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Difference Almost-Periodic Schrödinger Operators: Corollaries of Localization

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Abstract. We study the technique used in proving the exponential localization for one-dimensional difference Schrödinger operators with quasi-periodic potential. In this way we get some corollaries concerning the spectrum structure near the boundaries and the existence of bounded, non-exponentially decaying solutions of the equation on eigenvalues.

1. Introduction

In this paper we study the family of Schrödinger operators, acting in $\ell^2(\mathbb{Z}^1)$ as follows:

$$(\mathbf{H}_\varepsilon(\alpha)\psi)(n)=\varepsilon\cdot(\psi(n-1)+\psi(n+1))+\mathbf{V}(\alpha+n\cdot\omega)\cdot\psi(n), \tag{1.1}$$

with $\alpha\in\mathbf{S}^1$, $\omega\in\mathbf{R}^1$, $\mathbf{V}\in\mathbf{C}^2(\mathbf{S}^1)$, and ε being sufficiently small. One can consider $\mathbf{H}_\varepsilon(\alpha)$ as a metrically transitive operator in the sense of [5]. Indeed, $\mathbf{H}_\varepsilon(\mathbf{T}\alpha)=\mathbf{U}^{-1}\mathbf{H}_\varepsilon(\alpha)\mathbf{U}$, where $\mathbf{T}\alpha=(\alpha+\omega)\,(\mathrm{mod}\,1)$ and $\mathbf{U}$ is a unitary shift operator: $(\mathbf{U}f)(n)=f(n-1)$. For $\mathbf{V}(\alpha)=\cos(2\pi\alpha)$ we get the Almost-Mathieu operator as an important particular case. We shall use the results for (1.1) obtained by Sinai [1] and Fröhlich, Spencer, and Wittwer [3, 4]:

Main Theorem (Sinai, Fröhlich, Spencer, and Wittwer). *Let $\mathbf{V}\in\mathbf{C}^2(\mathbf{S}^1)$ have exactly two critical points, both being non-degenerate. Let $\omega\in[0;1]$ be a Diophantine number, i.e. a number, satisfying the condition $|\omega-p/q|\geqq\mathrm{const}\cdot q^{-\delta-2}$ for some $\delta>0$. Then there exists a positive number $\varepsilon_0=\varepsilon_0(\mathbf{V},\delta)$ such that for any ε, $|\varepsilon|<\varepsilon_0$ and a.e. $\alpha\in\mathbf{S}^1$ the operator $\mathbf{H}_\varepsilon(\alpha)$ has purely point spectrum. All its eigenfunctions decay exponentially. The support of the density of states for $\mathbf{H}_\varepsilon(\alpha)$ is a nowhere dense Cantor set of positive Lebesgue measure and the total Lebesgue measure of all spectral gaps for $\mathbf{H}_\varepsilon(\alpha)$ is less than* const $\cdot\,|\varepsilon|$.

We shall derive in this paper several corollaries from the Main Theorem:

Corollary 1. *For all $\varepsilon:|\varepsilon|<\varepsilon_0$ there exists a countable set $\mathscr{A}(\varepsilon)\subset S^1$ such that for every $\alpha\in\mathscr{A}$: $\lambda_{\max}=\sup\{\lambda:\ \lambda\in\mathrm{Sp}\,\mathbf{H}_\varepsilon(\alpha)\}$ is an eigenvalue of $\mathbf{H}_\varepsilon(\alpha)$.*

Remarks. a) For all α, $\mathrm{Sp}\,\mathbf{H}_\varepsilon(\alpha)$ does not depend on α (see [5]).
b) $\mathscr{A}$ is an exceptional set in the following sense: for a.e. $\alpha \in S^1$; $\lambda_{\max}$ is not an eigenvalue of $\mathbf{H}_\varepsilon(\alpha)$ [5].
c) The same result holds for $\lambda_{\min} = \inf\{\lambda : \lambda \in \mathrm{Sp}\,\mathbf{H}_\varepsilon(\alpha)\}$.
d) The countable set of exceptional phases $\mathscr{A}$ is a trajectory of $\mathbf{T}$ on $\mathbf{S}^1$.
e) For $\alpha \in \mathscr{A}$ essential support of eigenfunctions (in the sense of [1, 2]) corresponding to $\lambda_{\min}$ and $\lambda_{\max}$ is a one-point subset of $\mathbb{Z}^1$.

Corollary 2. *Let* $|\omega - p/q| \geqq \mathrm{const} \cdot q^{-2-\delta}$ *for every* $p, q \in \mathbb{Z}^1$ *and some* $\delta > 0$. *Then for every* $\varepsilon : |\varepsilon| < \varepsilon_0$ *one can choose such* $\alpha(\varepsilon) \in \mathbf{S}^1$; $\lambda(\varepsilon) \in \mathrm{Sp}\,\mathbf{H}_\varepsilon(\alpha)$ *that there exists a bounded solution of the equation* $\mathbf{H}_\varepsilon(\alpha)\psi = \lambda(\varepsilon) \cdot \psi$ *for which* $\limsup_{n \to \infty} |\psi(n)| > 0$.

Remarks. a) There exists a continual set of pairs (λ, α) for which the statement of Corollary 2 holds.
b) Corollary 2 is useful in connection with the following theorem, proven by Riedel [6]:

Theorem (Riedel). *Assume that the condition* L' *holds.*
L': *Every bounded solution* ψ *of the almost Mathieu equation* $\varepsilon \cdot (\psi(n-1) + \psi(n+1)) + \cos(2\pi(\alpha + \omega \cdot n)) \cdot \psi(n) = \lambda\psi(n)$; *decays exponentially as* $|n| \to +\infty$. *Then* $\mathrm{Sp}\,\mathbf{H}_\varepsilon(\alpha)$ *is not a Cantor set.*

Corollary 2 shows that generically the condition L' does not take place.

Corollary 3. *Let be* $|\alpha - p/q| \geqq \mathrm{const} \cdot q^{-2} \cdot (\ln q)^{-\beta}$; $\beta > 1$; $\forall p; q \in \mathbb{Z}$. *Then for every* $\sigma > 0$ *one can find such* $\alpha_\sigma \in S^1$; $\lambda_\sigma \in \mathrm{Sp}\,\mathbf{H}_\varepsilon(\alpha)$ *that*
a) the solution $\psi_\sigma(n)$ *of the equation* $H_\varepsilon(\alpha)\psi_\sigma = \lambda\psi_\sigma$ *has a polynomial rate of decay at infinity.*
b) $\limsup_{|n| \to +\infty} |\psi_\sigma(n) \cdot n^\sigma| = 0$.

The plan of the paper is the following. In Sect. 2 we discuss the main steps of the inductive procedure, suggested in [2]. Using the technique of [2] we prove in Sects. 3, 4, and 5 all Corollaries 1, 2, and 3.

2. The Main Steps of the Proof of the Main Theorem

a) Assume that for all $\alpha \in \mathbf{S}^1$ we constructed a family $\psi(\alpha)$ of eigenfunctions (EF's) having the eigenvalues (EV's) $\lambda(\alpha)$. Then all functions $\psi_n(\alpha) = \mathbf{U}^n \psi(\alpha + n \cdot \omega)$ $n = -\infty, \ldots, +\infty$ are EF's for the individual operator with EV's $\lambda(\alpha + n \cdot \omega)$. The main idea of [1, 2] was that in order to construct all EF's $\psi(\alpha)$ for a fixed $\alpha \in \mathbf{S}^1$ it is sufficient to construct for all $\alpha \in \mathbf{S}^1$ only those EF's for which zero point is a left boundary of an essential support of $\psi(\alpha)$. (In fact, ES is a finite subset of $\mathbb{Z}^1$, where EF takes values of order of 1; see [1, 2] for precise definitions.)

