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Factorization of Random Jacobi Operators and Bäcklund Transformations

Oliver Knill

Mathematikdepartement, ETH Zürich, CH-8092 Zürich, Switzerland

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Summary. We show that a positive definite random Jacobi operator L over an abstract dynamical system $T : X \to X$ can be factorized as $L = D^2$, where D is again a random Jacobi operator but defined over a new dynamical system $S : Y \to Y$ which is an integral extension of T. An isospectral random Toda deformation of L corresponds to an isospectral random Volterra deformation of D. The factorization leads to commuting Bäcklund transformations which can be written explicitly in terms of Titchmarsh-Weyl functions. In the periodic case, the Bäcklund transformations are time 1 maps of a Toda flow with a time dependent Hamiltonian.

1. Introduction

Bäcklund transformations for Toda lattices have been given in a nonexplicit form by Toda and Wedati [WT,T]. Adler [A] found that Bäcklund transformations have their origin in a factorization $L = AA^*$ in analogy to the Miura map for the KdV equation. It has been mentioned already by Moser [M] that the relation between the Kac v. Moerbeke system and the Toda lattice has its algebraic origin in a factorization $L = D^2$, where D is a matrix on a vector space with twice the dimension of the vector space on which L acts. In those papers D or A are given first and L is obtained by forming $L = AA^* = D^2$. Recently, the Poisson structure of the Bäcklund transformations was studied in [DL] for the periodic Toda lattice and also in the more general context of Toda equations on Lie groups.

In [K] we studied Toda lattices with random boundary conditions. They were obtained by making isospectral deformations of random Jacobi operators. The random Toda lattice is a generalization of both the periodic and the tied Toda lattice. It is defined over an arbitrary abstract dynamical system. We will show here that Bäcklund transformations can also be done in this case. They are generalizing the Bäcklund transformations known for periodic and aperiodic Toda lattices investigated in [T,A,DL]. What is new here, (beside the fact that we are working with random

Jacobi operators and not with finite dimensional matrices), is that we have explicit formulas for the transformations in terms of Titchmarsh-Weyl functions. These functions are Green functions and play an important role for the study of spectral problems [S] and inverse spectral problems [CK] of stochastic Jacobi matrices.

The commutativity of the Bäcklund transformations follows from the fact that in the periodic case, there is an interpolation of the transformations by time dependent Hamiltonian flows. The Toda flow deformation of the operator L gives a random Volterra flow for D. The factorization $L = D^2$ leads to a kind of supersymmetry for random Jacobi matrices.

2. Random Toda Flows

We redefine shortly the definitions in [K] needed here: An ergodic *dynamical system* (X, T, μ) is a probability space (X, μ) together with a measurable ergodic invertible map T on X that preserves the measure μ. The *crossed product* $\mathcal{X}$ of $L^\infty(X)$ with the dynamical system (X, T, μ) is a C^* algebra and consists of sequences $K_n \in L^\infty(X)$ with convolution multiplication

$$(KM)_n(x) = \sum_{k+m=n} K_k(x) M_m(T^k x)$$

and involution

$$(K^*)_n(x) = \overline{K_{-n}}(T^n x) .$$

An element $K \in \mathcal{X}$ is written in the form

$$K = \sum_{n \in \mathbb{Z}} K_n \tau^n ,$$

where τ is a symbol. The multiplication in $\mathcal{X}$ is the multiplication of power series with the additional rule $\tau^k K_n = K_n(T^k)\tau^k$ for shifting the τ's to the right and the requirement $\tau^* = \tau^{-1}$. The norm on $\mathcal{X}$ is given by

$$|||K||| = |\ ||K(x)||\ |_\infty ,$$

where $K(x)$ is the infinite matrix

$$[K(x)]_{mn} = K_{n-m}(T^m x) .$$

The multiplication and involution in $\mathcal{X}$ is defined such that

$$K \in \mathcal{X} \mapsto K(x) \in \mathcal{B}(l^2(\mathbb{Z}))$$

is an algebra homomorphism:

$$KL(x) = K(x)L(x),\ K^*(x) = K(x)^* .$$

According to Pastur's theorem (adapted to the present situation), the spectrum of $K(x)$ is the same for almost all $x \in X$. The algebra $\mathcal{X}$ has the trace $\mathrm{tr}(K) = \int_X K_0 \, d\mu$. An element K has the decomposition $K = K^- + K_0 + K^+$ defined by requiring

$K^{\pm} = \sum_{\pm n>0} K_n \tau^n$. With $\mathcal{L} \subset \mathcal{X}$ is denoted the real Banach space consisting of *random Jacobi operators*

$$L = a\tau + (a\tau)^* + b$$

if $a, b \in L^\infty(X, \mathbb{R})$. The number

$$M(L) = \exp\left(\int_X \log(a)\, d\mu\right)$$

is the *mass* of L. We say it has *positive definite mass* if there exists $\delta > 0$ such that $a(x) \geq \delta$ for almost all $x \in X$. For a Hamiltonian

$$H \in C^\omega(\mathcal{L}) = \{H(L) = \mathrm{tr}(h(L)) \mid h \ \text{ entire}, \ h(\mathbb{R}) = \mathbb{R}\}$$

the *random Toda lattice*

$$\dot{L} = [B_H(L), L],$$

with $B_H(L) = h'(L)^+ - h'(L)^-$ is an isospectral flow in $\mathcal{L}$. It reduces to the periodic Toda lattice in the case when $|X|$ is finite. The flows all commute and exist globally.

