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On Sums of q-Independent $SU_q(2)$ Quantum Variables

Romuald Lenczewski[⋆]

Hugo Steinhaus Center for Stochastic Methods, Instytut Matematyki, Politechnika Wrocławska, PL-50-370 Wrocław, Poland

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Abstract. A representation-free approach to the q-analog of the quantum central limit theorem for $\mathscr{C} = SU_q(2)$ is presented. It is shown that for certain functionals $\phi \in \mathscr{C}^*$ one can derive a version of a quantum central limit theorem (qclt) with $\sqrt{[N]}$ as a scaling parameter, which may be viewed as a q-analog of qclt.

1. Introduction

Limit theorems in quantum probability are related to the notion of independence. Depending on its kind we obtain various approaches to quantum limit theorems, in particular quantum central limit theorems (qclt).

The study of qclt's originated with the works of Giri and Waldenfels [4] for commuting independence and Waldenfels [11] for anticommuting independence. Those works gave boson and fermion versions of qclt. A general approach for coalgebras with independence introduced through the coproduct was presented by Schürmann [7]. In [2] Accardi and Lu proved a qclt for weakly dependent maps.

Voiculescu [10] developed a general theory for free products (free independence). Following his ideas, Speicher [9] proved a general limit theorem giving the free analogues of Gaussian and Poisson distributions. A q-example of Brownian was considered by Bożejko and Speicher [3].

Recently, a q-version of quantum central limit theorem (q-independence) and a q-version of white noise was presented by Schürmann [8]. His qclt was based on the qclt for coalgebras. He assumed that ϕ agrees with δ on $\mathscr{C}^{(0)}$, where $\mathscr{C} = \mathscr{C}^{(0)} \oplus \mathscr{C}^{(1)} \oplus \dots$ is a $\mathbf{N}$-graduation on $\mathscr{C}$ that is compatible with the coproduct and δ is a counit. Independently, in [6] we studied a q-analog of qclt for $SU_q(2)$ for q real positive. Our approach was group-theoretic and related to a certain group contraction of $SU_q(2)$. In our work the scaling q-qclt constant was not $\sqrt{N}$ but $\sqrt{[N]}$, where $[N]$ is the

⋆ e-mail ROMEK@PLWRTU11

q-analog of N, namely $[N]_{q^2} = \dfrac{q^{2N} - q^{-2N}}{q^2 - q^{-2}}$. Our proof was carried out for the two-dimensional spin representation of $SU_q(2)$. In [6] we showed that the algebra of q-commuting spins converges in law (see [1]) to the q-harmonic oscillator algebra in the way that corresponds to the group contraction introduced by Kulish [5].

The aim of this paper is to provide a representation-free version of a q-qclt for $SU_q(2)$ using a functional approach in the spirit of Waldenfels and Schürmann and show a connection of such an approach with out previous work. At the same time it is more general since it is representation free. It can be also viewed as a first step to the generalization for other quantum groups.

2. Preliminaries

Let be given a Hopf algebra over $\mathbf{C}$ generated by $\{J_+, J_-, t, t^{-1}\}$ which satisfy the following relations:

$$tt^{-1} = t^{-1}t = 1\,,$$
$$tJ_+t^{-1} = q^2 J_+\,, \qquad tJ_-t^{-1} = q^{-2}J_-\,,$$

where $q \in \mathbf{C}$, endowed with the coproduct Δ and the counit δ defined by:

$$\Delta(1) = 1 \otimes 1\,, \qquad \Delta(t) = t \otimes t\,, \qquad \Delta(t^{-1}) = t^{-1} \otimes t^{-1}\,,$$
$$\Delta(J_+) = J_+ \otimes t + t^{-1} \otimes J_+\,, \qquad \Delta(J_-) = J_- \otimes t + t^{-1} \otimes J_-\,,$$
$$\delta(1) = \delta(t) = \delta(t^{-1}) = 1\,, \qquad \delta(J_+) = \delta(J_-) = 0\,.$$

Note that our q would be q^2 or q^4 in some other works. For any $a \in \mathbf{C} - \{1, -1\}$ we denote $[N]_a = \dfrac{a^N - a^{-N}}{a - a^{-1}}$.

