are related to the conserved currents of so-called generalized Toda theories [23, 7]. Since finite W algebras are essentially a dimensional reduction of infinite W algebras, one expects that there are one-dimensional analogues of the ordinary two-dimensional Toda actions whose conserved currents are related to a finite W algebra. Indeed, the construction of these one-dimensional Toda actions is completely straightforward, and has the desired properties.

One starts with the action for a free particle moving on the group manifold $G = SL(n)$

$$S[g] = \frac{1}{2}\int dt \,\mathrm{Tr}\left(g^{-1}\frac{dg}{dt}g^{-1}\frac{dg}{dt}\right) \tag{3.1}$$

with equations of motion,

$$\frac{d}{dt}\left(g^{-1}\frac{dg}{dt}\right) = \frac{d}{dt}\left(\frac{dg}{dt}g^{-1}\right) = 0\,. \tag{3.2}$$

Therefore the quantities

$$J = \frac{dg}{dt}g^{-1} \equiv J^a t_a \quad \text{and} \quad \bar{J} = g^{-1}\frac{dg}{dt} \equiv \bar{J}^a t_a \tag{3.3}$$

are conserved. They form a Poisson algebra [24] $\{J^a, J^b\} = f^{ab}_c J^c$ with similar equations for $\bar{J}$. This is precisely the Kirillov Poisson bracket we used as a starting point for the construction of finite W algebras.

Finite W algebras were obtained by imposing a set of first class constraints. In terms of the decomposition $g = g_- \oplus g_0 \oplus g_+$, these constraints were $\pi_+(J) = t_+$, where $\pi_\pm$ are the projections on $g_\pm$. Here we want to impose the same constraints, together with similar constraints on $\bar{J}$, i.e. $\pi_-(\bar{J}) = t_-$. If $G_\pm$ denote the subgroups of G with Lie algebra $g_\pm$, and G_0 the subgroup with Lie algebra g_0, then almost every element g of G can be decomposed as $g_- g_0 g_+$, where $g_{\pm,0}$ are elements of the corresponding subgroups, because G admits a generalized Gauss decomposition $G = G_- G_0 G_+$.[2] Inserting this decomposition into the definition of J and $\bar{J}$ and using arguments similar to those used in [7, 5] one finds that the equations of motion for these constrained currents are equivalent to

$$\begin{aligned} 0 &= g_-^{-1}\frac{dJ}{dt}g_- = \frac{d}{dt}\left(\frac{dg_0}{dt}g_0^{-1}\right) + [g_0 t_- g_0^{-1}, t_+]\,, \\ 0 &= g_+\frac{d\bar{J}}{dt}g_+^{-1} = \frac{d}{dt}\left(g_0^{-1}\frac{dg_0}{dt}\right) + [t_-, g_0^{-1} t_+ g_0]\,, \end{aligned} \tag{3.4}$$

[2] Strictly speaking $G_- G_0 G_+$ is only dense in G but we will ignore this subtlety in the remainder

which are generalized finite Toda equations as will be shown in a moment. The corresponding action is

$$S[g_0] = \frac{1}{2}\int dt\,\mathrm{Tr}\left(g_0^{-1}\frac{dg_0}{dt}g_0^{-1}\frac{dg_0}{dt}\right) - \int dt\,\mathrm{Tr}(g_0 t_- g_0^{-1} t_+)\,. \tag{3.5}$$

This generalized finite Toda action describes a particle moving on G_0 in some background potential. Two commuting copies of the finite W algebra leave the action (3.5) invariant and act on the space of solutions of the equations of motion (3.4).[3] This action is only given infinitesimally, because we do not know how to exponentiate finite (or infinite) W algebras. One can, however, sometimes find subspaces of the space of solutions that constitute a minimal orbit of the W algebra, see for example [25] where this was worked out for the ordinary W_3 algebra.

For the principal embeddings of sl_2 in sl_n, the equations of motion reduce to ordinary finite Toda equations of the type

$$\frac{d^2 q_i}{dt^2} + \exp\left(\sum_{j=1}^{n-1} K_{ij} q_j\right) = 0\,, \tag{3.6}$$

where $i = 1, \ldots, n-1$, K_{ij} is the Cartan matrix of sl_n, and $g_0 = \exp(q_i H_i)$.

The general solution of the equations of motion (3.4) can be constructed as follows. Let $h_0^{(1)}, h_0^{(2)}$ be elements of G_0. Let X_0 be an arbitrary element of g_0. If $g_0(t)$ is defined by the Gauss decomposition

$$g_-(t)g_0(t)g_+(t) = h_0^{(1)}\exp t(X_0 + (h_0^{(1)})^{-1}t_+h_0^{(1)} + h_0^{(2)}t_-(h_0^{(2)})^{-1})h_0^{(2)}\,, \tag{3.7}$$

then $g_0(t)$ is the most general solution of (3.4). The easiest way to find the action of the finite W algebra on these solutions, is to construct the conserved charges associated to these finite W symmetries (which can be done via a time dependent Miura transformation), and to study the transformations they generate. This might provide a valuable tool in the study of the solutions (3.7). We leave a detailed investigation of this, as well as many other issues like the quantization of the action (3.1), to future study.

4. Quantization of Finite W Algebras

In quantum mechanics, quantization amounts to replacing Poisson brackets by commutators. Since finite W algebras are Poisson algebras, the question arises whether it is possible to quantize these Poisson algebras, to give finite quantum W algebras. In the infinite dimensional case (i.e. the usual infinite W algebras), this is known to be possible for the standard W_n algebras associated to the principal embeddings. The W_3 algebra constructed by Zamolodchikov is a quantization of the Poisson algebra one gets from hamiltonian reduction of the affine sl_3 algebra. The most difficult task in constructing infinite quantum W algebras, is to check that the resulting commutator satisfies the Jacobi identity, or, equivalently, to check that the operator product algebra is associative. Zamolodchikov did this explicitly for his W_3 algebra. It is clear that this will become very cumbersome for

[3] More precisely, the symmetries of (3.5) form an algebra that is on-shell isomorphic to a finite W algebra

higher W algebras, and that it is difficult to obtain generic results in this direct approach.

A different way to find (infinite) quantum W algebras has been pioneered by Feigin and Frenkel [11]. In this approach the quantum W algebra is described as the zeroth cohomology of a certain complex. The advantage of this approach is that one automatically knows that the resulting operator algebra will be associative. This procedure is closely related to BRST quantization, and is usually called "quantum hamiltonian reduction." We will employ this method to study the quantization of finite W algebras, related to arbitrary sl_2 embeddings of sl_n. Another advantage of this method is that it provides a functor from the category of representations of g to those of the quantum finite W algebras, and is thus very useful to study the representation theory of quantum finite W algebras.

