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An Algebraic Characterization of Vacuum States in Minkowski Space

Detlev Buchholz[1] **and Stephen J. Summers**[2]

[1] II. Institut für Theoretische Physik, Universität Hamburg, D-22761 Hamburg, Germany
[2] Department of Mathematics, University of Florida, Gainesville, FL 32611, USA

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Abstract. An algebraic characterization of vacuum states on nets of C^*-algebras over Minkowski space is given and space-time translations are reconstructed with the help of the modular structures associated with such states. The result suggests that a "condition of geometrical modular action" might hold in quantum field theories on a wider class of spacetime manifolds.

I. Introduction

In the algebraic setting of quantum field theory one commonly characterizes vacuum states on an algebra of observables $\mathscr{A}$ by their invariance and spectral properties with respect to the group of space-time translations [1, 2]. In this note we pose the question whether it is possible to characterize these states using only the net structure of the observables, i.e. the assignment $\mathscr{O} \to \mathscr{A}(\mathscr{O})$ of spacetime regions $\mathscr{O}$ to local subalgebras $\mathscr{A}(\mathscr{O})$ of $\mathscr{A}$. The existence of spacetime symmetries will not be assumed from the outset.

This question is motivated by the following considerations. First of all there is a matter of principle: it is believed that in algebraic quantum field theory the physical information of a model is encoded in the relative positions of the local algebras in a given net. One should therefore be able to characterize the vacuum and to determine the spacetime symmetries using only this net structure. Secondly, the question is of relevance to the theory of quantum fields on curved spacetimes, where one deals with physical systems for which, in general, one has little in the way of spacetime symmetries to simplify matters. Hence the characterization of physical states by means of the net structure is also of practical interest.

In the present approach we start from the property of isotony of any net of observables. This property imposes special relations between the modular structures associated to the local algebras and a given faithful state. Only these data, entirely determined by the net and the particular state chosen, will be used here. A remarkable fact is known in this connection: the modular objects associated to a vacuum state and algebras over Minkowski space corresponding to wedge-shaped regions $\mathscr{W}$, bounded by two characteristic planes, contain geometric information about the spacetime itself, as well as dynamical information about the representation [3, 4]. They act in a geometrical manner on the net and can be used to reconstruct the symmetries of Minkowski space, in other words, the Poincaré group.

We will show in this note that this specific geometric action is a distinctive feature of the modular structure associated with vacuum states. It thereby leads to a characterization of these states in terms of the net structure. Moreover, the space-time translations can be reconstructed from the modular objects and shown to be unique. These results suggest that a "condition of geometrical modular action" might be suitable for an algebraic characterization of states of physical interest in quantum field theories on more general spacetime manifolds. We comment on this idea at the end of this note.

II. The Vacuum in Minkowski Space

For the characterization of vacuum states in Minkowski space we proceed from a net $\mathscr{A}$ of C^*-algebras $\{\mathscr{A}(\mathscr{O})\}_{\mathscr{O}\in\mathfrak{R}}$ satisfying the condition of isotony, with $\mathfrak{R}$ equal to the set of all (open) wedges $\mathscr{W}$ and double cones $\mathscr{K}$ in $\mathbb{R}^4$. It is useful (and standard) to assume that the algebras $\mathscr{A}(\mathscr{O})$, $\mathscr{O}\in\mathfrak{R}$, are continuous from the inside in the sense that they are the C^*-inductive limits of all double cone algebras $\mathscr{A}(\mathscr{K})$ for which the closures of $\mathscr{K}$ are contained in the interior of $\mathscr{O}$.

We look upon $\mathscr{A}$ as some abstract net on $\mathbb{R}^4$ without any *a priori* dynamical information. In particular, we assume neither the existence of space-time translations nor that of causal properties (i.e. locality). These more detailed features will be attributed to the GNS representations of $\mathscr{A}$ induced by suitable states ω. It is our aim to characterize within this general setting those states which can be regarded as vacuum states of some relativistic quantum field theory in Minkowski space.

Given any state ω on $\mathscr{A}$, we consider its GNS representation $(\mathscr{H}, \pi, \Omega)$ and the corresponding net $\mathscr{R}$ of von Neumann algebras $\mathscr{R}(\mathscr{O}) = \bigcap_{\mathscr{W}\supseteq\mathscr{O}}\pi(\mathscr{A}(\mathscr{W}))''$, $\mathscr{O}\in\mathfrak{R}$. Note that $\mathscr{R}(\mathscr{O})\supseteq\pi(\mathscr{A}(\mathscr{O}))$ because of isotony. Anticipating the Reeh–Schlieder property of vacuum states, we shall be interested only in states ω whose GNS-vector Ω is cyclic and separating for the von Neumann algebras $\mathscr{R}(\mathscr{W})$ corresponding to arbitrary wedges $\mathscr{W}$. With this input the modular operators $\Delta_{\mathscr{W}}$ and modular conjugations $J_{\mathscr{W}}$ associated with $(\mathscr{R}(\mathscr{W}), \Omega)$ are well defined, and we shall state our conditions on ω in terms of these modular objects.

