Commun. Math. Phys. 156, 277–300 (1993)

Communications in Mathematical Physics

Isomorphism of Two Realizations of Quantum Affine Algebra $U_q(\widehat{\mathfrak{gl}(n)})$

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Received December 8, 1992

Abstract. We establish an explicit isomorphism between two realizations of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}(n)})$ given previously by Drinfeld and Reshetikhin–Semenov-Tian-Shansky. Our result can be considered as an affine version of the isomorphism between the Drinfield/Jimbo and the Faddeev–Reshetikhin–Takhtajan constructions of the quantum algebra $U_q(\mathfrak{gl}(n))$.

1. Introduction

The theory of quantum groups has become firmly established with the fundamental independent discovery of Drinfeld [D1] and Jimbo [J1] that the universal enveloping algebra $U(\mathfrak{g})$ of any Kac–Moody algebra $\mathfrak{g}$ admits as a Hopf algebra a certain q-deformation $U_q(\mathfrak{g})$. Their construction is given in terms of generators and relations and does not reveal the specific structure of the new Hopf algebras, in particular, when $\mathfrak{g}$ is a classical finite dimensional simple Lie algebra. In the latter case, Faddeev, Reshetikhin and Takhtajan [FRT1] gave a realization of $U_q(\mathfrak{g})$ by means of solutions of the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} , \tag{1.1}$$

where $R_{12} = R \otimes I$, etc., and $R \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$. This realization is a natural analogue of the matrix realization of the classical Lie algebras. Related constructions appeared previously in quantum field theory and statistical mechanics and provided main motivations for the subsequent discovery of Drinfeld and Jimbo (see [FRT1] for historical remarks).

It is well known [G] that the affine Kac–Moody algebra $\hat{\mathfrak{g}}$ associated to a simple Lie algebra $\mathfrak{g}$ admits a natural realization as a central extension of the corresponding loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. Faddeev, Reshetikhin and Takhtajan [FRT2] have shown how to extend their realization of $U_q(\mathfrak{g})$ to the quantum loop algebra $U_q(\mathfrak{g} \otimes [t, t^{-1}])$ using a solution of the Yang–Baxter equation depending on a parameter $z \in \mathbb{C}$,

$$R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(z)\,, \tag{1.2}$$

where $R(z)$ is a rational function of z with values in $\mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$.

The first realization of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ and its special degeneration called the Yangian has been obtained by Drinfield [D2]. Later Reshetikhin and Semenov-Tian-Shansky [RS] found a way to incorporate the central extension in the previous construction of [FRT2], thus obtaining the second realization of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$. In the case when $\mathfrak{g}$ is a classical finite dimensional simple Lie algebra, the construction in [RS] is an exact affine analogue of the construction in [FRT1].

The main goal of this paper is to establish an explicit relation between the two realizations of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$. We construct the isomorphism in the case when $\mathfrak{g} = \mathfrak{gl}(n)$. The extension of our relation to the other classical Lie algebras is straightforward though it requires some extra work and is not considered here.

We show that the realization of Drinfield's construction can be naturally established in the Gauss decomposition of a matrix composed of elements of the quantum affine algebra. This reflects the general principle that additive results in the classical case admit multiplicative formulation in the quantum case. The organization of the paper is the following.

In Sect. 2, we recall the isomorphism between Faddeev–Reshetikhin–Takhtajan and Drinfeld–Jimbo definitions of $U_q(\mathfrak{gl}(n))$.

In Sect. 3, we recall the two realizations of the quantum affine $U_q(\widehat{\mathfrak{gl}(n)})$ due to Drinfeld and Reshetikhin–Semenov-Tian-Shansky and formulate our main theorem.

In Sect. 4, we give the proof of surjectivity of the homomorphism in the case $n = 2$ and in Sect. 5 we complete the proof.

The relation between the two realizations is interesting from the structural point of view, and may also find important applications in constructions of representations of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}(n)})$ and isomorphisms between them. In the classical case, at least four models of representations are known: bosonic constructions [FK, S]; fermionic constructions [F, KP]; the algebraic version of the WZNW model [TK]; and the free boson realization [W, FeF]. There exist quantum analogue of all four constructions [FJ]; [H, DF2]; [FR]; [M] and [ABE]. The bosonic constructions use the Drinfeld realization. The other two are based on the Reshetikhin–Semenov-Tian-Shansky realization. Thus the connection we establish between the two realizations of $U_q(\widehat{\mathfrak{gl}(n)})$ should be crucial in obtaining isomorphisms between different models of representations.

II. Quantum algebra $U_q(\widehat{\mathfrak{gl}(n)})$

The Quantum Inverse Scattering Method was one of the principle motivations for the creation of Quantum Groups [D2, FRT1]. The basic formula in this method

$$RT_1T_2 = T_2T_1R\,, \tag{2.1}$$

where R is a solution of the Yang–Baxter equation (1.1), $T_1 = T \otimes I$, $T_2 = I \otimes T$ and $T = (t_{ij})_{i,j=1}^n$, was later interpreted as defining relations for the generators

t_{ij} of a Hopf algebra also called quantum group. In particular, the quantum group associated to $GL(n, \mathbb{C})$ is determined by the following matrix R depending on a complex parameter $q \neq 0$:

$$R = R_{12} = q \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + \sum_{\substack{i \neq j \\ i,j=1}}^{n} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{1 \leqq i < j \leqq n} E_{ij} \otimes E_{ji}\,, \tag{2.2}$$

where $E_{ij} \in \mathrm{End}(\mathbb{C}^n)$ are matrix units.

This matrix has a natural origin. In addition to Yang–Baxter equation, it also satisfies

$$R - PRP^{-1} = (q - q^{-1})P\,, \tag{2.3}$$

where P is the permutation operator, $P(v_1 \otimes v_2) = v_2 \otimes v_1$, for $v_1, v_2 \in \mathbb{C}^n$. These two properties ensure a representation of the Hecke algebra of type A_{m-1} on $\otimes^m \mathbb{C}^n$.

The matrix R also satisfies the following properties:

$$R^{-1} = R_{q \leftrightarrow q^{-1}}\,, \tag{2.4}$$

$$R^{t_1 t_2} = PRP = R_{21}\,, \tag{2.5}$$

where t_1, t_2 denote the transposition on the first or the second space respectively.

From now on, we will consider q to be a formal variable.

Definition 2.1. *$U(R)$ is an associative algebra with unit. It has generators l_{ij}^+, l_{ji}^-, $1 \leqq i \leqq j \leqq n$. Let $L^\pm = (l_{ij}^\pm)$, $1 \leqq i,j \leqq n$, with $l_{ij}^+ = l_{ji}^- = 0$ for $1 \leqq j < i \leqq n$. The defining relations are given in matrix form as follows:*

$$\begin{gathered} RL_1^\pm L_2^\pm = L_2^\pm L_1^\pm R\,, \quad RL_1^+ L_2^- = L_2^- L_1^+ R\,, \\ l_{ii}^- l_{ii}^+ = l_{ii}^+ l_{ii}^- = 1\,, \end{gathered} \tag{2.6}$$

where $L_1^\pm = L^\pm \otimes I$, $L_2^\pm = I \otimes L^\pm$.

Since $L^\pm$ are upper and lower triangular, respectively, and the diagonal elements of these matrix are invertible, $L^\pm$ have inverse $(L^\pm)^{-1}$ as a matrix with elements in $U(R)$. The relations between $L_1^\pm$ and $L_2^\pm$ immediately imply the following proposition.

Proposition 2.1. *Let $L'^\pm = (L^\pm)^{-1}$, $L''^\pm = ((L'^\pm)^t)^{-1}$. Then*

$$\begin{gathered} R_{21} L_1'^\pm L_2'^\pm = L_2'^\pm L_1'^\pm R_{21}\,, \quad R_{21} L_1'^+ L_2'^- = L_2'^- L_1'^+ R_{21}\,, \\ RL_1''^\pm L_2''^\pm = L_2''^\pm L_1''^\pm R\,, \quad RL_1''^+ L_2''^- = L_2''^- L_1''^+ R\,. \end{gathered} \tag{2.7}$$

Definition 2.2. *$U(R)$ is a Hopf algebra with comultiplication Δ defined by*

$$\Delta(L^\pm) = L^\pm \dot{\otimes} L^\pm\,,$$

or in terms of the generators $l_{ij}^\pm$

$$\Delta(l_{ij}^\pm) = \sum_{k=1}^{n} l_{ik}^\pm \otimes l_{kj}^\pm\,.$$

The antipode S is defined by:

$$S(L^\pm) = (L^\pm)^{-1}\,.$$

The counit is defined by:

$$\varepsilon(L) = I \ .$$

Originally, a wide class of examples of Hopf algebras associated to any simple Lie algebra $\mathfrak{g}$ was introduced independently by Drinfeld [D1] and Jimbo [J1]. In particular, we are most interested in the case when $\mathfrak{g} = \mathfrak{sl}(n)$, which is trivially extended to the case when $\mathfrak{g} = \mathfrak{gl}(n)$.

Definition 2.3. [J3]. *$U_q(\mathfrak{gl}((n))$ is an associative algebra over $\mathbb{C}$ generated by $q^{\pm H_j}$, e_i and f_i $(1 \leqq j \leqq n,\ 1 \leqq i \leqq n-1)$ with the defining relations*:

$$q^{H_i} q^{-H_i} = q^{-H_i} q^{H_i} = 1,$$

$$q^{H_i} e_j q^{-H_i} = q^{\delta_{ij}} q^{-\delta_{ij+1}} e_j\,, \quad q^{H_i} f_j q^{-H_i} = q^{-\delta_{ij}} q^{\delta_{ij+1}} f_j$$

$$e_i f_i - f_i e_i = \delta_{ij} \frac{q^{H_i - H_{i+1}} - q^{-H_i + H_{i+1}}}{q - q^{-1}}\,,$$

$$e_i e_j = e_j e_i, \quad f_i f_j = f_j f_i \quad (|i-j| \leqq 2)\,,$$

$$e_i^2 e_{i\pm1} - (q + q^{-1}) e_i e_{i\pm1} e_i + e_{i\pm1} e_i^2 = 0 \quad (1 \leqq i, i \pm 1 \leqq n)\,,$$

$$f_i^2 f_{i\pm1} - (q + q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0 \quad (1 \leqq i, i \pm 1 \leqq n)\,. \qquad (2.8)$$

The subalgebra generated by $e_i, f_i, q^{H_i - H_{i+1}}$ by definition is $U_q(\mathfrak{sl}(n))$. $U_q(\mathfrak{sl}(n))$ and $U_q(\mathfrak{gl}((n))$ are Hopf algebras with the comultiplication Δ, antipode S and counit ε defined as:

$$\Delta(q^{\pm H_i}) = q^{\pm H_i} \otimes q^{\pm H_i}, \quad \Delta(e_i) = 1 \otimes e_i + e_i \otimes q^{h_i}\,,$$

$$\Delta(f_i) = q^{-h_i} \otimes f_i + f_i \otimes 1,$$

$$S(e_i) = -\, e_i q^{-h_i}\,, \quad S(f_i) = -\, q^{h_i} f_i, \quad S(q^{\pm H_i}) = q^{\mp H_i}\,,$$

$$\varepsilon(e_i) = \varepsilon(f_i) = 0\,, \quad \varepsilon(q^{H_i}) = 1\,, \qquad (2.9)$$

where $h_i = H_i - H_{i+1}$.

The verification of the Hopf algebra axioms for $U_q(\mathfrak{sl}(n))$ and $U_q(\mathfrak{gl}(n))$ is straightforward. We call these Hopf algebras quantum algebras.

The important point of the approach in [FRT1] is an invariant, matrix-type description of the quantum algebra $U_q(\mathfrak{gl}(n))$. The authors constructed a surjective homomorphism from $U_q(\mathfrak{gl}(n))$ to $U(R)$ and stated that it is an isomorphism.

