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Quantum Group Symmetry of Partition Functions of IRF Models and its Application to Jones' Index Theory

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Abstract. For each Boltzmann weight of a face model, we associate two quantum groups (face algebras) which describe the dependence of the partition function on boundary value condition. Using these, we give a proof of (non-)flatness of A–D–E connections of A. Ocneanu, which is a crucial algebraic part of the classification of subfactors with Jones' index less than 4.

1. Introduction

The development of Jones' index theory have exhibited significant similarities to solvable lattice models (SLM). Jones' basic construction naturally gives a quotient of braid group algebra which is known as Temperley–Lieb algebra in SLM. More recently, A. Ocneanu announced the classification of certain class of II_1-subfactors, in which he reduced the problem to that of a certain kind of Boltzmann weights on graphs called *flat connections*. While his full paper has not been published, S. Popa obtained further deep analytic results.

Since flatness of connection is equivalent to certain conditions on values of its partition function, the classification can be viewed as a problem of SLM theory.

In this paper, we propose a new framework to deal with partition functions of SLM's via our notion of *face algebra*, which is a generalization of bialgebra. For each IRF model, we associate two face algebras $\mathfrak{H}_\nu$ ($\nu = 1, 2$) and a bilinear pairing $\langle\,,\rangle: \mathfrak{H}_1 \otimes \mathfrak{H}_2 \to \mathbb{C}$. Generators of $\mathfrak{H}_\nu$ are indexed by "boundary conditions" of finite size models and the values of the pairing are given by partition functions.

As an application, we compute partition functions of connections on A–D–E Dynkin diagrams under some boundary conditions. Thanks to the results of Kawahigashi [K], it gives a proof of flatness of these connections, which is different from that of [K] for D_n and Izumi's recent work [I] for E_8.

In Sect. 2, we fix some terminologies on IRF models which we use in this paper. In Sect. 3, we introduce a notion of face algebras, and construct these from IRF models. In Sect. 4, we show some relation in the face algebras which correspond to Boltzmann weights on non-oriented graphs. In Sect. 5, we construct some representations Σ_r of these algebras. In Sect. 6, we give a proof of flatness of

D_{2n}-connections using the relations of the algebras, which we give in Sect. 4. In Sect. 7, we give a proof of (non-)flatness of E_n-connections using Σ_r.

2. Partition Functions of IRF Models

Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be finite graphs with the common set of vertices $\mathscr{V}$, which is either oriented or non-oriented. When $\mathscr{G}_v$ $(v = 1, 2)$ is non-oriented, we will identify $\mathscr{G}_v$ with an oriented graph $\mathscr{G}'_v$ which is defined as follows: (1) Take an orientation on $\mathscr{G}_v$. Then (2) define an oriented graph $\mathscr{G}'_v$ by vertex$(\mathscr{G}'_v) = \mathscr{V}$ and edge$(\mathscr{G}'_v) = \{\mathbf{p}, \mathbf{p}^{\sim} \mid \mathbf{p} \in \text{edge}(\mathscr{G}_v)\}$, where $\mathbf{p}^{\sim}$ means the edge with its orientation reversed. For example, we identify the Dynkin diagram A_4 with the following graph.

$$\cdot \rightleftarrows \cdot \rightleftarrows \cdot \rightleftarrows \cdot$$

We say that a quadruple $\begin{pmatrix} & \mathbf{p} & \\ \mathbf{r} & & \mathbf{s} \\ & \mathbf{q} & \end{pmatrix}$ or a diagram

$$\begin{array}{ccc} i & \xrightarrow{\mathbf{p}} & j \\ {\scriptstyle \mathbf{r}}\downarrow & & \downarrow{\scriptstyle \mathbf{s}} \\ k & \xrightarrow[\mathbf{q}]{} & \ell \end{array}$$

is a *cell* (or a *face*) if $\mathbf{p}, \mathbf{q} \in \text{edge}(\mathscr{G}_1)$, $\mathbf{r}, \mathbf{s} \in \text{edge}(\mathscr{G}_2)$ and $\mathfrak{s}(\mathbf{p}) = i = \mathfrak{s}(\mathbf{r})$, $\mathfrak{r}(\mathbf{p}) = j = \mathfrak{s}(\mathbf{s})$, $\mathfrak{r}(\mathbf{r}) = k = \mathfrak{s}(\mathbf{q})$, $\mathfrak{r}(\mathbf{q}) = \ell = \mathfrak{r}(\mathbf{s})$, where $\mathfrak{s}(\mathbf{p})$ and $\mathfrak{r}(\mathbf{p})$ denote the source (i.e. start) and the range (i.e. end) of $\mathbf{p}$ respectively. We say that $(\mathscr{G}_1, \mathscr{G}_2, W)$ is a *graph coupling* if W: $(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}) \mapsto W\begin{pmatrix} & \mathbf{p} & \\ \mathbf{r} & & \mathbf{s} \\ & \mathbf{q} & \end{pmatrix}$ is a complex-valued function on edge$(\mathscr{G}_1)^2 \times$ edge$(\mathscr{G}_2)^2$ such that $W\begin{pmatrix} & \mathbf{p} & \\ \mathbf{r} & & \mathbf{s} \\ & \mathbf{q} & \end{pmatrix} = 0$ unless $\begin{pmatrix} & \mathbf{p} & \\ \mathbf{r} & & \mathbf{s} \\ & \mathbf{q} & \end{pmatrix}$ is a cell. We call W the *Boltzmann Weight* of $(\mathscr{G}_1, \mathscr{G}_2, W)$.

For $r > 0$, let $P^r(\mathscr{G}_v) = \coprod_{i,j \in \mathscr{V}} P^r_{i,j}(\mathscr{G}_v)$ be the set of *paths* on $\mathscr{G}_v$ of length r. That is, $\mathbf{p} \in P^r_{i,j}(\mathscr{G}_v)$ if $\mathbf{p}$ is a sequence $(\mathbf{p}_1, \ldots, \mathbf{p}_r)$ of edges of $\mathscr{G}_v$ such that $\mathfrak{s}(\mathbf{p}) := \mathfrak{s}(\mathbf{p}_1) = i$, $\mathfrak{r}(\mathbf{p}_1) = \mathfrak{s}(\mathbf{p}_2), \ldots, \mathfrak{r}(\mathbf{p}_{r-1}) = \mathfrak{s}(\mathbf{p}_r)$, $\mathfrak{r}(\mathbf{p}) := \mathfrak{r}(\mathbf{p}_r) = j$. Also we use the following notation:

$$P^r_{i,-}(\mathscr{G}_v) = \coprod_{j \in \mathscr{V}} P^r_{i,j}(\mathscr{G}_v), \quad P^r_{-,j}(\mathscr{G}_v) = \coprod_{i \in \mathscr{V}} P^r_{i,j}(\mathscr{G}_v),$$

$$P^0_{i,j}(\mathscr{G}_v) = \begin{cases} \{i\} & (i = j) \\ \emptyset & (i \neq j) . \end{cases}$$

For a path $\mathbf{p} \in P^r(\mathscr{G}_v)$, we define $\mathbf{p}_1, \ldots, \mathbf{p}_r \in \text{edge}(\mathscr{G}_v)$ by $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_r)$. If $\mathfrak{r}(\mathbf{p}) = \mathfrak{s}(\mathbf{r})$ for $\mathbf{p} \in P^r(\mathscr{G}_v)$ and $\mathbf{r} \in P^s(\mathscr{G}_v)$, we define the *composition* $\mathbf{p} \cdot \mathbf{r} \in P^{r+s}(\mathscr{G}_v)$ by $\mathbf{p} \cdot \mathbf{r} = (\mathbf{p}_1, \ldots, \mathbf{p}_r, \mathbf{r}_1, \ldots, \mathbf{r}_s)$.