We denote by $\mathscr{W}^{(0)}$ any wedge whose edge passes through the origin of $\mathbb{R}^4$, and by $x^{(0)}$, $y^{(0)}$, etc. any translation in the two-dimensional subspace $\mathbb{R}^2_{\mathscr{W}^{(0)}}$ generated by the two lightlike directions fixing the boundaries of $\mathscr{W}^{(0)}$. To simplify notation we denote the modular conjugations associated with $(\mathscr{R}(\mathscr{W}^{(0)} + z^{(0)}), \Omega)$ by $J_{z^{(0)}}$. We distinguish a special class of states ω on $\mathscr{A}$ by making the following assumption on the action induced by the corresponding modular conjugations on the net $\mathscr{R}$.

Assumption 1. For each wedge $\mathscr{W}^{(0)}$ and every $\mathscr{O}\in\mathfrak{R}$ one has the action

$$J_{z^{(0)}}\mathscr{R}(\mathscr{O})J_{z^{(0)}} = \mathscr{R}(\Lambda_{\mathscr{W}^{(0)}}\mathscr{O} + 2z^{(0)})\,, \tag{2.1}$$

where $\Lambda_{\mathscr{W}^{(0)}}$ is a reflection which is equal to -1 on $\mathbb{R}^2_{\mathscr{W}^{(0)}}$ and equal to 1 on the two dimensional subspace of $\mathbb{R}^4$ which forms the edge of $\mathscr{W}^{(0)}$.

It was shown in [3] that Assumption (1) holds in all irreducible vacuum representations of any net of local algebras that is locally associated with a quantum field, when one takes the modular objects associated with the vacuum state (see also [5]). Note that Assumption (1) *entails* the locality of the net $\mathscr{R}$. In fact, by the Tomita–Takesaki theory one has $J_{\mathscr{W}}\mathscr{R}(\mathscr{W})J_{\mathscr{W}} = \mathscr{R}(\mathscr{W})'$, for any wedge $\mathscr{W}$, and (2.1) implies that $J_W\mathscr{R}(\mathscr{W})J_{\mathscr{W}} = \mathscr{R}(\mathscr{W}')$, where $\mathscr{W}'$ denotes the spacelike complement of $\mathscr{W}$. Hence the net $\mathscr{R}$ satisfies duality (thus, locality) on the wedges and it is standard to derive from this the locality of the net $\{\mathscr{R}(\mathscr{O})\}_{\mathscr{O}\in\mathfrak{R}}$. In this sense any state ω on $\mathscr{A}$ satisfying Assumption (1) induces on $\mathbb{R}^4$, through the action of the associated modular conjugations, the causal structure of Minkowski space. By Assumption (1) we also have the following relation for each $\mathscr{W}^{(0)}$ and any $x^{(0)}$, $y^{(0)}$ as above,

$$J_{x^{(0)}}J_{y^{(0)}}\mathscr{R}(\mathscr{O})J_{y^{(0)}}J_{x^{(0)}} = \mathscr{R}(\mathscr{O} + 2x^{(0)} - 2y^{(0)}) . \tag{2.2}$$

Since the modular conjugations are antiunitary involutions, the products $J_{x^{(0)}}J_{y^{(0)}}$ are unitary operators. Relation (2.2) shows that these products induce translations of the local algebras, possibly accompanied by internal symmetry transformations. The latter possibility is excluded by our second assumption.

Assumption 2. For each fixed $\mathscr{W}^{(0)}$ and any $x^{(0)}$, $y^{(0)}$ as above, the unitaries $J_{x^{(0)}}J_{y^{(0)}}$ depend only on the difference $x^{(0)} - y^{(0)}$.

Once again, this assumption follows from what has been proven by Bisognano and Wichmann for nets in vacuum representations locally associated with Wightman fields [3], cf. also [4]. With this assumption it is meaningful to set

$$V(2[x^{(0)} - y^{(0)}]) \equiv J_{x^{(0)}}J_{y^{(0)}} . \tag{2.3}$$

Note that since the modular conjugations leave Ω invariant, these unitary operators do so, as well. We shall construct a representation of the translation group out of these operators.

Lemma 2.1. *Under Assumption* (2) *the map* $x^{(0)} \to V(x^{(0)})$ *gives a strongly continuous unitary representation of the additive group* $\mathbb{R}^2_{\mathscr{W}^{(0)}}$.