Remark. Actually, eigenfunctions with the above mentioned condition on ES will be constructed only for α belonging to some Cantor set of positive Lebesgue measure, which will be sufficient for our purposes.

Denote the ES of $\psi(\alpha)$ by $\mathscr{X}(\psi(\alpha))$ and

$$\mathrm{pos}(\psi(\alpha)) = \min\{k : k \in \mathscr{X}(\psi(\alpha))\}.$$

b) Exponential localization will be proven by induction, and on each step $s \geqq s_0$ of the inductive process we shall construct approximate eigenfunctions (AEF's) $\phi_\ell^s(\alpha)$ and approximate eigenvalues (AEV's) $\Lambda_\ell^s(\alpha)$; $1 \leqq \ell \leqq n_s$, $n_s \leqq \mathrm{const} \cdot s \cdot \ln s$ such that:

1.
$$\mathrm{pos}(\phi_\ell^s(\alpha)) = 0\,.$$

2.
$$(\mathbf{H}_\varepsilon(\alpha) - \Lambda_\ell^s(\alpha))\phi_\ell^s(\alpha) = \mathbf{F}_\ell^s(\alpha)\,, \tag{2.1}$$
$$\mathbf{F}_\ell^s(\alpha) = \mathbf{\Gamma}_\ell^s(\alpha) + h_\ell^s(\alpha)\,,$$
$$|h_\ell^s(n,\alpha)| \leqq \exp(-(3-\beta)s)\,; \qquad 0<\beta<1\,;$$

$\mathbf{\Gamma}_\ell^s(n,\alpha)$ is different from zero only at the points n, where

$$\mathrm{dist}(n, \mathscr{X}(\phi_\ell^s(\alpha))) - [2s/\ln(\varepsilon^{-1})] = 0; 1\,.$$

3.
$$|\phi_\ell^s(n,\alpha)| \leqq (a\varepsilon)^{\mathrm{dist}(n;\, \mathscr{X}(\phi_l^s(\alpha)))} \tag{2.2}$$

provided that

$$\mathrm{dist}(n, \mathscr{X}(\phi_\ell^s(\alpha))) \leqq [2s/\ln(\varepsilon^{-1})]\,.$$

4. $\bigcup_{n=-\infty}^{+\infty} \mathbf{U}^n(\phi^s(\alpha+n\omega))$ is a complete set of functions in $\ell^2(\mathbb{Z}^1)$, where we denoted $\phi^s(\alpha) = \bigcup_\ell \phi_\ell^s(\alpha)$.

We also define the many-valued function

$$\Lambda^s(\alpha)\colon\ \alpha \to \bigcup_\ell \Lambda_\ell^s(\alpha)\,.$$

5.
$$\mathrm{diam}(\mathscr{X}(\phi_\ell^s(\alpha)) \leqq \mathrm{const} \cdot s/\ln(\varepsilon^{-1})\,. \tag{2.3}$$

Remark. $\mathbf{U}^n(\phi_\ell^s(\alpha+n\omega))$ is also an AEF of $\mathbf{H}_\varepsilon(\alpha)$ in the same sense as in (2.1) with AEV $\Lambda_\ell^s(\alpha+n\omega)$ but $\mathrm{pos}(\mathbf{U}^n(\phi_\ell^s(\alpha+n\omega)) = n$.

c) Because we may assume ε to be sufficiently small, we can consider $\mathbf{H}_\varepsilon(\alpha)$ as a perturbation of the multiplicative operator

$$(\mathbf{H}_0(\alpha)\psi)(n) = \mathbf{V}(\alpha + n\cdot\omega)\cdot\psi(n)\,.$$

The EF's of $\mathbf{H}_0(\alpha)$ concentrated at one-point subsets of the lattice are AEF's of $\mathbf{H}_\varepsilon(\alpha)$ and EV's of $\mathbf{H}_0(\alpha)$, which are $\mathbf{V}(\alpha+m\cdot\omega)$; $m=-\infty, \ldots, +\infty$ are also AEV's of $\mathbf{H}_\varepsilon(\alpha)$. So at the initial step of induction $s_0 = [\ln(\varepsilon^{-1})/2]$:

$$\phi_1^{s_0}(n;\alpha) = \delta_{n;\,0}\,; \qquad \Lambda_1^{s_0}(\alpha) = \mathbf{V}(\alpha)$$

and

$$\begin{aligned}\mathbf{F}_1^{s_0}(\alpha) &= (\mathbf{H}_\varepsilon(\alpha) - \Lambda_1^{s_0}(\alpha))\phi_1^{s_0}(\alpha)\\ &= \varepsilon(\delta_{n,\,-1} + \delta_{n;\,1}) = o(\varepsilon)\,.\end{aligned} \tag{2.4}$$

The set of functions $\bigcup_{n=-\infty}^{+\infty} \{\mathbf{U}^n\phi_1^{s_0}(\alpha+n\cdot\omega)\}$ is a basis in $\ell^2(\mathbb{Z}^1)$.

d) Let us discuss the construction of the first non-trivial approximations $\phi^{s_0+1}(\alpha)$ and $\Lambda^{s_0+1}(\alpha)$. We shall use the perturbation formulas of the first order:

$$\begin{cases} \phi_1^{s_0+1}(\alpha)=\phi_1^{s_0}(\alpha)+\sum\limits_{t\neq 0}\dfrac{(\mathbf{F}_1^{s_0}(\alpha);\,\mathbf{U}^t\phi_1^{s_0}(\alpha+t\cdot\omega))}{\Lambda_1^{s_0}(\alpha)-\Lambda_1^{s_0}(\alpha+t\cdot\omega)} \\ \qquad\qquad \times\mathbf{U}^t\phi_1^{s_0}(\alpha+t\cdot\omega)\,; \\ \Lambda_1^{s_0+1}(\alpha)=\Lambda_1^{s_0}(\alpha)+(\mathbf{F}_1^{s_0}(\alpha),\phi_1^{s_0}(\alpha))\,; \\ \mathscr{X}(\phi_1^{s_0+1}(\alpha))=\mathscr{X}(\phi_1^{s_0})\,. \end{cases} \tag{2.5}$$

