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The Integrated Density of States for the Difference Laplacian on the Modified Koch Graph

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Abstract. We consider the integrated density of states $N(\lambda)$ of the difference Laplacian $-\Delta$ on the modified Koch graph. We show that $N(\lambda)$ increases only with jumps and a set of jump points of $N(\lambda)$ is the set of eigenvalues of $-\Delta$ with the infinite multiplicity. We establish also that

$$0 < C_1 \le \varliminf_{\lambda \to 0} \frac{N(\lambda)}{\lambda^{d_s/2}} < \varlimsup_{\lambda \to 0} \frac{N(\lambda)}{\lambda^{d_s/2}} \le C_2 < \infty ,$$

where $d_s = 2\log 5/\log(40/3)$ is the spectral dimension of MKG.

1. Introduction

In this paper, we consider the integrated density of states (IDS) $N(\lambda)$, $\lambda \in \mathbb{R}$ of the difference Laplacian $-\Delta$ on the modified Koch graph (MKG). The function N is defined as the normalized limit of the number of eigenvalues less than λ as the size of the finite graph being expanded to infinity. It turns out that N increases only with jumps and the set of jumps points of N is the set of eigenvalues with the infinite multiplicity $D_1 \cup D_2 \cup D_3$, where the set $\mathscr{F} = \bar{D}_2$ is the Julia set of the iteration of the rational function

$$R(z) = 9z(z-1)(z-4/3)(z-5/3)/(z-3/2) .$$

Moreover, the set $\mathscr{F}$ is the set of accumulation points for points from the set $D_1 \cup D_3$.

We shall see that the behavior of the function $N(\lambda)$ near zero is $\lambda^{d_s/2}$, $d_s = 2\log 5/\log(40/3)$, or more exactly, there exist two positive constants, C_1, C_2 such that

$$0 < C_1 \le \varliminf_{\lambda \to 0} \frac{N(\lambda)}{\lambda^{d_s/2}} < \varlimsup \frac{N(\lambda)}{\lambda^{d_s/2}} \le C_2 < \infty \tag{1.0}$$

i.e., the ratio $N(\lambda)/\lambda^{d_s/2}$ is oscillating and non-convergent as $\lambda \to 0$.

The number d_s denotes the so-called spectral dimension of the MKG. The power that is singled out is, unlike in the $\mathbb{R}^n$ case, not the Hausdorff dimension of the MKG $d_f = \log 5/\log 3$, but its spectral dimension d_s.

We will note that for the first time Rammal [R] discovered the high singularity of the IDS of the difference Laplacian on the Sierpinski gasket. Recently, Fukushima and Shima [FS] proved this fact for the differential Laplacian on the infinite Sierpinski gasket. Finally Fukushima [F] considered the asymptotic behavior of the IDS for the infinite nested fractals.

2. Preliminaries

Here we collect those preliminary notions and relations from [M] which we shall use in this paper.

Beginning with the line segment of length 1 in Fig. 1, we first replace it by five line segments of length 1/3 (Fig. 2), and then we replace each one of these by five segments of length 1/9 (Fig. 3). The limit set is the modified Koch curve.

Fig. 1

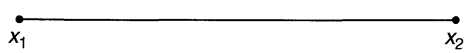


Fig. 2

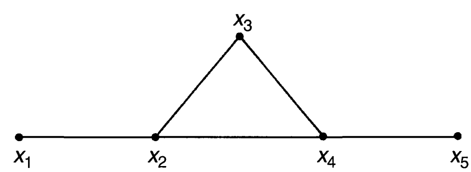
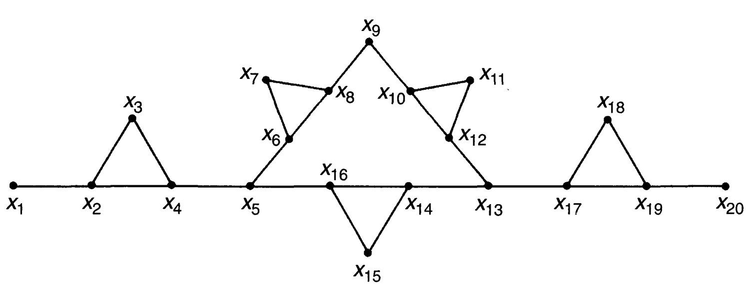


Fig. 3



We define the modified Koch graph somewhat more formally.