The N^{th} iteration of the coproduct Δ satisfies the following equation $\Delta_N = (\mathrm{Id} \otimes \Delta_{N-1}) \circ \Delta = (\Delta_{N-1} \otimes \mathrm{Id}) \circ \Delta$. Thus, it is easy to see that

$$\Delta_{N-1}(t) = t^{\otimes N}\,, \qquad \Delta_{N-1}(t^{-1}) = (t^{-1})^{\otimes N}\,,$$
$$\Delta_{N-1}(v) = \sum_{i=1}^{N} j_{i,N}(v)\,,$$

where $v \in \mathscr{V} = \mathscr{V}_+ \oplus \mathscr{V}_- = \mathbf{C}J_+ \oplus \mathbf{C}J_-$ and

$$j_{i,N}(v) = (t^{-1})^{\otimes(i-1)} \otimes v \otimes t^{\otimes(N-i)}$$

are canonical embeddings of v into $\bigotimes^{N} \mathscr{C}$. They neither commute nor anticommute, but q-commute, i.e. for $i < k$, we have

$$j_{i,N}(J_+) j_{k,N}(J_+) = q^4 j_{k,N}(J_+) j_{i,N}(J_+)\,,$$
$$q^4 j_{i,N}(J_-) j_{k,N}(J_-) = j_{k,N}(J_-) j_{i,N}(J_-)\,,$$

and $j_{i,N}(J_+)$ commutes with $j_{k,N}(J_-)$ for $i \neq k$. The $N-1^{\text{th}}$ iteration of Δ represents the sum of N random variables. We call the independence introduced through such coproduct by the *q-independence*. Note that Δ is a homomorphism of $\mathscr{C}$ into $\mathscr{C} \otimes \mathscr{C}$ and Δ_{N-1} is a homomorphism of $\mathscr{C}$ into $\bigotimes^{N} \mathscr{C}$.

Now, let us define the N^{th} convolution of $\phi \in \mathscr{C}^*$, i.e. $\phi_N^* = \phi^{\otimes N} \circ \Delta_{N-1}$. The qclt will consist in evaluating the limit of $\phi_N^*(v_1^N \dots v_p^N)$ for $v_1^N, \dots, v_p^N \in \mathscr{V} \cup \mathscr{T} \cup \mathscr{T}^{-1}$, where $\mathscr{T} = \{t\}$ and $\mathscr{T}^{-1} = \{t^{-1}\}$ and the superscript N denotes appropriate scaling.

In the usual qclt's those scaling constants are equal to $1/\sqrt{N}$. We shall assume that

$$v_k^N = \begin{cases} (1/\sqrt{[N]_{a^2}} v_k & \textit{if } v_k \in \mathscr{V} \\ a^{-N} v_k & \textit{if } v_k \in \mathscr{T} \\ a^N v_k & \textit{if } v_k \in \mathscr{T}^{-1} \end{cases},$$

where $\phi(t) = a$ (see [6]).

3. Partitions and Convolutions

Let us start with the notions related to the combinatorics of the problem. By an *ordered partition* S of an index set $I = \{1, \dots, p\}$ [the set of such partitions will be denoted $\mathscr{P}(I)$] we shall understand a sequence of nonempty disjoint subsets $(S_1, \dots, S_r)$ of I, such that $I = S_1 \cup \dots \cup S_r$. By a *signature* of partition S we will understand an r-tuple $(\alpha_1^S, \dots, \alpha_r^S)$, where α_i^S denotes the number of elements in S_i. For this r-tuple we shall use the abbreviated notation α^S or α if no confusion arises. By $\mathscr{P}^e(I)$ we shall understand partitions into subsets, each of which has an even number of elements. The signature of such a partition will be called *even*.