Remark. Actually, the sum in (2.5) contains only the finite number of terms and in the case $s=s_0+1$, in fact only two, corresponding to $t=\pm 1$. But the formulas (2.5) are applicable only under certain "non-resonant" conditions. More precisely, (2.5) makes no sense if the denominator is zero, while the corresponding numerator is nonzero, and the precision of that approximation is insufficient if the denominator is small enough. This is the reason why the resonant zone [the neighbourhoods of the points, where $\Lambda_1^{s_0}(\alpha)=\Lambda_1^{s_0}(\alpha+\omega)$; $\Lambda_1^{s_0}(\alpha)=\Lambda_1^{s_0}(\alpha-\omega)$] appear. The width of the resonant zone on the α-axis at the s^{th} step of induction has an order of e^{-s} (see [2]). The former and the latter resonant neighbourhoods play essentially different roles in inductive constructions. Recall that we should construct only those AEF $\phi_\ell^s(\alpha)$ for which $\mathrm{pos}(\phi_\ell^s(\alpha))=0$. As was shown in [1, 2] in resonant zones there appear two AEV's which are close to each other and close to $\Lambda_1^{s_0}(\alpha)$; $\Lambda_1^{s_0}(\alpha\pm\omega)$, but the corresponding AEF are linear combinations of $\phi_1^{s_0}(\alpha)$ and $\mathbf{U}^{\pm 1}\phi_1^{s_0}(\alpha\pm\omega)$ up to terms of higher order of smallness. The ES of new AEF is defined as a union of the ES's for $\phi_1^{s_0}(\alpha)$ and $\mathbf{U}^{\pm 1}\phi_1^{s_0}(\alpha\pm\omega)$. Therefore, in the case of $\mathbf{U}\phi_1^{s_0}(\alpha+\omega)$ the new AEF satisfies the condition $\mathrm{pos}(\phi_\ell^{s_0+1}(\alpha))=0$, while in the other case we have an AEF with $\mathrm{pos}=-1$. So we should exclude the RZ of the latter type from the domain of definition $\phi^{s_0+1}(\alpha)$, while in the former case there appears an additional AEF in the resonant zone (RZ). We shall follow this strategy at all inductive steps $s\geqq s_0$, so in the limit $s\to+\infty$ the domain of definition of any $\phi_\ell=\lim\limits_{s\to+\infty}\phi_\ell^s$ will be a nowhere dense Cantor set.

Let us write down the perturbation formulas, which are used in the RZ:

$$\begin{aligned}\varphi_\pm^{s_0+1}(\alpha)=\;&\mathbf{A}\pm(\alpha)\cdot\phi_1^{s_0}(\alpha)+\mathbf{B}_\pm(\alpha)\cdot\mathbf{U}\phi_1^{s_0}(\alpha+\omega)+\mathbf{A}_\pm(\alpha)\\ &\times\sum_{t\neq 0;\,1}\left(\frac{(\mathbf{F}_1^{s_0}(\alpha);\,\mathbf{U}^t\phi_1^{s_0}(\alpha+t\omega))}{\Lambda_1^{s_0}(\alpha)-\Lambda_1^{s_0}(\alpha+t\omega)}\right)\\ &\times\mathbf{U}^t\phi_1^{s_0}(\alpha+t\omega)+\mathbf{B}_\pm(\alpha)\\ &\times\sum_{t\neq 0;\,1}\left(\frac{(\mathbf{U}\mathbf{F}_1^{s_0}(\alpha+\omega);\,\mathbf{U}^t\phi_1^{s_0}(\alpha+t\omega))}{\Lambda_1^{s_0}(\alpha+\omega)-\Lambda_1^{s_0}(\alpha+t\omega)}\right)\\ &\times\mathbf{U}^t\phi_1^{s_0}(\alpha+t\omega)\,.\end{aligned} \tag{2.6}$$

$$\begin{aligned}\Lambda_\pm^{s_0+1}(\alpha)=\;&\tfrac{1}{2}\cdot[\Lambda_1^{s_0}(\alpha)+(\mathbf{F}_1^{s_0}(\alpha);\,\phi_1^{s_0}(\alpha))\\ &+\Lambda_1^{s_0}(\alpha+\omega)+(\mathbf{U}\mathbf{F}_1^{s_0}(\alpha+\omega);\,\mathbf{U}\phi_1^{s_0}(\alpha+\omega))]\\ &\pm\tfrac{1}{2}\cdot[(\Lambda_1^{s_0}(\alpha)+(\mathbf{F}_1^{s_0}(\alpha);\,\phi_1^{s_0}(\alpha)))\\ &-(\Lambda_1^{s_0}(\alpha+\omega)+(\mathbf{U}\mathbf{F}_1^{s_0}(\alpha+\omega);\,\mathbf{U}\phi_1^{s_0}(\alpha+\omega)))\\ &+4(\mathbf{F}_1^{s_0}(\alpha);\,\mathbf{U}\phi_1^{s_0}(\alpha+\omega))\cdot(\mathbf{U}\mathbf{F}_1^{s_0}(\alpha+\omega);\,\phi_1^{s_0}(\alpha))]^{1/2}\,,\end{aligned} \tag{2.7}$$

$$\begin{pmatrix}\mathbf{A}_+; & \mathbf{A}_-\\ \mathbf{B}_+; & \mathbf{B}_-\end{pmatrix}\in O(2)\,.$$

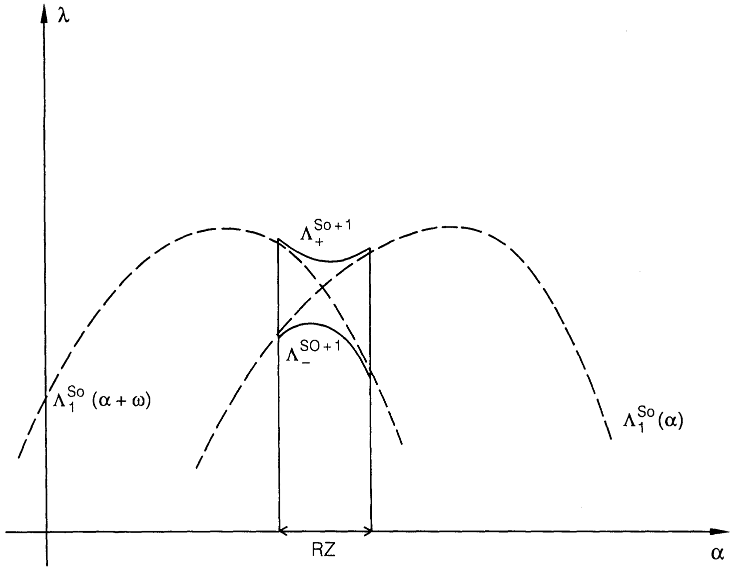

Fig. 1

(Figure 1 presents the graph of $\Lambda_{\pm}^{s_0+1}$ in RZ.)

e) The exponential decay of AEF's (and EF in the limit) is derived from the representation (see [1, 2]):

$$\varphi_i^{s+1}(n-1;\alpha)=\mathbf{M}^s(n;\lambda_i^s(\alpha);\alpha)\cdot\varphi_i^s(n;\alpha)+\underline{o}(\ell^{-(3-\beta)s}),$$

$$\operatorname{dist}(n,\mathscr{X}(\varphi_i^{s+1}(\alpha))\sim[2(s+1)/\ln(\varepsilon^{-1})],$$

where $\varphi_j^{s+1};\ \varphi_i^{s+1}\in\bigcup_n \mathbf{U}^n\phi^s(\alpha+n\omega)$,

$$\mathbf{M}^s(n;\lambda;\alpha)=\varepsilon\cdot\mathbf{G}^s(n-1;n-1;\lambda;\alpha)/(1+\varepsilon\cdot\mathbf{G}^s(n;n-1;\lambda;\alpha)$$

and

$$\mathbf{G}^s(x;y;\lambda;\alpha)=\sum_j\frac{\varphi_j^s(x;\alpha)\cdot\varphi_j^s(y;\alpha)}{\lambda-\lambda_j^s(\alpha)}\tag{2.8}$$

is an approximate Green function. Factors $\mathbf{M}^s(n;\lambda_i^s;\alpha)$ take values of order of ε at "most" points n. Besides that, for points n which are in the fixed neighbourhood of the ES we have:

$$|\varphi_i^{s+1}(n;\alpha)-\varphi_i^{s+1}(n;\alpha)|\leqq e^{-(3-\beta)s}\cdot\text{const}.\tag{2.9}$$