Theorem 2.1. *There is a Hopf algebra isomorphism between $U(R)$ and $U_q(\mathfrak{gl}(n))$ defined by*

$$l_{ii}^{+} = q^{H_i}, \quad l_{i,i+1}^{+} = (q - q^{-1}) q^{H_i} f_i$$

$$l_{i+1,i}^{-} = -\,(q - q^{-1}) e_i q^{-H_i} \qquad (2.10)$$

or in matrix form:

$$L^+ = \begin{pmatrix} q^{H_1} & & 0 \\ & \ddots & \\ 0 & & q^{H_n} \end{pmatrix} \begin{pmatrix} 1 & (q-q^{-1})f_1 & & * \\ & \ddots & \ddots & \\ & & & (q-q^{-1})f_{n-1} \\ 0 & & & 1 \end{pmatrix},$$

$$L^- = \begin{pmatrix} 1 & & & 0 \\ -(q-q^{-1})e_1 & & \ddots & \\ \ddots & & & \\ * & & -(q-q^{-1})e_{n-1} & 1 \end{pmatrix} \begin{pmatrix} q^{-H_1} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & q^{-H_n} \end{pmatrix}. \quad (2.11)$$

Proof of Theorem 2.1. One can directly check that the map (2.10) is a homomorphism from $U_q(\mathfrak{gl}(n))$ to $U(R)$ [FRT1]. Surjectivity of this map can be proved by induction on the height of roots.

The injectivity of the homomorphism is less trivial. Quantum algebra $U_q(\mathfrak{sl}(n))$ has a structure of a quasitriangular Hopf algebra [D2] with a universal R matrix in the form:

$$\mathfrak{R} = \sum_i a_i \otimes b_i\,, \quad (2.12)$$

where $a_i \in U_q\mathfrak{b}^+$, $b_i \in U_q\mathfrak{b}^-$ and $U_q\mathfrak{b}^\pm$ are Hopf subalgebras of $U_q(\mathfrak{sl}(n))$ generated, respectively, by e_i, $q^{\pm(H_i - H_{i+1})}$ or f_i, $q^{\pm(H_i - H_{i+1})}$.

Let V_λ be a highest weight representation of $\mathfrak{sl}(n)$, where λ is determined by its values on $h_i = H_i - H_{i+1}$. From the theory of Lusztig [L] and Rosso [R], we know that for any highest weight representation of $\mathfrak{sl}(n)$, we can construct a quantum deformation V_λ^q, which by definition is a representation of $U_q(\mathfrak{sl}(n))$.

Let $V = \mathbb{C}^n$ be the n-dimensional fundamental representation of $U_q(\mathfrak{sl}(n))$. On $V_\lambda^q \otimes V$, we set

$$L^{+V_\lambda^q} = (\pi_{V_\lambda^q} \otimes \pi_V)\mathfrak{R}_{21}, \quad L^{-V_\lambda^q} = (\pi_{V_\lambda^q} \otimes \pi_V)\mathfrak{R}^{-1}\,. \quad (2.13)$$

We also denote

$$L_1^{\pm V_\lambda^q} = L^{\pm V_\lambda^q} \otimes I_V, \quad L_2^{\pm V_\lambda^q} = P_{23} I_V \otimes L^{\pm V_\lambda^q} P_{23}\,, \quad (2.14)$$

and

$$R^{V_\lambda^q VV} = I_{V_\lambda^q} \otimes R\,, \quad (2.15)$$

where $R = (\pi_V \otimes \pi_V)\mathfrak{R}$ and P_{23} is the permutation operator of the last two components of the tensor product. Explicit form of the universal R-matrix of $U_q(\mathfrak{sl}(n))$ [R2] implies that this R is exactly the same as the one we defined in (2.2).

Since $\mathfrak{R}$ satisfies the Yang–Baxter equation, we obtain on $V_\lambda^q \otimes V \otimes V$,

$$R^{V_\lambda^q VV} L_1^{\pm V_\lambda^q} L_1^{\pm V_\lambda^q} = L_1^{\pm V_\lambda^q} L_2^{\pm V_\lambda^q} R^{V_\lambda^q VV}\,, \quad (2.16)$$

$$R^{V_\lambda^q VV} L_1^{+V_\lambda^q} L_2^{-V_\lambda^q} = L_2^{-V_\lambda^q} L_1^{+V_\lambda^q} R^{V_\lambda^q VV}\,. \quad (2.17)$$

From the explicit form of $\mathfrak{R}$ [R2], we also know that $L^{+V_\lambda^q}$ is upper triangular and $L^{-V_\lambda^q}$ is lower triangular, and the diagonal elements are in agreement with the first formula in (2.10). On the other hand, for any $r \in \mathbb{C}\backslash 0$, $r^{\pm 1} L^{\pm V_\lambda^q}$ also satisfy (2.16), (2.17). Thus we get a representation of $U(R)$ with the action of the central element $\prod l_{ii}^{\pm}$ by any nonzero constant. Via the homomorphism (2.10), we can get the representation of $U_q(\mathrm{gl}(n))$ on V_λ^q with the highest weight λ and an arbitrary action of the central element, which means that every highest weight representation of $U_q(\mathrm{gl}(n))$ can be pulled through $U(R)$. We denote by Λ a highest weight of $\mathrm{gl}(n)$, i.e. the pair λ and the value of the central element. Therefore the kernel of map (2.10) must also be inside I_Λ^q, the kernel of representation of $U_q(\mathrm{gl}(n))$ on the highest weight module V_Λ^q.

Classical theory of simple Lie algebras [Di] that $\bigcap_\lambda I_\lambda = 0$ immediately implies

$$\bigcap_\Lambda I_\Lambda = 0 \tag{2.18}$$

for $U(\mathrm{gl}(n))$, where I_λ (resp. I_Λ) is the kernel of the representation of $U(\mathfrak{sl}(n))$ (resp. $U(\mathrm{gl}(n))$ on the highest weight module with the highest weight λ (resp. Λ). On the other hand, we can identify $U_q(\mathrm{gl}(n))/(q-1)U_q(\mathrm{gl}(n)) \cong U_q(\mathrm{gl}(n))$. This follows from Drinfeld's theorem in [D4] and can be checked directly. Note, for example, that $q^{H_i} = e^{\log(1+(q-1))H_i} = e^{\left(\sum_{n>0}\frac{(q-1)^n}{n}\right)H_i} = \sum_{n>0}\frac{(q-1)^n}{n}B_n$, where B_n is a polynomial of H_i. Thus the kernel of the map (2.10) is inside the ideal generated by $q-1$. We will denote the map from $U_q(\mathrm{gl}(n))$ to $U_q(\mathrm{gl}(n))/(q-1)U_q(\mathrm{gl}(n)) \cong U(\mathrm{gl}(n))$ by ϕ_0. Let A be a nonzero element in this kernel, then $A = (q-1)^m \tilde{A}$, where $\phi_0(\tilde{A}) \neq 0$ as an element of $U(\mathrm{gl}(n))$, see [D4]. Since q is a formal parameter, $\pi_{V_\Lambda^q}(A) = 0$ implies $\pi_{V_\Lambda^q}(\tilde{A}) = 0$. We get $\phi_0(\tilde{A})$ as an element of $U(\mathrm{gl}(n))$ satisfies $\pi_{V_\Lambda}(\phi_0(\tilde{A})) = 0$ for any highest weight module Λ of $U(\mathrm{gl}(n))$. This contradicts (2.18). Therefore

$$\bigcap_\Lambda I_\Lambda^q = 0\,, \tag{2.19}$$

and the injectivity of map (2.10) is proven.

III. Quantum Affine Algebras and the Main Theorem

Drinfeld–Jimbo definition of quantum group by generators and relations is valid for an arbitrary generalized Cartan matrix. In particular, the choice of extended Cartan matrix of type $A_n^{(1)}$ yields the quantum affine algebra $U_q(\widehat{\mathfrak{sl}(n)})$. Drinfeld found in [D2] another realization of the quantum affine algebras, which to a certain degree plays the role of loop algebra realization in the underformed case. We extend Drinfeld's construction to the quantum affine algebra $U_q(\widehat{\mathrm{gl}(n)})$.

Definition 3.1. *$U_q(\widehat{\mathrm{gl}(n)})$ is an associative algebra with unit 1 and generators*

$$\{X_{ik}^{\pm}, k_{jl}^{+}, k_{jm}^{-}, q^{\pm\frac{1}{2}c} \mid i = 1, \ldots, n-1, j = 1, \ldots, n, k \in \mathbf{Z}, l \in \mathbf{Z}_+, m \in -\mathbf{Z}_+\} \tag{3.1}$$

satisfying relations in terms of the following generating functions in a formal variable z:

$$X_i^{\pm}(z) = \sum_{k\in\mathbf{Z}} X_{ik}^{\pm} z^{-k}, \quad k_j^{+}(z) = \sum_{l\in-\mathbf{Z}_+} k_{jl}^{+} z^{-l}\,,$$
$$k_j^{-}(z) = \sum_{m\in\mathbf{Z}_+} k_{jm}^{-} z^{-m}\,. \tag{3.2}$$

The generators $q^{\pm\frac{1}{2}c}$ are central and mutually inverse. The other relations are:

$$k_{j0}^{+}k_{j0}^{-} = k_{j0}^{-}k_{j0}^{\pm} = 1\ ,$$

$$k_i^{\pm}(z)k_j^{\pm}(w) = k_j^{\pm}(w)k_i^{\pm}(z)\ ,$$

$$k_i^{+}(z)k_i^{-}(w) = k_i^{-}(w)k_i^{+}(z)\ ,$$

$$\frac{z_{\mp} - w_{\pm}}{z_{\mp}q - w_{\pm}q^{-1}}\, k_i^{\mp}(z)k_j^{\pm}(w) = k_j^{\pm}(w)k_i^{\mp}(z)\frac{z_{\pm} - w_{\mp}}{z_{\pm}q - w_{\mp}q^{-1}}, \quad \text{if } j > i\ ,$$

$$\begin{cases} k_i^{\mp}(w)^{-1}X_j^{\mp}(z)k_i^{\pm}(z) = X_j^{\pm}(z) & \text{if } i - j \leqq -1 \text{ or} \\ k_i^{\pm}(w)^{-1}X_j^{\mp}(z)k_i^{\pm}(w) = X_j^{\mp}(z) & i - j \geqq 2 \end{cases},$$

$$k_i^{\pm}(z)^{-1}X_i^{-}(w)k_i^{\pm}(z) = \frac{z_{\mp}q - wq^{-1}}{z_{\mp} - w}X_i^{-}(w)\ ,$$

$$k_{i+1}^{\pm}(z)^{-1}X_i^{-}(w)k_{i+1}^{\pm}(z) = \frac{z_{\mp}q^{-1} - wq}{z_{\mp} - w}X_i^{-}(w)\ ,$$

$$k_i^{\pm}(z)X_i^{+}(w)k_i^{\pm}(z)^{-1} = \frac{z_{\pm}q - wq^{-1}}{z_{\pm} - w}X_i^{+}(w)\ ,$$

$$k_{i+1}^{\pm}(z)X_i^{+}(w)k_{i+1}^{\pm}(z)^{-1} = \frac{z_{\pm}q^{-1} - wq}{z_{\pm} - w}X_i^{+}(w)\ ; \tag{3.3}$$

$$(zq^{-1} - wq)X_i^{-}(z)X_i^{-}(w) = X_i^{-}(w)X_i^{-}(z)(zq - wq^{-1})\ ,$$

$$(zq - wq^{-1})X_i^{+}(z)X_i^{+}(w) = X_i^{+}(w)X_i^{+}(z)(zq^{-1} - wq)\ ,$$

$$(z - w)X_i^{+}(z)X_{i+1}^{+}(W) = (zq - wq^{-1})X_{i+1}^{+}(w)X_i^{+}(z)\ ,$$

$$(zq - wq^{-1})X_i^{-}(z)X_{i+1}^{-}(w) = (z - w)X_{i+1}^{-}(w)X_i^{-}(z)\ ,$$

$$[X_i^{\pm}(z), X_j^{\pm}(w)] = 0, \quad \text{for } A_{ij} = 0\ ,$$

$$\begin{aligned}[X_i^{+}(z)X_j^{-}(w)] = (q - q^{-1})\delta_{ij}\{&\delta(zw^{-1}q^{-c})k_{i+1}^{-}(w_{+})k_i^{-}(w_{+})^{-1} \\ &- \delta(zw^{-1}q^{c})k_{i+1}^{+}(z_{+})k_i^{+}(z_{+})^{-1}\}\end{aligned}$$

$$\begin{aligned}\{&X_i^{\pm}(z_1)X_i^{\pm}(z_2)X_j^{\pm}(w) - (q + q^{-1})X_i^{\pm}(z_1)X_j^{\pm}(w)X_i^{\pm}(z_2) \\ &+ X_j^{\pm}(w)X_1^{\pm}(z_1)X_i^{\pm}(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0, \quad \text{for } A_{ij} = -1\ .\end{aligned} \tag{3.4}$$

Here $z_{\pm} = q^{\pm\frac{c}{2}}z$, A_{ij} are the entries of the Cartan matrix for $\mathfrak{sl}(n)$, and

$$\delta(z) = \sum_{n\in\mathbf{Z}} z^n\ . \tag{3.5}$$

Drinfeld's realization of the subalgebra $U_q(\widehat{\mathfrak{sl}(n)})$ is given by $x_i^{\pm}(z) = (q - q^{-1})^{-1}X_i^{\pm}(zq^i)$, $\psi_i(z) = k_{i+1}^{-}(zq^i)k_i^{-}(zq^i)^{-1}$ and $\varphi_i(z) = k_{i-1}^{+}(zq^i)k_i^{+}(zq^i)^{-1}$ [D2], *see also* [FJ].