Let $\mathbf{p}, \mathbf{q} \in P^r(\mathscr{G}_1)$ $\mathbf{r}, \mathbf{s} \in P^s(\mathscr{G}_2)$ be paths such that $\mathfrak{s}(\mathbf{p}) = \mathfrak{s}(\mathbf{r})$, $\mathfrak{r}(\mathbf{p}) = \mathfrak{s}(\mathbf{s})$, $\mathfrak{r}(\mathbf{r}) = \mathfrak{s}(\mathbf{q})$, $\mathfrak{r}(\mathbf{q}) = \mathfrak{r}(\mathbf{s})$. We define the *partition function* $W\begin{pmatrix} & \mathbf{p} & \\ \mathbf{r} & & \mathbf{s} \\ & \mathbf{q} & \end{pmatrix}$ to be a complex number defined by the following formula:

$$W\begin{pmatrix} & \mathbf{p} & \\ \mathbf{r} & & \mathbf{s} \\ & \mathbf{q} & \end{pmatrix} = \sum_{\text{configurations}} \prod_{\text{cells}} W(\text{cell}),$$

where "configuration" means a choice of cells which fills the following $r\times s$-diagram:

$$\begin{array}{ccccccccc}
 & \xrightarrow{\mathbf{p}_1} & & \xrightarrow{\mathbf{p}_2} & \cdots & & \xrightarrow{\mathbf{p}_r} & & \\
\mathbf{r}_1\downarrow & \longrightarrow & \downarrow & \longrightarrow & \cdots & \downarrow & \longrightarrow & \downarrow \mathbf{s}_1 & \\
\downarrow & & \downarrow & & & & & \downarrow & \\
\vdots & & \vdots & & & & & \vdots & . \\
\mathbf{r}_s\downarrow & \xrightarrow{\mathbf{q}_1} & \downarrow & \longrightarrow & \cdots & \downarrow & \xrightarrow{\mathbf{q}_r} & \downarrow \mathbf{s}_s &
\end{array}$$

For convenience, we set $W\left(\mathbf{r}\,\genfrac{}{}{0pt}{}{\mathbf{p}}{\mathbf{q}}\,\mathbf{s}\right)=\delta_{\mathbf{pq}}$ (respectively $W\left(\mathbf{r}\,\genfrac{}{}{0pt}{}{\mathbf{p}}{\mathbf{q}}\,\mathbf{s}\right)=\delta_{\mathbf{rs}}$) if $s=0$ (respectively $r=0$). Also, we set $W\left(\mathbf{r}\,\genfrac{}{}{0pt}{}{\mathbf{p}}{\mathbf{q}}\,\mathbf{s}\right)=0$ for sequences $\mathbf{p},\mathbf{q},\mathbf{r},\mathbf{s}$ of edges which do not satisfy the above condition. We note that the partition function is characterized as an extension of the Boltzmann weight which satisfies the following algebraic relations:

$$W\left(\mathbf{r}\,\genfrac{}{}{0pt}{}{\mathbf{p}\cdot\mathbf{p}'}{\mathbf{q}\cdot\mathbf{q}'}\,\mathbf{s}\right)=\sum_{\mathbf{t}\in P^s(\mathscr{G}_2)} W\left(\mathbf{r}\,\genfrac{}{}{0pt}{}{\mathbf{p}}{\mathbf{q}}\,\mathbf{t}\right)W\left(\mathbf{t}\,\genfrac{}{}{0pt}{}{\mathbf{p}'}{\mathbf{q}'}\,\mathbf{s}\right), \tag{2.1}$$

$$W\left(\mathbf{r}\cdot\mathbf{r}'\,\genfrac{}{}{0pt}{}{\mathbf{p}}{\mathbf{q}}\,\mathbf{s}\cdot\mathbf{s}'\right)=\sum_{\mathbf{t}\in P^r(\mathscr{G}_1)} W\left(\mathbf{r}\,\genfrac{}{}{0pt}{}{\mathbf{p}}{\mathbf{t}}\,\mathbf{s}\right)W\left(\mathbf{r}'\,\genfrac{}{}{0pt}{}{\mathbf{t}}{\mathbf{q}}\,\mathbf{s}'\right), \tag{2.2}$$

where $\mathbf{p},\mathbf{q}\in P^r(\mathscr{G}_1)$, $\mathbf{r},\mathbf{s}\in P^s(\mathscr{G}_2)$, $\mathbf{p}',\mathbf{q}'\in P^t(\mathscr{G}_1)$, $\mathbf{r}',\mathbf{s}'\in P^t(\mathscr{G}_2)$.

Let $(\mathscr{G}_1,\mathscr{G}_2,W)$ be a graph coupling such that $\mathscr{G}_1=\mathscr{G}=\mathscr{G}_2$. We say that W is a *biunitary connection* if the following three properties are satisfied: (1) $\mathscr{G}$ is a bipartite non-oriented graph. (2) For each cell, W satisfies the following *renormalization rule*:

$$W\begin{pmatrix} i & \xrightarrow{\mathbf{p}} & j \\ \mathbf{r}\downarrow & & \downarrow\mathbf{s} \\ k & \xrightarrow[\mathbf{q}]{} & \ell \end{pmatrix}=\left(\frac{\mu(j)\mu(k)}{\mu(i)\mu(\ell)}\right)^{1/2}\overline{W\begin{pmatrix} j & \xrightarrow{\tilde{\mathbf{p}}} & i \\ \mathbf{s}\downarrow & & \downarrow\mathbf{r} \\ \ell & \xrightarrow[\tilde{\mathbf{q}}]{} & k \end{pmatrix}}$$

$$=\left(\frac{\mu(j)\mu(k)}{\mu(i)\mu(\ell)}\right)^{1/2}\overline{W\begin{pmatrix} k & \xrightarrow{\mathbf{q}} & \ell \\ \tilde{\mathbf{r}}\downarrow & & \downarrow\tilde{\mathbf{s}} \\ i & \xrightarrow[\mathbf{p}]{} & j \end{pmatrix}}\,. \tag{2.3}$$

where $\mu(-)$ denotes an entry of the Perron–Frobenius eigenvector of the adjacency matrix of each graph. (3) For each $\mathbf{q},\mathbf{q}',\mathbf{r},\mathbf{r}'\in\mathrm{edge}(\mathscr{G}_2)$, the following *biunitary axiom* is satisfied:

$$\sum_{\mathbf{p},\mathbf{s}} W\left(\mathbf{r}\,\genfrac{}{}{0pt}{}{\mathbf{p}}{\mathbf{q}}\,\mathbf{s}\right)\overline{W\left(\mathbf{r}'\,\genfrac{}{}{0pt}{}{\mathbf{p}}{\mathbf{q}'}\,\mathbf{s}\right)}=\delta_{\mathscr{r}(\mathbf{r})\mathscr{s}(\mathbf{q})}\delta_{\mathscr{r}(\mathbf{r}')\mathscr{s}(\mathbf{q}')}\delta_{\mathbf{rr}'}\delta_{\mathbf{qq}'}\,. \tag{2.4}$$

We say that $(\mathscr{G}_1,\mathscr{G}_2,W)$ is *flat with respect to* $*\in\mathscr{V}$ if $W\left(\mathbf{r}\,\genfrac{}{}{0pt}{}{\mathbf{p}}{\mathbf{q}}\,\mathbf{s}\right)=\delta_{\mathbf{pq}}\delta_{\mathbf{rs}}$ for each $\mathbf{p},\mathbf{q}\in P^r_{*,*}(\mathscr{G}_1)$ and $\mathbf{r},\mathbf{s}\in P^s_{*,*}(\mathscr{G}_2)$ $(r,s\geqq 0)$. It is known that W is flat if and only if

$$\sum_{\mathbf{t}\in P^s(\mathscr{G}_2)} W\left(\mathbf{r}\,\genfrac{}{}{0pt}{}{\mathbf{p}}{\mathbf{q}}\,\mathbf{t}\right)\overline{W\left(\mathbf{s}\,\genfrac{}{}{0pt}{}{\mathbf{p}'}{\mathbf{q}'}\,\mathbf{t}\right)}=\delta_{\mathbf{rs}}C_{\mathbf{pp}'\mathbf{qq}'} \tag{2.5}$$

for each $i \in \mathscr{V}$ and $\mathbf{p}, \mathbf{p}' \in P^r_{*,i}(\mathscr{G}_1)$, where $C_{\mathbf{pp'qq'}}$ denotes a constant which depends only on $\mathbf{p}, \mathbf{p}', \mathbf{q}, \mathbf{q}'$ (see [O1, K]).

Actually, the above definition of connections is only a special case of the original. We refer the reader to [O1, O2, K] for a precise definition. For a connection on graphs, one can construct an AFD II_1-subfactor. Conversely, a flat connection appears as an invariant of certain subfactors via Ocneanu's "Golois functor." Moreover classification of AFD II_1-subfactor with Jones' index less than 4 is reduced to that of flat connections on A–D–E diagrams.

Lemma 2.1. *If W is a connection, then its partition function also satisfies the renormalization rule* (2.3) *for each* $\mathbf{p} \in P^r_{i,j}(\mathscr{G}_1)$, $\mathbf{q} \in P^r_{k,\ell}(\mathscr{G}_1)$, $\mathbf{r} \in P^s_{i,k}(\mathscr{G}_2)$ *and* $\mathbf{s} \in P^s_{j,\ell}(\mathscr{G}_2)$.

