Form Factors, Deformed Knizhnik–Zamolodchikov Equations and Finite-Gap Integration

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Abstract. We study the limit of asymptotically free massive integrable models in which the algebra of nonlocal charges turns into affine algebra. The form factors of fields in that limit are described by KZ equations on level 0. We show the limit to be connected with finite-gap integration of classical integrable equations.

1. Motivations

The nonlocal symmetries of integrable models of quantum field theory in two dimensions were first studied several years ago [18]. The reason for that is in attempts to understand the quantization of asymptotically free models. Being almost forgotten for some time the nonlocal symmetries returned to the field rather indirectly, namely, through the finite-dimensional quantum group symmetries of CFT. Now it is understood that the integrability of massive models is closely connected with possessing infinite-dimensional algebra of nonlocal symmetries [1, 2, 10, 17, 23]. The local integrals constitute a center of it. The algebra of nonlocal symmetries is always a Hopf algebra, particles transform under its finite-dimensional representations while quasilocal fields constitute infinite-dimensional multiplets with highest vectors corresponding to the local fields. The S-matrix is nothing but the universal R-matrix specified onto finite-dimensional representations while the braiding of the multiplets of quasilocal fields is described by the universal R-matrix specified onto the tensor product of two Verma modules. Moreover, it was shown [23] that the form factors which put together particles and fields can be considered as solutions of deformed Knizhnik–Zamolodchikov equations [11] which is quite natural since these equations are used to relate finite-dimensional representations of deformed loop algebras with infinite-dimensional highest weight representations.

A question which should be asked is the following. Suppose we have an integrable model with symmetry under a certain deformation of the loop algebra. Then what is the meaning of the "classical limit" which moves the deformed algebra into an undeformed one? The answer to this question is not trivial as we shall see. For many models this limit does not look like it makes much sense. The point is

that the limit is related not to the rescaling of space coordinates (or momenta) as the conformal limit is, but rather to rescaling of rapidities of particles which are logarithms of momenta. In a sense the limit in question is even opposite to the conformal limit in which we throw away all the logarithms preserving powers; here we preserve logarithms, but throw away powers considering the asymptotic expansions. However, the limit seems to be reasonable for asymptotically free models. So, we return to the same idea as years ago: to use nonlocal symmetries in asymptotically free theories, certainly we return to this idea with new experience. It should be said also that there exist certain problems with correspondence between local integrals of motion for classical and quantum cases in the theory of asymptotically free models. The reason for that is very fundamental, that is why our consideration will not be a Hamiltonian one.

Particular (and, probably, personal) interest in this business is the following. We have very nice formulae for final physical quantities in both the classical and quantum theory of integrable models. However the existing way of quantization [8] (which has clear Hamiltonian meaning) does not allow to proceed directly from one to another, it starts with beautiful things (R-matrix, algebraic Bethe ansatz), but then leads through the usual jungle with renormalizations and all that. We would like to have direct way even if it has less clear meaning. Similar ideas were applied to the study of correlation functions of vertex models of statistical mechanics in recent works from the Kyoto school [14].

Let us consider a particular asymptotically free model. Namely, let it be the $su(2)$ chiral Gross–Neveu model. The spectrum of the model contains one two-component particle. The two-particle S-matrix is given by [27]:

$$S_{1,2}(\beta_1 - \beta_2) = \frac{\Gamma\left(-\dfrac{\beta_1 - \beta_2}{2\pi i}\right)\Gamma\left(\dfrac{1}{2} + \dfrac{\beta_1 - \beta_2}{2\pi i}\right)}{\Gamma\left(\dfrac{\beta_1 - \beta_2}{2\pi i}\right)\Gamma\left(\dfrac{1}{2} - \dfrac{\beta_1 - \beta_2}{2\pi i}\right)}\left[\frac{(\beta_1 - \beta_2) - \pi i P_{1,2}}{\beta_1 - \beta_2 - \pi i}\right],$$

where $P_{1,2}$ is a permutation. If we are interested in local operators which transform under the integer-spin representation of the global $SL(2)$ isotopic group all of them can be obtained as descendents with respect to different conservation laws of one nonlocal operator called $\mathfrak{A}(x_0, x_1)$. The connection with important local operators will be explained later. The operator $\mathfrak{A}(x_0, x_1)$ is scalar with respect to both isotopic and Lorenz transformations. In the form factor approach [25] it can be presented as follows:

$$\mathfrak{A}(x_0, x_1) = \sum_{m,n=0}^{\infty} \frac{1}{m;n!}$$
$$\times \int_{-\infty}^{\infty} d\alpha_1 \ldots \int_{-\infty}^{\infty} d\alpha_m \int_{-\infty}^{\infty} d\beta_1 \ldots \int_{-\infty}^{\infty} d\beta_n \, Z_{\varepsilon_1}^*(\alpha_1) \cdots Z_{\varepsilon_m}^*(\alpha_m)$$
$$\times F(\alpha_m, \ldots, \alpha_1 | \beta_1, \ldots, \beta_n)_{\delta_1, \ldots, \delta_n}^{\varepsilon_m, \ldots, \varepsilon_1} \, Z^{\delta_n}(\beta_n) \cdots, Z^{\delta_1}\beta_1)$$
$$\times \exp\left(\sum ip^{\mu}(\alpha_i)x_{\mu} - \sum ip^{\mu}(\beta_i)x_{\mu}\right), \qquad (1.1)$$

where Z^*, Z are Zamolodchikov–Faddeev creation-annihilation operators of particles, $p_{\mu}(\alpha)$ is the one-particle energy-momentum $(p_{\mu}(\alpha) = M(\exp(\alpha) + (-1)^{\mu}\exp(\alpha)))$, ε, $\delta = \pm$ are isotopic indices. F are form factors. They satisfy the

crossing symmetry requirement:

$$F(\alpha_m, \ldots, \alpha_1 | \beta_1, \ldots, \beta_n)_{\delta_1, \ldots, \delta_n}^{\varepsilon_m, \ldots, \varepsilon_1} = c_{\delta_1, \delta_1'} \cdots c_{\delta_n, \delta_n'}$$

$$\times F(\alpha_m, \ldots, \alpha_1, \beta_1 + \pi i - i0; \ldots, \beta_n + \pi i - i0)^{\varepsilon_m, \ldots, \varepsilon_1, \delta_1', \ldots, \delta_n'} \quad c = i\sigma^2 .$$

It should be mentioned that for the operator $\mathfrak{A}$ the two particle form factor has a double pole at $\beta_2 = \beta_1 + \pi i$, the corresponding singular part being normalized as $(\beta_2 - \beta_1 - \pi i)^{-2} c^{\varepsilon_1, \varepsilon_2}$.

The model allows abelian and non-abelian symmetries. The first are the local conservation laws with all possible odd spins. The one particle eigenvalues of the local integrals $I_s(s = 2m + 1, m = -\infty, \ldots, \infty)$ are equal to $\exp(s\alpha)$ the spectrum of them is additive which can be expressed in a more formal way by the following comultiplication formula:

$$\Delta(I_s) = I_s \otimes 1 + 1 \otimes I_s .$$

The nonlocal symmetries together with $SL(2)$-charges constitute the Yangian (Y) [1]. The generators of Y are J_0^a, J_1^a, $a = 1,2,3$ whose action onto one particle is described by σ^a, $\alpha\sigma^a$ respectively and the comultiplication is given by

$$\Delta(J_0^a) = J_0^a \otimes 1 + 1 \otimes J_0^a ,$$

$$\Delta(J_1^a) = J_1^a \otimes 1 + 1 \otimes J_1^a + \pi i f^{abc} J_0^b \otimes J_0^c . \tag{1.2}$$

Every operator acting in the space of states (H_p) which is the Fock space of particles has descendents with respect to both local and non-local symmetries defined through adjoint action. One can think of the space H_p as of

$$H_p = \bigoplus_{n=1}^{\infty} \int_{\beta_1 < \cdots < \beta_{2n}} V_{\beta_1} \otimes \cdots \otimes V_{\beta_{2n}} , \tag{1.3}$$

where V_{β_i} is the space of representation ρ_{β_i}. The operators $Z^*(\beta)$, $Z(\beta)$ either add or remove one space V_β. In particular the energy-momentum tensor and currents are the following descendents of the operator $\mathfrak{A}$:

$$T_{\mu,\nu}(x_0, x_1) = \varepsilon_{\mu,\mu'} \varepsilon_{\nu,\nu'} P_{\mu'} P_{\nu'} (\mathfrak{A}(x_0, x_1)), \quad j_\mu^a(x_0, x_1) = \varepsilon_{\mu,\mu'} P_{\mu'} J_1^a (\mathfrak{A}(x_0, x_1)) ,$$

where $P_0 = I_1 + I_{-1}$, $P_1 = I_1 - I_{-1}$. There are more general relations. For example the density of the local conservation law $I_s(d_s(x_0, x_1))$ is obtained in the following way:

$$d_s(x_0, x_1) = P_0 I_s(\mathfrak{A}(x_0, x_1)) .$$

The last relation is very much in common with the classical relation identifying the densities of local conservation laws with the derivatives of logarithm of τ-function. So, it is reasonable to think of the operator $\mathfrak{A}(x_0, x_1)$ as of the quantum analog of $\log(\tau)$. Certainly, this is a formal analogy and in order to make it more instructive we have to develop the relation with the classical τ-function. This is the main goal of the present paper. For the reasons which will be explained later the connection in question can hardly be achieved in Hamiltonian formalism, so, our strategy will be to find a "good" formula in classics in which all the objects involved in quantum formula (the operator $\mathfrak{A}$, the form factors, Zamolodchikov–Faddeev operators) will have their classical analogs. This is what we are going to do in that paper. But

before proceeding in that direction we have to remind certain facts about the symmetries of the quantum model.

2. The Yangian Symmetry

In this section the brief account of the recent developments concerning the dynamical symmetries of integrable models will be presented.

As it has been already mentioned the model under consideration allows nonlocal conservation laws J_1^a which together with isotopic charges J_0^a constitute the infinite dimensional algebra called Yangian [5]. It is explained in [17, 23] that for the application to the description of local and quasilocal operators in the theory one has to add one more generator (J_{-1}^a) to the algebra. The full algebra generated by J_0^a, J_1^a, J_{-1}^a is called the double of Yangian being denoted by $D(Y)$. One can think of $D(Y)$ as a deformation of the affine algebra $\widehat{sl}(2)$ with natural identification of the generators. This will be explained more precisely later. The Yangian Y is a subalgebra of $D(Y)$ which acting onto the fields in the theory creates descendents. Introducing the second half of the algebra was originally motivated by the problem of description of the commutation relations among these descendents [17]. The commutation relations are described as follows. Consider a local operator $\varphi(x_0, x_1)$ which can be for the simplicity taken as invariant under the isotopic algebra. Now create all posible descendents of the operator acting on it by an arbitrary number of the operators J_1^a:

$$\varphi(x_0, x_1)^{a_1, \ldots, a_n} = J_1^{a_1} \ldots J_1^{a_n}(\varphi(x_0, x_1)) .$$

We can combine these operators into Φ considering $a_1, \ldots, a_n$ as multiindex. So, Φ is an infinite column of operators which can be considered as belonging to $\mathrm{End}(H_p) \otimes W$, where H_p is the space of states, W is the Verma module created by action of the operator J_1^a. Now, if we consider two towers of operators $\Phi(x_0, x_1)$ and $\Phi(x_0, x_1')$, all the products of their elements can be combined in the following object: $\Phi_1(x_0, x_1)\Phi_2(x_0, x_1')$, where two copies of $W (W_1, W_2)$ are considered and $\Phi_1 \in \mathrm{End}(H_p) \otimes W_1$, $\Phi_2 \in \mathrm{End}(H_p) \otimes W_2$. The commutation relation in question can be written down in the following way [17]:

$$\Phi_1(x_0, x_1)\Phi_2(x_0, x_1') = R_{1,2}\Phi_1(x_0, x_1')\Phi_2(x_0, x_1), \quad \text{for} \quad x_1 > x_1' .$$

Here $R_{1,2}$ is $D(Y)$ R-matrix acting in the tensor product of two Verma modules. That means in these relations we effectively consider the local field φ as that annihilated by the operator J_{-1}^a.