Drinfeld [D2] stated that the algebra $U_q(\widehat{\mathfrak{sl}(n)})$ is isomorphic to the one constructed by generators and relations as in Definition 2.3 for $U_q(\widehat{\mathfrak{sl}(n)})$ for extended Cartan matrix of type $\hat{A}_{n-1}$.

Definition 3.2. [D5] *The algebra $U_q(\widehat{\mathrm{gl}(n)})$ is a Hopf algebrta with comultiplication Δ, antipode S and counit ε defined by:*

$$\begin{aligned}
&\Delta(k_i^-(z)) = k_i^-(zq^{1\otimes\frac{c}{2}})\otimes k_i^-(zq^{-\frac{c}{2}\otimes 1})\,,\\
&\Delta(k_i^+(z)) = k_i^+(zq^{-1\otimes\frac{c}{2}})\otimes k_i^+(zq^{\frac{c}{2}\otimes 1})\,,\\
&\Delta(x_i^+(z)) = x_i^+(z)\otimes 1 + \psi_i(zq^{c\otimes 1})\otimes x_i^+(zq^{\frac{c}{2}\otimes 1})\,,\\
&\Delta(x_i^-(z)) = 1\otimes x_i^-(z) + x_i^-(zq^{1\otimes c})\otimes \varphi_i(zq^{1\otimes\frac{c}{2}})\,,\\
&S(k_j^\pm(z)) = k_j^\pm(z)^{-1}, \quad S(x_i^+(z)) = -\,\psi_i(z)^{-1}x_i^+(z)\,,\\
&S(x_i^-(z)) = -\,x_i^-(z)\varphi_i(z)^{-1}, \quad \varepsilon(x_i^+(z)) = 0\,,\\
&\varepsilon(x_i^-(z)) = 0, \quad \varepsilon(k_i^\pm(z)) = 1\,.
\end{aligned} \tag{3.6}$$

One can directly check the axioms of Hopf algebra. We note however that it is still an open problem to find the complete Hopf algebra isomorphism between this construction and the Drinfeld–Jimbo construction [D5].

Let $R(z)$ be an element of $\mathrm{End}(\mathbb{C}^n\otimes\mathbb{C}^n)$ defined by

$$\begin{aligned}
R(z) = {}& \sum_{i=1}^{n} E_{ii}\otimes E_{ii} + \sum_{\substack{i\neq j\\ i,j=1}}^{n} E_{ii}\otimes E_{jj}\frac{z-1}{qz-q^{-1}}\\
&+ \sum_{\substack{i<j\\ i,j=1}}^{n} E_{ij}\otimes E_{ij}\frac{z(q-q^{-1})}{zq-q^{-1}} + \sum_{\substack{i>j\\ i,j=1}}^{n} E_{ij}\otimes E_{ji}\frac{(q-q^{-1})}{qz-q^{-1}}\,,
\end{aligned} \tag{3.7}$$

where q, z are formal variables. Then $R(z)$ satisfies the Yang–Baxter equation with a parameter:

$$R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(z)\,, \tag{3.8}$$

and R is unitary, namely

$$R_{21}(z)^{-1} = R(z^{-1})\,, \tag{3.9}$$

where $R_{12}(z)$, etc., are defined as in Sects. 1, 2.

The definition of $R(z)$ implies

$$\begin{aligned}
&\lim_{z\to 0} R(z) = \sum_{i=1}^{n} E_{ii}\otimes E_{ii} + \sum_{\substack{i\neq j\\ i,j=1}}^{n} qE_{ii}\otimes E_{jj} + q(q^{-1}-q)\sum_{\substack{i>j\\ i,j=1}}^{n} E_{ij}\otimes E_{ji} = qR_{21}^{-1}\,,\\
&\lim_{z\to\infty} R(z) = \sum_{i=1}^{n} E_{ii}\otimes E_{ii} + \sum_{\substack{i\neq j\\ i,j=1}}^{n} q^{-1}E_{ii}\otimes E_{jj} + q^{-1}(q-q^{-1})\sum_{\substack{i<j\\ i,j=1}}^{n} E_{ij}\otimes E_{ji}\\
&\qquad = q^{-1}R\,,
\end{aligned} \tag{3.10}$$

Faddeev, Reshetikhin and Takhtajan in [FRT2] defined a Hopf algebra using this element $R(z)$. Later Reshetikhin and Semenov-Tian-Shansky in [RS] obtained a central extension of this algebra. We denote this modified algebra $U(\tilde{R})$. The central extension is incorporated in shifts of the parameter z in $R(z)$. The algebra $U(\tilde{R})$ contains a subalgebra $U(R)$ because of the relation (3.10).

Definition 3.2. *$U(\tilde{R})$ is an associative algebra with generators $\{l_{ij}^{\pm}[\mp m], m \in \mathbf{Z}_+ \backslash 0$ and $l_{ij}^{+}[0]$, $l_{ji}^{-}[0]$, $1 \leqq j \leqq i \leqq n\}$. Let $l_{ij}^{\pm}(z) = \sum_{m=0}^{\infty} l_{ij}^{\pm}[\pm m] z^{\pm m}$, where $l_{ij}^{+}[0] = l_{ji}^{-}[0] = 0$, for $1 \leqq i < j \leqq n$. Let $L^{\pm}(z) = (l_{ij}^{\pm}(z))_{i,j=1}^{n}$. Then the defining relations are the following:*

$$l_{ii}^{+}[0] l_{ii}^{-}[0] = l_{ii}^{-}[0] l_{ii}^{+}[0] = 1 \, , \tag{3.11}$$

$$R\left(\frac{z}{w}\right) L_1^{\pm}(z)\, L_2^{\pm}(w) = L_2^{\pm}(w) L_1^{\pm}(z) R\left(\frac{z}{w}\right),$$

$$R\left(\frac{z_+}{w_-}\right) L_1^{+}(z)\, L_2^{-}(w) = L_2^{-}(w) L_1^{+}(z) R\left(\frac{z_-}{w_+}\right), \tag{3.12}$$

where $z_{\pm} = zq^{\pm \frac{c}{4}}$. For the first formula of (3.12), the expansion direction of $R\left(\frac{z}{w}\right)$ can be chosen in $\frac{z}{w}$ or $\frac{w}{z}$, but for the second formula of (3.12), the expansion direction is only in $\frac{z}{w}$.

$U(\tilde{R})$ is a Hopf algebra: its coproduct is defined by

$$\Delta(L^{\pm}(z)) = L^{\pm}(zq^{\pm(1 \otimes \frac{c}{4})}) \dot{\otimes} L^{\pm}(zq^{\mp(\frac{c}{4} \otimes 1)})$$

$$\text{or} \quad \Delta(l_{ij}^{\pm}(z)) = \sum_{k=1}^{n} l_{ik}^{\pm}(zq^{\pm(1 \otimes \frac{c}{4})}) \otimes l_{kj}^{\pm}(zq^{\mp(\frac{c}{4} \otimes 1)}) \, , \tag{3.13}$$

and its antipode is

$$S(L^{\pm}(z)) = L^{\pm}(z)^{-1} \, . \tag{3.14}$$

Note that the invertibility of $L^{\pm}(z)$ follows from the properties that $l_{ii}^{\pm}$ are invertible and $L^{\pm}(0)$ are upper triangular and lower triangular, respectively.

We remark that the original definition of Reshetikin–Semenov-Tian-Shansky contained an extra set of generators $\tilde{L}(z)$ satisfying additional relations below:

$$R^{VV*}\left(\frac{z}{w}\right) L_1^{\pm}(z) \tilde{L}_2^{\pm}(w) = \tilde{L}_2^{\pm}(w) L_1^{\pm}(z) R^{VV*}\left(\frac{z}{w}\right), \tag{3.15}$$

$$R^{VV*}\left(\frac{z_+}{w_-}\right) L_1^{+}(z) \tilde{L}_2^{-}(w) = \tilde{L}_2^{-}(w) L_1^{+}(z) R\left(\frac{z_-}{w_+}\right), \tag{3.16}$$

$$R^{V*V}\left(\frac{z_+}{w_-}\right) \tilde{L}_1^{+}(z) L_2(w) = L_2^{-}(w) \tilde{L}_1^{+}(z) R^{V*V}\left(\frac{z_-}{w_+}\right), \tag{3.17}$$

$$R^{V*V*}\left(\frac{z}{w}\right) \tilde{L}_1^{\pm}(z) \tilde{L}_2^{\pm}(w) = \tilde{L}_2^{\pm}(w) \tilde{L}_1^{\pm}(z) R^{V*V*}\left(\frac{z}{w}\right), \tag{3.18}$$

$$R^{V*V*}\left(\frac{z_+}{w_-}\right) \tilde{L}_1^{+}(z) \tilde{L}_2^{-}(w) = \tilde{L}_2^{-}(w) \tilde{L}_1^{+}(z) R^{V*V*}\left(\frac{z_-}{w_+}\right), \tag{3.19}$$

where $V = \mathbb{C}^n$ and

$$R^{V^*V}\left(\frac{z}{w}\right) = \left(R\left(\frac{z}{w}\right)^{-1}\right)^{t_1}, \quad R^{VV^*}\left(\frac{z}{w}\right) = \left(R\left(\frac{z}{w}\right)^{t_2}\right)^{-1},$$

$$R^{V^*V^*}\left(\frac{z}{w}\right) = R\left(\frac{z}{w}\right)^{t_1 t_2}. \tag{3.20}$$

It turns out that it is natural to impose one more relation

$$\tilde{L}(z) = ((L^{\pm}(z))^t)^{-1}. \tag{3.21}$$

Then all the relations (3.15)–(3.19) follow from (3.12) and we obtain our definition of $U(\tilde{R})$ [Re].