Remark. The formulas (2.8), (2.9) are obtained from the perturbation formulas (2.5)–(2.7) and the inductive assumptions b).

f) The Lebesgue measure $\operatorname{Mes}(\mathbf{A}_k)$ of the set of points $\alpha\in\mathbf{S}^1$ which take part in more than $k-1$ resonances for all "history" of the inductive procedure is estimated as:

$$\operatorname{Mes}(\mathbf{A}_k)\leqq\sum_{s=s_0}^{+\infty}\text{const}\cdot\exp\left(-\frac{1}{\delta}\exp\left(\frac{1}{\delta}\exp\left(\frac{1}{\delta}\dots\exp\left(\frac{s}{\delta}\right)\dots\right)\dots\right)\dots\right).$$

$$\longleftarrow k-1\longrightarrow$$

Therefore, $\sum_{k=1}^{+\infty} \mathrm{Mes}(\mathbf{A}_k) < \infty$, so according to the Borel-Cantelli lemma a.e. $\alpha \in \mathbf{S}^1$ takes part only in a finite number of resonances.

g) Define

$$\phi(\alpha) = \lim_{s \to +\infty} \phi^s(\alpha),$$

$$\Lambda(\alpha) = \lim_{s \to +\infty} \Lambda^s(\alpha).$$

For a.e. $\alpha \in \mathbf{S}^1$ the set $\bigcup_{n=-\infty}^{+\infty} \mathbf{U}^n \phi(\alpha + n\omega)$ is the basis of exponentially localized eigenfunctions of $\mathbf{H}_\varepsilon(\alpha)$ and $\bigcup_{n=-\infty}^{+\infty} \Lambda(\alpha + n \cdot \omega)$ is the corresponding set of EV's.

3. Proof of Corollary 1

According to the inductive procedure at every step $s \geqq s_0$ we have the real (in the general case multi-valued) function $\Lambda^s(\alpha)*$ with the domain of definition $\mathbf{D}_s \subset S^1$ such that $\mathbf{D}_{s+1} \subset \mathbf{D}_s$; $\forall s \geqq s_0$ and $\mathbf{D}_{s+1}$ obtained from $\mathbf{D}_s$ by deleting the resonant zones, corresponding to the resonance of $\Lambda^s_\ell(\alpha)$ and $\Lambda^s_\ell(\alpha + t \cdot \omega)$ for some t, $[-2s/\ln(\varepsilon^{-1})] \leqq t < 0$. $\Lambda^{s+1}(\alpha)$ is obtained from $\Lambda^s(\alpha)$ with the help of the perturbation formulas (2.5)–(2.7). Passing to the limit, define $\Lambda(\alpha) = \lim_{s \to +\infty} \Lambda^s(\alpha)$ with the domain of definition $\mathbf{D}_\varepsilon = \bigcap_s \mathbf{D}^s$. Then $\mathbf{D}_\varepsilon$ is a Cantor set of positive Lebesgue measure

$$\mathrm{Mes}(S^1 \backslash \mathbf{D}_\varepsilon) = \underline{o}(\varepsilon).$$

The form of the graph of $\Lambda^s(\alpha)$ for $s = s_0 + 2$ is presented in the Fig. 2. In the more general case $s > s_0 + 2$ the graph $\Lambda^s(\alpha)$ differs only by a larger number of resonant zones and resonances.

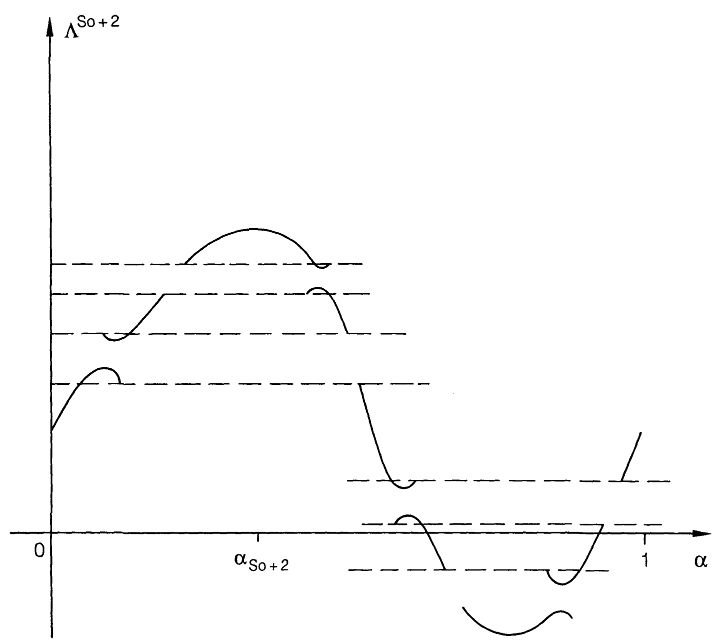

Fig. 2

The proof of Corollary 1 requires the analysis only of the behaviour of the top part of the graph $\Lambda^s(\alpha)$. Namely, let us denote by α_s the point, where $\Lambda^s(\alpha)$ takes its maximal value.

Proposition. *Let all conditions of the Main Theorem be valid and ε be sufficiently small: $|\varepsilon|<\varepsilon_0(\mathbf{V},\delta)$. Then there exists a positive constant $a_1(\omega)$ such that for every $s\geqq s_0$ the neighbourhood $O_s=(\alpha_s-a_1/s^\delta;\ \alpha_s+a_1/s^\delta)$ of the point α_s belongs to a non-resonant zone at the s^{th} step;*

$$\overline{O_{s+1}(\alpha_{s+1})}\subset O_s(\alpha_s)$$

and

$$\inf_{\alpha\in O_s(\alpha_s)}\left|\frac{d^2\Lambda^s(\alpha)}{d\alpha^2}\right|>\exp\cdot\left(-\frac{2}{3}s\right). \tag{3.1}$$

Remark. Assume that the proposition is already proven. Then the set of exceptional phases $\mathscr{A}$ is the trajectory $\{\mathbf{T}^n\alpha_{\text{exc}}\}_{n=-\infty}^{+\infty}$, where α_{exc} is the unique common point of all $O_s(\alpha_s)$; $s\geqq s_0$, i.e. $\alpha_{\text{exc}}=\bigcap_{s\geqq s_0} O_s(\alpha_s)$. □

Choose $a_1(\omega)>0$ such that for any $s\geqq s_0$ and any $\alpha\in O_s(\alpha_s)=(\alpha_s-a_1/s^\delta;\ \alpha_s+a_1/s^\delta)$ and all t, $|t|<[2s/\ln(\varepsilon^{-1})]$, we have

$$\alpha+t\omega\notin O_s(\alpha_s). \tag{3.2}$$

The existence of this positive constant follows from the Diophantine condition on ω.