3. Factorization of Random Jacobi Operators

3.1 Definition of the Titchmarsh-Weyl functions.

Given a random Jacobi operator $L \in \mathcal{L}$ with positive definite mass. For almost all $x \in X$, $L(x)$ is a bounded operator on $l^2(\mathbb{Z})$. We consider it also as a matrix acting algebraically on $\mathbb{R}^{\mathbb{Z}}$. Fix an energy E outside the spectrum of L. The time independent Schrödinger equation

$$L(x)u = Eu$$

admits a two dimensional family of solutions $\{u_n(x)\} \in \mathbb{R}^{\mathbb{Z}}$. If we fix for example u_0, u_1, all the other values u_n can be calculated recursively by

$$a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n = E u_n \ ,$$

where $a_n = a(T^n x)$ and $b_n = b(T^n x)$. With the vector

$$w_n(x) = (a_n(x) u_{n+1}(x), u_n(x)) \ ,$$

the Schrödinger equation can be written as the first order system

$$A_E(x) w_{-1}(x) = w_0(x) \ ,$$

where A_E is the *transfer cocycle*

$$A_E(x) = a^{-1}(T^{-1}x) \begin{pmatrix} E - b(x) & -a^2(T^{-1}x) \\ 1 & 0 \end{pmatrix} \ .$$

The name "cocycle" is usually used for the function $\mathbb{Z} \times X \to SL(2, \mathbb{C})$,

$$(x, n) \mapsto A_E^n(x) = A_E(T^{n-1}(x)) \cdots A_E(Tx) A_E(x) \ .$$

Claim . For E outside the spectrum of L, the cocycle A_E has a positive *Lyapunov exponent*

$$\lambda(A_E) = \lim_{n\to\infty} n^{-1} \int_X \log\left(||A_E^n(x)||\right) \, d\mu(x) .$$

Proof . The *Thouless formula*

$$\mathrm{Re}(\mathrm{tr}(\log(L - E))) = \log(M) + \lambda(A_E)$$

shows that the Lyapunov exponent $E \to \lambda(A_E)$ is harmonic outside the spectrum. Because $\det(A_E(x)) = 1$, the Lyapunov exponent takes values ≥ 0. According to the maximum principle for harmonic functions, the minimum 0 can not occur in the resolvent set. □

It follows from the *multiplicative ergodic theorem* (see [R]) that for E in the resolvent set, there exist one dimensional coinvariant stable und unstable vector spaces $W^\pm(x)$ such that

$$A_E(x)W^\pm(x) = W^\pm(Tx) .$$

For almost all $x \in X$ we can take a unit vector $w^\pm(x) \in W^\pm(x)$ and define $u^\pm(x)$ as the first coordinate of $w^\pm(x)$. Like this, there exist solutions $u^+(x), u^-(x) \in \mathbb{R}^\mathbb{Z}$ of $L(x)u = Eu$ satisfying $\{u_n^+(x)\} \in l^2(\mathbb{N})$ and $\{u_n^-(x)\} \in l^2(-\mathbb{N})$. The *Titchmarsh-Weyl functions* are

$$m^+(x) = a(x)\frac{u^+(Tx)}{u^+(x)},\ m^-(x) = a(x)\frac{u^-(Tx)}{u^-(x)} ,$$

$$n^+(x) = a(T^{-1}x)\frac{u^+(T^{-1}x)}{u^+(x)},\ n^-(x) = a(T^{-1}x)\frac{u^-(T^{-1}x)}{u^-(x)} .$$

They are measurable according to the multiplicative ergodic theorem and are allowed to take the value ∞ or $-\infty$.

Remark . Contrary to $u_n^+(x), u_n^-(x)$, which were defined pointwise for $x \in X$ and only up to a multiplication with a nonzero constant, $m^+(x)$ and $m^-(x)$ are uniquely defined measurable functions.

Remark . We use slightly different Titchmarsh-Weyl functions than in the literature. In [CL] for example

$$m^+(x) = -\frac{u^+(Tx)}{a(x)u^+(x)}$$

is used. Often, (for example in$[S, CFKS]$,) stochastic Jacobi matrices are discussed with $a(x) = 1$.

3.2 The Titchmarsh-Weyl functions as Green functions.

Related to the operator $L(x) \in \mathcal{B}(l^2(\mathbb{Z}))$ are the operators $L^\mathbb{N}(x) \in \mathcal{B}(l^2(\mathbb{N}))$ defined by

$$[L^\mathbb{N}(x)]_{ij} = [L(x)]_{ij},\quad i, j > 0$$

and $L^{-\mathbb{N}}(x) \in \mathcal{B}(l^2(-\mathbb{N}))$ by

$$[L^{-\mathbb{N}}(x)]_{ij} = [L(x)]_{ij},\quad i, j < 0 .$$

By the spectral theorem there are two probability measures $d\sigma^+, d\sigma^-$ on the real axes such that

$$[(L^{\mathbb{N}}(x) - E)^{-1}]_{11} = \int_{\mathbb{R}} \frac{d\sigma^+(x)(E')}{E' - E} ,$$
$$[(L^{-\mathbb{N}}(x) - E)^{-1}]_{-1,-1} = \int_{\mathbb{R}} \frac{d\sigma^-(x)(E')}{E' - E} .$$

Lemma 3.1.