Proof. From Assumption (2) it follows that

$$V(x^{(0)})V(y^{(0)}) = J_{\frac{1}{2}x^{(0)}}J_0 \cdot J_0 J_{-\frac{1}{2}y^{(0)}} = J_{\frac{1}{2}x^{(0)}}J_{-\frac{1}{2}y^{(0)}} = V(x^{(0)} + y^{(0)}) ,$$

$$V(x^{(0)})^* = J_0 J_{\frac{1}{2}x^{(0)}} = V(-x^{(0)}) = V(x^{(0)})^{-1} .$$

Thus $x^{(0)} \to V(x^{(0)})$ is a unitary representation of the group $\mathbb{R}^2_{\mathscr{W}^{(0)}}$. If $x^{(0)} \in \mathscr{W}^{(0)}$ tends to 0, we have $J_{x^{(0)}} \to J_0$ in the strong operator topology (we may appeal to the technical lemmas in the Appendix, since $\mathscr{W}^{(0)} + x^{(0)} \subseteq \mathscr{W}^{(0)}$) and consequently $V(x^{(0)}) \to 1$. Due to the identity $\|(V(-x^{(0)}) - 1)\Phi\| = \|(1 - V(x^{(0)}))\Phi\|$, it also follows that $V(-x^{(0)}) \to 1$. This establishes the continuity of the map $\lambda \to V(\lambda \cdot e^{(0)})$, $\lambda \in \mathbb{R}$, for any $e^{(0)} \in \mathscr{W}^{(0)}$. Using the previously verified group structure, it follows that $x^{(0)} \to V(x^{(0)})$ is continuous in the strong operator topology. □

For fixed $\mathscr{W}^{(0)}$ we shall call the collection of wedges $\{\mathscr{W}^{(0)} + z^{(0)} \mid z^{(0)} \in \mathbb{R}^2_{\mathscr{W}^{(0)}}\}$ a *coherent system of wedges*. As we have shown in Lemma 2.1, associated to any coherent system of wedges is a strongly continuous unitary representation of $\mathbb{R}^2_{\mathscr{W}^{(0)}}$. Introducing proper coordinates $x = (x_0, x_1, x_2, x_3)$ on $\mathbb{R}^4$ we pick the wedges

$\mathscr{W}_i^{(0)} = \{x \in \mathbb{R}^4 \mid x_i > |x_0|\}$, $i = 1, 2, 3$, and consider the corresponding representations $V_i(\mathbb{R}^2_{\mathscr{W}_i^{(0)}})$, $i = 1, 2, 3$.

Proposition 2.2. *Under Assumptions (1) and (2) the unitary operators* $\{V_1(x) \mid x = (x_0, x_1, 0, 0)\}$, $\{V_2(x) \mid x = (0, 0, x_2, 0)\}$, $\{V_3(x) \mid x = (0, 0, 0, x_3)\}$, *with* $x_0, x_1, x_2, x_3 \in \mathbb{R}$ *arbitrary, determine a strongly continuous, unitary representation* $V(\mathbb{R}^4)$ *of the translations on* $\mathbb{R}^4$ *that leaves* Ω *invariant and acts geometrically correctly on the net,* i.e.

$$V(x)\mathscr{R}(\mathcal{O})V(x)^{-1} = \mathscr{R}(\mathcal{O} + x), \quad \forall \mathcal{O} \in \mathfrak{R}, x \in \mathbb{R}^4 .$$

Proof. By (2.2) and the preceding lemma, the proposition will follow as soon as one sees that the three sets of unitary operators above mutually commute. To illustrate the simple idea of the proof, consider the operators $\{V_2(x) \mid x = (0, 0, x_2, 0)\}$. One has for the coherent system $\{\mathscr{W}_1^{(0)} + z^{(0)} \mid z^{(0)} \in \mathbb{R}^2_{\mathscr{W}_1^{(0)}}\}$ the elementary fact that $\mathscr{W}_1^{(0)} + z^{(0)} + x = \mathscr{W}_1^{(0)} + z^{(0)}$ for any $x = (0, 0, x_2, 0)$. Thus, by (2.2), one has $V_2(x)\mathscr{R}(\mathscr{W}_1^{(0)} + z^{(0)})V_2(x)^{-1} = \mathscr{R}(\mathscr{W}_1^{(0)} + z^{(0)})$ for each such translation x. Since, in addition, $V_2(x)\Omega = \Omega$ for all such x, $\{V_2(x) \mid x = (0, 0, x_2, 0)\}$ is a collection of unitary operators leaving the pairs $(\mathscr{R}(\mathscr{W}_1^{(0)} + z^{(0)}), \Omega)$ invariant. Hence $V_2(x)$ must commute with the modular objects associated with these pairs (see [6, Theorem 3.2.18]) and consequently with $V_1(y)$. The other cases are argued similarly. □

This result shows that in the GNS representation space of any state satisfying Assumptions (1) and (2) one has a representation of the translations. There is, of course, nothing unique about this construction, since many different choices of coherent systems of wedges can be made. We shall return to the question of uniqueness below.