Let $G = (\Lambda(G), E(G))$ be a connected infinite locally finite graph without loops with the vertex set $\Lambda(G)$ and the edge set $E(G)$. We use the following graph distance:

$$d(x, y) = \min\{k : \exists\{x_i\}_{i=0}^{i=k} : x_0 = x, x_k = y, 0 < i \leq k, (x_{i-1}, x_i) \in E(G)\},$$
$$d(x, x) = 0.$$

Let d_x denote the degree of the vertex $x \in \Lambda(G)$, i.e., be the largest number of the edges that meet at the point x. If D is a finite subgraph of G, then the degree of a vertex x in D will be denoted by $d_x(D)$. By ∂D we denote the boundary of the subgraph D, i.e.,

$$\partial D = \{x \in \Lambda(D), d_x(D) < d_x\},$$

and $\operatorname{int} D$ is the set of internal points of D, i.e.,

$$\operatorname{int} D = \{x \in \Lambda(D), d_x(D) = d_x\}.$$

We define the MKG by induction.

Definition 1. Let $G_1 = (\Lambda(G_1), E(G_1))$ be a graph having the set of vertices $\Lambda(G_1) = \{x_i\}_{i=1}^{i=5}$ and the set of edges

$$E(G_1) = \{(x_1, x_2), (x_2, x_3), (x_2, x_4), (x_3, x_4), (x_4, x_5)\}.$$

We introduce $\partial G_1 = \{x \in \Lambda(G_1), d_x(G_1) = 1\}$ and $\operatorname{int} G_1 = \{x \in \Lambda(G_1) > 1\}$. Now we define the graph G_2 as $G_2 = \bigcup_{i=1}^{5} G_1^i$, where G_1^i and G_1 are isomorphic graphs for any $i = 1, 2, 3, 4, 5$, $G_1^4 = G_1$, which satisfy the following conditions (conditions A):
1. $\operatorname{int} G_1^i \cap \operatorname{int} G_1^j = \emptyset$ for $i \neq j$,
2. $E(G_1^i) \cap E(G_1^j) = \emptyset$ for $i \neq j$,
3. if $\partial G_1^i = \langle x_1^i, y_1^i \rangle$, then $y_1^1 = x_1^2 = x_1^4$, $y_1^2 = x_1^3$, $y_1^3 = x_1^5 = y_1^4$ and $d_{x_1^1}(G_2) = d_{y_1^5}(G_2) = 1$.

Let $\partial G_2 = \langle x \in \Lambda(G_2), d_x(G_2) = 1 \rangle$.

Now we define the subgraph $G_{n+1} = \bigcup_{i=1}^{5} G_n^i$, where the G_n^i satisfy conditions A (with G_1^i replaced by G_n^i); see Fig. 4. Let $\partial G_{n+1} = (x \in \Lambda(G_{n+1}), d_x(G_{n+1}) = 1)$. Then the MKG is defined by the formula $G = \bigcup_{n=1}^{\infty} G_n$.

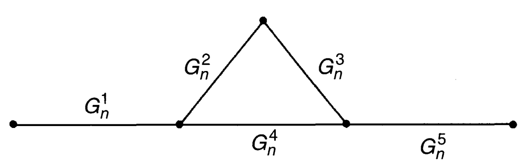


Fig. 4

Let us denote by $B_{x,N}$ or B_N the ball in G centered at x with radius N, i.e.

$$B_{x,N} = \{y \in \Lambda(G), d(x, y) \leq N\},$$

and by $b_{x,N} = |B_{x,N}|$ its cardinality.

The fractal (Hausdorff) dimension of a graph can be defined as the following:

Definition 2.

$$d(x) = \lim_{N \to \infty} \sup \frac{\log |b_{x,N}|}{\log N}.$$

It is easy to see that $d(x)$ is independent of x for MKG and we can use the common value d_f of $d(x)$'s as the fractal dimension. Moreover, $d_f = \log 5/\log 3$. We will note here that the Hausdorff dimension of the modified Koch curve is also $\log 5/\log 3$ [H].