We shall prove the proposition by induction. At the first step $s=s_0$ one can verify the validity of the proposition, provided ε is sufficiently small. Assume the proposition on the s^{th} step of induction is true. Take α^s_{res} which is the nearest center of the resonant zone on the s^{th} step, to α_s:

$$|\alpha_s-\alpha^s_{\text{res}}|=\min\{|\alpha_s-\alpha|,\ \text{where } \alpha \text{ is such that there exist } \ell>0;|t| \leqq[2s/\ln(\varepsilon^{-1})],\ \Lambda^s_\ell(\alpha)=\Lambda^s_\ell(\alpha+t\omega)\}.$$

It follows from (3.2) that $\alpha^s_{\text{res}}\notin O_s(\alpha_s)$. This is the origin where the resonant neighbourhood of the point α^s_{res} does not intersect $O_{s+1}(\alpha_{s+1})$. Actually, $\Lambda^s(\alpha)$ is a monotone function on the required interval and the following estimates hold:

$$\sup_{\alpha\in\mathbf{R}^s(\alpha^s_{\text{res}})}|\Lambda^s(\alpha)-\Lambda^s(\alpha^s_{\text{res}})|<e^{-s}$$

(this is a consequence of the definition of RZ) and

$$\begin{aligned}&|\Lambda^s(\alpha_s\pm a_1/s^\delta)-\Lambda^s(\alpha_{s+1}\pm a_1/(s+1)^\delta)|\\ &\geqq\inf_{\alpha\in O_s(\alpha_s)}\left|\frac{d^2\Lambda^s}{d\alpha^2}\right|\times\frac{1}{2}|\alpha_s-\alpha_{s+1}\pm(a_1/s^\delta-a_1/(s+1)^\delta)|^2\\ &\geqq\text{const}\cdot\exp\left(-\frac{2}{3}s\right)\cdot s^{-2-2\delta}.\end{aligned}$$

So, we get $\Lambda^{s+1}(\alpha)$ from $\Lambda^s(\alpha)$ in the top neighbourhood with the help of the non-resonant perturbation formulas (2.5):

$$\Lambda^{s+1}_\ell(\alpha)=\Lambda^s_\ell(\alpha)+(f^s_\ell(\alpha);\ \varphi^s_\ell(\alpha)).$$

Below we verify the validity of (3.1) on the $(s+1)^{\text{th}}$ step of inductive procedure.

Lemma.

$$\|(f_i^s(\alpha);\varphi_j^s(\alpha)\|_{\mathbf{C}^2}\leqq \text{const}\cdot\exp(-2s(1-\ln a/\ln(\varepsilon^{-1}))),\qquad \forall i,j.$$

The proof of this lemma is inductive, too. We have:

$$(\varphi_i^s(\alpha);f_i^s(\alpha))=(\varphi_i^s(\alpha);\mathbf{H}_\varepsilon\varphi_i^s-\lambda_i^s\varphi_i^s),$$

$$\varphi_i^s=\varphi_i^{s-1}+\sum_{j\neq i}\frac{(f_i^{s-1},\varphi_j^{s-1})}{\lambda_i^{s-1}-\lambda_j^{s-1}}\cdot\varphi_j^{s-1};$$

$$\lambda_i^s=\lambda_i^{s-1}+(f_i^{s-1};\varphi_i^{s-1}),$$

according to the perturbation formulas. Therefore,

$$\begin{aligned}
(f_i^s;\varphi_i^s)&=(f_i^{s-1};\varphi_i^{s-1})\\
&+\sum_{j\neq i}\frac{(f_i^{s-1};\varphi_j^{s-1})}{\lambda_i^{s-1}-\lambda_j^{s-1}}\cdot\lambda_j^{s-1}\cdot(\varphi_j^{s-1};\varphi_i^{s-1})\\
&+2\cdot\sum_{j\neq i}\frac{(f_i^{s-1};\varphi_j^{s-1})}{\lambda_i^{s-1}-\lambda_j^{s-1}}\cdot(f_j^{s-1};\varphi_i^{s-1})\\
&++\sum_{\substack{j_1\neq i\\ j_2\neq i}}\left(\frac{(f_i^{s-1};\varphi_{j_1}^{s-1})}{\lambda_i^{s-1}-\lambda_{j_1}^{s-1}}\cdot\lambda_{j_1}^{s-1}\cdot\frac{(f_i^{s-1};\varphi_{j_2}^{s-1})}{\lambda_i^{s-1}-\lambda_{j_2}^{s-1}}\right)\\
&\times(\varphi_{j_1}^{s-1};\varphi_{j_2}^{s-1})\\
&+\sum_{\substack{j_1\neq i\\ j_2\neq i}}\frac{(f_i^{s-1};\varphi_{j_1}^{s-1})}{\lambda_i^{s-1}-\lambda_{j_1}^{s-1}}\cdot\frac{(f_i^{s-1};\varphi_{j_2}^{s-1})}{\lambda_i^{s-1}-\lambda_{j_2}^{s-1}}\cdot(f_{j_1}^{s-1};\varphi_{j_2}^{s-1})\\
&-(f_i^{s-1};\varphi_i^{s-1})\cdot(\varphi_i^{s-1};\varphi_i^{s-1})\\
&-(f_i^{s-1};\varphi_i^{s-1})\cdot\sum_{j\neq i}\frac{(f_i^{s-1};\varphi_j^{s-1})}{\lambda_i^{s-1}-\lambda_j^{s-1}}\cdot(\varphi_j^{s-1};\varphi_i^{s-1})\\
&-\sum_{\substack{j_1\neq i\\ j_2\neq i}}\left((f_i^{s-1};\varphi_i^{s-1})\cdot\frac{(f_i^{s-1};\varphi_{j_1}^{s-1})}{\lambda_i^{s-1}-\lambda_{j_1}^{s-1}}\cdot\frac{(f_i^{s-1};\varphi_{j_2}^{s-1})}{\lambda_i^{s-1}-\lambda_{j_2}^{s-1}}\right)\\
&\times(\varphi_{j_1}^{s-1};\varphi_{j_2}^{s-1})\\
&-\sum_{j\neq i}\frac{(f_i^{s-1};\varphi_i^{s-1})}{\lambda_i^{s-1}-\lambda_j^{s-1}}\cdot(f_i^{s-1};\varphi_j^{s-1})\cdot(\varphi_i^{s-1};\varphi_j^{s-1})\\
&-\sum_{j\neq i}\lambda_i^{s-1}\cdot\frac{(f_i^{s-1};\varphi_j^{s-1})}{\lambda_i^{s-1}-\lambda_j^{s-1}}\cdot(\varphi_j^{s-1};\varphi_i^{s-1})\\
&-\sum_{\substack{j_1\neq i\\ j_2\neq i}}\lambda_i^{s-1}\cdot\frac{(f_i^{s-1};\varphi_{j_1}^{s-1})}{\lambda_i^{s-1}-\lambda_{j_1}^{s-1}}\cdot\frac{(f_i^{s-1};\varphi_{j_2}^{s-1})}{\lambda_i^{s-1}-\lambda_{j_2}^{s-1}}\cdot(\varphi_{j_1}^{s-1};\varphi_{j_2}^{s-1})\\
&=(f_i^{s-1};\varphi_i^{s-1})\cdot(1-(\varphi_i^{s-1};\varphi_i^{s-1}))+r_s.
\end{aligned}\tag{3.3}$$

The remainder sum r_s contains no more than const $\cdot\, s^2$ terms, whose C^2-norm is not more than

$$\text{const}\cdot\exp(-4s(1-\ln a/\ln(\varepsilon^{-1}))).\quad\square$$

The formulas (3.1), (3.3) give us the estimate of $|\alpha_{s+1}-\alpha_s|$. Namely,

$$|\alpha_{s+1}-\alpha_s|\cdot \inf_{O_{s+1}(\alpha_{s+1})}\left|\frac{d^2\Lambda^{s+1}}{d\alpha^2}\right| \leqq \left|\frac{d\Lambda^{s+1}}{d\alpha}(\alpha_{s+1})-\frac{d\Lambda^{s+1}}{d\alpha}(\alpha_s)\right| \leqq \mathrm{const}\cdot\exp(-2s(1-\ln a/\ln\varepsilon^{-1}))) \Rightarrow |\alpha_{s+1}-\alpha_s|\leqq \mathrm{const}\cdot e^{-s}.$$

The proposition is proven.