The representation $V(\mathbb{R}^4)$ fulfills every desideratum of the translations in a vacuum representation except possibly one – the spectrum condition. And, in fact, there *are* simple examples of nets and states satisfying Assumptions (1) and (2) which generate representations of the translations that violate the spectrum condition [7]. In order for the representation $V(\mathbb{R}^4)$ to satisfy this condition, it is necessary and sufficient that the modular groups $\Delta_{\mathscr{W}}^{i\lambda}$ act upon the translations $V(\mathbb{R}^2_{\mathscr{W}})$ in the geometrical manner of Lorentz boosts. More precisely, the following assumption has to be satisfied.

Assumption 3. For every wedge $\mathscr{W}^{(0)}$ and every positive lightlike vector $e^{(0)}$ such that $\mathscr{W}^{(0)} + e^{(0)} \subset \mathscr{W}^{(0)}$ there holds

$$\Delta_{\mathscr{W}^{(0)}}^{i\lambda} V(e^{(0)}) \Delta_{\mathscr{W}^{(0)}}^{-i\lambda} = V(e^{-2\pi\lambda} e^{(0)}), \quad \forall \lambda \in \mathbb{R} . \tag{2.4}$$

(From (2.4) it follows that also $\Delta_{\mathscr{W}^{(0)}}^{i\lambda} V(-e^{(0)}) \Delta_{\mathscr{W}^{(0)}}^{-i\lambda} = V(-e^{-2\pi\lambda} e^{(0)})$.)

Proposition 2.3. *Let* $V(\mathbb{R}^4)$ *be the representation of the translations on Minkowski space constructed in Proposition 2.2. Then* $V(\mathbb{R}^4)$ *satisfies the relativistic spectrum condition, i.e.*[1] $sp(V) \subset \bar{V}_+$, *if and only if relation (2.4) holds for all wedges.*[2]

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[1] V_+ (resp. V_-) denotes the forward (resp. backward) lightcone

[2] An analogous result that yields the spectrum condition for a given representation of the translations is given under somewhat more restrictive conditions in [8]

Remark. At the cost of a significantly longer argument [7], this result can be established requiring Eq. (2.4) only for the wedges in the three coherent systems used to construct the representation $V(\mathbb{R}^4)$.

Proof. From the work of Bisognano and Wichmann [3] and Borchers [4] it follows that in a vacuum representation (i.e. a translation-covariant representation having an invariant vector and satisfying the spectrum condition) Eq. (2.4) holds. Hence we need only give a proof of the implication in the other direction.

Choose a wedge $\mathscr{W}^{(0)}$ and a positive lightlike vector $e^{(0)}$ such that $\mathscr{W}^{(0)} + e^{(0)} \subset \mathscr{W}^{(0)}$. For simplicity of notation, the subscripts $\mathscr{W}^{(0)}$ on the modular objects Δ, J associated to the pair $(\mathscr{R}(\mathscr{W}^{(0)}), \Omega)$ will be suppressed in the following. First note that by assumption, $V(e^{(0)})\mathscr{R}(\mathscr{W}^{(0)})V(e^{(0)})^{-1} \subset \mathscr{R}(\mathscr{W}^{(0)})$, which entails $V(e^{(0)})\mathscr{R}(\mathscr{W}^{(0)})'V(e^{(0)})^{-1} \supset \mathscr{R}(\mathscr{W}^{(0)})'$ and therefore $\mathscr{R}(\mathscr{W}^{(0)})' \supset V(e^{(0)})^{-1}\mathscr{R}(\mathscr{W}^{(0)})'V(e^{(0)})$. Then, with $X(e^{(0)}) \equiv JV(e^{(0)})^{-1}J$, one has $V(e^{(0)})^{-1}J = JX(e^{(0)})$ and

$$\begin{aligned} X(e^{(0)})\mathscr{R}(\mathscr{W}^{(0)})X(e^{(0)})^{-1} &= JV(e^{(0)})^{-1}\mathscr{R}(\mathscr{W}^{(0)})'V(e^{(0)})J \subset J\mathscr{R}(\mathscr{W}^{(0)})'J \\ &= \mathscr{R}(\mathscr{W}^{(0)}) . \end{aligned} \tag{2.5}$$

Let now $A' \in \mathscr{R}(\mathscr{W}^{(0)})'$ and $B \in \mathscr{R}(\mathscr{W}^{(0)})$. By the Tomita–Takesaki theory, the function

$$z \to f(z) \equiv \langle \Delta^{-i\bar{z}}A'\Omega, V(-e^{(0)})\Delta^{-iz}B\Omega \rangle$$

is continuous and bounded on the strip $0 \leqq \mathrm{Im}(z) \leqq 1/2$ and analytic in its interior. Moreover, for $\lambda \in \mathbb{R}$ the following bounds hold:

$$|f(\lambda)| \leqq \|A'\Omega\| \cdot \|B\Omega\|$$

and

$$\begin{aligned} |f(\lambda + i/2)| &= |\langle J\Delta^{-i\lambda}A'^*\Omega, V(-e^{(0)})J\Delta^{-i\lambda}B^*\Omega \rangle| \\ &= |\langle J\Delta^{-i\lambda}A'^*\Omega, JX(e^{(0)})\Delta^{-i\lambda}B^*\Omega \rangle| \\ &= |\langle X(e^{(0)})\Delta^{-i\lambda}B^*\Delta^{i\lambda}X(e^{(0)})^{-1}\Omega, \Delta^{-i\lambda}A'^*\Delta^{i\lambda}\Omega \rangle| \\ &= |\langle \Delta^{-i\lambda}A'\Omega, X(e^{(0)})\Delta^{-i\lambda}B\Omega \rangle| \leqq \|A'\Omega\| \cdot \|B\Omega\| , \end{aligned}$$

where the fourth equality follows from relation (2.5) and $\Delta^{-i\lambda}\mathscr{R}(\mathscr{W}^{(0)})\Delta^{i\lambda} = \mathscr{R}(\mathscr{W}^{(0)})$. Hence by the Three-Line-Theorem one has

$$|f(z)| \leqq \|A'\Omega\| \cdot \|B\Omega\|, \quad 0 \leqq \mathrm{Im}(z) \leqq 1/2 .$$

Now pick any two vectors Φ, Ψ in $\mathscr{H}$ and let $A'_n \in \mathscr{R}(\mathscr{W}^{(0)})'$ and $B_n \in \mathscr{R}(\mathscr{W}^{(0)})$ be such that $A'_n\Omega \to \Phi$ and $B_n\Omega \to \Psi$ strongly. Then it follows from the preceding estimate that the sequence of analytic functions

$$z \to f_n(z) \equiv \langle \Delta^{-i\bar{z}}A'_n\Omega, V(-e^{(0)})\Delta^{-iz}B_n\Omega \rangle$$

converges *uniformly* on the strip $0 \leqq \mathrm{Im}(z) \leqq 1/2$. Thus the limit $f_\infty(z)$ is continuous and bounded on $0 \leqq \mathrm{Im}(z) \leqq 1/2$, analytic in the interior, and $|f_\infty(z)| \leqq \|\Phi\| \|\Psi\|$. Moreover, when λ is real, it follows from relation (2.4) that

$$f_\infty(\lambda) = \langle \Phi, V(-e^{-2\pi\lambda}e^{(0)})\Psi \rangle .$$

Since Φ and Ψ are arbitrary, one may conclude that the operator function $z \to V(-e^{-2\pi z}e^{(0)})$ is weakly continuous on $0 \leqq \mathrm{Im}(z) \leqq 1/2$, analytic on the interior, and bounded in norm by 1. Setting $z = i/4$ one finds in particular that $\| V(ie^{(0)}) \| \leqq 1$, and consequently $P \cdot e^{(0)} \geqq 0$, where P is the generator of $V(\mathbb{R}^4)$. This inequality holds for all positive lightlike vectors $e^{(0)}$, hence $\mathrm{sp}(V) \subseteq \bigcap_{e^{(0)}} \{p \in \mathbb{R}^4 | p \cdot e^{(0)} \geqq 0\} = \bar{V}_+$. □

The representations $V(\mathbb{R}^4)$ of the translations constructed above are uniquely fixed by the properties established so far, as can be extracted from the literature (cf. the remark below). For completeness we give here a proof of this fact.

Proposition 2.4. *Let $U(\mathbb{R}^4)$, $V(\mathbb{R}^4)$ be two continuous unitary representations of the translations on $\mathscr{H}$ which act geometrically correctly on the local net $\mathscr{R}$, leave Ω invariant, and satisfy the spectrum condition. Then $U(\mathbb{R}^4) = V(\mathbb{R}^4)$.*

Proof. Let $\Gamma(x) \equiv U(x)V(x)^{-1}$, $x \in \mathbb{R}^4$, then $\Gamma(x)\Omega = \Omega$ and $\Gamma(x)\mathscr{R}(\mathscr{K})\Gamma(x)^{-1} = \mathscr{R}(\mathscr{K})$ for any double cone $\mathscr{K}$. Hence for any test function $h(x)$ whose Fourier transform $\tilde{h}$ has support in some compact set $C \subset \mathbb{R}^4 \setminus \bar{V}_+$ and for any $A \in \mathscr{R}(\mathscr{K})$, the weak integral $B \equiv \int h(y)\Gamma(y)A\Gamma(y)^{-1}\,dy$ is an element of $\mathscr{R}(\mathscr{K})$ and the spectral support of the vector function $x \to U(x)B\Omega$ is contained in $\bar{V}_+ \cap (\bar{V}_+ - C)$.