$$m^+(x) = -a^2 \int_{\mathbb{R}} \frac{d\sigma^+(x)(E')}{E' - E} , \quad m^-(x) = -\left(\int_{\mathbb{R}} \frac{d\sigma^-(Tx)(E')}{E' - E} \right)^{-1} ,$$
$$n^+(x) = -\left(\int_{\mathbb{R}} \frac{d\sigma^+(T^{-1}x)(E')}{E' - E} \right)^{-1} , \quad n^-(x) = -a^2(T^{-1}x) \int_{\mathbb{R}} \frac{d\sigma^-(x)(E')}{E' - E} .$$

Proof . $u_0^+(x)$ or $u_0^-(x)$ can't be zero, or else $u_n^\pm = u_0^\pm(T^n x)$ is an eigenfunction for the matrix $L^{\pm\mathbb{N}}$. This is not possible since the spectra of $L^{\mathbb{N}}(x)$ and $L^{-\mathbb{N}}(x)$ are lying in an interval containing the whole spectrum of $L(x)$. Define the solutions $\{v_n^+(x)\}$ and $\{v_n^-(x)\}$ of $L(x)u = Eu$ by imposing the boundary conditions $v_0^+(x) = v_0^-(x) = 0$ and $v_1^+(x) = v_{-1}^-(x) = 1$. Because both $u_0^+(x)$ and $u_0^-(x)$ never can get zero, v^+, u^+ and v^-, u^- are two pairs of linearly independent solutions of $L(x)u = Eu$. This implies that the two *Wronskians*

$$[v^\pm(x), u^\pm(x)]_n = \det \begin{pmatrix} a_n(x)v_{n+1}^\pm(x) & a_n(x)u_{n+1}^\pm(x) \\ v_n^\pm(x) & u_n^\pm(x) \end{pmatrix} = \det(W_n^\pm(x))$$

are both different from zero. Because $\det(A_E(x)) = 1$ and

$$A_E^n(x)W_{-1}^\pm(x) = W_{n-1}^\pm(x) ,$$

the Wronskians are independent of n. Define symmetric matrices $G^+(x), G^-(x)$ by requiring that for $m \le n$,

$$[G^+(x)]_{mn} = -\frac{v_m^+(x)u_n^+(x)}{[v^+(x), u^+(x)]} ,$$
$$[G^-(x)]_{-m,-n} = -\frac{v_{-m}^-(x)u_{-n}^-(x)}{[v^-(x), u^-(x)]}$$

and $[G^\pm(x)]_{nm} = [G^\pm(x)]_{mn}$. For all $n, m \in \mathbb{N}$ one has

$$[G^+(x)]_{mn}(x) = [(L^{\mathbb{N}}(x) - E)^{-1}]_{mn},$$
$$[G^-(x)]_{-m,-n}(x) = [(L^{-\mathbb{N}}(x) - E)^{-1}]_{-m,-n} .$$

To verify this, use $L(x)v(x) = Ev$ as well as the symmetry of the matrices $G^\pm$. Especially

$$[(L^{\mathbb{N}}(x) - E)^{-1}]_{11} = [G^+(x)]_{11} = -\frac{u_1^+(x)}{a(x)u_0^+(x)}$$
$$= -\frac{m^+(x)}{a^2(x)} = -\frac{1}{n^+(Tx)}$$

and

$$[(L^{-\mathbb{N}}(x) - E)^{-1}]_{-1,-1} = [G^{-}(x)]_{-1,-1} = -\frac{u^{-}_{-1}(x)}{a(T^{-1}x)u_0^{-}(x)}$$
$$= -\frac{1}{m^{-}(T^{-1}x)} = -\frac{n^{-}(x)}{a^2(T^{-1}x)} .$$

$\square$

Remark. The lemma implies that $-m^{\pm}(x), -n^{\pm}(x)$ are *Herglotz functions*: they are mapping the upper complex half plane into itself.

3.3 Factorization of a random Jacobi operator.

We will simplify the notation, by often leaving away the x in the variables $m^{\pm}, n^{\pm}$, a, b. If we avoid the superscripts +, − in $m, n, etc.$, we mean that both equations (one with superscript + and one with superscript −) are true.
One can calculate the functions a, b back from m, n by

$$m + n = E - b ,$$
$$m \cdot n(T) = a^2.$$

We want to show now, how a random Jacobi operator L can be factorized as $L = D^2 + E$.

We take a special *integral extension* (Y, S, ν) of the dynamical system (X, T, μ) (see [CFS] for the general notion of an integral extension). It is defined like this: Y consists of two copies X_1, X_2 of the probability space (X, m). S is the identity map from X_1 to X_2 and the mapping T from X_2 to X_1. The S invariant measure ν is determined by $\nu(Y) = \mu(Y)/2$ for $Y \subset X_i$. Define on Y a new function c by requiring that for $x \in X = X_1$,

$$c(x) = -m(x), \ c(S^{-1}x) = -n(x) .$$

We have then for all $x \in X_1$,

$$c(x) + c(S^{-1}x) = -E + b(x) ,$$
$$c(x) \cdot c(Sx) = a^2(x) .$$

Because c is defined on Y, these formulas extend a, b to functions on Y. Define the C^* algebra $\mathcal{Y}$ analogously to $\mathcal{X}$ as the crossed product of $L^\infty(Y)$ through the dynamical system (Y, S, ν). The elements of $\mathcal{Y}$ can be represented as