4.1. Quantization. Let $(\mathscr{A}_0, \{\cdot,\cdot\})$ be a commutative associative Poisson algebra. A quantization of $(\mathscr{A}_0, \{\cdot,\cdot\})$ is an associative algebra $\mathscr{A}$ depending on a parameter $\hbar$ such that (i) $\mathscr{A}$ is a free $\mathbb{C}[[\hbar]]$ module, (ii) $\mathscr{A}/\hbar\mathscr{A} \simeq \mathscr{A}_0$ and (iii) if π denotes the natural map $\pi\colon \mathscr{A} \to \mathscr{A}/\hbar\mathscr{A} \simeq \mathscr{A}_0$, then $\{\pi(X), \pi(Y)\} = \pi((XY - YX)/\hbar)$. In most cases one has a set of generators for $\mathscr{A}_0$, and $\mathscr{A}$ is completely fixed by giving the commutation relations of these generators.

For example, let $\mathscr{A}_0$ be the Kirillov Poisson algebra of polynomial functions on a Lie algebra g, determined by Eq. (11). Then a quantization of this Poisson algebra is the algebra $\mathscr{A}$ generated by the J^a and $\hbar$, subject to the relations $[J^a, J^b] = \hbar f^{ab}_{\ c} J^c$. Obviously, the Jacobi identities are satisfied. Specializing to $\hbar = 1$, this algebra is precisely the universal enveloping algebra $\mathscr{U}g$ of g.

To find quantizations of finite W algebras, one can first reduce the sl_n Kirillov Poisson algebra, and then try to quantize the resulting algebras that we studied in the previous sections. On the other hand, one can also first quantize and then constrain. We will follow the latter approach, and thus study the reductions of the quantum Kirillov algebra

$$[J^a, J^b] = \hbar f_c^{ab} J^c \,. \tag{4.1}$$

We want to impose the same constraints on this algebra as we imposed previously on the Kirillov Poisson algebra, to obtain the quantum versions of the finite W algebras related to sl_2 embeddings. Imposing constraints on quantum algebras can be done using the BRST formalism [26]. In the infinite dimensional case, this has been done for the usual W_N algebras by Feigin and Frenkel [11]. We use the finite dimensional counterpart of this approach.

BRST quantization in the presence of second class constraints is more cumbersome than in the presence of first class constraints; it requires the introduction of extra auxiliary fields to change the second class constraints into first class constraints. However, it was shown above that for arbitrary embeddings of sl_2 one can always choose a set of constraints that is completely first class and leads to the same W algebras as the set $\{\phi^{j,m}_{(\mu)}\}$. To perform a BRST quantization of the finite W algebras we use these alternative systems of first class constraints.

4.2. The BRST *Complex.* Consider the map $\chi\colon g_+ \to \mathbb{C}$ defined by $\chi(E_{l+s_{k-1},l+s_k}) = 1$ for $l \geqq 1, 1 \leqq k \leqq n_l - 1$ and $\chi(E_{ij}) = 0$ otherwise. Because the constraints $\{J^{l+s_{k-1},r+s_k} - \delta^{r,l}\}_{l \geqq 1;\, 1 \leqq k \leqq n_l - 1;\, r > 0}$ are first class, χ defines a one-dimensional representation of g_+. In terms of χ, the constraints can be written as

$\pi_+(J) = \chi(\pi_+(J))$, where π_+ again denotes the projection $g \to g_+$. It is this form of the constraints that we will use. Furthermore we will take $\hbar = 1$ for simplicity; the explicit $\hbar$ dependence can be determined afterwards.

As before latin indices will be supposed to run over a basis t_a of the Lie algebra g, Greek indices run over a basis of g_+ and barred Greek indices (like $\bar{\alpha}$) run over a basis of $g_- \oplus g_0$. Indices can again be raised and lowered by use of the Cartan Killing metric. The basis elements t_a, t_α and $t_{\bar{\alpha}}$ are as before so chosen that they have a well defined degree with respect to ad_δ.

To set up the BRST framework we need to introduce anticommuting ghosts and antighosts c_α and b^α, associated to the constraints that we want to impose [26]. They satisfy $b^\alpha c_\beta + c_\beta b^\alpha = \delta^\alpha_\beta$ and generate the Clifford algebra $Cl(g_+ \oplus g_+^*)$. The quantum Kirillov algebra is just the universal enveloping algebra $\mathscr{U}g$, and the total space on which the BRST operator acts is $\Omega = \mathscr{U}g \otimes Cl(g_+ \oplus g_+^*)$. A $\mathbf{Z}$ grading on Ω is defined by $\deg(J^a) = 0$, $\deg(c_\alpha) = +1$ and $\deg(b^\alpha) = -1$, and we can decompose $\Omega = \bigoplus_k \Omega^k$ accordingly. The BRST differential on Ω is given by $d(X) = [Q, X]$, where Q is the BRST charge

$$Q = (J^\alpha - \chi(J^\alpha))c_\alpha - \frac{1}{2} f^{\alpha\beta}_{\ \ \gamma} b^\gamma c_\alpha c_\beta , \tag{4.2}$$

and $[\cdot,\cdot]$ denotes the graded commutator (as it always will from now on)

$$[A, B] = AB - (-1)^{\deg(A)\cdot\deg(B)} BA . \tag{4.3}$$

Note that $\deg(Q) = 1$.

This is the standard BRST complex associated to the first class constraints of the previous section. Of interest are the cohomology groups of this complex, $H^k(\Omega; d)$. The zeroth cohomology group is the quantization of the classical finite W algebra. Because the gauge group H in (39) acts properly on g_c, we expect the higher cohomologies of the BRST complex to vanish, as they are generically related to singularities in the quotient g_c/H. In the mathematics literature the cohomology of the BRST complex is called the Hecke algebra $\mathscr{H}(g, g_+, \chi)$ associated to g, g_+, χ. Hecke algebras related to arbitrary sl_2 embeddings have not been computed, apart from those related to the principal sl_2 embeddings. In that case it was shown by Kostant [27] that the only non-vanishing cohomology is $H^0(\Omega; d)$ and that it is isomorphic to the center of the universal enveloping algebra. Recall that the center of the $\mathscr{U}g$ is generated by the set of independent Casimirs of g. This set is closely related to the generators of standard infinite W_n-algebras; in that case there is one W field for each Casimir which form a highly nontrivial algebra [28]. We see that for finite W algebras the same generators survive, but that they form a trivial abelian algebra. For non-principal sl_2 embeddings however quantum finite W algebras are non-trivial.