We define the function m on G as $m(x) = d_x$ for every $x \in \Lambda$. Let

$$l_2(G, m) = \left\langle f = f(x), x \in \Lambda(G), \sum_{x \in \Lambda(G)} m(x)\,|f(x)|^2 < \infty \right\rangle .$$

Then the finite difference Laplacian Δ on the graph G is defined by the formula

$$(\Delta u)(x) = d_x^{-1} \sum_{t, d(x,t)=1} u(t) - u(x) .$$

It is easy to see that the operator Δ is a symmetric operator with respect to the product

$$(f, g) = \sum_{x \in \Lambda(G)} m(x) f(x) g(x) .$$

For a set $A \subset \Lambda$, $|A|$ will denote the number of points in A. We denote by $f|_A$ the restriction of a function f to the set A. It is easy to see that $|\Lambda(G_n)| = (3 \cdot 5^n + 5)/4$. Let $l_2(G_n) = \{g = g(x), x \in \Lambda(G_n), g|_{\partial G_n} = 0\}$ with the product

$$(u, v) = \sum_{x \in \Lambda(G_n)} d_x(G_n) u(x) v(x) , \qquad u, v \in l_2(G_n) . \tag{1.1}$$

Then we obtain that $\dim l_2(G_n) = 3(5^n - 1)/4$. We denote by Δ_n the operator Δ restricted to $l_2(G_n)$ with zero boundary conditions (the Dirichlet boundary conditions). In the sequel we denote $\Lambda(G_n)$ by Λ_n for $n \geq 1$.

Let $-\Delta_1$ be an operator on $l_2(G_1)$. We denote the function $f = f(x)$ on G_1 by $f = (f_{x_2}, f_{x_3}, f_{x_4})$, where $f_{x_i} = f(x_i)$, $i = 2, 3, 4$. By a straightforward calculation we have

Lemma 2.1. *The eigenvalue λ_i, $i = 1, 2, 3$, of $-\Delta_1$ and the corresponding eigenfunction φ_i, $i = 1, 2, 3$, are as follows:*

$$\lambda_1 = (5 - \sqrt{13})/6) , \qquad \lambda_2 = 4/3 , \qquad \lambda_3 = (5 + \sqrt{13})/6)$$

and

$$\varphi_1 = (2, -(1 - \sqrt{13}), 2) , \qquad \varphi_2 = (1, 0, -1) , \qquad \varphi_3 = (2, -(1 + \sqrt{13}), 2) .$$

By $\tau(-\Delta)$ we denote the spectrum of the operator $-\Delta$.

There is the following statement [M]:

Proposition 2.2. *The number $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of the operator $-\Delta_n$ with multiplicity $r_n(\lambda_i)$ and*

$$r_n(\lambda_i) = (5^{n-1} + 3)/4 \quad \text{for} \quad n \geq 2, i = 1, 2, 3 .$$

We introduce the rational function

$$R(x) = 9x(x - 1)(x - 4/3)(x - 5/3)/(x - 3/2)$$

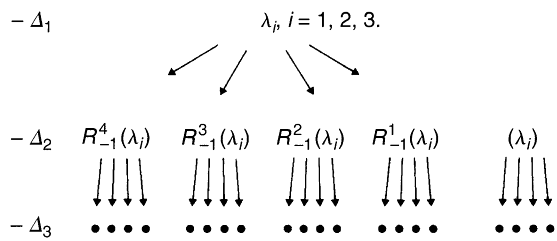
and R_{-1} is inverse to R. The main result which makes it possible to calculate all eigenvalues of the operator $-\Delta_{n+1}$ is the following:

Theorem 2.3 [M]. (i) *If λ_0, $\lambda_0 \neq \lambda_i$, $i = 1, 2, 3$, is an eigenvalue of the operator $-\Delta_{n+1}$ corresponding to the eigenfunction $f = f(x)$, $x \in \Lambda_{n+1}$, then the function $u = f(x)|_{\Lambda_n}$ is a solution of the problem*

$$-(\Delta_n u)(x) = R(\lambda_0) u(x) , \qquad u|_{\partial G_n} = 0 , \qquad x \in \Lambda_n .$$

(ii) *Let* $R(\lambda)$, $\lambda \neq \lambda_1, \lambda_2, \lambda_3$ *be an eigenvalue of the operator* $-\Delta_n$ *corresponding to the eigenfunction* $u(x)$, $x \in \Lambda_n$. *Then there exists a unique extension* $f = f(x)$, $x \in \Lambda_{n+1}$ *of* u *such that* f *is an eigenfunction of the operator* $-\Delta_{n+1}$ *with the eigenvalue* λ.
(iii) *Let* $\lambda \in \tau(-\Delta_n)$ *and* $\beta \in \{R_{-1}(\lambda)\}$. *Then the multiplicity of* λ *equals that of* β.

