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Generalized Hypergeometric Functions on the Torus and the Adjoint Representation of $U_q(sl_2)$

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Abstract. We study the homology groups with coefficient in local systems arising in the free field representation of minimal models of conformal field theory on an elliptic curve with punctures. We define an action of the quantum enveloping algebra $U_q(sl_2)$ on a space of relative cycles, extending results obtained previously for the sphere. Absolute cycles are identified with singular vectors. In the case of one puncture, we find that the resulting topological representation is essentially the adjoint representation.

1. Introduction

Recent study [1–3] indicates that there exists a dictionary between homology of certain configuration spaces with coefficients in local systems and representation theory of quantum enveloping algebras [4]. The examples of local systems providing such connections come from integral representation of conformal blocks of conformal field theory [5–14]. The idea is that (in some sense) the charges generating (half of) the quantum group symmetry in the free field representation in conformal field theory are given by integrals over screening operators [15–18]. In a previous paper [2], we have shown the existence of an action of $U_q(sl_2)$ on certain relative locally finite homology groups on configuration spaces on the sphere. In this case, the local system is given by the integrand of the free field representation of conformal blocks of the SU(2) WZW models or minimal models.

In this paper, we consider the situation of the torus, for which one knows explicit integral representations [19–21]. We restrict our attention to the case of minimal models, which is the simplest. The main difference is that the local system is not given by a line bundle as in the case of the sphere, but rather a vector bundle. From the point of view of free fields, this follows from the fact that the space of free field conformal blocks on the torus is higher dimensional.

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We find again an action of the quantum enveloping algebra of $sl_2(\mathbb{C})$ on relative cycles, in such a way that absolute cycles are identified with singular vectors, as in the case of the sphere. The resulting representation is a tensor product of Verma modules with $U_q(sl_2)$ with the adjoint action.

We hope that this work will lead to a clearer understanding of the role of quantum groups for higher genus Riemann surfaces.

2. Generalized Hypergeometric Functions on the Torus

In the free field representation of minimal models with central charge $c = 1 - 6(p' - p)^2/pp'$ on the torus $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ one is led to consider integrals of the form

$$G_C(z_1, \ldots, z_s|\tau) = \sum_{\mu=0}^{2p'p-1} \lambda_\mu \int_{C_\mu} \Delta_\mu(W|\tau) \prod_{1 \leqq i<j \leqq s+r+r'} E(z_i, z_j|\tau)^{\alpha_i\alpha_j} dz_{s+1} \wedge \cdots \wedge dz_{s+r+r'}\,, \quad (2.1)$$

where

$$\alpha_i = \begin{cases} \alpha_{n_i' n_i}, & 1 \leqq i \leqq s\,, \\ \alpha_{1,-1}, & s+1 \leqq i \leqq s+r\,, \\ \alpha_{-1,1}, & s+r+1 \leqq i \leqq s+r+r'\,, \end{cases}$$

$$\alpha_{m',m} = \frac{(1-m)p' - (1-m')p}{\sqrt{2p'p}}\,, \quad (2.2)$$

parametrize the exponents, $\alpha_\pm = \alpha_{\pm 1, \mp 1}$ belonging to the integrated screening variables, and

$$\Delta_\mu(W|\tau) = \frac{1}{\eta(\tau)} \exp\left\{\frac{\pi i}{2p'p}\mu(2W + \tau\mu)\right\}\theta_3(W + \tau\mu|2p'p\tau)\,,$$

$$W = \sqrt{2p'p}\sum_{i=1}^{s+r+r'} \alpha_i z_i\,, \quad (2.3)$$

$$E(z_i, z_j|\tau) = 2\pi i \frac{\theta_1(z_i - z_j|\tau)}{\theta_1'(0|\tau)}\,,$$

with

$$\eta(\tau) = e^{\frac{\pi i\tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i\tau n})\,,$$

$$\theta_3(z|\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i z n - \pi i \tau n^2}\,,$$

$$\theta_1(z|\tau) = -\sum_{n=-\infty}^{\infty} e^{2\pi i(z+\frac{1}{2})(n+\frac{1}{2}) - \pi i\tau(n+\frac{1}{2})^2}\,, \quad (2.4)$$

the Dedekind eta function and the Jacobi theta functions. The values of the exponents are constrained to satisfy

$$\sum_{i=1}^{s} (1 - n_i) + 2r = xp \,, \quad \sum_{i=1}^{s} (1 - n_i') + 2r' = xp' \,, \tag{2.5}$$

for some integer x, reflecting charge conservation. In the following we restrict our attention to the case when $r' = 0$ and $n_1' = \ldots = n_s' = 1$, admitting α_+ screening charges only. That is, we assume that

$$\alpha_i = \begin{cases} \dfrac{(1-n_i)p'}{\sqrt{2p'p}}, & 1 \leqq i \leqq s \,, \\ \dfrac{2p'}{\sqrt{2p'p}}, & s+1 \leqq i \leqq s+r \,, \end{cases} \tag{2.6}$$

with the neutrality condition

$$\sum_{i=1}^{s+r} \alpha_i = 0 \,. \tag{2.7}$$

When studying integrals of this form, one is entering the following kind of problems. To begin with, assume that $\alpha_1 = \cdots = \alpha_{n_1}, \alpha_{n_1+1} = \cdots = \alpha_{n_1+n_2}, \ldots, \alpha_{n_1+\cdots+n_{k-1}+1} = \cdots = \alpha_{n_1+\cdots+n_k}$, for some k and $(n_1, \ldots, n_k) \in \{1, 2, \ldots\}^k$, such that $n_1 + \cdots + n_{k-1} = s$ and $n_k = r$. Let $\Sigma = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ be the torus with modular parameter τ. Then we have a vector of multivalued forms with components

$$\begin{aligned} &\omega_\mu(z_1, \ldots, z_s | z_{s+1}, \ldots, z_{s+r} | \tau) \\ &\quad = \Delta_\mu(W|\tau) \prod_{1 \leqq i < j \leqq s+r} E(z_i, z_j|\tau)^{\alpha_i \alpha_j} dz_{s+1} \wedge \cdots \wedge dz_{s+r} \,, \end{aligned} \tag{2.8}$$

on the configuration space

$$X_n = (\Sigma^{s+r} \backslash \textstyle\bigcup_{i<j} \{z_i = z_j\})/S_{n_1} \times \cdots \times S_{n_k} \,. \tag{2.9}$$

Since $\Delta_{\mu+2p'p}(W|\tau) = \Delta_\mu(W|\tau)$, we can restrict μ to the range $0 \leqq \mu \leqq 2p'p - 1$. Other properties of $\Delta_\mu(W|\tau)$ are summarized in Appendix A.

We will often use the identification $\mu \mapsto (m, m')$, $\mu = m'p - mp'$, of $\mathbb{Z}/2pp'\mathbb{Z}$ with $\mathbb{Z}^2/\Lambda$, where Λ is the lattice generated by (p, p') and $(2p, 0)$.

Let $n' = (n_1, \ldots, n_{k-1})$ and

$$X_{n'} = (\Sigma^s \backslash \textstyle\bigcup_{i<j} \{z_i = z_j\})/S_{n_1} \times \cdots \times S_{n_{k-1}} \,. \tag{2.10}$$

The projection $p \colon X_n \to X_{n'}$ on the first s variables is a fibration with fibers

$$X_r(z_1, \ldots, z_s) = p^{-1}(z_1, \ldots, z_s) \,. \tag{2.11}$$

These are configuration spaces of r indistinguishable particles on the punctured torus $\Sigma \backslash \{z_1, \ldots, z_s\}$. Fix τ and $(z_1, \ldots, z_s) \in X_{n'}$ to obtain a vector (2.8) of multi-valued r-forms on (2.11). The positions $z_1, \ldots, z_s$, which are presently kept fixed, should not be confused with the positions of the screening charges $z_{s+1}, \ldots, z_{s+r}$. Let us suppress the dependence on the former and the modular parameter in our notation.

The r-forms (2.8) are multivalued on X_r, single valued on the universal covering space $\tilde{X}_r(*)$ with base point $*$, and define a $2p'p$-dimensional representation ρ of $\pi_1(X_r, *)$ through

$$\phi_\sigma^*(\omega_\mu) = \sum_{\nu=0}^{2p'p-1} \omega_\nu \rho_{\nu\mu}(\sigma), \qquad \sigma \in \pi_1(X_r, *), \tag{2.12}$$

with $\phi_\sigma(x) = x\sigma$ the right action of the fundamental group on the universal covering space. The representation matrices can be computed explicitly by analytic continuation. Let c_μ, $\mu = 0, \ldots, 2p'p - 1$ be singular r-chains in the universal covering space. The equivalence relation $\phi_\sigma(c_\mu) \sim \sum_\nu \rho_{\mu\nu} c_\nu$ is compatible with the pairing

$$\langle \omega, c \rangle = \sum_{\mu=0}^{2p'p-1} \int_{c_\mu} \omega_\mu . \tag{2.13}$$

In other words we can view c as a singular r-chain with coefficients in the space of local horizontal sections of the vector bundle of rank $2p'p$,

$$L_r = \tilde{X}_r(*) \times \mathbb{C}^{2p'p}/\sim \;,$$
$$(x\sigma, v) \sim (x, \rho(\sigma)v) . \tag{2.14}$$

We thus need to examine the singular homology group $H_r^{\mathrm{lf}}(X_r, L_r)$ with coefficients in the local system associated to the representation ρ of $\pi_1(X_r, *)$. As a support condition we will require that the chains are locally finite (lf), possibly infinite linear combinations of simplices, [23] on $X_r^\varepsilon = \{(w_1, \ldots, w_r) \in X_r \,|\, |w_i - z_j| \geqq \varepsilon\}$. Elements of $H_r^{\mathrm{lf}}(X_r, L_r)$ produce, when paired with (2.8) a generalized hypergeometric function on the torus (provided the integral is convergent).