Main Theorem. *$L^{\pm}(z)$ have the following unique decompositions:*

$$L^{\pm}(z) = \begin{pmatrix} 1 & & & 0 \\ e^{\pm}_{2,1}(z) & \ddots & & \\ e^{\pm}_{3,1}(z) & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots \\ e^{\pm}_{n,1}(z) & \cdots & e^{\pm}_{n,n-1}(z) & 1 \end{pmatrix} \begin{pmatrix} k^{\pm}_1(z) & & 0 \\ & \ddots & \\ & & \ddots & \\ 0 & & k^{\pm}_n(z) \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & f^{\pm}_{1,2}(z) & f^{\pm}_{1,3}(z) & \cdots & f^{\pm}_{1,n}(z) \\ & \ddots & \ddots & \ddots & \vdots \\ & & & & f^{\pm}_{n-1,n}(z) \\ & 0 & & & 1 \end{pmatrix}, \tag{3.22}$$

where $e^{\pm}_{i,j}(z), f^{\pm}_{j,i}(z)$ and $k^{\pm}_i(z)$ $(i > j)$ are elements in $U(\tilde{R})$ and $k^{\pm}_i(z)$ are invertible. Let

$$X^-_i(z) = f^+_{i,i+1}(z_+) - f^-_{i,i+1}(z_-),$$

$$X^+_i(z) = e^+_{i+1,i}(z_-) - e^-_{i+1,i}(z_+), \tag{3.23}$$

where $z_{\pm} = zq^{\pm\frac{c}{2}}$, then $q^{\pm\frac{1}{2}c}, X^{\pm}_i(z), k^{\pm}_j(z)$, $i = 1, \ldots, n-1, j = 1, \ldots, n$ satisfy the relations (3.3), (3.4) *of $U_q(\widehat{\mathfrak{gl}(n)})$. The homomorphism*

$$\Phi\colon U_q(\widehat{\mathfrak{gl}(n)}) \to U(\tilde{R}) \tag{3.24}$$

defined by (3.23) *is an isomorphism.*

Later we will use $f_i^+(z)$ to denote $f^+_{i,i+1}(z)$ and $e_i^+(z)$ to denote $e^+_{i+1,i}(z)$.

Since $k^{\pm}_i(z)$ are invertible, the elements $e^{\pm}_{i,j}(z), f^{\pm}_{j,i}(z)$ and $k^{\pm}_i(z)$ $(i > j)$ are uniquely expressed in terms of the matrix coefficients of $L^{\pm}(z)$ as in the scalar case. In view of this analogy we call (3.22) Gauss decomposition of $L^{\pm}(z)$.

We note that as a corollary of the Main Theorem and Definition 3.1, one gets a realization of $U_q(\widehat{\mathfrak{sl}(n)})$ as a subalgebra of $U(\tilde{R})$.

IV. Case $n = 2$

To verify that Φ is a homomorphism, we will use (3.22), (3.12) their inversions as follows:

$$L^{\pm}(z)^{-1} = \begin{pmatrix} 1 & -f_1^{\pm}(z) & & \\ \ddots & \ddots & & \\ & & & -f_{n-1}^{\pm}(z) \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} k_1^{\pm}(z)^{-1} & & 0 \\ & \ddots & \\ 0 & & k_n^{\pm}(z)^{-1} \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & & & 0 \\ -e_1^{\pm}(z) & & & \\ \ddots & \ddots & & \\ & & -e_{n-1}^{\pm}(z) & 1 \end{pmatrix} \tag{4.1}$$

$$L_1^{\pm}(w)^{-1} R_{21}\left(\frac{z}{w}\right) L_2^{\pm}(z) = L_2^{\pm}(z) R_{21}\left(\frac{z}{w}\right) L_1^{\pm}(w)^{-1}, \tag{4.2}$$

$$L_1^{-}(w)^{-1} R_{21}\left(\frac{z^{+}}{w^{-}}\right) L_2^{+}(z) = L_2^{+}(z) R_{21}\left(\frac{z_{-}}{w_{+}}\right) L_1^{-}(w)^{-1}, \tag{4.3}$$

$$R_{21}\left(\frac{z_{-}}{w_{+}}\right) L_2^{-}(z) L_1^{+}(w) = L_1^{+}(w) L_2^{-}(z) R_{21}\left(\frac{z_{+}}{w_{-}}\right), \tag{4.4}$$

$$L_1^{+}(w)^{-1} R_{21}\left(\frac{z_{-}}{w_{+}}\right) L_2^{-}(z) = L_2^{-}(z) R_{21}\left(\frac{z_{+}}{w_{-}}\right) L_1^{+}(w), \tag{4.5}$$

$$L_2^{\pm}(z)^{-1} (L_1^{\pm}(w))^{-1} R\left(\frac{z}{w}\right) = R_{21}\left(\frac{z}{w}\right) (L_1^{\pm}(w))^{-1} (L_2^{\pm}(z))^{-1}, \tag{4.6}$$

$$L_2^{+}(z)^{-1} L_1^{-}(w)^{-1} R_{21}\left(\frac{z_{+}}{w_{-}}\right) = R_{21}\left(\frac{z_{-}}{w_{+}}\right) (L_1^{-}(w))^{-1} (L_2^{+}(z))^{-1}. \tag{4.7}$$

The proof that Φ is a homomorphism is based on the induction with respect to n. We consider first the case $n = 2$,

$$L^{\pm}(z) = \begin{pmatrix} k_1^{\pm}(z) & k_1^{\pm}(z) f_1^{\pm}(z) \\ e_1^{\pm}(z) k_1^{\pm}(z) & k_2^{\pm}(z) + e_1^{\pm}(z) k_1^{\pm}(z) f_1^{\pm}(z) \end{pmatrix}, \tag{4.8}$$

$$(L_1^{\pm}(z))^{-1} = \begin{pmatrix} k_1^{\pm}(z)^{-1} + f_1^{\pm}(z) k_2^{\pm}(z)^{-1} e_1^{\pm}(z) & -f_1^{\pm}(z) k_2^{\pm}(z)^{-1} \\ -k_2^{\pm}(z)^{-1} e_1^{\pm}(z) & k_2^{\pm}(z)^{-1} \end{pmatrix}, \tag{4.9}$$

$$R_{21}\left(\frac{z}{w}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{z-w}{zq-wq^{-1}} & \dfrac{w(q-q^{-1})}{zq-wq^{-1}} & 0 \\ 0 & \dfrac{z(q-q^{-1})}{zq-wq^{-1}} & \dfrac{z-w}{zq-wq^{-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.10}$$

$$\left(R_{21}\left(\frac{w}{z}\right)\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{z-w}{zq-wq^{-1}} & \dfrac{z(q-q^{-1})}{zq-wq^{-1}} & 0 \\ 0 & \dfrac{w(q-q^{-1})}{zq-wq^{-1}} & \dfrac{z-w}{zq-wq^{-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = R\left(\frac{z}{w}\right) , \tag{4.11}$$

$$(L_1(w))^{-1} =$$

$$\begin{pmatrix} * & 0 & -f_1(w)k_2(w)^{-1} & 0 \\ 0 & * & 0 & -f_1(w)k_2(w)^{-1} \\ -k_2(w)^{-1}e_1(w) & 0 & k_2(w)^{-1} & 0 \\ 0 & -k_2(w)^{-1}e_1(w) & 0 & k_2(w)^{-1} \end{pmatrix}. \tag{4.12}$$

From (3.15), (3.16) and (4.2)–(4.7), we can write down all the relations between $k_1^\pm(z), k_2^\pm(z)$,

$$k_1^\pm(z)k_1^\pm(w) = k_1^\pm(z)k_2^\pm(z) , \tag{4.13}$$

$$k_2^\pm(z)k_2^\pm(w) = k_2^\pm(w)k_2^\pm(z) , \tag{4.14}$$

$$k_1^+(z)k_1^-(w) = k_1^-(w)k_1^+(z) , \tag{4.15}$$

$$k_2^+(z)k_2^-(w) = k_2^-(w)k_2^+(w) , \tag{4.16}$$

$$k_1^\pm(z)k_2^\pm(w) = k_2^\pm(w)k^\pm(z) , \tag{4.17}$$

$$\frac{z_+ - w_-}{z_+q - w_-q^{-1}} k_2^-(w)^{-1}k_1^+(z) = k_1^+(z)k_2^-(w)^{-1} \frac{z_- - w_+}{z_-q - w_+q^{-1}} , \tag{4.18}$$

$$\frac{z_- - w_+}{z_-q - w_+q^{-1}} k_2^+(w)^{-1}k_1^-(z) = k_1^-(z)k_2^+(w)^{-1} \frac{z_+ - w_-}{z_+q - w_-q^{-1}} . \tag{4.19}$$

Then, we derive the relations between $k_1^{\pm}(z)$ and $e_1^{\pm}(z)$ or $f_1^{\pm}(z)$.

$$k_1^{\pm}(z)k_1^{\pm}(w)f_1^{\pm}(w) = k_1^{\pm}(w)k_1^{\pm}(z)f_1^{\pm}(z)\frac{w(q-q^{-1})}{zq-wq^{-1}} + \frac{z-w}{zq-wq^{-1}}k_1^{\pm}(w)f_1^{\pm}(w)k_1^{\pm}(z)\,, \tag{4.20}$$

$$e_1^{\pm}(w)k_1^{\pm}(w)k_1^{\pm}(z) = e_1^{\pm}(z)k_1^{\pm}(z)k_1^{\pm}(w)\frac{z(q-q^{-1})}{zq-wq^{-1}} + \frac{z-w}{zq-wq^{-1}}k_1^{\pm}(z)e_1^{\pm}(w)k_1^{\pm}(w)\,, \tag{4.21}$$

$$k_1^{\pm}(z)k_1^{\mp}(w)f_1^{\mp}(w) = k_1^{\mp}(w)k_1^{\pm}(z)f_1^{\pm}(z)\frac{w_{\pm}(q-q^{-1})}{z_{\mp}q-w_{\pm}q^{-1}} + \frac{z_{\mp}-w_{\mp}}{z_{\mp}q-w_{\pm}q^{-1}}k_1^{\mp}(w)f_1^{\pm}(w)k_1^{\pm}(z)\,, \tag{4.22}$$

$$e_1^{\pm}(w)k_1^{\mp}(w)k_1^{\pm}(z) = e_1^{\pm}(z)k_1^{\pm}(z)k_1^{\mp}(w)\frac{z_{\pm}(q-q^{-1})}{z_{\pm}q-w_{\mp}q^{-1}} + \frac{z_{\pm}-w_{\mp}}{z_{\pm}q-w_{\mp}q}k_1^{\pm}(z)e_1^{\mp}(w)k_1^{\mp}(w)\,. \tag{4.23}$$

Thus

$$k_1^{\pm}(z)^{-1}f_1^{\pm}(w)k_1^{\pm}(z) = \frac{zq-wq^{-1}}{z-w}f_2^{\pm}(w) + \frac{w(q-q^{-1})}{w-z}f_1^{\pm}(z)\,, \tag{4.24}$$

$$k_1^{\pm}(z)e_1^{\pm}(w)k_1^{\pm}(z)^{-1} = \frac{zq-wq^{-1}}{z-w}e_1^{\pm}(w) + \frac{z(q-q^{-1})}{w-z}e_1^{\pm}(z)\,, \tag{4.25}$$

$$k_1^{\pm}(z)^{-1}f_1^{\mp}(w)k_1^{\pm}(z) = \frac{z_{\mp}q-w_{\pm}q^{-1}}{z_{\mp}-w_{\mp}}f_1^{\mp}(w) + \frac{w_{\pm}(q-q^{-1})}{w_{\pm}-z_{\mp}}f_1^{\pm}(z)\,, \tag{4.26}$$

$$k_1^{\pm}(z)e_1^{\mp}(w)k_1^{\pm}(z) = \frac{z_{\pm}q-w_{\mp}q^{-1}}{z_{\pm}-w_{\mp}}e_1^{\mp}(w) + \frac{z_{\pm}(q-q^{-1})}{w_{\mp}-z_{\pm}}e_1^{\pm}(z)\,. \tag{4.27}$$

So

$$k_1^{\pm}(z)^{-1}X_1^{-}(w)k_1^{\pm}(z) = \frac{z_{\mp}q-wq^{-1}}{z_{\mp}-w}X_1^{-}(w)\,, \tag{4.28}$$

$$k_1^{\pm}(z)X_1^{+}(w)k_1^{\pm}(z)^{-1} = \frac{z_{\pm}q-wq^{-1}}{z_{\pm}-w}X_1^{+}(w)\,. \tag{4.29}$$

