Global Existence and Asymptotic Behavior of Solutions for the Maxwell–Schrödinger Equations in Three Space Dimensions

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Dedicated to Professor R. Iino on his 70th birthday

Abstract. In this paper we study the global existence and asymptotic behavior of solutions for the Maxwell–Schrödinger equations under the Coulomb gauge condition in three space dimensions with the final states given at $t = +\infty$. This leads to the construction of the modified wave operator for certain scattered data. It is also shown that for the initial data in the range of the modified wave operator, the initial value problem of the Maxwell–Schrödinger equations has the global solutions in time.

1. Introduction and Main Results

In the present paper we consider the global existence and asymptotic behavior of solutions for the Maxwell–Schrödinger equations under the Coulomb gauge condition in three space dimensions:

$$\frac{\partial^2}{\partial t^2} A - \Delta A = -i\{\bar{\psi}(\nabla - iA)\psi - \psi(\nabla + iA)\bar{\psi}\}$$

$$-\frac{1}{2\pi}\frac{\partial}{\partial t}\nabla\left(\frac{1}{|x|} * |\psi|^2\right), \quad t > 0, \ x \in \mathbf{R}^3, \tag{1.1}$$

$$2i\frac{\partial}{\partial t}\psi + (\nabla - iA)^2\psi - \frac{1}{4\pi}\left(\frac{1}{|x|} * |\psi|^2\right)\psi = 0, \quad t > 0, \ x \in \mathbf{R}^3, \tag{1.2}$$

$$\operatorname{div} A = 0, \quad t \geq 0, \ x \in \mathbf{R}^3, \tag{1.3}$$

where $*$ denotes the convolution with respect to the spatial variables. Here, $A(t, x)$ is a function from $[0, \infty) \times \mathbf{R}^3$ to $\mathbf{R}^3$ which denotes the electromagnetic real vector potential, and $\psi(t, x)$ is a function from $[0, \infty) \times \mathbf{R}^3$ to $\mathbf{C}$ which denotes the complex scalar field of nonrelativistic charged particles. Equations (1.1)–(1.3) are the classical approximation to the quantum field equations for an electrodynamical nonrelativistic many body system.

In [11, 12], Nakamitsu and M. Tsutsumi studied the solvability of the initial value problem of the Maxwell–Schrödinger equations under the Lorentz gauge condition, instead of the Coulomb gauge condition. In [11, 12], it is proved that if the spatial dimensions are two or one, the initial value problem of the Maxwell–Schrödinger equations with the Lorentz gauge condition is globally solvable and that if the spatial dimensions are three, the initial value problem with the Lorentz gauge condition is locally solvable. However, there seem to be no results concerning the global existence and asymptotic behavior of solutions for (1.1)–(1.2) in three space dimensions until now. In the present paper, instead of the initial value problem we consider solving (1.1)–(1.3) with the final states given at $t = +\infty$. The difficulty of constructing the global solutions of (1.1)–(1.3) consists in the quadratic nonlinearity of (1.1)–(1.2). In [10] Klainerman introduced the notion of the null condition to show the global existence of small amplitude solutions for the wave equation with quadratic nonlinear term. Recently, Bachelot [1] and Georgiev [5] have improved the null condition technique to show the global existence of small amplitude solutions for the Dirac–Klein–Gordon equations and the Maxwell–Dirac equations, respectively, which are the classical field equations related to (1.1)–(1.3). However, the null condition technique does not seem to be directly applicable to (1.1)–(1.3), because the null condition technique is based on the Lorentz invariance of the equations. But the Schrödinger equation does not necessarily have the same invariance as the wave equation, and especially the Schrödinger equation is not invariant under the Lorentz transform.

While the initial value problem is considered in [1, 5 and 10], in [3] Flato, Simon and Taflin study the global existence and asymptotic behavior of solutions for the Maxwell–Dirac equations with the final states given at $t = +\infty$. This corresponds to the construction of the wave operator. But in fact, the wave operator cannot be constructed in the usual sense, because the phase of asymptotic profile is distorted by the long range effect arising from the quadratic nonlinearity of the Maxwell–Dirac equations. In [3] they take the phase shift of asymtotic profile into account in order to compare the asymptotic behavior of the Maxwell–Dirac system with that of the free dynamics (see also [5]). This corresponds to the construction of the so-called modified wave operator. In [3] the phase function indicating the phase shift of asymptotic profile is given as an approximate solution of the classical Hamilton–Jacobi equation associated with a relativistic electron in an external field. Recently, in [13] Ozawa has given a more explicit form of the phase function for the one dimensional Schrödinger equation with power nonlinearity of long range effect. In [8] Hayashi and Ozawa have also shown the existence of the modified wave operator for the derivative nonlinear Schrödinger equation, following [13]. In this paper we shall give an explicit form of the phase function indicating the phase shift of asymptotic profile and construct the modified wave operator of the Maxwell–Schrödinger equations for certain scattered data. We shall first eliminate the long range effect of the worst term by using the difference between the propagation properties of the Maxwell wave and the Schrödinger wave, following [14] and we shall next construct the phase function of the phase shift resulting from the long range effect of the rest term in a similar way to [13] and [8]. Since the long range effect of the Maxwell–Schrödinger equations is more complicated than that of the decoupled nonlinear Schrödinger equation, our proof is more complicated and delicate than that of [13] or [8].

Before we state the main results in this paper, we give several notations. For $1 \leqq p \leqq +\infty$, we denote the standard L^p space on $\mathbf{R}^3$ by L^p. For $m, s \in \mathbf{R}$, let H^m

and $H^{m,s}$ be the Banach spaces defined by the completion of the Schwartz space $\mathscr{S}(\mathbf{R}^3)$ in the norms

$$\|v\|_{H^m} = \|(1 - \Delta)^{m/2} v\|_{L^2},$$

$$\|v\|_{H^{m,s}} = \|(1 + |x|^2)^{s/2}(1 - \Delta)^{m/2} v\|_{L^2},$$

respectively. For $m > 0$, let $\dot{H}^m$ be the completion of $\mathscr{S}(\mathbf{R}^3)$ with respect to the semi-norm

$$\|v\|_{\dot{H}^m} = \|(- \Delta)^{m/2} v\|_{L^2}.$$

For $1 \leq p \leq \infty$ and a nonnegative integer k, $W^{k,p}$ denotes the Banach space of all functions in L^p with derivatives up to kth order in L^p. Let the norm in $W^{k,p}$ be defined as follows:

$$\|v\|_{W^{k,p}} = \left(\sum_{|\alpha| \leq k} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha v \right\|_{L^p}^p \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|v\|_{W^{k,\infty}} = \max_{|\alpha| \leq k} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha v \right\|_{L^\infty}, \quad p = \infty.$$

For $v \in \mathscr{S}'(\mathbf{R}^3)$, $\hat{v}$ denotes the Fourier transform of v. For a function v on $\mathbf{R}^3$, we denote the support of v by supp v. For $z \in \mathbf{C}$, we denote the complex conjugate of z by $\bar{z}$. Let $\omega = \sqrt{-\Delta}$ and $U(t) = e^{\frac{i}{2}\Delta t}$. Let $\theta(t)$ be a real-valued function in $C^\infty([0, \infty))$ such that $\theta(t) = 1$ for $t \geq 2$ and $\theta(t) = 0$ for $0 \leq t \leq 1$.

Now we state the main results in this paper.

Theorem 1.1. *Let $1 > \delta > 0$. Assume that $A_{+0} = (A_{+0}^1, A_{+0}^2, A_{+0}^3) \in \dot{H}^5 \cap \dot{H}^1 \cap W^{5,1}$, $A_{+1} = (A_{+1}^1, A_{+1}^2, A_{+1}^3) \in H^4 \cap W^{4,1}$ and div $A_{+0} = $ div $A_{+1} = 0$. We put $A_+(t) = (A_+^1(t), A_+^2(t), A_+^3(t)) = \cos \omega t A_{+0} + \omega^{-1} \sin \omega t A_{+1}$. Assume that $\psi_{+0} \in H^{6,7}$ and supp $\hat{\psi}_{+0}(\xi) \subset \{\xi; |\xi| \geq 1 + \delta\} \cup \{\xi; |\xi| \leq 1 - \delta\}$. We put $\psi_+(t) = U(t)\psi_{+0}$. Then, there exists an $\eta > 0$ such that if*

$$\|A_{+0}\|_{\dot{H}^5 \cap \dot{H}^1} + \|A_{+0}\|_{W^{5,1}} + \|A_{+1}\|_{H^4} + \|A_{+1}\|_{W^{4,1}} + \|\psi_{+0}\|_{H^{6,7}} \leq \eta, \quad (1.4)$$

(1.1)–(1.3) have the unique solutions $(A(t), \psi(t))$ satisfying

$$A(t) \in C([0, \infty); \dot{H}^3 \cap \dot{H}^1), \quad (1.5)$$

$$\frac{\partial}{\partial t} A(t) \in \bigcap_{k=0}^2 C^k([0, \infty); H^{2-k}), \quad (1.6)$$

$$\psi(t) \in \bigcap_{k=0}^1 C^k([0, \infty); H^{3-2k}), \quad (1.7)$$

$$(1 + t)^{1/2} \left\{ \|A(t) - A_+(t) - R(t)\|_{\dot{H}^2 \cap \dot{H}^1} + \sum_{j=1}^2 \left\| \left(\frac{\partial}{\partial t}\right)^j (A(t) - A_+(t) - R(t)) \right\|_{H^{2-j}} \right\}$$

$$+ \left\{ \|A(t) - A_+(t) - R(t)\|_{\dot{H}^3} + \sum_{j=1}^3 \left\| \left(\frac{\partial}{\partial t}\right)^j (A(t) - A_+(t) - R(t)) \right\|_{H^{3-j}} \right\}$$

$$+ \sum_{j=0}^{1} \left\| \left(\frac{\partial}{\partial t} \right)^j \{ \psi(t) - e^{i\varphi} \psi_+(t) \} \right\|_{H^{3-2j}} = O(t^{-1}(\log t)^2) \quad (t \to +\infty), \tag{1.8}$$

$$\left(\int_t^{\infty} \| \psi(s) - e^{i\varphi} \psi_+(s) \|_{W^{2,4}}^{8/3} ds \right)^{3/8} = O(t^{-1}(\log t)^2) \quad (t \to +\infty), \tag{1.9}$$

where η depends only on δ, $R(t) = (R^1(t), R^2(t), R^3(t))$ is the solution of the following linear problem:

$$\frac{\partial^2}{\partial t^2} R - \Delta R = \left[2|t|^{-3} \frac{x}{t} \left| \hat{\psi}_{+0} \left(\frac{x}{t} \right) \right|^2 \right.$$

$$- i|t|^{-3} \left\{ \overline{\hat{\psi}}_{+0} \left(\frac{x}{t} \right) \frac{1}{t} (\nabla \hat{\psi}_{+0}) \left(\frac{x}{t} \right) - \hat{\psi}_{+0} \left(\frac{x}{t} \right) \frac{1}{t} (\nabla \overline{\hat{\psi}}_{+0}) \left(\frac{x}{t} \right) \right\} \right] \theta(t)^2$$

$$+ \frac{1}{4\pi} \nabla \left[\frac{1}{|x|} * \nabla \cdot \left\{ 2|t|^{-3} \frac{x}{t} \left| \hat{\psi}_{+0} \left(\frac{x}{t} \right) \right|^2 \right. \right.$$

$$- i|t|^{-3} \left(\overline{\hat{\psi}}_{+0} \left(\frac{x}{t} \right) \frac{1}{t} (\nabla \hat{\psi}_{+0}) \left(\frac{x}{t} \right) \right.$$

$$\left. \left. - \hat{\psi}_{+0} \left(\frac{x}{t} \right) \frac{1}{t} (\nabla \overline{\hat{\psi}}_{+0}) \left(\frac{x}{t} \right) \right) \right\} \right] \theta(t)^2, \quad t > 0, \tag{1.10}$$

$$\left\| \frac{\partial}{\partial t} R(t) \right\|_{L^2}^2 + \sum_{j=1}^{3} \| \nabla R^j(t) \|_{L^2}^2 \to 0 \quad (t \to +\infty), \tag{1.11}$$

and

$$\varphi(t, x) = -\frac{1}{8\pi} \log t \int_{\mathbf{R}^3} \frac{1}{|y|} \left| \hat{\psi}_{+0} \left(\frac{x}{t} - y \right) \right|^2 dy + \sum_{j=1}^{3} \int_0^t R^j \left(s, \frac{s}{t} x \right) ds \frac{x_j}{t}. \tag{1.12}$$

Remark 1.1. (i) We note that for $k > l > 0$, $\dot{H}^k \nsubseteq \dot{H}^l \nsubseteq L^2$. But, if $v \in \dot{H}^k \cap \dot{H}^1$ for a positive integer k, then $\left(\frac{\partial}{\partial x} \right)^\alpha v \in L^2$ for any multi-index α with $1 \leq |\alpha| \leq k$. In addition, for the three dimensional case, $\dot{H}^1 \subset L^6$. For the homogeneous Sobolev space $\dot{H}^m(\mathbf{R}^n)$, see, e.g., [2, §6.3]. But note that the definition of $\dot{H}^m(\mathbf{R}^n)$ in the present paper is slightly different from the one in [2, §6.3], where $\dot{H}^m(\mathbf{R}^n)$ is defined as the set of all tempered distributions v such that $(-\Delta)^{m/2} v \in L^2(\mathbf{R}^n)$. If we take $v \in \dot{H}^m(\mathbf{R}^n)$ modulo polynomials of degree at most $[m - n/2]$ in the definition of [2, §6.3], then the definition in [2, §6.3] is identical to that in the present paper (see [2, problem 12 in §6.8]). Here $[m - n/2]$ is the largest integer that is not larger than $m - n/2$ and if $[m - n/2]$ is negative, we take zero as a polynomial of order $[m - n/2]$.

(ii) There exists the unique solution $R(t)$ of (1.10)–(1.11) under the assumptions of Theorem 1.1 (see Lemma 2.4 in §2). We note that $R(t)$ does not necessarily belong to L^2 for $t \geq 0$ and

$$\| R(t) \|_{L^\infty} = O(t^{-1}) \quad (t \to +\infty).$$

(iii) $A_+(t)$ and $\psi_+(t)$ are the free solutions of the Maxwell equations and the Schrödinger equation with the initial conditions $A_+(0) = A_{+0}$, $\dfrac{\partial}{\partial t} A_+(0) = A_{+1}$ and $\psi_+(0) = \psi_{+0}$, respectively. Furthermore,

$$\operatorname{div} A_+(t) = \frac{\partial}{\partial t} \operatorname{div} A_+(t) = 0, \quad t \geqq 0 . \tag{1.13}$$

(iv) Equation (1.8) shows that the phase function $\varphi(t, x)$ defined in (1.12) indicates the phase shift of $\psi(t, x)$ from the free dynamics. The first term at the right-hand side of (1.12) corresponds to the long range effect of the Coulomb term, that is, the last term at the left-hand side of (1.2). This part of (1.12) is the same as the phase function given in [13] for the decoupled nonlinear Schrödinger equation. The second term at the right-hand side of (1.12) corresponds to the long range effect resulting from the coupling of the Maxwell equations and the Schrödinger equation.

(v) The assumption on the support of $\hat\psi_{+0}$ ensures that the product of $A_+(t)$ and $\nabla\psi_+(t)$ decays fast enough as $t \to +\infty$. This fact is based on the difference between the propagation properties of the Maxwell wave and the Schrödinger wave (see Corollary 2.5(ii) in Sect. 2). The influence by the product of $A_+(t)$ and $\nabla\psi_+(t)$ is most difficult to control, although it does not seem to be so bad at first sight.

(vi) We consider the Maxwell–Schrödinger equations under the Coulomb gauge condition instead of the Lorentz gauge condition, because the former is more usual than the latter in the nonrelativistic case from a physical point of view. Our proof of Theorem 1.1 is also applicable to the case of the Lorentz gauge condition.

(vii) For the case of $t \to -\infty$, we can obtain the same result as Theorem 1.1 without change of the proof.

The following corollary is an immediate consequence of Theorem 1.1.

Corollary 1.2. (i) *Let* $\delta > 0$. *By* D_+ *we denote the set of all scattered states* $(A_{+0}, A_{+1}, \psi_{+0})$ *such that* $\operatorname{div} A_{+0} = \operatorname{div} A_{+1} = 0$, $\operatorname{supp} \hat\psi_{+0} \subset \{\xi; |\xi| \geqq 1 + \delta\} \cup \{\xi; |\xi| \leqq 1 - \delta\}$ *and* (1.4) *holds. Then, for* (1.1)–(1.3) *the modified wave operator* W_+: $(A_{+0}, A_{+1}, \psi_{+0}) \mapsto \left(A(0), \dfrac{\partial}{\partial t} A(0), \psi(0) \right)$ *is well defined on* D_+.