From Proposition 2.2 and Theorem 2.3 we get the following diagram of the eigenvalues of $-\Delta_n$.



Let us denote

$$R_0(z) = z\,, \qquad R_1(z) = R(z)\,, \qquad R_{n+1} = R_1(R_n(z))\,, \qquad n = 0, 1, 2, 3 \dots .$$

Definition 3 [B]. If $w = R_n(z)$, then we say that w is a successor of z and z is a predecessor of w of order n.

We denote by $D_i = \{R_{-n}(\lambda_i)\}$, $n \geq 0$ the set of all predecessors of λ_i, $i = 1, 2, 3$. It is easy to see that $D_i \subset \mathbb{R}$ for all i.

Let ζ be the maximal fixpoint of the function R, i.e. $\zeta = \max\{\theta : R(\theta) = \theta\}$. Then $1.75 \leq \zeta \leq 1.76$.

Theorem 2.4 [M]. *The following statements are true:*
(i) *Each point of* $D_1 \cup D_2 \cup D_3$ *is an eigenvalue of the operator* $-\Delta$ *with infinite multiplicity.*
(ii) *The spectrum* $\tau(-\Delta)$ *of the operator* $-\Delta$ *on* $l_2(G, m)$ *is*

$$\tau(-\Delta) = \mathscr{F} \cup D_1 \cup D_3 \subset [0, \zeta]\,, \qquad \mathscr{F} = \bar{D}_2\,.$$

(iii) *The spectrum* $\tau(-\Delta)$ *is a set of Lebesgue measure zero.*
(iv) *The Julia set* $\mathscr{F}$ *of the rational function* R *is a set of accumulation points of the set* $D_1 \cup D_3$.

We shall divide D_i into $D_i = \bigcup_{k=1}^{\infty} S_k(\lambda_i)$ such that $S_l(\lambda_i) \cap S_j(\lambda_i) = \emptyset$ if only $l \neq j$ and we define

$$S_k(\lambda_i) = \{\lambda \mid \lambda \in \tau(-\Delta_k) \backslash \tau(-\Delta_{k-1}), k \geq 2\}\,,$$
$$S_1(\lambda_i) = \{\lambda_i\}\,, \qquad i = 1, 2, 3\,.$$

3. The Integrated Density of States

We introduce the following function:

$$N_l(\lambda) = \#\{\lambda_k < \lambda \mid \lambda_k \text{ are eigenvalues of the } -\Delta_l\} \cdot 5^{-l} = n_l(\lambda) 5^{-l}.$$

Lemma 3.1. *There exists*

$$\tfrac{4}{3} \lim_{l \to \infty} N_l(\lambda) = N(\lambda) \tag{3.1}$$

at each continuity point λ of $N(\lambda)$. $N(\lambda)$ is called the integrated density of states.

Proof. We shall prove that the sequence $\{N_l(\lambda)\}$ are not decreasing for $l \geq 1$, i.e., $N_l(\lambda) \leq N_{l+1}(\lambda)$ for any $\lambda \in \mathbb{R}$. Let

$$\mathscr{F}_{l+1} = \{f \mid f \in l_2(G_{l+1}), f|_{\partial G_l^i} = 0, i = 1, 2 \ldots 5\}$$

and $-\Delta_l^i$ be the restriction of the $-\Delta$ on $l_2(G_l^i)$. Moreover, $-\Delta_l^4 = -\Delta_l$. We denote by $\bigoplus_{i=1}^{5} -\Delta_l^i = \Delta_{l+1}^0$ the direct sum of the operators $-\Delta_l^i$. We need the following functions:

$$n_{l+1}^0 = \#\{\lambda_k^0 < \lambda \mid \lambda_k^0 \text{ are eigenvalues of } -\Delta_{l+1}^0\}.$$

Because $\Delta_{l+1} = \Delta_{l+1}^0$ on the space $\mathscr{F}_{l+1}$ we have the inequality

$$N_{l+1} = \frac{n_{l+1}(\lambda)}{5^{l+1}} \geq \frac{n_{l+1}^0}{5^{l+1}} = \frac{5n_l}{5^{l+1}} = N_l(\lambda). \tag{3.2}$$