3. Local Systems over Configuration Spaces on the Torus

3.1. Braid Group on the Torus. Let $T = S^1 \times S^1$, $n \in \{1, 2, \ldots\}$, and define

$$\mathscr{C}_n(T) = (T^n \setminus \bigcup_{i<j} \{x_i = x_j\})/S_n , \tag{3.1}$$

the configuration space of n indistinguishable particles on the torus. Here S_n denotes the symmetric group acting as $\pi(x_1, \ldots, x_n) = (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$. Let

$$B_n(T, *) = \pi_1(\mathscr{C}_n(T), *) , \tag{3.2}$$

be the braid group on the torus. A convenient choice of base point is

$$* = \left[\left(\frac{1}{N}, \frac{1}{N}\right), \ldots, \left(\frac{n}{N}, \frac{n}{N}\right)\right] \tag{3.3}$$

for some $N > n$. For $1 \leqq i \leqq n$, define elements $\alpha_i, \beta_i \in B_n(T, *)$ as represented by the paths $[0, 1] \to \mathscr{C}(T)$,

$$\alpha_i(t) = \left[\ldots, \left(\frac{i}{N} + t, \frac{i}{N}\right), \ldots\right],$$
$$\beta_i(t) = \left[\ldots, \left(\frac{i}{N}, \frac{i}{N} + t\right), \ldots\right], \tag{3.4}$$

moving the particles in position i along an A- and B-cycle, respectively. For $1 \leqq i \leqq n-1$, define elements $\sigma_i \in B_n(T, *)$ as represented by

$$\sigma_i = [\ldots, x_i(t), x_{i+1}(t), \ldots] ,$$

$$x_i(t) = \frac{1}{2N}(2i+1, 2i+1) - \frac{1}{\sqrt{2N}}\Delta(t) ,$$

$$x_{i+1}(t) = \frac{1}{2N}(2i+1, 2i+1) + \frac{1}{\sqrt{2N}}\Delta(t) ,$$

$$\Delta(t) = \left(\cos\left(2\pi t - \frac{\pi}{4}\right), \sin\left(2\pi t - \frac{\pi}{4}\right)\right) , \tag{3.5}$$

implementing a counterclockwise exchange of the particle in position i with that in position $i+1$. It is convenient to introduce the abbreviations

$$\alpha = \sigma_1 \cdots \sigma_{n-1}\alpha_n, \qquad \beta = \beta_n , \tag{3.6}$$

in terms of which

$$\alpha_i = \sigma_{i-1}^{-1} \cdots \sigma_1^{-1}\alpha\sigma_{n-1} \cdots \sigma_i ,$$

$$\beta_i = \sigma_i^{-1} \cdots \sigma_{n-1}^{-1}\beta\sigma_{n-1}^{-1} \cdots \sigma_i^{-1} . \tag{3.7}$$

The group $B_n(T, *)$ is generated by α, β, and σ_i, $1 \leqq i \leqq n-1$. A detailed investigation of $B_n(T, *)$ can be found in [24].

3.2. Coloured Braid Groupoid and Local System. Let $n = (n_1, \ldots, n_k) \in \{1, 2, \ldots\}^k$, $|n| = n_1 + \cdots + n_k$, and define

$$\mathscr{C}_n(T) = (T^{|n|} \setminus \textstyle\bigcup_{i<j}\{x_i = x_j\})/S_{n_1} \times \cdots \times S_{n_k} , \tag{3.8}$$

the configuration space of particles with colours $\{1, \ldots, k\}$, identically coloured particles being indistinguishable. Let $* \in \mathscr{C}_n(T)$ be the base point (3.3). The orbit of $*$ under the action of $S_{|n|}$ can be identified with the right coset space $I_n = S_{|n|}/S_{n_1} \times \cdots \times S_{n_k}$. An element $[\pi] \in I_n$ can in turn be described by a colour map $\bar{\pi}$: $\{1, \ldots, |n|\} \to \{1, \ldots, k\}$. For $[\pi]$, $[\sigma] \in I_n$, let $B_{|n|}^{\bar{\sigma}, \bar{\pi}}(T, *)$ be the space of paths starting in $\pi *$ and ending in $\sigma *$, up to homotopies preserving the endpoints. Define

$$B_n(T, *) = \textstyle\bigcup_{[\pi], [\sigma] \in I_n} B_{|n|}^{\bar{\sigma}, \bar{\pi}}(T, *) , \tag{3.9}$$

the coloured braid groupoid on the torus. Multiplication is composition of paths. In particular, $B_{|n|}^{\overline{\mathrm{id}}, \overline{\mathrm{id}}}(T, *) = \pi_1(\mathscr{C}_n(T), *)$. The coloured braid groupoid $B_n(T, *)$ can be described in terms of the braid group $B_{|n|}(T, *)$. Let ψ: $B_{|n|}(T, *) \to S_{|n|}$ be the canonical homomorphism. There exist one-to-one maps

$$\phi_{\bar{\sigma}, \bar{\pi}}: \{g \in B_{|n|}(T, *) \,|\, [\sigma] = [\psi(g)\pi]\} \to B_{|n|}^{\bar{\sigma}, \bar{\pi}}(T, *) , \tag{3.10}$$

having the property

$$\phi_{\bar{v}, \bar{\sigma}}(g')\phi_{\bar{\sigma}, \bar{\pi}}(g) = \phi_{\bar{v}, \bar{\sigma}}(g'g) . \tag{3.11}$$

Using this, we can write down generators for $B_n(T, *)$. The generators are

$$\begin{aligned} \alpha^{[\pi]} &= \phi_{\overline{\varepsilon\pi}, \bar{\pi}}(\alpha) , \\ \beta^{[\pi]} &= \phi_{\bar{\pi}, \bar{\pi}}(\beta) , \\ \sigma_i^{[\pi]} &= \phi_{\overline{\tau_i \pi}, \bar{\pi}}(\sigma_i) , \quad 1 \leqq i \leqq |n| - 1 . \end{aligned} \tag{3.12}$$

ε is the cyclic permutation $\varepsilon(i) = i + 1 \bmod |n|$ and τ_i is the i^{th} transposition. A representation of $B_n(T, *)$ on a family of finite dimensional vector spaces $V_{\bar{\pi}}$ indexed by $[\pi] \in I_n$, is a family of maps

$$\rho_{\bar{\sigma}, \bar{\pi}} \colon B_{|n|}^{\bar{\sigma}, \bar{\pi}}(T, *) \to \mathrm{Hom}^*(V_{\bar{\pi}}, V_{\bar{\sigma}}) , \tag{3.13}$$

from $B_n(T, *)$ to $\bigcup_{[\pi], [\sigma] \in I_n} \mathrm{Hom}^*(V_{\bar{\pi}}, V_{\bar{\sigma}})$, the groupoid of invertible linear mappings between the vector spaces $V_{\bar{\pi}}$, satisfying the representation property[1]

$$\rho_{\bar{\nu}, \bar{\sigma}}(g') \rho_{\bar{\sigma}, \bar{\pi}}(g) = \rho_{\bar{\nu}, \bar{\sigma}}(g' g) . \tag{3.14}$$

The dimension of a representation ρ is $d = \dim(V_{\bar{\pi}})$. A d-dimensional representation ρ of $B_n(T, *)$ defines a flat rank d vector bundle over $\mathscr{C}_n(T)$ with distinguished trivializations over the points $\pi *$, $[\pi] \in I_n$.

The representation ρ restricted to $\pi_1(\mathscr{C}_n(T), *)$ gives a flat vector bundle $L = \tilde{\mathscr{C}}_n(T) \times_{\pi_1} V_{\overline{\mathrm{id}}}$. It comes with an identification of the fiber over $*$ with $V_{\overline{\mathrm{id}}}$. The identification of the fiber over $\pi *$ with $V_{\bar{\pi}}$ is uniquely given by the condition that the parallel transport along any path η from $*$ to $\pi *$ is $\rho_{\overline{\mathrm{id}}, \bar{\pi}}(\eta)$.

To do explicit calculations, it is convenient to introduce local trivializations of L. Define cells labeled by elements of I_n:

$$\mathscr{C}_{n, \bar{\pi}}(T) = \{[x_1, \ldots, x_{|n|}] \in \mathscr{C}_n(T) | 0 < x^1_{\pi(1)} < \cdots < x^1_{\pi(|n|)} < 1, 0 < x_i^2 < 1\} . \tag{3.15}$$

The union of the closures of these cells is $\mathscr{C}_n(T)$, and every cell contains precisely one of the points in the $S_{|n|}$ orbit of $*$. Since cells are contractible, we have an identification of the restriction of L to $\mathscr{C}_{n, \bar{\pi}}(T)$ with the trivial flat bundle $\mathscr{C}_{n, \bar{\pi}}(T) \times V_{\bar{\pi}}$. This trivialization will be used often below.

3.3. Torus with Punctures. Let $n' = (n_1, \ldots, n_{k-1})$, $s = |n'|$, and $r = n_k$. The projection $p \colon \mathscr{C}_n(T) \to \mathscr{C}_{n'}(T)$ on the first s variables is a fibration with fibres

$$\mathscr{C}_r(T \backslash \{x_1, \ldots, x_s\}) = p^{-1}(x_1, \ldots, x_s) . \tag{3.16}$$

Note that $\mathscr{C}_r(T \backslash \{x_1, \ldots, x_s\})$ is the configuration space of r indistinguishable particles on $T \backslash \{x_1, \ldots, x_s\}$, the punctured torus. Choose a base point $* = [x_1, \ldots, x_s] \in \mathscr{C}_{n, \overline{\mathrm{id}}}(T)$ and let $* = [x_{s+1}, \ldots, x_{s+r}]$ be the base point of $\mathscr{C}_r(T \backslash \{x_1, \ldots, x_s\})$. Then $B_{n', r}(T, *) = \pi_1(\mathscr{C}_r(T \backslash \{x_1, \ldots, x_s\}), *)$ is a subgroupoid of $B_n(T, *)$, and we have a homomorphism $B_{n', r}(T, *) \to B_n(T, *)$. The flat vector bundle corresponding to the pull back of a representation ρ is just the restriction to $\mathscr{C}_r(T \backslash \{x_1, \ldots, x_s\})$ of the flat vector bundle over $\mathscr{C}_n(T)$ associated to ρ.

[1] In the language of categories, $B_n(T, *)$ is the set of morphisms of a category whose objects are elements of I_n. A representation is a functor to the category of finite dimensional vector spaces

4. Topological Representations of $U_q(sl_2)$

4.1. Local System from Multivalued Forms. The multipliers of the multivalued r-forms upon analytic continuation define a particular representation of the coloured braid groupoid. This representation defines in turn a local system over the configuration space. The singular homology groups we investigate have coefficients in this local system.

Recall the basic data which we start from: $n = (n_1, \ldots, n_k) \in \{1, 2, \ldots\}^k$, $|n| = s + r$, $n_k = r$, and $(\alpha(1), \ldots, \alpha(k)) \in \mathbb{Q}^k$, $\alpha(k) = \alpha_+$ such that $\sum_{j=1}^{k} n_k \alpha(k) = 0$. Then $\alpha_i = \alpha(\overline{\mathrm{id}}(i))$. Remember that $\bar{\pi}$: $\{1, \ldots, s+r\} \to \{1, \ldots, k\}$ denotes the colour map associated with $[\pi] \in I_n$. Given these data, we consider

$$f_\mu^\Sigma(z_1, \ldots, z_{s+r}|\tau) = \Delta_\mu\left(\sqrt{2p'p}\sum_{i=1}^{s+r}\alpha_i z_i\,\middle|\,\tau\right)\prod_{1\leqq i<j\leqq s+r} E(z_i, z_j)^{\alpha_i\alpha_j}, \tag{4.1}$$

$0 \leqq \mu \leqq 2p'p - 1$. Fix the modular parameter τ. Then $f^\Sigma = (f_\mu^\Sigma)_{0\leqq\mu\leqq 2p'p-1}$ is a multivalued analytic function on the configuration space $\mathscr{C}_n(\Sigma)$ with values in $\mathbb{C}^{2p'p}$.