(ii) *Let* $R(W_+)$ *be the range of the modified wave operator* W_+ *given by part* (i). *Then, for any* $(A_0, A_1, \psi_0) \in R(W_+)$ *there exist the global solutions* $(A(t), \psi(t))$ *of* (1.1)–(1.3) *with* $\left(A(0), \dfrac{\partial}{\partial t} A(0), \psi(0) \right) = (A_0, A_1, \psi_0)$ *satisfying* (1.5)–(1.7). *Furthermore, these solutions satisfy* (1.8) *and* (1.9) *with* $(A_{+0}, A_{+1}, \psi_{+0}) = W_+^{-1}(A_0, A_1, \psi_0)$.

Remark 1.2. (i) Noting Remark 1.2(vii), we can also construct the modified wave operator W_-, which is a mapping from the scattered states as $t = -\infty$ to the interacting states at $t = 0$.

(ii) Corollary 1.2(ii) gives the global existence result for the initial value problem of (1.1)–(1.3). However, it is not clear what initial data belong to $R(W_+)$.

The proof of Theorem 1.1 is roughly described as follows. Since the last term at the right-hand side of (1.1) is difficult to treat as it is, we rewrite (1.1) by using (1.2).

We substitute (1.2) into (1.1) to obtain

$$\frac{\partial^2}{\partial t^2} A - \Delta A = - i(\bar{\psi} \nabla \psi - \psi \nabla \bar{\psi}) - 2A|\psi|^2$$

$$+ \frac{1}{4\pi} \nabla \left[\frac{1}{|x|} * \nabla \cdot \{ - i(\bar{\psi} \nabla \psi - \psi \nabla \bar{\psi}) - 2A|\psi|^2 \} \right], \quad t > 0, \ x \in \mathbf{R}^3 .$$

$$(1.14)$$

We also rewrite (1.2) as follows:

$$2i \frac{\partial}{\partial t} \psi + \Delta \psi = i(\operatorname{div} A)\psi + 2iA \cdot \nabla \psi$$

$$+ |A|^2 \psi + \frac{1}{4\pi} \left(\frac{1}{|x|} * |\psi|^2 \right) \psi, \quad t > 0, \ x \in \mathbf{R}^3 . \quad (1.15)$$

Equations (1.14)–(1.15) are equivalent to (1.1)–(1.2), if the solutions $(A(t), \psi(t))$ are smooth. It is well known that if the L^2 norm of the perturbed term is integrable in time over $[0, \infty)$, the solution of an evolution equation such as the Maxwell–Schrödinger equations behaves like a free solution as $t \to \infty$. Therefore, it is conjectured that the long range effect arises from the second term and the last term at the right-hand side of (1.15). The long range effect of the latter can be controlled in the same way as the case of the decoupled nonlinear Schrödinger equation (see Ozawa [13]). The long range effect of the former comes from the coupling of the Maxwell equations and the Schrödinger equation. Accordingly, our main task in the proof of Theorem 1.1 is to evaluate the long range effect of the second term $2iA \cdot \nabla \psi$ at the right-hand side of (1.15). For that purpose, we first assume that the support of $\hat{\psi}_{+0}$ does not include the unit sphere centered at the origin. This assumption ensures that the product of $A_+(t)$ and $\nabla \psi_+(t)$ decays faster as $t \to \infty$ than $A_+(t)$ alone or $\nabla \psi_+(t)$ alone and so the influence by the product of $A_+(t)$ and $\nabla \psi_+(t)$ decays fast enough as $t \to \infty$. This fact results from the difference between the propagation properties of the Maxwell wave and the Schrödinger wave (see Corollary 2.5(ii) in Sect. 2). After the influence by the product of $A_+(t)$ and $\nabla \psi_+(t)$ has become negligible, the long range effect of $2iA \cdot \nabla \psi$ still remains. Therefore, we need to introduce a long range correction in the definition of the wave operator. The solution $A(t)$ of the Maxwell part can be divided into three parts, that is, $A(t) = A_+(t) + R(t) + v(t)$. Here $A_+(t)$ is a free solution which $A(t)$ approaches asymptotically, $R(t)$ is a solution of (1.10)–(1.11) which has a long range effect on the solution of the Schrödinger part and $v(t)$ is a remainder which decays faster as $t \to \infty$ than $A_+(t)$ and $R(t)$. If we put

$$m(t, x) = (it)^{-3/2} e^{\frac{i|x|^2}{2t}} \hat{\psi}_{+0} \left(\frac{x}{t} \right) \theta(t) ,$$

we can take $m(t, x)$ as an asymptotic profile of $\psi_+(t)$ (see Lemma 2.1(iii) in Sect. 2). We put

$$A(t) = A_+(t) + R(t) + v(t) ,$$

$$\psi(t) = e^{i\varphi} m(t) + u(t) ,$$

where $\varphi(t, x)$ is defined in (1.12). If we rewrite (1.14)–(1.15) as the equations of $v(t)$ and $u(t)$, the terms of slow time decay vanish and the resulting equations of $v(t)$ and $u(t)$ with the final states given at $t = \infty$ become solvable in proper function spaces under the assumptions of Theorem 1.1.

Our plan in this paper is the following. In Sect. 2 we summarize several lemmas and one proposition needed for the proof of Theorem 1.1. In Sect. 3 we state the proof of Theorem 1.1.

We conclude this section with several notations given. For two three-dimensional complex vectors $a, b \in \mathbf{C}^3$, we put $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$. We abbreviate $\partial/\partial x_j, j = 1, 2, 3$ and $\partial/\partial t$ to $\partial_{x_j}, j = 1, 2, 3$ and ∂_t, respectively. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we put $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$. We denote the scalar product in L^2 by $(\cdot, \cdot)$. For $s \in \mathbf{R}$, let $[s]$ be the largest integer that is not larger than s. Let ρ be a nonnegative function in $C_0^\infty(\mathbf{R}^3)$ such that $\| \rho \|_{L^1} = 1$. For $\varepsilon > 0$, we put $\rho_\varepsilon(x) = \varepsilon^{-3} \rho\left(\dfrac{x}{\varepsilon}\right)$. In the course of calculations below, various constants are simply denoted by C.

2. Preliminary Results

In this section we give several lemmas and one proposition needed for the proof of Theorem 1.1.

We start with the estimates of the evolution operator for the free Schrödinger equation.

Lemma 2.1. *Let $n \geq 1$.*

(i) *Let p and q be two positive constants such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $2 \leq p \leq + \infty$. Then,*

$$\| U(t)v \|_{L^p(\mathbf{R}^n)} \leq (2\pi|t|)^{-\left(\frac{n}{2} - \frac{n}{p}\right)} \| v \|_{L^q(\mathbf{R}^n)}, \quad v \in L^q(\mathbf{R}^n), \ t \neq 0 . \tag{2.1}$$

(ii) *Let q and r be two positive constants such that $2 \leq q < + \infty$ for $n = 1, 2$, $2 \leq q < \dfrac{2n}{n-2}$ for $n \geq 3$ and $\left(\dfrac{n}{2} - \dfrac{n}{q}\right)r = 2$. Then,*

$$\left(\int\limits_t^\infty \left\| \int\limits_\tau^\infty U(\tau - s)f(s)\,ds \right\|_{L^q(\mathbf{R}^n)}^r d\tau \right)^{1/r} \leq K_1 \int\limits_t^\infty \| f(s) \|_{L^2(\mathbf{R}^n)}\,ds, \quad t \geq 0 , \tag{2.2}$$

where K_1 is a positive constant depending only on n and q.

(iii) *Let k be a nonnegative integer. Suppose that for $j + |\alpha| \leq k$, $(1 + |x|)^{j+2} \partial_x^\alpha \psi_{+0} \in L^2(\mathbf{R}^n)$ and $(1 + |x|)^{j+2} \partial_x^\alpha \psi_{+0} \in L^1(\mathbf{R}^n)$. We put $m(t, x) = (it)^{-n/2} e^{\frac{i|x|^2}{2t}} \hat{\psi}_{+0}\left(\dfrac{x}{t}\right) \theta(t)$. Then, for some $K_2 > 0$,*

$$\sum_{2j + |\alpha| + |\beta| \leq k} \left\| \left(\dfrac{x}{t}\right)^\beta \partial_x^\alpha \partial_t^j \{ U(t)\psi_{+0} - m(t) \} \right\|_{L^2(\mathbf{R}^n)}$$

$$\leq K_2 t^{-1} \sum_{j + |\alpha| \leq k} \| (1 + |x|)^{j+2} \partial_x^\alpha \psi_{+0} \|_{L^2(\mathbf{R}^n)}, \quad t \geq 1 , \tag{2.3}$$

$$\sum_{2j + |\alpha| + |\beta| \leq k} \left\| \left(\frac{x}{t} \right)^\beta \partial_x^\alpha \partial_t^j \{ U(t)\psi_{+0} - m(t) \} \right\|_{L^\infty(\mathbf{R}^n)}$$

$$\leq K_2 t^{-n/2 - 1} \sum_{j + |\alpha| \leq k} \| (1 + |x|)^{j+2} \partial_x^\alpha \psi_{+0} \|_{L^1(\mathbf{R}^n)}, \quad t \geq 1 , \tag{2.4}$$

where K_2 depends only on k and n.

Remark 2.1. Lemma 2.1(ii) implies that $m(t)$ is asymptotically equivalent to $U(t)\psi_{+0}$ as $t \to \infty$.

For the proofs of Lemma 2.1(i) and (ii), see, e.g., [17] and [6]. For the proof of Lemma 2.1(iii), see [14, Lemma 2.1(ii)]. Lemma 2.1(ii) in [14] treats only the case $|\beta| = 0$, but the proof for $|\beta| \neq 0$ is the same as that for $|\beta| = 0$.

The following lemma is concerned with the decay property of solution of the free wave equation.

Lemma 2.2. (i) *Let $n \geq 1$ and let L be a nonnegative integer. Then,*

$$\sum_{j + |\alpha| \leq L} \| \partial_x^\alpha \partial_t^j \cos \omega t u \|_{L^\infty(\mathbf{R}^n)}$$

$$\leq K_3(1 + t)^{-(n-1)/2} (\| u \|_{H^{L-[n/2]+1}(\mathbf{R}^n)} + \| u \|_{W^{L+[n/2]+1,1}(\mathbf{R}^n)}), \quad t > 0 , \tag{2.5}$$

$$\sum_{j + |\alpha| \leq L} \| \partial_x^\alpha \partial_t^j \omega^{-1} \sin \omega t u \|_{L^\infty(\mathbf{R}^n)}$$

$$\leq K_3(1 + t)^{-(n-1)/2} (\| u \|_{H^{L+[n/2]}(\mathbf{R}^n)} + \| u \|_{W^{L+[n/2],1}(\mathbf{R}^n)}), \quad t > 0 , \tag{2.6}$$

where K_3 depends only on L and n.

(ii) *Let n be odd spatial dimensions larger than one and let L be a nonnegative integer. For any δ with $1 > \delta > 0$, there exists a $K_5 > 0$ such that*

$$\sum_{j + |\alpha| \leq L} \| \partial_x^\alpha \partial_t^j \cos \omega t u \|_{L^\infty(M_\delta(t))} \leq K_4(1 + t)^{-2} \| u \|_{H^{L+[n/2]+1,2}}, \quad t > 0 , \tag{2.7}$$

$$\sum_{j + |\alpha| \leq L} \| \partial_x^\alpha \partial_t^j \omega^{-1} \sin \omega t u \|_{L^\infty(M_\delta(t))} \leq K_4(1 + t)^{-2} \| u \|_{H^{L+[n/2]+1,2}}, \quad t > 0 , \tag{2.8}$$

where $M_\delta(t) = \{ \xi \in \mathbf{R}^n; |\xi| \geq (1 + \delta)t \} \cup \{ \xi \in \mathbf{R}^n; |\xi| \leq (1 - \delta)t \}$ and K_4 depends only on L, δ and n.

Proof. For the proof of part (i), see [9, §1].

Part (ii) follows from the Huygens principle for the wave equation in odd spatial dimensions larger than one. We first choose two functions $\phi_1, \phi_2 \in C^\infty(\mathbf{R})$ such that

$$\phi_j(s) = \phi_j(-s), j = 1, 2, \phi_1(s) = 1 \text{ for } |s| \leq 1, \phi_1(s) = 0 \text{ for } |s| \geq \left(1 - \frac{\delta}{2} \right) \Big/ (1 - \delta),$$

$$\phi_s(s) = 1 \text{ for } |s| \geq 1 \text{ and } \phi_2(s) = 0 \text{ for } |s| \leq \left(1 + \frac{\delta}{2} \right) \Big/ (1 + \delta). \text{ We put}$$

$$\phi_\delta(t, x) = \phi_1 \left(\frac{|x|}{(1 - \delta)t} \right) + \phi_2 \left(\frac{|x|}{(1 + \delta)t} \right) .$$

We note that for any nonnegative integer k,

$$\sum_{j + |\alpha| \leq k} |\partial_x^\alpha \partial_t^j \phi_\delta(t, x)| \leq C_k, \quad (t, x) \in [0, \infty) \times \mathbf{R}^n .$$

Since the propagation speed of wave is one, we have by the Sobolev imbedding theorem and the Huygens principle

$$\sum_{j + |\alpha| \leq L} \| \partial_x^\alpha \partial_t^j \cos \omega t u \|_{L^\infty(M_\delta(t))}$$

$$\leq \sum_{j + |\alpha| \leq L} \| \partial_x^\alpha \partial_t^j (\phi_\delta \cos \omega t u) \|_{L^\infty(\mathbf{R}^n)}$$

$$\leq C \sum_{j + |\alpha| \leq L + [n/2] + 1} \| \partial_x^\alpha \partial_t^j (\phi_\delta \cos \omega t u) \|_{L^2(\mathbf{R}^n)}$$

$$\leq C \sum_{j + |\alpha| \leq L + [n/2] + 1} \| \partial_x^\alpha u \|_{L^2(|x| > \frac{\delta}{2} t)}$$

$$\leq C \sum_{j + |\alpha| \leq L + [n/2] + 1} \left(1 + \frac{\delta}{2} t\right)^{-2} \| (1 + |x|)^2 \partial_x^\alpha u \|_{L^2(|x| > \frac{\delta}{2} t)}$$

$$\leq C(1 + t)^{-2} \| u \|_{H^{L + [n/2] + 1, 2}(\mathbf{R}^n)}, \quad t \geq 1 . \tag{2.9}$$

By the Sobolev imbedding theorem we also have

$$\sum_{j + |\alpha| \leq L} \| \partial_x^\alpha \partial_t^j u \|_{L^\infty(\mathbf{R}^n)} \leq C \sum_{j + |\alpha| \leq L + [n/2] + 1} \| \partial_x^\alpha \partial_t^j \cos \omega t u \|_{L^2(\mathbf{R}^n)}$$

$$\leq C \sum_{j + |\alpha| \leq L + [n/2] + 1} \| \partial_x^\alpha u \|_{L^2(\mathbf{R}^n)}, \quad t \geq 0 . \tag{2.10}$$

Equations (2.9) and (2.10) show (2.7). The proof of (2.8) is the same as above. □

Remark 2.1. (i) The analogous result to Lemma 2.2(ii) also holds in even space dimensions (see [15, Theorem XI. 18(a)]), although the Huygens principle does not strictly hold in even space dimensions.

(ii) Lemmas 2.2(i) and (ii) imply that the solution of the wave equation decays faster inside and outside the light cone as $t \to \infty$ than on the light cone.

We next consider the following linear wave equation:

$$\partial_t^2 w - \Delta w = h(t, x), \quad t > 0, \quad x \in \mathbf{R}^3 , \tag{2.11}$$

$$\| \nabla w(t) \|_{L^2}^2 + \| \partial_t w(t) \|_{L^2}^2 \to 0 \quad (t \to \infty) . \tag{2.12}$$

We have the following proposition concerning the solvability of (2.11)–(2.12).

Proposition 2.3. *Let L be an arbitrary integer with $L \geq 2$. We assume that*

$$h(t) \in \bigcap_{j=0}^{L-1} C^j([0, \infty); H^{L-j})$$

and for some $C_0 > 0$,

$$\sum_{k=0}^{L-1} \sum_{j + |\alpha| = k} (1 + t)^{3/2 + k} \| \partial_x^\alpha \partial_t^j h(t) \|_{L^2} \leq C_0, \quad t > 0 .$$

Then, there exists a unique solution $w(t)$ of (2.11)–(2.12) satisfying

$$w(t) \in C([0, \infty); \dot{H}^L \cap \dot{H}^1) , \tag{2.13}$$

$$\partial_t w(t) \in \bigcap_{j=0}^{L-1} C^j([0, \infty); H^{L-1-j}) , \tag{2.14}$$

$$\sum_{j+|\alpha|=k} \| \partial_x^\alpha \partial_t^j w(t) \|_{L^2} \leqq K_5 (1+t)^{1/2-k} \sup_{t \geqq 0} \sum_{j+|\alpha|=k-1} (1+t)^{k+1/2} \| \partial_x^\alpha \partial_t^j h(t) \|_{L^2} ,$$

$$t > 0, \quad k = 1, \ldots, L , \tag{2.15}$$

$$\sum_{j+|\alpha|=k} \| \partial_x^\alpha \partial_t^j w(t) \|_{L^\infty} \leqq K_5 (1+t)^{-1-k}$$

$$\times \sup_{t \geqq 0} \sum_{l=k}^{k+1} \sum_{j+|\alpha|=l} (1+t)^{3/2+l} \| \partial_x^\alpha \partial_t^j h(t) \|_{L^2}, \quad t > 0, \quad k = 0, \ldots, L-2 , \tag{2.16}$$

where K_5 depends only on L.