3. Face Algebras

Let $\mathfrak{H}$ be an algebra over $\mathbb{C}$ which also has a coalgebra structure $(\mathfrak{H}, \Delta, \varepsilon)$. Let $\left\{ x\begin{pmatrix} i \\ j \end{pmatrix} \,\middle|\, i, j \in \mathscr{V} \right\}$ be elements of $\mathfrak{H}$ indexed by two elements of a finite non-empty set $\mathscr{V}$. We say that $\left(\mathfrak{H}, \left\{ x\begin{pmatrix} i \\ j \end{pmatrix} \right\} \right)$ is a *$\mathscr{V}$-face algebra* if the following conditions are satisfied:

$$\Delta(a)\Delta(b) = \Delta(ab)\,,$$

$$x\begin{pmatrix} i \\ j \end{pmatrix} x\begin{pmatrix} m \\ n \end{pmatrix} = \delta_{im}\delta_{jn} x\begin{pmatrix} i \\ j \end{pmatrix}, \quad \sum_{i,j \in \mathscr{V}} x\begin{pmatrix} i \\ j \end{pmatrix} = 1\,,$$

$$\Delta\left(x\begin{pmatrix} i \\ j \end{pmatrix} \right) = \sum_{k \in \mathscr{V}} x\begin{pmatrix} i \\ k \end{pmatrix} \otimes x\begin{pmatrix} k \\ j \end{pmatrix}, \quad \varepsilon\left(x\begin{pmatrix} i \\ j \end{pmatrix} \right) = \delta_{ij}\,,$$

$$\varepsilon(ab) = \sum_{i,j,k \in \mathscr{V}} \varepsilon\left(a x\begin{pmatrix} i \\ k \end{pmatrix} \right) \varepsilon\left(x\begin{pmatrix} k \\ j \end{pmatrix} b \right),$$

where $i, j, m, n \in \mathscr{V}$ and $a, b \in \mathfrak{H}$. When $\mathrm{card}(\mathscr{V}) = 1$, $\mathscr{V}$-face algebra is an equivalent notion of bialgebra.

Let $(\mathscr{G}_1, \mathscr{G}_2, W)$ be a graph coupling with the vertex set $\mathscr{V}$. Let $\tilde{\mathfrak{H}}_\nu$ $(\nu = 1, 2)$ be the linear span of the symbols $\left\{ x_\nu\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \,\middle|\, \mathbf{p}, \mathbf{q} \in P^r(\mathscr{G}_\nu), r \geqq 0 \right\}$. We define a face algebra structure on $\tilde{\mathfrak{H}}_\nu$ as follows:

$$x_\nu\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} x_\nu\begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix} = \delta_{\mathfrak{r}(\mathbf{p}),\, \mathfrak{s}(\mathbf{r})}\delta_{\mathfrak{r}(\mathbf{q}),\, \mathfrak{s}(\mathbf{s})} x_\nu\begin{pmatrix} \mathbf{p}\cdot\mathbf{r} \\ \mathbf{q}\cdot\mathbf{s} \end{pmatrix},$$

$$\Delta\left(x_\nu\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \right) = \sum_{\mathbf{t} \in P^r(\mathscr{G}_\nu)} x_\nu\begin{pmatrix} \mathbf{p} \\ \mathbf{t} \end{pmatrix} \otimes x_\nu\begin{pmatrix} \mathbf{t} \\ \mathbf{q} \end{pmatrix}, \quad \varepsilon\left(x_\nu\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \right) = \delta_{\mathbf{pq}}\,,$$

where $\mathbf{p}, \mathbf{q} \in P^r(\mathscr{G}_\nu)$, $\mathbf{r}, \mathbf{s} \in P^s(\mathscr{G}_\nu)$. We define a bilinear pairing $\langle\,,\rangle\colon \tilde{\mathfrak{H}}_1 \otimes \tilde{\mathfrak{H}}_2 \to \mathbb{C}$ by

$$\left\langle x_1\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}, x_2\begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix} \right\rangle = W\begin{pmatrix} & \mathbf{p} & \\ \mathbf{r} & & \mathbf{s} \\ & \mathbf{q} & \end{pmatrix}.$$

It follows easily from (2.1) and (2.2) that $\langle\,,\rangle$ is a face algebra pairing [H2], that is, we have:

$$\langle ab, c\rangle = \sum_{(c)} \langle a, c_{(1)}\rangle \langle b, c_{(2)}\rangle\,,$$
$$\langle a, cd\rangle = \sum_{(a)} \langle a_{(1)}, c\rangle \langle a_{(2)}, d\rangle\,, \tag{3.1}$$

$$\left\langle x_1\begin{pmatrix} i \\ j\end{pmatrix}, c\right\rangle = \sum_{k,\ell\in\mathscr{V}} \varepsilon\left(x_2\begin{pmatrix} i \\ k\end{pmatrix} c x_2\begin{pmatrix} \ell \\ j\end{pmatrix}\right),$$
$$\left\langle a, x_2\begin{pmatrix} i \\ j\end{pmatrix}\right\rangle = \sum_{k,\ell\in\mathscr{V}} \varepsilon\left(x_1\begin{pmatrix} i \\ k\end{pmatrix} a x_1\begin{pmatrix} \ell \\ j\end{pmatrix}\right), \tag{3.2}$$

where $i, j\in\mathscr{V}$, $a, b\in\tilde{\mathfrak{H}}_1$, $c, d\in\tilde{\mathfrak{H}}_2$, $\Delta(a) = \sum_{(a)} a_{(1)}\otimes a_{(2)}$ and $\Delta(c) = \sum_{(c)} c_{(1)}\otimes c_{(2)}$. Hence $\mathfrak{J}_1 := \{a\in\tilde{\mathfrak{H}}_1 \mid \langle a, \tilde{\mathfrak{H}}_2\rangle = 0\}$ (respectively $\mathfrak{J}_2 := \{c\in\tilde{\mathfrak{H}}_2 \mid \langle\tilde{\mathfrak{H}}_1, c\rangle = 0\}$) becomes both an ideal and a coideal of $\tilde{\mathfrak{H}}_1$ (respectively $\tilde{\mathfrak{H}}_2$), and $\mathfrak{H}_v := \tilde{\mathfrak{H}}_v/\mathfrak{J}_v$ $(v = 1, 2)$ becomes a face algebra. We call $\mathfrak{H}_1$ and $\mathfrak{H}_2$ *horizontal* and *vertical generating face algebra* of $(\mathscr{G}_1, \mathscr{G}_2, W)$ respectively. We denote the image of $x_v\begin{pmatrix} \mathbf{p} \\ \mathbf{q}\end{pmatrix}$ via the projection $\tilde{\mathfrak{H}}_v \twoheadrightarrow \mathfrak{H}_v$ again by $x_v\begin{pmatrix} \mathbf{p} \\ \mathbf{q}\end{pmatrix}$.

We say that a graph coupling $(\mathscr{G}_1, \mathscr{G}_2, W)$ is *self-dual* if $\mathscr{G}_1 = \mathscr{G}_2$ and $W\begin{pmatrix} & \mathbf{p} & \\ \mathbf{r} & & \mathbf{s} \\ & \mathbf{q} & \end{pmatrix} = W\begin{pmatrix} & \mathbf{r} & \\ \mathbf{p} & & \mathbf{q} \\ & \mathbf{s} & \end{pmatrix}$ for each $\mathbf{p}, \mathbf{q}\in P^1(\mathscr{G}_1)$ and $\mathbf{r}, \mathbf{s}\in P^1(\mathscr{G}_2)$. Since $\left\langle x_1\begin{pmatrix} \mathbf{p} \\ \mathbf{q}\end{pmatrix}, x_2\begin{pmatrix} \mathbf{r} \\ \mathbf{s}\end{pmatrix}\right\rangle = \left\langle x_1\begin{pmatrix} \mathbf{r} \\ \mathbf{s}\end{pmatrix}, x_2\begin{pmatrix} \mathbf{p} \\ \mathbf{q}\end{pmatrix}\right\rangle$, we have the following.

Lemma 3.1. *If $(\mathscr{G}_1, \mathscr{G}_2, W)$ is self-dual, then there exists an algebra, coalgebra isomorphism $\mathfrak{H}_1 \xrightarrow{\sim} \mathfrak{H}_2$ which sends $x_1\begin{pmatrix} \mathbf{p} \\ \mathbf{q}\end{pmatrix}$ to $x_2\begin{pmatrix} \mathbf{p} \\ \mathbf{q}\end{pmatrix}$.*

For a self-dual graph coupling, we identify $\mathfrak{H}_1$ with $\mathfrak{H}_2$ by the above lemma and denote $\mathfrak{H}$ and $x\begin{pmatrix} \mathbf{p} \\ \mathbf{q}\end{pmatrix}$ instead of $\mathfrak{H}_1$ and $x_1\begin{pmatrix} \mathbf{p} \\ \mathbf{q}\end{pmatrix}$ respectively.