$$K = \sum_n K_n \sigma^n ,$$

where $\sigma^2 = \tau$. Call ψ the map $\mathcal{Y} \to \mathcal{X}$

$$K = \sum_n K_n \sigma^n \mapsto \sum_n \tilde{K}_n \tau^n ,$$

where $\tilde{K}_n(x) = K_{2n}(x)$ for $x \in X_1 = X$. The mapping ψ gives for $x \in X_1$,

$$[\psi(K)(x)]_{nm} = [K(x)]_{2n,2m} \ .$$

Theorem 3.2.
a) The random Jacobi operators

$$D = (\sqrt{c}\sigma + \sigma^* \sqrt{c}) \in \mathcal{Y}$$

are bounded for E outside an interval containing the spectrum $\Sigma(L)$ and

$$\psi(D^2) = L - E \ .$$

D is selfadjoint if E is real and below $\Sigma(L)$.
b) The operators

$$BT_E^{\pm}(L) := \psi((D^{\pm})^2(S) + E)$$

have the same spectrum then L.

Proof .
a) If E is real and below the spectrum of L, we have from Lemma 3.1,

$$-m^{\pm}(x) > 0, \ -n^{\pm} > 0 \ .$$

If E is outside an interval containing the spectrum of L, $m^{\pm}$ take complex values in general. But they are bounded in modulus by the inverse of the distance from E to the interval containing the spectrum of L. The relation $\psi(D^2) = L - E$ follows from the definition:

$$\begin{aligned}\psi(D^2) &= \psi(\sqrt{c \cdot c(S)}\sigma^2 + (c + c(S^{-1})) + \sqrt{c(S^{-2}) \cdot c(S^{-1})}\sigma^{-2} \\ &= a\tau + b - E + a(T^{-1})\tau^* = L - E \ .\end{aligned}$$

b) S is ergodic as an integral extension of T [CFS] and D is an ergodic random Jacobi operator over the ergodic dynamical system (Y, S, ν). The spectrum $\Sigma(D(x))$ is constant almost everywhere. Especially it is translational invariant: the spectrum of D is the same as the spectrum of $D(S)$. Thus, also the spectrum of

$$L = \psi(D^2) + E$$

is the same as the spectrum of

$$L(S) = \psi(D^2(S)) + E \ .$$

□

For simplicity, we will leave away in the future the restriction map ψ and write just $L = D^2 + E$ instead of $L = \psi(D^2) + E$.

Remark . The requirement that L has positive definite mass could be weakened. If L has positive mass then $\int \log^+ \left(||A_E^{\pm}||\right) \, d\mu$ is finite and Oseledec's theorem is still applicable to define the Titchmarsh-Weyl functions. If the mass is zero and $a(x) > 0$ for almost all $x \in X$ then one can consider the cocycle $a(T^{-1}) \cdot A_E$ to find the Titchmarsh-Weyl functions. If $a(x) = 0$ on a set of positive measure, the Jacobi matrix $L(x)$ is block diagonal for almost all $x \in X$ and a decomposition $L = D^2 + E$ is then in general no longer possible.

4. Bäcklund Transformations

4.1 Bäcklund transformations as isospectral transformations.

Theorem 4.1. *For $H \in C^\omega(\mathcal{L})$, the Toda flow $\dot{L} = [B_H(L), L]$ is with $D^2 = L - E$ equivalent to the Volterra flow*

$$\dot{D} = [B_H(D^2 + E), D] .$$

The mapping

$$BT_E : L \mapsto L(S)$$

is a Bäcklund transformation: It is isospectral and commutes with each Toda flow.

In order to prove Theorem 4.1, we need a lemma

Lemma 4.2. *Given two random operators $D = d\sigma + \sigma^* d$, $R = r\sigma + \sigma^* r$ over the ergodic dynamical system (Y, S, ν). If d^2 is not constant on Y and $DR + RD = 0$ then $R = 0$.*

Proof . The equation

$$\begin{aligned} RD + DR &= (rd(S) + dr(S))\sigma^2 + 2(dr + (dr)(S^{-1})) \\ &\quad + \left((rd(S) + dr(S))\sigma^2\right)^* = 0 \end{aligned}$$

is equivalent to

$$dr + (dr)(S^{-1}) = 0 , \tag{1}$$

$$r \cdot d(S) + d \cdot r(S) = 0. \tag{2}$$

From (1), we get $dr = -dr(S^{-1})$ or $dr = dr(S^{-2})$. If S^2 is ergodic, then $dr = C_0 = const$ and from (1) follows $C_0 = 0$ and so $r = 0$. If S^2 is not ergodic, then $Y = X_1 \cup X_2$ and S^2 is ergodic on X_ι. This implies that

$$dr = -(dr)(S) = C_0 = const$$

when restricted to X_1. Equation (2) gives

$$C_0 \cdot \left(\frac{d(S)}{d} - \frac{d}{d(S)}\right) = C_0 \cdot \frac{(d(S)^2 - d^2)}{dd(S)} = 0$$

which implies that $C_0 = 0$ unless $d^2(S) = d^2$ almost everywhere. By ergodicity of S, the equation $d^2(S) = d^2$ is equivalent to $d^2 = const$ which was excluded by assumption. Therefore $r = 0$ and so $R = 0$. □

We prove Theorem 4.1:

Proof . If D fulfills the equation $\dot{D} = [B_H(D^2 + E), D]$ then $L(t) = D^2(t) + E$ satisfies the differential equation $\dot{L} = [B_H(L), L]$:

$$\begin{aligned} \frac{d}{dt}L(t) &= \frac{d}{dt}(D^2(t) + E) = \dot{D}D + D\dot{D} \\ &= [B_H(D^2 + E), D]D + D[B_H(D^2 + E), D] \\ &= [B_H(D^2 + E), D^2] = [B_H(L), L]. \end{aligned}$$