Proof. We first consider the following regularized problem:

$$\partial_t^2 w_{\varepsilon\lambda} - \Delta w_{\varepsilon\lambda} = (1 + \lambda t)^{-L-1} \rho_\varepsilon * h(t), \quad t > 0, \quad x \in \mathbf{R}^3 , \tag{2.17}$$

$$\| \nabla w_{\varepsilon\lambda}(t) \|_{L^2}^2 + \| \partial_t w_{\varepsilon\lambda}(t) \|_{L^2}^2 \to 0 \quad (t \to +\infty) , \tag{2.18}$$

where $0 < \lambda < 1$ and $0 < \varepsilon < 1$. The solution $w_{\varepsilon\lambda}(t)$ of (2.17)–(2.18) is given by

$$w_{\varepsilon\lambda}(t) = - \int_t^\infty (1 + \lambda s)^{-L-1} (\omega^{-1} \sin \omega(t-s)) \rho_\varepsilon * h(s) \, ds, \quad t \geqq 0 . \tag{2.19}$$

Formula (2.19) and a direct calculation yield

$$w_{\varepsilon\lambda}(t) \in \bigcap_{j=1}^\infty C^{L+1}([0, \infty); H^j) , \tag{2.20}$$

$$\sup_{t \geqq 0} \left[(1+t)^{L+1/2} \| w_{\varepsilon\lambda}(t) \|_{L^2} + \sum_{1 \leqq j+|\alpha| \leqq L+1} (1+t)^{L+3/2} \| \partial_x^\alpha \partial_t^j w_{\varepsilon\lambda}(t) \|_{L^2} \right] < \infty \tag{2.21}$$

for $0 < \lambda < 1$ and $0 < \varepsilon < 1$.

We next derive the a priori estimates independent of λ and ε for $w_{\varepsilon\lambda}$. We take the scalar product in L^2 between (2.17) and $\partial_t w_{\varepsilon\lambda}(t)$ to obtain

$$\frac{1}{2} \frac{d}{dt} \{ \| \partial_t w_{\varepsilon\lambda}(t) \|_{L^2}^2 + \| \nabla w_{\varepsilon\lambda} \|_{L^2}^2 \} = (1 + \lambda t)^{-L-1} (\rho_\varepsilon * h(t), \partial_t w_{\varepsilon\lambda}(t)), \quad t > 0 . \tag{2.22}$$

Integrating (2.22) over $[t, \infty)$, we have by Young's inequality

$$\| \partial_t w_{\varepsilon\lambda}(t) \|_{L^2}^2 + \| \nabla w_{\varepsilon\lambda}(t) \|_{L^2}^2 \leqq 2 \int_t^\infty (1 + \lambda s)^{-L-1} \| h(s) \|_{L^2} \| \partial_s w_{\varepsilon\lambda}(s) \|_{L^2} \, ds$$

$$\leqq 2 \int_t^\infty (1 + \lambda s)^{-L-1} (1+s)^{-2} \, ds$$

$$\times \left\{ \sup_{s \geq t} (1 + s)^{3/2} \| h(s) \|_{L^2} \right\}$$

$$\times \left\{ \sup_{s \geq t} (1 + s)^{1/2} \| \partial_s w_{\varepsilon\lambda}(s) \|_{L^2} \right\}$$

$$\leq C(1 + t)^{-1} \left\{ \sup_{s \geq t} (1 + s)^{3/2} \| h(s) \| \right\}$$

$$\times \left\{ \sup_{s \geq t} (1 + s)^{1/2} \| w_{\varepsilon\lambda}(s) \|_{L^2} \right\}$$

$$\leq C(1 + t)^{-1} \left\{ \sup_{s \geq t} (1 + s)^{3/2} \| h(s) \|_{L^2} \right\}^2$$

$$+ \frac{1}{2}(1 + t)^{-1} \left\{ \sup_{s \geq t} (1 + s)^{1/2} \| \partial_s w_{\varepsilon\lambda}(s) \|_{L^2} \right\}^2, \quad t \geq 0,$$

$$(2.23)$$

where C does not depend on ε and λ. Equation (2.23) gives us

$$\sup_{t \geq 0} (1 + t)^{1/2} \left\{ \| \partial_t w_{\varepsilon\lambda}(t) \|_{L^2} + \| \nabla w_{\varepsilon\lambda}(t) \|_{L^2} \right\} \leq C \sup_{t \geq 0} (1 + t)^{3/2} \| h(t) \|_{L^2}, \quad (2.24)$$

where C does not depend on ε and λ. In the same way as above we obtain

$$\sup_{t \geq 0} \sum_{j + |\alpha| = k} (1 + t)^{k - 1/2} \| \partial_x^\alpha \partial_t^j w_{\varepsilon\lambda}(t) \|_{L^2}$$

$$\leq C \sup_{t \geq 0} \sum_{j + |\alpha| = k - 1} (1 + t)^{k + 1/2} \| \partial_x^\alpha \partial_t^j h(t) \|_{L^2}, \quad k = 2, \ldots, L, \quad (2.25)$$

where C does not depend on ε and λ. Therefore, (2.24), (2.25) and the standard compactness argument show the existence of a solution to (2.11)–(2.12) satisfying (2.13) and (2.15) with the continuity in t of $w_{\varepsilon\lambda}(t)$ and its derivatives replaced by $L^\infty(0, \infty)$. The continuity in t follows from the regularity theorem of linear hyperbolic equation. Equation (2.16) follows immediately from (2.15) and the Gagliardo–Nirenberg inequality (see, e.g., [4]).

We finally prove the uniqueness. We suppose that $w_1(t)$ and $w_2(t)$ are two solutions of (2.11) and (2.12) satisfying (2.13)–(2.15). Then we put $z = w_1 - w_2$ to obtain

$$\partial_t^2 z - \Delta z = 0, \quad t > 0, \quad x \in \mathbf{R}^3, \quad (2.26)$$

$$\| \partial_t z(t) \|_{L^2}^2 + \| \nabla z(t) \|_{L^2}^2 \to 0 \quad (t \to + \infty). \quad (2.27)$$

The conservation law of energy for (2.26) shows

$$\| \partial_t z(t) \|_{L^2}^2 + \| \nabla z(t) \|_{L^2}^2 = \| \partial_t z(s) \|_{L^2}^2 + \| \nabla z(s) \|_{L^2}^2, \quad 0 \leq s, t < + \infty. \quad (2.28)$$

Letting $s \to + \infty$ in (2.28), we have by (2.27)

$$\| \partial_t z(t) \|_{L^2}^2 + \| \nabla z(t) \|_{L^2}^2 = 0$$

for any $t \geq 0$, which implies that $w_1(t) = w_2(t)$ for $t \geq 0$. The proof is complete. $\quad \square$

We next state the following lemma concerning the estimates of the phase function defined in (1.12).

Lemma 2.4. *We assume* $\psi_{+0} \in H^{6,7}$. *Then, there exists the unique solution* $R(t)$ *of* (1.10)–(1.11) *such that*

$$R(t) \in C([0, \infty); \dot{H}^7 \cap \dot{H}^1), \tag{2.29}$$

$$\partial_t R(t) \in \bigcap_{k=0}^{6} C^k([0, \infty); H^{6-k}), \tag{2.30}$$

$$\sup_{t \geq 0} \sum_{k=1}^{7} \left[(1+t)^{k-1/2} \sum_{j+|\alpha|=k} \| \partial_x^\alpha \partial_t^j R(t) \|_{L^2} \right]$$
$$+ \sup_{t \geq 0} \sum_{k=0}^{5} \left[(1+t)^{k+1} \sum_{j+|\alpha|=k} \| \partial_t^\alpha \partial_t^j R(t) \|_{L^\infty} \right] \leq K_6 \| \psi_{+0} \|_{H^{6,7}}^2 \tag{2.31}$$

for some $K_6 > 0$. *Furthermore, the phase function* $\varphi(t, x)$ *defined in* (1.12) *satisfies*

$$\sup_{t \geq 1} \sum_{k=0}^{5} \left[(1+t)^k (\log(1+t))^{-1} \sum_{j+|\alpha|=k} \left\| \left(1 + \frac{|x|}{t} \right)^{-j-1} \partial_x^\alpha \partial_t^j \varphi(t) \right\|_{L^\infty} \right]$$
$$\leq K_7 \| \psi_{+0} \|_{H^{6,7}}^2 \tag{2.32}$$

for some $K_7 > 0$.

Proof. We note that for some $C > 0$,

$$\| \hat{\psi}_{+0} \|_{W^{5,\infty}} \leq C \| \psi_{+0} \|_{H^{6,7}}. \tag{2.33}$$

We put

$$H(t, x) = h(t, x) + \frac{1}{4\pi} \nabla \left(\frac{1}{|x|} * \nabla \cdot h \right)(t, x),$$

$$h(t, x) = 2|t|^{-3} \frac{x}{t} \left| \hat{\psi}_{+0} \left(\frac{x}{t} \right) \right|^2 \theta(t)^2$$

$$- i|t|^{-3} \left\{ \bar{\hat{\psi}}_{+0} \left(\frac{x}{t} \right) \frac{1}{t} (\nabla \hat{\psi}_{+0}) \left(\frac{x}{t} \right) - \hat{\psi}_{+0} \left(\frac{x}{t} \right) \frac{1}{t} (\nabla \bar{\hat{\psi}}_{+0}) \left(\frac{x}{t} \right) \right\} \theta(t)^2.$$

We first verify that $H(t, x)$ satisfies the assumption in Proposition 2.4 with $L = 7$. Let k be any integer with $0 \leq k \leq 6$. Leibniz' rule shows that

$$\sum_{j+|\alpha|=k} |\partial_x^\alpha \partial_t^j h(t, x)|$$

is bounded by a sum of terms

$$C \left| \partial_x^{\alpha_1} \partial_t^{j_1} \left\{ |t|^{-3/2} \frac{x}{t} \hat{\psi}_{+0} \left(\frac{x}{t} \right) \theta(t)^2 \right\} \right| \left| \partial_x^{\alpha_2} \partial_t^{j_2} \left\{ |t|^{-3/2} \bar{\hat{\psi}}_{+0} \left(\frac{x}{t} \right) \right\} \right|,$$

$$C \left| \partial_x^{\alpha_1} \partial_t^{j_1} \left\{ |t|^{-3/2} \bar{\hat{\psi}}_{+0} \left(\frac{x}{t} \right) \frac{\theta(t)^2}{t} \right\} \right| \left| \partial_x^{\alpha_2} \partial_t^{j_2} \left\{ |t|^{-3/2} (\nabla \hat{\psi}_{+0}) \left(\frac{x}{t} \right) \right\} \right|,$$

where $j_1 + j_2 = j$ and $\alpha_1 + \alpha_2 = \alpha$. Since either $j_1 + |\alpha_1|$ or $j_2 + |\alpha_2|$ is less than $[k/2] + 1$ and $\operatorname{supp} \theta(t) \subset \{t \geqq 1\}$, we have by (2.33)

$$\sum_{j + |\alpha| = k} \| \partial_x^\alpha \partial_t^j h(t) \|_{L^2} \leqq C(1 + t)^{-3/2 - k} \| \hat{\psi}_{+0} \|_{W^{[k/2]+2, \infty}} \| \hat{\psi}_{+0} \|_{H^{k+1,k}}$$

$$\leqq C(1 + t)^{-3/2 - k} \| \psi_{+0} \|_{H^{6,7}}^2, \quad t \geqq 0 . \tag{2.34}$$

On the other hand, since the Fourier transform of $\dfrac{1}{4\pi |x|}$ is $|\xi|^{-2}$ and $\xi_j \xi_k / |\xi|^2 \in L^\infty$, we have

$$\left\| \frac{1}{4\pi} \nabla \left(\frac{1}{|x|} * \nabla \cdot h \right) \right\|_{L^2} \leqq C \| h \|_{L^2} . \tag{2.35}$$

Equations (2.34) and (2.35) show that $H(t, x)$ satisfies the assumption in Proposition 2.3 with $L = 7$. Therefore, by Proposition 2.3 we obtain the first half of Lemma 2.4.

We next prove (2.32). By the Gagliardo–Nirenberg inequality (see, e.g., [4]) and the Hardy–Littlewood–Sobolev inequality (see, e.g., [16]), we have

$$\left\| \int_{\mathbf{R}^3} \frac{1}{|y|} v(x - y) \, dy \right\|_{L^\infty} = \left\| \int_{\mathbf{R}^3} \frac{1}{|x - y|} v(y) \, dy \right\|_{L^\infty}$$

$$\leqq C \left\| \Delta \int_{\mathbf{R}^3} \frac{1}{|x - y|} v(y) \, dy \right\|_{L^2}^{1/2} \left\| \nabla \int_{\mathbf{R}^3} \frac{1}{|x - y|} v(y) \, dy \right\|_{L^2}^{1/2}$$

$$\leqq C \| v \|_{L^2}^{1/2} \| v \|_{L^{6/5}}^{1/2} . \tag{2.36}$$

Let k be any integer with $0 \leqq k \leqq 5$ and let $j + |\alpha| = k$. Since $\partial_t \hat{\psi}_{+0} \left(\dfrac{x}{t} - y \right) = -\dfrac{1}{t} \dfrac{x}{t} \cdot (\nabla \hat{\psi}_{+0}) \left(\dfrac{x}{t} - y \right)$, Leibniz' rule shows that

$$\left| \partial_x^\alpha \partial_t^j |\hat{\psi}_{+0} \left(\frac{x}{t} - y \right)|^2 \right|$$

$$\leqq C t^{-k} \sum_{|\alpha_1| + |\alpha_2| = k} \left(1 + \frac{|x|}{t} \right)^j \left| (\partial_x^{\alpha_1} \hat{\psi}_{+0}) \left(\frac{x}{t} - y \right) \right| \left| (\partial_x^{\alpha_2} \bar{\hat{\psi}}_{+0}) \left(\frac{x}{t} - y \right) \right|, \quad t \geqq 1 . \tag{2.37}$$

Therefore, we have by (2.37), (2.36) and (2.33),

$$\sum_{j + |\alpha| = k} \left\| \left(1 + \frac{|x|}{t} \right)^{-j} \partial_x^\alpha \partial_t^j \int_{\mathbf{R}^3} \frac{1}{|y|} \left| \hat{\psi}_{+0} \left(\frac{x}{t} - y \right) \right|^2 dy \right\|_{L^\infty}$$

$$\leqq C t^{-k} \sum_{|\alpha_1| + |\alpha_2| = k} \left\| \int_{\mathbf{R}^3} \frac{1}{|y|} \left| (\partial_x^{\alpha_1} \hat{\psi}_{+0}) \left(\frac{x}{t} - y \right) \right| \left| (\partial_x^{\alpha_2} \bar{\hat{\psi}}_{+0}) \left(\frac{x}{t} - y \right) \right| dy \right\|_{L^\infty}$$

$$\leqq Ct^{-k} \sum_{|\alpha_1| + |\alpha_2| = k} \left\| \left| \int_{\mathbf{R}^3} \frac{1}{|y|} \right| \partial_x^{\alpha_1} \hat{\psi}_{+0}(x - y) \left| \left| \partial_x^{\alpha_2} \overline{\hat{\psi}}_{+0}(x - y) \right| dy \right\|_{L^\infty}$$

$$\leqq Ct^{-k} \sum_{|\alpha_1| + |\alpha_2| = k} \| (\partial_x^{\alpha_1} \hat{\psi}_{+0})(\partial_x^{\alpha_2} \overline{\hat{\psi}}_{+0}) \|_{L^2}^{1/2} \| (\partial_x^{\alpha_1} \hat{\psi}_{+0})(\partial_x^{\alpha_2} \overline{\hat{\psi}}_{+0}) \|_{L^{6/5}}^{1/2}$$

$$\leqq Ct^{-k} \sum_{|\alpha_1| + |\alpha_2| = k} \| \partial_x^{\alpha_1} \hat{\psi}_{+0} \|_{L^\infty}^{1/2} \| \partial_x^{\alpha_2} \overline{\hat{\psi}}_{+0} \|_{L^2}^{1/2}$$

$$\times \| \partial_x^{\alpha_1} \hat{\psi}_{+0} \|_{L^\infty}^{1/6} \| \partial_x^{\alpha_1} \hat{\psi}_{+0} \|_{L^2}^{1/3} \| \partial_x^{\alpha_2} \overline{\hat{\psi}}_{+0} \|_{L^2}^{1/2}$$

$$\leqq Ct^{-k} \| \hat{\psi}_{+0} \|_{W^{k,\infty}}^{2/3} \| \hat{\psi}_{+0} \|_{H^k}^{4/3}$$

$$\leqq Ct^{-k} \| \psi_{+0} \|_{H^{6,7}}^2, \quad t \geqq 1 . \tag{2.38}$$

On the other hand, let k be any integer with $0 \leq k \leq 5$ and let $j + |\alpha| = k$. If $1 \leq k \leq 5$ and $j \neq 0$, we have

$$\left| \partial_x^\alpha \partial_t^j \int_0^t R\left(s, \frac{s}{t} x\right) ds \right| \leqq C \sum_{j + |\alpha| = k - 1} | \partial_x^\alpha \partial_t^j R(t, x) |$$

$$+ Ct^{-k} \left(1 + \frac{|x|}{t} \right)^j \sum_{|\alpha| = k} \int_0^t s^k \left| (\partial_x^\alpha R)\left(s, \frac{s}{t} x\right) \right| ds, \quad t \geqq 1 . \tag{2.39}$$