To compute the cohomology of $(\Omega; d)$, Feigin and Frenkel make the crucial observation that the operator d can be decomposed into two commuting pieces. Write $Q = Q_0 + Q_1$, with

$$Q_0 = J^\alpha c_\alpha - \frac{1}{2} f^{\alpha\beta}_{\ \ \gamma} b^\gamma c_\alpha c_\beta ,$$

$$Q_1 = -\chi(J^\alpha)c_\alpha , \tag{4.4}$$

and define $d_0(X) = [Q_0, X]$, $d_1(X) = [Q_1, X]$, then one can verify by explicit computation that $d_0^2 = d_0 d_1 = d_1 d_0 = d_1^2 = 0$. Associated to this decomposition is a bigrading of $\Omega = \bigoplus_{k,l} \Omega^{k,l}$ defined by

$$\begin{aligned} \deg(J^a) &= (k, -k), && \text{if } t_a \in g_k , \\ \deg(c_\alpha) &= (1-k, k), && \text{if } t_\alpha \in g_k , \\ \deg(b^\alpha) &= (k-1, -k), && \text{if } t_\alpha \in g_k , \end{aligned} \tag{4.5}$$

with respect to which d_0 has degree $(1, 0)$ and d_1 has degree $(0, 1)$. Thus $(\Omega^{k,l}; d_0; d_1)$ has the structure of a double complex. Explicitly, the action of d_0 and d_1 is given by

$$\begin{aligned} d_0(J^a) &= f^{\alpha a}_{\ \ b} J^b c_\alpha , \\ d_0(c_\alpha) &= -\frac{1}{2} f^{\beta\gamma}_{\ \ \alpha} c_\beta c_\gamma , \\ d_0(b^\alpha) &= J^\alpha + f^{\alpha\beta}_{\ \ \gamma} b^\gamma c_\beta , \\ d_1(J^a) &= d_1(c_\alpha) = 0 , \\ d_1(b^\alpha) &= -\chi(J^\alpha) . \end{aligned} \tag{4.6}$$

To simplify the algebra, it is advantageous to introduce

$$\hat{J}^a = J^a + f^{a\beta}_{\ \ \gamma} b^\gamma c_\beta . \tag{4.7}$$

Our motivation to introduce these new elements $\hat{J}^a$ is twofold: first, similar expressions were encountered in a study of the effective action for W_3 gravity [8], where it turned out that the BRST cohomology for the infinite W_3 algebra case could conveniently be expressed in terms of $\hat{J}$'s; second, such expressions were introduced for the J^a's that live on the Cartan subalgebra of g in [11], and simplified their analysis considerably. In terms of $\hat{J}$ we have

$$\begin{aligned} d_0(\hat{J}^a) &= f^{\alpha a}_{\ \ \bar{\gamma}} \hat{J}^{\bar{\gamma}} c_\alpha , \\ d_0(c_\alpha) &= -\frac{1}{2} f^{\beta\gamma}_{\ \ \alpha} c_\beta c_\gamma , \\ d_0(b^\alpha) &= \hat{J}^\alpha , \\ d_1(\hat{J}^a) &= -f^{a\beta}_{\ \ \gamma} \chi(J^\gamma) c_\beta , \\ d_1(c_\alpha) &= 0 , \\ d_1(b^\alpha) &= -\chi(J^\alpha) . \end{aligned} \tag{4.8}$$

The advantage of having a double complex is that we can apply the technique of spectral sequences [29] to it, in order to compute the cohomology of $(\Omega; d)$. The results from the theory of spectral sequences that we need are gathered in the next section.

4.3. Spectral Sequences for Double Complexes. Let $(\Omega^{p,q}; d_0; d_1)$ denote a double complex, where d_0 has degree $(1, 0)$ and d_1 has degree $(0, 1)$. The standard spectral sequence for this double complex is a sequence of complexes $(E_r^{p,q}; D_r)_{r \geqq 0}$, where

D_r is a differential of degree $(1-r, r)$, and is defined as follows: $E_0^{p,q} = \Omega^{p,q}$, $D_0 = d_0$, $D_1 = d_1$, and for $r \geqq 0$,

$$E_{r+1}^{p,q} = H^{(p,q)}(E_r; D_r) = \frac{\ker D_r\colon E_r^{p,q} \to E_r^{p+1-r,q+r}}{\operatorname{im} D_r\colon E_r^{p-1+r,q-r} \to E_r^{p,q}} \,. \tag{4.9}$$

The differential D_{r+1} for $r > 0$ is given by $D_{r+1}(\alpha) = d_1(\beta)$, where β is chosen such that $d_0\beta = D_r\alpha$. Such a β always exists and D_{r+1} is uniquely defined in this way. The usefulness of this spectral sequence is provided by the following [30]:

Theorem 2. *If* $\bigcap_q \bigoplus_{p,q \geqq s} \Omega^{p,q} = \{0\}$, *then* $E_\infty^{p,q} = \bigcap_r E_r^{p,q}$ *exists, and*

$$E_\infty^{p,q} \simeq \frac{F^q H^{(p+q)}(\Omega; d)}{F^{q+1} H^{(p+q)}(\Omega; d)} \,, \tag{4.10}$$

where

$$F^q H^{(p+q)}(\Omega; d) = \frac{\ker d\colon \bigoplus_{s \geqq q} \Omega^{p+q-s,s} \to \bigoplus_{s \geqq q} \Omega^{p+q+1-s,s}}{\operatorname{im} d\colon \bigoplus_{s \geqq q} \Omega^{p+q-1-s,s} \to \bigoplus_{s \geqq q} \Omega^{p+q-s,s}} \,. \tag{4.11}$$

This spectral sequence is especially useful if it collapses at the n^{th} term for some n, i.e. $D_r = 0$ for $r \geqq n$, because then $E_n \simeq E_\infty$ and one needs only to compute the first n terms of the spectral sequence.

If the double complex is also an algebra, i.e. there is a multiplication operator $m\colon \Omega^{p,q} \otimes_{\mathbb{C}} \Omega^{p',q'} \to \Omega^{p+p',q+q'}$, and d satisfies the Leibniz rule with respect to this multiplication, then (4.10) is also an equality of algebras. It is in general nontrivial to reconstruct the full algebra structure of $H^*(\Omega; d)$ from $E_\infty^{p,q}$, due to the quotient in the right-hand side of (4.10).