For a given partition S we define the following family of homomorphisms:

$$\tau_1^S(v_k) = \begin{cases} v_k & \text{if } k \in S_1 \\ t^{-1} & \text{if } k \in S_2 \cup \dots \cup S_r \end{cases},$$

$$\tau_j^S(v_k) = \begin{cases} t & \text{if } k \in S_1 \cup \dots \cup S_{j-1} \\ v_k & \text{if } k \in S_j \\ t^{-1} & \text{if } k \in S_{j+1} \cup \dots \cup V_r \end{cases},$$

$$\tau_r^S(v_k) = \begin{cases} t & \text{if } k \in S_1 \cup \dots \cup S_{r-1} \\ v_k & \text{if } k \in S_r \end{cases},$$

where $v_k \in \mathscr{V}$. Moreover, $\tau_j^S(t) = t$, $\tau_j^S(t^{-1}) = t^{-1}$. Another family of homomorphisms is given by

$$\sigma_j^S(v_k) = \begin{cases} t & \text{if } k \in S_1 \cup \dots \cup S_{j-1} \\ t^{-1} & \text{if } t \in S_j \cup \dots \cup S_r \end{cases},$$

where $j = 1, \dots, r+1$ and $v_k \in \mathscr{V}$. Again, we assume that they are identities on t, t^{-1}.

Now, we can state the following

Lemma 1. *Let ϕ be any functional on $\mathscr{C}$. Let τ_j^S, σ_j^S be the homomorphisms defined above. Then, for any $v_1, \dots, v_p \in \mathscr{V}$ we have*

$$\begin{aligned} \phi_N^*(v_1 \dots v_p) = \sum_{r=1}^{p} \sum_{1 \le i_1 < \dots < i_r \le N} \sum_{S=(S_1,\dots,S_r) \in \mathscr{P}(I)} & \times \phi(\sigma_1^S(v_1 \dots v_p))^{i_1 - 1} \dots \phi(\sigma_{r+1}^S(v_1 \dots v_p))^{N - i_r} \\ & \times \phi(\tau_1^S(v_1 \dots v_p)) \dots \phi(\tau_r^S(v_1 \dots v_p)) . \end{aligned}$$

Proof. We have

$$\phi_N^*(v_1 \ldots v_p) = \phi^{\otimes N}\left(\sum_{i_1,\ldots,i_p=1}^{N} j_{i_{1,N}}(v_1)\ldots j_{i_{p,N}}(v_p)\right).$$

We shall translate it now into the language of partitions. Let $S=(S_1,\ldots,S_r)$ be an ordered partition of $1,\ldots,p$ into r nonempty subsets. Using the explicit form of the canonical injections we get each term in the above sum in the following form:

$$((t^{-1})^{\otimes(i_1-1)}\otimes v_1 \otimes t^{\otimes(N-i_1)})\ldots((t^{-1})^{\otimes(i_p-1)}\otimes v_p \otimes t^{\otimes(N-i_p)}).$$

If we rearrange each p-tuple $(i_1,\ldots,i_p)$ in the ascending order, we get, say, an r-tuple $(k_1,\ldots,k_r)$, where $1\le k_1<\ldots<k_r\le N$. Then the above term can be written as

$$(\sigma_1^S(v))^{\otimes(k_1-1)}\otimes\tau_1^S(v)\otimes\ldots\otimes(\sigma_m^S(v))^{\otimes(k_m-k_{m-1}-1)}\otimes\tau_m^S(v)\otimes\ldots\otimes(\sigma_{r+1}^S(v))^{\otimes(N-k_r)},$$

where S is the partition in which S_i consists of indices j such that v_j appears at site k_i and we abbreviated $v=v_1\ldots v_p$. Applying the N^{th} tensorial power of ϕ ends the proof. □

When we introduce certain assumptions on ϕ we can get rid of the factors involving σ's. Thus, we get the following

Corollary 1. *Assume that $\phi(t)=a=\phi(t^{-1})^{-1}$, where $a\in\mathbf{C}$ and that ϕ is a homomorphism on $\mathbf{C}[t,t^{-1}]$. Let $v_1,\ldots,v_p\in\mathscr{V}$. Then*

$$\phi_N^*(v_1\ldots v_p)=a^{p(N+1)}\sum_{r=1}^{p}\sum_{1\le i_1<\ldots<i_r\le N}\sum_{S=(S_1,\ldots,S_r)} a^{-2\langle\alpha^S,i\rangle-\langle\alpha^S,r\rangle}$$
$$\times\prod_{j=1}^{r}\phi(\tau_j^S(v_1\ldots v_p)),$$

where $i=(i_1,\ldots,i_r)$ and $r=(r-1,r-3,\ldots,-(r-1))$ denote r-tuples and $\langle\cdot,\cdot\rangle$ is the usual scalar product.