Now we can say more about the non-exceptional phases. Let us recall the main result of the inductive procedure (see [1, 2] and the brief description in Sect. 2). For a.e. $\alpha\in\mathbf{S}^1$ there exists in $\ell^2(\mathbb{Z}^1)$ the orthogonal basis of eigenfunctions of $\mathbf{H}_\varepsilon(\alpha)$ such that for every EF $\phi_\ell(\alpha)$ with EV $\Lambda_\ell(\alpha)$ one can find the finite subset of $\mathbb{Z}^1$, named the essential support of $\phi_\ell(\alpha)$ and denoted by $\mathscr{X}(\phi_\ell(\alpha))$. The smallness of the value $\phi_\ell(n;\alpha)$ is estimated through the distance until n from $\mathscr{X}(\phi_\ell(\alpha))$:

$$|\phi_\ell(n;\alpha)|\leqq(a\cdot\varepsilon)^{\mathrm{dist}(n;\,\mathscr{X}(\phi_\ell(\alpha)))}.$$

Besides, for every $\phi_\ell(\alpha)$ with $\mathrm{pos}(\phi_\ell(\alpha))=m$ one can find a sequence of AEF's $\mathbf{U}^m\phi_\ell^s(\alpha+m\cdot\omega)$; $s\geqq s_0$ with AEV's $\Lambda_\ell^s(\alpha+m\omega)$ such that

$$\lim_{s\to+\infty}\Lambda_\ell^s(\alpha+m\omega)=\Lambda_\ell(\alpha),$$

$$\lim_{s\to+\infty}\mathbf{U}^m\phi_\ell^s(\alpha+m\omega)=\phi_\ell(\alpha),$$

and

$$\mathscr{X}(\mathbf{U}^m\phi_\ell^s(\alpha+m\cdot\omega))=\mathscr{X}(\phi_\ell(\alpha))$$

for all s, except the finite number of ones. Now take α, such that: 1) α satisfies the above-mentioned condition; 2) α does not belong to the countable set of exceptional phases. Then $\lambda_{\max}$ isn't an eigenvalue of $\mathbf{H}_\varepsilon(\alpha)$. Indeed, for such α $\alpha+m\omega\neq\alpha_{\mathrm{exc}}$. So there exists the number s' such that $\alpha+m\omega\notin O^{s'}(\alpha_{s'})$ and $\mathbf{U}^m\phi_\ell^s$ does not take part in any resonance for all $s\geqq s'$. It means that on some step s'', $s''>s'$ of inductive procedure a forbidden zone will appear, such that this forbidden zone lies in $O^{s'}(\alpha_{s'})$ and the length of the appearing gap on the spectral axis separates the values $\Lambda^s(\alpha_{\mathrm{exc}})$ and $\Lambda^s(\alpha+m\omega)$ for every $s\geqq s''$.

4. Proof of Corollary 2

Let us consider the complete set of AEF's

$$\{\varphi_\ell(\alpha)\}_{\ell=1}^{\infty}=\bigcup_{n=-\infty}^{+\infty}\phi^{s_1}(\alpha+n\omega)$$

of the operator $\mathbf{H}_\varepsilon(\alpha)$ on the s_1 th step of the inductive procedure. AEF's $\varphi_i(\alpha)$ and $\varphi_j(\alpha)$ are resonant in the neighbourhood $\mathbf{R}_1=\{\alpha\colon|\lambda_i(\alpha)-\lambda_j(\alpha)|<e^{-s_1}\}$. Let be $\mathrm{pos}(\varphi_i(\alpha))=0$, $\mathrm{pos}(\varphi_j(\alpha))=n_1>0$, then there exists ℓ, $1\leqq\ell\leqq n_s$:

$$\varphi_i(\alpha)=\phi_\ell^{s_1}(\alpha),\qquad \lambda_i(\alpha)=\Lambda_\ell^{s_1}(\alpha),$$

$$\varphi_j(\alpha)=\mathbf{U}^{n_1}\phi_\ell^{s_1}(\alpha+n_1\omega),\qquad \lambda_j(\alpha)=\Lambda_\ell^{s_1}(\alpha+n_1\omega).$$

Two new AEF's $\phi_{\pm}^{s_1+1}(\alpha)$ appear in the resonant zone $\mathbf{R}_1$ on the $(s+1)^{\text{th}}$ step of induction. These AEF's are the linear combinations of φ_i, φ_j to within the small terms of order $\underline{o}(\exp(-2s_1(1-\ln a/\ln\varepsilon^{-1}))$,

$$\phi_{\pm}^{s_1+1}(\alpha)=\mathbf{A}_{\pm}(\alpha)\cdot\varphi_i(\alpha)+\mathbf{B}_{\pm}(\alpha)\cdot\varphi_j(\alpha)+\ \dots \tag{4.1}$$

(the dots will systematically mean the smaller terms). The corresponding AEV's $\lambda_{\pm}(\alpha)$ are:

$$\lambda_{\pm}(\alpha)=\frac{s_{11}(\alpha)+s_{22}(\alpha)}{2}\pm\frac{1}{2}\sqrt{(s_{11}(\alpha)-s_{22}(\alpha))^2+4s_{12}^2(\alpha)}\,. \tag{4.2}$$

Therefore, $(s_{11}-\lambda)(s_{22}-\lambda)-s_{12}^2=0$, where

$$s_{11}(\alpha)=\lambda_i(\alpha)+(f_i(\alpha),\varphi_i(\alpha)),\qquad f_i(\alpha)=(\mathbf{H}_\varepsilon(\alpha)-\lambda_i(\alpha))\varphi_i(\alpha),$$
$$s_{22}(\alpha)=\lambda_j(\alpha)+(f_j(\alpha),\varphi_j(\alpha)),\qquad f_j(\alpha)=(\mathbf{H}_\varepsilon(\alpha)-\lambda_j(\alpha))\varphi_j(\alpha),$$
$$s_{12}(\alpha)=(f_i(\alpha),\varphi_j(\alpha))\,.$$

Below, we follow only one branch of $\lambda(\alpha)$, for definitiness, the plus branch of $\lambda(\alpha)$: $\lambda_+(\alpha)$. Denote by α_{int} the point of intersection of graphs of $s_{11}(\alpha)$ and $s_{22}(\alpha)$, and by α_{cr} the minimal point of $\lambda_+(\alpha)$. In fact, we intend to follow the countable number of resonances, taking place in reduced resonant zones. The reduced resonant zones will be embedded one into another and will be so small that $|\mathbf{A}(\alpha)/\mathbf{B}(\alpha)|$ will take values close to 1. The second resonance will appear in the neighbourhood of the point α, where $\lambda_+(\alpha)=\lambda_+(\alpha+n_2\omega)$. The corresponding inductive step will be denoted by s_2. Then

$$\begin{aligned}\phi_{+}^{s_2+1}&=\mathbf{A}_2(\alpha)\cdot(\mathbf{A}_1(\alpha)\cdot\varphi_i(\alpha)+\mathbf{B}_1(\alpha)\cdot\varphi_j(\alpha)+\ \dots)\\&\quad+\mathbf{B}_2(\alpha)\cdot(\mathbf{A}_1(\alpha+n_2\omega)\cdot\mathbf{U}^{n_2}\varphi_i(\alpha+n_2\omega)+\mathbf{B}_1(\alpha+n_2\omega)\cdot\\&\quad\times\mathbf{U}^{n_2}\varphi_j(\alpha+n_2\omega))+\ \dots\,.\end{aligned}$$