Let ϕ: $T \to \Sigma$, $\phi(x^1, x^2) = x^1 + \tau x^2$, to obtain a diffeomorphism ϕ: $\mathscr{C}_n(T) \to \mathscr{C}_n(\Sigma)$. Define $f^T = \phi^* f^\Sigma$. Fix a base point $* = [x_1, \ldots, x_{s+r}]$ in $Y_n := \mathscr{C}_n(T)$ such that $0 < x_1^1 < \cdots < x_{s+r}^1 < 1$. For $[\pi] \in I_n$, let $f^{T,\bar{\pi}}$ be the single-valued function on the universal covering space $\tilde{Y}_n(\pi*)$ with values in $\mathbb{C}^{2p'p}$, defined as the analytic continuation of f^T from the base point $\pi*$, where it takes the value $f^{T,\bar{\pi}}(\pi*) = f^{T,\overline{\mathrm{id}}}(*)$. An element $g \in B_{|n|}^{\bar{\sigma},\bar{\pi}}(T, *)$ induces a map λ_g: $\tilde{Y}_n(\pi*) \to \tilde{Y}_n(\sigma*)$ through $\lambda_g(x) = xg$. xg is represented by a path from $\pi*$ to $\sigma*$, composed with a path from $\sigma*$ to $p(x)$, p: $\tilde{Y}_n(\pi*) \to Y_n$ being the covering projection. Then

$$\lambda_g^*(f_\mu^{T,\bar{\pi}}) = \sum_{\nu=0}^{2p'p-1} f_\nu^{T,\bar{\sigma}} M_{\nu,\mu}^{\bar{\sigma},\bar{\mu}}(g) \tag{4.2}$$

defines a $2p'p$ dimensional representation of $B_n(T, *)$ on $V_{\bar{\pi}} = \mathbb{C}^{2p'p}$. An explicit calculation by analytic continuation yields

$$\begin{aligned}
M_{\nu,\mu}^{\overline{\varepsilon\pi},\bar{\pi}}(\alpha^{[\pi]}) &= \delta_{\nu,\mu}\, e^{2\pi i\frac{\alpha(\bar{\pi}(s+r))}{\sqrt{2p'p}}\mu}\,,\\
M_{\nu,\mu}^{\bar{\pi},\bar{\pi}}(\beta^{[\pi]}) &= \delta_{\nu,\mu+\sqrt{2p'p}\alpha(\bar{\pi}(s+r))}\, e^{-\pi i\alpha(\bar{\pi}(s+r))^2}\,,\\
M_{\nu,\mu}^{\overline{\tau_i\pi},\bar{\pi}}(\sigma_i^{[\pi]}) &= \delta_{\nu,\mu}\, e^{\pi i\alpha(\bar{\pi}(i))\alpha(\bar{\pi}(i+1))}\,.
\end{aligned} \tag{4.3}$$

If $\alpha(\bar{\pi}(s+r)) = \alpha_+$ is a screening charge, it follows that

$$\begin{aligned}
M_{\nu,\mu}^{\overline{\varepsilon\pi},\bar{\pi}}(\alpha^{[\pi]}) &= \delta_{\nu,\mu}\, e^{2\pi i\frac{\mu}{p}}\,,\\
M_{\nu,\mu}^{\bar{\pi},\bar{\pi}}(\beta^{[\pi]}) &= \delta_{\nu,\mu+2p'}\, e^{-2\pi i\frac{p'}{p}}\,.
\end{aligned} \tag{4.4}$$

These matrices deserve an abbreviation since they will occur frequently below. Let $q = \exp(\pi i p'/p)$ and

$$\begin{aligned}
A_{\nu,\mu} &= \delta_{\nu,\mu}\, q^{2\frac{\mu}{p'}}\,,\\
B_{\nu,\mu} &= \delta_{\nu,\mu+2p'}\, q^{-2}\,.
\end{aligned} \tag{4.5}$$

with the convention $q^{1/p'} = \exp(\pi i/p)$. If $\alpha(\bar\pi(i)) = (1 - n(\bar\pi(i)))p'/\sqrt{2p'p}$ and $\alpha(\bar\pi(i+1)) = \alpha_+$, it follows that

$$M^{\overline{\tau_i\pi},\bar\pi}_{\nu,\mu}(\sigma_i^{[\pi]}) = \delta_{\nu,\mu} q^{1-n(\bar\pi(i))} . \tag{4.6}$$

If $\alpha(\bar\pi(i)) = \alpha(\bar\pi(i+1)) = \alpha_+$, this representation matrix takes the form

$$M^{\overline{\tau_i\pi},\bar\pi}_{\nu,\mu}(\sigma_i^{[\pi]}) = \delta_{\nu,\mu} q^2 , \tag{4.7}$$

completing the description of the representation of $B_n(T, *)$ associated with the multipliers of f^T.

Recall that $n' = (n_1, \ldots, n_{k-1})$, $|n'| = s$, $p\colon \mathscr{C}_n(T) \to \mathscr{C}_{n'}(T)$, and $\mathscr{C}_r(T \backslash \{x_1, \ldots, x_s\}) = p^{-1}(x_1, \ldots, x_s)$. In the following, $[x_1, \ldots, x_s]$ will be fixed as above. Pulling back the above representation of $B_n(T, *)$ with the homomorphism $B_{n',r}(T, *) \to B_n(T, *)$, we obtain a representation of $B_{n',r}(T, *)$. We take the tensor product of this representation with the pull-back of the totally antisymmetric representation of S_r by the canonical homomorphism $B_{n',r}(T, *) \to S_r$. The result is the representation, denoted by ρ, associated with the multi-valued r-forms (2.8). This representation induces a local system (flat vector bundle) $\pi\colon L_r(x_1, \ldots, x_s) \to \mathscr{C}_r(T \backslash \{x_1, \ldots, x_s\})$.

Let $\varepsilon > 0$ be a small number, D_i^ε the open disc of radius ε centered at x_i, $i = 1, \ldots, s$, $Y^\varepsilon = T - \bigcup_{i=1}^s D_i$. Denote by Y_r^ε the configuration space $\mathscr{C}_r(Y^\varepsilon)$ of r indistinguishable points on Y^ε. Thus elements of Y_r^ε are subsets $Z \subset Y^\varepsilon$ of cardinality r. Fix points $y_-, y_+ \in \partial D_1^\varepsilon$ such that $y_+^1 < x_1 < y_-^1$, and define $Y_r^{\varepsilon\pm} = \{Z \in Y_r \mid y_\pm \in Z\}$. The bijections

$$\begin{aligned} \phi_\pm\colon Y_r^\varepsilon \backslash Y_r^{\varepsilon\pm} &\to Y_{r+1}^{\varepsilon\pm} , \\ Z &\mapsto Z \cup \{y_\pm\} \end{aligned} \tag{4.8}$$

lift to isomorphisms $\phi_\pm\colon L_r | Y_r^\varepsilon \backslash Y_r^{\varepsilon\pm} \to L_{r+1} | Y_{r+1}^{\varepsilon\pm}$. The lift is of course not unique. To fix it, it is sufficient to define the isomorphism from the fiber of the base point to the fiber of its image. We define it to be the identity map in the distinguished trivialization introduced above.

4.2. Families of Loops and Operators. Let $[x_1, \ldots, x_s] \in \mathscr{C}_{n',\mathrm{id}}(T)$, $\varepsilon > 0$, $Y_r^\varepsilon = \mathscr{C}_r(Y^\varepsilon)$, and $y^\pm \in \partial Y^\varepsilon$ be as above. The position of the punctures will be kept fixed in the following.

A non-intersecting family of loops (see [2] for details) based at y_- is a family $\gamma_0, \ldots, \gamma_{r-1}$ of smooth homotopically non-trivial embedded loops starting and ending at y_-, with no mutual intersections except at the endpoints. Homotopies of families of loops are defined. Non-intersecting families of loops can be represented by embeddings Γ of the open r-cube with open $(r-1)$-faces Q_r into X_r. Let $\tilde A_r^\varepsilon$ be the space of linear combinations $\sum \lambda_\Gamma [\Gamma]$ of homotopy classes $[\Gamma] = [\gamma_0, \ldots, \gamma_{r-1}]$ of non-intersecting families of loops, with coefficients λ_Γ in the space of horizontal sections of $\Gamma^* L_r$ over Q_r. Horizontal sections corresponding to homotopic families of loops are canonically identified by parallel transport, so the definition makes sense. The elements of $\tilde A_r^\varepsilon$ represent locally finite relative homology classes in $H_r^{\mathrm{lf}}(Y_r^\varepsilon, Y_r^{\varepsilon-}; L_r)$. We consider a quotient of $\tilde A_r^\varepsilon$ by a subspace which maps to zero in homology. Let $A_r^\varepsilon = \tilde{A}_r^\varepsilon/\sim$, where the equivalence relation $\sim$ is given by

I. $\lambda[\Gamma] \sim \pm f^*\lambda[\Gamma \circ f]$ for any orientation preserving $(+)$ or reversing $(-)$ isometry of the cube Q_r.
II. $\lambda[\gamma_0, \ldots, \gamma_i, \ldots, \gamma_{r-1}] = \lambda'[\gamma_0, \ldots, \gamma_i', \ldots, \gamma_{r-1}] + \lambda''[\gamma_0, \ldots, \gamma_i'', \ldots, \gamma_{r-1}]$, whenever γ_i is homotopic to the composition $\gamma_i' \circ \gamma_i''$ in such a way that if the homotopy is denoted by $h(\cdot, s)$, $s \in [0, 1]$, $\gamma_0, \ldots, h(\cdot, s), \ldots, \gamma_{r-1}$ is a non-intersecting family of loops for all $s \in [0, 1]$. The sections λ', λ'' are the restrictions of λ.
III. $\lambda[\gamma_0, \ldots, \gamma_{r-1}] \sim 0$, whenever at least p loops in the family are in the same class in $\pi_1(Y^\varepsilon, y_-)$.