Proposition 3.2. *Let $(\mathscr{G}_1, \mathscr{G}_2, W)$ be a self-dual graph coupling such that W satisfies the star-triangle relation (Yang–Baxter equation). Then the following relation holds in $\mathfrak{H}$:*

(a face version of L-operator relation [RTF, H3])

$$\sum_{\mathbf{r}\cdot\mathbf{s}\in P^2(\mathscr{G})} W\begin{pmatrix} & \mathbf{p} & \\ \mathbf{r} & & \mathbf{q} \\ & \mathbf{s} & \end{pmatrix} x\begin{pmatrix} \mathbf{a}\cdot\mathbf{b} \\ \mathbf{r}\cdot\mathbf{s}\end{pmatrix} - \sum_{\mathbf{c}\cdot\mathbf{d}\in P^2(\mathscr{G})} W\begin{pmatrix} & \mathbf{c} & \\ \mathbf{a} & & \mathbf{d} \\ & \mathbf{b} & \end{pmatrix} x\begin{pmatrix} \mathbf{c}\cdot\mathbf{d} \\ \mathbf{p}\cdot\mathbf{q}\end{pmatrix} = 0$$
$$(\mathbf{a}\cdot\mathbf{b}, \mathbf{p}\cdot\mathbf{q}\in P^2(\mathscr{G}))\,.$$

Proof. Let $R\begin{pmatrix} \mathbf{a}\cdot\mathbf{b} \\ \mathbf{p}\cdot\mathbf{q}\end{pmatrix}$ be the left-hand side of the above relation. Then we have

$$\Delta\left(R\begin{pmatrix} \mathbf{a}\cdot\mathbf{b} \\ \mathbf{p}\cdot\mathbf{q}\end{pmatrix}\right) = \sum_{\mathbf{i}\cdot\mathbf{j}\in P^2(\mathscr{G})} \left(R\begin{pmatrix} \mathbf{a}\cdot\mathbf{b} \\ \mathbf{i}\cdot\mathbf{j}\end{pmatrix} \otimes x\begin{pmatrix} \mathbf{i}\cdot\mathbf{j} \\ \mathbf{p}\cdot\mathbf{q}\end{pmatrix} + x\begin{pmatrix} \mathbf{a}\cdot\mathbf{b} \\ \mathbf{i}\cdot\mathbf{j}\end{pmatrix} \otimes R\begin{pmatrix} \mathbf{i}\cdot\mathbf{j} \\ \mathbf{p}\cdot\mathbf{q}\end{pmatrix}\right).$$

Hence, by (3.2), it suffices to verify $\left\langle R\begin{pmatrix}\mathbf{a}\cdot\mathbf{b}\\ \mathbf{p}\cdot\mathbf{q}\end{pmatrix}, x\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\end{pmatrix}\right\rangle = 0$ for $\mathbf{i}, \mathbf{j} \in P^1(\mathscr{G})$, which is equivalent to the star-triangle relation by the self-duality. □

4. Jones–Ocneanu's Connections

Let $\mathscr{G}$ be a connected non-oriented finite graph which has at least one edge. Let β be the Perron–Frobenius eigenvalue of the adjacency matrix of $\mathscr{G}$. Let ε be a solution of the equation $\varepsilon^2 + \varepsilon^{-2} + \beta = 0$. We define a self-dual biunitary connection $W = W_{\mathscr{G},\varepsilon}$ on $(\mathscr{G}, \mathscr{G})$ by

$$W\left(\begin{array}{ccc} i & \xrightarrow{\mathbf{p}} & j \\ \mathbf{r}\downarrow & & \downarrow\mathbf{s} \\ k & \xrightarrow[\mathbf{q}]{} & \ell \end{array}\right) = \delta_{\mathbf{pr}}\delta_{\mathbf{qs}}\varepsilon + \delta_{\mathbf{p},\mathbf{s}^\sim}\delta_{\mathbf{r},\mathbf{q}^\sim}\left(\frac{\mu(j)\mu(k)}{\mu(i)\mu(\ell)}\right)^{1/2}\varepsilon^{-1}$$

and call it *Jones–Ocneanu's connection* on $\mathscr{G}$. If $\mathscr{G}$ is one of the A–D–E Dynkin diagram, each connection on $\mathscr{G}$ is equivalent to $W_{\mathscr{G},\varepsilon}$ (cf. [O1, K]).

Lemma 4.1. *The Boltzmann weight $W_{\mathscr{G},\varepsilon}$ satisfies the star-triangle relation.*

We use this result only in case that $\mathscr{G}$ is bipartite. In this case, this lemma follows easily from Ocneanu–Sunder's path model for Jones' construction for multi-matrix algebras (cf. [GHJ, Chap. 2]). When $\mathscr{G}$ is not bipartite, this lemma is proved directly using the path model picture and we omit the details.

Let $\mathfrak{H} = \mathfrak{H}(\mathscr{G}, \varepsilon)$ be the generating face algebra of $(\mathscr{G}, \mathscr{G}, W_{\mathscr{G},\varepsilon})$.

Proposition 4.2. *For each $\mathbf{p}, \mathbf{q} \in P^1(\mathscr{G})$ and $j \in \mathscr{V}$, the following relations hold in $\mathfrak{H}(\mathscr{G}, \varepsilon)$:*

$$\sum_{\mathbf{s}\in P^1_{j,-}(\mathscr{G})} \mu(\mathfrak{r}(\mathbf{s}))^{1/2} x\begin{pmatrix}\mathbf{p}\cdot\mathbf{q}\\ \mathbf{s}\cdot\mathbf{s}^\sim\end{pmatrix} = \delta_{\mathbf{p},\mathbf{q}^\sim}\left(\frac{\mu(j)\mu(\mathfrak{r}(\mathbf{p}))}{\mu(\mathfrak{s}(\mathbf{p}))}\right)^{1/2} x\begin{pmatrix}\mathfrak{s}(\mathbf{p})\\ j\end{pmatrix},$$

$$\sum_{\mathbf{s}\in P^1_{j,-}(\mathscr{G})} \mu(\mathfrak{r}(\mathbf{s}))^{1/2} x\begin{pmatrix}\mathbf{s}\cdot\mathbf{s}^\sim\\ \mathbf{p}\cdot\mathbf{q}\end{pmatrix} = \delta_{\mathbf{p},\mathbf{q}^\sim}\left(\frac{\mu(j)\mu(\mathfrak{r}(\mathbf{p}))}{\mu(\mathfrak{s}(\mathbf{p}))}\right)^{1/2} x\begin{pmatrix}j\\ \mathfrak{s}(\mathbf{p})\end{pmatrix}. \tag{4.1}$$

Proof. Except for $\mathbf{p} = \mathbf{q}^\sim$, the relation follows from Proposition 3.2. Also, from Proposition 3.2, we see that

$$\det\begin{pmatrix}i\\ j\end{pmatrix} := \sum_{\mathbf{s}\in P^1_{j,-}(\mathscr{G})}\left(\frac{\mu(i)\mu(\mathfrak{r}(\mathbf{s}))}{\mu(j)\mu(\mathfrak{r}(\mathbf{p}))}\right)^{1/2} x\begin{pmatrix}\mathbf{p}\cdot\mathbf{p}^\sim\\ \mathbf{s}\cdot\mathbf{s}^\sim\end{pmatrix}$$
$$= \sum_{\mathbf{s}\in P^1_{i,-}(\mathscr{G})}\left(\frac{\mu(j)\mu(\mathfrak{r}(\mathbf{s}))}{\mu(i)\mu(\mathfrak{r}(\mathbf{q}))}\right)^{1/2} x\begin{pmatrix}\mathbf{s}\cdot\mathbf{s}^\sim\\ \mathbf{q}\cdot\mathbf{q}^\sim\end{pmatrix}$$

does not depend on the choice of $\mathbf{p} \in P^1_{i,-}(\mathscr{G})$ and $\mathbf{q} \in P^1_{j,-}(\mathscr{G})$. Since