In the case $s_1=s_0,\ n_1=1,\ \varphi_i(\alpha)=\delta_{n,0};\ \varphi_j(\alpha)=\delta_{n,1}$ we have

$$\begin{aligned}\phi_{+}^{s_2+1}&=\mathbf{A}_2(\alpha)\cdot\mathbf{A}_1(\alpha)\cdot\delta_{n,0}+\mathbf{A}_2(\alpha)\cdot\mathbf{B}_1(\alpha)\cdot\delta_{n,1}\\&\quad+\mathbf{B}_2(\alpha)\cdot\mathbf{A}_1(\alpha+n_2\omega)\cdot\delta_{n,n_2}+\mathbf{B}_2(\alpha)\cdot\mathbf{B}_1(\alpha+n_2\omega)\cdot\delta_{n,n_2+1}+\ \dots\,.\end{aligned} \tag{4.3}$$

It isn't difficult to understand that after the k^{th} resonance, $\phi_{+}^{s_k+1}(\alpha)$ is the linear combination of 2^k functions $\delta_{n,0},\delta_{n,1},\delta_{n,n_2},\delta_{n,n_2+1}$, and so on, with the coefficients

$$\prod_{t=1}^{k}\mathbf{A}_t(\alpha),\quad\left(\prod_{t=2}^{k}\mathbf{A}_t(\alpha)\right)\cdot\mathbf{B}_1(\alpha),\quad\left(\prod_{t=3}^{k}\mathbf{A}_t(\alpha)\right)\cdot\mathbf{B}_2(\alpha)\cdot\mathbf{A}_1(\alpha+n_2\omega)\dots$$

up to smaller terms. Suppose that there exist the limits

$$\lim_{k\to+\infty}\left(2^{k/2}\left(\prod_{t=1}^{k}\mathbf{A}_t(\alpha)\right)\right),\quad\lim_{k\to+\infty}\left(2^{k/2}\prod_{t=2}^{k}\mathbf{A}_t(\alpha)\right)\cdot\mathbf{B}_1(\alpha)\right), \tag{4.4}$$

and so on, uniformly with respect to all expressions which are bounded from zero and infinity at the point α, where α is the intersection of all constructed reduced resonant zones. Then, the needed bounded solution of the equation $\mathbf{H}_\varepsilon(\alpha)\psi=\lambda\psi$ is

$$\psi=\lim_{k\to+\infty}2^{k/2}\phi_{+}^{s_k+1}(\alpha),\qquad\lambda=\lim_{k\to+\infty}\lambda_{+}^{s_k+1}(\alpha)\,.$$

Note that $0 < \lim_{n\to\infty} \sup|\psi(n)| < +\infty$. Below we pay attention to the technical details (i.e. the construction of the reduced resonant zones and the estimates on smallness of $|\sqrt{2}\mathbf{A}_+|-1$, $|\sqrt{2}\mathbf{B}_+|-1$, $|\mathbf{A}_+/\mathbf{B}_+|-1$). The coefficients $\mathbf{A}_+(\alpha)$, $\mathbf{B}_+(\alpha)$ are defined as the solutions of the spectral problem:

$$\begin{cases} (s_{11}-\lambda_+)\mathbf{A}_+ + s_{12}\cdot\mathbf{B}_+ = 0, \\ s_{12}\cdot\mathbf{A}_+ + (s_{22}-\lambda_+)\mathbf{B}_+ = 0, \\ \mathbf{A}_+^2 + \mathbf{B}_+^2 = 1 \end{cases} \tag{4.5}$$

(see [2]). Since $(\mathbf{B}_+/\mathbf{A}_+)^2 = (s_{11}-\lambda_+)/(s_{22}-\lambda_+)$ we have to analyze the behavior of $(s_{11}-\lambda_+)/s_{22}-\lambda_+)(\alpha)$ in the resonant zone:

Lemma 4.1. a) *If* $s_{11}-s_{22}=\bar{o}(s_{12})$, *then* $|\mathbf{B}_+/\mathbf{A}_+| = 1 + \underline{o}\left(\frac{s_{11}-s_{22}}{s_{12}}\right)$.
b) *If* $s_{12}=\bar{o}(s_{11}-s_{22})$ *and* $\operatorname{sgn}(s_{11}-s_{22}) = -\operatorname{sgn}(s'_{11})$, *then* $|\mathbf{B}_+/\mathbf{A}_+| = (s_{12}/(s_{11}-s_{22})) + \bar{o}((s_{12}/(s_{11}-s_{22}))^2)$.
c) *If* $s_{11}-s_{22}$ *is of order of* s_{12}, *then* $|\mathbf{B}_+/\mathbf{A}_+|$ *takes values of order the constant, but sufficiently smaller than* 1, *i.e.*

$$s_{11}-s_{22}=k\cdot s_{12} \Rightarrow (s_{11}-\lambda_+)/(s_{22}-\lambda_+) = \frac{\sqrt{k^2+1}-k}{\sqrt{k^2+1}+k}.$$

This lemma follows from (4.2).

Remarks. 1. $\|s_{12}\|_{c^1} = \underline{o}(\exp(-2s(1-\ln a/\ln(\varepsilon^{-1}))))$.
2. Case a) takes place if α belongs to the $\bar{o}(s_{12})$-neighbourhood of the point α_{int}.
Case c) takes place if $|\alpha-\alpha_{\text{int}}|$ is comparable with s_{12}.
Case b) takes place if α doesn't belong to any neighbourhood of α_{int}, whose length is comparable with s_{12}. From one side, the next resonant interval has to be a neighbourhood of α_{cr}, a minimal point of λ_+. But on the other hand, we would like the points from the next resonant interval to satisfy the condition a) of Lemma 4.1. Differentiating (4.2), we get:

$$\lambda'_+ = s'_{22}\cdot\frac{s_{11}-\lambda_+}{s_{11}+s_{22}-2\lambda_+} + s'_{11}\cdot\frac{s_{22}-\lambda_+}{s_{11}+s_{22}-2\lambda_+} + \dots \tag{4.6}$$

Therefore, $(s_{11}-\lambda)/(s_{22}-\lambda)(\alpha_{\text{cr}}) = -s'_{11}/s'_{22}+\dots$, and, correspondingly,

$$(s_{11}-s_{22})(\alpha_{\text{cr}}) = (2\cdot(1-\mu)^2/((1+\mu)^2-(1-\mu)^2))\cdot s_{12}+\bar{o}(s_{12})$$

with $\mu = -(s'_{11}/s'_{22})|_{\alpha=\alpha_{\text{cr}}}$.