The identification III is peculiar to the case when q is a root of unity: if n loops, say $\gamma_0, \ldots, \gamma_{n-1}$ in a non-intersecting family $\gamma_0, \ldots, \gamma_{r-1}$ are homotopic to a loop γ, then the corresponding locally finite homology class is proportional to the class of a relative cycle parametrized by $t_0 < t_1 < \cdots < t_{n-1} < t_n, \ldots, t_{r-1} \in Q_r$ as

$$(t_0, \ldots, t_{r-1}) \mapsto (\gamma(t_0), \ldots, \gamma(t_{n-1}), \gamma(t_n), \ldots, \gamma(t_{r-1})) \,. \tag{4.9}$$

The proportionality factor is

$$\prod_{j=1}^{n} \frac{q^{2j}-1}{q^2-1}, \tag{4.10}$$

and vanishes if $n \geqq p$.

We define now operators $\hat{E}$, $\hat{F}$, and $\hat{K}^2$, acting on the space $\bigoplus_{r=0}^{\infty} A_r^\varepsilon$ and compute their commutation relations.

The operator $\hat{E}$ is a close relative of the boundary operator. Define

$$\hat{E}\colon \lambda_\Gamma[\gamma_0, \ldots, \gamma_{r-1}] \mapsto \sum_{i=0}^{r-1} (-1)^i \phi_-^{-1}(\lambda_\Gamma \circ e_{i,r}^+ - \lambda_\Gamma \circ e_{i,r}^-)[\gamma_0, \ldots, \hat{\gamma}_i, \ldots, \gamma_{r-1}] \,, \tag{4.11}$$

with $e_{i,r}^\pm\colon [0, 1]^{r-1} \to [0, 1]^r$ the standard face maps. $\hat{\gamma}_i$ denotes the omission of γ_i. Intuitively the i^{th} particle is moved to y_- and then taken out. The operator $\hat{F}$ adds a loop along the boundary of the hole around the first puncture.

$$\hat{F}\colon \lambda[\gamma_0, \ldots, \gamma_{r-1}] \mapsto \lambda'[\gamma_0, \ldots, \gamma_{r-1}, \gamma_C] \tag{4.12}$$

with $\gamma_C\colon [0, 1] \to Y^\varepsilon$, $\gamma_C(t) = x_1 + \frac{\varepsilon}{\sqrt{2}} \Delta(t)$. Here $\Gamma' = \gamma_0, \ldots, \gamma_{r-1}, \gamma_C$ and λ' is the horizontal section of $\Gamma'^* L_{r+1}^\varepsilon$ with $\phi_+\lambda = \lambda' \circ i$, where i is the inclusion $(t_0, \ldots, t_{r-1}, \frac{1}{2})$. This definition makes sense since we can assume that $\gamma_0, \ldots, \gamma_{r-1}$ do not intersect γ_C except at the endpoints. The operator $\hat{K}^2$ is simply defined as

$$\hat{K}^2|_{A_r^\varepsilon} = q^{-1-n_1(r)} \,. \tag{4.13}$$

Charge neutrality requires that $1 - n_1(r) = \sum_{i=2}^{s} (n_i - 1) - 2r$. Note that both the construction of $\hat{E}$ and $\hat{F}$ make use of the isomorphisms (4.8), relating local systems over different configuration spaces.

Theorem 4.1. *The operators $\hat{E}$, $\hat{F}$ and $\hat{K}^2$ satisfy the relations*

$$\hat{K}^2\hat{E} = q^2\hat{E}\hat{K}^2 ,$$
$$\hat{K}^2\hat{F} = q^{-2}\hat{F}\hat{K}^2 ,$$
$$[\hat{E}, \hat{F}] = \hat{K}^2 - \hat{K}^{-2} . \tag{4.14}$$

In other words, the operators $\hat{E}$, $\hat{F}$, $\hat{K}^2$ and $\hat{K}^{-2}$ define a representation of $U_q(sl_2)$ on $\bigoplus_{r=0}^{\infty} A_r^{\varepsilon}$.

Proof. The first and the second relation are immediate consequences of the definition of $\hat{E}$, $\hat{F}$ and $\hat{K}^2$. The third relation is best proved using an explicit trivialization. Without loss of generality, we can assume that $[x_1, \ldots, x_s, \gamma_0(\frac{1}{2}), \ldots, \gamma_{r-1}(\frac{1}{2})] \in \mathscr{C}_{n,\bar{\pi}}(T)$ for some $[\pi] \in I_n$. Denote by $\lambda(v)$ the section with the value v in the trivialization over $\mathscr{C}_{n,\bar{\pi}}(T)$. We can further assume that $[x_1, \ldots, x_s, \gamma_0(\frac{1}{2}), \ldots, y_-, \ldots, \gamma_{r-1}(\frac{1}{2})] \in \mathscr{C}_{n,\bar{\sigma}_i}(T)$ for some $[\sigma_i] \in I_n$, (y_- in position i). Let $\eta_i^{\pm}$ be the paths $t \mapsto [x_1, \ldots, x_s, \gamma_0(\frac{1}{2}), \ldots, \gamma_i(\frac{1}{2}(1 \pm t)), \ldots, \gamma_{r-1}(\frac{1}{2})]$. Using

$$\hat{E}\lambda(v)[\gamma_0, \ldots, \gamma_{r-1}] = \sum_{i=0}^{r-1} (-1)^i(\rho_{\bar{\sigma}_i,\bar{\pi}}(\eta_i^+) - \rho_{\bar{\sigma}_i,\bar{\pi}}(\eta_i^-))\lambda(v)[\gamma_0, \ldots, \hat{\gamma}_i, \ldots, \gamma_{r-1}] , \tag{4.15}$$

it follows that

$$\begin{aligned}
\hat{E}\hat{F}\lambda(v)[\gamma_0, \ldots, \gamma_{r-1}] &= \hat{E}\lambda(v)[\gamma_0, \ldots, \gamma_{r-1}, \gamma_C] \\
&= \sum_{i=0}^{r} (-1)^i(\rho_{\bar{\sigma}'_i,\bar{\pi}'}(\eta_i^{+\prime}) - \rho_{\bar{\sigma}'_i,\bar{\pi}'}(\eta_i^{-\prime})) \\
&\quad \lambda(v)[\gamma_0, \ldots, \hat{\gamma}_i, \ldots, \gamma_{r-1}, \gamma_C] \\
&= \sum_{i=0}^{r-1} (-1)^i(\rho_{\bar{\sigma}_i,\bar{\pi}}(\eta_i^+) - \rho_{\bar{\sigma}_i,\bar{\pi}}(\eta_i^-)) \\
&\quad \lambda(v)[\gamma_0, \ldots, \hat{\gamma}_i, \ldots, \gamma_{r-1}, \gamma_C] \\
&\quad + (-1)^r(\rho_{\bar{\sigma}'_r,\bar{\pi}'}(\eta_{r'}^+) - \rho_{\bar{\sigma}'_r,\bar{\pi}'}(\eta_{r'}^-))\lambda(v)[\gamma_0, \ldots, \gamma_{r-1}] \\
&= \hat{F}\hat{E}\lambda(v)[\gamma_0, \ldots, \gamma_{r-1}] \\
&\quad + (q^{1-n_1(r+1)} - q^{n_1(r+1)-1})\lambda(v)[\gamma_0, \ldots, \gamma_{r-1}] ,
\end{aligned} \tag{4.16}$$

where the primed quantities are defined with r replacing $r-1$, proving the third relation. □

4.3. The Torus with One Puncture. We have proved above that $\bigoplus_{r=0}^{\infty} A_r^{\varepsilon}$ comes equipped with the structure of a $U_q(sl_2)$ module. A legitimate question to address is what kind of module this is. In this section we will consider the torus with a single puncture and find the adjoint representation of $U_q(sl_2)$.

To describe A_r^{ε} as a space, we choose a basis as follows. Let γ_A and γ_B be loops in Y^{ε} based at y_- such that γ_A winds around an A-cycle and γ_B winds around a B-cycle. For $j_A, j_B \in \{0, 1, \ldots\}$ such that $j_A + j_B = r$, let $\gamma_B^{(1)}, \ldots, \gamma_B^{(j_B)}$ be homotopic deformations of γ_B and $\gamma_A^{(1)}, \ldots, \gamma_A^{(j_A)}$ homotopic deformations of γ_A such that $\gamma_B^{(i)}$ lies on the left of $\gamma_B^{(i+1)}$ and $\gamma_A^{(i)}$ above $\gamma_A^{(i+1)}$, and $\gamma_B^{(1)}, \ldots, \gamma_A^{(j_A)}$ is a non-intersecting family of loops, also denoted by $\gamma_B^{j_B}, \gamma_A^{j_A}$. This family of loops is drawn in Fig. 1. We choose representatives in such a way that $[x_1, \gamma_1(\frac{1}{2}), \ldots, \gamma_r(\frac{1}{2})] \in \mathscr{C}_{n, \overline{\mathrm{id}}}(T)$. We define $\lambda(v)$ to be the horizontal section over $[\gamma_B^{j_B}, \gamma_A^{j_A}]$ which takes the value v in the trivialization over $\mathscr{C}_{n, \overline{\mathrm{id}}}(T)$. Let e_μ, $1 \leqq \mu \leqq 2p'p$ be the standard basis of $\mathbb{C}^{2p'p}$. Then

$$A_r^\varepsilon = \bigoplus_{\mu=1}^{2p'p} \bigoplus_{j_A + j_B = r} \mathbb{C}\lambda(e_\mu)[\gamma_B^{j_B}, \gamma_A^{j_A}] . \tag{4.17}$$

In this basis, the operators $\hat{E}$, $\hat{F}$ and $\hat{K}^2$ are represented by the following matrices.

Lemma 4.2. *Applying $\hat{E}$, $\hat{F}$ and $\hat{K}^2$ to the elements of the basis* (4.17) *of A_r^ε we obtain*

$$\begin{aligned}
\hat{E}\lambda(e_\mu)[\gamma_B^{j_B}, \gamma_A^{j_A}] &= \frac{[j_B]_q}{q - q^{-1}} (q^{-4j_A - 3j_B + 3} B - q^{-j_B + 1}) \lambda(e_\mu)[\gamma_B^{j_B - 1}, \gamma_A^{j_A}] \\
&\quad + \frac{[j_A]_q}{q - q^{-1}} (q^{j_A + 2j_B + 1} A - q^{-j_A - 2j_B + 1}) \lambda(e_\mu)[\gamma_B^{j_B}, \gamma_A^{j_A - 1}] , \\
\hat{F}\lambda(e_\mu)[\gamma_B^{j_B}, \gamma_A^{j_A}] &= (q^{2j_A + 2j_B - 2} B^{-1} - q^{2j_A} B^{-1} A^{-1}) \lambda(e_\mu)[\gamma_B^{j_B + 1}, \gamma_A^{j_A}] \\
&\quad + (q^{2j_A} B^{-1} A^{-1} - A^{-1}) \lambda(e_\mu)[\gamma_B^{j_B}, \gamma_A^{j_A + 1}] , \\
\hat{K}^2 \lambda(e_\mu)[\gamma_B^{j_B}, \gamma_A^{j_A}] &= q^{-2j_A - 2j_B - 2} \lambda(e_\mu)[\gamma_B^{j_B}, \gamma_A^{j_A}] ,
\end{aligned} \tag{4.18}$$

with the convention that $[\gamma_B^{j_B}, \gamma_A^{j_A}] = 0$ *if* $j_B < 0$ *or* $j_A < 0$.