Proof. The proof rests on the evaluation of factors involving σ's using the homomorphism assumption. Thus,

$$\begin{aligned}&(\phi(\sigma_1^S(v)))^{i_1-1}\ldots(\phi(\sigma_{r+1}^S(v)))^{N-i_r}\\ &=a^{-p(i_1-1)}a^{(i_2-i_1-1)(\alpha_1^S-\ldots-\alpha_r^S)}\ldots a^{(N-i_r)(\alpha_1^S+\ldots+\alpha_r^S)}\\ &=a^{p(N+1)-2\langle\alpha^S,i\rangle-\langle\alpha^S,r\rangle},\end{aligned}$$

which ends the proof. □

Let us now introduce two different graduations on $\mathscr{C}$. Namely, let $d_c(J_-)=d_c(J_+)=1$ and $d_c(t)=d_c(t^{-1})=d_c(\mathbf{C})=0$. We denote the $\mathbf{N}$-graduation obtained by natural extension to all free products in $\mathscr{C}$ by $\mathscr{C}=\mathscr{C}^{(0)}\oplus\mathscr{C}^{(1)}\oplus\ldots$. It is compatible with the coproduct. Another graduation (not compatible with the coproduct) is the following $\mathbf{Z}$-graduation: $d_f(J_-)=d_f(J_+)=1$, $d_f(t)=1$, $d_f(t^{-1})=-1$, $d_f(\mathbf{C})=0$. It is also extended to all free products in $\mathscr{C}$ in the usual manner.

Extending the arguments in the preceding corollary to monomials involving t,t^{-1} we easily get

Corollary 2. *Let* $v_1, \ldots, v_s \in \mathscr{V} \cup \mathscr{T} \cup \mathscr{T}^{-1}$. *Let* $d_c(v_1 \ldots v_s) = p$ *and* $d_f(v_1 \ldots v_s) = m$. *Let* $\phi(t) = a = (\phi(t^{-1}))^{-1}$, *where* $a \in \mathbf{C}$. *Assume that* ϕ *is a homomorphism on* $\mathbf{C}[t, t^{-1}]$. *Then*

$$\phi_N^*(v_1 \ldots v_s) = a^{p(N+1)+N(m-p)} \sum_{r=1}^{p} a^{r(p-m)} \sum_{1 \le i_1 < \ldots < i_r \le N} \sum_{S=(S_1,\ldots,S_r) \in \mathscr{P}(I')} \times a^{-2\langle \alpha^S, i \rangle - \langle \alpha^S, r \rangle} \prod_{j=1}^{r} \phi(\tau_j^S(v_1 \ldots v_s)),$$

where it is understood that in the summation over partitions we take into account partitions of the index set I' *of subscripts of those* v_k*'s that are in* $\mathscr{V}$.

Proof. In the product of v_k's we have $s - p$ elements from $\mathscr{T}$ or $\mathscr{T}^{-1}$. Each t gives rise to an a in each tensorial slot whereas each t^{-1} produces an a^{-1}. This results in the factor of $a^{(m-p)(N-r)}$. That finishes the proof. □

If we assume that the functional ϕ vanishes on monomials that have an odd number of elements of d_c degree equal to one, we get immediately the following

Corollary 3. *Let* $\phi(\mathscr{E}^{(2j+1)}) = 0$ *and let all assumptions of Corollary 2 be satisfied. Then*

$$\phi_N^*(v_1 \ldots v_s) = a^{2k(N+1)+N(m-2k)} \sum_{r=1}^{k} a^{r(2k-m)} \sum_{1 \le i_1 < \ldots < i_r \le N} \sum_{S=(S_1,\ldots,S_r) \in \mathscr{P}^e(I')} \times a^{-2\langle \alpha^S, i \rangle - \langle \alpha^S, r \rangle} \prod_{j=1}^{r} \phi(\tau_j^S(v_1 \ldots v_s))$$

if $p = 2k$ *and equals zero otherwise, where in the sum over partitions we take into account partitions of even signature of the subset* I' *as in Corollary 2.*