It is explained in the papers [23] that the situation can be inversed. We considered the space of particles and came to the conclusion that the local and quasilocal operators acting in the space behave, as far as the commutation relations are considered, as being combined into Verma modules. The local operators themselves are identified with the highest vectors of the Verma modules. We can consider now the space of fields instead of the space of particles. The space of fields (H_f) is a suitable collection of Verma modules. The particular local field φ_i is identified with a highest vector $|0\rangle_i$ satisfying the requirement

$$J_{-1}^a|0\rangle_i = 0 .$$

We consider also a dual vacuum $\langle 0|$ which is defined by the relations

$$\langle 0|J_0^a = 0, \quad \langle 0|J_1^a = 0 \ .$$

The algebra $D(Y)$ allows besides the infinite dimensional representations finite-dimensional ones. We will be interested in the two-dimensional representations (ρ_β) depending on a parameter β; the detailed description is given in [23]. One can define the vertex operators $V^\varepsilon(\beta)$ which belong to $\mathrm{End}(H_f) \otimes C^2$ ($\varepsilon = \pm$ is C^2 index). Acting in H_f the operator $V^\varepsilon(\beta)$ transforms under the adjoint representation with respect to the representation ρ_β:

$$\mathrm{ad}_x(V^\varepsilon(\beta)) = \rho_\beta(x)^\varepsilon_{\varepsilon'} V^{\varepsilon'}(\beta) \ .$$

Then the vacuum expectations of the operators $V^\varepsilon(\beta)$ given by

$$\langle 0|V^{\varepsilon_1}(\beta)) \ldots V^{\varepsilon_{2n}}(\beta_{2n})|0\rangle_i$$

satisfy the deformed Knizhnik–Zamolodchikov equations; in the usual tensor notation they take the form:

$$\langle 0|V(\beta_1) \ldots V(\beta_j + 2\pi) \ldots V(\beta_{2n})|0\rangle_i$$
$$= S_{2n,j}(\beta_{2n} - \beta_j - 2\pi i) \ldots S_{j+1,j}(\beta_{j+1} - \beta_j - 2\pi i)$$
$$\times S_{1,j}(\beta_1 - \beta_j) \ldots S_{j-1,j}(\beta_{j-1} - \beta_j)$$
$$\times \langle 0|V(\beta_1) \ldots V(\beta_j) \ldots V(\beta_{2n})|0\rangle_i \ . \tag{2.1}$$

As it has been explained in [23, 24] these equations are consistent with the symmetry property

$$S_{i,i+1}(\beta_i - \beta_{i+1})\langle 0|V(\beta_1), \ldots, V(\beta_i), V(\beta_{i+1}), \ldots, V(\beta_{2n})|0\rangle_i$$
$$= P_{i,i+1}\langle 0|V(\beta_1), \ldots, V(\beta_{i+1}), V(\beta_i), \ldots, V(\beta_{2n})|0\rangle_i \ . \tag{2.2}$$

Being supplied with the symmetry property (2.2), Eq. (2.1) appear to be the same as the basic requirements for the form factors. There is an additional equation on residues in the form factor bootstrap approach [25] which we do not present here. This equation can be interpreted as a form of the operator product requirement [23]. It is responsible for the special choice of blocks of the vertex operators. Namely, the following remarkable identification can be done

$$F(\beta_1, \ldots, \beta_{2n})^{\varepsilon_1, \ldots, \varepsilon_{2n}} \sim \langle 0|V^{\varepsilon_1}(\beta_1) \ldots V^{\varepsilon_{2n}}(\beta_{2n})|0\rangle \ , \tag{2.3}$$

where $|0\rangle$ is the highest vector of spin zero. The sequence of the intermediate Verma modules in the RHS is taken as follows (we indicate the spins of the highest vectors)

$$0 \to 1/2 \to 1 \to 1/2 \to 1 \to \ldots \to 1/2 \to 0 \ . \tag{2.4}$$

We put asymptotic equivalence in (2.3) for the reasons explained in [23]; we shall briefly explain the point later. The form factor can be considered as a matrix element of the operator $\mathfrak{A}$ in the space of particles. More generally the relation (2.3) can be rewritten as follows:

$$\langle \mathrm{vac}|Z^{\varepsilon_1}(\beta_1) \ldots Z^{\varepsilon_{2n}}(\beta_{2n})\varphi_i(0)|\mathrm{vac}\rangle \sim \langle 0|V^{\varepsilon_1}(\beta_1) \ldots V^{\varepsilon_{2n}}(\beta_{2n})|0\rangle_i \ .$$

We do not expect other local operators than the descendents under the action of local integrals on $\mathfrak{A}$, and on the currents j_μ^a to exist in the theory. There is also

parafermionic field of spin $\frac{1}{4}$ with respect to Lorentz transformations (kink field). From the point of view of the representation theory these three fields correspond to spin 0, 1, 1/2 highest vectors respectively. The sequence of the Verma modules in form factors of these operators is the same as in (2.4), the only difference being that the sequence for currents (kinks) terminates at spin 1 (1/2) representation. This is a variant of "rationality" of the theory. The reason for the phenomenon to occur is supposed to be in the absence of mixing of a different solution of the deformed KZ equation by braiding which is discussed in detail in the papers [23, 24].

Now we are going to explain the asymptotical character of the equivalence in the relations (2.3). This matter was considered in [23, 24]. The situation is briefly as follows. On the LHS of (2.3) we have the analytical solutions of (2.1) while the RHS contains the vacuum expectation of $D(Y)$-vertex operators. The RHS can be in principle calculated directly (without use of KZ) using the definition of the vertex operators. It is quite clear that the only thing we are able to get in that way are certain power series in β's. But as it follows from the explicit formulae for the analytical solutions they are transcendental functions of β's (the explicit formulae are given in [25]); roughly, they have the properties of Γ-functions. The only reasonable connection between these two types of objects is an asymptotic one. One can have in mind the following analogy: Γ-function (RHS) and its asymptotic series (LHS) both satisfy the functional equation for the Γ-function (dKZ-equation). The difference between the function and its asymptotics in the case is due to the exponential in β contributions which are considered as the contribution due to the intermediate states created by local integrals which should be added to the $D(Y)$ [23].

3. The Classical Limit

As it has been already said the Yangian double is a deformation of the algebra $\widehat{sl}(2)$. Let us ignore for the moment the physical content, and consider the formal aspects of the limit. The typical relation for us will be (1.2). By rescaling of J_1^a the relation can be rewritten as

$$\Delta(J_0^a) = J_0^a \otimes 1 + 1 \otimes J_0^a \,,$$

$$\Delta(J_1^a) = J_1^a \otimes 1 + 1 \otimes J_1^a + \frac{h}{2} f^{abc} J_0^b \otimes J_0^c \,, \tag{3.1}$$

where h is an arbitrary constant. Certainly, the generator J_{-1}^a should be rescaled as well. Now consider h as the Planck constant and take the limit $h \to +i0$ (we prefer considering imaginary h, so, probably, the temperature is a better analog for it than the Planck constant). In the limit Eqs. (3.1) turn into trivial comultiplication formulae for the generators of $\widehat{sl}(2)$, all other $D(Y)$-relations turn into $\widehat{sl}(2)$-ones in the limit. Also the dKZ equations turn formally into the usual $\widehat{sl}(2)$ KZ equations on level zero. It is clear that the rescaling of the generators we did for the dKZ equations is equivalent to the rescaling of the rapidities: $\beta_i = \frac{2\pi i}{h} \lambda_i, h \to +i0$ while λ_i are fixed. So, formally for the functions

$$f(\lambda_1, \ldots, \lambda_{2n}) \simeq_{h \to +i0} C(h) F\left(\frac{2\pi i \lambda_1}{h}, \ldots, \frac{2\pi i \lambda_{2n}}{h}\right)$$

$(C(h)$ is a normalization constant) Eqs. (2.1) turn in this limit into

$$\left(\frac{d}{d\lambda_i} + \sum_{i \neq j} r_{i,j}(\lambda_i - \lambda_j)\right) f(\lambda_1, \ldots, \lambda_{2n}) = 0 \,,$$

where r is the classical r-matrix:

$$r_{i,j}(\lambda_i - \lambda_j) = \frac{\sigma_i^a \otimes \sigma_j^a}{\lambda_i - \lambda_j} \,.$$

The connection between the deformed and undeformed KZ equation is described explicitly in [24]. Let us outline the basic points.

First, the dKZ equation allows the same number of solutions as the undeformed one. The solution corresponding to the form factors is a special one. Its particular character is explained by (2.4), we shall also describe it from another point of view soon.

Second, the solutions are in one-to-one correspondence in the following sense. Consider some solution of dKZ for $\beta_1 < \beta_2 < \cdots < \beta_{2n}$, then its asymptotics $\beta_i = \frac{2\pi i}{h} \lambda_i, h \to + i0$ is described by a solution of KZ. In the paper [24] the explicit formulae are presented for the solution of dKZ which correspond to all possible solutions of KZ in that sense.

Third, there is an essential difference in the properties of the soution of dKZ in comparison with those of KZ: braiding does not mix different solutions of dKZ. The difference is explained by the asymptotic character of the correspondence above. Braiding does not commute with taking the asymptotics.

The third point mentioned makes a real difference between the deformed and undeformed case. In particular, in the deformed case it makes sense to consider the form factors of $\mathfrak{A}$ themselves without considering other solutions to dKZ equations. But taking the asymptotics of the form factors we have to consider not only the solution of KZ we obtain, but also those which are connected with it by braiding. We shall call this part of solutions of KZ the main part.

We do not give here the explicit formulae for the solutions of dKZ which can be found in [24, 25], but we have to write down explicitly the solutions of KZ. These formulae concern the particular case of zero central extension which causes essential simplification with respect to the general case. So, let us explain first how the simplifications appear. In Varchenko–Shechtman [22] formulae for the solutions of KZ one has the following structure:

$$\prod_{i<j} (\lambda_i - \lambda_j)^{\frac{1}{2(k+2)}} \int d\tau_1 \ldots d\tau_p \prod_{m,i} (\tau_m - \lambda_i)^{-\frac{1}{(k+2)}} \prod_{l<m} (\tau_l - \tau_m)^{\frac{2}{(k+2)}}$$

$$\times R(\tau_1, \ldots; \lambda_1, \ldots) \,, \qquad (3.2)$$

where τ's are integration variables (corresponding to screenings), R is a rational function of its variables. For $k = 0$ the exponents in (3.2) turn into $1/4$, $-1/2$, 1 respectively. The first one is not so essential, the second causes the hyperelliptic character of the integrals, the third causes the absence of branching between the integration variables which simplifies essentially the choice of contours of integration. Also one can present $\prod_{l<m} (\tau_l - \tau_m)$ as the Vandermonde determinant, and rewriting properly the function R perform the integration over the columns of the determinant getting a determinant of single integrals instead of the multi-integral of

the determinant. Let us present the result to which these manipulations should lead (which actually has not been obtained in that way, but directly from the classical limit the solutions of dKZ solutions).

Consider the components of $f(\lambda_1, \ldots, \lambda_{2n})$: $f(\lambda_1, \ldots, \lambda_{2n})^{\varepsilon_1, \ldots, \varepsilon_{2n}}$. We are looking for the singlet solutions of the equations which means in particular that $\sum \varepsilon_i = 0$. For each particular component $f(\lambda_1, \ldots, \lambda_{2n})^{\varepsilon_1, \ldots, \varepsilon_{2n}}$ the multiindex $\varepsilon_1, \ldots, \varepsilon_{2n}$ induces a partition of $B = \{1, \ldots, 2n\}$ into $T = \{i_k\}_{k=1}^n : \varepsilon_{i_k} = +$ and $T' = B \setminus T = \{j_k\}_{k=1}^n : \varepsilon_{j_k} = -$. Different solutions will be parametrized by the sets $\gamma_1, \ldots, \gamma_{n-1}$ which will be specified later. The solutions look as follows

$$f_{\gamma_1, \ldots, \gamma_{n-1}}(\lambda_1, \ldots, \lambda_{2n})^{\varepsilon_1, \ldots, \varepsilon_{2n}} = \prod_{i<j} (\lambda_i - \lambda_j)^{\frac{1}{4}}$$

$$\times \prod_{i \in T, j \in T'} (\lambda_i - \lambda_j)^{-1} \det \left\| \int_{\gamma_i} \zeta_j(\tau | T | T') \, d\tau \right\|_{(n-1) \times (n-1)} . \qquad (3.3)$$

ζ_j are the following differentials on the hyper-elliptic surface (HES) $w^2 = P(\tau) \equiv \prod (\tau - \lambda_i)$:

$$\zeta_j(\tau | T | T') = \frac{Q_j(\tau | T | T')}{\sqrt{P(\tau)}} ,$$

$$Q_j(\tau) = \prod_T (\tau - \lambda_l) \left[\frac{d}{d\tau} \frac{\prod_{T'}(\tau - \lambda_l)}{\tau^{n-j}} \right]_0$$

$$+ \prod_{T'} (\tau - \lambda_l) \left[\frac{d}{d\tau} \frac{\prod_T (\tau - \lambda_l)}{\tau^{n-j}} \right]_0 \Bigg\} , \qquad (3.4)$$

where $[\ ]_0$ means that only the polynomial part of the expression in brackets is taken. The differentials ζ_j are of the second kind: they have singularities at $\infty^{\pm}$, but their residues at the infinities are equal to zero. It should be mentioned that the singular part of the differential ζ_j is independent of the partition of Λ:

$$\zeta_j(\tau | T | T') - \zeta_j(\tau | T_1 | T'_1) = \text{of the first kind} .$$

The first kind of differentials in our case are of the type: $\sigma_j = \tau^{j-1}/\sqrt{P(\tau)}$, $1 \leq j \leq n - 1$. The countours $\gamma_1, \ldots, \gamma_{n-1}$ are arbitrary cycles on the HES (notice that its genus g equals $n - 1$).