Another useful tool in the computation of cohomology is the following version of the Künneth theorem

Lemma 2. *Let* $A = \bigoplus_k A^k$ *be a graded differential algebra over* $\mathbb{C}$ *with a differential* d *of degree* 1 *that satisfies the Leibniz rule, and assume that* A *has two graded subalgebras* $A_1 = \bigoplus_k A_1^k$ *and* $A_2 = \bigoplus_k A_2^k$ *such that* $d(A_1) \subset A_1$ *and* $d(A_2) \subset A_2$, *that* $A_2^k = \{0\}$ *for* k *sufficiently large, and that* $m\colon A_1 \otimes_{\mathbb{C}} A_2 \to A$ *given by* $m(a_1 \otimes a_2) = a_1 a_2$ *is an isomorphism of vector spaces. Then*

$$H^*(A; d) \simeq \{a_1 a_2 \,|\, a_1 \in H^*(A_1; d), a_2 \in H^*(A_2; d)\} \,. \tag{4.12}$$

Proof. Form the double complex $(\Omega^{p,q}; d_0; d_1)$ with $\Omega^{p,q} \simeq m(A_1^p \otimes A_2^q)$, $d_0(a_1 a_2) = d(a_1)a_2$, and $d_1(a_1 a_2) = (-1)^{\deg(a_1)} a_1 d(a_2)$. The spectral sequence for this double complex collapses at the E_2 term, and one finds $E_\infty^{p,q} = \{a_1 a_2 \,|\, a_1 \in H^p(A_1; d),\ a_2 \in H^q(A_2; d)\}$. The condition $A_2^k = 0$ for k sufficiently large guarantees that $F^q H^{p+q} = 0$ for q sufficiently large, and one can assemble $H^*(\Omega; d)$ from $E_\infty^{p,q}$ using (4.10). This leads to (4.12) on the level of vector spaces. Because $a_1 a_2$ is really a representative of an element of $H^*(A; d)$, it follows that (4.12) is also an isomorphism of algebras.

By induction, one can easily prove that the theorem still holds if instead of two subalgebras n subalgebras $A_1, \ldots, A_n$ are given, with $A \simeq A_1 \otimes \cdots \otimes A_n$. The condition $A_2^k = \{0\}$ for sufficiently large k is replaced by $A_2^k = \cdots = A_n^k = \{0\}$ for sufficiently large k. Without such a condition, it may not be so easy to reassemble H^* from E_∞. To illustrate some of the difficulties that can arise, let us give an example where H^* cannot be recovered directly from E_∞. This example is not related to the above theorem, but it represents a situation we will encounter in the computation of the BRST cohomology.

Consider the algebra $\Omega = \mathbb{C}[x, y]/(y^2 = 0)$, where x is an even generator of bidegree $(1, -1)$, and y is an odd generator of bidegree $(0, -1)$. The differentials d_0, d_1 are given by $d_{0,1}(x) = 0$, $d_0(y) = x$ and $d_1(y) = -1$. One immediately computes $H^k(\Omega; d) = \mathbb{C}[x]/(x-1)\mathbb{C}[x] \simeq \mathbb{C}$ for $k = 0$, and $H^k = 0$ otherwise. The spectral sequence associated to the double complex collapses at the first term, and one finds $E_\infty^{p,q} = \mathbb{C}\delta_{p,0}\delta_{q,0}$. Because $F^1 H^* = 0$, one deduces that $H^k(\Omega; d) \simeq \mathbb{C}\delta_{k,0}$. On the other hand, we could also have started with the mirror double complex obtained by interchanging d_0 and d_1 and the bigrading. Thus, we assign bidegree $(-1, 1)$ to x and bidegree $(-1, 0)$ to y. The spectral sequence associated to the mirror double complex also collapses at the first term, but now one finds $E_\infty^{p,q} = 0$. This is not in conflict with the previous computation, because we cannot a priori find a q for which $F^q H^{p+q} = 0$, and we can only conclude that $F^q H^{p+q} \simeq F^{q+1} H^{p+q}$. If we conpute explicitly with respect to this bigrading what $F^q H^{p+q}$ is, we find that it is only nontrivial for $p + q = 0$, and then $F^q H^0 = x^q\mathbb{C}[x]/(x-1)x^q\mathbb{C}[x] \simeq \mathbb{C}$ for $q \geqq 0$, and $F^q H^0 = \mathbb{C}[x]/(x-1)\mathbb{C}[x] \simeq \mathbb{C}$ for $q < 0$. This indeed yields $E_\infty^{p,q} = 0$. The lesson is that one should be careful in deriving $H^*(\Omega; d)$ from $E_\infty^{p,q}$.

Finally, let us present another fact that will be useful later.

Lemma 3. *Suppose A is a differential graded algebra, $A = \bigoplus_{n \geqq 0} A^n$, with a differential of degree 1. Assume furthermore that A has a filtration*

$$\{0\} = F^0 A \subset F^1 A \subset F^2 A \subset \cdots \subset A\,, \tag{4.13}$$

such that $F^p A F^q A \subset F^{p+q} A$, and that d preserves the filtration, $d(F^p A) \subset F^p A$. If $H^k(F^{p+1}A/F^p A; d) = 0$ unless $k = 0$, then we have the following isomorphism of vector spaces

$$H^0(A; d) \simeq \bigoplus_{p \geqq 0} H^0(F^{p+1}A/F^p A; d)\,. \tag{4.14}$$

Proof. One can assign a spectral sequence to such a filtered graded algebra [30], whose first term contains the cohomologies $H^k(F^{p+1}A/F^p A; d)$. If only $H^0 \neq 0$, then the spectral sequence collapses at the first term, and because the filtration is bounded from below ($\{0\} = F^0 A$), one can collect the vector spaces that make up $E_\infty^{*,*}$, to get the isomorphism (4.14).

4.4. The BRST *Cohomology.* The computation of the BRST cohomology is simplified considerably due to the introduction of the new set of generators $\hat{J}^a$. The simplification arises due to

Theorem 3. *If $(\Omega; d)$ denotes the* BRST *complex, with Ω generated by $\hat{J}^a$, c_α and b^α, and $d = d_0 + d_1$ given by (4.8), then $H^*(\Omega; d) \simeq H^*(\Omega_{\mathrm{red}}; d)$, where Ω_{red} is the subalgebra of Ω generated by $\hat{J}^{\bar{\alpha}}$ and c_α.*

Proof. Apply the Künneth Theorem 2 to $\Omega_{\mathrm{red}} \otimes (\bigotimes_\alpha \Omega_\alpha)$, where Ω_α is the algebra generated by $\hat{J}^\alpha$ and b^α. Note that $[\hat{J}^\alpha, b^\alpha] = 0$ and that the conditions of the Künneth theorem are satisfied. Therefore, $H^*(\Omega; d) \simeq H^*(\Omega_{\mathrm{red}}; d) \otimes (\bigotimes_\alpha H^*(\Omega_\alpha; d))$. Now $(\Omega_\alpha; d)$ is essentially the same complex as the one we examined in the last part of the previous section, and one easily proves that $H^k(\Omega_\alpha; d) \simeq \mathbb{C}\delta_{k,0}$. This shows $H^*(\Omega; d) \simeq H^*(\Omega_{\mathrm{red}}; d)$.