When, in addition, we assume that a functional vanishes on $\mathscr{E}^{(0)}$, we get the following

Lemma 2. *Assume that* $\psi \in \mathscr{E}^*$ *is such that* $\psi(\mathscr{E}^{(0)}) = \psi(\mathscr{E}^{(2j+1)}) = 0$. *Let all assumptions of Corollary 2 be satisfied. Then*

$$\psi_r^*(v_1 \ldots v_s) = \sum_{S=(S_1,\ldots,S_r) \in \mathscr{P}^e(I')} \prod_{j=1}^{r} \psi(\tau_j^S(v_1 \ldots v_s))$$

if $1 \le r \le k$, $d_c(v_1 \ldots v_s) = 2k$, *where the sum runs over partitions of even signature of the index set* I' *and equals zero otherwise.*

Proof. If $d_c(v_1 \ldots v_s) = 2k$ and $r > k$, then in each term of $\Delta_{r-1}(v)$ there is at least one slot in the tensorial power that is of degree d_c equal to 0 or 1 and thus makes the whole term vanish by assumption on ψ. The remaining part of the proof rests on the proof of Lemma 1. Namely, if $1 \le r \le k$, then by assumption on ψ nonzero contribution comes only from partitions into r subsets, each of which is of even and positive d_c degree. For other partitions, a factor of type $\psi(t^\gamma)$ with γ a nonzero integer appears, which makes the term vanish. □

4. Limits of Sums of q-Independent Variables

In this section we prove our main result. Contrary to [8] and the usual clt's, we choose the following scaling on $\mathscr{C}$: $v_k^N = 1/\sqrt{[N]_{a^2}}v_k$ if $v_k \in \mathscr{V}$, $v_k^N = a^{-N}v_k$ if $v_k \in \mathscr{T}$ and $v_k^N = a^N v_k$ if $v_k \in \mathscr{T}^{-1}$. Thus, we shall evaluate the limits of $\phi_N^*(v^N)$, where $v^N = v_1^N \dots v_s^N$.

Theorem 1. *Let $\phi(\mathscr{C}^{(2j+1)}) = 0$, $\phi(1) = 1$ and ϕ be a homomorphism on $\mathscr{C}[t, t^{-1}]$ with $\phi(t) = a \in \mathbf{R}^+ - \{1\}$. Let $v^N = v_1^N \dots v_s^N$, where $v_1, \dots, v_s \in \mathscr{V} \cup \mathscr{T} \cup \mathscr{T}^{-1}$. Let $d_c(v^N) = p$, $d_f(v^N) = m$. Then, if p is odd, $\lim_{N\to\infty} \phi_N^*(v^N) = 0$. If $p = 2k$, then*

$$\lim_{N\to\infty} \phi_N^*(v^N) = \sum_{r=1}^{k} \sum_{\substack{\alpha_1,\dots,\alpha_r \in 2\mathbf{N} \\ \alpha_1+\dots+\alpha_r=2k}} C_k^m(\alpha_1, \dots, \alpha_r|a)\,(\psi^{\otimes r} \circ \pi_{(\alpha_1,\dots,\alpha_r)} \circ \Delta_{r-1})\,(v)\,,$$

where $\psi \in \mathscr{C}$ is such that $\psi(\mathscr{C}^{(0)}) = \psi(\mathscr{C}^{(2j+1)}) = 0$ and agrees with ϕ on $\mathscr{C}^{(2j)}$, $C_k^m(\alpha_1, \dots, \alpha_r|a)$ are constants and $\pi_{(\alpha_1,\dots,\alpha_r)}$ is a canonical projection onto $\mathscr{C}^{(\alpha_1)} \otimes \dots \otimes \mathscr{C}^{(\alpha_r)}$.