$$\Delta\left(\det\begin{pmatrix}i\\ j\end{pmatrix}\right) = \sum_{\mathbf{u}\in P^2(\mathscr{G})}\left(\frac{\mu(i)}{\mu(j)\mu(\mathfrak{r}(\mathbf{p}))}\right)^{1/2} x\begin{pmatrix}\mathbf{p}\cdot\mathbf{p}^\sim\\ \mathbf{u}\end{pmatrix}\otimes\sum_{\mathbf{s}\in P^1_{j,-}(\mathscr{G})}\mu(\mathfrak{r}(\mathbf{s}))^{1/2} x\begin{pmatrix}\mathbf{u}\\ \mathbf{s}\cdot\mathbf{s}^\sim\end{pmatrix}$$
$$= \sum_{\mathbf{q}\in P^1(\mathscr{G})}\left(\frac{\mu(i)}{\mu(j)\mu(\mathfrak{r}(\mathbf{p}))}\right)^{1/2} x\begin{pmatrix}\mathbf{p}\cdot\mathbf{p}^\sim\\ \mathbf{q}\cdot\mathbf{q}^\sim\end{pmatrix}\otimes\sum_{\mathbf{s}\in P^1_{j,-}(\mathscr{G})}\mu(\mathfrak{r}(\mathbf{s}))^{1/2} x\begin{pmatrix}\mathbf{q}\cdot\mathbf{q}^\sim\\ \mathbf{s}\cdot\mathbf{s}^\sim\end{pmatrix}$$
$$= \sum_{k\in\mathscr{V}}\det\begin{pmatrix}i\\ k\end{pmatrix}\otimes\det\begin{pmatrix}k\\ j\end{pmatrix},$$

we have

$$\left\langle \det\begin{pmatrix} i \\ j \end{pmatrix} - x\begin{pmatrix} i \\ j \end{pmatrix}, x\begin{pmatrix} \mathbf{r}\cdot\mathbf{r}' \\ \mathbf{s}\cdot\mathbf{s}' \end{pmatrix} \right\rangle = \sum_{k\in\mathscr{V}} \left(\left\langle \det\begin{pmatrix} i \\ k \end{pmatrix}, x\begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix} \right\rangle \left\langle \det\begin{pmatrix} k \\ j \end{pmatrix}, x\begin{pmatrix} \mathbf{r}' \\ \mathbf{s}' \end{pmatrix} \right\rangle \right.$$

$$\left. - \left\langle x\begin{pmatrix} i \\ k \end{pmatrix}, x\begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix} \right\rangle \left\langle x\begin{pmatrix} k \\ j \end{pmatrix}, x\begin{pmatrix} \mathbf{r}' \\ \mathbf{s}' \end{pmatrix} \right\rangle \right).$$

Hence $\det\begin{pmatrix} i \\ j \end{pmatrix} - x\begin{pmatrix} i \\ j \end{pmatrix} = 0$ follows from $\left\langle \det\begin{pmatrix} i \\ j \end{pmatrix}, x\begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix} \right\rangle = \left\langle x\begin{pmatrix} i \\ j \end{pmatrix}, x\begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix} \right\rangle$ for $\mathbf{r}, \mathbf{s} \in P_1(\mathscr{G})$, which follows easily from (2.3) and (2.4). □

Remark. When $\mathscr{G}$ is the Dynkin diagram A_n, the relations given in the above proposition agree with the defining relation of the face algebra $\mathscr{S}\left(A_1, \exp\left(\frac{2\pi i}{n+1}\right)\right)$ which appeared in [H1]. In a forthcoming paper, we show that $\mathfrak{H}(\mathscr{G}, \varepsilon)$ is actually isomorphic to $\mathscr{S}\left(A_1, \exp\left(\frac{2\pi i}{n+1}\right)\right)$.

5. Representations of $\mathfrak{H}(\mathscr{G}, \varepsilon)$

Let V^s be the linear span of the symbols $\{u(\mathbf{p}) \mid \mathbf{p} \in P^s(\mathscr{G})\}$. Since $u(\mathbf{s}) \mapsto \sum_{\mathbf{r}\in P^s(\mathscr{G})} u(\mathbf{r}) \otimes x\begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix}$ defines a right $\mathfrak{H}(\mathscr{G}, \varepsilon)$-comodule structure on V^s, V^s has a left $\mathfrak{H}(\mathscr{G}, \varepsilon)$-module structure given by:

$$x\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} u(\mathbf{s}) = \sum_{\mathbf{r}\in P^s(\mathscr{G})} u(\mathbf{r}) \left\langle x\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}, x\begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix} \right\rangle = \sum_{\mathbf{r}\in P^s(\mathscr{G})} W\begin{pmatrix} & \mathbf{p} & \\ \mathbf{r} & & \mathbf{s} \\ & \mathbf{q} & \end{pmatrix} u(\mathbf{r}) .$$

Let B be a linear operator on V^2 defined by

$$B(u(\mathbf{p}\cdot\mathbf{s})) = \sum_{\mathbf{r}\cdot\mathbf{q}\in P^2(\mathscr{G})} W\begin{pmatrix} & \mathbf{p} & \\ \mathbf{r} & & \mathbf{s} \\ & \mathbf{q} & \end{pmatrix} u(\mathbf{r}\cdot\mathbf{q}) .$$

Then, as an immediate consequence of Proposition 3.2, we obtain the following.

Proposition 5.1. *The operator B commutes with the coaction and the action of $\mathfrak{H}(\mathscr{G}, \varepsilon)$ on V^2.*

We define an associative algebra $\Sigma(\mathscr{G}) = \bigoplus_{r\geqq 0} \bigoplus_{i,j\in\mathscr{V}} \Sigma^r_{i,j}(\mathscr{G})$ by

$$\Sigma(\mathscr{G}) = \left\langle \sigma(\mathbf{p}) (\mathbf{p}\in P^r(\mathscr{G}), r \geqq 0) \mid \sigma(\mathbf{p})\sigma(\mathbf{q}) = \delta_{\mathfrak{t}(\mathbf{p}), \mathfrak{s}(\mathbf{q})} \sigma(\mathbf{p}\cdot\mathbf{q}) ,\right.$$

$$\left. \sum_{\mathbf{p}\in P^1_{i,-}(\mathscr{G})} \mu(\mathfrak{t}(\mathbf{p}))^{1/2} \sigma(\mathbf{p}\cdot\mathbf{p}^{\sim}) = 0 \quad (i \in \mathscr{V}) \right\rangle ,$$

$$\Sigma^r_{i,j}(\mathscr{G}) = \operatorname{span}\{\sigma(\mathbf{p}) \mid \mathbf{p}\in P^r_{i,j}(\mathscr{G})\} .$$

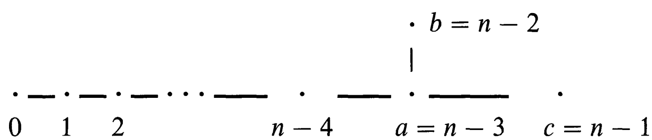
Since $\operatorname{Im}(B - \varepsilon\cdot\mathrm{id}) = \bigoplus_{i\in\mathscr{V}} \mathbb{C} \sum_{\mathbf{p}\in P^1_{i,-}(\mathscr{G})} \mu(\mathfrak{t}(\mathbf{p}))^{1/2} u(\mathbf{p}\cdot\mathbf{p}^{\sim})$ is a subcomodule of V^2, $\Sigma^r(\mathscr{G}) := \bigoplus_{i,j} \Sigma^r_{i,j}(\mathscr{G})$ has a quotient comodule structure of V^r. Hence $\Sigma^r(\mathscr{G})$

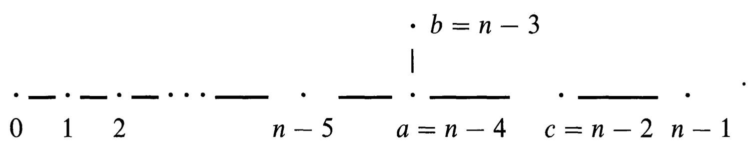
also has an $\mathfrak{H}$-module structure given by

$$x\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}\sigma(\mathbf{s}) = \sum_{\mathbf{r}\in P^r(\mathscr{G})} W\begin{pmatrix} & \mathbf{p} & \\ \mathbf{r} & & \mathbf{s} \\ & \mathbf{q} & \end{pmatrix}\sigma(\mathbf{r})\,. \tag{5.1}$$

6. Flatness of D_n-Connections

In this section and later, we work on the Dynkin diagrams D_n, E_6, E_7 and E_8, and denote a path $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_r)$ by $\mathbf{p} = (\mathscr{s}(\mathbf{p}_1), \ldots, \mathscr{s}(\mathbf{p}_r), \mathscr{r}(\mathbf{p}_r))$. We fix the following labeling of the vertices of the diagrams:

(D_n) 

(E_n) 