It can be shown that the asymptotics of the form factor F corresponds to the following special choice of $\gamma_1, \ldots, \gamma_{n-1}$: they are taken as canonical a-cycles $a_1, \ldots, a_g$ where the cycle a_i surrounds the cut between λ_{2i-1} and λ_{2i}, $i = 1, \ldots, n - 1$ (we suppose λ's to be ordered: $\lambda_1 < \lambda_2 < \cdots < \lambda_{2n}$).

Now, let us turn to the problem of braiding. Equations (5.3) are invariant under the permutation $\lambda_i \leftrightarrow \lambda_j$ and simultaneous permutation of the associated spaces. Let us denote the operation of the analytical continuation $\lambda_i \leftrightarrow \lambda_{i+1}$ and permutation of corresponding spaces by $B_{i,i+1}$. Then we are supposed to have a formula of the type:

$$B_{i,i+1} f_{\gamma_1, \ldots, \gamma_{n-1}}(\lambda_1, \ldots, \lambda_{2n})$$

$$= \sum_{\gamma'_1, \ldots, \gamma'_{n-1}} C^{\gamma'_1, \ldots, \gamma'_{n-1}}_{\gamma_1, \ldots, \gamma_{n-1}} f_{\gamma'_1, \ldots, \gamma'_{n-1}}(\lambda_1, \ldots, \lambda_{2n}) ,$$

where C are some constants. There is one interpretation of the braiding following from the formula (3.3): $\lambda_i \leftrightarrow \lambda_{i+1}$ corresponds to the certain element of the modular group $Sp(2g, Z)$ of the HES. Clearly under the braiding the determinants (3.3) transform under g^{th} exterior power of the vector representation of the group. It can be shown that this interpretation implies that the solution f corresponding to $\gamma_1, \ldots, \gamma_g = a_1, \ldots, a_g$ mixes by braiding only with those solutions which correspond to other choices of half-basis of homologies (g contours on the surface with zero intersection numbers). This is the main part of solutions of KZ defined above. There is the same number of elements in the main part of solutions of KZ defined above. There is the same number of elements in the main part as the number of independent half-bases, i.e. $C_{2g+2}^{g+1} - C_{2g+2}^g$ of them. On the other hand it is well known nowadays that the braiding is described by the finite-dimensional quantum group $SL(2)_q$ [26], in our case $q = -1$. The number of solution coincides with the multiplicity of the one-dimensional representation in the tensor product of $2n$ two-dimensional representations of $SL(2)_q$ which is in the generic situation $C_{2n}^n - C_{2n}^{n-1}$ (the same as above since $g = n - 1$). So, it is natural to suppose that the main part of solutions for the case $q = -1$ allows continuation to other q while the rest of solutions is special for $q = -1$. In what follows we shall need only the main part. There is a beautiful way of rewriting the formulae (3.3) for the case of the main part of solutions in terms of θ-functions. We shall consider that in the next section.

Now let us turn to the problem of physical interpretation of the classical limit in question. We have to understand what does the limit $D(Y) \to \widehat{sl}(2)$ mean. In that limit, in particular, the antipode-square automorphism $D(Y)$ turns into differentiation for $\widehat{sl}(2)$:

$$s^2 \to 1 + hD . \tag{3.5}$$

We do not denote D by familiar notation (L_{-1}) to avoid a confusion: D is not a derivative in the space-time. There is a strange thing about this limit. The antipode-square (s^2) is identified with the rotation of the space-time in the theory [23], for two-dimensional representations (particles) it corresponds to the shift of rapidity by $2\pi i$. So, the operation s^2 corresponds to in the space-time of QFT model is essentially finite being of topological character, hence the consideration of its infinitesimal limit (3.5) does not look very reasonable. However we will consider the limit because from the mathematical point of view the limits of certain objects (e.g. form factors) do make sense. Thinking more of this situation one comes upon an idea that in the limit in question the very notion of the space-time of the QFT model should be lost. Form factors are the objects referring to one point in space-time, so their limits in this strange situation might make sense because the notion of one point might remain (it simply appears to be isolated from the rest of the world). Certainly form factors also do not survive in full meaning. First, we deal not with a real limit, but with asymptotic equivalence. Second, the difference disappears between the two-dimensional representation and the dual to it which contained for $D(Y)$ the essential crossing shift, and the two-dimensional representation can not be considered as those corresponding to particles. Thus, the only reasonable guess we can make is that the QFT in the limit splits into a family of systems with finite degrees of freedom, the limits of form factors being somehow connected with these systems. The situation is difficult also because we do not expect any connection between the finite-dimensional system and QFT on the Hamiltonian level.

Let us consider the situation in the opposite direction. Suppose we have a certain family of classical systems every one of which has a finite number of degrees of freedom. These systems should be unified through the fact of possessing the symmetry under $\widehat{sl}(2)$. Now we perform a quantization of these systems which essentially leads only to the quantization of $\widehat{sl}(2)$ ($\widehat{sl}(2) \to D(Y)$). It should be mentioned that there is a jump in quantization of $\widehat{sl}(2)$: as far as we got a small h in $D(Y)$ it can be immediately rescaled to a finite one. So, the quantization provides a finite operation (s^2). Now we introduce a space-time, and identify this operation with the rotation of this space-time. The coordinates are introduced as those respecting this interpretation. As soon as it is done we get the notion of particle, and one space-time point we stared with appears to be able to interact with others through exchange of particles. This interpretation seems to be reasonable. In particular it solves the problem of disagreement between local conservation laws in classical and quantum integrable asymptotically free models. In a context close to the present consideration this problem was discussed in [23].

So, now we have to explain what classical systems with finite number of degrees of freedom and $\widehat{sl}(2)$-invariance we have in mind. The answer is implied by the structure of formulae of the solutions of KZ on level zero. They are connected with HES. So, it is natural to suppose that the classical systems in question are stationary finite-gap solutions [6, 13, 16] of the classical soliton equation with $\widehat{sl}(2)$ symmetry. These systems will be described in Sect. 5. But before doing that we have to explain the connection between the solutions of KZ on level zero and Riemann θ-functions.

4. KZ Equations on Level Zero and Riemann Theta-Functions

In this section we shall derive a formula which expresses the solutions of KZ equations on level zero in terms of Riemann θ-functions. Let us first fix the notations and introduce necessary definitions.

We consider the HES Σ of genus g with $2g + 2$ real branching points ordered as follows: $\lambda_1 < \cdots < \lambda_{2g+2}$. Let us put the cuts between the points λ_{2i-1}, λ_{2i} for $i = 1, \ldots, g + 1$. The a-cycles on the surface are taken in a canonical way: the cycle a_i surrounds the cut λ_{2i-1}, λ_{2i}, for $i = 1, \ldots, g$. The b-cycle b_i ($i = 1, \ldots, g$) starts from one bank of the cut λ_{2i-1}, λ_{2i}, reaches the cut λ_{2g+1}, λ_{2g+2} by one sheet, and then returns to the other bank of λ_{2i-1}, λ_{2i} by another sheet. There are g nonsingular differentials on the HES: $\sigma_j = \tau^{j-1}/\sqrt{P(\tau)}$. The normalized first kind differentials ω_j are linear combinations of σ's satisfying the condition:

$$\int_{a_i} \omega_j = \delta_{i,j} \, .$$

The matrix of periods Ω is defined as

$$\Omega_{ij} = \int_{b_i} \omega_j \, .$$

The matrix of periods is symmetric due to the Riemann bilinear identity.

The second kind of differentials possess singularities, but their residues at the singular points vanish. There is one type of the second kind of differentials of particular importance. The differentials of this type are obtained by erasing

the dependence on one argument of the two-differentials $\omega^2(x, y)$ defined on $\Sigma \times \Sigma$. The differential $\omega^2(x, y)$ possesses the only singularity at the diagonal being normalized as

$$\omega^2(x, y) = \frac{1}{(x - y)^2} \, dx \, dy, \quad x \sim y .$$

Being considered as a differential in one variable (say x) it satisfies the normalization condition:

$$\int_{a_i} \omega^2 = 0, \quad \forall i .$$

Finally, the differential ω^2 is symmetric: $\omega^2(x, y) = \omega^2(y, x)$.

The Riemann θ-function is defined as follows:

$$\theta(z \,|\, \Omega) = \sum_{m \in Z^g} \exp \left\{ \pi i m^t \Omega m + 2\pi i z^t m \right\} ,$$

where $z \in C^g$. The periodicity property says

$$\theta(z + \lambda' + \Omega \lambda'' \,|\, \Omega) = \exp \left\{ - \pi i \lambda''{}^t \Omega \lambda'' - 2\pi i \lambda'' z \right\} \theta(z \,|\, \Omega), \quad \lambda', \lambda'' \in Z^g .$$

The θ-function with characteristics is defined by:

$$\theta[\eta](z \,|\, \Omega) = \exp \left\{ \pi i \eta''{}^t \Omega \eta'' + 2\pi i (z + \eta')^t \eta'' \right\} \theta(z + \eta' + \Omega \eta'' \,|\, \Omega) ,$$

where $\eta = (\eta', \eta'')$ is a characteristic: the vectors η', η'' belong to Q^g (Q is the field of rational numbers). We will be interested in the case when η is a half-period characteristic which means that $\eta', \eta'' \in 1/2 Z^g$.

The definition of θ-function refers to the particular choice of a and b cycles. However, the θ-functions defined with respect to different choices of the homology bases are connected due to the modular property of θ-function. Suppose that we have two homology bases connected via a transformation from $\Gamma_{1,2} \in Sp(2g, Z)$:

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \quad \begin{pmatrix} d & c \\ b & a \end{pmatrix} \in Sp(2g, Z), \quad \begin{matrix} \text{diag } cd^t \\ \text{dig } ab^t \end{matrix} \equiv 0 \, (\text{mod } 2)$$

"diag" means the vector composed of the diagonal entries of the matrix. The transformation law for θ-functions says that [19]:

$$\theta[\xi](z' \,|\, \Omega') = \gamma (\det(M))^{1/2} \exp \left\{ \pi i \sum_{i < j} z_i z_j \frac{\partial \log(\det M)}{\partial \Omega_{i,j}} \right\} \theta[\eta](z \,|\, \Omega) , \qquad (4.1)$$

where $M = (c\Omega + d)$, $z = M^t z'$, $\Omega' = (a\Omega + b)(c\Omega + d)^{-1}$, the characteristics changes as follows

$$\begin{pmatrix} \xi' \\ \xi'' \end{pmatrix} = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix} \begin{pmatrix} \eta' \\ \eta'' \end{pmatrix} .$$

γ is irrelevant for our goals of 8th root of unity.

Let us denote the set $\{1, 2, \ldots, 2g + 2\}$ counting the branching points of Σ by B. The most significant property of HES is that they allow one-to-one correspondence between half-periods and the subsets of B with an even number of elements (mod identification $T \sim B \backslash T$) [19]. The correspondence is achieved as follows. Consider the subset $T(\# T = 2m)$ and divide it into two subsets T_1, T_2 such that

$\# T_1 = \# T_2 = m$. Now let $T_1 = \{i_1, \ldots, i_m\}$, $T_2 = \{j_1, \ldots, j_m\}$ and associate to T the half-period characteristic $\tilde{\eta}_T$ such that

$$\tilde{\eta}'_T + \Omega \tilde{\eta}''_T = \sum_k \int_{\lambda_{i_k}}^{\lambda_{j_k}} \omega \,, \tag{4.2}$$

where ω is considered as vector composed of first kind differentials. It can be easily shown that the ambiguity in dividing T into subsets and enumerating these subsets changes $\tilde{\eta}_T$ by a full period which can be ignored. It is convenient to measure the characteristics relative to the Riemann characteristic δ which is defined as

$$\delta = \begin{cases} \tilde{\eta}_U, & \text{if } g \equiv 1 \,(\text{mod } 2) \\ \tilde{\eta}_{\{U \setminus 1\}}, & \text{if } g \equiv 0 \,(\text{mod } 2) \,, \end{cases}$$

where $U = \{1, 3, 5, \ldots\}$. The characteristic η_T corresponding to the subset T such that $\# T \equiv g + 1 \,(\text{mod } 2)$ is defined by

$$\eta_T = \tilde{\eta}_{T \circ U} \,,$$

where $T \circ U = (T \cup U) \setminus (T \cap U)$.