Proof. It is immediate for p odd. For $p = 2k > 0$ we rewrite the expression from Corollary 3 in the following way. We split the summation over partitions from Corollary 3 into two sums: first over even signatures and second over partitions of the same signature. Thus, we obtain:

$$\phi_N^*(v^N) = \sum_{r=1}^{k} \sum_{\substack{\alpha_1,\dots,\alpha_r \in 2\mathbf{N} \\ \alpha_1+\dots+\alpha_r=2k}} \sum_{\substack{S=(S_1,\dots,S_r)\in\mathscr{P}^e(I') \\ \alpha^S=\alpha}} \sum_{1\le i_1<\dots<i_r\le N} \times\, a^{+2k(N+1)+(N-r)(m-2k)-2\langle\alpha,i\rangle-\langle\alpha,r\rangle} \prod_{j=1}^{r} \phi(\tau_j^S(v^N))\,.$$

Now, we need to evaluate the limit

$$C_k^m(\alpha_1, \dots, \alpha_r|a) = \lim_{N\to\infty} \frac{a^{+2k(N+1)+(N-r)(m-2k)-\langle\alpha,r\rangle}}{([N]_{a^2})^k a^{N(m-2k)}} F_N(\alpha_1, \dots, \alpha_r|a)\,,$$

where

$$F_N(\alpha_1, \dots, \alpha_r|a) = \sum_{1\le i_1<\dots<i_r\le N} a^{-2\langle\alpha,i\rangle}.$$

One can derive the following recurrence formula for F_N's:

$$F_N(\alpha_1, \dots, \alpha_r|a) = f(\alpha_r)\,(F_{N-1}(\alpha_1, \dots, \alpha_{r-1}+\alpha_r) - a^{-2\alpha_r N} F_{N-1}(\alpha_1, \dots, \alpha_{r-1}))$$

and $F_N(\alpha_1) = a^{-2\alpha}\dfrac{1-a^{-2\alpha_1 N}}{1-a^{-2\alpha_1}}$, where $f(\alpha) = \dfrac{a^{-2\alpha}}{1-a^{-2\alpha}}$. Assume first that $a < 1$. Then we get

$$L(\alpha_1, \dots, \alpha_r|a) = f(\alpha_r)\,(a^{2(\alpha_1+\dots+\alpha_r)} L(\alpha_1, \dots, \alpha_{r-1}+\alpha_r) - a^{2(\alpha_1+\dots+\alpha_{r-1})} L(\alpha_1, \dots, \alpha_{r-1}))\,,$$

where

$$L(\alpha_1,\ldots,\alpha_r|a) = \lim_{N\to\infty} \frac{F_N(\alpha_1,\ldots,\alpha_r)}{a^{-2N(\alpha_1+\ldots+\alpha_r)}}$$

and $L(\alpha_1) = -f(\alpha_1)$. From this one can prove by induction the following formula:

$$L(\alpha_1,\ldots,\alpha_r|a) = \sum_{M\in M_r^{\Delta}} (-1)^{\theta(M)} \prod_{i=1}^{r} f\Big(\sum_{k=1}^{r} M_{ik}\alpha_k\Big) a^{g_i(\alpha_1,\ldots,\alpha_r|M)},$$

where M_r^{Δ} denotes block-diagonal $r\times r$ matrices built of blocks having 1's on and above the main diagonal and zeros otherwise, $\theta(M)$ denotes the number of blocks in the matrix M and

$$g_i(\alpha_1,\ldots,\alpha_r|M) = 2\sum_{j=1}^{r}(r-j-1)\alpha_j + \sum_{i,j=1}^{r} M_{ij}\alpha_j .$$

Notice that M_r^{Δ} has 2^{r-1} elements. Thus, we finally get

$$C_k^m(\alpha_1,\ldots,\alpha_r|r) = (1-a^4)^k a^{-r(m-2k)-\langle\alpha,r\rangle} L(\alpha_1,\ldots,\alpha_r|a).$$

That finishes the proof for $a<1$ by Lemma 2 (or, rather its version, in which each signature produces a certain weight factor). For $a>1$ we put $b=1/a$. Then we can rewrite Corollaries 1–3 using b and we change the summation over $1\le i_1<\ldots<i_r\le N$ to the summation over $j_1 = N-i_r+1,\ldots,j_r = N-i_1+1$. Then the whole proof is analogous and we get

$$C_k^m(\alpha_1,\ldots,\alpha_r|a) = (1-a^{-4})^k a^{-r(m-2k)+\langle\alpha,r\rangle} L(\alpha_r,\ldots,\alpha_1|a^{-1}). \quad \square$$

Let us notice, that if we proceeded in the same way as above with $a=1$, (assuming that v_i's are from $\mathscr{V}$) then we would obtain the result of [8] and the only nonvanishing constants would be $C_k^{2k}(2,\ldots,2|a) = \frac{1}{k!}$, which would enable us to write the result in the convolution exponential form.