The reduced complex $(\Omega_{\rm red}; d)$ is described by the following set of relations:

$$\begin{aligned}
d_0(\hat{J}^{\bar{\alpha}}) &= f^{\alpha\bar{\alpha}}_{\bar{\gamma}}\hat{J}^{\bar{\gamma}}c_\alpha ,\\
d_0(c_\alpha) &= -\frac{1}{2}f^{\beta\gamma}{}_{\alpha}c_\beta c_\gamma ,\\
d_1(\hat{J}^{\bar{\alpha}}) &= -f^{\bar{\alpha}\beta}{}_{\gamma}\chi(J^\gamma)c_\beta ,\\
d_1(c_\alpha) &= 0 ,\\
[\hat{J}^{\bar{\alpha}}, \hat{J}^{\bar{\beta}}] &= f^{\bar{\alpha}\bar{\beta}}_{\bar{\gamma}}\hat{J}^{\bar{\gamma}} ,\\
[\hat{J}^{\bar{\alpha}}, c_\beta] &= -f^{\bar{\alpha}\gamma}{}_{\beta}c_\gamma ,\\
[c_\alpha, c_\beta] &= 0 .
\end{aligned} \tag{4.15}$$

Feigin and Frenkel [11] propose to use the spectral sequence for the double complex $(\Omega; d_0; d_1)$ to compute the BRST cohomology for the infinite case, to obtain the standard W algebras. They claim that in that case, the spectral sequence collapses at the second term, and use this to identify the W algebras as the centralizers of some vertex operators in a free field algebra. Because finite W algebras are the same as infinite W algebras, with all dependence on the co-ordinates suppressed (so that derivatives vanish), we would expect the same thing to happen for finite W algebras. However, this turns out not to be the case here.

Let us demonstrate what happens for the case of the principal sl_2 embedding in sl_2, i.e. the embedding is given by the identity map. The reduced algebra $\Omega_{\rm red}$ is generated by $\hat{H}$ of degree (0, 0), $\hat{F}$ of degree $(-1, 1)$ and c of degree (0, 1). The nontrivial relations between these generators are $[\hat{H}, \hat{F}] = -2F$ and $[\hat{H}, c] = -2c$. Furthermore, $d_1(\hat{F}) = d_1(c) = 0$, $d_1(\hat{H}) = -2c$, $d_0(\hat{H}) = d_0(c) = 0$ and $d_0(\hat{F}) = \hat{H}c$. To find $E_1^{*,*}$ we compute $d_0(\hat{H}^a\hat{F}^bc) = 0$ and

$$d_0(\hat{H}^a\hat{F}^b) = b\hat{H}^a(\hat{H} + b - 1)\hat{F}^{b-1}c . \tag{4.16}$$

Therefore in $H^*_{d_0}$ the identity $f(\hat{H})\hat{F}^bc = f(1-b)\hat{F}^bc$ is valid, and

$$E_1^{*,*} = \mathbb{C}[\hat{H}] \oplus \mathbb{C}[\hat{F}]c . \tag{4.17}$$

Next we compute the E_2 term of the spectral sequence. In E_1 we have

$$\begin{aligned}
d_1(f(\hat{H})) &= (f(\hat{H}) - f(\hat{H} + 2))c\\
&= (f(0) - f(2))c ,
\end{aligned} \tag{4.18}$$

from which it follows that

$$E_1^{*,*} = \{f(\hat{H}) \,|\, f(0) = f(2)\} \oplus \hat{F}\mathbb{C}[\hat{F}]c . \tag{4.19}$$

This does not yet look like the final answer [27], namely $H^k = 0$ for $k \neq 0$ and H^0 is isomorphic to the center of $\mathscr{U}g$, generated by the second casimir of sl_2. So let us compute D_2 to see whether the spectral sequence has already collapsed. Using the definition of D_2 given at the beginning of the previous section, we compute subsequently

$$d_1(f(\hat{H})) = (f(\hat{H}) - f(\hat{H}+2))c \,,$$

$$d_0\left(\frac{f(\hat{H}) - f(\hat{H}+2)}{\hat{H}}\hat{F}\right) = (f(\hat{H}) - f(\hat{H}+2))c \,,$$

$$\begin{aligned} d_1\left(\frac{f(\hat{H}) - f(\hat{H}+2)}{\hat{H}}\hat{F}\right) &= \left(\frac{f(\hat{H}) - f(\hat{H}+2)}{\hat{H}} - \frac{f(\hat{H}+2) - f(\hat{H}+4)}{\hat{H}+2}\right)\hat{F}c \\ &= (f(3) - f(-1))\hat{K}c \,, \end{aligned} \tag{4.20}$$

and see that $D_2(f(\hat{H})) = (f(3) - f(-1))\hat{K}c$. The spectral sequence does not collapse, and the next term in the sequence is

$$E_3^{*,*} = \{f(\hat{H}) \,|\, f(0) = f(2) \wedge f(-1) = f(3)\} \oplus \hat{F}^2\mathbb{C}[\hat{F}]c \,. \tag{4.21}$$

Continuing in this way one finds for the next terms in the spectral sequence

$$E_r^{*,*} = \{f(\hat{H}) \,|\, f(-l) = f(l+2), 0 \leqq l \leqq r-2\} \oplus \hat{F}^{r-1}\mathbb{C}[\hat{F}]c \,, \tag{4.22}$$

and finally

$$E_\infty^{*,*} = \{f(\hat{H}) \,|\, f(1+\hat{H}) = f(1-\hat{H})\} = \mathbb{C}[(\hat{H}-1)^2] \,. \tag{4.23}$$

This agrees with the result of Kostant [27]. The spectral sequence does not collapse at all, and it is clear that this is a rather cumbersome procedure to compute the BRST cohomology. Luckily, there is another spectral sequence one can associate to a double complex, namely the sequence associated to the mirror double complex obtained by interchanging the bigrading and d_0 and d_1. This spectral sequence turns out to be much simpler, and will be examined in the next section, where we use it to compute the BRST cohomology for arbitrary sl_2 embeddings.