If on the other hand $L(t)$ satisfies $\dot{L} = [B_H(L), L]$, then

$$\dot{D}D + D\dot{D} = [B_H(D^2 + E), D^2] = [B_H(D^2 + E), D]D + D[B_H(D^2 + E), D] ,$$

where $D(t)$ is defined by $L(t) = D(t)^2 + E$. With

$$R = \dot{D} - [B_H(D^2 + E), D]$$

we can write this as

$$RD + DR = 0 .$$

From Lemma 4.2, we have $R = 0$, unless D is constant. But in the later case $\dot{D} = \dot{L} = 0$ anyway.

Each Toda flow commutes with $L \mapsto L(T)$ and in the same way, each Volterra flow is commuting with $D \mapsto D(S)$. The just proved relation between the Toda and the Volterra flow shows that the Toda flows are commuting with $L \mapsto L(S)$ and the operators $BT_E^{\pm}L$ satisfies the same differential equation as L. A transformation with this property is called a Bäcklund transformation. $\square$

Example . In the case $h(L) = L^2/2$, the motion of D is given by the differential equation

$$\dot{c} = 2c(c(S) - c(S^{-1}))$$

which is called the *Volterra, Kac Moerbeke or Langmuir lattice*. It is a conservation law for the integral

$$\int_Y \log(c)\, d\nu$$

and in terms of the Titchmarsh-Weyl functions m, n it can be rewritten as

$$\dot{m} = 2m(n - n(T)) ,$$
$$\dot{n} = 2n(m(T^{-1}) - m) .$$

Also these differential equations are parametrized by a parameter E.

A historical remark. The Volterra system appeared first in 1931 in Volterra's work. He studied the evolution of a hierarchical system of competing individuals. Henon mentions in a letter (1973) to Flaschka the relation of the Toda lattice with the Volterra system. In 1975 the version with aperiodic boundary conditions was solved by Moser [M]. In the same paper, the relation with the Toda lattice is published.

Remark . Bäcklund transformations are also defined for complex values E outside the convex hull of the spectrum. They still preserve the spectrum, but the images are no more selfadjoint operators. The norm can blow up in an isospectral way. Indeed, if E is approaching a pole of $m^+(x)$ in a gap of the spectrum, then $||BT^+(E)L(x)|| \to \infty$.

4.2 Bäcklund transformations in the coordinates of Flaschka.

A Bäcklund transformation can also be described in the canonical coordinates $q, p \in L^\infty(X)$ if they exist. If $\log(a)$ is a *additive coboundary*: $\exists f \in L^\infty(X)$

$$\log(a) = f(T) - f ,$$

we can define q, p by

$$4a^2 = e^{q(T)-q},\ 2b = p .$$

We will see that there is an additional free parameter for doing the Bäcklund transformations in the coordinates q, p. The generating function and the implicit canonical transformations in the following proposition have been given by Toda and Wedati [WT] in the case of aperiodic Toda lattices where the Bäcklund transformations are not isospectral.

Theorem 4.3. *For E outside an interval containing the spectrum $\Sigma(L)$, the Bäcklund transformations $BT_E^{\pm}$*

$$b' = b + n - n(T),\ a'^2 = a^2 \frac{m(T)}{m}$$

can be written in the canonical variables q, p as canonical transformations $BT_E^{\pm}$: $(q, p) \to (q', p')$

$$p = \frac{\partial W}{\partial q} = -e^{q'-q-C} - e^{q-q'(T^{-1})+C} + 2E ,$$
$$p' = -\frac{\partial W}{\partial q'} = -e^{q'-q-C} - e^{q(T)-q'+C} + 2E$$

with a generating function

$$W(q, q') = \int_X e^{q'-q-C} - e^{q(T)-q'+C} - 2E \cdot (q' - q)\, d\mu .$$

where C is a parameter. Explicitly

$$q' = q + \log(2m) + C ,$$
$$p' = p + 2n - 2n(T) .$$

Proof. From $b' = b + n - n(T)$ we get $p' = p + 2n - 2n(T)$ which gives together with $b = -m - n + E$

$$p = -2m - 2n + 2E ,$$
$$p' = -2m - 2n(T) + 2E .$$

Taking the difference of these two equations gives

$$p' = p + 2n - 2n(T) .$$

From $a'^2 = a^2 \frac{m(T)}{m}$ we obtain

$$\log(4a'^2) = \log(4a^2) + \log(m(T)) - \log(m)$$

and

$$q'(T) - q(T) - \log(m(T)) = q' - q - \log(m) .$$

The ergodicity of T implies that

$$q' - q = \log(2m) + C ,$$

where C is a constant. Together with

$$\log(2m) + \log(2n(T)) = \log(4a^2) = q(T) - q$$

this gives

$$q' - q(T) = C - \log(2n(T)) ,$$

and so

$$\begin{aligned} p &= -2m - 2n + 2E = -e^{q'-q-C} - e^{q-q'(T^{-1})+C} + 2E , \\ p' &= -2m - 2n(T) + 2E = -e^{q'-q-C} - e^{q(T)-q'+C} + 2E . \end{aligned}$$