Proceeding now as in [9], one considers the commutator function $x \to K(x) \equiv \langle \Omega, [B^*, B(x)]\Omega \rangle$, where $B(x) \equiv U(x)BU(x)^{-1}$, and sets $K_\pm x) \equiv \pm\,\Theta(\pm x_0)K(x)$, where $\Theta(x_0)$ is the unit step function at 0. Because of locality, the Fourier transforms $\tilde{K}_\pm$ are analytic in the tubes $\mathscr{T}_\pm$ [3]. Hence, using the spectral support properties of $x \to U(x)B\Omega$, it follows from the Edge-of-the-Wedge-Theorem that $\tilde{K}$ is the discontinuity on the reals of some function analytic in $\mathscr{T}_+ \cup \mathscr{T}_- \cup \mathscr{N}$, where $\mathscr{N}$ is a complex neighborhood of the region $\mathbb{R}^4 \setminus ((\bar{V}_+ \cap (\bar{V}_+ - C)) \cup \bar{V}_-)$. This situation is encountered frequently in applications of the Jost–Lehmann–Dyson technique. And since no complete (double shelled) hyperboloid fits into the region of discontinuities $(\bar{V}_+ \cap (\bar{V}_+ - C)) \cup \bar{V}_-$, one must conclude that the analytic function (and hence K) is identically zero.

It follows from this fact and the spectrum condition for $U(\mathbb{R}^4)$ that $B\Omega = E(\{0\})B\Omega = 0$, where $E(\cdot)$ is the spectral resolution of $U(\mathbb{R}^4)$ and the spectral support properties of $x \to U(x)B\Omega$ have been used once again. As Ω is cyclic for $\mathscr{R}(\mathscr{W})$ and $\mathscr{R}(\mathscr{W})$ is generated by double cone algebras $\mathscr{R}(\mathscr{K})$, this equation implies that $\int h(y)\Gamma(y)\,dy = 0$. Interchanging the role of $U(\mathbb{R}^4)$ and $V(\mathbb{R}^4)$, one sees by the same argument that also $\int h(y)\Gamma(y)^*\,dy = 0$. Thus, taking the adjoint, one gets $\int \overline{h(y)}\Gamma(y)\,dy = 0$, where the Fourier transform of $\overline{h(x)}$ has support in the region $-C$. Since $C \subset \mathbb{R}^4 \setminus \bar{V}_+$ was arbitrary, one concludes that the Fourier transform of the operator function $x \to \Gamma(x)$ must have its support at the origin, and taking also into account that the function is bounded in norm, it follows that $\Gamma(x) = \Gamma(0) = 1$ for $x \in \mathbb{R}^4$. □

Remark. A very different argument establishing this result can be deduced from [4]. In the language established above, Borchers has shown that for any representation $V(\mathbb{R}^4)$ of the translations acting geometrically correctly on the net, having

[3] With $\mathscr{T}_+$ (resp. $\mathscr{T}_-$) we denote the forward (resp. backward) complex tube $\{z \in \mathbb{C}^4 | \mathrm{Im}(z) \in V_\pm\}$

an invariant vector Ω, and satisfying the spectrum condition, and for any coherent system of wedges $\{\mathscr{W}^{(0)} + z^{(0)} | z^{(0)} \in \mathbb{R}^2_{\mathscr{W}^{(0)}}\}$, one has

$$J_0 J_{z^{(0)}} = V(-2z^{(0)}) .$$

Since the left side of this equation is fixed by the net and the vector Ω, the right side likewise depends only on these data. Thus there is only one representation of the translations satisfying all requirements.

Summarizing our results, we have established the following theorem.

Theorem 2.5. *Let $\mathscr{A}$ be a net of C^*-algebras over $\mathbb{R}^4$, let ω be a state on $\mathscr{A}$ which satisfies Assumptions* (1) *to* (3), *and let $(\mathscr{H}, \pi, \Omega)$ be the corresponding* GNS *representation. Then there exists on $\mathscr{H}$ a continuous unitary representation $V(\mathbb{R}^4)$ of the translations which acts geometrically correctly on the net $\mathscr{R}$, leaves Ω invariant, and satisfies the spectrum condition. These properties uniquely fix the representation $V(\mathbb{R}^4)$.*

III. A Condition of Geometrical Modular Action

In the preceding section we have characterized the vacuum states on a given net of algebras over Minkowski space by conditions on the modular operators and conjugations associated with wedge regions. It is of significance here that these conditions involve only the wedge algebras $\mathscr{R}(\mathscr{W})$. The action of the modular objects on the double cone algebras is fixed by the relation $\mathscr{R}(\mathscr{K}) \equiv \bigcap_{\mathscr{W} \supseteq \mathscr{K}} \pi(\mathscr{A}(\mathscr{W}))''$, reflecting the fact that any double cone can be represented as an intersection of wedges. We call such a collection of regions whose intersections generate a given index set $\mathfrak{R}$ of subregions of spacetime a *generating family*.