Proof. The action of $\hat{E}$ is explicitly computed from

$$\begin{aligned}
\hat{E}\lambda(e_\mu)[\gamma_B^{j_B}, \gamma_A^{j_A}] &= \sum_{i=1}^{j_B} (-1)^i (\rho_{\bar{\sigma}_i, \overline{\mathrm{id}}}(\eta_i^+) - \rho_{\bar{\sigma}_i, \overline{\mathrm{id}}}(\eta_i^-)) \lambda(e_\mu)[\gamma_B^{j_B - 1}, \gamma_A^{j_A}] \\
&\quad + \sum_{i = j_B + 1}^{j_A + j_B} (-1)^i (\rho_{\bar{\sigma}_i, \overline{\mathrm{id}}}(\eta_i^+) - \rho_{\bar{\sigma}_i, \overline{\mathrm{id}}}(\eta_i^-)) \lambda(e_\mu)[\gamma_B^{j_B}, \gamma_A^{j_A - 1}] .
\end{aligned} \tag{4.19}$$

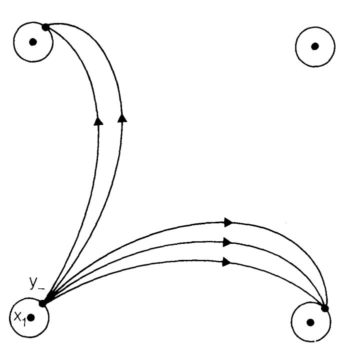

Fig. 1. The family of loops $\gamma_B^{j_B}, \gamma_A^{j_A}$

The paths $\eta_i^{\pm}$ have representation matrices

$$\rho_{\bar{\sigma}_i,\overline{\mathrm{id}}}(\eta_i^+) = \begin{cases} (-q^{-2})^{r-1}B(-q^{-2})^{r-i}, & 1 \leqq i \leqq j_B\,, \\ (-1)^{r-1}q^{n_1(r)-1}A(-q^{-2})^{r-i}, & j_B+1 \leqq i \leqq j_A+j_B\,, \end{cases}$$

$$\rho_{\bar{\sigma}_i,\overline{\mathrm{id}}}(\eta_i^-) = (-q^{-2})^{i-1}\,. \tag{4.20}$$

To compute the action of $\hat{F}$, we have to deform the added loop to the composition of two γ_A and two γ_B loops. The result is

$$\begin{aligned} \hat{F}\lambda(e_\mu)[\gamma_B^{j_B},\gamma_A^{j_A}] &= \rho_{\bar{\sigma}_1,\bar{\pi}}(\eta_1)\lambda(e_\mu)[\gamma_B^{j_B},\gamma_A^{j_A+1}] \\ &\quad + \rho_{\bar{\sigma}_2,\bar{\pi}}(\eta_2)\lambda(e_\mu)([\gamma_B^{(1)},\ldots,\gamma_B^{(j_B)},\gamma_A^{(1)},\ldots,\gamma_A^{(j_A)},\gamma_B^{(j_B+1)}] \\ &\quad - [\gamma_B^{(1)},\ldots,\gamma_B^{(j_B)},\gamma_A^{(2)},\ldots,\gamma_A^{(j_A+1)},\gamma_A^{(1)}]) \\ &\quad - \rho_{\bar{\sigma}_3,\bar{\pi}}(\eta_3)\lambda(e_\mu)[\gamma_B^{(2)},\ldots,\gamma_B^{(j_B+1)},\ldots,\gamma_A^{(1)},\ldots,\gamma_A^{(j_A)},\gamma_B^{(1)}]\,. \end{aligned} \tag{4.21}$$

A transition function is picked up when the cell on which the bundle is trivialized changes. The paths η_i have braid groupoid representation matrices

$$\begin{aligned} \rho_{\bar{\sigma}_1,\bar{\pi}(\eta_1)} &= A^{-1}\,, \\ \rho_{\bar{\sigma}_2,\bar{\pi}(\eta_2)} &= (-q^2)^{j_A}B^{-1}A^{-1}\,, \\ \rho_{\bar{\sigma}_3,\bar{\pi}(\eta_3)} &= (-1)^r q^{n_1(r+1)-1}AB^{-1}A^{-1}\,. \end{aligned} \tag{4.22}$$

The last representation matrix can be simplified using $AB = q^4BA$. Finally, the loops are reordered with the help of an isometry of the unit cube. The action of $\hat{K}^2$ follows immediately from its definition. □

The matrices acting on the sections are

$$Ae_\mu = q^{\frac{2\mu}{p'}}e_\mu, \quad Be_\mu = q^{-2}e_{\mu+2p'}\,. \tag{4.23}$$

Using these formulae, we conclude that the representation of $U_q(sl_2)$ on $\bigoplus A_r^\varepsilon$ can be split into a direct sum of isomorphic representations.

Define for $n = 0, 1, 2, \ldots$, $[n]_q = q^n - q^{-n}$, $[n]_q! = [n]_q[n-1]_q\cdots[2]_q[1]_q$, $[0]_q! = 1$.

Lemma 4.3. (1) *A^ε decomposes into a direct sum of subspaces $A^{\varepsilon,n'} = \bigoplus_{n=0}^{2p-1}\bigoplus_{j_A,j_B=0}^{p-1}\mathbb{C}\lambda(e_{-n'p+np'})[\gamma_B^{j_B},\gamma_A^{j_A}]$, $0 \leqq n' \leqq p'-1$, invariant under the action of $U_q(sl_2)$.* (2) *The action of $\hat{E}$, $\hat{F}$ and $\hat{K}^2$ on $A_r^{\varepsilon,n'}$ takes the explicit form*

$$\begin{aligned} \hat{E}\lambda(e_{-n'p+np'})[\gamma_B^{j_B},\gamma_A^{j_A}] &= \frac{[j_B]_q}{q-q^{-1}}(q^{-4j_A-3j_B+1}\lambda(e_{-n'p+(n+2)p'}) \\ &\quad - q^{-j_B+1}\lambda(e_{-n'p+np'}))[\gamma_B^{j_B-1},\gamma_A^{j_A}] \end{aligned}$$

$$+ q^{\frac{n'p}{p'}+n+1} \frac{[j_A]_q \left[j_A + 2j_B - \frac{n'p}{p'} + n \right]_q}{q - q^{-1}}$$

$$\times \lambda(e_{-n'p+np'})[\gamma_B^{j_B}, \gamma_A^{j_A-1}] ,$$

$$\hat{F}\lambda(e_{-n'p+np'})[\gamma_B^{j_B}, \gamma_A^{j_A}] = q^{2j_A + j_B + \frac{n'p}{p'} - n + 1} \left[j_B - \frac{n'p}{p'} + n - 1 \right]_q$$

$$\times \lambda(e_{-n'p+(n-2)p'})[\gamma_B^{j_B+1}, \gamma_A^{j_A}]$$

$$+ (q^{2j_A + 2\frac{n'p}{p'} - 2n + 2} \lambda(e_{-n'p+(n-2)p'})$$

$$- q^{2\frac{n'p}{p'} - 2n} \lambda(e_{-n'p+np'}))[\gamma_B^{j_B}, \gamma_A^{j_A+1}] ,$$

$$\hat{K}^2\lambda(e_{-n'p+np'})[\gamma_B^{j_B}, \gamma_A^{j_A}] = q^{-2j_A - 2j_B - 2}\lambda(e_{-n'p+np'})[\gamma_B^{j_B}, \gamma_A^{j_A}] . \tag{4.24}$$

Denote $A_r^{\varepsilon, n'} = A_r \cap A^{\varepsilon, n'}$. Let $U_q(sl_2)$ be the Hopf algebra with generators E, F, $K^{\pm 2}$, with the relations of Theorem 4.1, and coproduct $\Delta(E) = E \otimes K^2 + 1 \otimes E$, $\Delta(F) = F \otimes K^{-2} + 1 \otimes F$, $\Delta(K^{\pm 2}) = K^{\pm 2} \otimes K^{\pm 2}$ (see Sect. 5). Let I_p be the ideal generated by the central elements E^p, F^p and $(K^2)^{2p} - 1$, and $U_q^0(sl_2) = U_q(sl_2)/I_p$. $U_q(sl_2)$ acts on $U_q^0(sl_2)$ by the adjoint action. We define the idempotents

$$T_{n,\omega} = \frac{1}{2p} \sum_{m=0}^{2p-1} (\omega q^n)^{-m} K^{2m} , \quad n \in \frac{1}{p'} \mathbb{Z}, \quad \omega = \pm 1 . \tag{4.25}$$

As before we set $q^{1/p'} = \exp(\pi i/p)$. These idempotents have the following properties:

$$K^2 T_{n,\omega} = \omega q^n T_{n,\omega} ,$$

$$T_{n+p,\omega} = T_{n, q^p \omega} ,$$

$$T_{n+n'p/p',\omega} = T_{n,(-1)^{n'}\omega} , \quad n, n' \in \mathbb{Z} ,$$

$$T_{n+n'p/p',\omega} T_{m+n'p/p',\omega'} = \delta_{n,m} \delta_{\omega,\omega'} T_{n+n'p/p',\omega} , \quad 0 \leqq n, m \leqq p-1, \quad n' \in \mathbb{Z} . \tag{4.26}$$

For each $n' \in \mathbb{Z}$, the elements $T_{n+n'p/p',\omega}$, $n = 0, \dots, p-1, \omega = \pm 1$ build a basis of the subalgebra generated by K^2.

Definition 4.4. For $0 \leqq n' \leqq p' - 1$, define linear maps $\phi_{n'}$: $A^{\varepsilon, n'} \to U_q^0(sl_2)$ as follows:

$$\phi_{n'}\lambda(e_{-n'p+np'})[\gamma_B^{j_B}, \gamma_A^{j_A}] = q^{-\frac{1}{2} j_B(j_B-1) + (j_A-1)(-\frac{n'p}{p'}+n+1)}$$

$$\times \frac{[j_B]_q!}{(q-q^{-1})^{j_B}} F^{j_A} E^{p-1-j_B} T_{1+\frac{n'p}{p'}-n, 1} \tag{4.27}$$

Having introduced the maps $\phi_{n'}$, we are ready to state the main result of this section.