We verify

$$p = \frac{\partial W}{\partial q}, \; p' = -\frac{\partial W}{\partial q'} .$$

$\square$

4.3 Asymptotic behavior for $E \to -\infty$.

Proposition 4.4.
a)

$$\lim_{E \to -\infty} BT_E^+(L) = L(T) ,$$

$$\lim_{E \to -\infty} BT_E^-(L) = L .$$

b) For all E outside an interval containing $\Sigma(L)$,

$$BT_E^+ \circ BT_E^-(L) = L(T) .$$

Proof .
a) From Lemma 3.1, we see that the functions

$$\begin{aligned} \log\left(m^+(x) \cdot E\right) &= \log(a^2(x)) + \log\left([(1 - \frac{L^{\mathbb{N}}(x)}{E})^{-1}]_{11}\right) \\ &= \log\left(a^2(x)\right) + \log\left(\sum_{n=0}^{\infty} \frac{s_n^+(x)}{E^n}\right) , \\ \log\left(\frac{m^-(x)}{E}\right) &= -\log\left(E[(E - L^{\mathbb{N}}(Tx))^{-1}]_{-1,-1}\right) \\ &= -\log\left(\sum_{n=0}^{\infty} \frac{s_n^-(Tx)}{E^n}\right) \end{aligned}$$

with $s_n^+(x) = [(L^{\mathbb{N}}(x))^n]_{11}$ and $s_n^-(x) = [(L^{-\mathbb{N}}(x))^n]_{-1,-1}$ are analytic in a disc around ∞. We have thus the Taylor expansion in the variable $1/E$ (compare [CK])

$$\log\left(m^+(x)\cdot E\right) = \log(a^2(x)) + s_1^+(x)\frac{1}{E} + \left(s_2^+(x) - \frac{s_1^+(x)^2}{2}\right)\frac{1}{E^2} + \dots ,$$
$$\log\left(\frac{m^-(x)}{E}\right) = s_1^-(Tx)\frac{1}{E} + \left(s_2^-(Tx) - \frac{s_1^-(Tx)^2}{2}\right)\frac{1}{E^2} + \dots ,$$
$$\log\left(\frac{n^+(x)}{E}\right) = s_1^+(T^{-1}x)\frac{1}{E} + \left(s_2^+(T^{-1}x) - \frac{s_1^+(T^{-1}x)^2}{2}\right)\frac{1}{E^2} + \dots ,$$
$$\log\left(n^-(x)\cdot E\right) = \log\left(a^2(T^{-1}x)\right) + s_1^-(x)\frac{1}{E} + \left(s_2^-(x) - \frac{s_1^-(x)^2}{2}\right)\frac{1}{E^2} + \dots ,$$

leading to

$$\log\left(m^+(T)\right) - \log(m^+) = \log\left(a^2(T)\right) - \log(a^2) + (s_1^+(T) - s_1^+)\frac{1}{E} + \dots$$

and

$$\lim_{E\to-\infty} \frac{m^+(T)}{m^+} = \frac{a^2(T)}{a^2} .$$

Because $n^+ = E - b - m^+$ we get also

$$\lim_{E\to-\infty} n^+(T) - n^+ = b - b(T) .$$

From these two formulas $\lim_{E\to-\infty} BT_E^+(L) = L(T)$ follows. Similar, we deduce from the Taylor expansion that

$$\lim_{E\to-\infty} \frac{m^-(T)}{m^-} = 1, \quad \lim_{E\to-\infty} n^- = 0$$

and $\lim_{E\to-\infty} BT_E^-(L) = L$.
b) With $L = a\sigma + (a\sigma) + b$ and $L'' = BT_E^+ \circ BT_E^- L = a''\sigma + (a''\sigma)^* + b''$, we obtain

$$\begin{aligned}\log(a'') &= \log(a) + \frac{1}{2}\log\left(m^+(T)\right) - \frac{1}{2}\log(m^+) + \frac{1}{2}\log\left(n^-(T)\right) - \frac{1}{2}\log(n^-)\\ &= \log(a) + \frac{1}{2}\log\left(a^2(T)\right) - \frac{1}{2}\log(a^2) = \log(a(T)) ,\\ b'' &= b + n^+ - n^+(T) + m^- - m^-(T) = b + (E-b) - (E - b(T)) = b(T) .\end{aligned}$$

□

Each random Toda flow $L(t)$ is now embedded in a one parameter family of flows

$$t \mapsto BT_E L(t) ,$$

where E is a parameter. The random flow itself is obtained for $E \to -\infty$.

Remark . In the case $|X| < \infty$, the boundaries of the curves $E \mapsto BT_E^-(L)$ and $E \mapsto BT_E^+(L(T^{-1}))$ have nonempty intersection. This gives the possibility to deform L into $L(T)$ inside the isospectral set: For aperiodic dynamical systems, one can't expect that a deformation of L into $L(T)$ can always be done because in general, m^+ and m^- are different everywhere.