Then we write down the relations between $f_1^{\pm}(z)$, $f_2^{\pm}(w)$; $e_1^{\pm}(z)$, $e_2^{\pm}(w)$,

$$k_1^{\pm}(z)f_1^{\pm}(z)k_1^{\pm}(w)f_1^{\pm}(w) = k_1^{\pm}(w)f_1^{\pm}(w)k_1^{\pm}(z)f_1^{\pm}(z)\,, \tag{4.30}$$

$$k_1^{\pm}(z)f_1^{\pm}(z)k_1^{\mp}(w)f_1^{\mp}(w) = k_1^{\mp}(w)f_1^{\mp}(w)k_1^{\pm}(z)f_1^{\pm}(z)\,, \tag{4.31}$$

$$e_1^{\pm}(z)k_1^{\pm}(z)e_1^{\pm}(w)k_1^{\pm}(w) = e_1^{\pm}(w)k_1^{\pm}(w)e_1^{\pm}(z)k_1^{\pm}(z)\,, \tag{4.32}$$

$$e_1^{\pm}(z)k_1^{\pm}(z)e_1^{\mp}(w)k_1^{\mp}(w) = e_1^{\mp}(w)k_1^{\mp}(w)e_1^{\pm}(z)k_1^{\pm}(z)\,. \tag{4.33}$$

From above, we get

$$\frac{zq^{-1}-wq}{z-w}f_1^{\pm}(z)f_1^{\pm}(w)+\frac{z(q-q^{-1})}{z-w}f_1^{\pm}(w)f_1^{\pm}(w)$$
$$=\frac{zq-wq^{-1}}{z-w}f_1^{\pm}(w)f_1^{\pm}(z)+\frac{w(q-q^{-1})}{w-z}f_1^{\pm}(z)f_1^{\pm}(z)\,, \tag{4.34}$$

$$\frac{z_{\pm}q^{-1}-w_{\pm}q}{z_{\mp}-w_{\pm}}f_1^{\pm}(z)f_1^{\mp}(w)+\frac{z_{\mp}(q-q^{-1})}{z_{\mp}-w_{\pm}}f_1^{\mp}(w)f_1^{\mp}(w)$$
$$=\frac{z_{\mp}q-w_{\pm}q^{-1}}{z_{\mp}-w_{\pm}}f_1^{\mp}(w)f_1^{\pm}(z)+\frac{w_{\pm}(q-q^{-1})}{w_{\pm}-z_{\mp}}f_1^{\pm}(z)f_1^{\pm}(z)\,, \tag{4.35}$$

$$\frac{zq-wq^{-1}}{z-w}e_1^{\pm}(z)e_1^{\pm}(w)+\frac{z(q-q^{-1})}{w-z}e_1^{\pm}(z)e_1^{\pm}(z)$$
$$=\frac{zq^{-1}-wq}{z-w}e_1^{\pm}(w)e_1^{\pm}(z)+\frac{w(q-q^{-1})}{z-w}e_1^{\pm}(w)e_1^{\pm}(w)\,, \tag{4.36}$$

$$\frac{z_{\pm}q-w_{\mp}q^{-1}}{z_{\pm}-w_{\pm}}e_1^{\pm}(z)e_1^{\mp}(w)+\frac{z_{\pm}(q-q^{-1})}{w_{\mp}-z_{\pm}}e_1^{\pm}(z)e_1^{\pm}(z)$$
$$=\frac{z_{\pm}q^{-1}-w_{\mp}q}{z_{\pm}-w_{\mp}}e_1^{\mp}(w)e_1^{\pm}(z)+\frac{w_{\mp}(q-q^{-1})}{z_{\pm}-w_{\mp}}e_1^{\mp}(w)e_1^{\pm}(w)\,. \tag{4.37}$$

Thus

$$(zq^{-1}-wq)X_1^-(z)X_1^-(w)=X_1^-(w)X_1^-(z)(zq-wq^{-1})\,, \tag{4.38}$$
$$(zq-wq^{-1})X_1^+(z)X_1^+(w)=X_1^+(w)X_1^+(z)(zq^{-1}-wq)\,. \tag{4.39}$$

The relations between $f_1^{\pm}(z)$, $e_1^{\pm}(z)$ and $k_2^{\pm}(z)$ are

$$-f_1^{\pm}(z)k_2^{\pm}(z)^{-1}k_2^{\pm}(w)^{-1}=\frac{-z(q-q^{-1})}{zq-wq^{-1}}f_1^{\pm}(w)k_2^{\pm}(w)^{-1}k_2^{\pm}(z)^{-1}$$
$$-\frac{z-w}{zq-wq^{-1}}k_2^{\pm}(w)^{-1}f_1^{\pm}(z)k_2^{\pm}(z)^{-1}\,, \tag{4.40}$$

$$-k_2^{\pm}(w)^{-1}k_2^{\pm}(z)^{-1}e_1^{\pm}(z)=\frac{-w(q-q^{-1})}{zq-wq^{-1}}k_2^{\pm}(z)^{-1}k_2^{\pm}(w)^{-1}e_1^{\pm}(w)$$
$$-\frac{z-w}{zq-wq^{-1}}k_2^{\pm}(z)^{-1}e_1^{\pm}(z)k_2^{\pm}(w)^{-1}\,, \tag{4.41}$$

$$-f_1^{\pm}(z)k_2^{\pm}(z)^{-1}k_2^{\mp}(w)^{-1}=\frac{-z_{\mp}(q-q^{-1})}{z_{\mp}q-w_{\pm}q^{-1}}f_1^{\mp}(w)k_1^{\mp}(w)^{-1}k_2^{\pm}(z)^{-1}$$
$$-\frac{z_{\mp}-w_{\pm}}{z_{\mp}q-w_{\pm}q^{-1}}k_2^{\mp}(w)^{-1}f_1^{\pm}(z)k_2^{\pm}(z)^{-1}\,, \tag{4.42}$$

$$-k_2^{\mp}(w)^{-1}k_2^{\pm}(z)^{-1}e_1^{\pm}(z)=\frac{-w_{\mp}(q-q^{-1})}{z_{\pm}q-w_{\mp}q^{-1}}k_2^{\pm}(z)^{-1}k_2^{\mp}(w)^{-1}e_1^{\mp}(w)$$
$$-\frac{z_{\pm}-w_{\mp}}{z_{\pm}q-w_{\mp}q^{-1}}k_2^{\pm}(z)^{-1}e_1^{\pm}(z)k_2^{\mp}(w)^{-1}\,. \tag{4.43}$$

Then

$$k_2^{\pm}(w)^{-1}f_1^{\pm}(z)k_2^{\pm}(w)=\frac{zq-wq^{-1}}{z-w}f_1^{\pm}(z)+\frac{z(q-q^{-1})}{z-w}f_1^{\pm}(w)\,, \tag{4.44}$$

$$k_2^{\pm}(w)e_1^{\pm}(z)k_2^{\pm}(w)^{-1}=\frac{zq-wq^{-1}}{z-w}e_1^{\pm}(z)+\frac{w(q-q^{-1})}{z-w}e_1^{\pm}(w)\,, \tag{4.45}$$

$$k_2^{\mp}(w)^{-1}f_1^{\pm}(z)k_2^{\mp}(w)=\frac{z_{\mp}q-w_{\pm}q^{-1}}{z_{\mp}-w_{\pm}}f_1^{\pm}(z)+\frac{z_{\mp}(q-q^{-1})}{w_{\pm}-z_{\mp}}f_1^{\mp}(w)\,, \tag{4.46}$$

$$k_2^{\mp}(w)e_1^{\pm}(z)k_2^{\mp}(w)^{-1}=\frac{z_{\pm}q-w_{\mp}q^{-1}}{z_{\pm}-w_{\mp}}e_1^{\pm}(z)+\frac{w_{\pm}(q-q^{-1})}{w_{\mp}-z_{\pm}}e_1^{\pm}(w)\,. \tag{4.47}$$

Thus we get

$$k_2^{\pm}(z)^{-1}X_1^{-}(w)k_2^{\pm}(z)=\frac{z_{\mp}q^{-1}-wq}{z_{\pm}-w}X_1^{-}(w)\,, \tag{4.48}$$

$$k_2^{\pm}(z)X_1^{+}(w)k_2^{\pm}(z)^{-1}=\frac{z_{\pm}q-wq^{-1}}{z_{\pm}-w}X^{+}(w)\,, \tag{4.49}$$

$$\begin{aligned}&\frac{z(q-q^{-1})}{zq-wq^{-1}}(k_2^{\pm}(z)+e_1^{\pm}(z)k_1^{\pm}(z)f_1^{\pm}(z))k_1^{\pm}(w)\\&\quad+k_1^{\pm}(z)f_1^{\pm}(z)e_1^{\pm}(w)k_1^{\pm}(w)\frac{z-w}{zq-wq^{-1}}\\&=e_1^{\pm}(w)k_1^{\pm}(w)k_1^{\pm}(z)f_1^{\pm}(z)\frac{z-w}{zq-wq^{-1}}\\&\quad+(k_2^{\pm}(w)+e_1^{\pm}(w)k_1^{\pm}(w)f_1^{\pm}(w))k_1^{\pm}(z)\frac{z(q-q^{-1})}{zq-wq^{-1}}\,,\end{aligned} \tag{4.50}$$

$$\begin{aligned}&\frac{z_{\pm}(q-q^{-1})}{z_{\pm}q-w_{\mp}q}(k_2^{\pm}(z)+e_1^{\pm}(z)k_1^{\pm}(z)f_1^{\pm}(z))k_1^{\mp}(w)\\&\quad+k_1^{\pm}(z)f_1^{\pm}(z)e_1^{\mp}(w)k_1^{\mp}(w)\frac{z_{\pm}-w_{\pm}}{z_{\pm}q-w_{\mp}q^{-1}}\\&=e_1^{\mp}(w)k_1^{\mp}(w)k_1^{\pm}(z)f_1^{\pm}(z)\frac{z_{\pm}-w_{\pm}}{z_{\mp}q-w_{\pm}q^{-1}}\\&\quad+(k_2^{\mp}(w)+e_1^{\mp}(w)k_1^{\mp}(w)f_1^{\mp}(w))k_1^{\pm}(z)\frac{z_{\mp}(q-q^{-1})}{z_{\mp}q-w_{\pm}q^{-1}}\,.\end{aligned} \tag{4.51}$$

But from (3.15), (3.16) and (4.4), we also have

$$k_1^{\pm}(z) f_1^{\pm}(z) k_1^{\pm}(w) = k_1^{\pm}(w) k_1^{\pm}(z) f_1^{\pm}(z) \frac{z-w}{zq-wq^{-1}} + \frac{z(q-q^{-1})}{zq-wq^{-1}} k_1^{\pm}(w) f_1^{\pm}(w) k_1^{\pm}(z)\,, \tag{4.52}$$

$$k_1^{\pm}(z) f_1^{\pm}(z) k_1^{\mp}(w) = k_1^{\mp}(w) k_1^{\pm}(z) f_1^{\pm}(z) \frac{z_{\mp}-w_{\pm}}{z_{\mp}q-w_{\pm}q^{-1}} + \frac{z_{\mp}q-w_{\pm}q^{-1}}{z_{\mp}q-w_{\pm}q^{-1}} k_1^{\mp}(w) f_1^{\mp}(w) k_1^{\pm}(z)\,, \tag{4.53}$$

then

$$[f_1^{\pm}(z), e_1^{\pm}(w)] = \frac{z(q-q^{-1})}{z-w} k_2^{\pm}(w) k_1^{\pm}(w)^{-1} - k_2^{\pm}(z) k_1^{\pm}(z)^{-1} \frac{z(q-q^{-1})}{z-w}\,, \tag{4.54}$$

$$[f_1^{\pm}(z), e_1^{\mp}(w)] = \frac{z_{\mp}(q-q^{-1})}{z_{\mp}-w_{\pm}} k_2^{\mp}(w) k_1^{\mp}(w)^{-1} - k_2^{\pm}(z) k_1^{\pm}(z)^{-1} \frac{z_{\pm}(q-q^{-1})}{z_{\pm}-w_{\mp}}\,, \tag{4.54}$$

here the denominators are power series in $\frac{w}{z}$ and $\frac{z}{w}$ respectively. Thus

$$[X_1^{+}(z), X_1^{-}(w)] = (q-q^{-1})\{\delta(zw^{-1}q^{-c}) k_2^{-}(w_+) k_1^{-}(w_+)^{-1} - \delta(zw^{-1}q^{c}) k_2^{+}(z_+) k_1^{+}(z_+)^{-1}\}\,. \tag{4.56}$$

So we prove that the map Φ is a homomorphism in $n=2$ case. Surjectivity is immediate, and the injectivity will be proved in next section.