By considering the structure of string algebras, Kawahigashi [K] showed that the flatness of A_n-connection is obvious and that flatness of D_{2m}, E_6 and E_8-connection is equivalent to the single equation $W_{\mathscr{G},\varepsilon}\begin{pmatrix} & \mathbf{b} & \\ \mathbf{b} & & \mathbf{b} \\ & \mathbf{b} & \end{pmatrix} = 1$. Here and hereafter we set:

$$\mathbf{b} = (0, 1, 2, \ldots, a, b, a, \ldots, 2, 1, 0)\,,$$

$$\mathbf{c} = (0, 1, 2, \ldots, a, c, a, \ldots, 2, 1, 0)\,.$$

Lemma 6.1. (1) *Let $\mathscr{G}$ be either D_n or E_n ($n = 6, 7, 8$) and let* $\mathbf{r}$ *be either* $(0, 1, \ldots, i)$, $(i, i-1, \ldots, 0)$ $(1 \leqq i \leqq b)$ *or* $(0, 1, \ldots, a, c)$, $(c, a, a-1, \ldots, 0)$. *Then there exist linear maps $\Sigma(\mathscr{G}) \to \mathfrak{H}(\mathscr{G})$ which are given by $\sigma(\mathbf{p}) \mapsto x\begin{pmatrix}\mathbf{r}\\ \mathbf{p}\end{pmatrix}$ and $\sigma(\mathbf{p}) \mapsto x\begin{pmatrix}\mathbf{p}\\ \mathbf{r}\end{pmatrix}$.*
(2) *There exists a linear map $\Sigma^{2n-4}(D_n) \to \mathfrak{H}(D_n)$ which sends $\sigma(\mathbf{p})$ to $x\begin{pmatrix}\mathbf{b}\\ \mathbf{p}\end{pmatrix} - x\begin{pmatrix}\mathbf{c}\\ \mathbf{p}\end{pmatrix}$.*

Proof. Since $\mathbf{r}$ does not contain a sub-path of the form $\mathbf{p}\cdot\mathbf{p}^{\sim}$, part (1) follows from Proposition 4.2. Also, by Proposition 4.2, we have:

$$\sum_{\mathbf{s}\in P^1(\mathscr{G})} \mu(\mathscr{r}(\mathbf{s}))^{1/2}\left(x\begin{pmatrix}a, b, a\\ \mathbf{s}\cdot\mathbf{s}^{\sim}\end{pmatrix} - x\begin{pmatrix}a, c, a\\ \mathbf{s}\cdot\mathbf{s}^{\sim}\end{pmatrix}\right) = 0\,.$$

Part (2) follows from this relation and part (1). $\square$

Lemma 6.2. *The following formulas hold in $\mathfrak{H}(D_n, \varepsilon)$:*

$$x\begin{pmatrix}\mathbf{b}\\ \mathbf{b}\end{pmatrix} + x\begin{pmatrix}\mathbf{b}\\ \mathbf{c}\end{pmatrix} = x\begin{pmatrix}0\\ 0\end{pmatrix}, \quad x\begin{pmatrix}\mathbf{b}\\ \mathbf{b}\end{pmatrix} + x\begin{pmatrix}\mathbf{c}\\ \mathbf{b}\end{pmatrix} = x\begin{pmatrix}0\\ 0\end{pmatrix}.$$

Proof. By Proposition 4.2, we have

$$\mu(b)^{1/2}x\begin{pmatrix}\mathbf{b}\\ \mathbf{b}\end{pmatrix}+\mu(c)^{1/2}x\begin{pmatrix}\mathbf{b}\\ \mathbf{c}\end{pmatrix}+\mu(a-1)^{1/2}x\begin{pmatrix}\mathbf{b}\\ 0,1,\ldots,a,a-1,a,\ldots,0\end{pmatrix}$$
$$=\mu(b)^{1/2}x\begin{pmatrix}0,1,\ldots,a-1,a,a-1,\ldots,0\\ 0,1,\ldots,a-1,a,a-1,\ldots,0\end{pmatrix}. \tag{6.1}$$

Let i be a vertex of $\mathscr{G}$ such that $1 \leqq i \leqq a$. Since

$$\begin{aligned}\sigma(0,1,\ldots,i-1,i,i-1) &= \text{const.}\,\sigma(0,1,\ldots,i-1,i-2,i-1)\\ &\ \vdots\\ &= \text{const.}\,\sigma(0,1,0,1,\ldots,i-1)\\ &= 0,\end{aligned}$$

we have

$$x\begin{pmatrix}0,1,\ldots,i,i+1\\ 0,1,\ldots,i,i-1\end{pmatrix}=0 \tag{6.2}$$

by Lemma 6.1(1). Hence the third term of the left-hand side of (6.1) is 0. Using (6.2) and Proposition 4.2, we obtain

$$\begin{aligned}&x\begin{pmatrix}0,\ldots,i,\ldots,0\\ 0,\ldots,i,\ldots,0\end{pmatrix}\\ &=x\begin{pmatrix}0,\ldots,i,\ldots,0\\ 0,\ldots,i,\ldots,0\end{pmatrix}+\left(\frac{\mu(i-2)}{\mu(i)}\right)^{1/2}x\begin{pmatrix}0,\ldots,i-1,i,i-1,\ldots,0\\ 0,\ldots,i-1,i-2,i-1,\ldots,0\end{pmatrix}\\ &=x\begin{pmatrix}0,\ldots,i-1,\ldots,0\\ 0,\ldots,i-1,\ldots,0\end{pmatrix}.\end{aligned}$$

Hence by induction, the right-hand side of (6.1) is $\mu(b)^{1/2}x\begin{pmatrix}0\\ 0\end{pmatrix}$. □

Lemma 6.3. *Let $n \geqq 4$ be an even (respectively an odd) integer. Then for each $\mathbf{p}\in P_{b,c}^{2n-4}(D_n)$ (respectively $\mathbf{p}\in P_{b,b}^{2n-4}(D_n)$), we have* $x\begin{pmatrix}\mathbf{b}\\ \mathbf{p}\end{pmatrix}=x\begin{pmatrix}\mathbf{c}\\ \mathbf{p}\end{pmatrix}$.

Proof. By Lemma 6.1 (2), it suffices to show that $\Sigma_{b,c}^{2n-4}(D_n)=0$ for $n\in 2\mathbb{Z}$ and that $\Sigma_{b,b}^{2n-4}(D_n)=0$ for $n\in 2\mathbb{Z}+1$. Let $\mathbf{p}=(p_0,\ldots,p_{2n-4})$ be an element of $\Sigma_{b,c}^{2n-4}(D_n)$ ($n\in 2\mathbb{Z}$). We show $\sigma(\mathbf{p})=0$ by reverse induction on $m(\mathbf{p}):=\min\{p_i\}$. Suppose $m(\mathbf{p})=a$ and $\sigma(\mathbf{p})\neq 0$. Since $\sigma(b,a,b)=\sigma(c,a,c)=0$, $\mathbf{p}$ is of the form $(b,a,c,a,b,\ldots,c)$, which contradicts to $\text{length}(\mathbf{p})=2n-4$. Next suppose $m(\mathbf{p})=a-1$. Since $\sigma(p_{i-1},p_i,p_{i+1})=\text{const.}\,\sigma(a,b,a)+\text{const.}\,\sigma(a,c,a)$ for each i such that $p_i=m(\mathbf{p})$, $\sigma(\mathbf{p})$ is a linear combination of $\{\sigma(\mathbf{q})\mid m(\mathbf{q})=a\}$. Thus we also obtain $\sigma(\mathbf{p})=0$. Proof of the case $m(\mathbf{p})<a-1$ is similar. □

Theorem 6.4. *The Jones–Ocneanu's connection on D_n ($n\geqq 4$) is flat with respect to $*=0$ if and only if n is even.*