4.4 Commutation of Bäcklund transformations.

Assume the Hamiltonian $H_E(L) = \mathrm{tr}(h_E(L))$ is dependent on a parameter $E \in U \subset \mathbb{R}$ where U is an open interval in $\mathbb{R}$. Together with a smooth curve $t \to E(t)$ in U, we can define an isospectral deformation

$$\frac{d}{dt} L(t) = [B_{H(E(t))}(L), L] .$$

Theorem 4.5.
a) Assume $|X|$ is finite. For all $L \in \mathcal{L}$, there exist time dependent Hamiltonians $H_E^{\pm}(L) = \mathrm{tr}(h_E^{\pm}(L))$, such that $E \mapsto L(E) = BT^{\pm}(E)$ is the Toda orbit of

$$\frac{d}{dt} L(E(t)) = [B_{H^{\pm}(E(t))}(L), L] .$$

b) In general, for all real $E', E'' < \inf(\Sigma(L))$ and for all $\sigma, \mu \in \{+, -\}$,

$$BT^{\sigma}(E')BT^{\mu}(E'')L = BT^{\mu}(E'')BT^{\sigma}(E')L .$$

For the proof we need the following little lemma:

Lemma 4.6. *Given d linear independent constant real vector fields $f^{(i)}$ on the d dimensional torus $\mathbb{T}^d$.*
a) If the smooth vector field $\frac{d}{dE}\chi = f(E, \chi)$ is commuting with the vector fields $f^{(i)}$, then there exist $a_i(E, \chi) : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}$ independent of χ such that

$$f(E, \chi) = \sum_{i=1}^{d} a_i(E) f^{(i)} .$$

b) For any differentiable functions $a_i(E), b_i(E)$, the time dependent vector fields

$$F(E) = \sum_{i=1}^{d} a_i(E) f^{(i)}, \ G(E) = \sum_{i=1}^{d} b_i(E) f^{(i)}$$

are commuting.

Proof .
a)

$$0 = [f, f^{(i)}]_j = (\nabla f_j) \cdot f^{(i)} - f \cdot (\nabla f_j^{(i)}) = (\nabla f_j) \cdot f^{(i)}$$

implies

$$\nabla f_j = 0 ,$$

for all $j = 1, \ldots, d$.
b)

$$\begin{aligned}[F(E), G(E)] &= [\sum_i a_i(E) f^{(i)}, \sum_j b_j(E) f^{(j)}] \\ &= \sum_{i,j} a_i(E) b_j(E) \left[f^{(i)}, f^{(j)}\right] = 0.\end{aligned}$$

Now to the proof of Theorem 4.5

Proof . a) The set Iso(L) $\subset \mathcal{L}$ of Jacobi operators with the same spectrum and mass forms a d dimensional real torus $\mathbb{T}^d$, where $d \leq |X| - 1$ (see [vM 1]). There are d linearly independent real vector fields f_i on $\mathbb{T}^d$ which correspond to d different Toda flows [vM 2]. The real analytic curves

$$E \in [-\infty, \inf(\Sigma(L))] \mapsto L(E) = BT_E^{\pm}(L)$$

on the isospectral set Iso(L) correspond to real analytic curves $E \mapsto \chi^{\pm}(E)$ on the torus $\mathbb{T}^d$. Because these curves are smooth and passing through every point $\chi \in \mathbb{T}^d$, they are integral curves of time dependent vector fields

$$f^{\pm}(E, \chi) = \frac{d}{dE}\chi^{\pm}(E) .$$

We have seen that a Bäcklund transformation is commuting with each constant Toda flow. Therefore, the vector fields $f^{\pm}(E, \chi)$ and f_i are commuting. Application of Lemma 4.6 a) implies that $f^{\pm}(E, \chi)$ are independent of χ. In the original operator coordinates, this means that the time dependent Hamiltonian fields

$$\frac{d}{dE}L = [B_{H(E)}(L), L]$$

with Hamiltonian

$$H_E(L) = \mathrm{tr}(h_E(L)) = \sum_n h_{E,n}\mathrm{tr}(L^n)$$

have coefficients $h_{E,n}$, which are independent of L.
b) Assume first $|X|$ is finite. Assume $\sigma = +$ and $\mu = -$. The other cases go in the same way. Take from a) the time dependent vector fields $f^{\pm}(E)$ on $\mathbb{T}^d$ which are independent of the coordinate $\chi \in \mathbb{T}^d$. We know from Lemma 4.6 b) that for each E', E'', the flows of the vector fields

$$t \mapsto F^+(t) = f^+(E'/t) ,$$
$$t \mapsto F^-(t) = f^-(E''/t)$$

are commuting. As $BT^-_{-\infty}L = L$ (see Proposition 4.4), the transformation $BT^-_{E''}$ is obtained by integrating up the time dependent vector field f^-_E from $E = -\infty$ to $E = E''$ which is just the time 1 map of the flow given by the vector field $F^-(t)$. Because $BT^+_{-\infty}L = L(T)$ (again Proposition 4.4), the transformation $BT^+_{E'}$ is obtained by shifting $L \mapsto L(T)$ and then integrating up the vector field f^+_E from $E = -\infty$ to $E = E'$. This is a shift T followed with a time 1 map of the vector field $F^+(t)$. We have thus interpolated the Bäcklund transformations by Toda flows with time dependent Hamiltonians. From the commutation of the vector fields and the commutation of the Bäcklund transformations with the shift $T : L \mapsto L(T)$, the claim follows.