V. The Proof of the Theorem for the General *n*

We will start with the case $n=3$, the cubic relations appear first time in this discussion. Then we will prove the general case by induction on n. It will be convenient to distinguish $R(z)$ for different dimension n. In this section, we will use the notation $R_n(z)$ for this purpose, which should not be confused with the notation $R_{21}(z)$, etc., used in the previous sections.

Let us restrict (3.15) and (3.16) to $E_{ij} \otimes E_{kl}$, $i, j, k, l \leqq 2$, then we get

$$R_2\left(\frac{z}{w}\right) J_1^{\pm}(z) J_2^{\pm}(w) = J_2^{\pm}(w) J_1^{\pm}(z) R_2\left(\frac{z}{w}\right), \tag{5.1}$$

$$R_2\left(\frac{z_+}{w_-}\right) J_1^{+}(z) J_2^{-}(w) = J_2^{-}(w) J_1^{+}(z) R_2\left(\frac{z_-}{w_+}\right), \tag{5.2}$$

where we denote

$$J_1 = J \otimes I, \quad J_2 = I \otimes J ,$$

$$J(z) = \begin{pmatrix} 1 & 0 \\ e_1^{\pm}(z) & 1 \end{pmatrix} \begin{pmatrix} k_1^{\pm}(z) & 0 \\ 0 & k_2^{\pm}(z) \end{pmatrix} \begin{pmatrix} 1 & f_1^{\pm}(z) \\ 0 & 1 \end{pmatrix} . \tag{5.3}$$

Thus we are exactly in the setting of $n = 2$ case. Similarly consider (4.4) and (4.5) and restrict them to $E_{ij} \otimes E_{kl}$, $2 \leqq i, j, k, l \leqq 3$, then

$$\tilde{J}_1^{\pm}(z)^{-1}(\tilde{J}_2^{\pm})^{-1} R_2\left(\frac{z}{w}\right) = R_2\left(\frac{z}{w}\right)(\tilde{J}_2(w))^{-1}(\tilde{J}_1^{\pm}(z))^{-1} , \tag{5.4}$$

$$(\tilde{J}_1^{+}(z))^{-1}(\tilde{J}_2^{-}(w))^{-1} R_2\left(\frac{z_+}{w_-}\right) = R_2\left(\frac{z_-}{w_+}\right)(\tilde{J}_1^{-}(w))^{-1}(\tilde{J}_1^{+}(z))^{-1} , \tag{5.5}$$

where we denote

$$\tilde{J}_1 = \tilde{J} \otimes 1, \quad \tilde{J}_2 = 1 \otimes \tilde{J} ,$$

$$\tilde{J}(z) = \begin{pmatrix} 1 & 0 \\ e_2^{\pm}(z) & 1 \end{pmatrix} \begin{pmatrix} k_2^{\pm}(z) & 0 \\ 0 & k_3^{\pm}(z) \end{pmatrix} \begin{pmatrix} 1 & f_2^{\pm}(z) \\ 0 & 1 \end{pmatrix} . \tag{5.6}$$

It is also the same as in $n = 2$ case. We only need to check the relations between $k_1^{\pm}(z)$, $f_1^{\pm}(z)$, $e_1^{\pm}(z)$ and $k_3^{\pm}(z)$, $f_2^{\pm}(z)$, $e_2^{\pm}(z)$, the other relations are immediate from above observation and the results for $n = 2$ case.

Let

$$\begin{aligned} L^{\pm}(z) &= \begin{pmatrix} 1 & & 0 \\ e_1^{\pm}(z) & 1 & \\ e_{3,1}^{\pm}(z) & e_2^{\pm}(z) & 1 \end{pmatrix} \begin{pmatrix} k_1^{\pm}(z) & & 0 \\ & k_2^{\pm}(z) & \\ 0 & & k_3^{\pm}(z) \end{pmatrix} \begin{pmatrix} 1 & f_1^{\pm}(z) & f_{1,3}^{\pm}(z) \\ & 1 & f_2^{\pm}(z) \\ 0 & & 1 \end{pmatrix} \\ &= \begin{pmatrix} k_1^{\pm}(z) & k_1^{\pm}(z) f_1^{\pm}(z) & k_1^{\pm}(z) f_{1,3}^{\pm}(z) \\ e_1^{\pm}(z) k_1^{\pm}(z) & * & * \\ e_{3,1}^{\pm}(z) k_1^{\pm}(z) & * & * \end{pmatrix} . \end{aligned} \tag{5.7}$$

Let

$$\begin{aligned} x^{\pm} &= k_3^{\pm}(w)^{-1}(-e_{3,1}^{\pm}(w) + e_2^{\pm}(w) e_1^{\pm}(w)) , \\ y^{\pm} &= (-f_{1,3}^{\pm}(w) + f_1^{\pm}(w) f_2^{\pm}(w)) k_3^{\pm}(w)^{-1} , \end{aligned} \tag{5.8}$$

then

$$L_1^{\pm}(w)^{-1} = \begin{pmatrix} * & * & y^{\pm} I \\ * & * & -f_2^{\pm}(w) k_3^{\pm}(w)^{-1} I \\ x^{\pm} I & -k_3^{\pm}(w) e_2^{\pm}(w) I & k_3^{\pm}(w)^{-1} I \end{pmatrix}, \tag{5.9}$$

From (3.15), (4.2), (4.3) and (4.5) we have

$$k_1^\pm(z)k_3^\pm(w) = k_3^\pm(w)k_1^\pm(z)\,, \tag{5.10}$$

$$k_3^\pm(z)f_1^\pm(w) = f_1^\pm(w)k_3^\pm(z)\,, \tag{5.11}$$

$$k_3^\pm(z)e_1^\pm(w) = e_1^\pm(w)k_3^\pm(z)\,, \tag{5.12}$$

$$k_1^\pm(z)f_2^\pm(w) = f_2^\pm(w)k_1^\pm(z)\,, \tag{5.13}$$

$$k_1^\pm(z)e_2^\pm(w) = e_2^\pm(w)k_1^\pm(z)\,, \tag{5.14}$$

$$f_1^\pm(z)e_2^\pm(w) = e_2^\pm(w)f_1^\pm(z)\,, \tag{5.15}$$

$$f_2^\pm(z)e_1^\pm(w) = e_1^\pm(w)f_2^\pm(z)\,, \tag{5.16}$$

$$\frac{z_\pm - w_\mp}{z_\pm q - w_\mp q^{-1}}\,k_3^\mp(w)^{-1}k_1^\pm(z) = \frac{z_\mp - w_\pm}{z_\mp q - w_\pm q^{-1}}\,k_1^\pm(z)k_3^\mp(w)^{-1}\,, \tag{5.17}$$

$$k_3^\pm(z)f_1^\pm(w) = f_1^\mp(w)k_3^\pm(z)\,, \tag{5.18}$$

$$k_3^\pm(z)e_1^\mp(w) = e_1^\mp(w)k_3^\pm(z)\,, \tag{5.19}$$

$$k_1^\pm(z)f_2^\mp(w) = f_2^\mp(w)k_1^\mp(z)\,, \tag{5.20}$$

$$k_1^\pm(z)e_2^\mp(w) = e_2^\mp(w)k_1^\pm(z)\,, \tag{5.21}$$

$$f_1^\pm(z)e_2^\mp(w) = e_2^\mp(w)f_1^\pm(z)\,, \tag{5.22}$$

$$f_2^\pm(z)e_1^\mp(w) = e_1^\mp(w)f_2^\pm(z)\,. \tag{5.23}$$

Next we need to check the relations between $f_1^\pm(z)$ and $f_2^\pm(z)$, and the relations between $e_1^\pm(z)$ and $e_2^\pm(z)$. From the formulae (5.8), (4.2), (4.3) and (4.5), we have

$$\begin{aligned}
&\frac{w(q-q^{-1})}{zq - wq^{-1}}\,x^\pm k_1^\pm(z) - k_3^\pm(w)^{-1}e_2^\pm(w)e_1^\pm(z)k_2^\pm(z) \\
&\qquad + k_3^\pm(w)^{-1}e_{3,1}^\pm(z)k_1^\pm(z)\,\frac{z(q-q^{-1})}{zq - wq^{-1}} \\
&\qquad = -\frac{z - w}{zq - wq^{-1}}\,e_1^\pm(z)k_1^\pm(z)k_3^\pm(w)^{-1}e_2^\pm(w)\,,
\end{aligned} \tag{5.24}$$

$$\begin{aligned}
&\frac{w_\mp(q-q^{-1})}{z_\pm q - w_\mp q^{-1}}\,x^\mp k_1^\mp(z) - k_3^\mp(w)^{-1}e_2^\mp(w)e_1^\pm(z)k_1^\pm(z) \\
&\qquad + k_3^\mp(w)^{-1}e_{3,1}^\pm(z)k_1^\pm(z)\,\frac{z_\pm(q-q^{-1})}{z_\pm q - w_\mp q^{-1}} \\
&\qquad = -\frac{z_\mp - w_\pm}{z_\mp q - w_\pm q^{-1}}\,e_1^\pm(z)k_1^\pm(z)k_3^\mp(w)^{-1}e_2^\mp(w)\,.
\end{aligned} \tag{5.25}$$

We multiply (5.24) by $k_3^{\pm}(w)$ on the right side and by $k_1^{\mp}(z)^{-1}$ on the left side, and (5.25) by $k_3^{\mp}(w)$ on the right, and by $k_1^{\mp}(z)$ on the left. Then we get

$$e_1^{\mp}(z_{\mp})e_2^{\pm}(w_{\mp}) - e_1^{\pm}(z_{\mp})e_2^{\mp}(w_{\pm}) = \frac{zq - wq^{-1}}{z - w} e_2^{\mp}(w_{\mp})e_1^{\pm}(z_{\mp}) - \frac{zq - wq^{-1}}{z - w} e_2^{\mp}(w_{\pm})e_1^{\pm}(z_{\mp}) \,, \tag{5.26}$$

where the terms $\frac{zq - wq^{-1}}{z - w}$ are expanded in different directions as specified in Definition 3.2. From this, we get

$$(z - w)X_1^{+}(z)X_2^{+}(w) = (zq - wq^{-1})X_2^{+}(w)X_1^{+}(z) \,, \tag{5.27}$$

and we can use the same relations to get

$$\{X_1^{-}(z_1)X_1^{-}(z_2)X_2^{-}(w) - (q + q^{-1})X_1^{-}(z_1)X_2^{-}(w)X_1^{-}(z_2) + X_2^{-}(w)X_1^{-}(z_1)X_1^{-}(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0 \,, \tag{5.28}$$

$$\{X_2^{-}(z_1)X_2^{-}(z_2)X_1^{-}(w) - (q + q^{-1})X_2^{-}(z_1)X_1^{-}(w)X_2^{-}(z_2) + X_1^{-}(w)X_2^{-}(z_1)X_2^{-}(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0 \,, \tag{5.29}$$

As for $f_1^{\pm}(z)$, $f_2^{\pm}(z)$, we have

$$\begin{aligned} &-\frac{z - w}{zq - wq^{-1}} f_2^{\pm}(w)k_3^{\pm}(w)^{-1}k_1^{\pm}(z)f_1^{\pm}(z) \\ &= \frac{z(q - q^{-1})}{zq - wq^{-1}} k_1^{\pm}(z)y^{\pm} - k_1^{\pm}(z)f_1^{\pm}(z)f_2^{\pm}(w)k_3^{\pm}(w)^{-1} \\ &+ \frac{w(q - q^{-1})}{zq - wq^{-1}} k_1^{\pm}(z)f_{1,3}^{\pm}(z)k_3^{\pm}(w)^{-1} \,, \end{aligned} \tag{5.30}$$

$$\begin{aligned} &-\frac{z_{\pm} - w_{\mp}}{z_{\pm}q - w_{\mp}q^{-1}} f_1^{\mp}(w)k_3^{\mp}(w)^{-1}k_1^{\pm}(z)f_1^{\pm}(z) \\ &= \frac{z_{\mp}(q - q^{-1})}{z_{\mp}q - w_{\pm}q^{-1}} k_1^{\pm}(z)y^{\mp} - k_1^{\pm}(z)f_1^{\pm}(z)f_2^{\pm}(w)k_3^{\mp}(w)^{-1} \\ &+ \frac{w_{\pm}(q - q^{-1})}{z_{\mp}q - w_{\pm}q^{-1}} k_1^{\pm}(z)f_{1,3}^{\pm}(z)k_3^{\mp}(w)^{-1} \,, \end{aligned} \tag{5.31}$$

then we get

$$f_1^{\pm}(z_{\pm})f_2^{\pm}(w_{\pm}) - f_1^{\pm}(z_{\pm})f_2^{\mp}(w_{\mp}) = \frac{z - w}{zq - wq^{-1}} f_2^{\pm}(w_{\pm})f_1^{\pm}(z_{\pm}) - \frac{z - w}{zq - wq^{-1}} f_2^{\mp}(w_{\mp})f_1^{\pm}(z_{\pm}) \,. \tag{5.32}$$

Here also we should be careful with the expansion of coefficients of the equality.