Our nearest goal is to prove the following

Proposition. *Consider the solution of KZ on level zero which corresponds to $\gamma_1, \ldots, \gamma_g = a_1, \ldots, a_g$. The components of this solution are denoted by $f(\lambda_1, \ldots, \lambda_{2g+2})^{\varepsilon_1, \ldots, \varepsilon_{2g+2}}$. To every multiindex $\varepsilon_1, \ldots, \varepsilon_{2n}$ a partition of B corresponds such that $B = T \cup T'$, $i \in T$: $\varepsilon_i = +$, $i \in T'$: $\varepsilon_i = -$. Denote the elements of T by i_p, $p = 1, \ldots, g + 1$. The set T is associated with θ-characteristic η_T. The following relation holds:*

$$f(\lambda_1, \ldots, \lambda_{2g+2})^{\varepsilon_1, \ldots, \varepsilon_{2g+2}} = C^{-3} \theta[\eta_T](0)^4 \det \| \partial_p \partial_q \log \theta [\eta_T](0) \|_{g \times g} \,, \tag{4.3}$$

where C is the constant: $C = \prod_{i < j} (\lambda_i - \lambda_j)^{\frac{1}{4}} \Delta$, Δ is the following important in the future determinant:

$$\Delta = \det \left\| \int_{a_i} \sigma_j \right\|_{g \times g} \,.$$

The notations are used:

$$\partial_i F(\ldots, 0, \ldots) \equiv \frac{\partial}{\partial z_i} F(\ldots, z_i, \ldots)|_{z_i} = 0 \,.$$

To prove the proposition we have to explain certain facts concerning θ-functions on HES.

The Tomae formulae says that $\theta[\eta_T](0) = 0$ if $\# T \neq g + 1$, and if $\# T = g + 1$, then

$$\theta[\eta_T](0)^2 = \prod_{i < j, \, i, j \in T} (\lambda_i - \lambda_j)^{\frac{1}{2}} \prod_{i < j, \, i, j \in B \setminus T} (\lambda_i - \lambda_j)^{\frac{1}{2}} \Delta \,. \tag{4.4}$$

The following remarkable relation between the differentials ω^2 and the derivatives of the θ-function at $z = 0$ (θ-constants) takes place. If x, y, η are such that $\theta[\eta](\int_x^y \omega) = 0$ then [9]:

$$\omega^2(x, y) = - \sum_{i, j} \partial_i \partial_j \log \theta[\eta](0) \omega_i(x) \omega_j(y) \,. \tag{4.5}$$

To proceed further we need explicit formulae for certain differentials. The normalized first kind differential ω_i is given by the $g \times g$ determinant:

$$\omega_i(x) = (-1)^i \Delta^{-1} \det \begin{pmatrix} \sigma_1(x), & \int_{a_1} \sigma_1 & \cdots & \int_{a_{i-1}} \sigma_1, & \int_{a_{i+1}} \sigma_1, & \cdots & \int_{a_g} \sigma_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_g(x), & \int_{a_1} \sigma_g & \cdots & \int_{a_{i-1}} \sigma_g, & \int_{a_{i+1}} \sigma_g, & \cdots & \int_{a_g} \sigma_g \end{pmatrix}.$$

Let us consider now the differential $\omega^2(x, y)$ as a function of x specifying y to one of the branching points: $y = \lambda_i$. We take $\xi = \sqrt{(\tau - \lambda_i)}$ as a local parameter in the vicinity of λ_i. The differential $\omega^2(x, \lambda_i)$ is second kind differential with zero a-periods and fixed singularity: $\omega^2(x, \lambda_i) \sim \xi^{-2}$, $x \sim \lambda_i$. It can be presented as a $(g+1) \times (g+1)$ determinant:

$$\omega^2(x, \lambda_i) = \Delta^{-1} (P'(\lambda_i))^{\frac{1}{2}} \det \begin{pmatrix} \sigma_1(x), & \int_{a_1} \sigma_1 & \cdots & \cdots & \int_{a_g} \sigma_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_g(x), & \int_{a_1} \sigma_g & \cdots & \cdots & \int_{a_g} \sigma_g \\ \rho_i(x), & \int_{a_1} \rho_i & \cdots & \cdots & \int_{a_g} \rho_i \end{pmatrix},$$

where $\rho_i(\tau) = \dfrac{1}{(\tau - \lambda_i)\sqrt{P(\tau)}}$, $P'(\lambda_i) \equiv \prod_j'(\lambda_i - \lambda_j)$. From this formula one derives the following representation for $\omega^2(x, \lambda_i)$ when $x = \lambda_j$ in terms of a $g \times g$ determinant:

$$\omega^2(\lambda_j, \lambda_i) = \Delta^{-1} \left(\frac{P'(\lambda_i)}{P'(\lambda_j)} \right)^{\frac{1}{2}} \det \begin{pmatrix} \int_{a_1} \sigma_{2,j} & \cdots & \cdots & \int_{a_g} \sigma_{2,g} \\ \vdots & \vdots & \vdots & \vdots \\ \int_{a_1} \sigma_{g,j} & \cdots & \cdots & \int_{a_g} \sigma_{g,j} \\ \int_{a_1} \rho_{i,j} & \cdots & \cdots & \int_{a_g} \rho_{i,j} \end{pmatrix}, \qquad (4.6)$$

where $\sigma_{l,j} = \sigma_l - \lambda_j \sigma_{l-1}$, $\rho_{i,j} = \rho_i + (\lambda_j - \lambda_i)^{-1}\sigma_1$. Combining (4.6) and (4.5) one arrives at

$$\sum \partial_p \partial_q \theta[\eta_T](0) A_u^p A_v^q \lambda_i^u \lambda_j^v = \Delta^{-1} P'(\lambda_i) \det \begin{pmatrix} \int_{a_1} \sigma_{2,j} & \cdots & \cdots & \int_{a_g} \sigma_{2,g} \\ \vdots & \vdots & \vdots & \vdots \\ \int_{a_1} \sigma_{g,j} & \cdots & \cdots & \int_{a_g} \sigma_{g,j} \\ \int_{a_1} \rho_{i,j} & \cdots & \cdots & \int_{a_g} \rho_{i,j} \end{pmatrix}, \qquad (4.7)$$

where $i, j \in T$, the matrix A connects the normalized first kind of differentials with trivial ones:

$$\omega = A\sigma, \qquad A = \left\| \int_{a_i} \sigma_j \right\|^{-1}. \tag{4.8}$$

The reason for Eq. (4.7) to hold is that adding $\int_{\lambda_i}^{\lambda_j} \omega$ to η_T we get odd characteristics corresponding to $T \setminus \{i, j\}$, and $\theta[\eta_{T \setminus \{i,j\}}](0) = 0$.

Let us return to the solution of KZ equations corresponding to $\gamma_1, \ldots, \gamma_g = a_1, \ldots, a_g$. It is given by (3.3):

$$f(\lambda_1, \ldots, \lambda_{2g+2})^{\varepsilon_1, \ldots, \varepsilon_{2g+2}} = \prod_{i<j} (\lambda_i - \lambda_j)^{\frac{1}{4}}$$

$$\prod_{i \in T, j \in T'} (\lambda_i - \lambda_j)^{-1} \det \left\| \int_{a_i} \zeta_j(\tau \,|\, T \,|\, T') d\tau \right\|_{g \times g}, \tag{4.9}$$

where T, T' are associated to $\varepsilon_1, \ldots, \varepsilon_g$ in a usual way. Let us take g elements from T (say i_p, $p = 1, \ldots, g$) and multiply the determinant from (4.9) by Vandermonde composed of corresponding λ's the result being

$$\det \|F_{p,q}\|_{g \times g}, \qquad F_{p,q} = \int_{a_p} \sum_{m=1}^{g} \zeta_m \lambda_{i_q}{}^{g-m}.$$

The following two differentials are equivalent (differ by a total derivative):

$$\sum_{m=1}^{g} \zeta_m \lambda_{i_q}{}^{g-m} \sim \mu_q \equiv \prod_{j \in B \setminus T} (\lambda_{i_q} - \lambda_j) \frac{\prod_{i \in T, i \neq i_q}(\tau - \lambda_i)}{(\tau - \lambda_{i_q})\sqrt{P(\tau)}}.$$

Hence the determinant in (4.9) can be replaced by

$$\frac{1}{\prod_{i<i', i, i' \in T \setminus i_{g+1}}(\lambda_i - \lambda_{i'})} \det(C), \qquad C_{pq} = \int_{a_p} \mu_q. \tag{4.10}$$

Let us find the matrix X which satisfies the equation:

$$AX = C \tag{4.11}$$

with A and C given by (4.8), (4.10). Kramer's rule tells that

$$X_{p,q} = -(-1)^p \Delta^{-1} \det \begin{pmatrix} \int_{a_1} \sigma_1 & \cdots & \cdots & \cdots & \int_{a_g} \sigma_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \int_{a_1} \sigma_{p-1} & \cdots & \cdots & \cdots & \int_{a_g} \sigma_{p-1} \\ \int_{a_1} \sigma_{p+1} & \cdots & \cdots & \cdots & \int_{a_g} \sigma_{p+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \int_{a_1} \sigma_g & \cdots & \cdots & \cdots & \int_{a_g} \sigma_g \\ \int_{a_1} \mu_q & \cdots & \cdots & \cdots & \int_{a_g} \mu_q \end{pmatrix}.$$

Let us now consider the determinant of the matrix X. Multiplying it by Vandermonde composed of λ_{ik}, $k = 1, \ldots, g$ one gets

$$\det(X) = \frac{1}{\prod_{i < i', i, i' \in T \backslash i_{g+1}} (\lambda_i - \lambda_{i'})} \det(\tilde{X}),$$

where

$$\tilde{X}_{p,q} = \Delta^{-1} \det \begin{pmatrix} \int_{a_1} \sigma_{2,p} & \cdots & \cdots & \cdots & \int_{a_g} \sigma_{2,p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \int_{a_1} \sigma_{g,p} & \cdots & \cdots & \cdots & \int_{a_g} \sigma_{g,p} \\ \int_{a_1} \mu_q & \cdots & \cdots & \cdots & \int_{a_g} \mu_q \end{pmatrix}. \tag{4.12}$$

It is easy to show that μ_q in (4.12) can be replaced by $P'(\lambda_{i_q}) \rho_{i_p, i_q}$. Combining that with (4.7) and taking into account that $\det(A) = \Delta^{-1}$ one gets:

$$\det(X) = \prod_{i < i', i, i' \in T \backslash i_{g+1}} (\lambda_i - \lambda_{i'}) \det \| \partial_p \partial_q \log \theta[\eta_T](0) \|_{g \times g}.$$

Calculating determinants of RHS and LHS of formulae (4.11) and having in mind (4.8) we obtain:

$$\det \left\| \int_{a_i} \zeta_j \right\| = \Delta \det \| \partial_p \partial_q \log \theta[\eta_T](0) \|. \tag{4.13}$$

Together with the Tomae formulae it provides the representation for the solutions of KZ on level 0:

$$f(\lambda_1, \ldots, \lambda_{2g+2})^{\varepsilon_1, \ldots, \varepsilon_{2g+2}} = C^{-3} \theta[\eta_T](0)^4 \det \| \partial_p \partial_q \log \theta[\eta_T](0) \|_{g \times g}.$$

where T is related to $\varepsilon_1, \ldots, \varepsilon_{2g+2}$ as explained above, $C = \prod_{i<j} (\lambda_i - \lambda_j)^{\frac{1}{4}} \Delta$. That proves the proposition above.

Formula (4.3) has a very beautiful meaning. It relates the solution of KZ equations which are differential equations with respect to moduli of HES (the positions of branching points) in terms of derivatives on the Jacobian. It would be nice to prove directly that (4.3) satisfies KZ using the heat equation for the θ-function.