If $0 \leq k \leq 5$ and $j = 0$, we have

$$\sum_{|\alpha| = k} \left| \partial_x^\alpha \int_0^t R\left(s, \frac{s}{t} x\right) ds \right| \leqq Ct^{-k} \sum_{|\alpha| = k} \int_0^t s^k \left| (\partial_x^\alpha R)\left(s, \frac{s}{t} x\right) \right| ds, \quad t \geqq 1 . \tag{2.40}$$

Therefore, (2.31), (2.39) and (2.40) yield

$$\sum_{k=0}^5 \sum_{j + |\alpha| = k} (1 + t)^k (\log(1 + t))^{-1} \left\| \left(1 + \frac{|x|}{t} \right)^{-j} \partial_x^\alpha \partial_t^j \int_0^t R\left(s, \frac{s}{t} x\right) ds \right\|_{L^\infty}$$

$$\leqq C \sum_{k=1}^5 \sum_{j + |\alpha| = k} (1 + t)^k (\log(1 + t))^{-1} \left\| \left(1 + \frac{|x|}{t} \right)^{-j} \partial_x^\alpha \partial_t^j R(t) \right\|_{L^\infty}$$

$$+ C \sum_{k=1}^5 \sum_{|\alpha| = k} (1 + t)^k t^{-k} (\log(1 + t))^{-1} \int_0^t s^k \left\| (\partial_x^\alpha R)\left(s, \frac{s}{t} x\right) \right\|_{L^\infty} ds$$

$$+ C \sum_{k=0}^5 \sum_{|\alpha| = k} (1 + t)^k t^{-k} (\log(1 + t))^{-1} \int_0^t s^k \left\| (\partial_x^\alpha R)\left(s, \frac{s}{t} x\right) \right\|_{L^\infty} ds$$

$$\leqq C \sum_{k=1}^5 (1 + t)^k (\log(1 + t))^{-1} (1 + t)^{-k} \| \psi_{+0} \|_{H^{6,7}}^2$$

$$+ C \sum_{k=1}^5 (\log(1 + t))^{-1} \int_0^t s^k (1 + t)^{-k-1} ds \| \psi_{+0} \|_{H^{6,7}}^2$$

$$+ C \sum_{k=0}^5 (\log(1 + t))^{-1} \int_0^t s^k (1 + t)^{-k-1} ds \| \psi_{+0} \|_{H^{6,7}}^2$$

$$\leqq C \| \psi_{+0} \|_{H^{6,7}}^2, \quad t \geqq 1 . \tag{2.41}$$

Equation (2.32) follows immediately from (2.38), (2.41) and the definition of $\varphi(t, x)$. $\square$

The following corollary is an immediate consequence of Lemmas 2.1(iii), 2.2(ii) and 2.4.

Corollary 2.5. *Let $\delta > 0$ and let all the assumptions in Theorem 1.1 be satisfied. We put*

$$m(t, x) = (it)^{-3/2} e^{\frac{i|x|^2}{2t}} \hat{\psi}_{+0}\left(\frac{x}{t}\right) \theta(t) .$$

Let $\psi_+(t)$, $A_+(t)$ and $\varphi(t, x)$ be defined as in Theorem 1.1.
  (i) *The following relations hold.*

$$\sum_{j=0}^{1} \| \partial_t^j \{e^{i\varphi}\psi_+(t) - e^{i\varphi}m(t)\} \|_{H^{3-2j}} = O(t^{-1}) \quad (t \to \infty) , \tag{2.42}$$

$$\left(\int_t^{\infty} \sum_{2j + |\alpha| \leq 2} \| \partial_x^\alpha \partial_s^j \{e^{i\varphi}\psi_+(s) - e^{i\varphi}m(s)\} \|_{L^4}^4 ds \right)^{1/4} = O(t^{-3/2}) \quad (t \to \infty) . \tag{2.43}$$

  (ii) *There exists a $K_8 > 0$ such that*

$$\sum_{2j + |\alpha| \leq 3} \| \partial_x^\alpha \partial_t^j \{e^{i\varphi}A_+(t)m(t)\} \|_{L^2}$$

$$\leq K_8(1 + t)^{-2}(1 + \|\psi_{+0}\|_{H^{6,7}}^2)^3 (\|A_{+0}\|_{\dot{H}^5 \cap \dot{H}^1} + \|A_{+0}\|_{W^{5,1}}$$

$$+ \|A_{+1}\|_{H^4} + \|A_{+1}\|_{W^{4,1}}) \|\psi_{+0}\|_{H^{6,7}}, \quad t \geq 0 , \tag{2.44}$$

where K_8 depends only on δ.

Proof. We first show (i). Equation (2.42) follows directly from Lemma 2.1(iii) and Lemma 2.4. On the other hand, (2.3), (2.4), the interpolation and Lemma 2.4 yield

$$\sum_{2j + |\alpha| \leq 2} \| \partial_t^\alpha \partial_t^j \{e^{i\varphi}\psi_+(t) - e^{i\varphi}m(t)\} \|_{L^4} \leq Ct^{-7/4}, \quad t \geq 1 ,$$

which implies (2.43).
  We next show (ii). Since

$$\operatorname{supp} m(t, x) \subset \{x \in \mathbf{R}^3; |x| \geq (1 + \delta)t\} \cup \{x \in \mathbf{R}^3; |x| \leq (1 - \delta)t\}$$

for each $t > 0$, Lemma 2.2(ii) and Lemma 2.4 give us (2.44). $\square$

Remark 2.2. The assumption on the support of $\hat{\psi}_{+0}$ is not needed for the proof of Corollary 2.5(i), but it is indispensable to the proof of Corollary 2.5(ii).
  We conclude this section by giving the following lemma.

Lemma 2.6. *Let s be a real number with $s > 1$ and let k be a positive integer. Then, for some $C_{sk} > 0$,*

$$\int_t^{\infty} (1 + \tau)^{-s}(\log(2 + \tau))^k \, d\tau \leq C_{sk}(1 + t)^{-s+1}(\log(2 + t))^k, \quad t \geq 0 , \tag{2.45}$$

where C_{sk} depends only on s and k. Furthermore, we can choose $C_{sk} = 5$ in (2.45) for $(s, k) = (4, 3)$, $(3, 4)$, $(7/2, 3)$ and $(2, 2)$.

  Lemma 2.6 follows immediately from the integration by parts.

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1 by using the results obtained in Sect. 2.

As stated in Sect. 1, we use (1.2) to rewrite (1.1) and (1.2) as (1.14) and (1.15). We next put

$$A(t) = A_+(t) + R(t) + v(t) , \tag{3.1}$$

$$\psi(t) = e^{i\varphi} m(t) + u(t) , \tag{3.2}$$

where $A_+(t)$, $R(t)$ and $\varphi(t, x)$ are defined in Theorem 1.1 and

$$m(t, x) = (it)^{-3/2} e^{\frac{i|x|^2}{2t}} \hat{\psi}_{+0}\left(\frac{x}{t}\right) \theta(t) . \tag{3.3}$$

We substitute (3.1) and (3.2) into (1.14) and (1.15) to obtain

$$\partial_t^2 v - \Delta v = f(t) + F(t, v, u), \quad t > 0, \quad x \in \mathbf{R}^3 , \tag{3.4}$$

$$2i\partial_t u + \Delta u = 2i(A_+ + R + v) \cdot \nabla u$$
$$+ g(t) + G_1(t, v, u) + G_2(t, v, u), \quad t > 0, \quad x \in \mathbf{R}^3 , \tag{3.5}$$

$$\|\partial_t v(t)\|_{L^2}^2 + \sum_{j=1}^{3} \|\nabla v_j\|_{L^2}^2 \to 0 \quad (t \to \infty) , \tag{3.6}$$

$$\|u(t)\|_{L^2} \to 0 \quad (t \to \infty) . \tag{3.7}$$

Here

$$f(t) = f_0(t) + \frac{1}{4\pi} \nabla \left\{ \frac{1}{|x|} * (\nabla \cdot f_0(t)) \right\} ,$$

$$F(t, v, u) = F_0(t, v, u) + \frac{1}{4\pi} \nabla \left\{ \frac{1}{|x|} * (\nabla \cdot F_0(t, v, u)) \right\} ,$$

$$g(t) = -i\Delta\varphi e^{i\varphi} m + |\nabla\varphi|^2 e^{i\varphi} m - ie^{i\varphi} \frac{1}{t} \nabla\varphi \cdot \left\{ (it)^{-3/2} e^{\frac{i|x|^2}{2t}} (\nabla\hat{\psi}_{+0})\left(\frac{x}{t}\right) \theta(t) \right\}$$

$$- ie^{i\varphi} \frac{1}{t^2} (it)^{-3/2} e^{\frac{i|x|^2}{2t}} (\Delta\hat{\psi}_{+0})\left(\frac{x}{t}\right) \theta(t)$$

$$+ i\operatorname{div}(A_+ + R) e^{i\varphi} m - 2e^{i\varphi} A_+ \cdot \left(\frac{x}{t} m\right) - 2e^{i\varphi}(A_+ + R) \cdot \nabla\varphi m$$

$$+ 2ie^{i\varphi} \frac{1}{t} (A_+ + R) \cdot \left\{ (it)^{-3/2} e^{\frac{i|x|^2}{2t}} (\nabla\hat{\psi}_{+0})\left(\frac{x}{t}\right) \theta(t) \right\}$$

$$+ |A_+ + R|^2 e^{i\varphi} m - ie^{i\varphi} (it)^{-3/2} e^{\frac{i|x|^2}{2t}} \hat{\psi}_{+0}\left(\frac{x}{t}\right) \frac{d}{dt} \theta(t) ,$$

$$G_1(t, v, u) = i\operatorname{div}(A_+ + R + v) u - 2e^{i\varphi} v \cdot \left(\frac{x}{t} m\right) - 2e^{i\varphi} v \cdot \nabla\varphi m$$

$$+ 2ie^{i\varphi}\frac{1}{t}v\cdot\left\{(it)^{-3/2}e^{\frac{i|x|^2}{2t}}(\nabla\hat{\psi}_{+0})\left(\frac{x}{t}\right)\theta(t)\right\}$$

$$+ 2(A_+ + R)\cdot ve^{i\varphi}m + 2(A_+ + R)\cdot vu$$

$$+ 2|v|^2 e^{i\varphi}m + 2|v|^2 u ,$$

$$G_2(t, v, u) = \frac{1}{2\pi}\left\{\frac{1}{|x|}*\Re(e^{i\varphi}m\bar{u})\right\}e^{i\varphi}m$$

$$+ \frac{1}{4\pi}\left\{\frac{1}{|x|}*|m|^2\right\}u + \frac{1}{4\pi}\left\{\frac{1}{|x|}*|u|^2\right\}e^{i\varphi}m$$

$$+ \frac{1}{2\pi}\left\{\frac{1}{|x|}*\Re(e^{i\varphi}m\bar{u})\right\}u + \frac{1}{4\pi}\left\{\frac{1}{|x|}*|u|^2\right\}u ,$$

where

$$f_0(t) = 2(\nabla - A_+ - R)|m|^2 ,$$

$$F_0(t, v, u) = 2\Im(e^{-i\varphi}\bar{m}\nabla u) + 2\nabla\varphi\Re(e^{i\varphi}m\bar{u})$$

$$+ 2\Im(e^{i\varphi}\nabla m\bar{u}) + 2\Im(\bar{u}\nabla u) - 2v|m|^2$$

$$- 4(A_+ + R + v)\Re(e^{i\varphi}m\bar{u}) - 2(A_+ + R + v)|u|^2 .$$

We shall solve (3.4)–(3.7) by the contraction mapping principle. Let η be a positive constant satisfying (1.4). By Lemmas 2.2, 2.4, Corollary 2.5(ii) and (2.35), we obtain

$$\sum_{j=0}^{1}\sum_{|\alpha|\leq 2-j}\|\partial_x^\alpha\partial_t^j f(t)\|_{L^2} \leq C(1+\eta)^2\eta^2(1+t)^{-5/2}\log(2+t), \quad t\geq 0 , \quad (3.8)$$

$$\sum_{2j+|\alpha|\leq 3}\|\partial_x^\alpha\partial_t^j g(t)\|_{L^2} \leq C(1+\eta)^{10}\eta(1+t)^{-2}(\log(2+t))^2, \quad t\geq 0 , \quad (3.9)$$

where C does not depend on η. Therefore, if we choose $\eta > 0$ such that

$$\eta \leq 1 , \tag{3.10}$$

we have by (3.8) and (3.9)

$$\left(\sum_{j=0}^{1}\sum_{|\alpha|\leq 2-j}\|\partial_x^\alpha\partial_t^j f(t)\|_{L^2}^2\right)^{1/2} \leq M\eta(1+t)^{-5/2}\log(2+t), \quad t\geq 0 , \quad (3.11)$$

$$\max\left\{\left(\sum_{J+|\alpha|\leq 3}\|\partial_x^\alpha\partial_t^j g(t)\|_{L^2}^2\right)^{1/2}, \sum_{j+|\alpha|\leq 3}\|\partial_x^\alpha\partial_t^j g(t)\|_{L^2}\right\}$$

$$\leq M\eta(1+t)^{-2}(\log(2+t))^2, \quad t\geq 0 \tag{3.12}$$

for some $M \geq 0$, where M is independent of η for $0 < \eta \leq 1$. We put

$$X = \left\{[v(t), u(t)]; v(t)\in C([0,\infty); \dot{H}^3 \cap \dot{H}^1) ,\right.$$

$$\partial_t v(t)\in C([0,\infty); H^2) ,$$

$$u(t) \in \bigcap_{j=0}^{1} C^j([0, \infty); H^{3-2j}),$$

$$\sup_{t \geq 0} \left[(1+t)^{3/2} (\log(2+t))^{-2} \left(\sum_{1 \leq |\alpha| \leq 2} \| \partial_x^\alpha v(t) \|_{L^2}^2 \right. \right.$$

$$\left. \left. + \sum_{|\alpha| \leq 1} \| \partial_x^\alpha \partial_t v(t) \|_{L^2}^2 \right)^{1/2} \right] \leq 11 M\eta,$$

$$\sup_{t \geq 0} \left[(1+t)(\log(2+t))^{-2} \left(\sum_{|\alpha|=3} \| \partial_x^\alpha v(t) \|_{L^2}^2 \right. \right.$$

$$\left. \left. + \sum_{|\alpha|=2} \| \partial_x^\alpha \partial_t v(t) \|_{L^2}^2 \right)^{1/2} \right] \leq 11 M\eta,$$

$$\sup_{t \geq 0} \left[(1+t)(\log(2+t))^{-2} \left(\sum_{|\alpha| \leq 2} \| \partial_x^\alpha u(t) \|_{L^2}^2 \right)^{1/2} \right] \leq 6 M\eta,$$

$$\sup_{t \geq 0} \left[(1+t)(\log(2+t))^{-2} \left(\sum_{|\alpha| \leq 1} \| \partial_x^\alpha \partial_t u(t) \|_{L^2}^2 \right)^{1/2} \right] \leq 21 M\eta,$$

$$\sup_{t \geq 0} \left[(1+t)(\log(2+t))^{-2} \sum_{|\alpha|=1} \| \Lambda \partial_x^\alpha u(t) \|_{L^2} \right] \leq 3(14\sqrt{3}+1) M\eta,$$

$$\sup_{t \geq 0} \left[(1+t)(\log(2+t))^{-2} \sum_{|\alpha| \leq 2} \left(\int_t^\infty \| \partial_x^\alpha u(s) \|_{L^4}^{8/3} ds \right)^{3/8} \right] \leq 6 K_1 M\eta \right\},$$

where K_1 is defined in Lemma 2.1 (ii). We introduce the metric $d([v_1, u_1], [v_2, u_2])$ into X as follows:

$$d([v_1, u_1], [v_2, u_2]) = \sum_{1 \leq |\alpha| \leq 2} \sup_{t \geq 0} [(1+t)^{3/2}(\log(2+t))^{-2} \| \partial_x^\alpha(v_1(t) - v_2(t)) \|_{L^2}]$$

$$+ \sum_{|\alpha| \leq 1} \sup_{t \geq 0} [(1+t)^{3/2}(\log(2+t))^{-2} \| \partial_x^\alpha \partial_t(v_1(t) - v_2(t)) \|_{L^2}]$$

$$+ \sum_{|\alpha|=3} \sup_{t \geq 0} [(1+t)(\log(2+t))^{-2} \| \partial_x^\alpha(v_1(t) - v_2(t)) \|_{L^2}]$$

$$+ \sum_{|\alpha|=2} \sup_{t \geq 0} [(1+t)(\log(2+t))^{-2} \| \partial_x^\alpha \partial_t(v_1(t) - v_2(t)) \|_{L^2}]$$

$$+ \sum_{2j+|\alpha| \leq 3} \sup_{t \geq 0} [(1+t)(\log(2+t))^{-2} \| \partial_x^\alpha \partial_t(u_1(t) - u_2(t)) \|_{L^2}]$$

$$+ \sum_{|\alpha| \leq 2} \sup_{t \geq 0} \left[(1+t)(\log(2+t))^{-2} \left(\int_t^\infty \| \partial_x^\alpha(u_1(s) - u_2(s)) \|_{L^4}^{8/3} ds \right)^{3/8} \right]$$

for $[v_1, u_1], [v_2, u_2] \in X$. We note that X is a complete metric space with the metric d. We first define the nonlinear mapping $N_1[v, u]$ for $[v, u] \in X$. For $[v, u] \in X$, we

consider the linear problem:

$$\partial_t^2 w - \Delta w = f(t) + F(t, v, u), \quad t \geqq 0, \quad x \in \mathbf{R}^3, \tag{3.13}$$

$$\sum_{j=1}^{3} \| \nabla w_j(t) \|_{L^2}^2 + \| \partial_t w(t) \|_{L^2}^2 \to 0 \quad (t \to \infty). \tag{3.14}$$