From this, we get that

$$(zq - wq^{-1})X_1^{-}(z)X_2^{-}(w) = X_2^{-}(w)X_1^{-}(z)(z - w) \tag{5.33}$$

and

$$\{X_1^+(z_1)X_1^+(z_2)X_2^+(w) - (q+q^{-1})X_1^+(z_1)X_2^+(w)X_1^+(z_2) + X_2^+(w)X_1^+(z_1)X_1^+(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0\,, \quad (5.34)$$

$$\{X_2^+(z_1)X_2^+(z_2)X_1^+(w) - (q+q^{-1})X_2^+(z_1)X_1^+(w)X_2^+(z_2) + X_1^+(w)X_2^+(z_1)X_2^+(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0\,. \quad (5.35)$$

Thus we proved that Φ is a homomorphism for $n=3$. We note that in the process of proof we have seen that $f_{1,3}^{\pm}(z)$, and $e_{3,1}^{\pm}(z)$ are generated by $k_1^{\pm}(z)$, $k_2^{\pm}(z)$, $k_3^{\pm}(z)$ $f_1^{\pm}(z)$, $e_1^{\pm}(z)$, $f_2^{\pm}(z)$ and $e_2^{\pm}(z)$, which shows that Φ is surjective.

Now we proceed to the proof of surjectivity for general n.

Just as in $n=3$ case, we first restrict (3.15) and (3.16) to $E_{ij} \otimes E_{kl}$, $1 \leqq i, j, k, l \leqq n-1$, then get

$$R_{n-1}\left(\frac{z}{w}\right) J_1^{\pm}(z) J_2^{\pm}(w) = J_2^{\pm}(w) J_1^{\pm}(z) R_{n-1}\left(\frac{z}{w}\right), \quad (5.36)$$

$$R_{n-1}\left(\frac{z_+}{w_-}\right) J_1^{+}(z) J_2^{-}(w) = J_2^{-}(w) J_1^{+}(z) R_{n-1}\left(\frac{z_-}{w_+}\right), \quad (5.37)$$

$$J^{\pm}(z) = \begin{pmatrix} 1 & & 0 \\ e_1^{\pm}(z) & \ddots & \\ & \ddots & \\ & e_{n-2}^{\pm}(z) & 1 \end{pmatrix} \begin{pmatrix} k_1^{\pm}(z) & & 0 \\ & \ddots & \\ & & \ddots \\ 0 & & k_{n-1}^{\pm}(z) \end{pmatrix} \begin{pmatrix} 1 & e_1^{\pm}(z) & \\ & \ddots & \ddots \\ & & e_{n-2}^{\pm}(z) \\ 0 & & 1 \end{pmatrix}. \quad (5.38)$$

Similarly restricting (4.6) and (4.7) to $E_{ij} \otimes E_{kl}$, $2 \leqq i, j, l \neq n$, then

$$R_{n-1}\left(\frac{z}{w}\right) (\tilde{J}_2^{\pm}(w))^{-1} (\tilde{J}_1^{\pm}(z))^{-1} = (\tilde{J}_1^{\pm}(z))^{-1} (\tilde{J}_2^{\pm}(w))^{-1} R_{n-1}\left(\frac{z}{w}\right), \quad (5.39)$$

$$R_{n-1}\left(\frac{z_-}{w_+}\right) (\tilde{J}_2^{-}(w))^{-1} (\tilde{J}_1^{+}(z))^{-1} = (\tilde{J}_1^{+}(z))^{-1} (\tilde{J}_2^{-}(w))^{-1} R_{n-1}\left(\frac{z_+}{w_-}\right), \quad (5.40)$$

$$\tilde{J}^{\pm}(z) = \begin{pmatrix} 1 & & 0 \\ e_2^{\pm}(z) & \ddots & \\ \ddots & \ddots & \\ & & e_{n-1}^{\pm}(z) \end{pmatrix} \begin{pmatrix} k_2^{\pm}(z) & & 0 \\ & \ddots & \\ & & \ddots \\ 0 & & k_n^{\pm}(z) \end{pmatrix} \begin{pmatrix} 1 & f_2^{\pm}(z) & & \\ & \ddots & \ddots & \\ & & \ddots & f_{n-1}^{\pm}(z) \\ 0 & & & 1 \end{pmatrix}. \quad (5.41)$$

By induction, we know all the commutator relations we need exccept those between $f_1^{\pm}(z)$, $k_1^{\pm}(z)$, $e_1^{\pm}(z)$ and $f_n^{\pm}(z)$, $k_n^{\pm}(z)$, $e_n^{\pm}(z)$. We now use the formulae (4.2), (4.3) and (4.5). First we write down $L^{\pm}(z)$,

$$L_1^{\pm}(z) = \begin{pmatrix} k_1^{\pm}(z) & k_1^{\pm}(z) f_1^{\pm}(z) & \vdots & \vdots & \cdots \\ e_1^{\pm}(z) k_1^{\pm}(z) & \vdots & \vdots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \quad (5.42)$$

and

$$L_1^{\pm}(z)^{-1} = \begin{pmatrix} \cdots & \vdots & \vdots & \vdots \\ \cdots & \vdots & \vdots & -f_{n-1}^{\pm}(z)k_n^{\pm}(z)^{-1} \\ \cdots & \vdots & -k_{n-1}^{\pm}(z)^{-1}e_1^{\pm}(z) & k_n^{\pm}(z)^{-1} \end{pmatrix}. \tag{5.43}$$

Then we obtain

$$k_1^{\pm}(z)k_n^{\pm}(w) = k_n^{\pm}(w)k_1^{\pm}(z)\,, \tag{5.44}$$

$$k_1^{\pm}(z)e_{n-1}^{\pm}(w) = e_{n-1}^{\pm}(w)k_1^{\pm}(z)\,, \tag{5.45}$$

$$k_1^{\pm}(z)f_{n-1}^{\pm}(w) = f_{n-1}^{\pm}(w)k_1^{\pm}(z)\,, \tag{5.46}$$

$$k_n^{\pm}(z)f_1^{\pm}(w) = f_1^{\pm}(w)k_n^{\pm}(z)\,, \tag{5.47}$$

$$k_n^{\pm}(z)e_1^{\pm}(w) = e_1^{\pm}(w)k_n^{\pm}(z)\,, \tag{5.48}$$

$$\frac{z_{\pm} - w_{\mp}}{z_{\pm}q - w_{\mp}q^{-1}}\,k_n^{\mp}(w)^{-1}k_1^{\pm}(z) = k_1^{\pm}(z)k_n^{\mp}(w)^{-1}\,\frac{z_{\mp} - w_{\pm}}{z_{\mp}q - w_{\pm}q^{-1}}\,, \tag{5.49}$$

$$k_n^{\pm}(z)f_1^{\mp}(w) = f_1^{\mp}(w)k_n^{\pm}(z)\,, \tag{5.50}$$

$$k_n^{\pm}(z)e_1^{\mp}(w) = e_1^{\mp}(w)k_n^{\pm}(z)\,, \tag{5.51}$$

$$k_1^{\pm}(z)f_{n-1}^{\mp}(w) = f_{n-1}^{\mp}(w)k_1^{\pm}(z)\,, \tag{5.52}$$

$$k_1^{\pm}(z)e_{n-1}^{\mp}(w) = e_{n-1}^{\mp}(w)k_1^{\pm}(z)\,, \tag{5.53}$$

$$f_1^{\pm}(z)f_{n-1}^{\pm}(w) = f_{n-1}^{\pm}(w)f_1^{\pm}(z)\,, \tag{5.54}$$

$$f_1^{\pm}(z)e_{n-1}^{\pm}(w) = e_{n-1}^{\pm}(w)f_1^{\pm}(z)\,, \tag{5.55}$$

$$f_1^{\pm}(z)f_{n-1}^{\mp}(w) = f_{n-1}^{\mp}(w)f_1^{\pm}(z)\,, \tag{5.56}$$

$$f_1^{\pm}(z)e_{n-1}^{\pm}(w) = e_{n-1}^{\mp}(w)f_1^{\pm}(z)\,, \tag{5.57}$$

$$e_1^{\pm}(z)f_{n-1}^{\pm}(w) = f_{n-1}^{\pm}(w)e_1^{\pm}(z)\,, \tag{5.58}$$

$$e_1^{\pm}(z)f_{n-1}^{\mp}(w) = f_{n-1}^{\mp}(w)e_1^{\pm}(z)\,, \tag{5.59}$$

$$e_1^{\pm}(z)e_{n-1}^{\pm}(w) = e_{n-1}^{\pm}(w)e_1^{\pm}(z)\,, \tag{5.60}$$

$$e_1^{\pm}(z)e_{n-1}^{\mp}(w) = e_{n-1}^{\mp}(w)e_1^{\pm}(z)\,. \tag{5.61}$$

To prove the surjectivity of the map Φ, we only need to show that $e_{n,1}^{\pm}(z)$ and $f_{1,n}^{\pm}(z)$ are generated by $k_i^{\pm}(z)$, $e_i^{\pm}(z)$ and $f_i^{\pm}(z)$. Since all other elements $e_{i,j}^{\pm}(z)$ and $f_{i,j}^{\pm}(z)$ are generated by $k_i^{\pm}(z)$, $e_i^{\pm}(z)$ and $f_i^{\pm}(z)$ by the induction. From (4.2) and (4.3), we can get the relations between $e_{n-1,1}^{\pm}$ and $e_{n,n-1}^{\pm}$, and the relations between $f_{1,n-1}^{\pm}(z)$ and $f_{n-1,n}^{\pm}(z)$, which also contain $e_{n,1}^{\pm}(z)$ and $f_{1,n}^{\pm}(z)$. These formulae are similar to (5.24), (5.25), (5.30) and (5.31) in the case when $n = 3$. They imply that $f_{1,n}^{\pm}(z)$ and $e_{n,1}^{\pm}(z)$ are generated by $k_i^{\pm}(z)$ $f_i^{\pm}(z)$ and $e_i^{\pm}(z)$, therefore the algebra $U(\tilde{R})$ is generated by $k_i^{\pm}(z)$, $f_i^{\pm}(z)$, $e_i^{\pm}(z)$. Thus we proved that Φ is a surjective map.

Now, we proceed to the last step to prove the injectivity of Φ, which essentially is the same as the proof for the case of $U_q(\mathrm{gl}(n))$.