Theorem 4.5. (1) *For $X \in U_q(sl_2)$, and $n' = 0, \ldots, p' - 1$, the diagram*

$$\begin{array}{ccc} A^{\varepsilon, n'} & \xrightarrow{\hat{X}} & A^{\varepsilon, n'} \\ \phi_{n'} \downarrow & & \downarrow \phi_{n'} \\ U_q^0(sl_2) & \xrightarrow{\mathrm{ad}_X} & U_q^0(sl_2) \end{array} \tag{4.28}$$

commutes. That is, $\phi_{n'}$ is a homomorphism from $A^{\varepsilon, n'}$ to the module $U_q^0(sl_2)$ with the adjoint action. (2) *If p' is odd, $\phi_{n'}$ is an isomorphism. If p' is even, $\phi_{n'}$ is two-to-one, with image the submodule $\{X \in U_q^0(sl_2) | K^{2p}X = (-1)^{n'}X\}$.*

Proof. (1) is checked by an explicit calculation using Lemma 4.3 and Lemma 4.2. To prove (2), we notice that $F^j E^l T_{n,\omega}, j, l, n = 0, \ldots, p-1, \omega = \pm 1$, build a basis of $U_q(sl_2)$. For p' odd, we see from Definition 4.4 using the third of (4.26), that $\phi_{n'}$ is bijective. If p' is even, the image is the subspace spanned by the basis vectors with $\omega = (-1)^{n'}$. □

Let us work out the interplay between topological and algebraic objects a little further. We have introduced $\hat{F}$: $A_r^\varepsilon \to A_{r+1}^\varepsilon$ as the operator which adds a γ_C-loop and identifies the section as described above, using the point y_+. With any loop γ: $[0, 1] \to Y^\varepsilon$ based at y_- such that $\gamma(\frac{1}{2}) = y_+$, we can associate an operator $\hat{L}(\gamma)$: $A_r^\varepsilon \to A_{r+1}^\varepsilon$, $\lambda[\gamma_0, \ldots, \gamma_{r-1}] \mapsto \lambda'[\gamma_0, \ldots, \gamma_{r-1}, \gamma]$, such that $\phi_+\lambda = \lambda' \circ i$, generalizing $\hat{F} = \hat{L}(\gamma_C)$. Two special cases are $\hat{F}_L = \hat{L}(\gamma_A)$ and $\hat{F}_R = \hat{L}(\gamma_D)$. See Fig. 2 for a graphical representation.

Theorem 4.6. (1) *For $0 \leqq n' \leqq p' - 1$, $\hat{F}_{L,R}$ maps $A_r^{\varepsilon, n'}$ to $A_{r+1}^{\varepsilon, n'}$ and $\hat{F} = \hat{F}_L - \hat{F}_R$.* (2) *The diagrams*

$$\begin{array}{ccc} A_r^{\varepsilon, n'} & \xrightarrow{\hat{F}_L} & A_r^{\varepsilon, n'} \\ \phi_{n'} \downarrow & & \downarrow \phi_{n'} \\ U_q^0(sl_2) & \xrightarrow{X \mapsto FXK^2} & U_q^0(sl_2) \end{array} \tag{4.29}$$

and

$$\begin{array}{ccc} A_r^{\varepsilon, n'} & \xrightarrow{\hat{F}_R} & A_r^{\varepsilon, n'} \\ \phi_{n'} \downarrow & & \downarrow \phi_{n'} \\ U_q^0(sl_2) & \xrightarrow{X \mapsto XFK^2} & U_q^0(sl_2) \end{array} \tag{4.30}$$

commute.

Proof. (1) γ_C is homotopic to the composition $(\gamma_D^{-1}) \circ \gamma_A$. Using the equivalence relations imposed on $A_{r+1}^{\varepsilon, n'}$ it follows that

$$\lambda[\gamma_0, \ldots, \gamma_{r-1}, \gamma_C] = \lambda'[\gamma_0, \ldots, \gamma_{r-1}, \gamma_A] + \lambda''[\gamma_0, \ldots, \gamma_{r-1}, \gamma_D] , \tag{4.31}$$

the sections being identified as above. (2) The counterpart of $\hat{F}_L$ on $U_q^0(sl_2)$ follows from

$$\lambda(e_{-n'p+np'})'[\gamma_B^{j_B}, \gamma_A^{j_A+1}] = A^{-1}\lambda(e_{-n'p+np'})[\gamma_B^{j_B}, \gamma_A^{j_A+1}] \tag{4.32}$$

using the explicit form (4.27) of $\phi_{n'}$. The action of $\hat{F}_R$ on $U_q^0(sl_2)$ is computed with $F = \hat{F}_L - \hat{F}_R$, and $\mathrm{ad}_F(X) = FXK^2 - XFK^2$. □

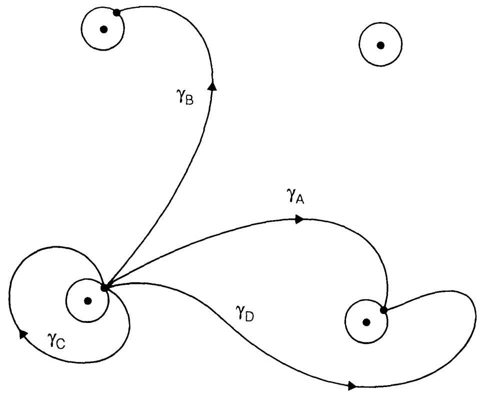

Fig. 2. The loops $\gamma_A, \ldots, \gamma_D$

4.4. The Torus with Many Punctures. Combining the above results with previous work on topological representations of $U_q(sl_2)$ on the disc, the representation on $\bigoplus_{r=0}^{\infty} A_r^\varepsilon$ can be identified. The result is a tensor product of $U_q^0(sl_2)$ Verma modules, one for every additional puncture, with the algebra itself. The latter is understood as the representation space for the adjoint action.

The starting point is again an explicit description of A_r^ε as a space in terms of a basis. Fix a non-intersecting family of loops

$$[\gamma_2^{j_2}, \ldots, \gamma_s^{j_s}, \gamma_B^{j_B}, \gamma_A^{j_A}] := $$
$$[\gamma_2^{(1)}, \ldots, \gamma_s^{(j_2)}, \ldots, \gamma_s^{(1)}, \ldots, \gamma_s^{(j_s)}, \gamma_B^{(1)}, \ldots, \gamma_B^{(j_B)}, \gamma_A^{(1)}, \ldots, \gamma_A^{(j_A)}] . \quad (4.33)$$

It is understood that $\gamma_i^{(k)}$, $2 \leqq i \leqq s$ and $1 \leqq k \leqq j_i$, are homotopic deformations of γ_i such that $\gamma_i^{(k+1)}$ lies inside $\gamma_i^{(k)}$. See Fig. 3.

Let this family be parametrized such that

$$[x_1, \ldots, x_s, \gamma_2^{(1)}(\tfrac{1}{2}), \ldots, \gamma_A^{(j_A)}(\tfrac{1}{2})] \in \mathscr{C}_{n, \overline{\mathrm{id}}}(T) . \quad (4.34)$$

Define a horizontal section over this family, denoted by $\lambda(v)$ giving it the value v in the distinguished trivialization over $\mathscr{C}_{n, \overline{\mathrm{id}}}(T)$. Then

$$A_r^\varepsilon = \bigoplus_{\mu=1}^{2p'p} \bigoplus_{j_2+\cdots+j_A=r} \mathbb{C}\lambda(e_\mu)[\gamma_2^{j_2}, \ldots, \gamma_s^{j_s}, \gamma_B^{j_B}, \gamma_A^{j_A}] . \quad (4.35)$$

Generalizing the case with one puncture, we define the following map.

Definition 4.7. For $0 \leqq n' \leqq p' - 1$, define maps $\phi_{n'}^{(s)}: A^{\varepsilon, n'} \to V(n_2) \otimes \cdots \otimes V(n_s) \otimes U_q^0(sl_2)$ as follows:

$$\phi_{n'}^{(s)}(\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \ldots, \gamma_s^{j_s}, \gamma_B^{j_B}, \gamma_A^{j_A}])$$
$$:= F^{j_2} v_0(n_2) \otimes \cdots \otimes F^{j_s} v_0(n_s) \otimes \phi_{n'}^{(1)}(\lambda(e_{-n'p+np'})[\gamma_B^{j_B}, \gamma_A^{j_A}]) , \quad (4.36)$$

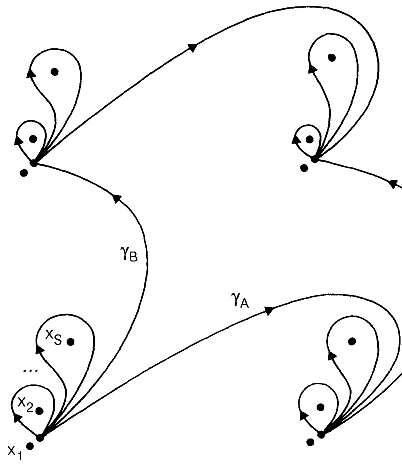

Fig. 3. The family of loops of Eq. (4.33)

where $\phi_{n'}^{(1)}$ denotes the map of Definition 4.4 for the torus with one puncture. $V(h)$ is the $U_q^0(sl_2)$ Verma module generated from a singular vector $v_0(h)$ with $Ev_0(h) = 0$ and $K^2 v_0(h) = q^{h-1} v_0(h)$.