We shall introduce the notion of the reduced resonant zone in order to make the ratio s'_{11}/s'_{22} sufficiently close to -1. Consider the first resonance. The resonant zone is, according to the definition

$$\mathbf{R}_1 = \{\alpha\colon |\Lambda^{s_1}_{\ell_1}(\alpha) - \Lambda^{s_1}_{\ell_1}(\alpha+n_1(\omega)| < e^{-s_1}\}.$$

We define the reduced resonant zone as

$$\tilde{\mathbf{R}}_1 = \left\{\alpha\colon |\alpha-\alpha^1_{\text{cr}}| < \frac{1}{s_1}(\mathbf{F}^{s_1}_{\ell_1}(\alpha);\ \mathbf{U}^{n_k}\phi^{s_1}_{\ell_1}(\alpha+n_k\omega))\right\},$$

where $\alpha^1_{\rm cr}$ is uniquely determined by the condition $\frac{d}{d\alpha}(\lambda^{s_1+1}_{+})(\alpha_{\rm cr})=0$ and $(\mathbf{F}^{s_1}_{\ell_1}; U^{n_k}\phi^{s_1}_{\ell_1}(\alpha+n_k\omega))$ play the same role as s_{12} above. Further, we shall take into account only resonances, appearing in $\tilde{\mathbf{R}}_1$. Consider the second resonant step s_2. Then, in the neighbourhood of the intersection of the graphs $\Lambda^{s_2}_{\ell_2}(\alpha)$ and $\Lambda^{s_2}_{\ell_2}(\alpha+n_2\omega)$, $|n_2|<2s(\ln(\varepsilon^{-1}))$, we define

$$\mathbf{R}_2=\{\alpha\colon |\Lambda^{s_2}_{\ell_2}(\alpha)-\Lambda^{s_2}_{\ell_2}(\alpha+n_2\omega)|<e^{-s_2}\}$$

and

$$\tilde{\mathbf{R}}_2=\left\{\alpha\colon |\alpha-\alpha^2_{\rm cr}|<\frac{1}{s_2}(\mathbf{F}^{s_2}_{\ell_2}(\alpha); \mathbf{U}^{n_2}\phi^{s_2}_{\ell_2}(\alpha+n_2\omega))\right\}$$

[recall that $\Lambda^{s_2}_{\ell_2}(\alpha)$ is obtained from $\lambda^{s_1+1}_{+}$ with the help of the unresonant perturbation formulas (2.5) during $s_1<s<s_2$]. Then $\mathbf{R}_k$, $\tilde{\mathbf{R}}_k$, $k>2$ are defined in the same way. Under the construction $\tilde{\mathbf{R}}_k\subset\mathbf{R}_k$, $(\tilde{\mathbf{R}}_{k+1})\subset\tilde{\mathbf{R}}_k$, $\forall k>0$. We have two new AEF's in $\phi^{s_k+1}(\alpha)$, provided $\alpha\in\mathbf{R}_k$:

$$\phi^{s_k+1}_{\pm}=\mathbf{A}_{k,\pm}(\alpha)\cdot\phi^{s_k}_{\ell_k}(\alpha)+\mathbf{B}_{k,\pm}(\alpha)\cdot\mathbf{U}^{n_k}\phi^{s_k}_{\ell_k}(\alpha+n_k\omega)+\ldots$$

if, besides that, we have $\alpha\in\tilde{\mathbf{R}}_k$, then $|\mathbf{A}_{k,\pm}|$, $|\mathbf{B}_{k,\pm}|$ are close to $1/\sqrt{2}$:

Lemma 4.2. *Let $\alpha\in\tilde{\mathbf{R}}_k$. Then the ratio of derivatives of $\lambda^{s_k+1}_{+}(\alpha)$ in symmetric points α and $2\alpha^k_{\rm cr}-\alpha$ is close to -1 in the following sense:*

$$\left|\frac{d}{d\alpha}(\lambda^{s_k+1}_{+})|_{\alpha}\Big/\frac{d}{d\alpha}(\lambda^{s_k+1}_{+})|_{2\alpha^k_{\rm cr}-\alpha}+1\right|<\frac{\rm const}{s_k}.$$

The lemma follows from the estimates on $\frac{d^2}{d\alpha^2}(\lambda^{s_k+1}_{+})$ which one can obtain, differentiating (4.6):

$$\begin{aligned}\lambda''=s''_{22}\cdot\frac{(s_{11}-\lambda)}{(s_{11}+s_{22}-2\lambda)}+s''_{11}\cdot\frac{(s_{22}-\lambda)}{(s_{11}+s_{22}-2\lambda)}+\frac{s'_{22}\cdot(s'_{11}-s'_{22})(s_{11}-\lambda)}{(s_{11}+s_{22}-2\lambda)^2}\\+\frac{s'_{11}\cdot(s'_{22}-s'_{11})\cdot(s_{22}-\lambda)}{(s_{11}+s_{22}-2\lambda)^2}-\frac{s'_{22}\cdot(s_{11}-\lambda)(s'_{11}-s'_{22})\cdot(s_{11}-s_{22})}{(s_{11}+s_{22}-2\lambda)^3}\\-\frac{s'_{11}\cdot(s_{22}-\lambda)\cdot(s'_{11}-s'_{22})\cdot(s_{11}-s_{22})}{(s_{11}+s_{22}-2\lambda)^3}+\ldots.\end{aligned}$$

The main terms are the third and the forth ones. Using the lemma one has:

$$\left|\mathbf{A}_{k+1}(\alpha)\pm\frac{1}{\sqrt{2}}\right|<\frac{\rm const}{s_k},\quad\left|\mathbf{B}_{k+1}(\alpha)\pm\frac{1}{\sqrt{2}}\right|<\frac{\rm const}{s_k}$$

with $\alpha\in\tilde{\mathbf{R}}_k$. Thanks to the superexponential increasing of s_k, all limits in (4.4) successfully exist. Thanks to the Diophantine condition on ω, all unresonant perturbations of λ^{s_k+1} on the steps $s_k<s<s_{k+1}$ are negligibly small. □

5. Proof of Corollary 3

The construction of the corresponding solution is quite similar to one presented in Sect. 4. It is based on the same method of reduced resonant zones. We can define

$$\tilde{\mathbf{R}}_k = \{\alpha\colon |\alpha - \alpha_{\mathrm{cr}}^k| < (s_k)^d \cdot (\mathbf{F}_{\ell_k}^{s_k}(\alpha), \mathbf{U}^{n_k}\phi_{\ell_k}^{s_k}(\alpha + n_k\omega))\},$$

where d is a positive sufficiently large constant, depending on coefficient β in Diophantine condition (1.3). Lemma 4.1 b) ensures $\mathbf{A}_k(\alpha), \mathbf{B}_k(\alpha)$ taking values $\pm\left(1 - \frac{\text{const}}{(s_k)^\gamma}\right)$, $\pm\frac{\text{const}}{(s_k)^{\gamma/2}}$ with $\gamma(d,\beta) > 0$. Let us consider

$$\psi_k = \mathbf{U}^{-\left(\sum_{\ell=1}^{k} n'_\ell\right)}\left(\phi_+^{s_k+1}\left(\alpha - \left(\sum_{\ell=1}^{k} n'_\ell\right)\cdot\omega\right)\right),$$

where

$$n'_\ell = \begin{cases} n_\ell & \text{if } |\mathbf{A}_\ell(\alpha)| \text{ is a small term and } |\mathbf{B}_\ell(\alpha)| \text{ is near } 1, \\ 0 & \text{if } |\mathbf{A}_\ell(\alpha)| \text{ is near 1 and } |\mathbf{B}_\ell(\alpha)| \text{ is a small term.} \end{cases}$$

If we choose a subsequence $k_i \to +\infty$ so that $\left\{\left(\left(\sum_{\ell=1}^{k_i} n'_\ell\right)\cdot\omega\right) \bmod 1\right\}$ converge to a limiting point, one can find the polynomially decreasing solutions as

$$\psi = \lim_{k_i \to +\infty} \psi_{k_i}; \qquad \lambda = \lim_{k_i \to +\infty} \lambda_+^{s_k+1}\left(\alpha - \left(\sum_{\ell=1}^{k} n'_\ell\right)\omega\right). \quad \square$$

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