Now suppose that we took a solution corresponding to the other choice of half basis: $\gamma_1, \ldots, \gamma_g = a'_1, \ldots, a'_g$. Denote this solution by $f_{A'}(\lambda_1, \ldots, \lambda_{2g+2})$. Clearly, all the reasonings above are applicable to $f_{A'}$, the difference being that the final formula has to contain θ-functions defined with respect to the half-basis A'. In order to rewrite the answer in terms of canonical θ-functions (those corresponding to A) we have to use the formula (4.1). After some simple computations one gets:

$$f_{A'}(\lambda_1, \ldots, \lambda_{2g+2})^{\varepsilon_1, \ldots, \varepsilon_{2g+2}}$$

$$= \det(M) C^{-3} \theta[\eta_T](0)^4 \det \left\| \partial_p \partial_q \log \theta[\eta_T](0) + \pi i \frac{\partial \log(\det M)}{\partial \Omega_{i,j}} \right\|,$$

where

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \quad \begin{pmatrix} d & c \\ b & a \end{pmatrix} \in \Gamma_{1,2} \, .$$

The matrix M is defined as $M = (c\Omega + d)$.

In Sect. 6 we shall need another representation for the solutions of KZ in terms of θ-functions. The author was unable to prove this representation completely, so we shall formulate it as a conjecture and present reasonings in favour of it.

Conjecture 1. For every half-basis A' a polynomial $Q_{A'}$ in ∂_i of total degree $2g$ exists such that

$$f_{A'}(\lambda_1, \cdots, \lambda_{2g+2})^{\varepsilon_1, \ldots, \varepsilon_{2g+2}} = C^{-3}\theta[\eta_T](0)^2 Q_{A'}\theta[\eta_T](0)^2 \, , \qquad (4.14)$$

where the connection between $\varepsilon_1, \ldots, \varepsilon_{2g+2}$ and T is as usual, the coefficients of $Q_{A'}$ might be complicated, but they do not depend on T.

Let us explain why we assume the representation exists. To this end we have to understand what kind of constants depending on T is a candidate for being presentable in the form $\theta[\eta_T](0)^2 Q\theta[\eta_T](0)^2$ for some polynomial in ∂_i and T corresponding to even non-singular characteristics ($\#T = g + 1$). Riemann θ-functions on HES satisfy many relations which follow from the combination of Riemann relations with the peculiar properties of HES. In particular, the following relations hold for $\theta[\eta](z)$ (η is even, nonsingular, i.e. such that $\theta[\eta](0) \neq 0$) [19]:

$$\sum_{S \subset B, \#S = g+1, 1 \in S} (-1)^{S \cap (U \circ T)} \theta[\eta_S](z)^2 \theta[\eta_S](0)^2 = 0 \, , \qquad (4.15)$$

where T is an arbitrary subset of B satisfying the requirements: $\#T \equiv g + 1 \pmod 2$, $\#T \neq g + 1$, $1 \in T$. We put the requirement $1 \in S$ in order to avoid the summation over $B \backslash S$. So, if the set of constants is presentable in the form $Q\theta[\eta_S](0)^2$ then they should satisfy this system of relations being placed instead of $\theta[\eta_S](z)^2$. The opposite should also be true: if a set of constants enumerated by S satisfies the system (4.15) then they can be presented in the form $Q\theta[\eta_S](0)^2$. In the elliptic case ($g = 1$) there is only one set T satisfying the requirements above: $T = B$. For this set the relation (4.15) turns into

$$\theta_{0,0}(0)^2 \theta_{0,0}(z)^2 = \theta_{1,0}(0)^2 \theta_{1,0}(z)^2 + \theta_{0,1}(0)^2 \theta_{0,1}(z)^2$$

in the usual notation for the elliptic θ-functions with characteristics [19]:

$$\theta_{1,0}(z) = \theta[\eta_{\{1,2\}}](z), \quad \theta_{0,0}(z) = \theta[\eta_{\{1,3\}}](z), \quad \theta_{0,1}(z) = \theta[\eta_{\{1,4\}}](z) \, .$$

Let us prove that the solutions of KZ for an arbitrary choice of A' satisfy the relations (4.15). First, let us check that $\theta[\eta_S](0)^2$ given by Tomae formulae satisfy the requirements. Substitute $\theta[\eta_S](0)^2$ given by (4.4) into (4.15) instead of both $\theta[\eta_S](0)^2$ and $\theta[\eta_S](z)^2$ and divide the relation by $\Delta^2 \prod_{i<j}(\lambda_i - \lambda_j)$ the result being

$$\sum_{S \subset B, \#S = g+1, 1 \in S} (-1)^{S \cap (U \circ T)} \frac{1}{\prod_{i \in S, j \in B \backslash S}(\lambda_i - \lambda_j)} = 0 \, . \qquad (4.16)$$

These identities are proven by consideration of residues at $\lambda_i = \lambda_j$, $\forall i, j$.

Now let us present the solution of KZ (4.9) as a multidimensional integral:

$$f_{A'}(\lambda_1, \ldots, \lambda_{2g+2})^{\varepsilon_1, \ldots, \varepsilon_{2g+2}} = \prod_{i<j} (\lambda_i - \lambda_j)^{\frac{1}{4}}$$

$$\times \prod_{i \in S, j \in B \backslash S} (\lambda_i - \lambda_j)^{-1} \int_{a_1'} d\tau_1 \frac{1}{\sqrt{P(\tau_1)}}$$

$$\ldots \int_{a_g'} d\tau_g \frac{1}{\sqrt{P(\tau_g)}} \det \|Q_i(\tau_j|S|S')\|_{g \times g} ,$$

where Q_i are given by (3.4), S is related to $\varepsilon_1, \ldots, \varepsilon_{2g+2}$ as usual. Let us denote $\det \|Q_i(\tau_j)\|_{g \times g}$ by $X_g(\tau_1, \ldots, \tau_g|S|S')$ indicating the dependence on genus g and subsets S, S' (recall that $S' = B \backslash S$). We shall prove that the solutions of KZ satisfy (4.15) being substituted as $\theta[\eta_S](0)^2 \theta[\eta_S](z)^2$ if we prove that $\prod_{i \in S, j \in S'} (\lambda_i - \lambda_j)^{-1} X_g(\tau_1, \ldots, \tau_g|S|S')$ satisfy them. So, we have to prove that

$$\sum_{S \subset B, \#S = g+1, 1 \in S} (-1)^{S \cap (U \circ T)} \prod_{i \in S, j \in S'} (\lambda_i - \lambda_j)^{-1} X_g(\tau_1, \cdots, \tau_g|S|S') = 0 . \quad (4.17)$$

In fact the proof of the identity does not differ much from the proof of (4.16). Again it is sufficient to check the cancellation of the residues at $\lambda_i = \lambda_j$ because $X_g(\tau_1, \ldots, \tau_g|S|S')$ is of degree g with respect to any λ. Suppose $l \in S$, $m \in S'$ then the polynomial $X_g(\tau_1, \ldots, \tau_g|S|S')$ satisfies the following recurrent relation (classical version of the relations from [25]):

$$X_g(\tau_1, \ldots, \tau_g|S|S')|_{\lambda_l = \lambda_m} = \prod_{k=1}^{g} (\tau_k - \lambda_l) \sum_{k=1}^{g} (-1)^k \left\{ \frac{\partial}{\partial \tau_k} \prod_{p \in B \backslash \{l, m\}} (\tau_k - \lambda_p) \right\}$$

$$\times X_{g-1}(\tau_1, \ldots, \tau_{k-1}, \tau_{k+1}, \ldots, \tau_g|S \backslash l|S' \backslash m) .$$

The point is that the coefficients in the relation are independent of S. That provides the possibility of inductive proof of (4.17).

It would be nice to have explicit formulae for the polynomials Q in (4.14). The author has not succeeded to get them. Presumably the formulae can be found from (4.3) using Fay identities [9].

To finish this section let us present explicit formulae for the elliptic case. The canonic way for construction of the homology basis is as follows: the cycle a surrounds the cut between λ_1, λ_2, the cycle b starts from the upper bank of this cut, reaches the upper bank of the cut λ_3, λ_4 by one sheet, then moves to another sheet and starting with the lower bank of λ_3, λ_4 returns to the lower bank λ_1, λ_2. There are two independent choices of the a-cycle: $a = a$ or $a' = b$ which are connected by the following element from $Sp(2, Z)$:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

So, there are two independent solutions to KZ

$$f_a(\lambda_1, \cdots, \lambda_4)^{\varepsilon_1, \ldots, \varepsilon_4} = C^{-3} \theta[\eta_T](0)^4 \partial^2 \log \theta[\eta_T](0) ,$$

$$f_{a'}(\lambda_1, \cdots, \lambda_4)^{\varepsilon_1, \ldots, \varepsilon_4} = C^{-3} \theta[\eta_T](0)^4 K [\partial^2 \log \theta[\eta_T](0) + K^{-1}] , \quad (4.18)$$

where K is full elliptic integral: $K = \dfrac{1}{\pi i} \int\limits_{\lambda_1}^{\lambda_2} \dfrac{d\tau}{\sqrt{P(\tau)}}$, the correspondence between $\varepsilon_1, \ldots, \varepsilon_4$, T and conventional notations for elliptic θ-functions are as follows:

$$\{ + + - - \}, \{ - - + + \} \to \{1,2\} \to \{1,0\} \,,$$
$$\{ + - - + \}, \{ - + + - \} \to \{1,4\} \to \{0,1\} \,,$$
$$\{ + - + - \}, \{ - + - + \} \to \{1,3\} \to \{0,0\} \,. \tag{4.19}$$

The solution can be also rewritten as (4.14) with

$$Q_a = \partial^2, \quad Q_{a'} = K\partial^2 + 1$$

since $\partial\theta[\eta](0) = 0$ for even η.

5. Pragmatic View of Finite Gap Integration

Consider an integrable equation with infinitely many times $t_1, t_2, \ldots$. To any time t_i the M-operator $\mathfrak{M}_i$ is attached. The M-operators satisfy the zero curvature condition:

$$\frac{\partial}{\partial t_i}\mathfrak{M}_j - \frac{\partial}{\partial t_j}\mathfrak{M}_i = [\mathfrak{M}_i, \mathfrak{M}_j] \,. \tag{5.1}$$

We consider the $\widehat{sl}(2)$-invariant case (one can think of the nonlinear Schrödinger model for example), so $\mathfrak{M}_i$ is a traceless 2×2 matrix depending on the spectral parameter λ such that

$$\operatorname{tr}\mathfrak{M}_i(\lambda) = 0, \quad \mathfrak{M}_i(\lambda) = \sum_{k=0}^{i} \lambda^k m_{i,k} \,,$$

also we normalize it by the requirement $m_{i,i} = \sigma^3$.

The coefficients of the M-operators are dynamical variables. Finite-gap integration [6, 13, 16] deals with the situation of stationary solutions which means that there is a time t_n on which the dynamical variables do not depend:

$$\frac{\partial}{\partial t_n}\mathfrak{M}_i = 0, \quad \forall i \,.$$

That means that the M-operator $\mathfrak{M}_n$ satisfies the equation:

$$\frac{\partial}{\partial t_i}\mathfrak{M}_n = [\mathfrak{M}_i, \mathfrak{M}_n], \quad \forall i \,,$$

hence the determinant of $\mathfrak{M}_n$ is an integral of motion with respect to all the times. Certainly this determinant can be presented as follows:

$$\det\mathfrak{M}_n(\lambda) = \prod_{k=1}^{2n} (\lambda - \lambda_k) \,.$$

That suggests that the problem in question is closely related to the problem of parametrization of all matrices $M(\lambda)$ (we omit the index n) which satisfy the following requirements: $M(\lambda)$ is a traceless 2×2 matrix depending on λ as a

polynomial of degree n with fixed leading coefficient (equal to σ^3) and given determinant $(\det M(\lambda) = \prod_{k=1}^{2n}(\lambda - \lambda_k))$. This problem was considered in the last century (Jacobi, Riemann); the solution can be found for example in Mumford's book [19]. Let us describe it in the terms appropriate for our further goals.