Thus the lemma is proved. □

Remark. We note that $\operatorname{codim} \mathscr{F}_{l+1} = 3$ in the space $l_2(G_{l+1})$ and $\mathscr{F}_{l+1}$ is the invariant space under Δ_{l+1}^0, so we have

$$n_{l+1}(\lambda) \leq n_{l+1}^0 + 3. \tag{3.3}$$

By (3.3) we get

$$N_{l+1}(\lambda) \leq N_l(\lambda) + \frac{3}{5^{l+1}},$$

and consequently

$$N_l(\lambda) \geq -\frac{3}{5^{l+1}} + N_{l+1}(\lambda) \geq \frac{3}{4} N(\lambda) - \frac{3}{5^l} \cdot \frac{1}{4}$$

that gives us the following inequality

$$N(\lambda) \leq \frac{4}{3} N_l(\lambda) + \frac{1}{5^l}. \tag{3.4}$$

We note also that

$$\tfrac{4}{3} N_l(\lambda) \leq N(\lambda) \quad \text{for any } l \geq 1 \text{ and } \lambda \in \mathbb{R}. \tag{3.5}$$

The number 4/3 in (3.1) is necessary so that $0 \leq N(\lambda) \leq 1$.

Proposition 3.2. *The following statements are true:*

(i) *The function $N(\lambda)$ is the nondecreasing function of λ and* $0 \leq N(\lambda) \leq 1$, $\lambda \in \mathbb{R}$, $N(0) = 0$.

(ii) *The function N is the continuous function for any $\lambda \in \mathbb{R} \setminus \bigcup_{i=1}^{3} D_i$. If $\lambda \in S_k(\lambda_i)$, then we have*

$$N(\lambda + 0) - N(\lambda - 0) = 5^{-k}/3\,, \tag{3.6}$$

where

$$N(\lambda \pm 0) = \lim_{t \to \lambda \pm 0} N(t)$$

and

$$N(\lambda + 0) = N(\lambda_0) = \lim_{l \to \infty} N_l(\lambda_0)\,.$$

(iii) $\operatorname{supp} N = \tau(-\Delta)$.

Proof. The statement (i) follows from the definition of the function N and Theorem 2.4 (ii).

(ii) At first, let $\lambda_0 \in S_k(\lambda_1) \cup S_k(\lambda_3) \subset D_1 \cup D_3$. There exists an interval (c, d) such that $(c, d) \cap \tau(-\Delta) = \lambda_0$ and $(c, d) \cap \tau(-\Delta_n) = \lambda_0$ for any $n \geq k$. If we take arbitrary numbers $\lambda_1, \lambda_2 \in (c, d)$ such that $\lambda_1 < \lambda_0 < \lambda_2$, then we obtain from Proposition 2.2,

$$n_l(\lambda_2) - n_l(\lambda_1) = \begin{cases} (5^{l-k} + 3)/4 & \text{if } l > k\,. \\ 1 & \text{if } l = k\,. \end{cases}$$

Thus, we get

$$N(\lambda_2) - N(\lambda_1) = \lim_{l \to \infty} \frac{(5^{l-k} + 3)/4}{\frac{3}{4} \cdot 5^l} = \frac{5^{-k}}{3} \tag{3.7}$$

and formula (3.6) is proved for $\lambda_0 \subset D_1 \cup D_3$.

Let $\lambda_0 \in S_k(\lambda_2)$ and λ_n^-, λ_n^+ are nearest points to λ_0 from $\tau(-\Delta_n)$ such that $\lambda_n^- < \lambda_0 < \lambda_n^+$, $n \geq k$. Because $\lambda_0 \in \mathscr{F}$, we obtain that $\lambda_n^\pm \to \lambda_0$ as $n \to \infty$. We note that

$$C_l^- = \tfrac{4}{3} N_l(\lambda_l^-) = \tfrac{4}{3} N_l(\lambda_{l+1}^-) \leq \tfrac{4}{3} N_{l+1}(\lambda_{l+1}^-) = C_{l+1}^-\,,$$

$$C_l^+ = \tfrac{4}{3} N_l(\lambda_0) = \tfrac{4}{3} N_l(\lambda_l^+ - 0) = \tfrac{4}{3} N_l(\lambda_{l+1}^+ - 0) \leq \tfrac{4}{3} N_{l+1}(\lambda_{l+1}^+ - 0) = C_{l+1}^+\,,$$

and let

$$C^\pm = \lim_{l \to \infty} C_l^\pm\,.$$

We shall prove that $C^\pm = N(\lambda_0 \pm 0)$. Because N is the monotony function, there exists $\lim\limits_{\lambda \to \lambda_0 \pm 0} N(\lambda) = N(\lambda_0 \pm 0)$ and by using the following inequality:

$$\begin{aligned} |N(\lambda_0 - 0) - \tfrac{4}{3} N_l(\lambda_l^-)| &\leq |N(\lambda_0 - 0) - N(\lambda_l^-)| + |N(\lambda_l^-) - N_l(\lambda_l^-)| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,, \qquad l \gg 1 \end{aligned}$$

we obtain $C^- = N(\lambda_0 - 0)$. Analogously to (3.7), we have

$$\lim_{n \to \infty} C_n^+ - C_n^- = 5^{-k}/3\,.$$

It is easy to see that the sum of all jumps of N equals

$$3\left(\frac{5^{-1}}{3} + 4 \cdot \frac{5^{-2}}{3} + \ldots 4^n \,\frac{5^{-n-1}}{3} + \ldots\right) = 1\,. \tag{3.8}$$

If $C^+ < N(\lambda_0 + 0)$ then this statement contradicts (3.8).

Finally, we shall prove the continuity of the function N in all points $\lambda \in \mathbb{R}\setminus \bigcup_{i=1}^{3} D_i$. Let λ_0 be such a point. There exists the sequence $\{\lambda_i\}$, $\lambda_i \in D_2$ such that $\lambda_i \to \lambda_0$ as $i \to \infty$. As above, we note $N(\lambda_0) = N(\lambda_0 + 0)$ and the equality $N(\lambda_0 + 0) = N(\lambda_0 - 0)$ follows from the sum (3.8).
(iii) Let (a, b) be an arbitrary interval such that $(a, b) \subset \mathbb{R}\setminus\tau(-\Delta)$. If we can find $t_1, t_2 \in (a, b)$ such that $N(t_1) < N(t_2)$ then there exists $l_0 \in \mathbb{N}$ that we have $N_{l_0}(t_1) < N_{l_0}(t_2)$. From this fact we obtain that there is a number $\lambda_0 \in \tau(-\Delta_{l_0}) \cap [t_1, t_2]$ and consequently we have $\lambda_0 \in \tau(-\Delta)$ that contradicts our supposition. That is why we have

$$\operatorname{supp} N \subset \tau(-\Delta)\,.$$

Now, we shall prove that $\tau(-\Delta) \subset \operatorname{supp} N$. Let $\lambda_0 \in \mathscr{F}$. There exists a sequence $\{\lambda_i\}$, $\lambda_i \in D_2$ such that $\lambda_i \to \lambda_0$ as $i \to \infty$. If we take an arbitrary $\varepsilon > 0$, we have from (ii) that $N(\lambda_0 + \varepsilon) - N(\lambda_0 - \varepsilon) > 0$. The proposition is proved. □

4. Schröder's Equation and König's Function

Let $R^i_{-1}(x)$, $i = 1, 2, 3, 4$ be the roots of the equation $R(t) = x$, $x \in [0, \zeta]$ such that

$$R^1_{-1}(x) < R^2_{-1}(x) < R^3_{-1}(x) < R^4_{-1}(x)\,.$$

We denote by $\Psi = \Psi(x)$ the inverse function to $R{:}\,[0, R^1_{-1}(\zeta)] \to [0, \zeta]$ and consequently $\Psi{:}\,[0, \zeta] \to [0, R^1_{-1}(\zeta)]$.

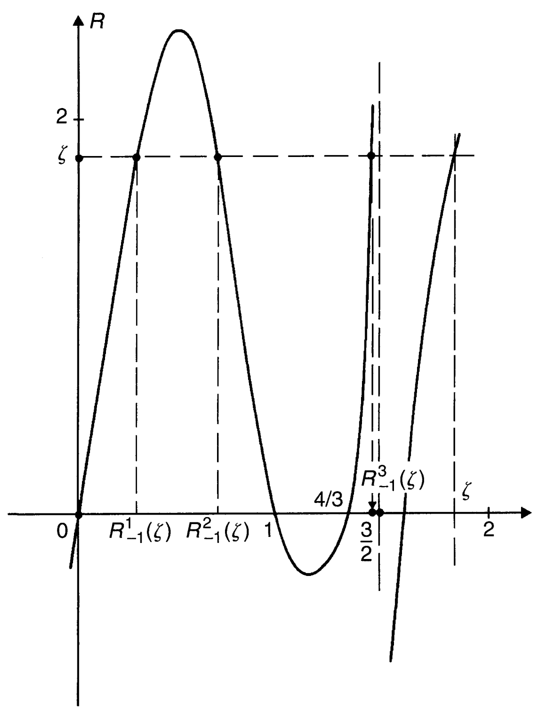


Fig. 5

The iterates $\Psi^{(n)}$ of the function Ψ are defined by

$$\Psi^{(0)}(x) = x\,, \qquad \Psi^{(n+1)}(x) = \Psi(\Psi^{(n)}(x))\,, \qquad x \in [0,\zeta]\,.$$