Proof. Suppose n is an even integer. By Lemma 6.2, we have the following formulas:

$$W\left(\begin{array}{ccc} & \mathbf{b} & \\ \mathbf{b} & & \mathbf{b} \\ & \mathbf{b} & \end{array}\right) + W\left(\begin{array}{ccc} & \mathbf{b} & \\ \mathbf{b} & & \mathbf{b} \\ & \mathbf{c} & \end{array}\right) = \left\langle x\begin{pmatrix}\mathbf{b}\\ \mathbf{b}\end{pmatrix} + x\begin{pmatrix}\mathbf{b}\\ \mathbf{c}\end{pmatrix}, x\begin{pmatrix}\mathbf{b}\\ \mathbf{b}\end{pmatrix}\right\rangle$$

$$= \left\langle x\begin{pmatrix}0\\ 0\end{pmatrix}, x\begin{pmatrix}\mathbf{b}\\ \mathbf{b}\end{pmatrix}\right\rangle$$

$$= 1\,,$$

$$W\left(\begin{array}{ccc} & \mathbf{b} & \\ \mathbf{b} & & \mathbf{b} \\ & \mathbf{c} & \end{array}\right) + W\left(\begin{array}{ccc} & \mathbf{b} & \\ \mathbf{c} & & \mathbf{b} \\ & \mathbf{c} & \end{array}\right) = \left\langle x\begin{pmatrix}\mathbf{b}\\ \mathbf{c}\end{pmatrix}, x\begin{pmatrix}\mathbf{b}\\ \mathbf{b}\end{pmatrix} + x\begin{pmatrix}\mathbf{c}\\ \mathbf{b}\end{pmatrix}\right\rangle$$

$$= 0\,.$$

Since

$$W\left(\begin{array}{ccc} & (0,1,\ldots,b) & \\ \mathbf{b} & & \mathbf{p} \\ & (0,1,\ldots,c) & \end{array}\right) = \left\langle x\begin{pmatrix}0,1,\ldots,b\\ 0,1,\ldots,c\end{pmatrix}, x\begin{pmatrix}\mathbf{b}\\ \mathbf{p}\end{pmatrix}\right\rangle$$

$$= W\left(\begin{array}{ccc} & (0,1,\ldots,b) & \\ \mathbf{c} & & \mathbf{p} \\ & (0,1,\ldots,c) & \end{array}\right)$$

$$(\mathbf{p} \in P_{b,c}^{2n-4}(\mathscr{G}))$$

by Lemma 6.3, we obtain

$$W\left(\begin{array}{ccc} & \mathbf{b} & \\ \mathbf{b} & & \mathbf{b} \\ & \mathbf{c} & \end{array}\right) = \sum_{\mathbf{p} \in \mathrm{Path}^{2n-4}(\mathscr{G})} W\left(\begin{array}{ccc} & (0,1,\ldots,b) & \\ \mathbf{b} & & \mathbf{p} \\ & (0,1,\ldots,c) & \end{array}\right) W\left(\begin{array}{ccc} & (b,\ldots,1,0) & \\ \mathbf{p} & & \mathbf{b} \\ & (c,\ldots,1,0) & \end{array}\right)$$

$$= W\left(\begin{array}{ccc} & \mathbf{b} & \\ \mathbf{c} & & \mathbf{b} \\ & \mathbf{c} & \end{array}\right).$$

Solving these three equations, we obtain $W\left(\begin{array}{ccc} & \mathbf{b} & \\ \mathbf{b} & & \mathbf{b} \\ & \mathbf{b} & \end{array}\right) = 1$. When n is odd, a similar calculation gives $W\left(\begin{array}{ccc} & \mathbf{b} & \\ \mathbf{b} & & \mathbf{b} \\ & \mathbf{b} & \end{array}\right) = 1/2$. □

7. Flatness of E_n-Connections

In [K], Kawahigashi gave some numerical computations of partition functions, which were done by a C program on a Sun. They show that E_n-connections with a wrong choice of $*$ do not satisfy the criterion (2.5) of flatness. Here we compute exact values of these using representations of $\mathfrak{H}(E_n, \varepsilon)$ on $\Sigma(E_n)$. Unfortunately, our method needs tedious calculation.

Proposition 7.1 (cf. [K]). *We have the following formulas. Therefore, the E_6 (respectively E_7, E_8) connection is not flat with respect to $* = 3$ (respectively $* = 0, 6, 4$, $* = 7, 5$).*

(E_6)

$$W\begin{pmatrix} 3 & 2 & 1 \\ 2 & & 2 \\ 1 & 2 & 3 \end{pmatrix} \cdot W\overline{\begin{pmatrix} 3 & 2 & 1 \\ 2 & & 2 \\ 4 & 2 & 3 \end{pmatrix}} = \frac{i}{2},$$

(E_7)

$$W\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & & & & 3 \\ 2 & & & & 2 \\ 3 & & & & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix} \cdot W\overline{\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & & & & 3 \\ 2 & & & & 2 \\ 3 & & & & 1 \\ 5 & 3 & 2 & 1 & 0 \end{pmatrix}} = q^{23}\left(\frac{[4]^5[6]}{[2]^4[3]^8}\right)^{1/2}$$

$$W\begin{pmatrix} 6 & 5 & 3 & 4 \\ 2 & & & 3 \\ 3 & & & 5 \\ 4 & 3 & 5 & 6 \end{pmatrix} \cdot W\overline{\begin{pmatrix} 6 & 5 & 3 & 4 \\ 5 & & & 3 \\ 3 & & & 5 \\ 2 & 3 & 5 & 6 \end{pmatrix}} = q^6(1 + q^4 - q^6 + q^8)\left(\frac{[2]^3[6]}{[3][4]^4}\right)^{1/2}$$

$$W\begin{pmatrix} 4 & 3 & 2 \\ 3 & & 3 \\ 2 & 3 & 4 \end{pmatrix} \cdot W\overline{\begin{pmatrix} 4 & 3 & 2 \\ 3 & & 3 \\ 5 & 3 & 4 \end{pmatrix}} = q^{-15}(1 + q^4 + q^6 - 2q^8)\left(\frac{[3][16]^4}{[2][4][6]^4}\right)^{1/2},$$

(E_8)

$$W\begin{pmatrix} 7 & 6 & 4 & 5 \\ 6 & & & 6 \\ 4 & & & 4 \\ 3 & 4 & 6 & 7 \end{pmatrix} \cdot W\overline{\begin{pmatrix} 7 & 6 & 4 & 5 \\ 6 & & & 6 \\ 4 & & & 4 \\ 5 & 4 & 6 & 7 \end{pmatrix}} = q^{18}\left(\frac{[3]^2[9]^2[14]^2}{[2]^7[4][5][13]^2}\right)^{1/2}$$

$$W\begin{pmatrix} 5 & 4 & 6 \\ 4 & & 4 \\ 3 & 4 & 5 \end{pmatrix} \cdot W\overline{\begin{pmatrix} 5 & 4 & 6 \\ 4 & & 4 \\ 6 & 4 & 5 \end{pmatrix}} = -q^{-5}(q - q^{-1})(1 + q^6 + q^8)\left(\frac{[4][7]^3}{[2]^3[5]^4}\right)^{1/2}.$$

Here, for example, the first multiplicand of the left-hand side of the last formula is $W\left((5,4,3)\ \begin{matrix}(5,4,6)\\ \overline{(3,4,5)}\end{matrix}\ (6,4,5)\right)$, $q = \exp(\pi\sqrt{-1}/12)$ (respectively $\exp(\pi\sqrt{-1}/18)$, $\exp(\pi\sqrt{-1}/30)$) and $[n] = (q^n - q^{-n})/(q - q^{-1})$.

Proof. We give a proof of the third formula. Using (5.1) and the relations of $\Sigma(E_7)$, we obtain the following formulas:

$$x\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}\sigma(4,3,5,6) = -q^3\sigma(3,4,3,5) + \sigma(3,5,6,5)\,,$$

$$x\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}\sigma(3,4,3,5) = -q^3\sigma(5,3,4,3) + \frac{q}{[2]}\sigma(5,6,5,3)\,,$$

$$x\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}\sigma(3,5,6,5) = \frac{[16]}{q[2]}\sigma(5,6,5,3)\,,$$

$$x\begin{pmatrix} 6 & 5 \\ 4 & 3 \end{pmatrix}\sigma(5,3,4,3) = \frac{-q^4}{[2]}\sigma(6,5,3,4)\,,$$

$$x\begin{pmatrix} 6 & 5 \\ 4 & 3 \end{pmatrix}\sigma(5,6,5,3) = \frac{1}{q[2]}\sigma(6,5,3,4)\,,$$

$$x\begin{pmatrix} 6 & 5 \\ 2 & 3 \end{pmatrix}\sigma(5,3,4,3) = q^2\left(\frac{[3]}{[2][4]}\right)^{1/2}\sigma(6,5,3,2)\,,$$

$$x\begin{pmatrix} 6 & 5 \\ 2 & 3 \end{pmatrix}\sigma(5,6,5,3) = q^{-1}\left(\frac{[3]}{[2][4]}\right)^{1/2}\sigma(6,5,3,2)\,.$$

Combining these, we get

$$W\begin{pmatrix} 6 & 5 & 3 & 4 \\ 5 & & & 3 \\ 3 & & & 5 \\ 4 & 3 & 5 & 6 \end{pmatrix}\sigma(6,5,3,4) = x\begin{pmatrix} 6,5,3,4 \\ 4,3,5,6 \end{pmatrix}\sigma(4,3,5,6)$$

$$= q^{-12}\frac{(1+q^4+q^6+2q^8)[16]}{[2][6]}\sigma(6,5,3,4)\,,$$

$$W\begin{pmatrix} 6 & 5 & 3 & 4 \\ 5 & & & 3 \\ 3 & & & 5 \\ 2 & 3 & 5 & 6 \end{pmatrix}\sigma(6,5,3,2) = q^3\left(\frac{[2][3][16]^2}{[4][6]^2}\right)^{1/2}\sigma(6,5,3,2)\,.$$

Since $\sigma(6,5,3,4)$, $\sigma(6,5,3,2) \neq 0$, this proves the third formula of the theorem. $\square$

Theorem 7.2. *The Jones–Ocneanu's connection on E_6 and E_8 are flat with respect to $* = 0$.*

Proof. Similar computations to the above proposition yields the following formulas.