Let $N_1[v, u]$ be a mapping from $[v, u] \in X$ to a solution $w(t) = (w_1(t), w_2(t), w_3(t))$ of (3.13)–(3.14). We next define the nonlinear mapping $N_2[v, u]$ as follows. For $[v, u] \in X$, we consider the following linear problem:

$$2i\partial_t z + \Delta z = 2i(A_+ + R + v) \cdot \nabla z$$

$$+ g(t) + G_1(t, v, u) + G_2(t, v, u), \quad t \geqq 0, \quad x \in \mathbf{R}^3, \tag{3.15}$$

$$\| z(t) \|_{L^2} \to 0 \quad (t \to \infty). \tag{3.16}$$

Let $N_2[v, u]$ be a mapping from $[v, u] \in X$ to a solution $z(t)$ of (3.15)–(3.16). We put $N[v, u] = [N_1[v, u], N_2[v, u]]$. By the definition of N, we see that the fixed point $[u, v]$ of the nonlinear mapping N in X are the solutions of (3.4)–(3.7). Now we show that if $\eta > 0$ is sufficiently small, the nonlinear mapping N is a contraction from X into X.

We first evaluate $w = N_1[v, u]$. By Lemmas 2.2(i), 2.4 and the Gagliardo–Nirenberg inequality (see, e.g., [4]) we have

$$\sum_{|\alpha| \leqq 1} \| \partial_x^\alpha F_0(t, v, u) \|_{L^2}$$

$$\leqq C \sum_{|\alpha| \leqq 1} [\| \partial_x^\alpha \{ \mathfrak{J}(e^{-i\varphi} \bar{m} \nabla u) + \nabla \varphi \mathfrak{R}(e^{i\varphi} m \bar{u}) + \mathfrak{J}(e^{i\varphi} \nabla m \bar{u}) \} \|_{L^2}$$

$$+ \| \partial_x^\alpha (\bar{u} \nabla u) \|_{L^2} + \| \partial_x^\alpha (v|m|^2) \|_{L^2} + \| \partial_x^\alpha \{ (A_+ + R + v) \mathfrak{R}(e^{i\varphi} m \bar{u}) \} \|_{L^2}$$

$$+ \| \partial_x^\alpha \{ (A_+ + R + v)|u|^2 \} \|_{L^2}]$$

$$\leqq C(1 + \| \nabla \varphi(t) \|_{W^{1,\infty}})^3 \| m(t) \|_{W^{2,\infty}} \| u(t) \|_{H^1}$$

$$+ C(\| u(t) \|_{W^{1,4}} \| \nabla u(t) \|_{L^4} + \| u(t) \|_{L^4} \| \nabla u(t) \|_{W^{1,4}})$$

$$+ C(\| v(t) \|_{L^\infty} \| m(t) \|_{W^{1,\infty}} \| m(t) \|_{H^1} + \| v(t) \|_{\dot{H}^1} \| m(t) \|_{L^\infty}^2)$$

$$+ C(\| A_+(t) \|_{W^{1,\infty}} + \| R(t) \|_{W^{1,\infty}}) \| m(t) \|_{W^{1,\infty}} \| u(t) \|_{H^1}$$

$$+ C(\| v(t) \|_{L^\infty} \| m(t) \|_{W^{1,\infty}} \| u(t) \|_{H^1} + \| v(t) \|_{\dot{H}^1} \| m(t) \|_{L^\infty} \| u(t) \|_{L^\infty})$$

$$+ C(\| A_+(t) \|_{W^{1,\infty}} + \| R(t) \|_{W^{1,\infty}}) \| u(t) \|_{W^{1,4}}^2$$

$$+ C(\| v(t) \|_{L^\infty} \| u(t) \|_{W^{1,4}}^2 + \| v(t) \|_{\dot{H}^1} \| u(t) \|_{L^\infty}^2)$$

$$\leqq C(1 + \eta^2)^3 \eta^2 (1 + t)^{-5/2} (\log(2 + t))^2 + C \| u(t) \|_{W^{2,4}}^2$$

$$+ C\eta^2 (1 + t)^{-3/2} \| v(t) \|_{\dot{H}^2}^{1/2} \| v(t) \|_{\dot{H}^1}^{1/2}$$

$$+ C\eta^3 (1 + t)^{-4} (\log(2 + t))^2 + C(1 + \eta) \eta^2 (1 + t)^{-3} (\log(2 + t))^2$$

$$+ C\eta^2 (1 + t)^{-5/2} (\log(2 + t))^2 \| v(t) \|_{\dot{H}^2}^{1/2} \| v(t) \|_{\dot{H}^1}^{1/2}$$

$$+ C\eta^3(1 + t)^{-7/2}(\log(2 + t))^4 + C(1 + \eta)\eta(1 + t)^{-1}\|u(t)\|^2_{W^{1,4}}$$

$$+ C\|v(t)\|^{1/2}_{\dot{H}^2}\|v(t)\|^{1/2}_{\dot{H}^1}\|u(t)\|^2_{W^{1,4}}$$

$$+ C\eta(1 + t)^{-3/2}(\log(2 + t))^2\|u(t)\|_{\dot{H}^2}\|u(t)\|_{\dot{H}^1}$$

$$\leqq C(1 + \eta)^6\eta^2(1 + t)^{-5/2}(\log(2 + t))^2 + C(1 + \eta)^2\|u(t)\|^2_{W^{2,4}}, \quad t \geqq 0 \quad (3.17)$$

for $[v, u] \in X$. In the same way as above, we have

$$\sum_{|\alpha| = 2}\|\partial_x^\alpha F_0(t, v, u)\|_{L^2} \leqq C(1 + \eta)^6\eta^2(1 + t)^{-5/2}(\log(2 + t))^2 + C\|u(t)\|^2_{W^{2,4}}$$

$$+ C\eta(1 + t)^{-1}(\log(2 + t))^2\|u(t)\|_{W^{1,4}}, \quad t \geqq 0 \quad (3.18)$$

for $[v, u] \in X$.

We consider the regularized problem associated with (3.13)–(3.14):

$$\partial_t^2 w_{\varepsilon\lambda} - \Delta w_{\varepsilon\lambda} = (1 + \lambda t)^{-4}\rho_\varepsilon * \{f(t) + F(t, v, u)\}, \quad t \geqq 0, \quad x \in \mathbf{R}^3, \quad (3.19)$$

$$\sum_{j=1}^3\|\nabla w_{\varepsilon\lambda j}(t)\|^2_{L^2} + \|\partial_t w_{\varepsilon\lambda}(t)\|^2_{L^2} \to 0 \quad (t \to \infty), \quad (3.20)$$

where $0 < \lambda < 1$ and $0 < \varepsilon < 1$. The solution $w_{\varepsilon\lambda}(t)$ of (3.19)–(3.20) is given by

$$w_{\varepsilon\lambda}(t) = -\int_t^\infty (1 + \lambda s)^{-4}(\omega^{-1}\sin\omega(t - s))\rho_\varepsilon * \{f(s) + F(s, v, u)\}\,ds, \quad t \geqq 0. \quad (3.21)$$

Formula (3.21) and a direct calculation yield

$$w_{\varepsilon\lambda}(t) \in \bigcap_{j=1}^\infty C^2([0, \infty); H^j), \quad (3.22)$$

$$\sum_{j=1}^2\sum_{|\alpha| \leqq 5-j}\sup_{t \geqq 0}[(1 + \lambda t)^4\|\partial_x^\alpha\partial_t^j w_{\varepsilon\lambda}(t)\|_{L^2} + (1 + \lambda t)^3\|w_{\varepsilon\lambda}(t)\|_{L^2}] < +\infty \quad (3.23)$$

for $0 < \lambda < 1$ and $0 < \varepsilon < 1$.

We next derive the a priori estimates independent of ε and λ for $w_{\varepsilon\lambda}$. Let α be an arbitrary multi-index with $|\alpha| \leqq 1$. We apply ∂_x^α to (3.19) and take the scalar product in L^2 between the resulting equation and $\partial_t\partial_x^\alpha w_{\varepsilon\lambda}(t)$ to obtain

$$\frac{1}{2}\frac{d}{dt}\left\{\|\partial_t\partial_x^\alpha w_{\varepsilon\lambda}(t)\|^2_{L^2} + \sum_{j=1}^3\|\nabla\partial_x^\alpha w_{\varepsilon\lambda j}(t)\|^2_{L^2}\right\}$$

$$= (1 + \lambda t)^{-4}(\rho_\varepsilon * \{\partial_x^\alpha f(t) + \partial_x^\alpha F(t, v, u)\}, \partial_t\partial_x^\alpha w_{\varepsilon\lambda}(t)), \quad t \geqq 0. \quad (3.24)$$

Integrating (3.24) over $[t, \infty)$, we have by (3.11), (3.17), (2.36) and (3.23)

$$\|\partial_t\partial_x^\alpha w_{\varepsilon\lambda}(t)\|^2_{L^2} + \sum_{j=1}^3\|\nabla\partial_x^\alpha w_{\varepsilon\lambda j}(t)\|^2_{L^2}$$

$$= -2\int_t^\infty (1 + \lambda s)^{-4}(\rho_\varepsilon * \{\partial_x^\alpha f(s) + \partial_x^\alpha F(s, v, u)\}, \partial_s\partial_x^\alpha w_{\varepsilon\lambda}(s))\,ds$$

$$\leqq 2\int_t^\infty \{\|\partial_x^\alpha f(s)\|_{L^2} + \|\partial_x^\alpha F(s, v, u)\|_{L^2}\}\|\partial_s\partial_x^\alpha w_{\varepsilon\lambda}(s)\|_{L^2}\,ds. \quad (3.25)$$

Since α is an arbitrary multi-index with $|\alpha| \leq 1$, we take the summation of (3.25) over $|\alpha| \leq 1$ to obtain

$$\sum_{|\alpha| \leq 1} \| \partial_x^\alpha \partial_t w_{\varepsilon\lambda}(t) \|_{L^2}^2 + \sum_{1 \leq |\alpha| \leq 2} \| \partial_x^\alpha w_{\varepsilon\lambda}(t) \|_{L^2}^2$$

$$\leq 2 \sum_{|\alpha| \leq 1} \int_t^\infty \{ \| \partial_x^\alpha f(s) \|_{L^2} + \| \partial_x^\alpha F(s, v, u) \|_{L^2} \} \| \partial_x^\alpha \partial_s w_{\varepsilon\lambda}(s) \|_{L^2} \, ds$$

$$\leq 2 \int_t^\infty \left\{ \left(\sum_{|\gamma| \leq 1} \| \partial_x^\alpha f(s) \|_{L^2}^2 \right)^{1/2} + \left(\sum_{|\alpha| \leq 1} \| \partial_x^\alpha F(s, v, u) \|_{L^2}^2 \right)^{1/2} \right\}$$

$$\times \left(\sum_{|\alpha| \leq 1} \| \partial_x^\alpha \partial_s w_{\varepsilon\lambda}(s) \|_{L^2}^2 \right)^{1/2} \, ds, \quad t \geq 0 \, . \tag{3.26}$$

At the last inequality of (3.26) we have used the Hölder inequality for sequence. Equations (3.11), (3.17), (2.36), (3.26) and Lemma 2.6 yield

$$\sum_{|\alpha| \leq 1} \| \partial_x^\alpha \partial_t w_{\varepsilon\lambda}(t) \|_{L^2}^2 + \sum_{1 \leq |\gamma| \leq 2} \| \partial_x^\alpha w_{\varepsilon\lambda}(t) \|_{L^2}^2$$

$$\leq 2 \int_t^\infty M\eta (1 + s)^{-4} (\log(2 + s))^3 \, ds$$

$$\times \sup_{t \geq 0} \left[(1 + t)^{3/2} (\log(2 + t))^{-2} \left(\sum_{|\gamma| \leq 1} \| \partial_x^\alpha \partial_t w_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2} \right]$$

$$+ C \int_t^\infty \{ (1 + \eta)^6 \eta^2 (1 + s)^{-4} (\log(2 + s))^2$$

$$+ (1 + \eta)^2 \| u(s) \|_{W^{2,4}}^2 (1 + s)^{-3/2} (\log(2 + s))^2 \} \, ds$$

$$\times \sup_{t \geq 0} \left[(1 + t)^{3/2} (\log(2 + t))^{-2} \left(\sum_{|\gamma| \leq 1} \| \partial_x^\alpha \partial_t w_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2} \right]$$

$$\leq \left\{ 10M\eta (1 + t)^{-3} (\log(2 + t))^4 + C(1 + \eta)^6 \eta^2 (1 + t)^{-3} (\log(2 + t))^4 \right.$$

$$\left. + C(1 + \eta)^2 \left(\int_t^\infty \| u(s) \|_{W^{2,4}}^{8/3} \, ds \right)^{3/4} \left(\int_t^\infty (1 + s)^{-6} (\log(2 + s))^8 \, ds \right)^{1/4} \right\}$$

$$\times \sup_{t \geq 0} \left[(1 + t)^{3/2} (\log(2 + t))^{-2} \left(\sum_{|\alpha| \leq 1} \| \partial_x^\alpha \partial_t w_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2} \right]$$

$$\leq \{ 10M + C_1 (1 + \eta)^6 \eta \} \eta (1 + t)^{-3} (\log(2 + t))^4$$

$$\times \sup_{t \geq 0} \left[(1 + t)^{3/2} (\log(2 + t))^{-2} \left(\sum_{|\alpha| \leq 1} \| \partial_x^\alpha \partial_t w_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2} \right], \quad t \geq 0 \, , \tag{3.27}$$

where C_1 does not depend on ε and λ. If we choose $\eta > 0$ so small that

$$C_1 (1 + \eta)^6 \eta \leq M \, , \tag{3.28}$$

then we obtain by (3.27)

$$\sup_{t \geq 0}\left[(1 + t)^{3/2}(\log(2 + t))^{-2} \left(\sum_{1 \leq |\alpha| \leq 2} \| \partial_x^\alpha w_{\varepsilon\lambda}(t) \|_{L^2}^2 \right. \right.$$

$$\left. \left. + \sum_{|\alpha| \leq 1} \| \partial_x^\alpha \partial_t w_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2} \right] \leq 11\eta M \ . \tag{3.29}$$

In the same way as above, we have by (3.11), (3.18), (2.36) and Lemma 2.6

$$\sum_{|\alpha| = 2} \| \partial_x^\alpha \partial_t w_{\varepsilon\lambda}(t) \|_{L^2}^2 + \sum_{|\alpha| = 3} \| \partial_x^\alpha w_{\varepsilon\lambda}(t) \|_{L^2}^2$$

$$\leq \{ 10M + C_2(1 + \eta)^6\eta \}\eta(1 + t)^{-2}(\log(2 + t))^4$$

$$\times \sup_{t \geq 0}\left[(1 + t)(\log(2 + t))^{-2} \left(\sum_{|\alpha| = 2} \| \partial_x^\alpha \partial_t w_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2} \right], \quad t \geq 0 \ , \tag{3.30}$$

where C_2 does not depend on ε and λ. If we choose $\eta > 0$ so small that

$$C_2(1 + \eta)^6\eta \leq M \ , \tag{3.31}$$

then we obtain by (3.30)

$$\sup_{t \geq 0}\left[(1 + t)(\log(2 + t))^{-2} \left(\sum_{|\alpha| = 3} \| \partial_x^\alpha w_{\varepsilon\lambda}(t) \|_{L^2}^2 \right. \right.$$

$$\left. \left. + \sum_{|\alpha| = 2} \| \partial_x^\alpha \partial_t w_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2} \right] \leq 11M\eta \ . \tag{3.32}$$

From (3.19), (3.8), (3.17), (3.18), (3.29) and (3.32), it follows that for some $C > 0$,

$$\sup_{t \geq 0}\left[(1 + t)^{3/2}(\log(2 + t))^{-2} \| \partial_t^2 w_{\varepsilon\lambda}(t) \|_{L^2} \right.$$

$$\left. + (1 + t)(\log(2 + t))^{-2} \sum_{j + |\alpha| = 1} \| \partial_x^\alpha \partial_t^{2 + j} w_{\varepsilon\lambda}(t) \|_{L^2} \right] \leq C \ , \tag{3.33}$$

where C does not depend on ε and λ. When we consider passage to the limit as $\varepsilon \to + 0$ and $\lambda \to + 0$, (3.29), (3.32), (3.33), a compactness argument and the regularity theorem of linear hyperbolic equation show the existence of the solution $w(t)$ to (3.13)–(3.14) such that

$$w(t) \in C([0, \infty); \dot{H}^3 \cap \dot{H}^1) \ , \tag{3.34}$$

$$\partial_t w(t) \in \bigcap_{j=0}^{2} C^j([0, \infty); H^{2 - j}) \ , \tag{3.35}$$

$$\sup_{t \geq 0}\left[(1 + t)^{3/2}(\log(2 + t))^{-2} \left(\sum_{1 \leq |\alpha| \leq 2} \| \partial_x^\alpha w(t) \|_{L^2}^2 \right. \right.$$

$$\left. \left. + \sum_{|\alpha| \leq 1} \| \partial_x^\alpha \partial_t w(t) \|_{L^2}^2 \right)^{1/2} \right] \leq 11M\eta \ , \tag{3.36}$$

$$\sup_{t \geq 0}\left[(1+t)(\log(2+t))^{-2}\left(\sum_{|\alpha|=3}\|\partial_x^\alpha w(t)\|_{L^2}^2\right.\right.$$

$$\left.\left.+\sum_{|\alpha|=2}\|\partial_x^\alpha \partial_t w(t)\|_{L^2}^2\right)^{1/2}\right] \leq 11M\eta, \tag{3.37}$$

$$\sup_{t \geq 0}\left[(1+t)^{3/2}(\log(2+t))^{-2}\|\partial_t^2 w(t)\|_{L^2}\right.$$

$$\left.+(1+t)(\log(2+t))^{-2}\sum_{j+|\alpha|=1}\|\partial_x^\alpha \partial_t^{2+j}w(t)\|_{L^2}\right] \leq C. \tag{3.38}$$

The uniqueness of the solution follows from the same argument as in the proof of Proposition 2.3.