4.5. The Mirror Spectral Sequence. The main result of this section is

Theorem 4. *As before let $g_{lw} \subset g$ be the kernel of the map $\mathrm{ad}_{t_-} : g \to g$. Then the* BRST *cohomology is given by the following isomorphisms of vector spaces*

$$H^k(\Omega; d) \simeq (\mathcal{U}g_{lw})\delta_{k,0} \,. \tag{4.24}$$

Proof. The E_1 term of the mirror spectral sequence is given by the d_1 cohomology of Ω_{red}. To compute the cohomology we use Lemma 3. The filtration on Ω_{red} is: $F^p\Omega_{\mathrm{red}}$ is spanned as a vector space by $\{\hat{J}^{\bar{\alpha}_1}\hat{J}^{\bar{\alpha}_2}\ldots\hat{J}^{\bar{\alpha}_r}c_{\beta_1}c_{\beta_2}\ldots c_{\beta_s} \,|\, r+s \leqq p\}$. Thus $F^p\Omega_{\mathrm{red}}/F^{p-1}\Omega_{\mathrm{red}}$ is spanned by the products of precisely p $\hat{J}$'s and c's, and in this quotient $\hat{J}$ and c (anti)commute with each other. Now let us rewrite $d_1(\hat{J}^{\bar{\alpha}})$ as

$$\begin{aligned} d_1(\hat{J}^{\bar{\alpha}}) &= -\,\mathrm{Tr}([\chi(J^\gamma)t_\gamma, t^{\bar{\alpha}}]t^\beta c_\beta) \\ &= -\,\mathrm{Tr}([t_+, t^{\bar{\alpha}}]t^\beta c_\beta) \,. \end{aligned} \tag{4.25}$$

From this it is clear that $d_1(\hat{J}^{\bar{\alpha}}) = 0$ for $t^{\bar{\alpha}} \in g_{hw}$. Furthermore, since $t_{\bar{\alpha}} \in g_0 \oplus g_-$ and $\dim(g_{lw}) = \dim(g_0)$, it follows that for each β there is a linear combination $a(\beta)_{\bar{\alpha}} \hat{J}^{\bar{\alpha}}$ with $d_1(a(\beta)_{\bar{\alpha}} \hat{J}^{\bar{\alpha}}) = c_\beta$. This proves that

$$\bigoplus_{p>0} \frac{F^p \Omega_{\mathrm{red}}}{F^{p-1} \Omega_{\mathrm{red}}} \simeq \bigotimes_{t_{\bar{\alpha}} \in g_{lw}} \mathbb{C}[\hat{J}^{\bar{\alpha}}] \bigotimes_{t_{\bar{\alpha}} \notin g_{lw}} (\mathbb{C}[\hat{J}^{\bar{\alpha}}] \oplus d_1(\hat{J}^{\bar{\alpha}}) \mathbb{C}[\hat{J}^{\bar{\alpha}}]) . \tag{4.26}$$

Using the Künneth theorem (Lemma 2) for (4.26), we find that

$$H^k(\Omega_{\mathrm{red}}; d_1) = \bigotimes_{t_{\bar{\alpha}} \in g_{lw}} \mathbb{C}[\hat{J}^{\bar{\alpha}}] \delta_{k,0} = (\mathscr{U} g_{lw}) \delta_{k,0} . \tag{4.27}$$

Because there is only cohomology of degree 0, the mirror spectral sequence collapses, and $E_\infty = E_1$. Because $\Omega_{\mathrm{red}}^{k,l} = 0$ for $l > 0$, we can find $H^*(\Omega_{\mathrm{red}}; d)$ from E_∞. The theorem now follows directly from Theorem 3.

4.6. Reconstructing the Quantum Finite W Algebra. We succeeded in computing the BRST cohomology on the level of vector spaces; as expected, there is only cohomology of degree zero, and furthermore, the elements of g_{lw} are in one-to-one correspondence with the components of g that made up the lowest weight gauge in Sect. 1. Therefore $H^*(\Omega; d)$ really is a quantization of the finite W algebra. What remains to be done is to compute the algebraic structure of $H^*(\Omega; d)$. The only thing that (4.24) tells us is that the product of two elements a and b of bidegree $(-p, p)$ and $(-q, q)$ is given by the product structure on $\mathscr{U} g_{lw}$, modulo terms of bidegree $(-r, r)$ with $r < p + q$. To find these lower terms we need explicit representatives of the generators of $H^0(\Omega; d)$ in Ω. Such representatives can be constructed using the so-called tic-tac-toe construction [29]: take some $\phi_0 \in g_{lw}$, of bidegree $(-p, p)$. Then $d_0(\phi)$ is of bidegree $(1 - p, p)$. Since $d_1 d_0(\phi_0) = -d_0 d_1(\phi_0) = 0$, and there is no d_1 cohomology of bidegree $(1 - p, p)$, $d_0(\phi_0) = d_1(\phi_1)$ for some ϕ_1 of bidegree $(1 - p, p - 1)$. Now repeat the same steps for ϕ_1, giving a ϕ_2 of bidegree $(2 - p, p - 2)$, such that $d_0(\phi_1) = d_1(\phi_2)$. Note that $d_1 d_0(\phi_1) = -d_0 d_1(\phi_1) = -d_0^2(\phi) = 0$. In this way we find a sequence of elements ϕ_l of bidegree $(l - p, p - l)$. The process stops at $l = p$. Let $W(\phi) = \sum_{l=0}^{p} (-1)^l \phi_l$. Then $dW(\phi) = 0$, and $W(\phi)$ is a representative of ϕ_0 in $H^0(\Omega; d)$. The algebra structure of $H^0(\Omega; d)$ is then completely determined by looking at the commutation relations of $W(\phi)$ in Ω, where ϕ_0 runs over a basis of g_{lw}.

Since the space g_{lw} is finite dimensional and is spanned by the elements $\{t_{\bar{\alpha}} \,|\, \mathrm{ad}_{t_-} t_{\bar{\alpha}} = 0\}$ the finite W algebra is finitely generated. A set of generators is $\{W(t_{\bar{\alpha}})\}_{t_{\bar{\alpha}} \in g_{lw}}$. In principle the algebra of these generators closes only modulo d exact terms, but since we computed the d cohomology on a reduced complex generated by $\hat{J}^{\bar{\alpha}}$ and c_α, and this reduced complex is zero at negative ghost number, there simply aren't any d exact terms at ghost number zero. Thus the algebra closes in itself. This is the quantum finite W algebra.

Let us now give an example of the construction described above.