The matrix in question is degenerate at the points $\lambda = \lambda_j$, also it is traceless, hence $M(\lambda_j)$ is a Jordan cell:

$$M(\lambda_j) = \psi_j \otimes \bar\psi_j \quad \bar\psi_j \equiv \psi_j^t \sigma^2$$

for some vector ψ_j. Let us parametrize $M(\lambda)$ by the set $\psi_j, j = 1, \ldots, 2n$ (these vectors are not independent as we shall see soon). Construct the interpolation

$$M'(\lambda) = \sum_{k=1}^{2n} \frac{\prod_{p \neq k}(\lambda - \lambda_p)}{\prod_{p \neq k}(\lambda_k - \lambda_p)} M(\lambda_k) \, .$$

Definitely, M' is degenerated at the given points with given values, but its degree in λ equals $2n - 1$ instead of n required. So, we have to kill $n - 1$ leading coefficients of M', also we have to take into account that λ^n is supposed to enter with given coefficient σ^3. After some simple manipulations these requirements lead to the following system of quadratic relations for the components a_i, b_i of the vectors ψ_i:

$$\sum_{k \in T} \frac{1}{\prod'_{j \in T}(\lambda_k - \lambda_j)} a_k^2 = 0 \, ,$$

$$\sum_{k \in T} \frac{1}{\prod'_{j \in T}(\lambda_k - \lambda_j)} b_k^2 = 0 \, ,$$

$$\sum_{k \in T} \frac{1}{\prod'_{j \in T}(\lambda_k - \lambda_j)} a_k b_k = 1 \tag{5.2}$$

for any subset T of $n + 1$ elements of the set $\{1, 2, \ldots, 2n\}$. This system of equations leaves only $n - 1$ independent parameters (which could be, certainly, calculated from the very beginning). The important point is that due to Riemann identities and Tomae formulae the relations allow a parametrization in Riemann θ functions on the hyperelliptic surface $\tau^2 = P(\lambda) \equiv \prod_{k=1}^{2n}(\lambda - \lambda_k)$. The solution to the system (5.2) looks as follows:

$$a_j = C \frac{\theta[\eta_j](r + z)}{\theta[\delta](z)\theta[\eta_j](gr)}, \quad b_j = C \frac{\theta[\eta_j](r - z)}{\theta[\delta](z)\theta[\eta_j](gr)} \, , \tag{5.3}$$

where η_j is the theta-characteristic corresponding to the branching point λ_j: $\eta_j = \tilde\eta_{1,j}$ with $\tilde\eta_T$ defined by (4.2). The variable $z = (z_1, \ldots, z_g)$ is the parameter on the Jacobian (genus $g = n - 1$),

$$r = \int_{\lambda_1}^{\infty^+} \omega \, ,$$

∞^+ is one of two infinities on the surface, ω is the vector composed of the first kind of differentials, δ is a Riemann constant, finally, C is the same as in (4.3).

We would like to make two comments on the above formulae in order to clarify them from two points of view. The first point is the connection with the more familiar in the context of finite-gap integration object, namely with the

Baker–Akhiezer function. The BA function is an eigenvector of $M(\lambda)$:

$$M(\lambda)\psi(\lambda) = m(\lambda)\psi(\lambda) \,,$$

where m is the corresponding eigenvalue. Certainly, for generic λ there are two solutions to this equation which means that the BA function is defined on the surface Σ. But for $\lambda = \lambda_i$ the matrix $M(\lambda)$ is degenerate, it has only one eigenvector corresponding to zero eigenvalue. This eigenvector is exactly our ψ_i. Thus,

$$\psi_i = \psi(\lambda_i) \,.$$

The BA-function $\psi(x)$ ($x = \lambda^{\pm}$ is a point on the surface) is written in terms of θ-functions through $\theta(\int^x \omega + \cdots)$, [16]. When x coincides with one of the branching points this expression turns into the θ-function with corresponding half-period characteristics because the integral of ω taken between two branching points on the hyperellilptic surface is a half-period.

The second point is the direct connection with well known addition theorems for θ-functions in the elliptic case ($n = 2$, there are four branching points). In that case the following identification can be done with the usual θ-functions [19]:

$$\theta[\eta_1](z) = \theta_{0,0}(z), \quad \theta[\eta_2](z) = \theta_{0,1}(z) \,,$$

$$\theta[\eta_3](z) = \theta_{1,1}(z), \quad \theta[\eta_4](z) = \theta_{1,0}(z) \,.$$

Also the following simple variant of Tomae formulae holds:

$$\theta_{0,0}(0)^2 = [(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)]^{\frac{1}{2}}K, \quad \theta_{0,1}(0)^2 = [(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)]^{\frac{1}{2}}K \,,$$

$$\theta_{1,0}(0)^2 = [(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)]^{\frac{1}{2}}K, \quad \theta_{1,1}(0)^2 = 0 \,,$$

where K is the full elliptic integral $K = \dfrac{1}{\pi i}\int_a d\tau/\sqrt{P(\tau)}$, a-cycle surrounds the cut between the points λ_1 and λ_2. The ratios of θ-functions at r are easy to calculate:

$$\theta[\eta_j](r)^2 = \operatorname{Const} P'(\lambda_j)^{-\frac{1}{2}}, \quad P'(\lambda_j) \equiv \prod_{k \neq j} (\lambda_j - \lambda_k) \,.$$

Taking all that into account one realizes that for the parametrization (5.3) considered the equations (5.2) turn, for example for $T = \{1, 2, 4\}$ into

$$\theta_{0,0}(0)^2 \theta_{0,0}(r + z)^2 - \theta_{0,1}(0)^2 \theta_{0,1}(r + z)^2 - \theta_{1,0}(0)^2 \theta_{1,0}(r + z)^2 = 0 \,,$$

$$\theta_{0,0}(0)^2 \theta_{0,0}(r - z)^2 - \theta_{0,1}(0)^2 \theta_{0,1}(r - z)^2 - \theta_{1,0}(0)^2 \theta_{1,0}(r - z)^2 = 0 \,,$$

$$\theta_{0,0}(0)^2 \theta_{0,0}(r + z)\theta_{0,0}(r - z) - \theta_{0,1}(0)^2 \theta_{0,1}(r + z)\theta_{0,1}(r - z)$$

$$- \theta_{1,0}(0)^2 \theta_{1,0}(r + z)\theta_{1,0}(r - z) = \theta_{1,1}(z)^2 \theta_{1,1}(r)^2 \,.$$

Equations (5.2) for other subsets T produce other known identities for θ-functions. In the case of generic n the situation is similar: in parametrization (5.3) Eqs. (5.2) turn into certain special cases of the Riemann identities (Frobenius formulae [19]).

Returning to the integrable models we conclude the following: the formulae (5.3) provide the parametrization of $M(\lambda)$ on the torus (Jacobian), that is why the times t_i should be related to z_j as

$$t_i = \sum_{j=1}^{g} c_i^j z_j$$

the constants c_i^j refer to the particular integrable equation which is not essential for us.

Let us consider now the same situation from a different point of view. What will follow is an extraction from the Adler–Reyman–Semenov-Tian-Shansky approach [7, 21]. Consider the loop algebra $\mathfrak{g} = \widehat{sl}(2)$ with generators J_m^a, $a = 1, 2, 3$; m is an integer. The generators satisfy the relations:

$$[J_m^a, J_n^b] = f_c^{a,b} J_{n+m}^c .$$

The algebra contains two subalgebras: $\mathfrak{g}_-$ generated by J_m^a with $m < 0$ and $\mathfrak{g}_+$ generated by J_m^a with $m \geq 0$. Evidently, the algebra allows finite-dimensional representation ρ_λ:

$$\rho_\lambda(J_m^a) = \lambda^m \sigma^a .$$

By $G = \widehat{SL}(2)$ we denote a group whose Lie algebra coincides with $\mathfrak{g}$. This group contains two subgroups:

$$G_+ = \exp(\mathfrak{g}_+), \quad G_- = \exp(\mathfrak{g}_-) .$$

Consider now the matrix $M(\lambda)$ of the same type as above (traceless, polynomial in λ of degree n with fixed leading coefficient) and introduce the action of G on $M(\lambda)$ as follows:

$$g(M(\lambda)) = \rho_\lambda(g) M(\lambda) \rho_\lambda(g)^{-1}, \quad \text{for } g \in G . \tag{5.4}$$

The subgroups G_+, G_- acting on $M(\lambda)$ generate the orbits O_+, O_- of the form:

$$g_+(M(\lambda)) = \sum_{k=0}^{\infty} m_k' \lambda^k ,$$

$$g_-(M(\lambda)) = \sum_{k=-\infty}^{n} m_k'' \lambda^k, \quad m_n'' = \sigma_3 .$$

The algebraic interpretation of the integrable models is explained as the problem of the description of the intersection J of the orbits O_+, O_-. Clearly this intersection is composed of the matrix of the same type as $M(\lambda)$ itself, so the connection with the matters discussed above is manifest. How to describe the intersection in question in an algebraic way? If $M' \in J$ then there exist g_+, g_- such that

$$g_+(M(\lambda)) = \rho_\lambda(g_+) M(\lambda) \rho_\lambda(g_+)^{-1}$$
$$= g_-(M(\lambda)) = \rho_\lambda(g_-) M(\lambda) \rho_\lambda(g_-)^{-1} . \tag{5.5}$$

It is evident from these equations that the matrix $g(\lambda) = \rho_\lambda(g_-)^{-1} \rho_\lambda(g_+)$ commutes with $M(\lambda)$. The only possibility for the matrix to commute with $M(\lambda)$ which is supposed to be not degenerate for generic λ is to be a function of $M(\lambda)$. Fortunately, additional dependence on λ makes the situation nontrivial. We can consider $g(\lambda)$ of the form:

$$g(\lambda) = \exp\left\{\sum t_i M_i(\lambda)\right\} , \tag{5.6}$$

where $M_i(\lambda) = \lambda^{-i} M(\lambda)$.

Now let us inverse the reasonings which means to start with $g(\lambda)$ in the form (5.6), and to try to construct J. Evidently to this end we need to solve the following Riemann problem: present

$$g(\lambda) = \exp\left\{\sum t_i M_i(\lambda)\right\} = \rho_\lambda(g_-)^{-1} \rho_\lambda(g_+) , \tag{5.7}$$

where the Laurent series for $\rho_\lambda(g_+)$ $(\rho_\lambda(g_-))$ contain only positive (negative) powers. It is clear that only those M_i are essential for which $0 < i < n$, others can be directly moved to either g_+ or g_-, and do not contribute to (5.5). So, we deal with the dependence on $n - 1$ times which can be, actually, identified with the parameters on the Jacobian considered above.

Suppose the problem (5.7) is solved. Then we can introduce "times" dependent matrices $M(\lambda, t_1, \ldots, t_g)$ as being dressed according to (5.5) by g_+ or g_- from (5.7). It is easy to show that the projections of $\lambda^{-i}M(\lambda, t_1, \ldots, t_g)$ onto $\mathfrak{g}_+$ denoted by $\mathfrak{M}_i$ satisfy the equations:

$$\frac{\partial}{\partial t_i}\mathfrak{M}_j - \frac{\partial}{\partial t_j}\mathfrak{M}_i = [\mathfrak{M}_i, \mathfrak{M}_j]$$

which coincide with (5.1).

We would like to emphasize the importance of the vectors ψ_i from the algebraic point of view. The BA function itself does not transform in a reasonable way under the transformations (5.4), but its values at the branching points (ψ_i) do transform under finite-dimensional representation of G when M transforms under (5.4):

$$\psi_i \to G(\lambda_i)\psi_i .$$

This nice property of ψ_i will be important in what follows.

To finish this section let us write down explicitly the equations in terms of m_i^a:

$$\frac{\partial}{\partial t_k}m_j^a = \sum_{q=\max(0, k+j-n)}^{j} f^{abc}m_{j-q+k}^b m_q^c . \tag{5.8}$$

6. Tau-Function and KZ Equations

The most mysterious object in the theory of classical integrable equations is that of the τ-function [3, 4, 12, 20]. Hirota observed that the integrable classical equations can be rewritten as follows. Let us consider a certain function τ, depending upon the arguments $t = \{t_1, t_2, \ldots\}$ and introduce the notations:

$$D_{i_1} \ldots D_{i_k}\tau \cdot \tau(t) \equiv \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}}\tau(t + x)\tau(t - x)|_{x=0} . \tag{6.1}$$

Then according to Hirota the integrable equations can be written as

$$P(D)\tau \cdot \tau = 0 \tag{6.2}$$

for a certain "polynomial" in D. We shall be interested mostly not in the τ-function itself, but in the linear space of functions $\mathfrak{B}$ which is defined as a space spanned by the functions $[D_{i_1} \ldots D_{i_k}\tau \cdot \tau(t)]/\tau(t)^2$. Hirota equations (6.1) imply certain linear dependence in the space.