(E_6)

$$W\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & & & 2 \\ 2 & & & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} = \varepsilon^{-9}, \quad W\begin{pmatrix} 2 & 4 & 2 & 3 \\ 4 & & & 2 \\ 2 & & & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} = 0, \quad W\begin{pmatrix} 5 & 4 & 2 & 3 \\ 4 & & & 2 \\ 2 & & & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} = \varepsilon^{-9} q^3 ,$$

(E_8)

$$W\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & & & & & 4 \\ 2 & & & & & 3 \\ 3 & & & & & 2 \\ 4 & & & & & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix} = \varepsilon^{-25}, \quad W\begin{pmatrix} 2 & 3 & 4 & 6 & 4 & 5 \\ 3 & & & & & 4 \\ 4 & & & & & 3 \\ 6 & & & & & 2 \\ 4 & & & & & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix} = 0 ,$$

$$W\begin{pmatrix} 4 & 5 & 4 & 6 & 4 & 5 \\ 6 & & & & & 4 \\ 7 & & & & & 3 \\ 6 & & & & & 2 \\ 4 & & & & & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix} = -\frac{\varepsilon^{-25} q^3 [7]}{[2]^3 [5]} , \quad W\begin{pmatrix} 4 & 6 & 7 & 6 & 4 & 5 \\ 6 & & & & & 4 \\ 7 & & & & & 3 \\ 6 & & & & & 2 \\ 4 & & & & & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix} = \frac{\varepsilon^{-25} q^{19} [7]}{[2]^3 [4]} ,$$

$$W\begin{pmatrix} 4 & 5 & 4 & 6 & 4 & 5 \\ 5 & & & & & 4 \\ 4 & & & & & 3 \\ 6 & & & & & 2 \\ 4 & & & & & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix} = -\frac{\varepsilon^{-25} q^{17} [9]}{[2]^2 [3] [5]} . \tag{7.1}$$

Using these, the self-duality of W and Lemma 6.1(1), we obtain the exact value of $W\begin{pmatrix} & \mathbf{p} & \\ \mathbf{r} & & \mathbf{b}' \\ & \mathbf{b}' & \end{pmatrix}$ for each paths $\mathbf{p}$ and $\mathbf{r}$, where $\mathbf{b}' = (3, 2, 1, 0)$ for E_6 and $\mathbf{b}' = (5, 4, 3, 2, 1, 0)$ for E_8. For example, if $\mathbf{p}, \mathbf{r} = (4, 5, 4, 3, 4, 5)$, this value is equal to "the value of (7.1)" $\times \mu(6)/\mu(3)$, since $x\begin{pmatrix} 4, 5, 4, 3, 4, 5 \\ \mathbf{b}' \end{pmatrix} = (-1)(\mu(6)/\mu(3))^{1/2} x\begin{pmatrix} 4, 5, 4, 6, 4, 5 \\ \mathbf{b}' \end{pmatrix}$. Now a straightforward computation using Lemma 2.1 and the self-duality yields $W\begin{pmatrix} & \mathbf{b} & \\ \mathbf{b} & & \mathbf{b} \\ & \mathbf{b} & \end{pmatrix} = 1$. □

Note. In a forthcoming paper, we will show the next two fundamental results. Let $\mathscr{G}$ and ε be as in Sect. 4.

(1) The face algebra $\mathfrak{H}(\mathscr{G}, \varepsilon)$ has an antipode, that is, there exists a linear operator S on $\mathfrak{H}(\mathscr{G}, \varepsilon)$ such that

$$\sum_{(a)} S(a_{(1)})a_{(2)} = \mathscr{E}(a), \quad \sum_{(a)} a_{(1)}S(a_{(2)}) = \mathscr{E}'(a) ,$$
$$\sum_{(a)} \mathscr{E}(a_{(1)})S(a_{(2)}) = \sum_{(a)} S(a_{(1)})\mathscr{E}'(a_{(2)}) = S(a) ,$$

where $\Delta(a) = \Sigma_{(a)} a_{(1)} \otimes a_{(2)}$ and $\mathscr{E}$ and $\mathscr{E}'$ denote linear operators defined by

$$\mathscr{E}(a) = \sum_{i,j,k \in \mathscr{V}} \varepsilon\left(ax\begin{pmatrix} i \\ k \end{pmatrix}\right) x\begin{pmatrix} j \\ k \end{pmatrix}, \quad \mathscr{E}'(a) = \sum_{i,j,k \in \mathscr{V}} \varepsilon\left(x\begin{pmatrix} k \\ i \end{pmatrix} a\right) x\begin{pmatrix} k \\ j \end{pmatrix} .$$

Explicitly, S is given by

$$S\left(x\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}\right) = \left(\frac{\mu(\mathfrak{s}(\mathbf{q}))\mu(\mathfrak{r}(\mathbf{p}))}{\mu(\mathfrak{s}(\mathbf{p}))\mu(\mathfrak{r}(\mathbf{q}))}\right)^{1/2} x\begin{pmatrix} \mathbf{q}^{\sim} \\ \mathbf{p}^{\sim} \end{pmatrix} . \tag{7.2}$$

(1) Let $\mathfrak{F}(\mathscr{G})$ be a face algebra generated by elements $\left\{ x\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix};\, \mathbf{p}, \mathbf{q} \in P^r(\mathscr{G}), r \geqq 0\right\}$ with the defining relation (4.1). Then $\mathfrak{F}(\mathscr{G})$ is a co-quasitriangular face algebra, that is, there exist elements $\mathscr{R}^+$ and $\mathscr{R}^-$ of $(\mathfrak{F}(\mathscr{G}) \otimes \mathfrak{F}(\mathscr{G}))^*$ such that

$$\mathscr{R}^+\mathscr{R}^- = (m^{\mathrm{op}})^*(1), \quad \mathscr{R}^-\mathscr{R}^+ = m^*(1) ,$$
$$\mathscr{R}^+ m^*(1) = \mathscr{R}^+, \quad m^*(1)\mathscr{R}^- = \mathscr{R}^-$$
$$\mathscr{R}^+ m^*(f) = (m^{\mathrm{op}})^*(f)\mathscr{R}^+ \quad (f \in \mathfrak{F}(\mathscr{G})^*)$$
$$(m \otimes \mathrm{id})^*(\mathscr{R}^+) = \overset{+}{\mathscr{R}}_{13}\overset{+}{\mathscr{R}}_{23}, \quad (\mathrm{id} \otimes m)^*(\mathscr{R}^+) = \overset{+}{\mathscr{R}}_{13}\overset{+}{\mathscr{R}}_{12} ,$$

where m denotes the product of $\mathfrak{F}(\mathscr{G})$ and $\langle \overset{+}{\mathscr{R}}_{ij}, a_1 \otimes a_2 \otimes a_3 \rangle = \langle \mathscr{R}^+, a_i \otimes a_j \rangle \varepsilon(a_k)$ for $\{i, j, k\} = \{1, 2, 3\}$. Explicitly, $\mathscr{R}^{\pm}$ is given by

$$\left\langle \mathscr{R}^+, x\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \otimes x\begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix}\right\rangle = W\begin{pmatrix} & \mathbf{q} & \\ \mathbf{r} & & \mathbf{s} \\ & \mathbf{p} & \end{pmatrix}, \quad \left\langle \mathscr{R}^-, x\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \otimes x\begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix}\right\rangle = W_{\varepsilon^{-1}}\begin{pmatrix} & \mathbf{p} & \\ \mathbf{s} & & \mathbf{r} \\ & \mathbf{q} & \end{pmatrix} .$$

The face algebra $\mathfrak{F}(\mathscr{G})$ also has an antipode which is given by (7.2).

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Note added in proof. After this work has been submitted, Prof. Kawahigashi pointed out to the author that $W_{\mathscr{G},\varepsilon}$ is not a biunitary connection if $\beta > 2$ (cf. §4). However, Proposition 4.2 is still valid, because both (2.3) and (2.4) hold if $\bar{W}$ is replaced by $W_{1/\varepsilon}$. The author is grateful to Prof. Kawahigashi for this comment.

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