Since the pertinent conditions in our investigation were of a purely geometrical nature, it seems natural to apply this approach to quantum field theories on other spacetime manifolds $\mathscr{M}$ with the intent to characterize physically significant states. In this more general setting one still deals with nets of C^*-algebras $\{\mathscr{A}(\mathscr{O})\}_{\mathscr{O} \in \mathfrak{R}}$, where $\mathfrak{R}$ is a suitable family of subregions of $\mathscr{M}$ [2]. As in the case of Minkowski space theories, one may restrict attention to the algebras affiliated with some generating family $\mathfrak{G} \subset \mathfrak{R}$. For, on the one hand, one expects that only the modular objects of algebras associated with certain special spacetime regions will have an action which can be interpreted in geometrical terms. On the other hand, the action of the modular objects on a net of algebras over $\mathscr{M}$ is completely fixed, once it has been specified on the algebras indexed by a generating family $\mathfrak{G}$.

With these remarks in mind, we propose the following less stringent version of Assumption (1) for the characterization of special states ω analogous to the vacuum. Again, this condition is expressed in terms of the GNS representation $(\mathscr{H}, \pi, \Omega)$ induced by ω.

Condition. Let $\mathfrak{G}$ be the given generating family of regions. Then the collection of algebras $\mathscr{R}(\mathscr{G}) \equiv \pi(\mathscr{A}(\mathscr{G}))''$, $\mathscr{G} \in \mathfrak{G}$, is stable under the action of the modular conjugations $J_{\mathscr{G}}$ affiliated with $(\mathscr{R}(\mathscr{G}), \Omega)$, $\mathscr{G} \in \mathfrak{G}$. More precisely, for every pair of regions $\mathscr{G}_1, \mathscr{G}_2 \in \mathfrak{G}$ there is some region $\mathscr{G}_1 \circ \mathscr{G}_2 \in \mathfrak{G}$ such that

$$J_{\mathscr{G}_1} R(\mathscr{G}_2) J_{\mathscr{G}_1} = \mathscr{R}(\mathscr{G}_1 \circ \mathscr{G}_2) . \tag{3.1}$$

(Setting $\mathscr{R}(\bigcap_\iota \mathscr{G}_\iota) \equiv \bigcap_\iota \mathscr{R}(\mathscr{G}_\iota)$, the geometric action of the modular conjugations can then be extended to the algebras affiliated with any region in $\mathfrak{R}$.)

Remark. Note that this condition is weaker than Assumption (1) since the particular form of the geometric action has not been specified. An even weaker version is obtained by demanding that the product of any two modular conjugations $J_{\mathscr{G}}$, $\mathscr{G} \in \mathfrak{G}$, (and not necessarily each modular conjugation individually) acts in the given geometrical manner on the algebras $\mathscr{R}(\mathscr{G})$, $\mathscr{G} \in \mathfrak{G}$. In other words, the action induced by products of any two modular conjugations associated with algebras in $\{\mathscr{R}(\mathscr{G})\}_{\mathscr{G} \in \mathfrak{G}}$ leaves the collection $\{\mathscr{R}(\mathscr{G})\}_{\mathscr{G} \in \mathfrak{G}}$ itself stable. This additionally weakened form of the condition of geometric modular action is suitable for covering a yet wider class of physical systems, including KMS states.

At this stage the condition of geometric modular action is stated in rather general terms[4] and therefore should be understood as providing only a framework for future consideration. Each choice of a manifold $\mathscr{M}$ and of a generating family $\mathfrak{G}$ of subregions fixes, in principle, some class of quantum field theories and their underlying elementary states. One should neither assume from the outset that the maps induced by the modular conjugations on the collection of regions $\mathfrak{G}$ are point transformations, nor should one in general expect that there is an analogue to Assumption (2) or (3). Nevertheless, the condition does impose stringent restrictions on a theory, respectively the underlying states.

It is an interesting problem to determine the manifolds, the causal structures (which, in view of relation (2.6), are fixed by defining the causal complement of $\mathscr{G} \in \mathfrak{G}$ by $\mathscr{G}' \equiv \mathscr{G} \circ \mathscr{G}$), and the spacetime symmetries which are compatible with our condition. We view this as a step towards an algebraic characterization of quantum field theories on arbitrary spacetime manifolds and their respective elementary systems.

As a first exercise in this program we have tried to recover the results of the preceding section, starting from this more general point of view. To this end we have reconsidered the case of a net over the manifold $\mathbb{R}^4$ which is based on the generating family $\mathfrak{G}$ of all wedges $\mathscr{W}$. If this net satisfies the condition given above, it follows without further specification of the assignment $(\mathscr{W}_1, \mathscr{W}_2) \to \mathscr{W}_1 \circ \mathscr{W}_2$ that the geometrical action induced on $\mathbb{R}^4$ by the modular conjugations is given by point transformations which are elements of the extended Poincaré group (cf. [10]). From this one infers that the causal structure induced on $\mathbb{R}^4$ coincides with that of Minkowski space. We note that the modular structure can also be used to induce a metric d on $\mathbb{R}^4$,

$$d(x, y) \equiv \sup \sum_n 2^{-n} \| (J_{\mathscr{W}_x} - J_{\mathscr{W}_y}) \Phi_n \| ,$$

where $\{\Phi_n\}_{n \in \mathbb{N}}$ is some orthonormal basis in $\mathscr{H}$ and the supremum is to be taken with respect to certain pairs $(\mathscr{W}_x, \mathscr{W}_y)$ of coherent wedges whose edges pass through x and y, respectively.