Theorem 4.8. (1) For $0 \leqq n' \leqq p' - 1$, *the maps* $\phi_{n'}^{(s)}: A^{\varepsilon, n'} \to V(n_2) \otimes \cdots \otimes V(n_s) \otimes U_q^0(sl_2)$ *are one-to-one and onto if* p' *is odd, and two-to-one if* p' *is even.* (2) *For* $X \in U_q^0(sl_2)$ *let*

$$\Delta^{(s)}(X) = \sum_i X_i^{(1)} \otimes \cdots \otimes X_i^{(s)} . \tag{4.37}$$

Then the diagram

$$\begin{array}{ccc} A^{\varepsilon, n'} & \xrightarrow{\qquad\qquad \hat{X} \qquad\qquad} & A^{\varepsilon, n'} \\ \phi_{n'}^{(s)} \downarrow & & \downarrow \phi_{n'}^{(s)} \\ \bigotimes_{j=1}^{s} V(n_j) \otimes U_q^0(sl_2) & \xrightarrow{\sum_i X_i^{(1)} \otimes \cdots \otimes X_i^{(s-1)} \otimes \mathrm{ad}(X_i^{(s)})} & \bigotimes_{j=1}^{s} V(n_j) \otimes U_q^0(sl_2) \end{array} \tag{4.38}$$

commutes. That is, the topological action of $U_q(sl_2)$ *on* $A^{\varepsilon, n'}$ *is given by the coproduct on the tensor product of Verma modules with the algebra itslef.*

Proof. We will give an explicit proof for $s = 2$. The generalization to $s > 2$ is obvious and will be omitted. Since

$$\begin{aligned} &\hat{E}\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A}] = \\ &\quad \frac{[j_2]_q [n_2 - j_2]_q}{q - q^{-1}} \lambda(e_{-n'p+np'})[\gamma_2^{j_2 - 1}, \gamma_B^{j_B}, \gamma_A^{j_A}] \\ &\quad + q^{n_2 - 1 - 2jx}\left(\frac{[j_B]_q}{q - q^{-1}} (q^{-4j_A - 3j_B + 3} B - q^{1 - j_B}) \lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B - 1}, \gamma_A^{j_A}] \right. \\ &\quad \left. + \frac{[j_A]_q}{q - q^{-1}} (q^{j_A + 2j_B + 1} A - q^{-j_A - 2j_B + 1}) \lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A - 1}] \right) \end{aligned} \tag{4.39}$$

it follows that

$$\phi_{n'}^{(2)}(\hat{E}\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A}])$$
$$= (E \otimes \mathbf{1} + K^2 \otimes E)\phi_{n'}^{(2)}(\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A}]) \,. \tag{4.40}$$

Where we identify the coproduct $\Delta(E) = E \otimes \mathbf{1} + K^2 \otimes E$. To compute the action of $\hat{F}$, the added loop has to be homotopically deformed and split into a composition of loops γ_2, γ_B, and γ_A. By a deformation procedure it is shown that

$$\begin{aligned}
&\hat{F}\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A}] \\
&\quad = q^{2j_A + 2j_B + 2}\lambda(e_{-n'p+np'})[\gamma_2^{j_2+1}, \gamma_B^{j_B}, \gamma_A^{j_A}] \\
&\qquad + (q^{2j_A}B^{-1}A^{-1} - q^{2j_A + 2j_B - 2}B^{-1})\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B+1}, \gamma_A^{j_A}] \\
&\qquad + (A^{-1} - q^{2j_A}B^{-1}A^{-1})\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A+1}] \,.
\end{aligned} \tag{4.41}$$

As a consequence

$$\phi_{n'}^{(2)}(\hat{F}\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A}])$$
$$= (F \otimes K^{-2} + \mathbf{1} \otimes F)\phi_{n'}^{(2)}(\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A}]) \,, \tag{4.42}$$

where $\Delta(F) = F \otimes K^{-2} + \mathbf{1} \otimes F$. Finally

$$\hat{K}^2\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A}]$$
$$= q^{n_2 - 1 - 2j_x}q^{-2j_A - 2j_B - 2}\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A}] \,. \tag{4.43}$$

So that

$$\phi_{n'}^{(2)}(\hat{K}^2\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A}])$$
$$= K^2 \otimes K^2\phi_{n'}^{(2)}(\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A}]) \tag{4.44}$$

proves the assertion since $\Delta(K^2) = K^2 \otimes K^2$. The "right" action of $\hat{K}^2$ is a consequence of the charge neutrality condition. □

Thus we have proved that the topological action of $U_q(sl_2)$ on the torus with many punctures algebraically reproduces the coproduct.

5. On the Adjoint Representation

Let $q = \exp(i\pi p'/p)$, where p and p' are relative prime integers with $p \geqq 2$. $U_q(sl_2(\mathbb{C}))$ is defined as the unital algebra over $\mathbb{C}$ generated by E, F and $K^{\pm 2}$ subject to the relations

$$K^{\pm 2}K^{\mp 2} = 1, \quad K^2E = q^2EK^2 \,,$$
$$K^2F = q^{-2}FK^2, \quad [E, F] = K^2 - K^{-2} \,. \tag{5.1}$$

In the following, we will consider the quotient $U_q^0(sl_2(\mathbb{C})) = U_q(sl_2(\mathbb{C}))/I_p$ obtained by dividing by the ideal I_p generated by the central elements $(K^2)^{2p} - 1$, E^p and F^p. From $U_q(sl_2(\mathbb{C}))$ it inherits the coproduct

$$\begin{aligned} \Delta(K^{\pm 2}) &= K^{\pm 2} \otimes K^{\pm 2} , \\ \Delta(E) &= E \otimes 1 + K^2 \otimes E , \\ \Delta((F) &= F \otimes K^{-2} + 1 \otimes F , \end{aligned} \tag{5.2}$$

and the antipode

$$\begin{aligned} S(E) &= - K^{-2} E , \\ S(F) &= - F K^2 , \\ S(K^{\pm 2}) &= K^{\mp 2} . \end{aligned} \tag{5.3}$$

Theorem 5.1. *The monomials* $F^j E^l K^{2n}$, $0 \leqq j, l \leqq p-1, 0 \leqq n \leqq 2p-1$, *form a* PBW-*basis of* $U_q^0(sl_2(\mathbb{C}))$.

Using the notation $\Delta(X) = \sum_i X_i' \otimes X_i''$, the adjoint representation of $U_q(sl_2(\mathbb{C}))$ acting on $U_q^0(sl_2(\mathbb{C}))$ is given by

$$\mathrm{ad}_X(Y) = \sum_i X_i' Y S(X_i'') . \tag{5.4}$$

In particular, the action of the generators E, F and $K^{\pm 2}$ is

$$\begin{aligned} \mathrm{ad}_E(X) &= EX - K^2 X K^{-2} E , \\ \mathrm{ad}_F(X) &= FXK^2 - XFK^2 , \\ \mathrm{ad}_{K^{\pm 2}}(X) &= K^{\pm 2} X K^{\mp 2} . \end{aligned} \tag{5.5}$$

In order to identify the $U_q(sl_2(\mathbb{C}))$-modules $A^{\varepsilon, n'}$, $0 \leqq n' \leqq p' - 1$, with $U_q^0(sl_2(\mathbb{C}))$ we introduce for each n' the new basis $F^j E^l T_{n + \frac{n'p}{p'}, \omega}$, $0 \leqq j, l, n \leqq p-1, \omega = \pm 1$. In addition to (4.26), the idempotents $T_{n,\omega}$ have the properties

$$E T_{n,\omega} = T_{n+2,\omega} E, \quad F T_{n,\omega} = T_{n-2,\omega} F , \tag{5.6}$$

which, together with

$$\begin{aligned} [F^n, E] &= F^{n-1} [n]_q \frac{q^{n-1} K^{-2} - q^{-n+1} K^2}{q - q^{-1}} , \\ [E^n, F] &= E^{n-1} [n]_q \frac{q^{n-1} K^2 - q^{-n+1} K^{-2}}{q - q^{-1}} , \end{aligned} \tag{5.7}$$

and (5.1), allow us to compute explicitly the action of the generators. The result is

Lemma 5.2. *Let* $0 \leqq n' \leqq p' - 1$. *The action of* E, F *and* K^2 *on the basis* $F^j E^l T_{n+\frac{n'p}{p'},\omega}$ $0 \leqq j, l, n \leqq p-1, \omega = \pm 1$, *is given explicitly by*

$$\mathrm{ad}_E(F^j E^l T_{n+\frac{n'p}{p'},\omega}) = F^j E^{l+1}\left(T_{n+\frac{n'p}{p'},\omega} - q^{2(l-j)} T_{n+\frac{n'p}{p'}-2,\omega}\right)$$

$$- \omega \frac{[j]_q \left[j - n - 2l - \frac{n'p}{p'} - 1\right]_q}{q - q^{-1}} F^{j-1} E^l T_{n+\frac{n'p}{p'},\omega} ,$$

$$\mathrm{ad}_F(F^j E^l T_{n+\frac{n'p}{p'},\omega}) = F^{j+1} E^l (\omega q^{n+\frac{n'p}{p'}} T_{n+\frac{n'p}{p'},\omega} - \omega q^{n+\frac{n'p}{p'}+2} T_{n+\frac{n'p}{p'}+2,\omega})$$

$$- q^{n+\frac{n'p}{p'}+2} \frac{[l]_q \left[l + n + \frac{n'p}{p'} + 1\right]_q}{q - q^{-1}} F^j E^{l-1} T_{n+\frac{n'p}{p'}+2,\omega} ,$$

$$\mathrm{ad}_{K^2}(F^j E^l T_{n+\frac{n'p}{p'},\omega}) = q^{2(l-j)} F^j E^l T_{n+\frac{n'p}{p'},\omega} . \tag{5.8}$$

6. Conjecture on Locally Finite Homology

We conclude by stating a conjecture on the locally finite middle-dimensional homology groups with coefficients in the local systems L_r. Let as before q be a root of unity, and p be the smallest positive integer such that $q^{2p} = 1$. Define quantum binomial coefficients as

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \lim_{\varepsilon \downarrow 0} \frac{[n]_{q_\varepsilon}!}{[n-m]_{q_\varepsilon}! [m]_{q_\varepsilon}!}, \quad q_\varepsilon = q(1+\varepsilon) . \tag{6.1}$$

Let A be the associative $\mathbb{Z}$-graded algebra with unit, generated by K^2, K^{-2} of degree zero, E of degree -1, and F_n of degree n, $n = 0, 1, \ldots$, with relations

$$K^2 E K^{-2} = q^2 E, \quad K^2 F_n K^{-2} = q^{-2n} F_n ,$$

$$E F_n - F_n E = F_{n-1}(q^{-n+1} K^2 - q^{n-1} K^{-2}), \quad n \geqq 1 ,$$

$$F_n F_m = \begin{bmatrix} n+m \\ m \end{bmatrix}_q F_{n+m} ,$$

$$K^{\pm 2} K^{\mp 2} = K^{\mp 2} K^{\pm 2} = F_0 = 1 . \tag{6.2}$$

This is "half" of Lusztig's construction of the quantum group at root of unity. It is obtained by formally setting $F_n = F^n [1]_q / [n]_q!$, for q generic, and taking the limit when q goes to a root of unity.

There is a homomorphism ι: $U_q(sl_2) \to A$ given by $E \mapsto E$, $K^{\pm 2} \mapsto K^{\pm 2}$, and $F \mapsto F_1$. Thus $U_q(sl_2)$ acts on A via the adjoint action $U_q(sl_2) \times A \to A$, $(x, a) \mapsto \sum_j \iota(x'_j) a_\iota(S(x''_j))$, where $\Delta(x) = \sum x'_j \otimes x''_j$. The $U_q(sl_2)$ module A is $\mathbb{Z}$ graded for the grading of $U_q(sl_2)$ defined by $\deg(E) = -1$, $\deg(F) = 1$, $\deg(K^{\pm 2}) = 0$.