4.7. Example. Consider again the embedding associated to the following partition of the number 3: $3 = 2 + 1$. We constructed the classical W associated to this embedding earlier. We shall now quantize this Poisson algebra by the methods developed above. Take the following basis of sl_3:

$$r_a t_a = \begin{pmatrix} \frac{r_4}{6} - \frac{r_5}{2} & r_2 & r_1 \\ r_6 & -\frac{r_4}{3} & r_3 \\ r_8 & r_7 & \frac{r_4}{6} + \frac{r_5}{2} \end{pmatrix} . \tag{4.28}$$

Remember that (in the present notation) the sl_2 embedding is given by $t_+ = t_1$, $t_0 = -t_5$ and $t_- = t_8$. The nilpotent subalgebra g_+ is spanned by $\{t_1, t_3\}$, g_0 by $\{t_2, t_4, t_5, t_6\}$ and g_- by $\{t_7, t_8\}$. The d_1 cohomology of Ω_{red} is generated by $\{\hat{J}^4, \hat{J}^7, \hat{J}^6, \hat{J}^8\}$, and using the tic-tac-toe construction one finds representatives for these generators in $H^0(\Omega_{\text{red}}; d)$:

$$\begin{aligned} W(\hat{J}^4) &= \hat{J}^4 , \\ W(\hat{J}^6) &= \hat{J}^6 , \\ W(\hat{J}^7) &= \hat{J}^7 - \frac{1}{2}\hat{J}^2\hat{J}^5 - \frac{1}{2}\hat{J}^4\hat{J}^2 + \frac{1}{2}\hat{J}^2 , \\ W(\hat{J}^8) &= \hat{J}^8 + \frac{1}{4}\hat{J}^5\hat{J}^5 + \hat{J}^2\hat{J}^6 - \hat{J}^5 . \end{aligned} \tag{4.29}$$

Let us introduce another set of generators

$$\begin{aligned} C &= -\frac{4}{3}W(\hat{J}^8) - \frac{1}{9}W(\hat{J}^4)W(\hat{J}^4) - 1 , \\ H &= -\frac{2}{3}W(\hat{J}^4) - 1 , \\ E &= W(\hat{J}^7) , \\ F &= \frac{4}{3}W(\hat{J}^6) . \end{aligned} \tag{4.30}$$

The commutation relations between these generators are given by

$$\begin{aligned} [H, E] &= 2E , \\ [H, F] &= -2F , \\ [E, F] &= H^2 + C , \\ [C, E] &= [C, F] = [C, H] = 0 . \end{aligned} \tag{4.31}$$

These are precisely the same as the relations for the finite $W_3^{(2)}$ algebra given in [14]. Notice that in this case the quantum relations are identical to the classical ones. The explicit $\hbar$ dependence can be recovered simply by multiplying the right-hand sides of (4.31) by $\hbar$. In this example the quantum relations are the same as the classical relations. This is not always true however, since in general there will be quantum corrections, i.e. terms of order $\hbar^2$ or higher. An example of such a case is given in the appendix.

5. The Representation Theory of Finite W Algebras

The next important topic in the theory of finite W algebras is their representation theory. This representation theory will presumably play an important role in the representation theory of ordinary W algebras as was already mentioned in the introduction. It will be possible to construct the finite W representations by a quantum version of the generalized Miura map. This will give us the $\mathscr{W}(i)$ representations as the Miura transform of g_0 representations.

If we denote by $X^{0,0}$ the component of an element X of bidegree (0, 0), so that $W^{0,0}(\phi) = (-1)^p \phi_p$, then

$$[W^{0,0}(\phi), W^{0,0}(\phi')] = [W(\phi), W(\phi')]^{0,0}, \tag{5.1}$$

and therefore $W(\phi) \to W(\phi)^{0,0}$ is a homomorphism of algebras. We now have the following important theorem which is a quantum version of the generalized Miura map.

Theorem 5. *(Quantum Miura Transformation). The map $W(\phi) \to W(\phi)^{0,0}$, or, equivalently, the map $H^0(\Omega; d) \to H^0(\Omega; d)^{0,0}$, is an isomorphism of algebras.*

Proof. Now that we know the cohomology of $(\Omega; d)$, let us go back to the original double complex $(\Omega_{\mathrm{red}}; d_0; d_1)$. Because there is only cohomology of degree 0, we know that the $E_\infty^{p,q}$ term of the spectral sequence associated to $(\Omega_{\mathrm{red}}; d_0; d_1)$ must vanish unless $p + q = 0$. If we look at $d_0(\hat{J}^{\bar{\alpha}}) = f^{\beta\bar{\alpha}}_{\bar{\gamma}} \hat{J}^{\bar{\gamma}} c_\beta$, we see that $d_0(\hat{J}^{\bar{\alpha}}) = 0$ if and only if $\bar{\alpha} \in g_0^*$. From this it is not difficult, repeating arguments similar to those in the proof of Theorem 4, to prove that the only nonvanishing piece of $\bigoplus_r E_1^{r,-r}$ is in $E_1^{0,0}$. This implies that $E_\infty^{p,q}$ is only nonzero for $p = q = 0$. Because $H^0(\Omega; d) = E_\infty^{0,0} = H^0(\Omega; d)^{0,0}$ as vector spaces, it follows that the map $H^0(\Omega; d) \to H^0(\Omega; d)^{0,0}$ can have no kernel, and is an isomorphism.

The quantum Miura transformation gives a faithful realization of the quantum W algebra in $\mathscr{U}g_0$. As g_0 is nothing but a direct sum of simple Lie algebras (up to $u(1)$ terms) its representation theory is just the standard representation theory of (semi)simple Lie algebras. If ρ is a representation of g then the composition of ρ with the Miura map is a representation of the finite W algebra. In this way we get the representation theory of finite W algebras from the representation theory of the grade zero subalgebras associated to the different sl_2 embeddings.

Since $\mathscr{U}(g_0)$ is abelian for the principal sl_2 embeddings, this implies that in those cases the quantum finite W algebras are also abelian, something which was already proven by Kostant [27]. To get some interesting novel structure, one should therefore consider nonprincipal sl_2 embeddings.

Again let us consider the example of $3 = 2 + 1$. The expressions for the quantum Miura transformation of this algebra are obtained from (4.29) by restricting these expressions to the bidegree (0, 0) part. If we introduce $s = (\hat{J}^4 + 3\hat{J}^5)/4$, $h = (\hat{J}^5 - \hat{J}^4)/4$, $f = 2\hat{J}^6$ and $e = \hat{J}^2/2$, then h, e, f form an sl_2 Lie algebra,

$[h, e] = e$ and $[h, f] = -f$ and $[e, f] = 2h$ while s commutes with everything. In terms of s and h, e, f, the quantum Miura transformation reads

$$\begin{aligned} C &= -\frac{4}{3}\left(h^2 + \frac{1}{2}ef + \frac{1}{2}fe\right) - \frac{4}{9}s^2 + \frac{4}{3}s - 1\,, \\ H &= 2h - \frac{2}{3}s + 1\,, \\ E &= -2(s - h - 1)e\,, \\ F &= \frac{2}{3}f\,. \end{aligned} \tag{5.2}$$

Notice that C contains the second Casimir of sl_2, which is what one would expect because it commutes with everything. Every $g_0 = sl_2 \oplus u(1)$ module gives, using the expressions (5.2), a module for the finite quantum algebra $W_3^{(2)}$. So if we have a representation of $sl_2 \oplus u(1)$ in terms of $n \times n$ matrices, we immediately get a representation of $\bar{W}_3^{(2)}$ in terms of $n \times n$ matrices.