In general, let $L^{(N)}(x)$ be a periodic approximation of period N such that for $-N/2 \leq i, j < N/2$,

$$[L^{(N)}(x)]_{i+N,j+N} = [L^{(N)}(x)]_{ij} = [L(x)]_{ij} .$$

Then

$$(L^{(N)}(x))^{\mathbb{N}} \to L^{\mathbb{N}}(x)$$

in the strong operator topology (the strong and weak operator topologies coincide on the space of tridiagonal operators) and so in the strong resolvent sense (see [RS] p. 292). This implies, that the Green functions of $(L^{(N)})^{\mathbb{N}}$ are converging to the Green functions of $L^{\mathbb{N}}(x)$. Therefore, the Titchmarsh-Weyl functions of $L^{(\mathbb{N})}$ are converging pointwise to the Titchmarsh-Weyl functions of $L(x)$ and so

$$BT_E L^{(N)}(x) \to BT_E L(x)$$

in the weak operator topology. This gives

$$BT^{\sigma}_{E'} BT^{\mu}_{E''} L^{(N)}(x) \to BT^{\sigma}_{E'} BT^{\mu}_{E''} L(x)$$

in the weak operator topology and finally

$$\begin{aligned} BT^{\sigma}_{E'} BT^{\mu}_{E''} L(x) &= \lim_{N\to\infty} BT^{\sigma}_{E'} BT^{\mu}_{E''} L^{(N)}(x) \\ &= \lim_{N\to\infty} BT^{\mu}_{E''} BT^{\sigma}_{E'} L^{(N)}(x) = BT^{\mu}_{E''} BT^{\sigma}_{E'} L(x) . \end{aligned}$$

□

5. Symmetries

5.1 A simple version of super symmetry.

The factorization $L - E = D^2$ leads to the simplest version of *super symmetry*: Define the elements

$$H = \begin{pmatrix} L-E & 0 \\ 0 & L(S)-E \end{pmatrix}, Q = \begin{pmatrix} 0 & D\sigma \\ (D\sigma)^* & 0 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in $M(2, \mathcal{L})$. The property

$$Q^2 = H, P^2 = 1, \{Q, P\} = QP + PQ = 0$$

is called *super symmetry* ([CFKS] p.121). One calls the operators

$$Q^+ = \begin{pmatrix} 0 & D\sigma \\ 0 & 0 \end{pmatrix}, \; Q^- = \begin{pmatrix} 0 & 0 \\ (D\sigma)^* & 0 \end{pmatrix}$$

super charge operators. They satisfy

$$(Q^+)^2 = (Q^-)^2 = 0, \; Q^+ + Q^- = Q .$$

The eigenspace of the eigenvalue 1 of P is the space of *Bosonic states* and the eigenspace to the eigenvalue -1 is the space of *Fermionic states*. The operator $L - E$ is the restriction of H on the Bosonic states and the Bäcklund transformation $L(S) - E$ is the restriction of H on the Fermionic states. D is also called *charge operator*. This suggests to denote the invariants

$$C^-(E) = \exp\left(\int_X \log\sqrt{m^-}\, d\mu\right) = \exp\left(\frac{-w(E)}{2}\right)$$

the *charge* of the operator D^- and

$$C^+(E) = \exp\left(\int_X \log\sqrt{m^+}\,d\mu\right) = M\cdot\exp\left(\frac{w(E)}{2}\right)$$

the charge of the "antioperator" D^+.

5.2 CPT symmetry.

We have seen that we can write the first Toda lattice as the differential equation

$$\begin{aligned}\dot{m} &= 2m(n - n(T)) ,\\ \dot{n} &= 2n(m(T^{-1}) - m)\end{aligned}$$

in $L^\infty(X) \times L^\infty(X)$. Define the transpositions

$$\begin{aligned}&C : m^\pm \leftrightarrow n^\pm ,\\ &P : T \leftrightarrow T^{-1} ,\\ &T : t \leftrightarrow -t.\end{aligned}$$

One can see that the above equations for the motion of the Titchmarsh-Weyl functions m, n satisfy the symmetry CPT in that applying the transformation $C \circ P \circ T$ leaves the equations invariant. We could call the transformations a change of *Charge, Parity and Time*. The name charge is matching with the habit to call a Dirac operator like $D = \sqrt{c}\sigma + (\sqrt{c}\sigma)^*$ the *charge* operator. Notice also that the linearization of the first Toda equation, the *random wave equation*

$$\begin{aligned}\dot{m} &= n - n(T) ,\\ \dot{n} &= m(T^{-1}) - m\end{aligned}$$

has this CPT symmetry while the continuous analogue on $\mathbb{R}$

$$\begin{aligned}\dot{m} &= n_x ,\\ \dot{n} &= m_x\end{aligned}$$

has more symmetry, namely C and TP where $P : x \mapsto -x$. The discretization changes the symmetry. The doubling of the lattice simplifies the Toda equation to

$$\dot{c} = 2c(c(S) - c(S^{-1}))$$

and the C transformation which was an involution before becomes now the shift $C : c \mapsto c(S)$. We have still TP symmetry where $P : S \mapsto S^{-1}$. The doubling of the lattice changed also the symmetry.

6. Some Questions

• How does the time dependent Hamiltonian flow interpolating Bäcklund transformations look like explicitly? Does such a Hamiltonian flow exist in the aperiodic case also?

• Under which conditions is $L \mapsto L(T)$ the time one map of a Hamiltonian isospectral flow in $\mathcal{L}$? For which ergodic automorphisms $\tilde{T}$ commuting with T can one connect L with $L(\tilde{T})$ inside the isospectral set?
• Can one find analogous factorizations and Bäcklund transformations of higher dimensional random Jacobi operators in the crossed product $\mathcal{X}$ of $L^\infty(X)$ with a $\mathbb{Z}^d$ dynamical system? Jacobi operators are then of the form

$$\sum_{i=1}^{d} \left(a_i\tau^i + (a_i\tau^i)^*\right) + b \; .$$

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