Let A_N be the quotient of A by the ideal generated by the central element E^{N_p}, $N = 1, 2, \ldots$. The algebra $U_q(sl_2)$ actrs on A_N since E^{N_p} commutes with the action on A. Multiplication by E^p defines embeddings of $U_q(sl_2)$ modules

$$\cdots \hookrightarrow A_N \hookrightarrow A_{N+1} \hookrightarrow \cdots . \tag{6.3}$$

These maps are of degree zero for the shifted degree on A_N $\overline{\deg}(x) = \deg(x) + Np$. Define the graded $U_q(sl_2)$ module A_∞ to be the direct limit of the modules A_N with the shifted degree. A basis of A_∞ is given by the classes of

$$F_j E^{Np-l-1} T_{n-1,\omega} \in A_N \,, \quad j, l = 0, 1, \ldots, \quad n = 0, 1, \ldots, p-1,$$
$$\omega = \pm 1 \,. \tag{6.4}$$

In this expression N is any number such that $Np - l - 1 \geqq 0$. The degree of (6.4) is $j + l - 1$. Denote by A^d_∞ the subspace of homogeneous elements of degree d.

An alternative description of the $U_q(sl_2)$ module A_∞ was essentially suggested to us by D. Kazhdan: Let Z be the subalgebra of the center of $U_q(sl_2)$ generated by E^p. Then $A_\infty = A \otimes_Z \mathbb{C}[t]$, with adjoint action of $U_q(sl_2)$, where E^p acts on A by multiplication and on $\mathbb{C}[t]$ as d/dt. The isomorphism relating the two definitions is $\mathrm{cl}(x \in A_N) \mapsto x \otimes t^{N-1}/(N-1)!$.

For simplicity, we state our conjecture in the case of p' odd.

Conjecture 6.1. Suppose that p' is odd.

(i) The action of $U_q(sl_2)$ on families of loop extends to an action on $H^{\mathrm{lf}}_r(X_r, X^-_r; L_r)$ and there is a degree zero isomorphism of graded $U_q(sl_2)$ modules

$$H^{\mathrm{lf}}_r(X_r, X^-_r; L_r) \simeq \bigoplus_{n'=0}^{p'-1} A^{r-1}_\infty \,.$$

(ii) There is a degree zero isomorphism of graded vector spaces

$$H^{\mathrm{lf}}_r(X_r) \simeq \bigoplus_{n'=0}^{2p'-1} \mathrm{Ker}(E\colon A^{r-1}_\infty \to A^{r-2}_\infty) \,.$$

This conjecture is parallel to the one formulated in [2] for the case of the sphere. To prove it one should understand better locally finite homology. In the remaining of this section we describe these isomorphisms. Let, as in 4.3, $\gamma_A, \gamma_B\colon [0, 1] \to X$ be A and B loops on the one-holed torus X based at a point y_- on the boundary of the hole. Consider the locally compact cells in X_r,

$$C_{l,r} = \{(\gamma_B(t_1), \ldots, \gamma_B(t_l), \gamma_A(t_{l+1}), \ldots, \gamma_A(t_r)) \in X_r |$$
$$0 < t_1 < \cdots < t_l < 1, 0 < t_{l+1} < \cdots < t_r < 1\}^- \,, \tag{6.5}$$

where $^-$ denotes closure in X_r. We orient $C_{l,r}$ using the standard orientation of the parameter space $\mathbb{R}^r \ni t$, and choose as section over it the section taking the value e_μ over a point in one of the cells defined in 3.2, where a trivialization is fixed. The class in $H^{\mathrm{lf}}_r(X_r, X^-_r; L_r)$ represented by $C_{l,r}$ with this section will be denoted by

$C_{\mu,l,r}$. The $U_q(sl_2)$ module $\bigoplus_{r=0}^{\infty} H_r^{\mathrm{lf}}(X_r, X_r^-; L_r)$ is a direct sum of submodules labeled by $n' = 0, \ldots, p'-1$, spanned by $C_{\mu,l,r}$ with $\mu = -n'p + np'$, $n = 0, \ldots, 2p-1$, $l = 0, 1, \ldots$ Each of these submodules is isomorphic to A_∞. The isomorphism is

$$C_{\mu,l,r} \mapsto \eta F_{r-l} E^{pN-l-1} T_{1-\mu/p',1}, \tag{6.6}$$

for some root of unity η depending on the choice of trivialization. The isomorphism in (ii) is obtained through the identification of $\mathrm{Ker}(E)$ with $\mathrm{Ker}(\partial_*)$.

The space of cycles obtained here is bigger than the space of cycles relevant for conformal field theory. The cycles for conformal field theory should be computed using the cohomological methods of [25], but the details remain to be understood.

By construction, there is a projective action of the mapping class group $\mathrm{PSL}(2,\mathbb{Z})$ on relative homology, which commutes with the action of the quantum group. We hope to describe this action elsewhere.

Appendix A. Properties of $\Delta_\mu(W|\tau)$

Define

$$\begin{aligned}\Delta_\mu(W|\tau) &:= \frac{e^{\frac{\pi i}{2p'p}\mu(2W+\tau\mu)}}{\eta(\tau)}\,\theta_3(W+\tau\mu|2p'p\tau) \\ &= \frac{1}{\eta(\tau)}\sum_{-\infty}^{\infty} e^{2\pi i W\left(n+\frac{\mu}{2p'p}\right)+\pi i 2p'p\tau\left(n+\frac{\mu}{2p'p}\right)^2}, \\ W &:= p'\sum_{i=1}^{s}(1-n_i)z_i + 2p'\sum_{i=s+1}^{s+r} z_i\,. \end{aligned}\tag{A.1}$$

A straightforward computation yields

$$\begin{aligned}\Delta_\mu(W+lp'|\tau) &= e^{\frac{\pi i}{p}\mu l}\Delta_\mu(W|\tau)\,, \\ \Delta_\mu(W+lp'\tau|\tau) &= e^{\frac{-\pi i}{p}\left(lW+\frac{1}{2}l^2p'\tau\right)}\Delta_{\mu+lp'}(W|\tau)\,,\end{aligned}\tag{A.2}$$

and

$$\begin{aligned}\Delta_\mu(W|\tau+1) &= e^{\pi i\left(\frac{\mu^2}{2p'p}-\frac{1}{12}\right)}\Delta_\mu(W|\tau)\,, \\ \Delta_\mu\left(W\middle|\frac{-1}{\tau}\right) &= \frac{1}{\sqrt{2p'p}}\,e^{\frac{\pi i}{2p'p}\tau W^2}\sum_{\nu=0}^{2p'p-1} e^{\frac{\pi i}{p'p}\mu\nu}\Delta_{2p'p-\nu}(W\tau|\tau)\,.\end{aligned}\tag{A.3}$$

To verify the last identity, note that

$$\begin{aligned}\sum_{\mu=0}^{2p'p-1} e^{\frac{-\pi i}{p'p}\mu\nu}\Delta_\mu(W|\tau) &= \frac{1}{\eta(\tau)}\theta_3\left(\frac{W-\nu}{2p'p}\middle|\frac{\tau}{2p'p}\right), \\ \Delta_\mu(W|\tau) &= \frac{1}{2p'p}\sum_{\nu=0}^{2p'p-1} e^{\frac{\pi i}{p'p}\mu\nu}\frac{1}{\eta(\tau)}\theta_3\left(\frac{W-\nu}{2p'p}\middle|\frac{\tau}{2p'p}\right),\end{aligned}\tag{A.4}$$

and

$$\eta\left(\frac{-1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)\,,$$

$$\theta_3\left(z\left|\frac{-1}{\tau}\right.\right) = \sqrt{-i\tau}\; e^{\pi i z^2 \tau}\theta_3(z\tau|\tau)\,. \tag{A.5}$$

Appendix B. Properties of $\theta_1(z|\tau)$

Following Jacobi one defines

$$\theta_1(z|\tau) := -\sum_{n=-\infty}^{\infty} e^{2\pi i\left(z+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)+\pi i\tau\left(n+\frac{1}{2}\right)^2}\,. \tag{B.1}$$

In terms of an infinite product

$$\theta_1(z|\tau) = 2e^{\frac{\pi i\tau}{6}}\eta(\tau)\sin\pi z\prod_{n=1}^{\infty}(1-2e^{2\pi i\tau n}\cos 2\pi z + e^{4\pi i\tau n})\,. \tag{B.2}$$

It satisfies the following identities

$$\begin{aligned}
\theta_1(z+1|\tau) &= -\theta_1(z|\tau)\,,\\
\theta_1(z+\tau|\tau) &= -e^{2\pi i z - \pi i\tau}\theta_1(z|\tau)\,,\\
\theta_1(-z|\tau) &= -\theta_1(z|\tau)\,,\\
\theta_1(z|\tau+1) &= \sqrt{i}\,\theta_1(z|\tau)\,,\\
\theta_1\left(z\left|\frac{-1}{\tau}\right.\right) &= \sqrt{i\tau}\; e^{\pi i z^2\tau}\theta_1(z\tau|\tau)\,.
\end{aligned} \tag{B.3}$$

It has simple zeros on the lattice $\mathbb{Z}\oplus\mathbb{Z}\tau$ and no others. Consider the fractional power $\theta_1(z|\tau)^\alpha$, $\alpha\in\mathbb{Q}\backslash\mathbb{Z}$, a multi-valued function on $\mathbb{C}\backslash\mathbb{Z}\oplus\mathbb{Z}\tau$. Upon analytic continuation along straight paths from z to $z+1$ and $z+\tau$ respectively, it has the property that

$$\begin{aligned}
\theta_1(z+1|\tau)^\alpha &= e^{2\pi i\alpha\varphi_A(z)}\theta_1(z|\tau)^\alpha\,,\\
\theta_1(z+\tau|\tau)^\alpha &= e^{2\pi i\alpha\varphi_B(z)-2\pi i\alpha z-\pi i\alpha\tau}\theta_1(z|\tau)^\alpha\,,
\end{aligned} \tag{B.4}$$

with

$$\begin{aligned}
\varphi_A(z) &= -n-\tfrac{1}{2}\,,\\
\varphi_B(z) &= m+\tfrac{1}{2}\,,\\
z\in\{x+y\tau\in\mathbb{C}\,|\,m<x&<m+1, n<y<n+1\}\,,
\end{aligned} \tag{B.5}$$

as follows from an explicit calculation. Note that $\varphi_\#(z)$ is constant on every translated fundamental domain.

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