We shall denote by $\theta_n(x) = \Psi^{(n)}(x)$ and $\tilde{\theta}_n = (R'(0))^n\,\theta_n$, $R'(0) = 40/3$.

Lemma 4.1. *There exists*

$$\lim_{n\to\infty} \tilde{\theta}_n(x) = \varphi(x) \tag{4.1}$$

for all $x \in [0,\zeta]$.

Proof. We note that $\theta_{n+1}(x) = \Psi(\theta_n(x))$, $x \in [0,\zeta]$ and then $R(\theta_{n+1}) = \theta_n$. Thus, we have $\tilde{\theta}_n = (R'(0))^n\,R(\theta_{n+1}) = \tilde{\theta}_{n+1} d_n(\theta_{n+1})$, where

$$d_n = \frac{(1-\theta_{n+1})\,(1-\frac{3}{4}\,\theta_{n+1})\,(1-\frac{3}{5}\,\theta_{n+1})}{(1-\frac{2}{3}\,\theta_{n+1})}\,.$$

It is clear that $d_n < 1$ for $\theta_{n+1} > 0$ because $(1-\theta_{n+1})\,(1-\frac{2}{3}\,\theta_{n+1})^{-1} < 1$ and $d_n = 1$ if $x = 0$. That is why $\tilde{\theta}_n(x) < \tilde{\theta}_{n+1}(x)$ for any $x \in (0,\zeta]$ and $\tilde{\theta}_n(0) = \tilde{\theta}_{n+1}(0) = 0$.

The statement (4.1) will be proved if we show that there exists a number C such that $\tilde{\theta}_n(x) \le C$ for all $x \in [0,\zeta]$ and $n \ge 1$. We note

$$\frac{\tilde{\theta}_n}{\tilde{\theta}_{n+1}} = \frac{(R'(0))^n\,\theta_n}{(R'(0))^{n+1}\,\theta_{n+1}} = d_n\,,$$

and consequently

$$\frac{\theta_{n+1}}{\theta_n} = (d_n \cdot R'(0))^{-1} \le C_1 (R'(0))^{-1} = \left(\tfrac{40}{3}\right)^{-1} \cdot C_1\,.$$

We can write $\tilde{\theta}_n$ as

$$\tilde{\theta}_n = R'(0)\,\frac{\theta_n}{\theta_{n-1}} \cdot \frac{\theta_{n-1}}{\theta_{n-2}}\,R'(0)\ldots R'(0)\,\frac{\theta_2}{\theta_1} \cdot \theta_1 R'(0)\,, \tag{4.2}$$

then

$$\prod_{n=1}^{\infty} R'(0)\,\frac{\theta_n}{\theta_{n-1}} = \prod_{n=1}^{\infty} d_n^{-1} \le C < \infty \tag{4.3}$$

because $d_n = 1 + \alpha(\theta_n)$, $\alpha(\theta_n) \le C_2\theta_n \le C_3\left(\frac{3}{40}\right)^n$. The lemma is proved. □

Proposition 4.2. *The function* $\varphi(x)$ *is the smooth strictly increasing function on* $[0,\zeta]$ *and* φ *is the exactly one König's solution of Schröder's equation* (4.4), i.e.

$$\varphi(\Psi(x)) = s\varphi(x)\,, \qquad s = \left(\tfrac{40}{3}\right)^{-1}, \qquad x \in [0,\zeta] \tag{4.4}$$

and

$$\varphi(0) = 0\,, \qquad \varphi'(0) = 1\,.$$

Proof. The continuity of the function φ follows from (4.2), (4.3). By (4.1) we obtain also

$$\lim_{n\to\infty} \left(\tfrac{40}{3}\right)^{n+1} \theta_n(\Psi(x)) = \tfrac{40}{3}\,\varphi(\Psi(x)) = \varphi(x)\,, \qquad x \in [0,\zeta]\,.$$

The equality $\varphi(0) = 0$ follows from the definition of the function φ. We note also that $-x\Psi(x) < 0$ and $(\Psi(x) - x)(-x) > 0$, $x \in (0,\zeta)$. The proof of Proposition 4.2 follows right now from [K] (Theorem 6.1, p. 137).