We next evaluate $z = N_2[v, u]$. By Lemmas 2.2(i), 2.4 and the Gagliardo–Nirenberg inequality (see, e.g., [4]) we have

$$\sum_{|\alpha|\leq 2}\|\partial_x^\alpha G_1(t, v, u)\|_{L^2}$$

$$\leq C(\|A_+\|_{W^{3,\infty}} + \|R\|_{W^{3,\infty}})\|u\|_{H^2}$$

$$+ C(\|\operatorname{div} v\|_{H^2}\|u\|_{L^\infty} + \|\operatorname{div} v\|_{W^{1,4}}\|u\|_{W^{1,4}} + \|\operatorname{div} v\|_{L^\infty}\|u\|_{H^2})$$

$$+ C(1 + \|\nabla\varphi\|_{W^{1,\infty}})^2\left(\|v\|_{L^6}\left\|\frac{x}{t}m\right\|_{W^{2,3}} + \|v\|_{\dot{H}^2\cap\dot{H}^1}\left\|\frac{x}{t}m\right\|_{W^{1,\infty}}\right)$$

$$+ C(1 + \|\nabla\varphi\|_{W^{1,\infty}})^2(\|v\|_{L^\infty}\|\nabla\varphi\|_{W^{2,\infty}}\|m\|_{H^2}$$

$$+ \|v\|_{\dot{H}^2\cap\dot{H}^1}\|\nabla\varphi\|_{W^{1,\infty}}\|m\|_{H^1})$$

$$+ C(1 + \|\nabla\varphi\|_{W^{1,\infty}})^2\frac{\theta(t)}{t}\left(\|v\|_{L^\infty}\left\|(it)^{-3/2}e^{\frac{i|x|^2}{2t}}(\nabla\hat\psi_{+0})\left(\frac{x}{t}\right)\right\|_{H^2}\right.$$

$$\left.+ \|v\|_{\dot{H}^2\cap\dot{H}^1}\left\|(it)^{-3/2}e^{\frac{i|x|^2}{2t}}(\nabla\hat\psi_{+0})\left(\frac{x}{t}\right)\right\|_{W^{1,\infty}}\right)$$

$$+ C(1 + \|\nabla\varphi\|_{W^{1,\infty}})^2\{(\|A_+\|_{W^{2,\infty}} + \|R\|_{W^{2,\infty}})\|v\|_{L^\infty}\|m\|_{H^2}$$

$$+ (\|A_+\|_{W^{1,\infty}} + \|R\|_{W^{1,\infty}})\|v\|_{\dot{H}^2\cap\dot{H}^1}\|m\|_{L^\infty}\}$$

$$+ C(\|A_+\|_{W^{2,\infty}} + \|R\|_{W^{2,\infty}})\|v\|_{L^\infty}\|u\|_{H^2}$$

$$+ C(\|A_+\|_{W^{1,\infty}} + \|R\|_{W^{1,\infty}})\left(\sum_{j=1}^3\|\nabla v_j\|_{L^4}\right)\|u\|_{W^{1,4}}$$

$$+ C(1 + \|\nabla\varphi\|_{W^{2,\infty}})^2\left(\|v\|_{L^\infty}^2\|m\|_{H^2} + \|v\|_{L^\infty}\|v\|_{\dot{H}^1}\|m\|_{W^{1,\infty}}\right.$$

$$\left.+ \sum_{j=1}^3\|\nabla v_j\|_{L^4}^2\|m\|_{L^\infty} + \|v\|_{L^\infty}\|v\|_{\dot{H}^2}\|m\|_{L^\infty}\right)$$

$$+ C\Bigg(\|v\|_{L^\infty}^2 \|u\|_{H^2} + \|v\|_{L^6} \|v\|_{\dot{H}^1} \|u\|_{W^{1,3}}$$

$$+ \sum_{j=1}^{3} \|\nabla v_j\|_{L^4}^2 \|u\|_{L^\infty} + \|v\|_{L^\infty} \|v\|_{\dot{H}^1} \|u\|_{L^\infty} \Bigg)$$

$$\leqq C(1+\eta)^3 \eta^2 (1+t)^{-2} (\log(2+t))^2, \quad t \geqq 0 \tag{3.39}$$

for $[v, u] \in X$. In the same way as above we have

$$\sum_{|\alpha| \leqq 1} \|\partial_x^\alpha \partial_t G_1(t, v, u)\|_{L^2} \leqq C(1+\eta)\eta^2 (1+t)^{-2} (\log(2+t))^2, \quad t \geqq 0 \tag{3.40}$$

for $[v, u] \in X$.

On the other hand, by (2.37), the Hardy–Littlewood–Sobolev inequality (see, e.g., [16]) and the Gagliardo–Nirenberg inequality (see, e.g., [4]) we have

$$\sum_{|\alpha| \leqq 2} \|\partial_x^\alpha G_2(t, v, u)\|_{L^2} \leqq C \left\| \frac{1}{|x|} * \Re(e^{i\varphi} m\bar{u}) \right\|_{W^{2,\infty}} \|e^{i\varphi} m\|_{H^2}$$

$$+ C \left\| \frac{1}{|x|} * |m|^2 \right\|_{W^{2,\infty}} \|u\|_{H^2} + C \left\| \frac{1}{|x|} * |u|^2 \right\|_{W^{2,6}} \|e^{i\varphi} m\|_{W^{2,3}}$$

$$+ C \left\| \frac{1}{|x|} * \Re(e^{i\varphi} m\bar{u}) \right\|_{W^{2,\infty}} \|u\|_{H^2} + C \left\| \frac{1}{|x|} * |u|^2 \right\|_{W^{2,\infty}} \|u\|_{H^2}$$

$$\leqq C(1 + \|\nabla\varphi\|_{W^{1,\infty}})^4 \|m\bar{u}\|_{H^2}^{1/2} \|m\bar{u}\|_{W^{2,6/5}}^{1/2} \|m\|_{H^2}$$

$$+ C \|m\bar{m}\|_{H^2}^{1/2} \|m\bar{m}\|_{W^{2,6/5}}^{1/2} \|u\|_{H^2}$$

$$+ C(1 + \|\nabla\varphi\|_{W^{1,\infty}})^2 \|u\bar{u}\|_{W^{2,6/5}}^{1/2} \|m\|_{W^{2,3}}$$

$$+ C(1 + \|\nabla\varphi\|_{W^{1,\infty}})^2 \|m\bar{u}\|_{H^2}^{1/2} \|m\bar{u}\|_{W^{2,6/5}}^{1/2} \|u\|_{H^2}$$

$$+ \|u\bar{u}\|_{H^2}^{1/2} \|u\bar{u}\|_{W^{2,6/5}}^{1/2} \|u\|_{H^2}$$

$$\leqq C(1 + \|\nabla\varphi\|_{W^{1,\infty}})^4 \|m\|_{W^{2,\infty}}^{1/2} \|u\|_{H^2}^{1/2} \|m\|_{W^{2,3}}^{1/2} \|u\|_{H^2}^{1/2} \|m\|_{H^2}$$

$$+ C \|m\|_{W^{2,\infty}}^{1/2} \|m\|_{H^2}^{1/2} \|m\|_{W^{2,3}}^{1/2} \|m\|_{H^2}^{1/2} \|u\|_{H^2}$$

$$+ C(1 + \|\nabla\varphi\|_{W^{1,\infty}})^2 (\|u\|_{H^2} \|\bar{u}\|_{L^3} + \|u\|_{H^1} \|\bar{u}\|_{W^{1,3}}$$

$$+ \|u\|_{L^3} \|\bar{u}\|_{H^2}) \|m\|_{W^{2,3}}$$

$$+ C(1 + \|\nabla\varphi\|_{W^{1,\infty}})^2 \|m\|_{W^{2,\infty}}^{1/2} \|u\|_{H^2}^{1/2} \|m\|_{W^{2,3}}^{1/2} \|u\|_{H^2}^{1/2} \|u\|_{H^2}$$

$$+ C(\|u\|_{H^2} \|\bar{u}\|_{L^\infty} + \|u\|_{W^{1,4}} \|\bar{u}\|_{W^{1,4}} + \|u\|_{L^\infty} \|\bar{u}\|_{H^2})^{1/2}$$

$$\times (\|u\|_{H^2} \|\bar{u}\|_{L^3} + \|u\|_{H^1} \|\bar{u}\|_{W^{1,3}} + \|u\|_{L^3} \|\bar{u}\|_{H^2})^{1/2} \|u\|_{H^2}$$

$$\leqq C(1+\eta)^4 \eta^3 (1+t)^{-2} (\log(2+t))^2, \quad t \geqq 0 \tag{3.41}$$

for $[v, u] \in X$. In the same way as above we have

$$\sum_{|\alpha| \leqq 1} \|\partial_x^\alpha \partial_t G_2(t, v, u)\|_{L^2} \leqq C(1+\eta)^4 \eta^3 (1+t)^{-2} (\log(2+t))^2, \quad t \geqq 0 \tag{3.42}$$

for $[v, u] \in X$.

We first consider the regularized problem associated with (3.15)–(3.16):

$$2i\partial_t z_{\varepsilon\lambda} + \Delta z_{\varepsilon\lambda} = (1 + \lambda t)^{-4}[2i\rho_\varepsilon * \{(A_+ + R + v) \cdot \nabla(\rho_\varepsilon * z_{\varepsilon\lambda})\}$$

$$+ \rho_\varepsilon * g(t) + \rho_\varepsilon * G_1(t, v, u) + \rho_\varepsilon * G_2(t, v, u)], \quad t \geq 0, \quad x \in \mathbf{R}^3,$$

(3.43)

$$\|z_{\varepsilon\lambda}(t)\|_{L^2} \to 0 \quad (t \to \infty),$$

(3.44)

where $0 < \varepsilon \leq 1$ and $0 < \lambda < 1/4$. By using the contraction mapping principle, we easily see that for each ε and λ, there exists a $T > 0$ such that (3.43)–(3.44) has the unique solution $z_{\varepsilon\lambda}(t)$ satisfying

$$z_{\varepsilon\lambda}(t) \in \bigcap_{j=1}^{\infty} C^2([T, \infty); H^j),$$

(3.45)

$$\sup_{t \geq T} \left[(1 + \lambda t)^4 \sum_{j=0}^{2} \|\partial_x^\alpha \partial_t^j z_{\varepsilon\lambda}(t)\|_{L^2} \right] < +\infty.$$

(3.46)

Since the initial value problem of (3.43) can easily be solved globally in time, the above solution $z_{\varepsilon\lambda}(t)$ can be extended to the time interval $[0, \infty)$.

We next derive the a priori estimates independent of ε and λ for $z_{\varepsilon\lambda}$. Let α be an arbitrary multi-index with $|\alpha| \leq 2$. We apply ∂_x^α to (3.43) and take the scalar product in L^2 between the resulting equation and $\partial_x^\alpha z_{\varepsilon\lambda}(t)$ to obtain

$$i\frac{d}{dt}\|\partial_x^\alpha z_{\varepsilon\lambda}(t)\|_{L^2}^2 = \|\nabla\partial_x^\alpha z_{\varepsilon\lambda}(t)\|_{L^2}^2$$

$$+ 2i(1 + \lambda t)^{-4}(\rho_\varepsilon * \{(A_+ + R + v) \cdot \nabla\partial_x^\alpha(\rho_\varepsilon * z_{\varepsilon\lambda})\}, \partial_x^\alpha z_{\varepsilon\lambda})$$

$$+ (1 + \lambda t)^{-4} \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ |\alpha_2| \leq |\alpha| - 1}} C_{\alpha_1\alpha_2}(2i\rho_\varepsilon * \{\partial_x^{\alpha_1}(A_+ + R + v) \cdot \nabla\partial_x^{\alpha_2}(\rho_\varepsilon * z_{\varepsilon\lambda})\}, \partial_x^\alpha z_{\varepsilon\lambda})$$

$$+ (1 + \lambda t)^{-4}(\rho_\varepsilon * (\partial_x^\alpha g + \partial_x^\alpha G_1(t, v, u) + \partial_x^\alpha G_2(t, v, u)), \partial_x^\alpha z_{\varepsilon\lambda}), \quad t \geq 0. \quad (3.47)$$

Here, the summation at the right-hand side of (3.47) is empty, when $|\alpha| = 0$. By the integration by parts we have

$$(\rho_\varepsilon * \{(A_+ + R + v) \cdot \nabla\partial_x^\alpha(\rho_\varepsilon * z_{\varepsilon\lambda})\}, \partial_x^\alpha z_{\varepsilon\lambda})$$

$$= ((A_+ + R + v) \cdot \nabla(\rho_\varepsilon * \partial_x^\alpha z_{\varepsilon\lambda}), \rho_\varepsilon * \partial_x^\alpha z_{\varepsilon\lambda})$$

$$= -\frac{1}{2}(\text{div}(A_+ + R + v)(\rho_\varepsilon * \partial_x^\alpha z_{\varepsilon\lambda}), \rho_\varepsilon * \partial_x^\alpha z_{\varepsilon\lambda}).$$

(3.48)