Example. To disentangle the statement of the theorem let us consider the lowest degree moment for which the calculations are different from the case $a=1$. Thus, let $v_1,\ldots,v_4\in\mathscr{V}$. Then we obtain

$$\begin{aligned}\lim_{N\to\infty} \phi_N^*(v_1^N v_2^N v_3^N v_4^N) &= C_2^4(4|a)\psi(v_1v_2v_3v_4)\\ &+ C_2^4(2,2|a)\,(\psi(t^2v_3v_4)\psi(v_1v_2t^{-2})\\ &+ \psi(tv_2tv_4)\psi(v_1t^{-1}v_3t^{-1})\\ &+ \psi(tv_2v_3t)\psi(v_1t^{-2}v_4) + \psi(v_1t^2v_4)\psi(t^{-1}v_2v_3t^{-1})\\ &+ \psi(v_1tv_3t)\psi(t^{-1}v_2t^{-1}v_4) + \psi(v_1v_2t^2)\psi(t^{-2}v_3v_4)),\end{aligned}$$

where $C_2^4(4|a) = \dfrac{(a^2-a^{-2})^2}{a^{-4}-a^4}$, $C_2^4(2,2|a) = \frac{1}{1+a^{-4}}$ for $a<1$ and $C_2^4(4|a) = \dfrac{(a^{-2}-a^2)^2}{a^4-a^{-4}}$, $C_2^4(2,2|a) = \dfrac{1}{1+a^4}$ for $a>1$.

To establish a connection with our previous work (see [6]) we need a number of additional assumptions. Note that we have not commuted $\mathscr{T}$ and $\mathscr{T}^{-1}$ with $\mathscr{V}$ yet (except when making a comment on q-commutation in Preliminaries). If we do that, we obtain each moment that appears in Corollaries 1–3 in the form:

$$\phi(\tau_j^S(v)) = \phi(v' t^{\gamma(S|v)}),$$

where $\gamma(S|v)$ is an integer that depends on the partition S and v' is an element of $\mathscr{C}$ on which d_f and d_c agree. If, in addition we assume that ϕ is $(\mathscr{T},\mathscr{T}^{-1})$*-right independent*, i.e.

$$\phi(v' t^{\gamma}) = \phi(v')\phi(t^{\gamma})$$

then, together with the homomorphism assumption on ϕ, we obtain a version of Theorem 1, in which ψ is also $(\mathscr{T},\mathscr{T}^{-1})$-right independent. Let's also assume that

$$\phi(\langle J_+^2 \rangle) = \phi(\langle J_-^2 \rangle) = 0,$$

where $\langle v \rangle$ denotes the two-sided ideal generated by v. This assumption corresponds to the spin two-dimensional representation of $SU_q(2)$ (see [6]). Then, the result of [6] can be obtained if we add the relation $[J_+, J_-] = \dfrac{t^2 - t^{-2}}{q^2 - q^{-2}}$ and put $\phi(t) = q$. As it was shown there we then get convergence in law of $SU_q(2)$ to the q-oscillator and in that sense the result obtained therein is a special case in this investigation.

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Note added in proof. The form of coefficients $L(\alpha_1, \ldots, \alpha_r|a)$ given in the article was derived by induction irrespective of $f(\alpha)$. Prof. M. Rahman indicated to me, which I gratefully acknowledge, that in the case of $f(\alpha) = \dfrac{a^{-2\alpha}}{1 - a^{-2\alpha}} = \dfrac{1}{a^{2\alpha} - 1}$, a much simpler formula can be given:

$$L(\alpha_1, \ldots, \alpha_2|q) = \frac{a^{-2(\alpha_1 + \ldots + \alpha_r)}}{(a^{-2\alpha_1} - 1)(a^{-2\alpha_1 - 2\alpha_2} - 1)\ldots(a^{-2\alpha_1 - \ldots -2\alpha_r} - 1)}.$$

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