5.1. Fock Realizations of Quantum Finite W Algebras. Using the quantum Miura transformation we can turn any Fock realization of g_0 into a Fock realization of the corresponding finite W algebra. The Fock realizations of simple Lie algebras are well known [31]. For example, the Fock realization of sl_2 is given by

$$\begin{aligned} \sigma_\Lambda(e) &= \frac{d}{dz}\,, \\ \sigma_\Lambda(f) &= (\Lambda, \alpha)z - z^2\frac{d}{dz}\,, \\ \sigma_\Lambda(h) &= \frac{1}{2}(\Lambda, \alpha) - z\frac{d}{dz}\,, \end{aligned} \tag{5.3}$$

where α is the root of sl_2. The expressions for arbitrary sl_n can be found in [15] and will not be given here.

Using this one is immediately able to write down a Fock realization of the algebra g_0 (since it is essentially a direct sum of sl_k algebras). Then using the quantum Miura transformation one thus arrives at a Fock realization of the finite W algebra in question. Let us now explicitly do this in the example of $\bar{W}_3^{(2)}$. Inserting the expressions (5.3) into the Miura map (5.2) one finds

$$\begin{aligned} \sigma_\Lambda(H) &= (\Lambda, \alpha) - \frac{2}{3}s + 1 - 2z\frac{d}{dz}\,, \\ \sigma_\Lambda(E) &= 2\left(1 - s + \frac{1}{2}(\Lambda, \alpha)\right)\frac{d}{dz} - 2z\frac{d^2}{dz^2}\,, \\ \sigma_\Lambda(F) &= \frac{2}{3}(\Lambda, \alpha)z - \frac{2}{3}z^2\frac{d}{dz}\,, \\ \sigma_\Lambda(C) &= -\frac{1}{3}(\Lambda, \alpha)^2 - \frac{2}{3}(\Lambda, \alpha) - \frac{4}{9}s^2 + \frac{4}{3}s - 1 \end{aligned} \tag{5.4}$$

(where we consider s to be a number). This realization is equal to the zero mode structure of the free field realization of the infinite $W_3^{(2)}$ algebra constructed in [32]. Note however that the derivation is completely different since in [32] (using standard methods) the expressions in terms of free fields were obtained by constructing the generators of the commutant of certain screening charges. Constructing this commutant is in general however rather cumbersome. The method we presented above is more direct and works for arbitrary embeddings (and realizations).

The Fock realizations (5.4) contain for certain values of s and (Λ, α) the finite dimensional representations of $\bar{W}_3^{(2)}$ which were constructed in [14]. Before we show this let us recall the results of [14].

Theorem 6. *Let d be a positive integer and x a real number.*

1. *For every pair (p, x) the algebra $\bar{W}_3^{(2)}$ has a unique highest weight representation $W(d; x)$ of dimension d with highest weight $j(d; x) = d + x - 1$ and central value $c(d; x) = \frac{1}{3}(1 - d^2) - x^2$.*
2. *Let $k \in \{1, \dots, d-1\}$ then $W(d; \frac{2}{3}k - \frac{1}{3}d)$ is reducible and its invariant subspace is isomorphic as a representation to $W(d - k; -\frac{1}{3}(k + d))$.*
3. *The representation $W(d; x)$ is unitary* iff $x > \frac{1}{3}d - \frac{2}{3}$.

Now one can easily check that for $d = (\Lambda, \alpha)$ and $s = \frac{3}{2}(1 - x)$ the subspace

$$V = \left\{ P(z) \in \mathbf{C}[z] \,\middle|\, \left(\frac{d}{dz}\right)^d P(z) = 0 \right\} \tag{5.5}$$

of $\mathbf{C}(z)$ is isomorphic as a representation to $W(d; x)$.

6. Discussion

In this paper we have studied finite W algebras in great detail and it turns out that they are very rich in their structure. There are several issues that deserve further study. For example it would be very interesting to calculate the orbit of finite W transformations on the solution space of the finite dimensional generalized Toda systems we encountered in this paper. These systems were already derived in [33] as a static and spherically symmetric solutions of the self dual Yang–Mills equations. In that paper some special solutions of generalized Toda theories were constructed but as far as we know the general solution space is not known. Since finite W algebras act on this solution space, transforming one solution into another, it may be possible to generate the entire solution space by the finite W action (remember that the symmetry group of the free particle on the group also acts transitively on the space of solutions).

Closely related to this problem is the problem of finding the symplectic orbits of finite W algebras (cf. [34], where a characterization of these was given for the infinite standard W_N algebras). Remember that the Kirillov Poisson structure is not associated to a symplectic form but that the Lie algebra splits up into union of symplectic orbits. It is well known that these orbits are just the coadjoint orbits of the group action on the Lie algebra. It may be interesting to apply the procedure of geometric quantization to the symplectic orbits of classical finite W algebras and see if one can reproduce the representations $W(d; x)$. Of course this would be equivalent to finding a Borel–Weil like theorem for finite W algebras.

Another interesting problem is finding comultiplications for finite W algebras, in order to be able to define tensor products. This is a difficult problem, since the natural comultiplication on the universal enveloping algebra does not induce one on the W algebras.

Many of the techniques developed in this paper can equally well be applied to ordinary W algebras. We will come back to this in a future publication.

7. Appendix

In this appendix we give an example of a quantum finite W algebra in which the quantum relations have obtained quantum corrections, i.e. terms of order $\hbar^2$ or higher. It has 7 generators $\{H, E, F, G^+, G^0, G^-, C\}$. The generators $\{H, E, F\}$ form an sl_2 subalgebra, C commutes with everything and the generators $\{G^-, G^0, G^+\}$ form a triplet under the adjoint action of $\{H, E, F\}$, i.e. $[H, G^i] = iG^i$, where $i \in \{-, 0, +\}$, $[E, G^-] = \hbar G^0$, $[E, G^0] = 2\hbar G^+$, $[F, G^+] = \hbar G^0$, $[F, G^0] = 2\hbar G^-$. These relations are the same as the classical ones. The quantum corrections appear in the relations

$$[G^0, G^+] = \hbar\left(-CE + EH^2 + \frac{1}{2}E^2F + \frac{1}{2}EFE\right) - 2\hbar^3 E\,,$$

$$[G^0, G^-] = \hbar\left(-CF + FH^2 + \frac{1}{2}F^2E + \frac{1}{2}FEF\right) - 2\hbar^3 F\,,$$

$$[G^+, G^-] = \hbar\left(-CH + H^3 + \frac{1}{2}HEF + \frac{1}{2}HFE\right) - 2\hbar^3 H\,.$$

The algebra described above is the finite W algebra associated to the partition $4 = 2 + 2$.

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