We take the imaginary part of (3.47) to obtain by (3.48)

$$\frac{d}{dt}\|\partial_x^\alpha z_{\varepsilon\lambda}(t)\|_{L^2}^2$$

$$= -(1 + \lambda t)^{-4}(\text{div}(A_+ + R + v)(\rho_\varepsilon * \partial_x^\alpha z_{\varepsilon\lambda}), \rho_\varepsilon * \partial_x^\alpha z_{\varepsilon\lambda})$$

$$+ 2(1 + \lambda t)^{-4} \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ |\alpha_2| \leq |\alpha| - 1}} C_{\alpha_1\alpha_2}\Re(\rho_\varepsilon * \{\partial_x^{\alpha_1}(A_+ + R + v) \cdot \nabla\partial_x^{\alpha_2}(\rho_\varepsilon * z_{\varepsilon\lambda})\}, \partial_x^\alpha z_{\varepsilon\lambda})$$

$$+ (1 + \lambda t)^{-4}\Im(\rho_\varepsilon * (\partial_x^\alpha g + \partial_x^\alpha G_1 + \partial_x^\alpha G_2), \partial_x^\alpha z_{\varepsilon\lambda})$$

$$\leqq (\| \operatorname{div} A_+ \|_{L^\infty} + \| \operatorname{div} R \|_{L^\infty} + \| \operatorname{div} v \|_{L^\infty}) \| \partial_x^\alpha z_{\varepsilon\lambda} \|_{L^2}^2$$

$$+ C(\| A_+ \|_{W^{|\alpha|, \infty}} + \| R \|_{W^{|\alpha|, \infty}}) \| z_{\varepsilon\lambda} \|_{H^{|\alpha|}}^2$$

$$+ C \| \partial_x^\alpha v \|_{L^3} \| \nabla z_{\varepsilon\lambda} \|_{L^6}^2 \| \partial_x^\alpha z_{\varepsilon\lambda} \|_{L^2} + C \| v \|_{W^{|\alpha|-1, \infty}} \| z_{\varepsilon\lambda} \|_{H^{|\alpha|}}^2$$

$$+ (\| \partial_x^\alpha g \|_{L^2} + \| \partial_x^\alpha G_1 \|_{L^2} + \| \partial_x^\alpha G_2 \|_{L^2}) \| \partial_x^\alpha z_{\varepsilon\lambda} \|_{L^2}, \quad t \geqq 0 , \quad\quad (3.49)$$

if $1 \leqq |\alpha| \leqq 2$. We take the summation of (3.49) with respect to α and use (3.12), (3.39), (3.41), the Gagliardo–Nirenberg inequality and Lemmas 2.2(i), 2.4 to obtain

$$\frac{d}{dt} \left(\sum_{1 \leqq |\alpha| \leqq 2} \| \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)$$

$$\leqq C(1 + \eta)\eta(1 + t)^{-1} \sum_{|\alpha| \leqq 2} \| \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2}^2$$

$$+ C\eta^2 (1 + t)^{-5/4} (\log(2 + t))^2 \sum_{|\alpha| \leqq 2} \| \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2}^2$$

$$+ C(1 + \eta)\eta(1 + t)^{-1} \sum_{|\alpha| \leqq 2} \| \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2}^2$$

$$+ C\eta(1 + t)^{-5/4} (\log(2 + t))^2 \sum_{|\alpha| \leqq 2} \| \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2}^2$$

$$+ \sum_{1 \leqq |\alpha| \leqq 2} \| \partial_x^\alpha g \|_{L^2} \| \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2}^2$$

$$+ C(1 + \eta)^3 \eta^2 (1 + t)^{-2} (\log(2 + t))^2 \sum_{|\alpha| \leqq 2} \| \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2}^2$$

$$+ C(1 + \eta)^4 \eta^3 (1 + t)^{-2} (\log(2 + t))^2 \sum_{|\alpha| \leqq 2} \| \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2}^2, \quad t \geqq 0 . \quad\quad (3.50)$$

If $|\alpha| = 0$, we have by (3.47), (3.48) with $|\alpha| = 0$ and Lemmas 2.2(i), 2.4

$$\frac{d}{dt} \| z_{\varepsilon\lambda}(t) \|_{L^2}^2 = - (1 + \lambda t)^{-4} (\operatorname{div}(A_+ + R + v)(\rho_\varepsilon * z_{\varepsilon\lambda}), z_{\varepsilon\lambda})$$

$$+ (1 + \lambda t)^{-4} \mathfrak{J}(\rho_\varepsilon * (g + G_1 + G_2), z_{\varepsilon\lambda})$$

$$\leqq C(1 + \eta)\eta(1 + t)^{-1} \| z_{\varepsilon\lambda}(t) \|_{L^2}^2 + C \| v \|_{\dot{H}^2}^{1/2} \| v \|_{\dot{H}^1}^{1/2} \| z_{\varepsilon\lambda}(t) \|_{L^2}^2$$

$$+ (\| g \|_{L^2} + \| G_1 \|_{L^2} + \| G_2 \|_{L^2}) \| z_{\varepsilon\lambda}(t) \|_{L^2}$$

$$\leqq C(1 + \eta)\eta(1 + t)^{-1} \| z_{\varepsilon\lambda}(t) \|_{L^2}^2$$

$$+ C\eta(1 + t)^{-3/2} (\log(2 + t))^2 \| z_{\varepsilon\lambda}(t) \|_{L^2}^2$$

$$+ (\| g \|_{L^2} + \| G_1 \|_{L^2} + \| G_2 \|_{L^2}) \| z_{\varepsilon\lambda}(t) \|_{L^2}, \quad t \geqq 0 . \quad\quad (3.51)$$

Equations (3.50), (3.51), (3.12), the Hölder inequality for sequence and Lemma 2.6 yield

$$\sum_{|\alpha| \leqq 2} \| \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2}^2 \leqq C(1 + \eta)^6 \eta \int_t^\infty (1 + s)^{-1} \sum_{|\alpha| \leqq 2} \| \partial_x^\alpha z_{\varepsilon\lambda}(s) \|_{L^2}^2 \, ds$$

$$+ \int_t^\infty \sum_{|\alpha| \leqq 2} \| \partial_x^\alpha g(s) \|_{L^2} \| \partial_x^\alpha z_{\varepsilon\lambda}(s) \|_{L^2} \, ds$$

$$\leqq C(1 + \eta)^6 \eta \int\limits_t^\infty (1 + s)^{-3}(\log(2 + s))^4 \, ds$$

$$\times \left\{ \sup_{t \geqq 0} \left[(1 + t)(\log(2 + t))^{-2} \left(\sum_{|\alpha| \leqq 2} \| \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2} \right] \right\}^2$$

$$+ \int\limits_t^\infty \left(\sum_{|\alpha| \leqq 2} \| \partial_x^\alpha g(s) \|_{L^2}^2 \right)^{1/2} \left(\sum_{|\alpha| \leqq 2} \| \partial_x^\alpha z_{\varepsilon\lambda}(s) \|_{L^2}^2 \right)^{1/2} ds$$

$$\leqq C_3(1 + \eta)^6 \eta (1 + t)^{-2}(\log(2 + t))^4$$

$$\times \left\{ \sup_{t \geqq 0} \left[(1 + t)(\log(2 + t))^{-2} \left(\sum_{|\alpha| \leqq 2} \| \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2} \right] \right\}^2$$

$$+ 5M\eta(1 + t)^{-2}(\log(2 + t))^4$$

$$\times \sup_{t \geqq 0} \left[(1 + t)(\log(2 + t))^{-2} \left(\sum_{|\alpha| \leqq 2} \| \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2} \right], \quad t \geqq 0 \, , \tag{3.52}$$

where C_3 does not depend on ε and λ. If we choose $\eta > 0$ so small that

$$C_3(1 + \eta)^6 \eta \leqq \frac{1}{2} \, , \tag{3.53}$$

then we have by (3.52)

$$\sup_{t \geqq 0} \left[(1 + t)(\log(2 + t))^{-2} \left(\sum_{|\alpha| \leqq 2} \| \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2} \right] \leqq 6M\eta \, . \tag{3.54}$$

Differentiating (3.43) in t, we have

$$2i\partial_t(\partial_t z_{\varepsilon\lambda}) + \Delta(\partial_t z_{\varepsilon\lambda}) = 2i(1 + \lambda t)^{-4} \rho_\varepsilon * \{ (A_+ + R + v) \cdot \nabla(\rho_\varepsilon * \partial_t z_{\varepsilon\lambda}) \}$$

$$- 8i\lambda(1 + \lambda t)^{-5} \rho_\varepsilon * \{ (A_+ + R + v) \cdot \nabla(\rho_\varepsilon * \partial_t z_{\varepsilon\lambda}) \}$$

$$+ 2i(1 + \lambda t)^{-4} \rho_\varepsilon * \{ (\partial_t A_+ + \partial_t R + \partial_t v) \cdot \nabla(\rho_\varepsilon * z_{\varepsilon\lambda}) \}$$

$$+ \partial_t[(1 + \lambda t)^{-4} \rho_\varepsilon * \{ g + G_1 + G_2 \}], \quad t \geqq 0 \, . \tag{3.55}$$

We take the scalar product in L^2 between (3.55) and $\partial_t z_{\varepsilon\lambda}(t)$ and take the imaginary part of the resulting equality to obtain by (3.48), (3.54), (3.42), Lemmas 2.2(i), 2.4 and the Gagliardo–Nirenberg inequality

$$\frac{d}{dt} \| \partial_t z_{\varepsilon\lambda}(t) \|_{L^2}^2 \leqq (\| \operatorname{div} A_+ \|_{L^\infty} + \| \operatorname{div} R \|_{L^\infty} + \| \operatorname{div} v \|_{L^\infty}) \| \partial_t z_{\varepsilon\lambda} \|_{L^2}^2$$

$$+ C(\| A_+ \|_{L^\infty} + \| R \|_{L^\infty} + \| v \|_{L^\infty}) \| \nabla z_{\varepsilon\lambda} \|_{L^2} \| \partial_t z_{\varepsilon\lambda} \|_{L^2}$$

$$+ C(\| \partial_t A_+ \|_{L^\infty} + \| \partial_t R \|_{L^\infty} + \| \partial_t v \|_{L^\infty}) \| \nabla z_{\varepsilon\lambda} \|_{L^2} \| \partial_t z_{\varepsilon\lambda} \|_{L^2}$$

$$+ (\| g \|_{L^2} + \| G_1 \|_{L^2} + \| G_2 \|_{L^2}) \| \partial_t z_{\varepsilon\lambda} \|_{L^2}$$

$$+ (\| \partial_t g \|_{L^2} + \| \partial_t G_1 \|_{L^2} + \| \partial_t G_2 \|_{L^2}) \| \partial_t z_{\varepsilon\lambda} \|_{L^2}$$

$$\leqq C(1 + \eta)\eta(1 + t)^{-1} \| \partial_t z_{\varepsilon\lambda} \|_{L^2}^2$$

$$+ C(1 + \eta)^5 \eta^2 (1 + t)^{-2} (\log(2 + t))^2 \| \partial_t z_{\varepsilon\lambda} \|_{L^2}$$

$$+ \left(\sum_{j=0}^{1} \| \partial_t^j g \|_{L^2} \right) \| \partial_t z_{\varepsilon\lambda} \|_{L^2}, \quad t \geqq 0 . \tag{3.56}$$

Here, we have used the fact that $0 < \lambda < \frac{1}{4}$ at the first inequality of (3.56). Let α be an arbitrary multi-index with $|\alpha| = 1$. The application of ∂_x^α to (3.55) yields

$$2i\partial_t(\partial_x^\alpha z_{\varepsilon\lambda}) + \Delta(\partial_x^\alpha z_{\varepsilon\lambda}) = 2i(1 + \lambda t)^{-4} \rho_\varepsilon * \{(A_+ + R + v) \cdot \nabla(\rho_\varepsilon * \partial_x^\alpha \partial_t z_{\varepsilon\lambda})\}$$

$$+ 2i(1 + \lambda t)^{-4} \rho_\varepsilon * \{(\partial_x^\alpha A_+ + \partial_x^\alpha R + \partial_x^\alpha v) \cdot \nabla(\rho_\varepsilon * \partial_t z_{\varepsilon\lambda})\}$$

$$- 8i\lambda(1 + \lambda t)^{-5} \rho_\varepsilon * \{(\partial_x^\alpha A_+ + \partial_x^\alpha R + \partial_x^\alpha v) \cdot \nabla(\rho_\varepsilon * z_{\varepsilon\lambda})$$

$$+ (A_+ + R + v) \cdot \nabla(\rho_\varepsilon * \partial_x^\alpha z_{\varepsilon\lambda})\}$$

$$+ 2i(1 + \lambda t)^{-4} \rho_\varepsilon * \{(\partial_x^\alpha \partial_t A_+ + \partial_x^\alpha \partial_t R + \partial_x^\alpha \partial_t v) \cdot \nabla(\rho_\varepsilon * z_{\varepsilon\lambda})$$

$$+ (\partial_t A_+ + \partial_t R + \partial_t v) \cdot \nabla(\rho_\varepsilon * \partial_x^\alpha z_{\varepsilon\lambda})\}$$

$$+ \partial_x^\alpha \partial_t [(1 + \lambda t)^{-4} \rho_\varepsilon * \{g + G_1 + G_2\}], \quad t \geqq 0 . \tag{3.57}$$

We take the scalar product in L^2 between (3.57) and $\partial_x^\alpha \partial_t z_{\varepsilon\lambda}(t)$ and take the imaginary part of the resulting equality to obtain by (3.48), (3.54), (3.42), Lemmas 2.2(i), 2.4 and the Gagliardo–Nirenberg inequality

$$\frac{d}{dt} \| \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2}^2 \leqq (\| \operatorname{div} A_+ \|_{L^\infty} + \| \operatorname{div} R \|_{L^\infty} + \| \operatorname{div} v \|_{L^\infty}) \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda} \|_{L^2}^2$$

$$+ C(\| \partial_x^\alpha A_+ \|_{L^\infty} + \| \partial_x^\alpha R \|_{L^\infty} + \| \partial_x^\alpha v \|_{L^\infty}) \| \nabla \partial_t z_{\varepsilon\lambda} \|_{L^2} \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda} \|_{L^2}$$

$$+ C(\| \partial_x^\alpha A_+ \|_{L^\infty} + \| \partial_x^\alpha R \|_{L^\infty} + \| \partial_x^\alpha v \|_{L^\infty}) \| \nabla z_{\varepsilon\lambda} \|_{L^2} \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda} \|_{L^2}$$

$$+ C(\| A_+ \|_{L^\infty} + \| R \|_{L^\infty} + \| v \|_{L^\infty}) \| \nabla \partial_x^\alpha z_{\varepsilon\lambda} \|_{L^2} \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda} \|_{L^2}$$

$$+ C(\| \partial_x^\alpha \partial_t A_+ \|_{L^\infty} + \| \partial_x^\alpha \partial_t R \|_{L^\infty}) \| \nabla z_{\varepsilon\lambda} \|_{L^2} \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda} \|_{L^2}$$

$$+ C \| \partial_x^\alpha \partial_t v \|_{L^4} \| \nabla z_{\varepsilon\lambda} \|_{L^4} \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda} \|_{L^2}$$

$$+ C(\| \partial_t A_+ \|_{L^\infty} + \| \partial_t R \|_{L^\infty} + \| \partial_t v \|_{L^\infty}) \| \nabla \partial_x^\alpha z_{\varepsilon\lambda} \|_{L^2} \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda} \|_{L^2}$$

$$+ (\| \partial_x^\alpha g \|_{L^2} + \| \partial_x^\alpha G_1 \|_{L^2} + \| \partial_x^\alpha G_2 \|_{L^2}) \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda} \|_{L^2}$$

$$+ (\| \partial_x^\alpha \partial_t g \|_{L^2} + \| \partial_x^\alpha \partial_t G_1 \|_{L^2} + \| \partial_x^\alpha \partial_t G_2 \|_{L^2}) \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda} \|_{L^2}$$

$$\leqq C(1 + \eta)\eta(1 + t)^{-1} \sum_{|\beta| = 1} \| \partial_x^\beta \partial_t z_{\varepsilon\lambda} \|_{L^2}^2$$

$$+ C(1 + \eta)\eta^2 (1 + t)^{-2} (\log(2 + t))^2 \sum_{|\beta| = 1} \| \partial_x^\beta \partial_t z_{\varepsilon\lambda} \|_{L^2}$$

$$+ C(1 + \eta^5)\eta^2 (1 + t)^{-2} (\log(2 + t))^2 \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda} \|_{L^2}$$

$$+ \left\{ \sum_{j=0}^{1} \| \partial_x^\alpha \partial_t^j g \|_{L^2} \right\} \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda} \|_{L^2}, \quad t \geqq 0 . \tag{3.58}$$

Here, we have used the fact that $0 < \lambda < 1/4$ at the first inequality of (3.58). Equations (3.56), (3.58), (3.12), the Hölder inequality for sequence and Lemma 2.6 yield

$$\sum_{|\alpha| \leq 1} \| \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2}^2 \leq C(1+\eta)\eta \int_t^\infty (1+s)^{-1} \sum_{|\alpha| \leq 1} \| \partial_x^\alpha z_{\varepsilon\lambda}(s) \|_{L^2}^2 \, ds$$

$$+ C(1+\eta)^5 \eta^2 \int_t^\infty (1+s)^{-2} (\log(2+s))^2 \sum_{|\alpha| \leq 1} \| \partial_x^\alpha z_{\varepsilon\lambda}(s) \|_{L^2} \, ds$$

$$+ \int_t^\infty \left\{ \sum_{j=0}^1 \| \partial_t^j g(s) \|_{L^2} \right\} \| \partial_t z_{\varepsilon\lambda}(s) \|_{L^2} \, ds$$

$$+ \int_t^\infty \left[\sum_{|\alpha| \leq 1} \| \partial_x^\alpha \partial_s g(s) \|_{L^2} \| \partial_x^\alpha \partial_s z_{\varepsilon\lambda}(s) \|_{L^2} \right.$$

$$\left. + \sum_{|\alpha| \leq 1} \| \partial_x^\alpha g(s) \|_{L^2} \| \partial_x^\alpha \partial_s z_{\varepsilon\lambda}(s) \|_{L^2} \right] ds$$

$$\leq C_4 (1+\eta)\eta (1+t)^{-2} (\log(2+t))^4$$

$$\times \left\{ \sup_{t \geq 0} \left[(1+t)(\log(2+t))^{-2} \left(\sum_{|\alpha| \leq 1} \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2} \right] \right\}^2$$

$$+ C_5 (1+\eta)^5 \eta^2 (1+t)^{-2} (\log(2+t))^2$$

$$\times \sup_{t \geq 0} \left[(1+t)(\log(2+t))^{-2} \left(\sum_{|\alpha| \leq 1} \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2} \right]$$

$$+ 10M\eta (1+t)^{-2} (\log(2+t))^4$$

$$\times \sup_{t \geq 0} \left[(1+t)(\log(2+t))^{-2} \right.$$

$$\left. \times \left(\sum_{|\alpha| \leq 1} \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2} \right], \quad t \geq 0 , \tag{3.59}$$

where C_4 and C_5 do not depend on ε and λ. If we choose $\eta > 0$ so small that

$$C_4 (1+\eta)\eta \leq \frac{1}{2} , \tag{3.60}$$

$$C_5 (1+\eta)^5 \eta \leq \frac{1}{2} M , \tag{3.61}$$

then we obtain by (3.58)

$$\sup_{t \geq 0} \left[(1+t)(\log(2+t))^{-2} \left(\sum_{|\alpha| \leq 1} \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2} \right] \leq 21M\eta . \tag{3.62}$$