The main achievement of the Kyoto school in that direction is the formula expressing the τ-function in terms of vacuum expectation in the space of highest weight representation for the algebra $\widehat{sl}(2)$ on level 1 [4]. Consider the Cartan subalgebra generated by $I_k \equiv J_k^3$, $k > 0$. Then the formula takes place

$$\tau(t) = \langle\Omega|g\exp\left(\sum_k t_k I_k\right)|\Omega\rangle ,$$

where $|\Omega\rangle$ is the highest vector of the representation on level 1, $\langle\Omega|$ is the dual vector, g is the element of the central extended group $\widehat{SL}(2)$ which specifies a particular solution of the equation. We want to relate the KZ equations on level zero to the τ-function. The formula (6.2) is both good and bad for our goals. The good thing is that (6.2) links the integrable equations to the highest weight representations which are involved in KZ. The bad thing is that the formula (6.2) on the one hand and our KZ on another deal with different central extensions of $\widehat{sl}(2)$. We shall take the good thing as a hint of the possible connection between $k = 0$ KZ and τ-function, and forget about the bad thing. It should be said also that for the case we are interested in (that of finite-gap integration) the description of g is rather complicated and indirect.

What is τ-function in the finite-gap case? The answer is well known: it is essentially the θ-function $\theta[\delta](z)$, $t_i = \sum_k c_i^k z_k$, where c are some coefficients. How is it connected with the BA-function? The usual answer is that it coincides with the value of the BA-function at a certain point (∞). This answer does not satisfy us because we do not like to use the BA-function at a generic point for its unclear algebraic properties, we want to deal with the values of the BA-function at the branching points (ψ_j) only.

The space $\mathfrak{B}$ is finite-dimensional in the finite-gap case. Let us consider the example $g = 1$. All the times t_i are proportional to one variable on the Jacobian z. The τ-function is $\theta_{1,1}(z)$. The space $\mathfrak{B}$ is generated by 1 and $\dfrac{D^2\tau\cdot\tau}{\tau^2} = \dfrac{\partial^2}{\partial z^2}\log\theta_{1,1}(z)$. The Hirota equation reads as

$$D^4\tau\cdot\tau + c_1 D^2\tau\cdot\tau + c_2\tau^2 = 0$$

which turns into the well-known differential equation for θ-function (equivalent to the equation for the Weierstrass $\mathfrak{P}$-function)

$$\frac{\partial^4}{\partial z^4}\log\theta_{1,1}(z) + 6\left(\frac{\partial^2}{\partial z^2}\log\theta_{1,1}(z)\right)^2 + c_1\frac{\partial^2}{\partial z^2}\log\theta_{1,1}(z) + c_2 = 0$$

for some c_1, c_2.

Our further strategy will be in putting together the space $\mathfrak{B}$, the vectors ψ_i and the solutions of $k = 0$ KZ equations, the idea being the following: ψ_i transforms under the finite-dimensional representation of $\widehat{sl}(2)$, τ is connected with the highest weight representation, hence they should be put together via KZ equations. To start let us consider the following object:

$$v(z) \equiv \bar{\psi}_{1,\varepsilon_1}(z) \ldots \bar{\psi}_{2n,\varepsilon_{2n}}(z)\langle 0|V^{\varepsilon_1}(\lambda_1) \ldots V^{\varepsilon_{2n}}(\lambda_{2n})|0\rangle ,$$

it should be said that $v(z)$ is not a function but rather a set of functions because different choices of blocks of vertex operators (intermediate Verma modules) are possible. This is the same as different choices of cycles A' for $f_{A'}(\lambda_1, \ldots, \lambda_{2n})^{\varepsilon_1, \ldots, \varepsilon_{2n}}$:

$$\langle 0|V^{\varepsilon_1}(\lambda_1) \ldots V^{\varepsilon_{2n}}(\lambda_{2n})|0\rangle_{\text{some block}} = f_{\text{some }A'}(\lambda_1, \ldots, \lambda_{2n})^{\varepsilon_1, \ldots, \varepsilon_{2n}} .$$

So, if necessary we shall indicate a particular component of v as $v_{A'}$. The first important property of the functions $v(z)$ is that their dependence on z is governed by "free" dynamics in Verma module.

Let us take the matrix $M(\lambda)$ as described in the previous section:

$$M(\lambda) = \sum_{i=0}^{n} \lambda^i m_i^a \sigma^a, \quad m_n^a = \delta_{a,3}, \quad \det(M(\lambda)) = \prod_{i=1}^{2n} (\lambda - \lambda_j).$$

Associate to the matrix $M(\lambda)$ the element of $\mathfrak{g}$

$$\widehat{M} = \sum_{i=0}^{n} m_i^a J_i^a$$

such that $M(\lambda) = \rho_\lambda(\widehat{M})$. Similarly, the algebra elements $\widehat{M}_j$ can be introduced:

$$\widehat{M}_j = \sum_{i=0}^{n} m_i^a J_{i-j}^a, \quad \rho_\lambda(\widehat{M}_j) = M_j(\lambda).$$

Clearly, $\widehat{M}_j$ commute among themselves:

$$[\widehat{M}_i, \widehat{M}_j] = 0.$$

Let us consider $M(\lambda)$ as a starting point for the procedure of the previous section, i.e. $M(\lambda)$ is the stationary M-operator corresponding to the moment $t_i = 0$. To the matrix $M(\lambda)$ we attach the set of vectors ψ_i, $i = 1, \ldots, 2n$ as it is explained above. The following important statement holds.

Proposition. *The z-dependence of $v(z)$ is governed by "free" dynamics in the Verma module:*

$$v(z) = \bar{\psi}_{1,\varepsilon_1}(z) \ldots \bar{\psi}_{2n,\varepsilon_{2n}}(z)\langle 0| V^{\varepsilon_1}(\lambda_1) \ldots V^{\varepsilon_{2n}}(\lambda_{2n})|0\rangle$$

$$= \bar{\psi}_{1,\varepsilon_1}(0) \ldots \bar{\psi}_{2n,\varepsilon_{2n}}(0)\langle 0| V^{\varepsilon_1}(\lambda_1) \ldots V^{\varepsilon_{2n}}(\lambda_{2n})\exp\left(\sum_{i=1}^{n-1} \widehat{M}_i t_i\right)|0\rangle, \quad (6.3)$$

where z and t are connected by some linear transformation: $t_i = c_i^j z_j$.

In what follows we shall allow ourselves inaccuracy using both notations: $\psi_i(z)$ and $\psi_i(t)$ assuming that they coincide when z and t are properly related. As it was explained in the previous section the functions $\bar{\psi}_i$ satisfy the following equation:

$$\frac{\partial}{\partial t_k} \bar{\psi}_i(t) = \bar{\psi}_i(t)\mathfrak{M}(\lambda_i, t),$$

where the t-dependence of $\mathfrak{M}$ is due to (5.8). This equation allows us to express all the higher derivatives of $\bar{\psi}$ at the moment $t = 0$ in terms of derivatives of $\mathfrak{M}$. Let us prove Eqs. (6.3) comparing all the derivatives at $t = 0$ of LHS and RHS.

With the first derivative it is trivial, in LHS one has:

$$\frac{\partial}{\partial t_i}(\bar{\psi}_{1,\varepsilon_1}(t) \ldots \bar{\psi}_{2n,\varepsilon_{2n}}(t)\langle 0| V^{\varepsilon_1}(\lambda_1) \ldots V^{\varepsilon_{2n}}(\lambda_{2n})|0\rangle)|_{t=0}$$

$$= \sum_{p=1}^{2n} \bar{\psi}_{1,\varepsilon_1}(0) \ldots \bar{\psi}_{p-1,\varepsilon_{p-1}}(0)\bar{\psi}_{p,\varepsilon_p'}(0)\bar{\psi}_{p+1,\varepsilon_{p+1}}(0) \ldots \bar{\psi}_{2n,\varepsilon_{2n}}(0)$$

$$\times \mathfrak{M}_{i,\varepsilon_p}^{\varepsilon_p'}(\lambda_p)\langle 0| V^{\varepsilon_1}(\lambda_1) \ldots V^{\varepsilon_{2n}}(\lambda_{2n})|0\rangle.$$

In RHS one has:

$$\frac{\partial}{\partial t_i}\bar\psi_{1,\varepsilon_1}(0)\ldots\bar\psi_{2n,\varepsilon_{2n}}(0)\langle 0|V^{\varepsilon_1}(\lambda_1)\ldots V^{\varepsilon_{2n}}(\lambda_{2n})\exp\left(\sum_{i=1}^{n-1}\widehat M_i t_i\right)|0\rangle|_{t=0}$$

$$=\bar\psi_{1,\varepsilon_1}(0)\ldots\bar\psi_{2n,\varepsilon_{2n}}(0)\langle 0|V^{\varepsilon_1}(\lambda_1)\ldots V^{\varepsilon_{2n}}(\lambda_{2n})\widehat M_{i,+}|0\rangle$$

$$=\sum_{p=1}^{2n}\bar\psi_{1,\varepsilon_1}(0)\ldots\bar\psi_{p-1,\varepsilon_{p-1}}(0)\bar\psi_{p,\varepsilon_p'}(0)\bar\psi_{p+1,\varepsilon_{p+1}}(0)\ldots\bar\psi_{2n,\varepsilon_{2n}}(0)$$

$$\times\mathfrak{M}_{i,\varepsilon_p}^{\varepsilon_p'}(\lambda_p)\langle 0|V^{\varepsilon_1}(\lambda_1)\ldots V^{\varepsilon_{2n}}(\lambda_{2n})|0\rangle\ .$$

Evaluating the RHS we used the notation $\widehat M_{i,+}$ for the projection of $\widehat M_i$ onto $\mathfrak{g}_+$; recall that $J_m^a|0\rangle = 0$, $m < 0$. We moved $\widehat M_{i,+}$ to the left using the properties of the vertex operators, $\widehat M_{i,+}$ annihilates the left vacuum, also $\mathfrak{M}_i(\lambda_p) = \rho_{\lambda_p}(\widehat M_{i,+})$. A less trivial computation is the one for the second derivatives. On the LHS one gets

$$\frac{\partial}{\partial t_i}\frac{\partial}{\partial t_j}(\bar\psi_{1,\varepsilon_1}(t)\ldots\bar\psi_{2n,\varepsilon_{2n}}(t)\langle 0|V^{\varepsilon_1}(\lambda_1)\ldots V^{\varepsilon_{2n}}(\lambda_{2n})|0\rangle)|_{t=0}$$

$$=\sum_{p<q}^{2n}\bar\psi_{1,\varepsilon_1}(0)\ldots\bar\psi_{p-1,\varepsilon_{p-1}}(0)\bar\psi_{p,\varepsilon_p'}(0)\bar\psi_{p+1,\varepsilon_{p+1}}(0)\ldots$$

$$\times\bar\psi_{q-1,\varepsilon_{q-1}}(0)\bar\psi_{q,\varepsilon_q'}(0)\bar\psi_{q+1,\varepsilon_{q+1}}(0)\bar\psi_{2n,\varepsilon_{2n}}(0)$$

$$\times\{\mathfrak{M}_{i,\varepsilon_p''}^{\varepsilon_p'}(\lambda_p)\mathfrak{M}_{j,\varepsilon_q}^{\varepsilon_q''}(\lambda_q)+\mathfrak{M}_{i,\varepsilon_q''}^{\varepsilon_q'}(\lambda_q)\mathfrak{M}_{j,\varepsilon_p}^{\varepsilon_p''}(\lambda_p)\}\langle 0|V^{\varepsilon_1}(\lambda_1)\ldots V^{\varepsilon_{2n}}(\lambda_{2n})|0\rangle$$

$$+\sum_{p=1}^{2n}\bar\psi_{1,\varepsilon_1}(0)\ldots\bar\psi_{p-1,\varepsilon_{p-1}}(0)\bar\psi_{p,\varepsilon_p'}(0)\bar\psi_{p+1,\varepsilon_{p+1}}(0)\ldots\bar\psi_{2n,\varepsilon_{2n}}(0)$$

$$\times\{\mathfrak{M}_{i,\varepsilon_p''}^{\varepsilon_p'}(\lambda_p)\mathfrak{M}_{j,\varepsilon_p}^{\varepsilon_p''}(\lambda_p)+\partial_i\mathfrak{M}_{j,\varepsilon_p}^{\varepsilon_p'}(\lambda_p)\}\langle 0|V^{\varepsilon_1}(\lambda_1)\ldots V^{\varepsilon_{2n}}(\lambda_{2n})|0\rangle\ .$$

To transform the RHS one has to deal with $\widehat M_i\widehat M_j|0\rangle$ which can be evaluated due to Eqs. (5.8):

$$\widehat M_i\widehat M_j|0\rangle = (\widehat M_{i,+}+\widehat M_{i,-})\widehat M_j|0\rangle = \widehat M_{i,+}\widehat M_{j,+}|0\rangle + \partial_i\widehat M_{j,+}|0\rangle\ . \qquad (6.4)$$

Now moving $\widehat M_{i,+}\widehat M_{j,+}$ and $\partial_i\widehat M_{j,+}$ to the left one gets (6.4). The consideration of the general case is quite similar to the case of the second derivative which is the most demonstrative one.