From the theory of Lusztig [L], we know that the highest weight representation of $\widehat{\mathfrak{sl}(n)}$ admits a quantum deformation. Let $V^q_{\lambda,c}$ be a highest weight representation of $U_q(\widehat{\mathfrak{sl}(n)})$ with central extension c and let $V\otimes\mathbb{C}[z,z^{-1}]=\mathbb{C}^n\otimes\mathbb{C}[z,z^{-1}]$ be the evaluation representation of $U_q(\widehat{\mathfrak{sl}(n)})$ on the fundamental representation of $U_q(\mathfrak{sl}(n))$ [J3]. The algebra $U_q(\widehat{\mathfrak{sl}(n)})$ possesses a universal R matrix $\mathfrak{R}(z)$ [D2].

Let

$$L^{+V^q_{\lambda,c}}=(\pi_{V^q_{\lambda,c}}\otimes\pi_V)\mathfrak{R}_{21}(zq^{\frac{c}{2}}),\quad L^{-V^q_{\lambda,c}}=(\pi_{V^q_{\lambda,c}}\otimes\pi_V)\mathfrak{R}(z^{-1}q^{\frac{-c}{2}}),\tag{5.62}$$

and

$$R^{VV}(z)=(\pi_V\otimes\pi_V)\mathfrak{R}(z).\tag{5.63}$$

We know $R^{VV}(z)=f(z)R(z)$, where $R(z)$ is defined in (3.7) and $f(z)$ is a function of z [FR].

Set

$$L_1^{\pm V^q_{\lambda,c}}=L^{\pm V^q_{\lambda,c}}\otimes I_V,\quad L_2^{\pm V^q_{\lambda,c}}=P_{23}L^{\pm V^q_{\lambda,c}}\otimes I_VP_{23},\tag{5.64}$$

$$R^{V^q_{\lambda,c}VV}(z)=I_{V^q_{\lambda,c}}\otimes R^{VV}(z),\tag{5.65}$$

where P_{23} is the permutation opeator on the last two components.

The Yang–Baxter equation implies

$$R^{V^q_{\lambda,c}VV}\left(\frac{z}{w}\right)L_1^{\pm V^q_{\lambda,c}}(z)L_2^{\pm V^q_{\lambda,c}}(w)=L_2^{\pm V^q_{\lambda,c}}(w)L_1^{\pm V^q_{\lambda,c}}(z)R^{V^q_{\lambda,c}VV}\left(\frac{z}{w}\right),\tag{5.66}$$

$$R^{V^q_{\lambda,c}VV}\left(\frac{z_+}{w_-}\right)L_1^{+V^q_{\lambda,c}}(z)L_2^{-V^q_{\lambda,c}}(w)=L_2^{-V^q_{\lambda,c}}(w)L_1^{+V^q_{\lambda,c}}(z)R^{V^q_{\lambda,c}VV}\left(\frac{z_-}{w_+}\right).\tag{5.67}$$

In the other hand, we can define a Heisenberg algebra generated by $h(n)$, $n\in\mathbb{Z}\backslash 0$ satisfying the following relation:

$$f\left(\frac{z_+}{w_-}\right)^{-1}q^{\sum_{m\in\mathbb{Z}_+}h(m)z^m}q^{\sum_{n\in\mathbb{Z}_+}h(-n)w^{-n}}=f\left(\frac{z_-}{w_+}\right)^{-1}q^{\sum_{n\in\mathbb{Z}_+}h(-n)w^{-n}}q^{\sum_{m\in\mathbb{Z}_+}h(m)z^m}.\tag{5.68}$$

We define a Fock space representation of our Heisenberg algebra, which we denote by V_c. It is generated by creation operators $h(n)$, $n\in\mathbb{Z}_+$, applied to the vacuum vector $1\in V_c$; the annihilation operators $h(-n)$, $n\in\mathbb{Z}_+$, by definition annihilate the vacuum vector.

Note that when $c=0$ the Heisenberg algebra degenerates into an infinite-dimensional abelian algebra and we can choose for V_c its trival representation. Now let us look at the representation on $V^q_{\lambda,c}\otimes V_c$. We set

$$L^{0+V^q_{\lambda,c}}(z)=r_0q^{\sum_{m\in\mathbb{Z}_+}h(m)z^m}\otimes L^{+V^q_{\lambda,c}}(z),\tag{5.69}$$

$$L^{0-V^q_{\lambda,c}}(w)=r_0^{-1}q^{\sum_{m\in\mathbb{Z}_+}h(-n)w^{-n}}\otimes L^{-V^q_{\lambda,c}}(w).\tag{5.70}$$

Here r_0 is in $\mathbb{C}\backslash 0$. Then we obtain

$$R\left(\frac{z}{w}\right)L_1^{0\pm V^q_{\lambda,c}}(z)L_2^{0\pm V^q_{\lambda,c}}(w)=L_2^{0\pm V^q_{\lambda,c}}(w)L_1^{0\pm V^q_{\lambda,c}}(z)R\left(\frac{z}{w}\right),\tag{5.71}$$

$$R\left(\frac{z_+}{w_-}\right)L_1^{0+V^q_{\lambda,c}}(z)L_2^{0-V^q_{\lambda,c}}(w)=L_2^{0-V^q_{\lambda,c}}(w)L_1^{0+V^q_{\lambda,c}}(z)R\left(\frac{z_-}{w_+}\right),\tag{5.72}$$

Thus we get a representation of $U(\tilde{R})$ with the action of the central elment $\prod l_{ii}^+$ by any constant. Via Φ it also gives a representation of $U_q(\widehat{\mathfrak{gl}(n)})$ with a highest weight Λ and central extension c, where Λ as in Sect. 3 denotes a pair (λ, γ). This can be seen by the first term of $\mathfrak{R}(z)$[RS]. Therefore the highest weight representation of $U_q(\widehat{\mathfrak{gl}(n)})$ can be pulled through $U(\tilde{R})$. It implies that the kernel of Φ must be inside $I^q_{\Lambda,c}$, the kernel of the representation of $U_q(\widehat{\mathfrak{gl}(n)})$ on the highest weight module $V^q_{\Lambda,c}$.

From [D4], we know $U_q(\widehat{\mathfrak{gl}(n)})/(q-1)U_q(\widehat{\mathfrak{gl}(n)}) \cong U(\widehat{\mathfrak{gl}(n)})$, so the kernel of the map Φ is inside $(q-1)U_q(\widehat{\mathfrak{gl}(n)})$. Using the same argument as for $U_q(\mathfrak{gl}(n))$ in Sect. 3 we can show that

$$\bigcap_{\Lambda,c} I^q_{\Lambda,c} = 0\,, \tag{5.73}$$

which follows the fact [GK] that

$$\bigcap_{\Lambda,c} I_{\Lambda,c} = 0\,. \tag{5.74}$$

Here $I_{\Lambda,c}$ is the kernel of the representation $U(\widehat{\mathfrak{gl}(n)})$ on the highest weight module with the highest weight Λ and central extension c. Thus we prove the injectivity of Φ.

Acknowledgements. We would like to thank V.G. Drinfeld and N.Yu. Reshetikhin for informative and stimulating discussions. I.B. Frenkel acknowledges the support of NSF Grant DMS-8906772 and the Guggenheim Foundation Fellowship.

References

[ABE] Abada, A., Bougourzi, H., El Gradechi, M.A.: Deformation of the Wakimoto construction. CRM-1829, 1992

[D1] Drinfeld, V.G.: Hopf algebras and the Quantum Yang–Baxter equation. Sov. Math. Doklady **32**, 254–258 (1985)

[D2] Drinfeld, V.G.: Quantum groups. Proc. ICM-86 (Berkeley), Vol. 1, New York: Academic Press 1986, pp. 789–820

[D3] Drinfeld, V.G.: New realization of Yangian and quantum affine algebra. Sov. Math. Doklady **36**, 212–216 (1988)

[D4] Drinfeld, V.G.: Quasi-Hopf algebras. Leningrad Math. J. **1**, 1419–1457 (1990)

[D5] Drinfeld, V.G.: Private communication

[DF] Ding, J. Frenkel, I.B.: Spinor and oscillator representations of quantum algebras in invariant form. Preprint, Yale University

[DF2] Ding, J., Frenkel, I.B.: In preparation

[Di] Dixmier, J.: Enveloping algebras. Amsterdam: North-Holland Publishing Company, 1977

[FRT1] Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: Quantization of Lie groups and Lie algebras. Algebra and Analysis (Russian) **1.1**, 118–206 (1989)

[FRT2] Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: Quantization of Lie groups and Lie algebras, Yang-Baxter equation in Integrable Systems. Advanced Series in Mathematical Physics, Vol. **10**, Singapore: World Scientific 1989, pp. 299–309

[FeF] Feigin, B.L., Frenkel, E.V.: The family of representations of affine Lie algebras. Ups. Mat. Nauk **43**, 227–228 (1988)

[FF] Feingold, A., Frenkel, I.B.: Classical affine algebras. Adv. Math. **56**, 117–172 (1985)

[F] Frenkel, I.B.: Spinor representation of affine Lie algebras. Proc. Natl. Acad. Sci. USA **77**, 6303–6306 (1980)

[FK] Frenkel, I.B., Kac, V.G.: Baqsic representations of affine Lie algebras and dual resonance model. Invent. Math. **62**, 23–66 (1980)

[FJ] Frenkel, I.B., Jing, N.: Vertex representation of quantum affine algebras. Proc. Natl. Acad. Sci. USA **85**, 9373–9377 (1988)

[G] Garland, H.: The arithmetic theory of loop groups. Publ. Math. IHES **52**, 5–136 (1980)

[GK] Gabber, O., Kac, V.G.: On defining relations of certain infinite-dimensional Lie algebras. Bull. Am. Math. Soc. **5**, 185–189 (1981)

[H] Hayashi, T.: Q-analogues of Clifford and Weyl algebras – Spinor and Oscillator representations of quantum enveloping algebra. Commun. Math. Phys. **127**, 129–144 (1990)

[J1] Jimbo, M.: A q-difference analogue of $U(g)$ and the Yang–Baxter equation. Lett. Math. Phys. **10**, 63–69 (1985)

[J2] Jimbo, M.: Quantum R-matrix for the generalized Toda system. Commun. Math. Phys. **102**, 537–548 (1986)

[J3] Jimbo, M.: A q-analog of $U(\mathrm{gl}(N+1))$, Hecke algebras, and the Yang–Baxter equation. Lett. Math. Phys. **11**, 247–252 (1986)

[KP] Kac, V.G., Peterson, D.H.: Spinor and wedge representations of infinite-dimensional Lie algebras and groups. Proc. Natl. Acad. Sci. USA **78**, 3308–3312 (1981)

[L] Lusztig, G.: Quantum deformations of certain simple modules over enveloping algebras. Adv. Math. **70**, 237–249 (1988)

[M] Matsuo, A.: Free field representation of quantum affine algebra $U_q(\widehat{\mathfrak{sl}})$. Preprint, 1992

[Re] Reshetikhin, N.Yu.: Private communication

[R1] Rosso, M.: Finite dimensional representations of the quantum analogue of the universal enveloping algebgra of a complex simple Lie algebra. Commun. Math. Phys. **117**, 558–593 (1988)

[R2] Rosso, M.: An analogue of P.B.W. theorem and the universal R-matrix for $U_h\mathfrak{sl}(N+1)$. Commun. Math. Phys. **124**, 307–318 (1989)

[RS] Reshetikhin, N.Yu., Semenov-Tian-Shansky, M.A.: Central extensions of quantum current groups. Lett. Math. Phys. **19**, 133–142 (1990)

[S] Segal, G.: Unitary representation of some infinite dimensional groups. Commun. Math. Phys. **80**, 301–342 (1981)

[TK] Tsuchiya, A., Kanie, Y.,: Vertex operators in conformal field theory on P^1 and monodromy representation of braid group. Adv. Stud. Pure Math. **16**, 297–372 (1988)

[W] Wakimoto, M.: Fock representations of affine Lie algebra $A_1^{(1)}$. Commun. Math. Phys. **104**, 605–609 (1986)

Communicated by N.Yu. Reshetikhin