[4] In fact, the condition is meaningful for nets based on any partially ordered index set

At this point in our understanding, it seems unlikely that there always exist space-time translations in a theory satisfying these general conditions. Yet if one does presume that Assumption (2) is satisfied, one can exhibit once again a continuous unitary representation of the translations in the GNS representation induced by ω. (And note that Assumption (2) implies that the metric d above is invariant under these space-time translations.) Then, as above, Assumption (3) implies that the translations so constructed satisfy the relativistic spectrum condition. A detailed account of these results, including also a discussion of Lorentz transformations, will be published elsewhere [7].

Appendix

We present here some technical lemmas required in the main text of this paper.

Lemma A.1 [11]. *Let A be a closed, densely defined linear (or antilinear) operator on a Hilbert space $\mathscr{H}$ with domain $D(A)$. Let $D_n \subset D(A)$, $n \in \mathbb{N}$, determine an increasing sequence of dense vector spaces such that $D \equiv \bigcup_{n=1}^{\infty} D_n$ is a core for A. Let $A_n \equiv A | D_n$ and let $A = VH$, $A_n = V_n H_n$ be the corresponding polar decompositions. Then $V_n \to V$ in the strong operator topology and $H_n \to H$ in the strong resolvent sense.*

Corollary A.2. *Let $\{\mathscr{M}_n\}_{n \in \mathbb{N}}$* be an increasing sequence of von *Neumann algebras on a Hilbert space $\mathscr{H}$ such that $\mathscr{M} = (\bigcup \mathscr{M}_n)''$, and let Ω be a cyclic and separating vector for $\mathscr{M}$ and $\mathscr{M}_n$, $\forall n \in \mathbb{N}$. If $\Delta^{1/2}$, $\Delta_n^{1/2}$, J, J_n are the corresponding modular objects, then $J_n \to J$ and $\Delta_n^{i\lambda} \to \Delta^{i\lambda}$ strongly for any $\lambda \in \mathbb{R}$.*

Proof. Let $S = \overline{S_0}$, with $S_0 A\Omega \equiv A^*\Omega$, $\forall A \in \mathscr{M}$. One needs only to show that if $D_n \equiv \mathscr{M}_n \Omega$, $n \in \mathbb{N}$, then the set $D \equiv \bigcup_{n=1}^{\infty} D_n$ is a core for S. Let $\Psi \in D(S)$. Then since $\mathscr{M}\Omega$ is a core for S, there exists a sequence $\{M_n\}_{n \in \mathbb{N}} \subset \mathscr{M}$ such that $M_n \Omega \to \Psi$ and $SM_n\Omega = M_n^*\Omega \to S\Psi$ strongly. On the other hand, since $\mathscr{M} = (\bigcup \mathscr{M}_n)''$, it is also true that $\bigcup \mathscr{M}_n$ is dense in $\mathscr{M}$ in the strong *-topology (cf. Theorem 2.6 in [12]). In other words, there exist sequences $\{M_{nm}\}_{m \in \mathbb{N}} \subset \bigcup \mathscr{M}_n$ such that $M_{nm}\Omega \to M_n\Omega$ and $M_{nm}^*\Omega \to M_n^*\Omega$ strongly, for every $n \in \mathbb{N}$. Thus, for suitable subsequences $\{N_n = M_{nm(n)}\} \subset \bigcup \mathscr{M}_n$ one has $N_n\Omega \to \Psi$ and $SN_n\Omega = N_n^*\Omega \to S\Psi$. □

In the main text we make use of these facts as follows: according to our assumption the wedge algebras $\mathscr{A}(\mathscr{W})$ are the inductive limits of double cone algebras $\mathscr{A}(\mathscr{K})$, where the closures of $\mathscr{K}$ are contained in the interior of $\mathscr{W}$. Hence if $\mathscr{W}_x \subset \mathscr{W}$ determines an increasing sequence of wedges such that $\bigcup_x \mathscr{W}_x = \mathscr{W}$, then $\{\bigcup \mathscr{R}(\mathscr{W}_x)\}'' = \mathscr{R}(\mathscr{W})$. From the corollary it then follows that $J_{\mathscr{W}_x}$ converges strongly to $J_{\mathscr{W}}$.

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Note added in proof. In a recent article Wiesbrock [13] introduced the notion of half sided modular inclusions of von Neumann algebras and exhibited a unitary representation of $\mathbb{R}$ with positive generator which is canonically associated with such inclusions. Making use of this observation Borchers [14] was able to establish the main theorem of the present paper under somewhat weaker assumptions. His result supports our suggestion in Sect. 3 that the condition of Geometric Modular Action comprises the dynamical information of a quantum field theory.

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