5. Bounds of the IDS

Let $\lambda_n^i = \Psi^{(n-1)}(\lambda_i)$, $i = 1,2,3$. It is clear that $\lambda_n^1 = \inf \tau(-\Delta_n)$. Due to Lemma 4.1 and Proposition 4.2, we have

Proposition 5.1.

$$\lim_{n\to\infty} \lambda_n^i (R'(0))^{n-1} = \varphi(\lambda_i) \tag{5.1}$$

and

$$\varphi(\lambda_1) < \varphi(\lambda_2) < \varphi(\lambda_3)\,.$$

Let λ_{n+1}^4 be the 4th eigenvalue of the operator $-\Delta_{n+1}$, then $\lambda_n^1 = \lambda_{n+1}^4$.

Lemma 5.2. *Let $\lambda \in [\lambda_{n+1}^1, \lambda_n^1]$. Then the following statement is true:*

$$\frac{4}{3\cdot 5^{n+1}} \le N(\lambda) \le \frac{3}{5^n}\,. \tag{5.2}$$

Proof. We get from (3.4),

$$N(\lambda) \le \frac{4}{3}\, N_n(\lambda) + \frac{1}{5^n} \le \frac{3}{5^n}\,.$$

The lower bound follows from (3.5), i.e. $\frac{4}{3}\cdot\frac{1}{5^{n+1}} \le \frac{4}{3}\, N_{n+1}(\lambda) \le N(\lambda)$.

The lemma is proved. □

The main result of this section are bounds of the function

$$N_s(\lambda) = N(\lambda)/\lambda^{d_s/2}\,,$$

where $d_s = 2\log 5/\log(40/3)$ is a so-called spectral dimension of the MKG. We shall prove that $N_s(\lambda)$ is oscillating and non-convergent as $\lambda \to 0$.

Theorem 5.3.

$$\frac{4}{3\cdot 25}\,\varphi(\lambda_1)^{d_s/2} \le \varliminf_{\lambda\to 0} N_s(\lambda) < \varlimsup_{\lambda\to 0} N_s(\lambda) \le 3\cdot\varphi(\lambda_1)^{d_s/2}\,. \tag{5.3}$$

Proof. Let $\lambda \in [\lambda_{n+1}^1, \lambda_n^1]$. By (5.2) we get

$$\frac{4}{3\cdot 5^{n+1}}\,(\lambda_n^1)^{-d_s/2} \le \frac{N(\lambda)}{(\lambda_n^1)^{d_s/2}} \le \frac{N(\lambda)}{\lambda^{d_s/2}} \le \frac{N(\lambda)}{(\lambda_{n+1}^1)^{d_s/2}} \le \frac{3}{5^n}\,(\lambda_{n+1}^1)^{-d_s/2}\,. \tag{5.4}$$

We note that $\left(\frac{40}{3}\right)^{-d_s/2} = \frac{1}{5}$ and from Proposition 5.1 we have

$$\lim_{n\to\infty} (\lambda_{n+1}^1)^{d_s/2}\Big(\big(\tfrac{40}{3}\big)^n\Big)^{d_s/2} = \varphi(\lambda_1)^{d_s/2}\,.$$

Now, let $n\to\infty$ in the inequality (5.4), then we get

$$\frac{4}{3\cdot 25}\,\varphi(\lambda_1)^{-d_s/2} \le \frac{N(\lambda)}{\lambda^{d_s/2}} \le 3\cdot\varphi(\lambda_1)^{-d_s/2}\,.$$

To prove the strict inequality in (5.3) we shall take the sequences $\{\lambda_k^i\}$, $i = 1,2,3$, $k = 1,2,\ldots$. By (3.6) we get

$$\lim_{k\to\infty} \frac{N(\lambda_k^i + 0) - N(\lambda_k^i - 0)}{(\lambda_k^i)^{d_s/2}} = \lim_{k\to\infty} \frac{5^{-k}}{3(\lambda_k^i)^{d_s/2}} = \frac{1}{15\varphi(\lambda_i)^{d_s/2}}\,.$$

The theorem is proved. □

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