Furthermore, from (3.43), (3.54), (3.62), (3.39), (3.41), the Hölder inequality for sequence and the Gagliardo–Nirenberg inequality it follows that

$$
\sum_{|\alpha|=1} \| \Delta \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2} \leq 2 \sum_{|\alpha|=1} \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda}(t) \|_{L^2}
$$
$$
+ 2 \sum_{|\alpha|=1} \| \partial_x^\alpha \rho_\varepsilon * \{ (A_+ + R + v) \cdot \nabla(\rho_\varepsilon * z_{\varepsilon\lambda}) \} \|_{L^2}
$$
$$
+ \sum_{|\alpha|=1} \| \partial_x^\alpha \rho_\varepsilon * \{ g + G_1 + G_2 \} \|_{L^2}
$$
$$
\leq 2\sqrt{3} \left(\sum_{|\alpha|=1} \| \partial_x^\alpha \partial_t z_{\varepsilon\lambda}(t) \|_{L^2}^2 \right)^{1/2}
$$
$$
+ C(\| A_+ \|_{W^{1,\infty}} + \| R \|_{W^{1,\infty}} + \| v \|_{W^{1,\infty}}) \| z_{\varepsilon\lambda} \|_{H^2}
$$
$$
+ \sum_{|\alpha|=1} (\| \partial_x^\alpha g \|_{L^2} + \| \partial_x^\alpha G_1 \|_{L^2} + \| \partial_x^\alpha G_2 \|_{L^2})
$$
$$
\leq [(42\sqrt{3} + 1) M\eta + C_6(1 + \eta)\eta(M\eta)
$$
$$
+ C_7 \{ 1 + (1 + \eta)^5 \} \eta^2] (1 + t)^{-1} (\log(2 + t))^2, \quad t \geq 0 ,
$$
$$
\tag{3.63}
$$

where C_6 and C_7 do not depend on ε and λ. If we choose $\eta > 0$ so small that

$$
C_6(1 + \eta)\eta \leq 1 , \tag{3.64}
$$
$$
C_7 \{ 1 + (1 + \eta)^5 \} \eta \leq M , \tag{3.65}
$$

then we obtain by (3.62),

$$
\sup_{t \geq 0} \left[(1 + t)(\log(2 + t))^{-2} \sum_{|\alpha|=1} \| \Delta \partial_x^\alpha z_{\varepsilon\lambda}(t) \|_{L^2} \right] \leq 3(14\sqrt{3} + 1)M\eta . \tag{3.66}
$$

Let α be an arbitrary multi-index with $|\alpha| \leq 2$. The application of ∂_x^α to (3.43) yields

$$
2i\partial_t(\partial_x^\alpha z_{\varepsilon\lambda}) + \Delta(\partial_x^\alpha z_{\varepsilon\lambda}) = (1 + \lambda t)^{-4} \partial_x^\alpha [2i\rho_\varepsilon * \{ (A_+ + R + v) \cdot \nabla(\rho_\varepsilon * z_{\varepsilon\lambda}) \}
$$
$$
+ \rho_\varepsilon * g + \rho_\varepsilon * G_1 + \rho_\varepsilon * G_2], \quad t \geq 0, \quad x \in \mathbf{R}^3 . \tag{3.67}
$$

Since $z_{\varepsilon\lambda}(t) \to 0$ in H^2 as $t \to \infty$, we have by (3.67) and Duhamel's principle

$$
\partial_x^\alpha z_{\varepsilon\lambda}(t) = \frac{i}{2} \int_t^\infty (1 + \lambda s)^{-4} U(t - s) [2i\rho_\varepsilon * \partial_x^\alpha \{ (A_+ + R + v) \cdot \nabla(\rho_\varepsilon * z_{\varepsilon\lambda}) \}
$$
$$
+ \rho_\varepsilon * \partial_x^\alpha g + \rho_\varepsilon * \partial_x^\alpha G_1 + \rho_\varepsilon * \partial_x^\alpha G_2] \, ds, \quad t \geq 0 . \tag{3.68}
$$

We take the $L^{8/3}(t, \infty; L^4)$ norm of (3.68) to obtain by Lemma 2.1(ii) with $q = 4$ and $r = 8/3$,

$$
\left(\int_t^\infty \| \partial_x^\alpha z_{\varepsilon\lambda}(s) \|_{L^4}^{8/3} \, ds \right)^{3/8} \leq 2K_1 \int_t^\infty \| \partial_x^\alpha \{ (A_+ + R + v) \cdot \nabla(\rho_\varepsilon * z_{\varepsilon\lambda}) \} \|_{L^2} \, ds
$$
$$
+ K_1 \int_t^\infty \| \partial_x^\alpha g \|_{L^2} \, ds + K_1 \int_t^\infty \| \partial_x^\alpha G_1 \|_{L^2} \, ds
$$
$$
+ K_1 \int_t^\infty \| \partial_x^\alpha G_2 \|_{L^2} \, ds, \quad t \geq 0 , \tag{3.69}
$$

where K_1 is defined in Lemma 2.1(ii). The summation of (3.69) over $|\alpha| \leqq 2$ gives us

$$\sum_{|\alpha| \leqq 2} \left(\int_t^\infty \| \partial_x^\alpha z_{\varepsilon\lambda}(s) \|_{L^4}^{8/3} \, ds \right)^{3/8}$$

$$\leqq 2K_1 \int_t^\infty \sum_{|\alpha| \leqq 2} \| \partial_x^\alpha \{ (A_+ + R + v) \cdot \nabla(\rho_\varepsilon * z_{\varepsilon\lambda}) \} \|_{L^2} \, ds$$

$$+ K_1 \int_t^\infty \sum_{|\alpha| \leqq 2} \| \partial_x^\alpha g \|_{L^2} \, ds + K_1 \int_t^\infty \sum_{|\alpha| \leqq 2} \| \partial_x^\alpha G_1 \|_{L^2} \, ds$$

$$+ K_1 \int_t^\infty \sum_{|\alpha| \leqq 2} \| \partial_x^\alpha G_2 \|_{L^2} \, ds, \quad t \geqq 0 \, . \tag{3.70}$$

Lemmas 2.2(i), 2.3, (3.54), (3.64) and the Gagliardo–Nirenberg inequality yield

$$\sum_{|\alpha| \leqq 2} \| \partial_x^\alpha \{ (A_+ + R + v) \cdot \nabla(\rho_\varepsilon * z_{\varepsilon\lambda}) \}(s) \|_{L^2}$$

$$\leqq C \Bigg[(\| A_+ \|_{L^\infty} + \| R \|_{L^\infty} + \| v \|_{L^\infty}) \| z_{\varepsilon\lambda} \|_{H^3}$$

$$+ \sum_{|\alpha| = 1} (\| \partial_x^\alpha A_+ \|_{L^\infty} + \| \partial_x^\alpha R \|_{L^\infty} + \| \partial_x^\alpha v \|_{L^\infty}) \| z_{\varepsilon\lambda} \|_{H^2}$$

$$+ \sum_{|\alpha| = 2} (\| \partial_x^\alpha A_+ \|_{L^\infty} + \| \partial_x^\alpha R \|_{L^\infty}) \| z_{\varepsilon\lambda} \|_{H^1}$$

$$+ \sum_{|\alpha| = 2} \| \partial_x^\alpha v \|_{L^4} \| z_{\varepsilon\lambda} \|_{W^{1,4}} \Bigg]$$

$$\leqq C(1 + \eta)\eta^2 (1 + s)^{-2} (\log(2 + s))^2, \quad s \geqq 0 \, . \tag{3.71}$$

By (3.70), (3.71), (3.12), (3.39), (3.41) and Lemma 2.6 we have

$$\sum_{|\alpha| \leqq 2} \left(\int_t^\infty \| \partial_x^\alpha z_{\varepsilon\lambda}(s) \|_{L^4}^{8/3} \, ds \right)^{3/8}$$

$$\leqq CK_1(1 + \eta)\eta^2 \int_t^\infty (1 + s)^{-2} (\log(2 + s))^2 \, ds$$

$$+ K_1 M \eta \int_t^\infty (1 + s)^{-2} (\log(2 + s))^2 \, ds$$

$$+ CK_1(1 + \eta)^3 \eta^2 \int_t^\infty (1 + s)^{-2} (\log(2 + s))^2 \, ds$$

$$+ K_1(1 + \eta)^4 \eta^3 \int_t^\infty (1 + s)^{-2} (\log(2 + s))^2 \, ds$$

$$\leqq \{ 5K_1 M \eta + C_8(1 + \eta)^5 \eta(K_1 \eta) \}(1 + t)^{-1} (\log(2 + t))^2, \quad t \geqq 0 \, , \tag{3.72}$$

where C_8 does not depend on ε and λ. If we choose $\eta > 0$ so small that

$$C_8(1 + \eta)^5 \eta \leqq M \, , \tag{3.73}$$

then we obtain by (3.71)

$$\sup_{t \geq 0} \left[(1 + t)(\log(2 + t))^{-2} \sum_{|\alpha| \leq 2} \left(\int_t^\infty \| \partial_x^\alpha z_{\varepsilon\lambda}(s) \|_{L^4}^{8/3} \, ds \right)^{3/8} \right] \leq 6K_1 M\eta \ . \quad (3.74)$$

When we consider passage to the limit as $\varepsilon \to +0$ and $\lambda \to +0$, (3.54), (3.62), (3.66), (3.74), a compactness argument and the regularity theorem of linear theory show the existence of the solution $z(t)$ to (3.15)–(3.16) such that

$$z(t) \in \bigcap_{j=0}^1 C^j([0, \infty); H^{3-2j}) \ , \quad (3.75)$$

$$\sup_{t \geq 0} \left[(1 + t)(\log(2 + t))^{-2} \left(\sum_{|\alpha| \leq 2} \| \partial_x^\alpha z(t) \|_{L^2}^2 \right)^{1/2} \right] \leq 6M\eta \ , \quad (3.76)$$

$$\sup_{t \geq 0} \left[(1 + t)(\log(2 + t))^{-2} \left(\sum_{|\alpha| \leq 1} \| \partial_x^\alpha \partial_t z(t) \|_{L^2}^2 \right)^{1/2} \right] \leq 21M\eta \ , \quad (3.77)$$

$$\sup_{t \geq 0} \left[(1 + t)(\log(2 + t))^{-2} \left(\sum_{|\alpha| = 1} \| \Delta \partial_x^\alpha z(t) \|_{L^2}^2 \right)^{1/2} \right] \leq 3(14\sqrt{3} + 1)M\eta \ , \quad (3.78)$$

$$\sup_{t \geq 0} \left[(1 + t)(\log(2 + t))^{-2} \sum_{|\alpha| \leq 2} \left(\int_t^\infty \| \partial_x^\alpha z(s) \|_{L^4}^{8/3} \, ds \right)^{3/8} \right] \leq 6K_1 M\eta \ . \quad (3.79)$$

We now prove the uniqueness of the solution $z(t)$ satisfying (3.75)–(3.79). We suppose that $z_1(t)$ and $z_2(t)$ are two solutions of (3.15)–(3.16) satisfying (3.75)–(3.79). If we put $Z(t) = z_1(t) - z_2(t)$, then $Z(t)$ satisfies (3.15) with $g(t) \equiv G_1(t, v, u) \equiv G_2(t, v, u) \equiv 0$. Accordingly, in the same way as the case of the regularized problem we can prove that if $\eta > 0$ is chosen so small that (3.53) holds, $Z(t)$ satisfies (3.54) with $4M\eta$ replaced by zero. This implies the uniqueness.

Thus, if we choose $\eta > 0$ so small that (3.10), (3.28), (3.31), (3.53), (3.60), (3.61), (3.64), (3.65) and (3.73) hold, then (3.34)–(3.38) and (3.75)–(3.79) show that $N[v, u]$ is a mapping from X to X. In the same way as above, we can prove that if $\eta > 0$ is chosen sufficiently small, then

$$d(N[v_1, u_1], N[v_2, u_2]) \leq \frac{1}{2} d([v_1, u_1], [v_2, u_2]), \quad [v_1, u_1], [v_2, u_2] \in X \ .$$

Consequently, we conclude that if $\eta > 0$ is sufficiently small, then $N[v, u]$ is a contraction mapping from X to X. Therefore, there exists a unique fixed point $[v, u] \in X$ such that $[v, u] = N[v, u]$. Since $[v, u]$ belongs to the image of N, v and u must satisfy (3.34)–(3.38) and (3.75)–(3.79), respectively. Furthermore, by Corollary 2.5(i) and the definition of the mapping N we can conclude that if we define $A(t)$ and $\psi(t)$ as in (3.1) and (3.2), respectively, then $A(t)$ and $\psi(t)$ are the solutions of (1.14)–(1.15) satisfying (1.5)–(1.9).

We next show that $A(t)$ satisfies the Coulomb gauge condition (1.3). We first note that $(4\pi|x|)^{-1}$ is the fundamental solution of $-\Delta$. The application of the divergence operator to (1.14) yields

$$\partial_t^2(\operatorname{div} A) - \Delta(\operatorname{div} A) = 0, \quad t > 0, \quad x \in \mathbf{R}^3 \ . \quad (3.80)$$

Since $\operatorname{div} A$ is a solution of (3.80), div satisfies the energy identity:

$$\|\partial_t \operatorname{div} A(t)\|_{L^2}^2 + \|\nabla \operatorname{div} A(t)\|_{L^2}^2 = \|\partial_t \operatorname{div} A(s)\|_{L^2}^2 + \|\nabla \operatorname{div} A(s)\|_{L^2}^2$$

$$= \|\partial_t \operatorname{div} A_+(s) + \partial_t \operatorname{div} R(s)$$

$$+ \{\partial_t \operatorname{div} A(s) - \partial_t \operatorname{div} A_+(s) - \partial_t \operatorname{div} R(s)\}\|_{L^2}^2$$

$$+ \|\nabla \operatorname{div} A_+(s) + \nabla \operatorname{div} R(s)$$

$$+ \{\nabla \operatorname{div} A(s) - \nabla \operatorname{div} A_+(s)$$

$$- \nabla \operatorname{div} R(s)\}\|_{L^2}^2, \quad 0 < t, \ s < \infty. \tag{3.81}$$

Letting $s \to \infty$ in (3.81), we have by (1.8), (2.31) and (1.13)

$$\|\partial_t \operatorname{div} A(t)\|_{L^2}^2 + \|\nabla \operatorname{div} A(t)\|_{L^2}^2 = 0, \quad t \geqq 0,$$

which implies that $A(t)$ satisfies the Coulomb gauge condition (1.3). We can easily verify that (1.1)–(1.2) are equivalent to (1.14)–(1.15) for the solutions in the class of (1.5)–(1.7). Therefore, we conclude that $(A(t), \psi(t))$ are the solutions of (1.1)–(1.3) satisfying (1.5)–(1.9).

It remains only to prove the uniqueness of the solutions satisfying (1.5)–(1.9). We omit the proof of the uniqueness, because the uniqueness follows from a similar (but slightly simpler) argument to above. For the details, see [13] and [14].

Concluding Remarks. (i) In the proof of Theorem 1.1 we have not used the gauge condition (1.3) at all, although the use of (1.3) makes the proof a little simpler. This is because we intend to make it clear that the gauge condition (1.3) does not play any essential role in the proof of Theorem 1.1. But the gauge condition plays an important role in the paper [5], which treats the initial value problem of the Maxwell–Dirac equations. It seems important to investigate the relation between the Maxwell–Schrödinger equations and the gauge condition.

(ii) The global existence of small amplitude solutions for the initial value problem of the covariant nonlinear wave equation has been studied through the $L^\infty - L^2$ estimates with weights related to the generators of the Poincaré group (see, e.g., [1, 5 and 10]). Recently in [7], Hayashi has proved the global existence of small amplitude solutions for the initial value problem of the Schrödinger equation with quadratic nonlinearity by using the $L^\infty - L^2$ estimates with weights related to the generators of the Schrödinger group. It is conjectured that the global existence of small amplitude solutions for the initial value problem of the Maxwell–Schrödinger equations could be proved through the $L^\infty - L^2$ estimates with weights related to the generators of the Poincaré group and the Schrödinger group, although the Poincaré group and the Schrödinger group are not necessarily compatible.

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