Thus, we have shown that the functions $v(z)$ possess the nice property (6.3) which makes them similar to the τ-function.

It should be mentioned that the starting $M(\lambda)$ can be taken in an especially simple way. For example we can divide the set of λ_i into two subsets ($B = S \cup S'$, $\#S = \#S' = n$) and define M as follows:

$$M(\lambda) = \begin{pmatrix}\prod_{i\in S}(\lambda-\lambda_i)+\prod_{i\in S'}(\lambda-\lambda_i), & \prod_{i\in S}(\lambda-\lambda_i)-\prod_{i\in S'}(\lambda-\lambda_i) \\ \prod_{i\in S'}(\lambda-\lambda_i)-\prod_{i\in S}(\lambda-\lambda_i), & -\prod_{i\in S}(\lambda-\lambda_i)-\prod_{i\in S'}(\lambda-\lambda_i)\end{pmatrix}\ .$$

In that case the vectors ψ_i are also simple.

We have to connect the functions $v(z)$ with some familiar objects. There should be a clever way to get the result which will be announced soon. But we do not know that way which makes us proceed to the calculations. Let us consider the elliptic

case. Recall that the covectors $\bar{\psi}_i$ are given in the elliptic case by:

$$\bar{\psi}_1(z) \simeq \left(\frac{\theta_{0,0}(r+z)}{\theta_{1,1}(z)}, \frac{\theta_{0,0}(r-z)}{\theta_{1,1}(z)} \right),$$

$$\bar{\psi}_2(z) \simeq \left(\frac{\theta_{0,1}(r+z)}{\theta_{1,1}(z)}, \frac{\theta_{0,1}(r-z)}{\theta_{1,1}(z)} \right),$$

$$\bar{\psi}_3(z) \simeq \left(\frac{\theta_{1,1}(r+z)}{\theta_{1,1}(z)}, \frac{\theta_{1,1}(r-z)}{\theta_{1,1}(z)} \right),$$

$$\bar{\psi}_4(z) \simeq \left(\frac{\theta_{1,0}(r+z)}{\theta_{1,1}(z)}, \frac{\theta_{1,0}(r-z)}{\theta_{1,1}(z)} \right).$$

In what follows we shall perform calculations up to constants which will be outlined by using $\simeq$. There are two solutions to KZ equations in the elliptic case which are given by

$$f_a(\lambda_1, \ldots, \lambda_4)^{\varepsilon_1, \ldots, \varepsilon_4} \simeq \theta[\eta_T](0)^2 \partial^2 \theta[\eta_T](0)^2,$$

$$f_{a'}(\lambda_1, \ldots, \lambda_4)^{\varepsilon_1, \ldots, \varepsilon_4} \simeq \theta[\eta_T](0)^2 [K\partial^2 + 1] \theta[\eta_T](0)^2,$$

the correspondence between $\varepsilon_1, \ldots, \varepsilon_4$, T and conventional notations for elliptic θ-functions is given by (4.19). This formulae suggest to generalize the expression for $v(z)$ substituting in it $\theta[\eta_T](0)^2 \theta[\eta_T](x)^2$ instead of the solutions of KZ. Doing that one gets the following

$$v(z, x) \simeq \theta_{1,1}(z)^{-4}$$

$$\times [\theta_{1,0}(0)^2 \theta_{1,0}(x)^2 \{\theta_{0,0}(r+z)\theta_{0,1}(r+z)\theta_{1,1}(r-z)\theta_{1,0}(r-z)$$

$$+ \theta_{0,0}(r-z)\theta_{0,1}(r-z)\theta_{1,1}(r+z)\theta_{1,0}(r+z)\}$$

$$+ \theta_{0,1}(0)^2 \theta_{0,1}(x)^2 \{\theta_{0,0}(r+z)\theta_{0,1}(r-z)\theta_{1,1}(r-z)\theta_{1,0}(r+z)$$

$$+ \theta_{0,0}(r-z)\theta_{0,1}(r+z)\theta_{1,1}(r+z)\theta_{1,0}(r-z)\}$$

$$+ \theta_{0,0}(0)^2 \theta_{0,0}(x)^2 \{\theta_{0,0}(r+z)\theta_{0,1}(r-z)\theta_{1,1}(r+z)\theta_{1,0}(r-z)$$

$$+ \theta_{0,0}(r-z)\theta_{0,1}(r+z)\theta_{1,1}(r-z)\theta_{1,0}(r+z)\}].$$

This expression can be simplified via Riemann identities, the result being quite simple:

$$v(z, x) \simeq \frac{\theta_{1,1}(z+x)\theta_{1,1}(z-x)}{\theta_{1,1}(z)^2},$$

which leads to the following nice formulae for $v_a(z)$, $v_{a'}(z)$:

$$v_a(z) \simeq \frac{D^2 \tau \cdot \tau}{\tau^2}(z), \quad v_{a'}(z) \simeq K \frac{D^2 \tau \cdot \tau}{\tau^2}(z) + 1,$$

where $\theta_{1,1}$ is denoted by τ, D is Hirota derivative.

Thus by convoluting $\bar{\psi}_i$ and the solutions of KZ in the elliptic case we got exactly the basis of the space $\mathfrak{B}$. Combining that with the proposition above we also realize that the dynamics in the space $\mathfrak{B}$ is governed by the free dynamics in the Verma module. We would suppose that this nice connection holds in the generic

case as well. In order to prove that we have to use Conjecture 1 from Sect. 4 together with the following

Conjecture 2. The following identity (up to neglectable constants) for the θ-functions on HES holds:

$$\sum_{T \subset B, \, \# T = g+1} \theta[\eta_T](0)^2 \theta[\eta_T](x)^2 \prod_{j=1}^{2g+2} \theta[\eta_j](r + \varepsilon_j z)$$

$$\simeq \theta[\delta](z)^{2g} \theta[\delta](z + x) \theta[\delta](z - x) ,$$

where T and $\{\varepsilon_i\}$ are connected in usual way.

The author strongly believes in this conjecture, still he has been unable to prove it.

If conjectures 1, 2 are both true in the generic case then the following relation should take place:

$$\mathfrak{B} \ni \frac{Q_{A'} \tau \cdot \tau}{\tau^2} = \bar{\psi}_{1,\varepsilon_1} \ldots \bar{\psi}_{2n,\varepsilon_{2n}} f_{A'} (\lambda_1, \ldots, \lambda_{2n})^{\varepsilon_1, \ldots, \varepsilon_{2n}} , \tag{6.5}$$

where $\tau(z) = \theta[\delta](z)$. We would suppose also that there are enough θ-constants hidden in $f_{A'}$, i.e. that all the independent Hirota derivatives of τ can be obtained by linear combinations of (6.5) for a different A'.

The formula (6.5) establishes the relation between $\bar{\psi}_i$ which transform under finite-dimensional representations of $\widehat{sl}(2)$ and the space of Hirota derivatives of the τ-function (the space of classical fields) via the solutions of KZ equations on level 0 which we can call classical form factors. This is exactly the formula we were looking for; its similarity with quantum formula (1.1) is manifest.

To finish this section we would like to present one more interesting formula:

$$\langle 0 | V^{\varepsilon_1}(\lambda_1) \cdots V^{\varepsilon_{2n}}(\lambda_{2n}) \widehat{M}^{2n} \exp(\sum t_i \widehat{M}_i) | 0 \rangle$$

$$= \psi_1^{\varepsilon_1}(0) \ldots \psi_{2n}^{\varepsilon_{2n}}(0) \{ \bar{\psi}_{1,\varepsilon_1'}(t) \ldots \bar{\psi}_{2n,\varepsilon_{2n}'}(t) \langle 0 | V^{\varepsilon_1'}(\lambda_1) \cdots V^{\varepsilon_{2n}'}(\lambda_{2n}) | 0 \rangle |_{t=0} \}$$

$$= \psi_1^{\varepsilon_1}(0) \ldots \psi_{2n}^{\varepsilon_{2n}}(0) v(t) .$$

This formula is due to the fact that

$$\rho \lambda_i (\widehat{M})^2 = (\psi_i \otimes \bar{\psi}_i)^2 = 0 .$$

Notice that as usual $\widehat{M}$ and $\lambda_1, \ldots, \lambda_{2n}$ must be connected: $\det(\rho \lambda_i(\widehat{M})) = \prod (\lambda - \lambda_i)$.

7. Conclusions

Let us return to the very beginning of the paper. We were wondering whether a formula similar to (1.1) exists in classics. We have answered this question in the last section of the paper. Really, the formula (6.5) is very much similar to (1.1). It connects classical local fields (Hirota derivatives of the τ-function) with the tensor product of finite-dimensional representations of the affine algebra via the classical form factors (the solutions of KZ on level 0). So, the similarity of the formulae (1.1) and (6.5) is described by:

$$\psi_i, \bar{\psi}_i \leftrightarrow Z(\beta_i), Z^*(\beta_i)$$

$$\mathfrak{B} \leftrightarrow \mathfrak{A}$$

$$f \leftrightarrow F .$$

The quantum formula needs also the summation over all particle states which can be interpreted as summation over all HES, the number of particles being related to the genus of surface, the rapidity being considered as the positions of the branching points, i.e. the moduli of the surfaces. Notice that in the classical limit the difference between the finite-dimensional representation and conjugate one (which were responsible for particles and antiparticles in the quantum case) essentially disappears.

So, we would suppose the following procedure of quantization to be possible. We start with the family of finite-gap solutions of the integrable model with $\widehat{sl}(2)$ symmetry. Different solutions are parametrized by the intersections of the orbits of $\mathfrak{g}_\pm$ as it is explained in Sect. 5. These intersections are parametrized by the matrix $M(\lambda)$, but we prefer to parametrize them by the set $\{\psi_i\}$ connected with $M(\lambda)$ as it is explained above. The vectors ψ_i transform under the finite-dimensional representation of the affine algebra. We can take

$$H_{cl} = \bigoplus_{n=1}^{\infty} \int_{\lambda_1 < \cdots < \lambda_{2n}} V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{2n}} \tag{7.1}$$

(V_λ is the space of the representation ρ_λ) as a completion of the manifold of classical finite-gap solutions. Not every vector from this space is good for the classical solutions, but only those which can be presented in the form $\psi_i \otimes \cdots \otimes \psi_{2n}$ with the set $\{\psi_i\}$ associated to a matrix $M(\lambda)$ as described above. For every particular element of this form the classical fields are given by convolution with the classical form factor. In fact the space (7.1) coincides with the space of states of the quantum model H_p(1.3). Also the quantum form factor is a quantization of the classical one which is essentially due to the quantization of the algebra underlying the theory ($\widehat{sl}(2) \to D(Y)$). So, we propose that the following way for quantization should be possible: we extend the set of classical solutions to the space H_{cl} and identify it with the space of states of the quantum model, the rest of the quantization is in quantizing the symmetry algebra. It should be strongly emphasized, however, that the space-time of the quantum model has nothing to do with the classical "times" t_i; it appears as a result of quantization as it is explained in Sect. 3.

Let us finish with several remarks. The finite-gap solution constitutes in the classical theory a nice but small subset of solutions. Generally, there are infinite-gap solutions in which the finite-gap ones are of measure zero. The infinite-gap solutions are rather ugly ones, no reasonable theory is available for them. So, since for the quantization only finite-gap solutions are needed, we have a nice example of the usual phenomena: quantization takes everything good from the classical theory and forgets about bad things typical for it. There is a puzzling connection between the matters discussed in the present paper and those considered in [15] in the connection with the theory of strings (this remark is due to H. Ooguri). The constructions of the present paper can be generalized to the $su(N)$ case for which the algebra $\widehat{sl}(N)$ and the Riemann surfaces with the branching points of Nth order are responsible; the form factors of the corresponding $su(N)$-chiral Gross–Neveu are given in [25]. But probably this is not the best way for the generalization. We would better proceed to the consideration of arbitrary Riemann surfaces which are associated to the KP-equation. This case should cover all the $su(N)$-invariant Gross–Neveu models and, probably, will lead to something essentially new.

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